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On the representations and \mathbb{Z}_2 -equivariant normal form for solenoidal Hopf-zero singularities

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HIGHLIGHTS

- Constructing the Poisson algebra for a Lie subalgebra of Hopf-zero singularities.
- Significant representations of solenoidal family of Hopf-zero singularities.
- Unique normal form classification of solenoidal family of Hopf-pitchfork singularities.
- \mathbb{Z}_2 -equivariant normal form of modified Chua's oscillator.

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ABSTRACT

In this paper, we deal with the solenoidal conservative Lie algebra associated to the classical normal form of Hopf-zero singular system. We concentrate on the study of some representations and \mathbb{Z}_2 -equivariant normal form for such singular differential equations. First, we list some of the representations that this Lie algebra admits. The vector fields from this Lie algebra could be expressed by the set of ordinary differential equations where the first two of them are in the canonical form of a one-degree of freedom Hamiltonian system and the third one depends upon the first two variables. This representation is governed by the associated Poisson algebra to one sub-family of this Lie algebra. Euler's form, vector potential, and Clebsch representation are other representations of this Lie algebra that we list here. We also study the non-potential property of vector fields with Hopf-zero singularity from this Lie algebra. Finally, we examine the unique normal form with non-zero cubic terms of this family in the presence of the symmetry group \mathbb{Z}_2 . The theoretical results of normal form theory are illustrated with the modified Chua's oscillator.

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1. Introduction

This investigation is a continuation of [1] in which the maximal solenoidal conservative Lie algebra of classical normal form of Hopf-zero singularities (in the sense of [2]) was introduced. This Lie algebra was denoted by \mathcal{L} . The simplest normal form, simplest parametric normal form, and radius of convergence corresponding to the second level normal form of this type of singularities were explored there. This present paper has two purposes: first, we intend to list some of the representations that the vector fields from \mathcal{L} admit. Solenoidal property of vector fields from \mathcal{L} allows us to express these vector fields by local Euler's potentials, vector potential, and local Clebsch potentials, see [3,4]. The second

purpose is to explore the \mathbb{Z}_2 -equivariant unique normal form of the solenoidal conservative family associated to Hopf-pitchfork singularities. We denote the Lie algebra of such a family by $\mathcal{L}^{\mathbb{Z}_2}$. Now, we elaborate in general terms our results in the following.

In the framework of classical mechanics, the Poisson structure has a key role in the description of Hamiltonian dynamics (see preface of [5]). In the literature of normal form theory, this structure is employed to facilitate the normal form study of singularities, an exhaustive treatment of this subject can be found in [6,7]. The first issue we address here involves constructing the Poisson algebra \mathcal{P} for a Lie subalgebra of \mathcal{L} , in order to give a representation for \mathcal{L} . Having established the Poisson structure we will be able to present the vector fields from this Lie algebra by a set of ordinary differential equations, the two first of them are in the canonical

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form of a one-degree of freedom Hamiltonian system and the third one depends upon the two first variables. Noting that, for a given three-dimensional solenoidal vector field which possesses one-parameter symmetry group with solenoidal and infinitesimal generators, one would be able to present the system in the normal form as given in [3]. Hence, besides the Poisson structure following [1, Remark 2.4] and [3] that form could be derived.

Solenoidal dynamical systems are important in a huge variety of applications. These vector fields, also known as incompressible vector fields, occur in many settings: velocity field of an incompressible fluid, magnetic field, *etc.*, see for instance [8,9]. Exploiting tools from geometric dynamics, solenoidal vector fields may be represented in some interesting and physically significant representations such as Euler's form and vector potential (see [4] and references therein). In the following, we briefly review these representations. If v be a solenoidal vector field then there exists a vector potential \mathbf{A} such that $v = \nabla \times \mathbf{A}$. From the point of view of electromagnetism, \mathbf{A} is called magnetic vector potential [10]. Euler's form is another representation that solenoidal vector fields admit (see [11, page 22] and [12, page 48]). Based on this form, one can find two independent invariant functions α and β from which the vector field v is derivable via $v = \nabla \alpha \times \nabla \beta$. Applications of these representations can be found in various books and papers, for instance, see [13,4,14,10]. In [4] the application of Euler's form in the partial differential equation has been discussed. For the significance of vector potential in quantum theory, we refer the reader to [13].

In [15] the authors express a Lie algebra of completely integrable solenoidal triple-zero singularities via Euler's form and vector potential. As in their studies, due to the solenoidal property of \mathcal{L} , here we shall present any vector field in this Lie algebra using vector potentials and Euler's form. Further, the non-potential property of this family with Hopf-zero singularity is examined. Thus, these vector fields cannot be derived by the gradient of a scalar valued function, instead, these vector fields are expressible in terms of the lamellar and complex lamellar vector fields. More precisely, any vector field $v \in \mathcal{L}$ could be expressed by $v = f_1 \nabla f_2 + \nabla f_3$ where f_1, f_2 , and f_3 are scalar valued functions. This representation is named Monge representation or Clebsch representation (see [11, page 27] and [12, Section 2.4]). For the application of this representation in thermodynamics, we refer to [12, Subsection 9.11].

Progress towards deriving these representations has significant practical implications for magnetic fields since these vector fields are solenoidal. For instance, in [8,9] these representations have been employed for studying magnetic reconnection at three-dimensional null points. In [16] the authors used the magnetic fields to interpret the solar flares. Hence, the results of the present paper should provide the researchers in this area with sufficiently powerful tools to analyze the motion of magnetic fields. As a matter of fact, each of these representation gives an interpretation about the structure of magnetic fields. Thereby, from these researches, we believe that these studies would be remarkably useful in application. In the remainder of introduction, we focus on the second aim of the paper.

In order to apply the general methods of bifurcation theory to singularities, it is necessary to apply normal form theory. Roughly speaking, normal form theory is to simplify the nonlinear part of vector fields with permissible transformations, see [17,18] and [19, chapters 9–13]. In this paper, we are also interested in treating the simplest normal form classification of solenoidal family associated to the classical normal form of Hopf-pitchfork singularities. The problem of the normal form of singular dynamical systems has been studied by many authors. Before explaining our results in detail, we explain some of this previous work in the following.

Baider and Sanders [6,7] studied the unique normal form of Bogdanov–Takens and Hamiltonian Bogdanov–Takens singularities. The paper [20] studied the infinite level normal form of the Lie algebra of quasi-Eulerian Hopf-zero vector fields. For the conservative–nonconservative decomposition of the classical normal form of Hopf-zero dynamical systems and complete classifications of the simplest normal form of this singularity, the reader is referred to [21]. In [22] the \mathbb{Z}_2 -equivariant normal form for Hopf-zero vector fields is computed under the assumption that the cubic terms be non-zero, for results on the bifurcation of Hopf-pitchfork singularities, see [23–25]. More studies regarding the normal form of dynamical system could be found in [26,27,15,28].

In the studies mentioned above and in the literature, the researchers did not investigate the problem of classifying the unique normal form for \mathbb{Z}_2 -equivariant solenoidal conservative Hopf-zero vector fields. As announced at the beginning of this section, our aim is to treat the unique normal form of the following system

$$\begin{cases} \frac{dx}{dt} = 2a_1^0 x \rho^2 + \sum a_k^l (k-2l+1) x^{2l+1} \rho^{2(k-2l)}, \\ \frac{d\rho}{dt} = -\frac{a_1^0 \rho^3}{2} - \sum a_k^l \frac{(2l+1)}{2} x^{2l} \rho^{2(k-2l)+1}, \\ \frac{d\theta}{dt} = 1 + \sum b_k^l x^{2l} \rho^{2(k-2l)}, \end{cases} \quad (1.1)$$

where constants a_k^l, b_k^l are real numbers, $0 \leq 2l \leq k, 0 \leq k$, $b_0^0 = a_0^0 = 0$, and $a_1^0 \neq 0$. The associated first integral of this system is

$$s(x, \rho) := a_1^0 x \rho^4 + \sum a_k^l x^{2l+1} \rho^{2(k-2l+1)},$$

with $0 \leq 2l \leq k$. In this work, the unique normal form study of the above system proceeds in a manner parallel to the study of the solenoidal Hopf-zero vector fields without symmetry [1]. We use naturally their algebraic structures to construct the Lie algebra $\mathcal{L}^{\mathbb{Z}_2}$. We wish to stress that, despite the fact that $\mathcal{L}^{\mathbb{Z}_2}$ is the Lie subalgebra of \mathcal{L} , the unique normal form of (1.1) cannot be obtained from the unique normal form of volume preserving Hopf-zero vector fields given there. The leading term that plays a dominant role in our normal form study is cubic, whereas there the quadratic term was the leading term. Hence, the normalization problem that is performed here differs from that was studied in [1]. Finally, we shall present the unique normal form of (1.1) in four different representations, see Theorem 4.4.

1.1. Outline of the paper

This paper has the following organization. In Section 2, first, we recall some notations and definitions that are needed throughout the paper. Then, we provide the associated Poisson algebra \mathcal{P} for one Lie subalgebra of \mathcal{L} . Following the Poisson algebra, a form for solenoidal Hopf-zero vector fields using Hamilton's equations as introduced in [3] is presented. It is also in this section that the representations such as Euler's form, vector potential, and Clebsch representation for Lie algebra \mathcal{L} are given.

In Section 3, we introduce and formulate the Lie algebra $\mathcal{L}^{\mathbb{Z}_2}$. We also recall the general framework for computing the normal form for Γ -equivariant singularities required to study the unique normal form of the class of singularities under consideration.

In Section 4, we study the unique normal form of (1.1). One symmetry of unique normal form is detected. We also present the four alternative representations of the unique normal form rely on the results on Section 2.

The final section is dedicated to make symbolic computations of our normal form study. Some sufficient conditions on the coefficients of any Hopf-pitchfork system, under which the lower order truncation of the classical normal form of the original system takes the form (1.1) are given. Moreover, the modified Chua's oscillator serves to demonstrate our main results in the normal form. All of the computations are performed using Maple [29].

2. Representations of Lie algebra \mathcal{L}

2.1. Preliminaries

In this section, we present the main results regarding the ways that the vector fields from \mathcal{L} would be expressible. The Poisson algebra associated to the sub-family of \mathcal{L} and tools from geometric dynamics are adapted to obtain these forms. Before going to the main results, we give a review of some definitions and facts from the Lie algebra \mathcal{L} which are fundamental to what follows. Furthermore, after we establish some notation, we recall some general information required to study the representations of vector fields in \mathcal{L} .

The classical normal form of solenoidal conservative Hopf-zero vector field in cylindrical coordinates is given by

$$\begin{cases} \frac{dx}{dt} = \sum (k-l+1) a_k^l x^{l+1} \rho^{2(k-l)}, \\ \frac{d\rho}{dt} = -\sum \frac{(l+1)}{2} a_k^l x^l \rho^{2(k-l)+1}, \\ \frac{d\theta}{dt} = \sum b_n^m x^m \rho^{2(n-m)}, \end{cases} \quad (2.1)$$

where $-1 \leq l \leq k$, $0 \leq k$, $0 \leq m \leq n$, $b_0^{-1} = a_0^0 = 0$, and $a_k^l, b_n^m \in \mathbb{R}$, see [1, Equation 1.1]. This class of vector fields was derived by \mathfrak{sl}_2 -decomposition of the classical normal form of Hopf-zero bifurcation [21]. And indeed, the above system could be formulated using the Lie algebraic structure as follows.

We recall [1,21] that the maximal Lie algebra of solenoidal Hopf-zero classical normal form is given by $\mathcal{L} = \mathcal{F} \oplus \mathcal{T}$, where

$$\mathcal{F} = \text{span} \left\{ \sum a_k^l F_k^l \mid 0 \leq k, -1 \leq l \leq k, a_k^l \in \mathbb{R} \right\},$$

$$\mathcal{T} = \text{span} \left\{ \sum b_k^l \Theta_k^l \mid 0 \leq l \leq k, b_k^l \in \mathbb{R} \right\},$$

and

$$F_k^l = x^l (y^2 + z^2)^{k-l} \left((k-l+1)x \frac{\partial}{\partial x} - \frac{(l+1)}{2} y \frac{\partial}{\partial y} - \frac{(l+1)}{2} z \frac{\partial}{\partial z} \right), \quad -1 \leq l \leq k, \quad (2.2)$$

$$\Theta_k^l = x^l (y^2 + z^2)^{k-l} \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right), \quad 0 \leq l \leq k. \quad (2.3)$$

Furthermore, the algebra of the first integral for \mathcal{F} is as $\langle x^{l+1} (y^2 + z^2)^{k-l+1} \rangle_{-1 \leq l \leq k}$ and for \mathcal{T} is as $\langle x, (y^2 + z^2) \rangle$. Expressed in terms of the cylindrical polar coordinates (x, ρ, θ) , the preceding vector fields will be given by the expressions

$$F_k^l = x^l \rho^{2(k-l)} \left((k-l+1)x \frac{\partial}{\partial x} - \frac{(l+1)}{2} \rho \frac{\partial}{\partial \rho} \right), \quad (2.4)$$

$$\Theta_k^l = x^l \rho^{2(k-l)} \frac{\partial}{\partial \theta}. \quad (2.5)$$

Using the structures given above, system (2.1) can be recast to

$$v = \sum_{k=0}^{\infty} \sum_{l=-1}^k a_k^l F_k^l + \sum_{n=0}^{\infty} \sum_{m=0}^n b_n^m \Theta_n^m.$$

As mentioned before, the simplest normal form and simplest parametric normal form of foregoing system with the assumption $a_0^{-1} \neq 0$ were explored using \mathfrak{sl}_2 -style in [1]. Now, we fix some notation.

Notation 2.1. The following notation is used throughout the paper.

- Define $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ by $v = v_1 \cdot \mathbf{e}_x + v_2 \cdot \mathbf{e}_y + v_3 \cdot \mathbf{e}_z$. We denote $F_k^l = dx(F_k^l) \frac{\partial}{\partial x} + dy(F_k^l) \frac{\partial}{\partial y} + dz(F_k^l) \frac{\partial}{\partial z}$.
- The symbol ∇ indicates the gradient operator of the vector field.
- The Pochhammer k -symbol notation for any $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$, is given by

$$(a)_b^k := \prod_{j=0}^{k-1} (a + jb).$$

Definition 2.2. Consider the dynamical system $\dot{x} = v$, with $x \in \mathbb{R}^3$.

- The vector field v is said to be potential, or locally potential if $\nabla \times v = 0$ for all $x \in \mathbb{R}^3$. The system is named non-potential, otherwise, see [30, page 1].
- A vector field v which derives from the gradient of a function is called lamellar (also known as gradient, or globally potential) vector field. The function is called the potential function. It follows that a vector field is lamellar if and only if the system is potential, see [11, page 23] and [30, page 1].
- The three-dimensional vector field v which has the representation of the form $v = f \nabla g$, in which f and g are functions, is named complex lamellar. The vector field v is complex lamellar if and only if $v \cdot (\nabla \times v) = 0$, that is, this type of vector field is orthogonal to its curl, see [11, page 23].

2.2. Poisson structure

As mentioned, the algebra of the first integral for \mathcal{F} is spanned by $\langle x^{l+1} (y^2 + z^2)^{k-l+1} \rangle_{-1 \leq l \leq k}$. In what follows, we extend this algebra to the Poisson structure for \mathcal{F} by equipping it with an appropriate Poisson bracket. In accordance with this Poisson algebra, one representation of any vector fields in \mathcal{F} by the Hamiltonian equation is given.

Performing the change of variable $r = \rho^2$ into the vector field given by (2.4) we obtain

$$F_k^l = x^l r^{k-l} \left((k-l+1)x \frac{\partial}{\partial x} - (l+1)r \frac{\partial}{\partial r} \right). \quad (2.6)$$

Now, define

$$\mathcal{P} := \left\{ \sum c_k^l f_k^l \mid -1 \leq l \leq k \right\}, \quad (2.7)$$

where $f_k^l := x^{l+1} r^{k-l+1}$ is the first integral of F_k^l . Now, we have the following result.

Theorem 2.3. Consider the vector field F_k^l given by (2.6) and the algebra \mathcal{P} given by (2.7). The following statements hold.

1. $(\mathcal{P}, \{\cdot, \cdot\})$ is Poisson algebra where the Poisson bracket is given by

$$\{f, g\} := \frac{\partial f}{\partial r} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial r} \frac{\partial f}{\partial x}, \quad \text{for all } f, g \in \mathcal{P}. \quad (2.8)$$

2. Hamilton's equations of F_k^l are

$$F_k^l = \{f_k^l, x\} \frac{\partial}{\partial x} + \{f_k^l, r\} \frac{\partial}{\partial r}. \quad (2.9)$$

3. $(\mathcal{P}, \{\cdot, \cdot\})$ and $(\mathcal{F}, [\cdot, \cdot])$ are isomorphic Lie algebras where Lie isomorphism is defined by $\varphi : (\mathcal{P}, \{\cdot, \cdot\}) \rightarrow (\mathcal{F}, [\cdot, \cdot])$, with $\varphi(f_k^l) = F_k^l$.

Proof. One can readily check that $(\mathcal{P}, \{\cdot, \cdot\})$ is a Poisson algebra (see [31, Section 5] and [5, Chapter one]). A direct calculation using the Hamiltonian function given by the first item and Poisson bracket given by (2.8) establish (2.9) (see also preface of [5]). The last claim follows from the straightforward calculation of structure constants given by (2.8) and [1, Lemma 2.5]. \square

Corollary 2.4. The class of Hopf-zero singular system given by (2.1) may be represented through appropriate coordinates transformations to the form

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H(x, r)}{\partial r} = \{H(x, r), x\}, \\ \frac{dr}{dt} = -\frac{\partial H(x, r)}{\partial x} = \{H(x, r), r\}, \\ \frac{d\theta}{dt} = G(x, r), \end{cases} \quad (2.10)$$

where $H(x, r) := \sum a_k^l x^{l+1} r^{k-l+1}$ is a constant of the motion and $G(x, r) := 1 + \sum b_n^m x^m r^{n-m}$.

Proof. The corollary can be verified in two ways: (1) recast the dynamical system (1.1) by employing the coordinate change $r = \rho^2$. Then, taking Hamilton's equations corresponding to F_k^l given by Eq. (2.9) into account, immediately verify the statement of the corollary. Then (2) follows from [1, Remark 2.4] and the procedure given in the proof of [3, Theorem 2.2] with the slight modification. In fact, replace J by $2J$, into that proof. First by employing the change of variables $y = \rho \cos(\theta)$, $z = \rho \sin(\theta)$, and $x = x$, the differential equation (2.1) goes over into the system

$$\begin{cases} \frac{dx}{dt} = \frac{\partial K(x, \rho)}{2J \partial \rho}, \\ \frac{d\rho}{dt} = -\frac{\partial K(x, \rho)}{2J \partial x}, \\ \frac{d\theta}{dt} = G(x, \rho), \end{cases}$$

where $J = \rho$ is the Jacobian of the cylindrical transformation, $K(x, \rho) := \sum a_k^l x^{l+1} \rho^{2(k-l+1)}$, and $G(x, \rho) := 1 + \sum b_n^m x^m \rho^{2(n-m)}$. To proceed further, carrying out the transformation $r = \int 2J = \rho^2$ into the former system. This turns the above system into the form given by (2.10), see also [3, Example 3]. The corollary follows. \square

2.3. Euler's form for \mathcal{L}

Any solenoidal vector field may be represented by Euler's form as

$$v = h(g_1, g_2) \nabla g_1 \times \nabla g_2, \quad (2.11)$$

where h , g_1 , and g_2 are scalar valued functions. We remark that g_1 and g_2 may not be defined everywhere on the domain of v . The functions g_1 and g_2 are first integrals or vector sheets of vector field v and named Euler's potentials. The geometrical meaning of Euler's form is that the vector lines of solenoidal vector fields are the intersection of level surfaces of Euler's potentials, see [11, page 22], Euler's Theorem [12, page 48], and [16].

In the following result, we express the vector fields in \mathcal{L} by local Euler's potentials.

Theorem 2.5. The Euler's forms of F_k^l and Θ_k^l are given as follows

$$F_k^l = \nabla \left(\frac{1}{2} \arctan\left(\frac{z}{y}\right) \right) \times \nabla (x^{l+1}(y^2 + z^2)^{k-l+1}), \quad \text{for } y \neq 0, \quad (2.12)$$

$$\Theta_k^l = \nabla (x^{l+1}) \times \nabla \left(\frac{-(y^2 + z^2)^{k-l+1}}{2(l+1)(k-l+1)} \right). \quad (2.13)$$

Proof. We prove this theorem only for F_k^l . The proof for Θ_k^l can be done analogously. Due to the solenoidal property of vector fields in \mathcal{L} and Euler's Theorem [12, page 48], F_k^l may be written as (2.11).

Setting the first integrals of F_k^l as local Euler's potentials, in fact, $g_1 := x^{l+1}(y^2 + z^2)^{k-l+1}$ and $g_2 := \frac{1}{2} \arctan(\frac{z}{y})$. Now, we show that $h(f_1, f_2)$ given by (2.11) equals one. By straightforward calculation, one has

$$\begin{aligned} \nabla \left(\frac{1}{2} \arctan\left(\frac{z}{y}\right) \right) &= \frac{-z}{2(y^2 + z^2)} \cdot \mathbf{e}_y + \frac{y}{2(y^2 + z^2)} \cdot \mathbf{e}_z, \\ \nabla (x^{l+1}(y^2 + z^2)^{k-l+1}) &= x^l(y^2 + z^2)^{k-l+1} \left((l+1)(y^2 + z^2) \right. \\ &\quad \cdot \mathbf{e}_x + 2(k-l+1)xy \cdot \mathbf{e}_y \\ &\quad \left. + 2(k-l+1)xz \cdot \mathbf{e}_z \right). \end{aligned}$$

By taking the cross product of the above vector fields and considering (2.2) one finds

$$\nabla \left(\frac{1}{2} \arctan\left(\frac{z}{y}\right) \right) \times \nabla (x^{l+1}(y^2 + z^2)^{k-l+1}) = F_k^l.$$

This implies that $h(f_1, f_2) = 1$ and concludes our assertion. \square

2.4. Vector potential for \mathcal{L}

Another way that solenoidal vector field could be constructed, is by means of the vector potential. These vector fields are expressible from another vector field by taking its curl. To find more information about the physical significant of vector potential, see [32–34].

Theorem 2.6. The vector fields F_k^l and Θ_k^l could be expressed in the following form.

$$\begin{aligned} F_k^l &= \nabla \times \left(\frac{1}{2} x^{l+1}(y^2 + z^2)^{k-l} (-z \cdot \mathbf{e}_y + y \cdot \mathbf{e}_z) \right), \\ \Theta_k^l &= \nabla \times \left(\frac{-x^l(y^2 + z^2)^{k-l+1}}{2(k-l+1)} \cdot \mathbf{e}_x \right). \end{aligned}$$

Proof. The proof follows from [16, Equation 5] and Euler's form given by Theorem 2.5. \square

The vector potential is not unique since the curl of a gradient is zero. In the following theorem we use an alternative approach to construct a vector potential associated to the solenoidal vector fields F_k^l and Θ_k^l .

Theorem 2.7. The vector fields F_k^l and Θ_k^l can be produced in the following form as

$$F_k^l = \nabla \times \mathbf{A}_k^l, \quad (2.14)$$

$$\Theta_k^l = \nabla \times \mathbf{B}_k^l, \quad (2.15)$$

where the vector potentials \mathbf{A}_k^l and \mathbf{B}_k^l are defined as follows

$$\begin{aligned} \mathbf{A}_k^l &:= \sum_{j=0}^{k-l} \frac{y^{2j} z^{2(k-l-j)+1} x^l \binom{k-l}{j}}{2(k-l-j)+1} \\ &\quad \times \left(\frac{l+1}{2} y \cdot \mathbf{e}_x + (k-l+1)x \cdot \mathbf{e}_y \right), \\ \mathbf{B}_k^l &:= \frac{x^l}{2(k-l+1)} ((y^2 + z^2)^{k-l+1} - y^{2(k-l+1)}) \\ &\quad \cdot \mathbf{e}_x - \frac{x^{l+1} y^{2k-2l+1}}{l+1} \cdot \mathbf{e}_y. \end{aligned}$$

Proof. To prove the theorem, we follow the approach given in [35]. By solving the equality $F_k^l = \nabla \times \mathbf{A}_k^l$, we obtain the following

equalities

$$\begin{aligned} \frac{\partial}{\partial z} \mathbf{A}_k^l \cdot \mathbf{e}_x &= dy(F_k^l), & -\frac{\partial}{\partial z} \mathbf{A}_k^l \cdot \mathbf{e}_y &= dx(F_k^l), \\ \frac{\partial}{\partial x} \mathbf{A}_k^l \cdot \mathbf{e}_y - \frac{\partial}{\partial y} \mathbf{A}_k^l \cdot \mathbf{e}_x &= dz(F_k^l). \end{aligned} \quad (2.16)$$

The first identity of (2.16) implies that

$$\mathbf{A}_k^l \cdot \mathbf{e}_x = -\frac{l+1}{2} x^l \sum_{j=0}^{k-l} \frac{\binom{k-l}{j} y^{2j+1} z^{2(k-l-j)+1}}{2(k-l-j)+1} + p_1(x, y), \quad (2.17)$$

analogously the second identity of (2.16) implies that

$$\begin{aligned} \mathbf{A}_k^l \cdot \mathbf{e}_y &= -(k-l+1) x^{l+1} \sum_{j=0}^{k-l} \frac{\binom{k-l}{j} y^{2j} z^{2(k-l-j)+1}}{2(k-l-j)+1} \\ &\quad + p_2(x, y), \end{aligned} \quad (2.18)$$

where $p_1(x, y)$ and $p_2(x, y)$ are functions. Now by substituting (2.17) and (2.18) into the last identity of (2.16) one can find, $\frac{\partial}{\partial z} p_1(x, y) - \frac{\partial}{\partial x} p_2(x, y) = 0$. Thus, without loss of generality we may assume that $p_1(x, y) = p_2(x, y) = 0$, the equality (2.14) concludes. By repeating this procedure for Θ_k^l we can verify equality (2.15). \square

2.5. Clebsch representation for \mathcal{L}

In this part, we shall study the following problems: namely, we shall study the non-potential property of Hopf-zero singularities from \mathcal{L} , and we shall present an alternative representation that is the Clebsch representation of any vector fields from \mathcal{L} . Based on this representation for a given vector field $v \in \mathbb{R}^3$, for any point $x \in \mathbb{R}^3$ in which $\nabla \times v \neq 0$, one can find three scalar valued functions f_1, f_2 , and f_3 such that

$$v = f_1 \nabla f_2 + \nabla f_3. \quad (2.19)$$

These scalar valued functions are called Clebsch potentials of v . Noting that f_1, f_2 and f_3 may not be defined everywhere on the domain of v . Moreover, this representation shows that any vector fields may be presented by summation of the lamellar field and complex lamellar field, see Definition 2.2. For those interested in knowing that how this representation is constructed, we include the following discussion from the proof of Clebsch's Theorem of [12, Section 2.4].

Since $\nabla \times v$ is a solenoidal vector field then, there exist local Euler's potentials f_1 and f_2 such that $\nabla \times v = \nabla f_1 \times \nabla f_2$. Making use of the vector calculus identity we have $\nabla \times (v - f_1 \nabla f_2) = 0$. Hence, $v - f_1 \nabla f_2$ is irrotational vector field, then there exists potential function f_3 such that $v - f_1 \nabla f_2 = \nabla f_3$, see Definition 2.2. Then, the relation (2.19) follows.

For the Euler's form (and consequently the representation given by vector potential) to be possible, the vector field v has to be solenoidal. We remark that deriving the vector field v from Clebsch potentials f_1, f_2 , and f_3 through the relation (2.19), is not related to the solenoidal property of vector field v . In fact, any vector field may be represented in the Clebsch representation, see [12, Section 2.4].

Theorem 2.8. Any $v \in \mathcal{L}$ with Hopf-zero singularity is the non-potential vector field.

Proof. Recall from [1], the grading function for generators of \mathcal{L} given by $\delta(F_k^l) = \delta(\Theta_k^l) = k$. Suppose that the non-zero Hopf-zero vector field $v = \sum_{j=0}^{\infty} v_j \in \mathcal{L}$ where $\delta(v_j) = j$ is given. Assume that the claim of the theorem does not hold. Similar to argument

given in [15], since the vector fields with different grades do not have any monomial in common, it implies that $\nabla \times v = 0$ if and only if $\nabla \times v_j = 0$, for all $j \in \mathbb{N}_0$. Thus without loss of generality, it suffices to show that $\nabla \times v_j = 0$. Taking into account the defined grading function, the vector field v_j may be represented by

$$v_j := \sum_{i=-1}^{\lfloor \frac{j}{2} \rfloor} a_i F_j^i + \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} b_i \Theta_j^i,$$

where a_i and b_i are real constants for all i . Applying the curl operator to v_j yields

$$\begin{aligned} \nabla \times v_j &= \sum_{i=-1}^{\lfloor \frac{j}{2} \rfloor} (0, 0, a_i \frac{i(i+1)}{2} x^{i-1} \rho^{2(j-i)+1} \\ &\quad + 2a_i(j-i)(j-i+1) x^{i+1} \rho^{2(j-i)-1}) \\ &\quad - \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} (ib_i x^{i-1} \rho^{2(j-i)}, 0, 2(j-i+1)b_i x^i \rho^{2(j-i)-1}). \end{aligned}$$

Hence, the only way that the foregoing relation vanishes is either $v_j = 0$ or $v_0 = x \frac{\partial}{\partial x} - \frac{1}{2} \rho \frac{\partial}{\partial \rho}$. This implies that either $v = 0$ or the vector field is not Hopf-zero singularity, which is in contradiction to our assumption. Hence, the claim is proved. \square

In the terminology of Definition 2.2, this result shows that the solenoidal family of Hopf-zero singularities are not lamellar vector fields. In this sense, these vector fields cannot be expressed by the gradient of the scalar valued function. In what follows, the Clebsch representation of \mathcal{L} is given.

Theorem 2.9. The following hold.

- For given $F_k^l \in \mathcal{T}$ there exist Clebsch potentials f_1, f_2 , and f_3 such that

$$F_k^l = f_1 \nabla f_2 + \nabla f_3, \quad (2.20)$$

where for $l \neq 0$,

$$\begin{aligned} f_1 &:= x^l, \\ f_2 &:= -\frac{(l+1)}{4(k-l+1)} (y^2 + z^2)^{k-l+1} \\ &\quad - \frac{(k-l+1)}{l} x^2 (y^2 + z^2)^{k-l}, \end{aligned} \quad (2.21)$$

$$f_3 := \frac{1}{l} (y^2 + z^2)^{k-l} (k-l+1) x^{l+2}.$$

Otherwise

$$\begin{aligned} f_1 &:= -x, & f_2 &:= (k+1)x(y^2 + z^2)^k, \\ f_3 &:= -(k+1)x^2(y^2 + z^2)^k + \frac{1}{4(k+1)}(y^2 + z^2)^{k+1}. \end{aligned}$$

- For each vector field $\Theta_k^l \in \mathcal{T}$ there exist local Clebsch potential g_1 and global Clebsch potentials g_2, g_3 as follows

$$\begin{aligned} g_1 &:= \frac{z}{y}, & g_2 &:= y^2 x^l (y^2 + z^2)^{k-l}, \\ g_3 &:= -yzx^l (y^2 + z^2)^{k-l}, & \text{for } y \neq 0, \end{aligned}$$

such that

$$\Theta_k^l = g_1 \nabla g_2 + \nabla g_3.$$

Proof. Following the discussions at the beginning of this part, first we need to find the Euler's form for $\nabla \times F_k^l$. Writing $\nabla \times F_k^l$, given

by the preceding theorem using Cartesian coordinates gives

$$\nabla \times F_k^l = (y^2 + z^2)^{k-l-1} x^{l-1} \left(\frac{l(l+1)}{2} (y^2 + z^2) + 2(k-l)(k-l+1)x^2 \right) (z \cdot \mathbf{e}_y - y \cdot \mathbf{e}_z).$$

Suppose that $l \neq 0$, then by straightforward calculation one can verify that $\nabla \times F_k^l = \nabla f_1 \times \nabla f_2$, where f_1, f_2 are given by Eq. (2.21). In order that (2.20) to be true the following equalities must hold

$$dx(F_k^l) - f_1 \nabla f_2 \cdot \mathbf{e}_x = \nabla f_3 \cdot \mathbf{e}_x, \quad (2.22)$$

$$dy(F_k^l) - f_1 \nabla f_2 \cdot \mathbf{e}_y = \nabla f_3 \cdot \mathbf{e}_y. \quad (2.23)$$

Eq. (2.22) follows after some computations that

$$f_3 = x^l (y^2 + z^2)^{k-l} \left(\frac{(k-l+1)x^2}{l+1} - \frac{y^2 + z^2}{4(k-l+1)} \right) + h(y, z).$$

Substituting f_3 in (2.23) results in $h(y, z) = 0$. In this way, we obtain the expression (2.20). If $l = 0$ one can check that

$$\begin{aligned} F_k^0 &= (y^2 + z^2)^k \left((k+1)x \cdot \mathbf{e}_x - \frac{1}{2}y \cdot \mathbf{e}_y - \frac{1}{2}z \cdot \mathbf{e}_z \right) \\ &= -x \nabla ((k+1)x(y^2 + z^2)^k) \\ &\quad + \nabla \left(-(k+1)x^2(y^2 + z^2)^k + \frac{1}{4(k+1)}(y^2 + z^2)^{k+1} \right). \end{aligned}$$

The proof for Θ_k^l is analogous to the proof of F_k^l . \square

3. Solenoidal conservative \mathbb{Z}_2 -equivariant Lie algebra

In this section, we shall introduce the solenoidal conservative Lie algebra associated to the classical normal form of Hopf-pitchfork singularities (1.1). We also recall the theory of unique normal form at the end of this part.

To start, we recall the following definition from [36, Chapter XII].

Definition 3.1. Let $\dot{x} = v$ with $x \in \mathbb{R}^n$, be an autonomous dynamical system where v , is smooth. Let Γ , be a compact Lie group in $\text{Gl}(n)$. This system is called Γ -equivariant if $v(\gamma x) = \gamma v$ for all $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$.

Define

$$H_k^l := (k-2l+1)x^{2l+1}\rho^{2(k-2l)}\frac{\partial}{\partial x} - \frac{(2l+1)}{2}x^{2l}\rho^{2(k-2l)+1}\frac{\partial}{\partial \rho}, \quad (3.1)$$

$$\Theta_k^l := x^{2l}\rho^{2(k-2l)}\frac{\partial}{\partial \theta}, \quad (3.2)$$

where $0 \leq 2l \leq k$. These vector fields are invariant under the linear map

$$\mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto (-x, -y, -z). \quad (3.3)$$

However, the foregoing vector fields are \mathbb{Z}_2 -equivariant version of those that are given by Eqs. (2.4) and (2.5), in order to avoid any confusion, we would like to change the notations. Thereby, instead of F -terms and Θ -terms we shall write H -terms and Θ -terms.

Denote

$$\mathcal{F}^{\mathbb{Z}_2} := \left\langle \sum a_k^l H_k^l \mid a_k^l \in \mathbb{R}, 0 \leq 2l \leq k, 1 \leq k \right\rangle \quad (3.4)$$

and

$$\mathcal{F}^{\mathbb{Z}_2} := \left\langle \sum b_k^l \Theta_k^l \mid b_k^l \in \mathbb{R}, 0 \leq 2l \leq k, 0 \leq k \right\rangle. \quad (3.5)$$

Then we define the maximal Lie algebra of solenoidal conservative \mathbb{Z}_2 -equivariant classical normal form of Hopf-zero vector fields by $\mathcal{L}^{\mathbb{Z}_2} := \mathcal{F}^{\mathbb{Z}_2} \oplus \mathcal{F}^{\mathbb{Z}_2}$. Since $\mathcal{L}^{\mathbb{Z}_2}$ is a Lie subalgebra of \mathcal{L} , then $\mathcal{L}^{\mathbb{Z}_2}$ inherits the geometrical properties such as conservation, incompressibility, and rotationality, from \mathcal{L} . See also [1, Theorem 2.4].

Our next task is to present the structure constants for $\mathcal{L}^{\mathbb{Z}_2}$.

Lemma 3.2. The following relations always hold.

$$[H_k^l, H_n^m] = ((2m+1)(k+2) - (2l+1)(n+2)) H_{k+n}^{l+m},$$

$$[H_k^l, \Theta_n^m] = (2m(k+2) - n(2l+1)) \Theta_{k+n}^{l+m},$$

$$[\Theta_k^l, \Theta_n^m] = 0.$$

Proof. The proof follows from [1, Lemma 2.5]. \square

The following specific cases of preceding result will be useful in the rest of the paper.

$$[H_1^0, H_n^m] = (6m - n + 1) H_{n+1}^m, \quad (3.6)$$

$$[H_1^0, \Theta_n^m] = (6m - n) \Theta_{n+1}^m. \quad (3.7)$$

The remainder of this section is devoted to revisiting a concise but detailed outline of the theory and method concerning finding the unique normal form for Γ -equivariant singularities.

- Denote $\text{ad}(u)v$ for $\text{ad}(u)v = uv - vu$, where u and v are two arbitrary vector fields.
- Denote \mathcal{L}^Γ for the space of all Γ -equivariant vector fields.
- Denote \mathcal{L}_k^Γ for the space of Γ -equivariant graded Lie algebra with grade k . Γ -equivariant graded Lie algebra means that $\text{ad}(v_i)v_j \in \mathcal{L}_{i+j}^\Gamma$, for arbitrary $v_i \in \mathcal{L}_i^\Gamma$ and $v_j \in \mathcal{L}_j^\Gamma$.

Consider the differential equation $v_s = \sum_{i=0}^\infty v_i$, where $v_s \in \mathcal{L}^\Gamma$. First, we define a linear map to find the first level normal form as follows

$$d_s^{k,1} : \mathcal{L}_k^\Gamma \rightarrow \mathcal{L}_k^\Gamma,$$

$$d_s^{k,1}(Y_k) := \text{ad}(Y_n)v_0.$$

The first level normal form is given by $v^{(1)} = \sum_{i=0}^\infty v_i^{(1)}$, where $v_i^{(1)} \in \mathcal{C}^{k,1}$ for all i and $\mathcal{C}^{k,1}$ is the complement space to $\text{Im}(d_s^{k,1})$. Note that in order to preserve the symmetric structure of v_s , the transformations should be taken from \mathcal{L}_k^Γ . Proceeding inductively, we define

$$d_s^{k,n} : \mathcal{L}_k^\Gamma \times \ker d_s^{k,n-1} \rightarrow \mathcal{L}_k^\Gamma,$$

$$d_s^{k,n}(Y_s^k, Y_s^{k-1}, \dots, Y_s^{k-n+1}) := \sum_{i=0}^{n-1} \text{ad}(Y_s^{k-i})v_i, \quad \text{for any } n \leq k.$$

Then, there exists the complement subspace $\mathcal{C}^{k,n}$ such that $\text{Im}(d_s^{k,n}) \oplus \mathcal{C}^{k,n} = \mathcal{L}_k^\Gamma$, where $\mathcal{C}^{k,n}$ follows the normal form style. Then the n -level normal form of v_s is given by $w_s = \sum_{i=0}^\infty w_i$, where $w_i \in \mathcal{C}^{k,n}$ for all i . For the more detailed treatment of this theory, we refer to [37,18,17] and [19, sections 9–13].

4. Normal form

In this section, we examine the unique normal form of dynamical system (1.1). We closely follow the approach in [21,6,1]. In

terms of H-terms and Θ -terms as introduced in (3.1) and (3.2), this system leads to the following expression

$$w^{(1)} := \Theta_0^0 + b_1^0 \Theta_1^0 + a_1^0 H_1^0 + \sum a_k^l H_k^l + \sum b_k^l \Theta_k^l, \quad (4.1)$$

for $0 \leq 2l \leq k$, $1 \leq k$, where $a_1^0 \neq 0$, a_k^l , and b_k^l are real constants which could be computed explicitly in terms of the coefficients of the original system using our Maple program. We shall depart from the foregoing system.

Lemma 4.1. *Given the dynamical system defined by (4.1), there exists a sequence of \mathbb{Z}_2 -equivariant transformations that send the system to the following second level normal form*

$$w^{(2)} := \Theta_0^0 + b_0^{(2)} \Theta_1^0 + a_1^{(2)} H_1^0 + \sum_{i=1}^{\infty} a_i^{(2)} H_{2i}^i + \sum_{i=1}^{\infty} b_i^{(2)} \Theta_{2i}^i, \quad (4.2)$$

in which all of the coefficients are real constants.

Proof. Let δ be grading function defined by $\delta(H_n^m) := n$, $\delta(\Theta_n^m) := n + 1$. Using the structure constants given by (3.6) and (3.7) one can deduce that $C^{k,2} = \text{span}\{\Theta_1^0, H_{2m}^m, \Theta_{2m}^m \mid m \geq 1\}$ for any $k \geq 1$ and the lemma follows. \square

Similar to [1], one can remove Θ_0^0 using the linear change of variables, for further details, see [17, Lemma 5.3.6] and [21]. In what follows, we intend to simplify the second level normal form; $w^{(2)}$. First, nonetheless, we need some notational conventions.

Suppose that there exists a non-zero H_{2l}^l in $w^{(2)}$ for some l and there exists a non-zero Θ_{2k}^k in $w^{(2)}$ for some k . Then, define

$$r := \min\{l \mid a_l^{(2)} \neq 0, l \geq 1\}, \quad s := \min\{k \mid b_k^{(2)} \neq 0, k \geq 1\}.$$

Define the following grading function

$$\delta(H_n^m) := r(n - 2m) + m, \quad \delta(\Theta_n^m) := r(n - 2m) + r + m + 1.$$

Let us denote the leading order term of (4.2) by \mathbb{H}_r . According to the above grading function, we have that

$$\mathbb{H}_r := H_1^0 + a_r^{(2)} H_{2r}^r. \quad (4.3)$$

This vector field plays a prominent role in the sequel. By performing re-scaling x in (4.2) as

$$x \rightarrow \left| \frac{a_r}{a_r^{(2)}} \right|^{\frac{1}{2r}} x,$$

then \mathbb{H}_r could be replaced by

$$\mathbb{H}_r := H_1^0 + a_r H_{2r}^r, \quad (4.4)$$

which implies that a_r could be taken as an arbitrary real constant. Henceforth, for simplicity, we set $a_r = 1$.

For further reduction of (4.2), the following lemma is quite useful.

Lemma 4.2. *For each $H_n^m \in \mathcal{F}^{\mathbb{Z}_2}$ and $\Theta_n^m \in \mathcal{T}^{\mathbb{Z}_2}$ there exist transformations \mathcal{S}_n^m and \mathcal{R}_n^m , such that the following hold.*

$$[\mathcal{S}_n^m, \mathbb{H}_r] + H_n^m = \frac{(-1)^n (4mr - 2nr + 4m - n + 1)_{4r+1}^{n-2m}}{(6m - n + 2)_{4r+1}^{n-2m}} \times H_{2(nr+m-2mr)}^{nr+m-2mr}, \quad (4.5)$$

$$[\mathcal{R}_n^m, \mathbb{H}_r] + \Theta_n^m = \frac{(-1)^n (4mr - 2nr + 4m - n + 2r + 1)_{4r+1}^{n-2m-1}}{2(6m - n + 1)_{4r+1}^{n-2m-1}} \times \Theta_{2(nr+m-2mr)}^{nr+m-2mr}. \quad (4.6)$$

Proof. Define

$$\mathcal{S}_n^m := \sum_{j=0}^{n-2m-1} \frac{(-1)^{j+1} (4mr - 2nr + 4m - n + 1)_{4r+1}^j}{(6m - n + 2)_{4r+1}^{j+1}} H_{2jr-j+n-1}^{jr+m},$$

$$\mathcal{R}_n^m := \sum_{j=0}^{n-2m-1} \frac{(-1)^{j+1} (4mr - 2nr + 4m - n + 2r + 1)_{4r+1}^j}{2(6m - n + 1)_{4r+1}^{j+1}} \times \Theta_{2jr-j+n-1}^{jr+m}.$$

Then, substitute the previous transformations into the left hand side of (4.5) and (4.6), respectively. The right hand side could be obtained readily. \square

The following theorem is the main result of this section.

Theorem 4.3. *The unique normal form of system (1.1) under assumption $a_1^0 \neq 0$ is given by*

$$w^{(\infty)} := \Theta_0^0 + H_1^0 + H_{2r}^r + b_0 \Theta_1^0 + \sum_{i=r+1}^{\infty} a_i H_{2i}^i + \sum_{i=s}^{\infty} b_i \Theta_{2i}^i, \quad (4.7)$$

or equivalently in the cylinder coordinates $w^{(\infty)}$ takes the form

$$\begin{cases} \frac{dx}{dt} = 2x\rho^2 + x^{2r+1} + \sum_{i=r+1}^{\infty} a_i x^{2i+1}, \\ \frac{d\rho}{dt} = -\frac{1}{2}\rho^3 - \frac{(2r+1)}{2}x^{2r}\rho - \sum_{i=r+1}^{\infty} \frac{(2i+1)}{2}a_i x^{2i}\rho, \\ \frac{d\theta}{dt} = 1 + b_0\rho^2 + b_s x^{2s} + \sum_{i=s+1}^{\infty} b_i x^{2i}, \end{cases}$$

where $b_i = 0$ for all $i \equiv_{4r+1}(s+r)$ and $s \not\equiv_{4r+1} 0$. The corresponding first integral of the unique normal form is given by

$$s^{(\infty)}(x, \rho) := x\rho^4 + \rho^2 x^{2r+1} + \sum_{i=r+1}^{\infty} a_i \rho^2 x^{2i+1}. \quad (4.8)$$

Proof. Due to special structure constants given by Eqs. (3.6) and (3.7) we conclude that the generators of $\ker(\text{ad}_{H_1^0})$ are $\{H_{6k+1}^k, \Theta_{6k}^k\}$ for all $k \in \mathbb{N}$. Applying the method given at the end of Section 3 and using Lemma 4.2, we have that

$$\begin{aligned} [H_{6k+1}^k, H_{2r}^r] + [\mathcal{S}_{2r+6k+1}^{r+k}, \mathbb{H}_r] &= \frac{(4r+1)(-2k)_1^{4k+2}}{(4k+1)!} \\ &\times H_{8rk+2k+2r}^{4rk+k+2r} = 0, \\ [H_{6k+1}^k, \Theta_{2s}^s] + [\mathcal{R}_{6k+2s+1}^{s+k}, \mathbb{H}_r] &= \frac{-2s(2k+1)(2s-2k-8rk)_{4r+1}^{4k}}{(4s)_{4r+1}^{4k}} \\ &\times \Theta_{8rk+2k+2s+2r}^{4rk+k+s+r}. \end{aligned}$$

Thus, the above relations imply that H_{6k+1}^k cannot eliminate any H-terms and $\Theta_{2m}^m \in \text{Im}(d_s^{m,s+1})$ for any $m \equiv_{4r+1}(r+s)$, where $s \not\equiv_{4r+1} 0$. Furthermore, one has

$$\begin{aligned} [\Theta_{6k}^k, H_{2r}^r] + [\mathcal{R}_{6k+2s}^{k+s}, \mathbb{H}_r] &= \frac{-k(4r+1)(1-2k)_1^{4k-1}}{(4k-1)!} \Theta_{8rk+k+2r}^{4rk+k+2r} = 0, \end{aligned}$$

which turns out that Θ_{6k}^k generates a symmetry for the unique normal form of (1.1). This completes the proof. \square

We close this section by giving four representations of (4.7) based on the given discussions in Section 2.

Theorem 4.4. *The unique normal form of (1.1) can be expressed in the following representations.*

1. The unique normal form can be presented in the following form

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H(x, r)}{\partial r}, \\ \frac{dr}{dt} = -\frac{\partial H(x, r)}{\partial x}, \\ \frac{d\theta}{dt} = 1 + b_0 \rho^2 + b_s x^{2s} + \sum_{i=s+1}^{\infty} b_i x^{2i}, \end{cases}$$

in which $H(x, r) := x r^2 + a_r r x^{2r+1} + \sum_{i=r+1}^{\infty} a_i x^{2i+1} r$ and $r = \rho^2$.

2. Euler's form for $y \neq 0$

$$\begin{aligned} w^{(\infty)} &= \frac{1}{2} \nabla (x(y^2 + z^2)^2) \times \nabla \left(\arctan\left(\frac{z}{y}\right) \right) - \frac{1}{2} \nabla (y^2 + z^2) \\ &\quad \times \nabla x + \frac{b_0}{4} \nabla ((y^2 + z^2)^2) \times \nabla x \\ &\quad + \sum_{i=r}^{\infty} \frac{a_i}{2} \nabla ((y^2 + z^2) x^{2i+1}) \times \nabla \left(\arctan\left(\frac{z}{y}\right) \right) \\ &\quad - \sum_{i=s}^{\infty} \frac{b_i}{2(2i+1)} \nabla ((y^2 + z^2)) \times \nabla (x^{2i+1}). \end{aligned}$$

3. Vector potential

$$w^{(\infty)} = \nabla \times \mathbf{A},$$

where

$$\begin{aligned} \mathbf{A} &= -\frac{1}{2} x(y^2 + z^2)(z \cdot \mathbf{e}_y - y \cdot \mathbf{e}_z) - \frac{1}{2} (y^2 + z^2) \\ &\quad \cdot \mathbf{e}_x - \frac{b_0(y^2 + z^2)^2}{4} \cdot \mathbf{e}_x \\ &\quad - \sum_{i=r}^{\infty} \frac{a_i}{2} x^{2i+1} (\mathbf{e}_y + y \cdot \mathbf{e}_z) - \sum_{i=s}^{\infty} \frac{b_i}{2} (y^2 + z^2) x^{2i} \cdot \mathbf{e}_x. \end{aligned}$$

4. Monge representation or Clebsch representation for $y \neq 0$

$$\begin{aligned} w^{(\infty)} &= -\left(\nabla \left(x^2 - \frac{1}{4} (y^2 + z^2) \right) + x \nabla x \right) \\ &\quad - \left(\nabla (zy(y^2 + z^2)) - \frac{z}{y} \nabla (y^2 (y^2 + z^2)) \right) \\ &\quad - \left(\nabla (zy) - \frac{z}{y} \nabla (y^2) \right) + \sum_{i=r}^{\infty} a_i \left(\frac{1}{2i} \nabla (x^{2i+2}) \right) \\ &\quad - x^{2i} \nabla \left(\frac{1}{4} (2i+1) (y^2 + z^2) + \frac{1}{2i} x^2 \right) \\ &\quad - \sum_{i=s}^{\infty} b_i \left(\nabla (zyx^{2i}) - \frac{z}{y} \nabla (y^2 x^{2i}) \right). \end{aligned}$$

For all representations given above the constants b_i for all $i \in \mathbb{N}_0$, satisfy the conditions that are given in [Theorem 4.3](#).

Proof. Follow [Corollary 2.4](#), [Theorems 2.5](#), [2.6](#) and [2.9](#), respectively. See [\[15\]](#) for the relevant result. \square

5. Practical formulas

In this section, we would like to give some fruitful formulas which are fundamentally significant for applications. First, some necessary relations between the coefficients of given Hopf-pitchfork differential system are provided. These relations guarantee that the classical normal form of the given system up to third order belongs to $\mathcal{L}^{\mathbb{Z}_2}$. Then, several formulas for the unique normal form's coefficients of the system $w \in \mathcal{L}^{\mathbb{Z}_2}$ are given. Note that all

of the results in this part are valid up to third order truncation. It is quite possible to derive these computations for any finite order using our Maple program. Finally, we illustrate our results with the modified Chua's circuit.

Consider the Hopf-pitchfork differential system governed by

$$\begin{cases} \frac{dx}{dt} = \sum a_{i,j,k} x^i y^j z^k, \\ \frac{dy}{dt} = z + \sum b_{i,j,k} x^i y^j z^k, \\ \frac{dz}{dt} = -y + \sum c_{i,j,k} x^i y^j z^k, \end{cases} \quad (5.1)$$

where $i + j + k = 3$ and the monomials $x^i y^j z^k$ are assumed to be odd functions. If the following relations between the coefficients of cubic terms of system (5.1) hold, then the classical normal form of this system up to third order belongs to $\mathcal{L}^{\mathbb{Z}_2}$.

$$a_{1,0,2} = -2c_{0,2,1} - 6c_{0,0,3} - 6b_{0,3,0} - 2b_{0,1,2} - 2a_{1,2,0}, \quad (5.2)$$

$$c_{2,0,1} = -b_{2,1,0} - 3a_{3,0,0}. \quad (5.3)$$

Following [Theorem 4.3](#), the unique normal form of (5.1) under the above conditions is as

$$\begin{cases} \frac{dx}{dt} = a_0 2x\rho^2 + a_1 x^3, \\ \frac{d\rho}{dt} = -\frac{a_0}{2} \rho^3 - a_1 \frac{3}{2} x^2 \rho, \\ \frac{d\theta}{dt} = 1 + b_0 \rho^2 + b_1 x^4, \end{cases}$$

where the coefficients of previous system are given explicitly by

$$\begin{aligned} a_0 &:= \frac{-1}{4} (c_{0,2,1} + 3c_{0,0,3} + 3b_{0,3,0} + b_{0,1,2}), \\ b_0 &:= \frac{1}{8} (-3c_{0,3,0} - c_{0,1,2} + b_{0,2,1} + 3b_{0,0,3}), \\ a_1 &:= a_{3,0,0}, \quad b_1 := -\frac{1}{2} (c_{2,1,0} + b_{2,0,1}). \end{aligned}$$

The above relations, Eqs. (5.2), and (5.3) are obtained using our Maple program and employing beneficial formulas regarding the coefficients of the classical normal form of Hopf-zero singularities derived in [\[22\]](#).

We finish with an example.

Example 5.1. Consider modified Chua's oscillator defined by the set of ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = -\gamma x - \beta y, \\ \frac{dy}{dt} = z - y + x + \mu_1 z^2 y, \\ \frac{dz}{dt} = \alpha (-cz + y - az^3) + \mu_2 z^2 y. \end{cases} \quad (5.4)$$

This system possesses the \mathbb{Z}_2 -symmetry given in (3.3). If one puts $(\mu_1, \mu_2) = (0, 0)$ in (5.4), then one gets the Chua's oscillator. The linearization of this system at origin has eigenvalues $0, \pm i\omega_0$ when

$$\begin{aligned} (c, \beta) &= \left(-\frac{\gamma+1}{\alpha}, -\frac{\gamma(\alpha+\gamma+1)}{\gamma+1} \right), \\ \frac{(\gamma+1)^3 + \alpha(2\gamma+1)}{\gamma+1} &> 0, \end{aligned} \quad (5.5)$$

where $\omega_0^2 := \frac{-(\gamma+1)^3 + \alpha(2\gamma+1)}{\gamma+1}$, $a \neq 0$, $\alpha \neq 0$, and $\gamma \approx 0$, see [\[38,22\]](#). The following linear transformations and rescaling the

time

$$\begin{aligned}x &\rightarrow \frac{(\gamma + 1)}{(\alpha + \gamma + 1)\gamma} \left(\omega_0 (\gamma + 1)x + \left(\alpha + (\gamma + 1)^2 \right) y \right. \\&\quad \left. - \frac{\gamma \alpha}{\gamma + 1} z \right), \\y &\rightarrow \frac{(\gamma + 1)}{(\alpha + \gamma + 1)} \left(\frac{\omega_0}{\gamma} x + y + z \right), \\z &\rightarrow y + z, \\t &\rightarrow \omega_0 t,\end{aligned}$$

bring the linear part of (5.4) to the Jordan canonical form. If in addition to the conditions already cited (5.5), we require

$$\begin{aligned}\mu_1 &:= \frac{3\omega_0 \alpha^2 a (2\alpha + (\gamma + 1)^2)}{(2\alpha - (\gamma + 1)^2) (\alpha - 2(\gamma + 1)^2)}, \\ \mu_2 &:= \frac{3\omega_0 \alpha^2 a (\gamma + 1) (3\alpha - (\gamma + 1)^2)}{(2\alpha - (\gamma + 1)^2) (\alpha - 2(\gamma + 1)^2)},\end{aligned}$$

where $\alpha \neq 2(\gamma + 1)^2$, then the classical normal form of modified Chua's oscillator up to third order belongs to $\mathcal{L}^{\mathbb{Z}_2}$. The previous relations are derived with the aid of Eqs. (5.2) and (5.3). Following Theorem 4.3, the unique normal form of (5.4) is given by

$$\begin{aligned}C^{(\infty)} &:= \frac{\alpha^3 a (\alpha + (\gamma + 1)^2)}{\omega_0 (\alpha + \gamma + 1) (2\alpha - (\gamma + 1)^2)} \\&\quad \times \left(\frac{3(\gamma + 1)}{2\gamma} H_1^0 + \frac{2\alpha \gamma}{(\gamma + 1)(\alpha + \gamma + 1)} H_2^1 \right) \\&\quad + \frac{3\alpha^3 a ((4\gamma + 5)(\gamma + 1)^2 + \alpha(3\gamma + 5))}{2(\alpha + \gamma + 1) (2\alpha - (\gamma + 1)^2) (\alpha - 2(\gamma + 1)^2)} \\&\quad \times \left(\frac{(\gamma + 1)^2}{4\gamma} \Theta_1^0 + \frac{\alpha}{(\alpha + \gamma + 1)} \Theta_2^1 \right).\end{aligned}$$

We refer the reader to [1] for relevant results.

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