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LOCAL THEORY FOR SPATIO-TEMPORAL CANARDS AND DELAYED BIFURCATIONS*

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MARTIN WECHSELBERGER[§]

Abstract. We present a rigorous framework for the local analysis of canards and slow passages through bifurcations in a wide class of infinite-dimensional dynamical systems with time-scale separation. The framework is applicable to models where an infinite-dimensional dynamical system for the fast variables is coupled to a finite-dimensional dynamical system for slow variables. We prove the existence of center-manifolds for generic models of this type, and study the reduced, finite-dimensional dynamics near bifurcations of (possibly) patterned steady states in the layer problem. Theoretical results are complemented with detailed examples and numerical simulations covering systems of local and nonlocal reaction-diffusion equations, neural field models, and delay-differential equations. We provide analytical foundations for numerical observations recently reported in the literature, such as spatio-temporal canards and slow passages through Hopf bifurcations in spatially extended systems subject to slow parameter variations. We also provide a theoretical analysis of slow passage through a Turing bifurcation in local and nonlocal models.

Key words. spatio-temporal canards, delayed bifurcations, PDEs, center manifold, nonlocal equations, infinite-dimensional systems

AMS subject classifications. 34E17, 35R15, 37L10

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1. Introduction. Many physical and biological systems consist of processes that evolve on disparate time and/or length scales and the observed dynamics in such systems reflect these multiple-scale features as well. Mathematical models of such multiple-scale systems are considered singular perturbation problems with two-scale problems as the most prominent.

In the context of ordinary differential equations (ODEs), singular perturbation problems are usually discussed under the assumption that there exists a coordinate system such that observed slow and fast dynamics are represented by corresponding slow and fast variables globally, i.e., the system of ODEs under consideration is given in the *standard (fast) form*

$$(1.1) \quad \begin{aligned} \dot{u} &= F(u, v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u, v, \mu, \varepsilon), \end{aligned}$$

where $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, F and G are smooth functions on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}_{>0}$, and $0 < \varepsilon \ll 1$ is the timescale separation parameter. The *geometric singular perturbation theory (GSPT)* to analyze such finite-dimensional singular perturbation problems is

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well established. It was pioneered by Fenichel in the 1970s [39] and is based on the notion of *normal hyperbolicity*¹ which refers to a spectral property of the equilibrium set S of the *layer problem* (also known as the *fast subsystem*) obtained in the limit $\varepsilon \rightarrow 0$ in (1.1). In GSPT, this set $S := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : F(u, v, \mu, 0) = 0\}$ is known as the *critical manifold* since it is assumed to be an m -dimensional differentiable manifold. Normal hyperbolicity refers then to the property that the (point) spectrum of the critical manifold is bounded away from the imaginary axis (on every compact subset of S).

Loss of normal hyperbolicity is a key feature in finite-dimensional singular perturbation problems for rhythm generation as observed, for example, in the famous van der Pol relaxation oscillator. An important question in this context is how the transition from an excitable to a relaxation oscillatory state occurs in such a system. The answer is (partially) given by the *canard phenomenon* which was discovered by French mathematicians who studied the van der Pol relaxation oscillator with constant forcing [12]. They showed an explosive growth of limit cycles from small Hopf-like to relaxation-type cycles in an exponentially small interval of the system parameter. This parameter-sensitive behavior is hard to observe, and the corresponding solutions in phase space resemble (with a good portion of imagination) the shape of a “duck” which explains the origin of the nomenclature. Note, this phenomenon is degenerate since it only exists in one-parameter families of slow-fast vector fields in $\mathbb{R}^2$. Furthermore, it is always associated with a nearby *singular* Hopf bifurcation, i.e., frequency and amplitude of the Hopf cycles depend on the singular perturbation parameter $\varepsilon \ll 1$. A seminal work on van der Pol canards [12] was obtained by Benoît et al. through *nonstandard analysis* methods and soon after similar results were reached using standard matched asymptotics techniques by Eckhaus [35]. A decade later, further results were obtained using *geometric desingularization* or *blowup* by Dumortier and Roussarie [33], and soon after by Krupa and Szmolyan [56, 57, 58].

Fortunately, this degenerate situation does not occur in systems with two (or more) slow variables where canards are generic, i.e., their existence is insensitive to small parameter perturbations. Benoît [11] was the first to study generic canards in $\mathbb{R}^3$. He also observed how a certain class of generic canards (known as canards of *folded-node* type) cause unexpected rotational properties of nearby solutions. Extending GSPT to canard problems in $\mathbb{R}^3$, Szmolyan and Wechselberger [74] provided a detailed geometric study of generic canards. In particular, Wechselberger [83] then showed that rotational properties of folded-node type canards are related to a complex local geometry of invariant manifolds near these canards and associated bifurcations of these canards. Coupling this local canard structure with a global return mechanism can explain complex oscillatory patterns known as mixed-mode oscillations; see, e.g., [83, 18, 29]. This is closely related to canards of folded-node and *folded-saddle-node* type. Firing threshold manifolds or, more broadly, transient separatrices in slowly modulated excitable systems form another important class of applications [87, 86] which is a different type of canard mechanism which includes *folded-saddle* canards.

A necessary (but not sufficient) condition for canards is the crossing of a real eigenvalue in the point spectrum of S which leads geometrically to a folded critical manifold. Another important singular perturbation phenomenon is the *delayed loss of stability through a Hopf bifurcation* [65, 66, 46] associated with the crossing of a complex conjugate pair of eigenvalues in the spectrum of S . We note that this phenomenon refers to a Hopf bifurcation in the layer problem which is, in general,

¹Precise definitions for this and other GSPT concepts can be found in the appendix.

different than a singular Hopf bifurcation in the full system (1.1) with $0 < \varepsilon \ll 1$, as the two bifurcations are not necessarily related. A prime application of this phenomenon is associated with *elliptic bursters* in neurons; see, for instance, [52].

Canard theory has been extended to arbitrary finite dimensions in [84], i.e., dimensional restrictions on the slow variable subspace are not necessary to study these problems. Slow-fast theory for ODEs also plays an important role in the construction of heterogeneous (and possibly relative) equilibria in PDEs posed on the real line. This construction relies on a spatial-dynamic ODE formulation [70]: stationary states and traveling waves are identified with homo- or hetero-clinic connections of an ODE in the spatial variable, which may exhibit spatial-scale separation. These *global orbits* may display canard segments, and can be constructed using GSPT theory for ODEs [32, 19, 61, 45, 20, 4, 21].

A further interesting class of intermediate problems between ODEs and the ones considered here arise when the spatial-dynamical system obtained in traveling wave problems is itself infinite dimensional, or does not have an obvious phase space. In this context, a GSPT may not be readily available, and recent contributions in this area have been provided by Hupkes and Sandstede for lattice equations [50, 49], and by Faye and Scheel for nonlocal equations with convolutions [38].

More generally, the literature on infinite-dimensional slow-fast dynamical systems is less developed than its ODE counterpart. The papers by Bates and Jones [7] and by Bates, Lu, and Zeng [9], include historical background and references to the state-of-the-art literature (as of the end of the 1980s and 1990s, respectively) on invariant manifolds for infinite-dimensional systems, including, in particular, [25, 78, 8, 82, 77, 47]. In addition, Bates, Lu, and Zeng [9] provided a seminal contribution for the persistence of invariant manifolds for problems posed on Banach spaces. The theory presented therein is general, albeit the derivation of GSPT for infinite-dimensional equations is problem dependent (see, for instance, the one derived by Menon and Haller for the Maxwell–Bloch equations [62]).

The construction of orbits displaying canard segments or delayed bifurcations in the infinite-dimensional setting remains an open problem, with two important obstacles: (i) the loss of normal hyperbolicity, and (ii) the connection of slow and fast orbit segments with the view of obtaining a global, possibly periodic, orbit. The present paper addresses the former, and provides a general local theory for infinite-dimensional problems of the following type:

$$(1.2) \quad \begin{aligned} \dot{u} &= Lu + R(u, v, \mu, \varepsilon) := F(u, v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u, v, \mu, \varepsilon), \end{aligned}$$

where the fast variables u belong to a Banach space X , and the slow variables v to $\mathbb{R}^m$ for some $m \in \mathbb{N}$. The vector $\mu \in \mathbb{R}^p$, $p \in \mathbb{N}$, refers to a set of (possible) additional control parameters.² We say that (1.2) is a system of m -slow, ∞ -fast differential equations.

As we discuss below, systems of the form (1.2) are prevalent in the limited literature currently available on canards and delayed bifurcations in infinite-dimensional dynamical systems. The first contribution to this topic was given in the early 1990s by Su [73], who studied a system of type (1.2) in which the fast variables $u = (u_1, u_2)$ evolve according to the FitzHugh–Nagumo model, with diffusivity in the voltage u_1 , and diffusionless recovery variable u_2 , subject to a heterogeneous, slowly increasing

²Precise definitions of the infinite-dimensional dynamical system under study will be given in section 3.

current with amplitude v . For this setup, Su proved the existence of orbits displaying a delayed passage through a Hopf bifurcation. Two decades later, de Maesschalck, Kaper, and Popović [28] proved existence of trajectories modeled around the canard phenomenon in the nonlinear example

$$\varepsilon \partial_t u = \varepsilon^\mu \partial_{xx} u + V(u, x, t, \varepsilon)u, \quad \mu > 0,$$

using the specific scaling in conjunction with the method of lower and upper solutions. An intuition of this group was to bypass the difficulties associated with the loss of normal hyperbolicity, by hard wiring a slow, nonmonotonic evolution in the function V , as opposed to prescribing a coupled dynamic for a slow variable v , as we do here.

In 2015, Tzou, Ward, and Kolokolnikov [76] studied slow passages through a Hopf bifurcation in a reaction-diffusion PDE with a slowly varying parameter, using asymptotic methods which Breña-Medina et al. concurrently adopted to investigate slow-fast orbits connecting patterned, metastable states in plant dynamics [15, 5]. In the same year Chen et al. gave numerical evidence of slow passages through Turing bifurcations in an advection-reaction-diffusion equation with slowly varying parameters subject to noise, with applications in vegetation patterns [22].

In 2016, Krupa and Touboul studied canards in a delay-differential equation (DDE) with slowly varying parameters. This work differs from the other mentioned in this literature review, because it does not deal with a spatially extended system, albeit the fast variable u lives in a Banach space.

In 2017, Avitabile, Desroches and Knobloch introduced canards in spatially extended systems (termed *spatio-temporal canards*), producing numerical evidence of cycles and transients containing canard segments of folded-saddle and folded-node type. They studied a 2-slow ∞ -fast system coupled to a neural field equation posed on the real line and on the unit sphere. Analytical predictions for spatio-temporal canards were derived in a regime where interfacial dynamics provides a dimensionality reduction of the problem, amenable to standard GSPT analysis.

More recently, in 2018, two papers obtained results similar to the ones in Su [73] but on different models, and proposing different techniques: Bilinsky and Baer [13] used a WKB expansion to study spatially dependent buffer points (buffer curves) in heterogeneous reaction-diffusion equations; Kaper and Vo [54] found numerical evidence of delayed-Hopf bifurcations in several examples of heterogeneously, slowly driven reaction-diffusion systems, and give a formal, accurate, asymptotic argument to compute the associated buffer curves.

The main contribution of this article is the derivation of a local theory for canards and slow passages through bifurcations in ∞ -fast m -slow systems: not only is this a natural and necessary step for the construction of global orbits, but it is an achievable target for generic models. A key ingredient of our framework is a center-manifold reduction of (1.2), which is an infinite-dimensional system. A general center-manifold theory is available (see a recent literature review by Roberts [69]), but has not been used for infinite-dimensional problems of type (1.1) to overcome the loss of normal hyperbolicity. We proceed systematically, as follows:

1. The layer problem $\dot{u} = F(u, v, \mu, 0)$ is a differential equation in X , with parameters in $\mathbb{R}^{m+p}$. We assume that this system admits a bifurcation of a steady state u_* at $(v, \mu) = (v_*, \mu_*)$. Thus the linear operator $D_u F(u_*, v_*, \mu_*, 0)$ has a nonempty center spectrum and normal hyperbolicity fails. Note that u_* is constant in time, but need not be constant in space (u_* is a heterogeneous steady state of the layer problem).

2. Under the assumption that $D_u F$ has a spectral gap, and its center spectrum has finite dimension $n \in \mathbb{N}$, we prove the existence of a (μ, ε) -dependent center manifold *of the original problem* (1.2).
3. We perform a center-manifold reduction: in a neighborhood of $(u_*, v_*, \mu_*, 0) \in X \times \mathbb{R}^{m+p+1}$, the m -slow ∞ -fast system reduces to an m -slow n -fast one. The latter inherits the slow-fast structure of the former, because our reduction acts trivially on the slow variables v .
4. We derive a normal form for the reduced system, which still exhibits loss of normal hyperbolicity at the origin. Existing ODE results can however be used to prove the local existence of canards and slow passages through bifurcations.

Center-manifold theory is at the core of the procedure described above, and our treatment relies on existing tools on this topic. In particular, we adopt the notation and formalism developed by Vanderbauwhede and Iooss [78], and expanded in great detail in a book by Haragus and Iooss [44]. We build our results around the ones presented in the latter, to which we refer for further reading.

The theory presented here justifies rigorously numerical evidence presented in recent literature on the subject, and provides new results: we do not assume weak diffusivity in the linear operators, nor locality of the linear/nonlinear operators, and we provide theory and numerical examples for generic systems of integro-differential equations, local and nonlocal reaction-diffusion problems, and DDEs. The center-manifold reduction can be carried out for generic singularities, and this enables us to present theory for delayed passages through Turing bifurcations, which had not been studied analytically before, and are specific to PDEs.

We aim to present content in a format that is hopefully useful to readers working in both finite- and infinite-dimensional slow-fast problems, and this impacts on the material and style of the paper. For instance, center-manifold reductions for infinite-dimensional systems rely on checking a series of technical assumptions on the layer problem, hence we took some measures to make the material self-contained, and to template our procedure for a relatively large class of problems. We believe that several applications should be covered by our treatment, or can be derived with minor modifications. For these reasons we structured the paper as follows: in section 2 we provide a suite of numerical examples that show canard phenomena and related slow passage through Hopf and Turing bifurcations in PDEs, DDEs, and integro-differential equations; in section 3 we introduce the notation and functional-analytic setup used in the following sections; section 4 exposes steps 1–3 of the procedure described above for generic m -slow, ∞ -fast systems; section 5 describes step 4 in the procedure for fold, Hopf, and Turing bifurcations in generic systems; in section 6 we go through several cycles of steps 1–4, showing how they can be used in the concrete applications presented in section 2; we conclude in section 7.

2. Numerical examples. We provide numerical examples of systems of ∞ -fast, m -slow variables to which our analytical framework is applicable. In this section we discuss evidence obtained for slow passage through saddle-node, Hopf, and Turing bifurcations in various models. All computations involve numerical bifurcation analysis of steady states or periodic orbits of the layer problem, and time stepping of the full problem. All computations in spatially extended systems are performed using a MATLAB suite for generic problems developed in [67] (see [2] for a recent tutorial). Computations for the DDE are performed using DDE-BIFTOOL [37].

2.1. Folded-saddle and folded-node canards in a neural field model. We begin with an integro-differential equation from mathematical neuroscience, namely,

a neural field model posed on a compact domain $\Omega \subset \mathbb{R}^d$,

$$(2.1) \quad \begin{aligned} \partial_t u &= -u + \int_{\Omega} w(\cdot, y) \theta(u(y, t), v_1) d\rho(y) && \text{in } \Omega \times \mathbb{R}_{>0}, \\ \dot{v}_1 &= \varepsilon \left(v_2 + c \int_{\Omega} \theta(u(y, t), v_1) d\rho(y) \right) && \text{in } \mathbb{R}_{>0}, \\ \dot{v}_2 &= \varepsilon \left(-v_1 + a + b \int_{\Omega} \theta(u(y, t), v_1) d\rho(y) \right) && \text{in } \mathbb{R}_{>0}, \end{aligned}$$

where θ models a firing rate function and w the synaptic connections (see [26] for a recent review on neural fields). The firing rate function is commonly modeled via a sigmoidal or a Heaviside function:

$$\theta_s(u, h) = \frac{1}{1 + \exp(-\mu(u - h))}, \quad \theta_H(u, h) = H(u - h).$$

In contrast to standard neural field models, the firing rate threshold v_1 oscillates slowly and harmonically in time if $b = c = 0$, and its evolution is coupled to the fast neural field activity variable u if b or c are nonzero.

In [3], spatio-temporal canards were introduced and analyzed using interfacial dynamics, valid in the case of Heaviside firing rate θ_H posed on one-dimensional domains. If $\Omega \subset \mathbb{R}$ and $\theta = \theta_H$, one can construct even solutions $u(x, t) = u(-x, t)$ from their v_1 -level set on $\mathbb{R}_{\geq 0}$,

$$(2.2) \quad A(t) = \{x \in \mathbb{R}_{\geq 0} : u(x, t) = v_1(t)\}.$$

If the v_1 level set has a single connected component, then $A(t)$ is a scalar function, and an exact evolution equation in the variable $(A, v_1, v_2) \in \mathbb{R}^3$ can be derived for the model and studied using GSPT for ODEs. Solutions $(A(t), v_1(t), v_2(t))$ with canard segments correspond to spatio-temporal canards of the original model.

In the present paper we study the case $\theta = \theta_s$ on generic domains Ω for which we report numerical simulations adapted from [3] in Figures 1 and 2: spatio-temporal canards of folded-saddle and folded-node type are predicted from the theory, are found numerically with a sigmoidal firing rate (Figure 1), and they also persist in higher spatial dimensions, where the theory does not apply (Figure 2). In the present paper we will introduce a rigorous treatment of this problem, valid for generic firing rates, kernels, and domains.

The functional setting for this problem will be the Banach space of continuous, real-valued functions defined on Ω . The theory developed in the following sections is also applicable to the Swift–Hohemberg equation subject to slow parameter variation, as studied in [40], albeit a natural functional setting for this problem requires Hilbert spaces. We shall give below several examples for problems on Hilbert spaces.

2.2. Slow passage through Hopf bifurcations in PDEs and DDEs. A second class of examples pertains to slow passages through Hopf bifurcations in infinite-dimensional systems. A first numerical example is given by the following reaction-diffusion PDE with slowly varying parameters:

$$(2.3) \quad \begin{aligned} \partial_t u_1 &= d_1 \partial_x^2 u_1 + u_1 - u_1^3/3 - u_2 + v && \text{in } (0, 2\pi) \times \mathbb{R}_{>0}, \\ \partial_t u_2 &= u_1 + c - bu_2 && \text{in } (0, 2\pi) \times \mathbb{R}_{>0}, \\ \dot{v} &= \varepsilon && \text{in } \mathbb{R}_{>0}, \end{aligned}$$

FIG. 1. Examples of spatio-temporal canards in (2.1). (a) Periodic orbit containing a folded-saddle canard segment obtained with $\theta = \theta_s$, $w(x, y) = \kappa_1 \exp(-|x - y|)(\kappa_2 + \kappa_3 \cos(y/\kappa_4))$, $\Omega = \mathbb{R}/2\mathbb{Z}$, $l = 100$, $a = 0.5$, $b = c = 0$, $\kappa_1 = 0.5$, $\kappa_2 = 1$, $\kappa_3 = 0.5$, $\kappa_4 = 1$, $\mu = 50$. Left: orbit in the (A, v_1, v_2) -space, with A defined in (2.2), superimposed on the critical manifold S_0 of the fast subsystem associated with (2.1) in the variables (A, v_1, v_2) . Right: corresponding spatio-temporal solution, in which the level sets are shown for illustrative purposes. (b) Examples of an orbit containing a folded-node canard segment, obtained from the example in (a) by setting $a = c = 1$, $b = 0$. Adapted from [3].

FIG. 2. Example of solutions containing spatio-temporal canards of folded-saddle type for the neural field (2.1) posed on $\Omega = \mathbb{S}^2$. Left bifurcation diagram of the fast subsystem. Center: trajectory of the model with $\varepsilon \ll 1$, near a fold of the fast subsystem. Right: exemplary patterns. We refer the reader to [3], from where this figure is adapted, for details about parameters and computations.

FIG. 3. Slow passage through a Hopf bifurcation in the Fitzhugh–Nagumo model (2.3). Parameters: $d_1 = 0.1$, $b = 0.1$, $c = 0.05$, $\varepsilon = 0, 5 \times 10^{-3}$.

subject to periodic boundary conditions

$$u_1(0, t) = u_1(2\pi, t), \quad u_2(0, t) = u_2(2\pi, t), \quad t \in \mathbb{R}_{>0},$$

where v plays the role of an external current. In this case we will show that the fast subsystem admits a homogeneous steady state undergoing a Hopf bifurcation, and will apply our theory to conclude that system (2.3) displays slow passage through a Hopf bifurcation, of which we give numerical evidence in Figure 3. This example shows a slow passage through a bifurcation of a *homogeneous* steady state, because the boundary conditions allow it. We highlight that the theory presented below is equally applicable to slow passages through *heterogeneous* (patterned) steady states, of which the neural field problem is an example.

The analytical treatment of this problem will be done on Sobolev spaces, and the presentation of this example assumes no prior knowledge of center-manifold reduction for PDEs, but only basic notions in functional analysis. The theory developed in the next sections is applicable to generic reaction–diffusion PDEs and, in particular, to the examples considered in [54].

A separate analytical treatment is instead required for the slow passage through a Hopf bifurcation in DDEs, for which we consider the model problem

$$(2.4) \quad \begin{aligned} \dot{u}(t) &= v(t)u(t) - u^3(t) - u(t - \tau) + d && \text{in } \mathbb{R}_{>0}, \\ \dot{v}(t) &= \varepsilon && \text{in } \mathbb{R}_{>0}. \end{aligned}$$

The fast subsystem of (2.4) is therefore the one-component DDE, in which the trivial equilibrium undergoes a first Hopf bifurcation at $v = -3/4$. The slow drift on α induces the slow passage presented in Figure 4.

2.3. Slow passage through Turing bifurcations in local and nonlocal reaction–diffusion models. Since center-manifold reductions are possible for generic

FIG. 4. Slow passage through a Hopf bifurcation in the DDE (2.4). Parameter values: $\varepsilon = 0.01$, $\tau = 4$, $d = 0.01$.

bifurcations in PDE or integro-differential equations, we consider the case of a slow passage through a Turing bifurcation. In general, we will study problems of the following type:

$$(2.5) \quad \begin{aligned} \partial_t u &= D \partial_x^2 u + f(u, v) && \text{in } \Omega \times \mathbb{R}_{>0}, \\ \dot{v} &= \varepsilon && \text{in } \mathbb{R}_{>0}, \end{aligned}$$

subject to suitable boundary conditions, where u denotes the concentrations of q reactants, and $D = \text{diag}\{d_1, \dots, d_q\}$ with $d_i > 0$ for all $i \in \mathbb{N}_q$, is a diffusion matrix. While developing a theory for generic models of this type is possible, we concentrate on two main examples, which can easily be generalized.

2.3.1. Nonlocal reaction-diffusion model. The first example under consideration is a model with nonlocal reaction, which has been studied for $\varepsilon = 0$ in the context of population dynamics [41, 81]:

$$(2.6) \quad \begin{aligned} \partial_t u &= d \partial_x^2 u + (v - b)u - u \int_{\Omega} w(\cdot - y)u(y, t) dy && \text{in } \Omega \times \mathbb{R}_{>0}, \\ \dot{v} &= \varepsilon && \text{in } \mathbb{R}_{>0}, \end{aligned}$$

where $\Omega = \mathbb{R}/2\mathbb{Z}$, and w is an interaction kernel. The model undergoes a Turing-like bifurcation of a homogeneous steady state, and we show numerical results of a slow passage through this bifurcation (see Figure 5). We will derive theoretical results for this one-component, nonlocal problem, and present a strategy that is easily adaptable to generic d -component reaction-diffusion equations subject to Dirichlet, Neumann, or mixed homogeneous boundary conditions, under mild hypotheses on f . This strategy can be used also in reaction-diffusion problems undergoing other bifurcations.

2.3.2. Schnakenberg model. As an example of a local reaction-diffusion equation, we consider the following Schnakenberg model, which undergoes a sub- or super-

FIG. 5. Slow passage through a Turing-like bifurcation for the nonlocal reaction-diffusion equation (2.6) (see [41, 81] for details). (a) Bifurcation diagram of the fast subsystem, where $\varepsilon = 0$ and v is a parameter (blue); the homogeneous steady state undergoes a Turing-like bifurcation at $v \approx 1.5$. A time simulation with slowly varying v , where $\|u\|_2$ changes slowly, displays a slow passage through the bifurcation (red). (b) Spatio-temporal profile of the time simulation. On the vertical axis we report both t and v , which increase linearly. Computations are performed with interaction function $w(x) = (2h)^{-1} \mathbb{1}_{(-h,h)}(x)$ and parameters $h = 3$, $d = 0.05$, $b = 1$, $l = 5$, and $\varepsilon = 0, 10^{-4}$.

critical Turing bifurcation, as discussed in [16]:

$$\begin{aligned}
 \partial_t u_1 &= d_1 \partial_x^2 u_1 + v u_1^2 u_2 - (c + r) u_1 + h u_2, & x \in (0, l), \\
 \partial_t u_2 &= d_2 \partial_x^2 u_2 - v u_1^2 u_2 + c u_1 - h u_2 + b, & x \in (0, l), \\
 \dot{v} &= \varepsilon,
 \end{aligned}
 \tag{2.7}$$

subject to Neumann boundary conditions

$$\begin{aligned}
 \partial_x u_1(0, t) &= \partial_x u_1(l, t) = 0, \\
 \partial_x u_2(0, t) &= \partial_x u_2(l, t) = 0.
 \end{aligned}$$

Numerical results for this model are reported in Figure 6.

3. General setup for infinite-dimensional problems. This section provides the specific setup for ∞ -fast, m -slow systems necessary such that center-manifold theory from [44, 78] can be adopted.

3.1. Notation. We will use a unique symbol, $|\cdot|$, to denote the 2-norm in $\mathbb{R}^n$, for any integer n . We indicate by X, Y, Z Banach spaces on the field $\mathbb{R}$ or $\mathbb{C}$, endowed with norms $\|\cdot\|_X, \|\cdot\|_Y$, and $\|\cdot\|_Z$, respectively. Throughout this article, we will assume the continuous embeddings $Z \hookrightarrow Y \hookrightarrow X$.

For a given Banach space X , we denote by $C_\eta(\mathbb{R}, X)$ the following weighted space:

$$(3.1) \quad C_\eta(\mathbb{R}, X) = \left\{ u \in C^0(\mathbb{R}, X) : \|u\|_{C_\eta(\mathbb{R}, X)} = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|u(t)\|_X < \infty \right\}$$

which is a Banach space equipped with the norm $\|\cdot\|_{C_\eta(\mathbb{R}, X)}$.

Furthermore, we indicate by $\mathcal{L}(Z, X)$ the set of bounded linear operators from Z to X and by Z^*, X^* the dual spaces of Z and X , respectively. The adjoint of a linear operator $L \in \mathcal{L}(Z, X)$ is the operator $L^* \in \mathcal{L}(X^*, Z^*)$ satisfying

$$(3.2) \quad \langle f, Lz \rangle_X = \langle L^* f, z \rangle_{Z^*} \quad \text{for all } z \in Z \text{ and for all } f \in X^*,$$

FIG. 6. Slow passage through a supercritical (a) and subcritical (b) Turing bifurcation of the Schnakenberg model (2.7). In the subcritical setting, the slow passage is through a first Turing bifurcation of the homogeneous steady state, albeit the solution jumps to a stable branch emanating from a second Turing bifurcation, which restabilizes after a fold and a secondary bifurcation. Parameters: $c = r = h = b = 1$, $d_2 = 10$, and (a) $d_1 = 0.26$, (b) $d_1 = 0.1497$.

where $\langle \cdot, \cdot \rangle_X : X^* \times X \rightarrow \mathbb{R}$ is the canonical duality pairing between X and its dual X^* (a similar definition holds for $\langle \cdot, \cdot \rangle_Z$).

The resolvent set of a linear operator $L : Z \subset X \rightarrow X$ is the set of complex numbers $\rho(L) = \{\lambda \in \mathbb{C} : (\lambda \text{id} - L) \text{ is invertible and } (\lambda \text{id} - L)^{-1} \in \mathcal{L}(X)\}$, and the spectrum of L is the complement of the resolvent set $\sigma(L) = \mathbb{C} \setminus \rho(L)$. A complex number λ is called an eigenvalue of L if $\ker(\lambda \text{id} - L) \neq \{0\}$ and the non-null-elements of this kernel are called the eigenvectors of L .

The k th Fréchet derivative of a nonlinear operator $R \in C^k(Z, Y)$ will be denoted by $D^k R$. For the Taylor expansions of $R \in C^k(Z, Y)$ around $0 \in Z$ we will use the compact notation

$$R(u) = \sum_{n=0}^k R_n(u^{(n)}) + o(\|u\|_Z^k),$$

where $u^{(n)} = (u, \dots, u) \in (Z)^n$ represents $u \in Z$ repeated n times and the n -linear operator $R_n \in \mathcal{L}^n(Z, X) = \mathcal{L}(\prod_{j=1}^n Z, X) \cong \mathcal{L}(Z, \mathcal{L}(Z, \dots, \mathcal{L}(Z, X)))$ with n recursions

[24, Lemma 1.6], acts as

$$R_n(u^{(n)}) = \frac{1}{n!} D^n R(0) u^{(n)}.$$

3.2. Systems of m -slow, ∞ -fast differential equations. For ease of reference, we recall system (1.2), i.e., we fix $m, p \in \mathbb{N}$, and study differential equations posed on $X \times \mathbb{R}^m$, featuring $p+1$ control parameters of the following type:

$$(3.3) \quad \begin{aligned} \dot{u} &= Lu + R(u, v, \mu, \varepsilon) := F(u, v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u, v, \mu, \varepsilon), \end{aligned}$$

where L is a linear operator while R , F , and G are nonlinear operators. More precisely we assume $L: Z \rightarrow X$, $R: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow Y$, $F: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow X$, and $G: Z \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}^m$. We have collected the $p+1$ control parameters in the vector $(\mu, \varepsilon) \in \mathbb{R}^p \times \mathbb{R}$. This choice reflects the special role played by the parameter ε , which is taken to be small and determines a separation of time scales between the *fast* variables u and the *slow* variables v . We are interested in applications where X is infinite dimensional, therefore we say that (3.3) is a system of m -slow, ∞ -fast differential equations. We note that, in general, owing to the splitting $F(u) = Lu + R(u)$ one poses $Z \subset Y \subset X$ [78, 44] and, depending on the problem, we may have $Y = X$ or $Y \neq X$. As we shall mention below, the choice $Y \neq X$, when possible, simplifies the verification of certain hypotheses.

It is sometimes advantageous to write the system (3.3) in a more compact form. To this end, we introduce the spaces $\tilde{X} = X \times \mathbb{R}^m$, $\tilde{Y} = Y \times \mathbb{R}^m$, and $\tilde{Z} = Z \times \mathbb{R}^m$, endowed with the corresponding natural norms and embeddings, and rewrite (3.3) as a differential equation on $\tilde{X}$,

$$(3.4) \quad \frac{d}{dt} \tilde{u} = \tilde{L} \tilde{u} + \tilde{R}(\tilde{u}, \mu, \varepsilon) := \tilde{F}(\tilde{u}, \mu, \varepsilon),$$

where $\tilde{u} = (u, v)$ and

$$\tilde{L} = \begin{pmatrix} L & D_v R(0, 0, 0, 0) \\ 0 & 0 \end{pmatrix}, \quad \tilde{R}(\tilde{u}, \mu, \varepsilon) = \begin{pmatrix} R(u, v, \mu, \varepsilon) - D_v R(0, 0, 0, 0)v \\ \varepsilon G(u, v, \mu, \varepsilon) \end{pmatrix}.$$

4. Center-manifold reduction. We aim to perform a center-manifold reduction of (3.3) near bifurcation points of equilibria of the fast subsystem

$$(4.1) \quad \dot{u} = F(u, v, \mu, 0) = Lu + R(u, v, \mu, 0)$$

in which $(v, \mu) \in \mathbb{R}^m \times \mathbb{R}^p$ is fixed. As we shall see below, this in turn will provide a reduction of (3.3) to a set of m -slow, n -fast ODEs, which can subsequently be studied with standard tools for slow-fast ODEs. Solutions to (4.1) are defined below.

DEFINITION 4.1 (solution). *A solution of the differential equation (4.1) is a function $u: I \rightarrow Z \hookrightarrow X$ defined on an interval $I \subset \mathbb{R}$ such that $u \in C(I, Z) \cap C^1(I, X)$, and equality (4.1) holds in X for all $t \in I$.*

4.1. Preliminary hypotheses. In order to perform the reduction of the fast subsystem, we employ the framework introduced by Haragus and Iooss in [44] and in [79, 78]. We report below a set of hypotheses that are necessary to perform the center-manifold reduction. These hypotheses are adapted (or reproduced) from [44]. We begin with an assumption on the vector field (3.3), and on a subset of its equilibria.

Hypothesis 4.2 (vector field). The operators L , R in (3.3) satisfy the following conditions:

- (i) $L \in \mathcal{L}(Z, X)$.
- (ii) There exist an integer $k \geq 2$ and neighborhoods $V_u \subset X$, $V_v \subset \mathbb{R}^m$, $V_\mu \subset \mathbb{R}^p$, $V_\varepsilon \subset \mathbb{R}$ of 0 such that $R \in C^k(V_u \times V_v \times V_\mu \times V_\varepsilon, Y)$ and

$$R(0, 0, 0, 0) = 0, \quad D_u R(0, 0, 0, 0) = 0.$$

Hypothesis 4.2 implies that $0 \in X \times \mathbb{R}^m$ is an equilibrium of (3.4) for $(\mu, \varepsilon) = 0$, that is, $0 \in X$ is an equilibrium of the fast subsystem (4.1) for $(v, \mu) = 0$. Note that Hypothesis 4.2(ii) implies $F(0, 0, 0, 0) = 0$, while in general $G(0, 0, 0, 0) \neq 0$.

Hypothesis 4.3 (spectral decomposition). The spectrum $\sigma(L)$ of L can be written as follows:

$$\sigma(L) = \sigma_u \cup \sigma_c \cup \sigma_s,$$

where

$$\begin{aligned} \sigma_u &:= \{\lambda \in \sigma(L) : \operatorname{Re} \lambda > 0\}, \\ \sigma_c &:= \{\lambda \in \sigma(L) : \operatorname{Re} \lambda = 0\}, \\ \sigma_s &:= \{\lambda \in \sigma(L) : \operatorname{Re} \lambda < 0\}, \end{aligned}$$

satisfy the following assumptions:

- (i) The set σ_c consists of a finite number $n > 0$ of eigenvalues with finite algebraic multiplicities.
- (ii) There exists a positive constant $\gamma > 0$ such that

$$\inf_{\lambda \in \sigma_u} \operatorname{Re} \lambda > \gamma, \quad \sup_{\lambda \in \sigma_s} \operatorname{Re} \lambda < -\gamma.$$

If Hypotheses 4.2 and 4.3 hold, we can introduce the spectral projector on the center subspace, $P_c \in \mathcal{L}(X, Z)$ associated with σ_c and its complementary projector $P_{su} = \operatorname{id}_X - P_c \in \mathcal{L}(X) \cap \mathcal{L}(Y) \cap \mathcal{L}(Z)$. We therefore have the decomposition

$$X = E_c \oplus X_{su}, \quad E_c = \operatorname{range} P_c = \ker P_{su} \subset Z, \quad X_{su} = \operatorname{range} P_{su} = \ker P_c \subset X;$$

we set $Z_{su} = P_{su}Z \subset Z$ and $Y_{su} = P_{su}Y \subset Y$, and we let $L_c \in \mathcal{L}(E_c)$, $L_{su} \in \mathcal{L}(Z_{su}, Y_{su})$, be the restrictions of L to E_c and Z_{su} , respectively. We are now ready to state one final hypothesis necessary for the center-manifold reduction.

Hypothesis 4.4 (linear equation in the hyperbolic subspace). For any $\eta \in [0, \gamma]$, where γ is the spectral gap defined in Hypothesis 4.3 and any $f \in C_\eta(\mathbb{R}, Y_{su})$, the linear problem

$$(4.2) \quad \dot{u}_{su} = L_{su}u_{su} + f(t)$$

has a unique solution $u_{su} = K_{su}f \in C_\eta(\mathbb{R}, Z_{su})$. The linear map K_{su} belongs to $\mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su}))$, and there exists $\Pi \in C([0, \gamma], \mathbb{R})$ such that

$$(4.3) \quad \|K_{su}\|_{\mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su}))} \leq \Pi(\eta).$$

Checking this hypothesis in applications may be nontrivial, as it requires to prove that there is a gain in regularity of solution $u(t) \in Z$ to (4.2), when the forcing term $f(t)$ belongs to Y . Note that this forcing term mimics the effect of the nonlinearity

and, as such, shares it range. In some cases, this hypothesis can be substituted by an estimate on the resolvent set of L , and in the examples treated below we will give pointers to the literature for these cases. In particular, when $Y \neq X$, a necessary condition to check the validity of Hypothesis 4.4 is available [44, section 2.2.3]. In this paper we focus on the more challenging case $Y = X$, and present examples where a necessary condition is available, and examples where it is not (hence Hypothesis 4.4 must be checked directly).

4.2. Center-manifold results. We now proceed to derive a center-manifold existence result for (3.3) in a neighborhood of $0 \in X \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$.

THEOREM 4.5 (parameter-dependent center manifold). *Assume Hypotheses 4.2 to 4.4. Further, assume $G \in C^k(V_u \times V_v \times V_p \times V_\varepsilon, \mathbb{R}^m)$. Then the m -slow, ∞ -fast system (3.3) admits a finite-dimensional center manifold. More precisely, there exists a map $\Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su})$ with*

$$\Psi(0, 0, 0, 0) = 0, \quad D_u \Psi(0, 0, 0, 0) = 0,$$

and a neighborhood $O_u \times O_v \times O_\mu \times O_\varepsilon$ of $0 \in Z \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}$ such that, for all $(\mu, \varepsilon) \in O_\mu \times O_\varepsilon$, the manifold

$$(4.4) \quad M_c(\mu, \varepsilon) = \{(u_c, v) + (\Psi(u_c, v, \mu, \varepsilon), 0) : u_c \in E_c, v \in O_v\}$$

has the following properties:

(i) $M_c(\mu, \varepsilon)$ is locally invariant, that is, if (u, v) is a solution of (3.3) such that $(u(0), v(0)) \in M_c(\mu, \varepsilon) \cap (O_u \times O_v)$ for all $t \in [0, T]$, then $(u(t), v(t)) \in M_c(\mu, \varepsilon)$ for all $t \in [0, T]$.

(ii) $M_c(\mu, \varepsilon)$ contains the set of bounded solutions of (3.3) staying in $O_u \times O_v$ for all $t \in \mathbb{R}$, that is, if (u, v) is a solution of (3.3) satisfying $(u(t), v(t)) \in O_u \times O_v$ for all $t \in \mathbb{R}$, then $(u(0), v(0)) \in M_c(\mu, \varepsilon)$.

Proof. We use the compact formulation of the ∞ -fast, m -slow system (3.3) as a differential equation on $\tilde{X} = X \times \mathbb{R}^m$,

$$(4.5) \quad \frac{d}{dt} \tilde{u} = \tilde{L} \tilde{u} + \tilde{R}(\tilde{u}, \mu, \varepsilon),$$

where $\tilde{u} = (u, v)$ and

$$\tilde{L} = \begin{pmatrix} L & D_v R(0, 0, 0, 0) \\ 0 & 0 \end{pmatrix}, \quad \tilde{R}(\tilde{u}, \mu, \varepsilon) = \begin{pmatrix} R(u, v, \mu, \varepsilon) - D_v R(0, 0, 0, 0)v \\ \varepsilon G(u, v, \mu, \varepsilon) \end{pmatrix}.$$

We aim to show that, if Hypotheses 4.2 to 4.4 hold, then the same hypotheses hold for $\tilde{L}, \tilde{R}$, on Banach spaces $\tilde{X}, \tilde{Y}, \tilde{Z}$, upon defining suitable projectors $\tilde{P}_c, \tilde{P}_{su}$. The assertion is then a consequence of Theorem 3.3 of [44], applied to (4.5). We proceed to verify the hypotheses in 3 steps:

Step 1. Verification of Hypothesis 4.2 for system (4.5). It is immediate to prove that Hypothesis 4.2 and $G \in C^k(V_u \times V_v \times V_p \times V_\varepsilon, \mathbb{R}^m)$ imply that Hypothesis 4.2 holds also for $\tilde{L}, \tilde{R}$ in system (4.5), and we omit this proof.

Step 2. Verification of Hypothesis 4.3 for system (4.5). The expression of $\tilde{L}$ makes it easy to compute its resolvent as function of the one of L . We conclude that $\sigma_c(\tilde{L}) = \sigma_c(L) \cup \{0\}$. One consequence is that the spectrum of $\tilde{L}$ admits the decomposition

$$\sigma(\tilde{L}) = \sigma_u \cup \tilde{\sigma}_c \cup \sigma_s,$$

where σ_s, σ_u are the stable and unstable spectra of L , respectively, and where $\tilde{\sigma}_c$ contains $n + m$ purely imaginary eigenvalues, where n is as in Hypothesis 4.3(i). This guarantees that $\tilde{\sigma}_c$ contains finitely many eigenvalues. Next, we define the projectors associated with $\tilde{\sigma}_c$ as follows:

$$(4.6) \quad \tilde{P}_c = \begin{pmatrix} P_c & 0 \\ 0 & \text{id}_{\mathbb{R}^m} \end{pmatrix}, \quad \tilde{P}_{su} = \begin{pmatrix} P_{su} & 0 \\ 0 & 0_{\mathbb{R}^m} \end{pmatrix},$$

where P_c, P_{su} are the spectral projectors associated with the center eigenspace of L . We have $\tilde{X} = \tilde{E}_c \oplus \tilde{X}_{su}$, where

$$\tilde{E}_c = \text{range } \tilde{P}_c, \quad \tilde{X}_{su} = \ker \tilde{P}_c, \quad \dim \tilde{E}_c = \dim E_c + m < \infty.$$

Since $\tilde{E}_0$ is finite dimensional, we conclude that the $\tilde{n}$ eigenvalues in $\tilde{\sigma}_c$ have finite multiplicities, hence Hypothesis 4.3(i) holds for $\tilde{L}$. Furthermore, since $\tilde{L}$ and L have the same stable and unstable spectra, their spectral gaps coincide, that is, Hypothesis 4.3(ii) for L implies that the same hypothesis holds for $\tilde{L}$ with the same γ .

Step 3. Verification of Hypothesis 4.4 for system (4.5). We set

$$\tilde{Y}_{su} = P_{su}Y = Y_{su} \times \{0_{\mathbb{R}^m}\}, \quad \tilde{Z}_{su} = P_{su}Z = Z_{su} \times \{0_{\mathbb{R}^m}\},$$

and we denote by $\tilde{L}_{su}$ the restriction of $\tilde{L}$ to $\tilde{Z}_{su}$. To prove the assertion we must show that Hypothesis 4.4 implies that the same hypothesis holds for the following differential equation on $\tilde{X}_{su}$:

$$(4.7) \quad \frac{d}{dt} \tilde{u}_{su} = \tilde{L}_{su} \tilde{u}_{su} + \tilde{f}(t),$$

where $\tilde{f} = (f, 0_{\mathbb{R}^m}) \in C_\eta(\mathbb{R}, \tilde{Y}_{su})$ with f and η given in Hypothesis 4.4. In the remainder of the proof, we use the symbol 0 in place of $0_{\mathbb{R}^m}$ to simplify the notation. We consider the linear mapping

$$\tilde{K}_{su}: Z_{su} \times \mathbb{R}^m \rightarrow Z_{su} \times \mathbb{R}^m, \quad (U, V) \mapsto (K_{su}U, 0).$$

It suffices to prove the following claims:

1. $\tilde{K}_{su}\tilde{f} = (K_{su}f, 0)$ is an element of $C_\eta(\mathbb{R}, \tilde{Z}_{su})$.
2. $\tilde{K}_{su}\tilde{f}$ is the unique solution to (4.7).
3. Let $\Pi(\eta)$ be fixed as in Hypothesis 4.4, then the following bound holds:

$$\|\tilde{K}_{su}\tilde{f}\|_{\mathcal{L}(C_\eta(\mathbb{R}, \tilde{Y}_{su}), C_\eta(\mathbb{R}, \tilde{Z}_{su}))} \leq \Pi(\eta).$$

To prove claim 1, we recall that $\tilde{Z}_{su} = Z_{su} \times \{0\}$, therefore, if $U \in Z_{su}$, then $(U, 0) \in Z_{su}$ and $\|U\|_{Z_{su}} = \|(U, 0)\|_{\tilde{Z}_{su}}$. By Hypothesis 4.4, $(K_{su}f)(t) \in Z_{su}$ for all $t \in \mathbb{R}$, and the mapping $t \mapsto (K_{su}f)(t)$ is continuous. This implies $\tilde{K}_{su}\tilde{f} = (K_{su}f, 0) \in C^1(\mathbb{R}, \tilde{Z}) \subset C^0(\mathbb{R}, \tilde{Z})$ and

$$\begin{aligned} \|\tilde{K}_{su}\tilde{f}\|_{C_\eta(\mathbb{R}, \tilde{Z}_{su})} &= \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|(\tilde{K}_{su}\tilde{f})(t)\|_{\tilde{Z}_{su}} \\ &= \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|(K_{su}f)(t)\|_{Z_{su}} \\ &= \|K_{su}f\|_{C_\eta(\mathbb{R}, Z_{su})}, \end{aligned}$$

which is finite by Hypothesis 4.4. Therefore $\tilde{K}_{su}\tilde{f} \in C_\eta(\mathbb{R}, \tilde{Z}_{su})$.

In order to prove claim 2 we note that

$$\begin{aligned} \frac{d}{dt} \tilde{K}_{su} \tilde{f} &= \frac{d}{dt} \begin{pmatrix} K_{su} f \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{su} K_{su} f + f(t) \\ 0 \end{pmatrix} && \text{by Hypothesis 4.4,} \\ &= \tilde{L}_{su} \tilde{K}_{su} \tilde{f} + \tilde{f}(t), \end{aligned}$$

therefore $\tilde{K}_{su} \tilde{f}$ solves (4.7). Uniqueness is proved by showing that any solution $\tilde{U} = (U, V) \in C_\eta(R, \tilde{Z}_{su})$ to (4.7) is equal to $\tilde{K}_{su} \tilde{f}$. Since $\tilde{Z}_{su} = Z_{su} \times \{0\}$, we have $V(t) \equiv 0$ for all $t \in \mathbb{R}$, hence, $\tilde{U} = (U, 0)$. Since $\tilde{U}$ solves (4.7), we have

$$\frac{d}{dt} U = L_{su} U + f(t),$$

therefore, U solves (4.2), and Hypothesis 4.4 guarantees $U = K_{su} f$. We conclude $\tilde{U} = (K_{su} f, 0) = \tilde{K}_{su} \tilde{f}$, hence, $\tilde{K}_{su} \tilde{f}$ is the unique solution to (4.7).

To prove claim 3, we estimate

$$\begin{aligned} \|\tilde{K}_{su}\|_{\mathcal{L}(C_\eta(\mathbb{R}, \tilde{Y}_{su}), C_\eta(\mathbb{R}, \tilde{Z}_{su}))} &= \sup \{ \|\tilde{K}_{su} \tilde{U}\|_{C_\eta(\mathbb{R}, \tilde{Z}_{su})} : \|\tilde{U}\|_{C_\eta(\mathbb{R}, \tilde{Y}_{su})} = 1 \} \\ &= \sup \{ \|\tilde{K}_{su}(U, V)\|_{C_\eta(\mathbb{R}, \tilde{Z}_{su})} : \|(U, V)\|_{C_\eta(\mathbb{R}, \tilde{Y}_{su})} = 1 \} \\ &= \sup \{ \|K_{su} U\|_{C_\eta(\mathbb{R}, \tilde{Z}_{su})} : \|(U, V)\|_{C_\eta(\mathbb{R}, \tilde{Y}_{su})} = 1 \} \\ &= \sup \{ \|K_{su} U\|_{C_\eta(\mathbb{R}, Z_{su})} : \|(U, V)\|_{C_\eta(\mathbb{R}, \tilde{Y}_{su})} = 1 \} \\ &\leq \|K_{su}\|_{\mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su}))} \sup \{ \|U\|_{C_\eta(\mathbb{R}, Y_{su})} : \|(U, V)\|_{C_\eta(\mathbb{R}, \tilde{Y}_{su})} = 1 \} \\ &\leq \|K_{su}\|_{\mathcal{L}(C_\eta(\mathbb{R}, Y_{su}), C_\eta(\mathbb{R}, Z_{su}))} \\ &\leq \Pi(\eta), \end{aligned}$$

which completes this step.

Following the 3 steps above, the assertion is a consequence of the parameter-dependent version of the center-manifold theorem [44, Theorem 3.3] for system (4.5). This theorem guarantees the existence of neighborhoods $O_{\tilde{u}} \times O_\mu \times O_\varepsilon$ of $0 \in \tilde{Z} \times \mathbb{R}^q \times \mathbb{R}$, a reduction function $\tilde{\Psi} \in C^k(\tilde{E}_c \times \mathbb{R}^q \times \mathbb{R}, \tilde{Z}_h)$, and a center manifold

$$M_c(\mu, \varepsilon) = \{ \tilde{u}_c + \tilde{\Psi}(\tilde{u}_c, \mu, \varepsilon) : \tilde{u}_c \in \tilde{E}_c \},$$

that satisfy the assertion. The expression above, however, is seemingly different from (4.4). To recover (4.4) and the assertion in terms of a reduction function $\Psi \in C^k(E_c \times \mathbb{R}^m, \mathbb{R}^q \times \mathbb{R}, Z_{su})$, we note that $\tilde{\Psi}$ has values in $\tilde{Z}_{su} = Z_{su} \times \{0\}$ hence $\tilde{\Psi} = (\Psi, 0)$ for some $\Psi \in C^k(E_c \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}, Z_{su})$, and this completes the proof. $\square$

After the existence of a center manifold has been derived, we proceed to write the reduced equation on the center manifold, as the following corollary states.

COROLLARY 4.6. *Assume the hypotheses of Theorem 4.5. Let $T > 0$, let $(\mu, \varepsilon) \in O_\mu \times O_\varepsilon$, and let (u, v) be a solution to (3.3) which belongs to $M_c(\mu, \varepsilon)$ for $t \in [0, T]$. Then $u = u_c + \Psi(u_c, v, \mu, \varepsilon)$ with (u_c, v) satisfying*

$$(4.8) \quad \begin{aligned} \dot{u}_c &= P_c F(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon). \end{aligned}$$

Proof. Let $\tilde{u} = (u, v)$ be a solution to (3.4), the compact formulation of the m -slow ∞ -fast problem, and assume $\tilde{u}$ stays in M_c for $t \in [0, T]$. Then

$$\tilde{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_c + \Psi(u_c, v, \mu, \varepsilon) \\ v \end{pmatrix}, \quad u_c = P_c u.$$

From (3.4) and the definition of $\tilde{P}_c$ and $\tilde{F}$ we obtain

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} u_c \\ v \end{pmatrix} &= \tilde{P}_c \frac{d}{dt} (\tilde{u}_c + \tilde{\Psi}(\tilde{u}_c, \mu, \varepsilon)) \\ &= \tilde{P}_c \tilde{F}(\tilde{u}_c + \tilde{\Psi}(\tilde{u}_c, \mu, \varepsilon), \mu, \varepsilon) \\ &= \tilde{P}_c \tilde{F}((u_c + \Psi(u_c, v, \mu, \varepsilon), v), \mu, \varepsilon) \\ &= \begin{pmatrix} P_c F(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon) \\ \varepsilon G(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon) \end{pmatrix}. \end{aligned} \quad \square$$

Importantly, the evolution equation (4.8) is posed on a finite-dimensional state space, whose dimension is $\dim E_c + m$. Owing to the particular form of the right-hand side of (3.4), this finite-dimensional dynamical system has m -slow and $(\dim E_c)$ -fast variables.

Remark 4.7. The center-manifold reduction theory outlined here applies directly to the finite-dimensional case, i.e., all results have a finite-dimensional version for a slow-fast system (1.1). We will highlight these results in later sections where we discuss folds and delayed Hopf (and Turing) bifurcations in detail.

4.3. Center-manifold reduction with symmetries. When the fast subsystem admits symmetries, it is expected that they influence the center-manifold reduction. The following result combines a parameter-dependent center-manifold reduction for the ∞ -fast, m -slow dynamical system (which has been developed above) with equivariant center-manifold theory for infinite dimensional dynamical systems [42, 23]. We give this result without proof, as it is a minor modification of Theorem 4.5.

THEOREM 4.8. *Assume there is a linear operator $T \in \mathcal{L}(X) \cap \mathcal{L}(Z)$ which commutes with the fast vector field*

$$TL = LT, \quad TR(u, v, \mu, \varepsilon) = R(Tu, v, \mu, \varepsilon),$$

and such that the restriction T_c of T to the subspace E_c is an isometry. Under the assumptions in Theorem 4.5, there exists a reduction function Ψ which commutes with T , that is, $T\Psi(u_c, v, \mu, \varepsilon) = \Psi(T_c u_c, v, \mu, \varepsilon)$ for all $u_c \in X_c$, and such that the vector field in the reduced equation (4.8) commutes with (T_c, id) .

5. Reduction around bifurcations of the fast subsystem. We now specialize the result of the previous section, and find reductions of the full systems around saddle-node, Hopf, and Turing bifurcations of the fast subsystem. We will also derive normal form calculations for all these cases.

5.1. Reduction around a fold of the critical manifold. Before deriving a normal form around the fold of the fast subsystem, we make additional assumptions that are verified at the saddle-node bifurcation point.

Hypothesis 5.1 (fold of the critical manifold). Let $(e_i)_{i=1}^m$ be the canonical basis in $\mathbb{R}^m$.

(i) The center spectrum σ_c of L is given by $\sigma_c = \{0\}$. The eigenvalue 0 has multiplicity 1 and eigenvector $\zeta \in Z$.

(ii) We have $P_c(D_u^2 R(0, 0, 0, 0)\zeta^{(2)}) \neq 0$.

(iii) There exists $1 \leq j \leq m$ such that $P_c(D_v R(0, 0, 0, 0)e_j) \neq 0$.

Remark 5.2. When L has an adjoint operator L^* (this happens for instance when Z is dense in X) then there is a general expression for the center projector, namely, $P_c(u) = \langle \zeta^*, u \rangle_X \zeta$, where ζ^* is the eigenvector of L^* for the eigenvalue 0 such that $\langle \zeta^*, \zeta \rangle_X = 1$.

LEMMA 5.3 (normal form for fold). *Assume Hypothesis 5.1(i) and the hypotheses of Corollary 4.6, and let O_μ , O_ε be defined as in Corollary 4.6. There exist neighborhoods O_A , O_B of 0 in $\mathbb{R}$ and $\mathbb{R}^m$, respectively, such that, for any $(\mu, \varepsilon) \in O_\mu \times O_\varepsilon$ there is a change of variables, defined in $O_A \times O_B$, which transforms the reduced equation (4.8) into*

$$(5.1) \quad \begin{aligned} \dot{A} &= \alpha(B) + \beta A^2 + \gamma(\mu, \varepsilon) + O(A^2(\|B\| + |\mu| + |\varepsilon|) + (\|B\| + |\mu| + |\varepsilon|)^2), \\ \dot{B} &= \varepsilon G(A\zeta + \Psi(A, B, \mu, \varepsilon), B, \mu, \varepsilon), \end{aligned}$$

where $A(t) \in \mathbb{R}$, $B(t) \in \mathbb{R}^m$, and

$$\begin{aligned} \alpha(B)\zeta &= P_c(D_v R(0, 0, 0, 0)B), \\ \beta\zeta &= P_c(D_u^2 R(0, 0, 0, 0)\zeta^{(2)}), \\ \gamma(\mu, \varepsilon)\zeta &= P_c(D_\mu R(0, 0, 0, 0)\mu + D_\varepsilon R(0, 0, 0, 0)\varepsilon). \end{aligned}$$

If, in addition, Hypotheses 5.1(ii) and 5.1(iii) hold, we have $\alpha \neq 0$ and $\beta \neq 0$, respectively.

Proof. We fix $(\mu, \varepsilon) \in O_\mu \times O_\varepsilon$. From $X_c = \text{span}(\zeta)$, we write $(u_c, v) = (A\zeta, B)$, where $A \in \mathbb{R}$, $B \in \mathbb{R}^m$. This parametrization is defined in a neighborhood $O_A \times O_B$ of $0 \in \mathbb{R} \times \mathbb{R}^m$. From (4.8) and the definition of F , we obtain

$$(5.2) \quad \begin{aligned} \dot{A}\zeta &= P_c R(A\zeta + \Psi(A\zeta, B, \mu, \varepsilon), B, \mu, \varepsilon), \\ \dot{B} &= \varepsilon G(A\zeta + \Psi(A\zeta, B, \mu, \varepsilon), B, \mu, \varepsilon). \end{aligned}$$

The second equation in the system above is already written as in (5.1) of the statement, therefore, we concentrate henceforth only on the first equation.

In the remainder of this proof we set $\|\cdot\| = \|\cdot\|_Z$, and with a small abuse of notation we use $|\cdot|$ for the 2-norm in both $\mathbb{R}^p$ and $\mathbb{R}^m$. We consider the Taylor expansion of R around the origin

$$R(u, v, \mu, \varepsilon) = \sum_{0 \leq i+j+r+s \leq 2} R_{ijrs}(u^{(i)}, v^{(j)}, \mu^{(r)}, \varepsilon^{(s)}) + O\left(\sum_{i+j+r+s=3} \|u\|^i \|v\|^j |\mu|^r \varepsilon^s\right),$$

and we recall that $R_{0000} = 0$, $R_{1000}(u) = 0$ by Hypothesis 4.2, hence,

$$\begin{aligned} R(u, v, \mu, \varepsilon) &= R_{0100}(v) + R_{0010}(\mu) + R_{0001}(\varepsilon) + R_{2000}(u^{(2)}) \\ &\quad + O\left(\sum_{\substack{i+j+r+s=2, \\ i \neq 2}} \|u\|^i \|v\|^j |\mu|^r \varepsilon^s\right). \end{aligned}$$

Using the following identities

$$\begin{aligned} R_{0100}(v) &= D_v R(0, 0, 0, 0)v, & R_{0010}(\mu) &= D_\mu R(0, 0, 0, 0)\mu, \\ R_{0001}(\varepsilon) &= D_\varepsilon R(0, 0, 0, 0)\varepsilon, & R_{2000}(u^{(2)}) &= \frac{1}{2} D_u^2 R(0, 0, 0, 0)u^{(2)}, \end{aligned}$$

and recalling that $\Psi = O(\|u\|^2 + |v|^2 + |\mu| + \varepsilon)$, we obtain

$$\begin{aligned} \dot{A}\zeta &= P_c(D_v R(0, 0, 0, 0)B) + A^2 P_c(D_u^2 R(0, 0, 0, 0)\zeta^{(2)}) \\ &\quad + P_c(D_\mu R(0, 0, 0, 0)\mu + D_\varepsilon R(0, 0, 0, 0)\varepsilon) \\ &\quad + O\left(\sum_{\substack{i+j+r+s=2, \\ i \neq 2}} \|u\|^i |v|^j |\mu|^r \varepsilon^s\right) \end{aligned}$$

which, combined with the second equation in (5.2), proves that (5.1) holds in $O_A \times O_B$. Hypothesis 5.1(iii) guarantees $\beta \neq 0$ and Hypothesis 5.1(iv) implies that α is nonzero in $O_B \setminus \{0\}$. $\square$

Remark 5.4. Note that, with a further change of coordinates, (5.1) can be transformed into

$$\begin{aligned} \frac{d}{dt}A &= \alpha \tilde{B}_1 + \beta A^2 + O\left(\sum_{i+j+r+s=2, i \neq 2} A^i |\tilde{B}|^j |\mu|^r \varepsilon^s\right), \\ \frac{d}{dt}\tilde{B} &= \varepsilon \tilde{G}(A\zeta + \Psi(A, \tilde{B}, \mu, \varepsilon), \tilde{B}, \mu, \varepsilon). \end{aligned}$$

The reduction above leads to studying an ODE with 1-fast and m -slow variables, near the fold. This system is amenable to applying existing results on canards in this context. We mention here classical results on canards for $m = 1$ [12, 33, 56, 58], for $m = 2$ near a folded singularity [74, 83, 18], and their generalization to the case of m -slow variables [84].

5.2. Reduction around a Hopf bifurcation in a fast subsystem.

Hypothesis 5.5 (Hopf bifurcation). The center spectrum σ_c of L consists of a simple pair of complex conjugate purely imaginary eigenvalues, $\sigma_c = \{i\omega, -i\omega\}$, $\omega > 0$, with associated eigenvectors $\{\zeta, \bar{\zeta}\}$.

LEMMA 5.6 (normal form of Hopf bifurcation). *Assume Hypothesis 5.5 and the hypotheses of Corollary 4.6. Then there exist neighborhoods $W_u, W_v, W_\mu, W_\varepsilon$ in $O_u, O_v, O_\mu, O_\varepsilon$, respectively, such that for any $(\varepsilon, \mu) \in W_\mu \times W_\varepsilon$ there exists a (μ, ε) -dependent polynomial change of variables*

$$(5.3) \quad \begin{aligned} u_c &= A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon), \\ v &= B \end{aligned}$$

with $A \in \mathbb{C}$, $B \in \mathbb{R}^m$, $\Phi(\cdot, v, \mu, \varepsilon)$ a polynomial of degree k , $\Phi(0, 0, 0, 0) = 0$, $D_u \Phi(0, 0, 0, 0) = 0$, and whose monomials of degree q are functions of (v, μ, ε) of class C^{k-q} , which transforms the reduced equation (4.8) for $(u_c, v) \in W_u \times W_v$ into

$$(5.4) \quad \begin{aligned} \dot{A} &= i\omega A + \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|B| + |\mu| + \varepsilon + |A|^2)^2), \\ \dot{B} &= \varepsilon G(A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon), B, \mu, \varepsilon), \end{aligned}$$

where $\alpha: \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{C}$ is multilinear and $\beta \in \mathbb{R}$.

Proof. In this case, (4.8) leads to

$$(5.5) \quad \begin{aligned} \dot{u}_c &= L_c u_c + P_c R(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u_c + \Psi(u_c, v, \mu, \varepsilon), v, \mu, \varepsilon) \end{aligned}$$

with $L_c u_c \neq 0$. The operators L_c and L have an identical center spectrum, and by Hypothesis 5.5 we have $E_c = \text{span}\{\zeta, \bar{\zeta}\}$, hence, $\dim E_c = 2$. As in the case of saddle-node bifurcations, we write a normal form for the first of the two equations above, when v is fixed. We set $v = B$ and consider

$$\dot{u}_c = L_c u_c + P_c R(u_c + \Psi(u_c, B, \mu, \varepsilon), B, \mu, \varepsilon).$$

We use a parameter-dependent normal form result [44, Theorem 2.2, Chapter 3] on the 2-dimensional system above, with parameters $(B, \mu, \varepsilon) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$. There exist neighborhoods $W_u, W_v, W_\mu, W_\varepsilon$ in $O_u, O_v, O_\mu, O_\varepsilon$, respectively, and a polynomial Φ such that, for any $(B, \mu, \varepsilon) \in W_v \times W_\mu \times W_\varepsilon$, the change of variables $u_c = \nu_c + \Phi(\nu_c, B, \mu, \varepsilon)$ transforms the system above into the normal form

$$\dot{\nu}_c = \Lambda \nu_c + N(\nu_c, B, \mu, \varepsilon) + o(|(B, \mu, \varepsilon)|^q)$$

for any integer q with $2 \leq q \leq k$, where k is fixed as in Hypothesis 4.2(ii). The fact that Φ is a polynomial, and the properties listed in the statement for Φ are also a consequence of [44, Theorem 2.2, Chapter 3]. We aim to recover the normal form for a Hopf bifurcation, hence, it is convenient to identify $\mathbb{R}^2$ with the diagonal $\{(z, \bar{z}) : z \in \mathbb{C}\}$, thereby representing ν_c as $\nu_c(t) = A(t)\zeta + \bar{A}(t)\bar{\zeta}$ and obtaining [44, Lemma 1.7, Chapter 3]

$$\frac{d}{dt} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} + \begin{pmatrix} \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 \\ \alpha(B, \mu, \varepsilon)\bar{A} + \beta \bar{A}|A|^2 \end{pmatrix} + O((|B| + |\mu| + \varepsilon + |A|^2)^2)$$

where α is a linear form and β a real number. We have therefore shown the existence of a change of coordinates of type (5.3) which transforms the reduced equation (5.5) into (5.4). $\square$

Remark 5.7. We highlight that, in contrast to the reduction around a saddle-node bifurcation, we do not include here explicit expressions for the coefficients of the normal form α, β , and we do not include in Hypothesis 5.5 nondegeneracy conditions to guarantee $\alpha, \beta \neq 0$. These conditions are known for ODEs [60], and computing the coefficients is strongly problem dependent, therefore, we omit such calculations in the present paper.

The center-manifold reduction leads to a C^k (and not analytic) ODE with 2-fast and m -slow variables. Hence, this system is amenable to applying existing C^k results on delayed loss of stability through a Hopf bifurcation. We refer to classical results for $m = 1$, e.g., [65, 66, 46]. These C^k results guarantee the existence of a short delay ($O(\varepsilon \ln(\varepsilon))$) due to the slow passage. By (5.3), we deduce that to first order the fast variables on the center manifold are approximated by $A(t)\xi(x) + \bar{A}(t)\bar{\xi}(x)$, where $\xi(x)$ is the unstable Hopf mode, and where $A(t)$ has the dynamics of a slow passage through a Hopf bifurcation. The existence of long $O(1)$ delays associated with slow passage through a Hopf bifurcation in a PDE in the analytic context is still an open problem.

5.3. Reduction around Turing bifurcation. In order to perform a reduction around a Turing bifurcation for the applications, we make some additional assumptions concerning the symmetry of the problem.

Hypothesis 5.8 ($O(2)$ -equivariance). There exists a one-parameter continuous family of linear maps T_φ on X for $\varphi \in \mathbb{R}/2l\mathbb{Z}$, $l \in \mathbb{R}$, and a symmetry S on X such that

- (i) $T_\varphi \circ T_\psi = T_{\varphi+\psi}$ and $ST_\varphi = T_{-\varphi}S$ for $\varphi, \psi \in \mathbb{R}/2l\mathbb{Z}$;
- (ii) $T_0 = \text{id}$ and $S^2 = \text{id}$;
- (iii) the fast vector field $u \mapsto F(u, v, \mu, \varepsilon)$ commutes with T_φ, S .

LEMMA 5.9 (normal form of Pitchfork $O(2)$ bifurcation). Assume Hypothesis 5.8, and the hypotheses of Corollary 4.6. Further, assume that 0 is a double eigenvalue of L and no other eigenvalue is in σ_c for $(\varepsilon, \mu) = (0, 0)$. Assume that the action of T_φ on X_c is not trivial. Then, there exist neighborhoods $O_u, O_v, O_\mu, O_\varepsilon$ of 0 in $\mathbb{R}^2, \mathbb{R}^m, \mathbb{R}^p$, and $\mathbb{R}$, respectively, such that for any $(\varepsilon, \mu) \in O_\mu \times O_\varepsilon$, in a suitable basis $(\zeta, \bar{\zeta})$ of $\ker L$

$$(5.6) \quad \begin{aligned} u_c &= A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon), \\ v &= B \end{aligned}$$

with $A \in \mathbb{C}$, $B \in \mathbb{R}^m$, $\Phi(\cdot, v, \mu, \varepsilon)$ a polynomial of degree k , $\Phi(0, 0, 0, 0) = 0$, $D_u\Phi(0, 0, 0, 0) = 0$, and whose monomials of degree q are functions of (v, μ, ε) of class C^{k-q} . The reduced equation (4.8) for $(u_c, v) \in O_u \times O_v$ has, at third order in $(A, \bar{A})$, the expression

$$(5.7) \quad \begin{aligned} \dot{A} &= \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|B| + |\mu| + \varepsilon + |A|^2)^2), \\ \dot{B} &= \varepsilon G(A\zeta + \bar{A}\bar{\zeta} + \Phi(A\zeta + \bar{A}\bar{\zeta}, B, \mu, \varepsilon), B, \mu, \varepsilon), \end{aligned}$$

where α, β are real multilinear forms.

Proof. This proof follows closely the one of Lemma 5.6 except that at the end, we have to use the $O(2)$ -pitchfork normal form instead of the Hopf one. As shown in [44, section 1.2.4], one can find a suitable basis $\zeta, \bar{\zeta}$ such that $T_\varphi\zeta = e^{ik\varphi}\zeta$ for some fixed $k \in \mathbb{N}$ and $S\zeta = \bar{\zeta}$. In this basis, we represent ν_c as $\nu_c(t) = A(t)\zeta + \bar{A}(t)\bar{\zeta}$ and obtain at third order

$$\dot{A} = \alpha(B, \mu, \varepsilon)A + N_3(A, \bar{A}, \mu, \varepsilon) + O((|v| + |\mu| + \varepsilon + |A|^2)^2),$$

where N_3 is a cubic polynomial in $(A, \bar{A})$ which commutes with the action of the symmetry group. Thus $N_3(e^{i\varphi}A, e^{-i\varphi}\bar{A}, \mu, \varepsilon) = e^{i\varphi}N_3(A, \bar{A}, \mu, \varepsilon)$ and $\bar{N}_3(A, \bar{A}, \mu, \varepsilon) = N_3(\bar{A}, A, \mu, \varepsilon)$. This gives

$$\dot{A} = \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|v| + |\mu| + \varepsilon + |A|^2)^2),$$

where α, β are real valued. □

This lemma applies to the case of a PDE equivariant with respect to translations (which take the role of T_φ) in one unbounded spatial direction and possesses a reflection symmetry in this direction. The reduction above leads to studying an ODE with 1-fast and m -slow variables, that is (5.7). Results on slow passages through transcritical and pitchfork bifurcations for ODEs of this type are available in the literature [57, 14]. For example, if $m = 1$ and the coefficients α and β are such that the layer

problem undergoes a pitchfork bifurcation at the origin, then the results in [57] imply that (5.7) has a delayed pitchfork bifurcation. By (5.6), we deduce that to first order the fast variables on the center manifold are approximated by $A(t)\xi(x) + \bar{A}(t)\bar{\xi}(x)$, where $\xi(x)$ are Turing modes, and where $A(t)$ has the dynamics of a slow passage through a pitchfork. This phenomenon is what we call a slow passage through a Turing bifurcation.

6. Applications. In this section we collect applications of center-manifold reductions and normal forms to several examples presented in section 2. We highlight that the order in which the material is presented does not follow the one used in section 2, because we have chosen to start from simple examples and work towards more advanced material. In subsection 6.1 we present preparatory results for Hilbert spaces, which will be used in the following calculations. We then work through an example of slow passage through a Hopf bifurcation in (2.3) in subsection 6.2, assuming no prior knowledge of center-manifold reductions for infinite-dimensional systems. In subsection 6.3, we discuss slow passage through a Turing bifurcation in the nonlocal reaction–diffusion equation (2.6), and give pointers for performing the same analysis in local reaction–diffusion problems. The technical treatment of these sections depends on the fact that Hilbert spaces have been used in the accompanying examples. This motivates our presentation of the example in subsection 6.4, which is posed on Banach spaces and discusses canards in neural-field models. Towards the end of this section, in subsection 6.5, we discuss slow passage through Hopf in the DDE (2.4).

6.1. Preparatory results. We begin by introducing two lemmas that will be useful in the upcoming calculations.

LEMMA 6.1. *Let $\Omega = (-l, l)$ for some $l > 0$, and let $X = L^2(\Omega, \mathbb{C})$ and $Z = H_{\text{per}}^2(\Omega, \mathbb{C})$, endowed with the standard norms. Further consider the Laplacian operator $\partial_x^2 \in \mathcal{L}(Z, X)$. If $z \in \rho(\partial_x^2)$, then*

$$\|(z - \partial_x^2)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{|\operatorname{Im} z|}.$$

Proof. We denote by $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$ the standard norm and associated inner product on X , respectively. Let $z \in \rho(\partial_x^2)$. For any $f \in X$, there is a unique solution $u \in Z$ to $\partial_x^2 u - zu = f$. Multiplying the equation by the complex conjugate $\bar{u}$ of u , integrating over Ω , and using integration by parts and boundary conditions we obtain

$$\|\partial_x u\|_X^2 - z \|u\|_X^2 = \langle f, u \rangle_X.$$

Taking the imaginary part of the equation above we obtain the bound

$$|\operatorname{Im} z| \|u\|_X^2 = |\operatorname{Im} \langle f, u \rangle_X| \leq \|f\|_X \|u\|_X, \quad \text{hence,} \quad \|u\|_X \leq \frac{\|f\|_X}{|\operatorname{Im} z|}$$

and

$$\|(z - \partial_x^2)^{-1}\|_{\mathcal{L}(X)} = \sup_{\|f\|_X \neq 0} \frac{\|(z - \partial_x^2)^{-1} f\|_X}{\|f\|_X} \leq \frac{1}{|\operatorname{Im} z|}. \quad \square$$

Remark 6.2. We note that Lemma 6.1 can be derived in a similar fashion also when the problem is subject to Neumann or Dirichlet boundary conditions. We omit these cases for simplicity.

LEMMA 6.3. Let $A \in \mathcal{L}(Z, X)$, $B \in \mathcal{L}(X)$, and assume there exist $\omega_0, K > 0$ such that $\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq K/|\omega|$ for all $\omega \in \mathbb{R}$ such that $|\omega| > \omega_0$. Then there exists $C > 0$ such that

$$\|(i\omega - A - B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\omega|} \quad \text{for all } |\omega| > \max(\omega_0, \|B\|_{\mathcal{L}(X)} K).$$

Proof. Since $i\omega \in \rho(A)$ and $\|B\|_{\mathcal{L}(X)} K < |\omega|$, then $\text{id} - B(i\omega - A)^{-1}$ is invertible. We use the following identity

$$(i\omega - A - B)^{-1} = (i\omega - A)^{-1} [\text{id} - B(i\omega - A)^{-1}]^{-1}$$

and the main hypotheses to derive the bound

$$\|(i\omega - A - B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{K}{|\omega|} \frac{1}{1 - \|B\|_{\mathcal{L}(X)} K/|\omega|} \leq \frac{C}{|\omega|}. \quad \square$$

6.2. Analytical results for slow passage through a Hopf bifurcation in the FitzHugh–Nagumo PDE. We now show how to apply our framework to the FitzHugh–Nagumo model:

$$(6.1) \quad \begin{aligned} \partial_t u_1 &= \partial_x^2 u_1 + u_1 - u_1^3/3 - u_2 + v, & (x, t) &\in (0, 2\pi) \times \mathbb{R}_{>0}, \\ \partial_t u_2 &= u_1 + c - bu_2, & (x, t) &\in (0, 2\pi) \times \mathbb{R}_{>0}, \\ \dot{v} &= \varepsilon, & t &\in \mathbb{R}_{>0} \end{aligned}$$

subject to periodic boundary conditions

$$u_1(0, t) = u_1(2\pi, t), \quad u_2(0, t) = u_2(2\pi, t), \quad t \in \mathbb{R}_{>0}.$$

We make the following assumption, in order to guarantee the existence of a homogeneous steady state undergoing a Hopf bifurcation.

Hypothesis 6.4. There exist $u_{*1}, u_{*2}, v_*, b_*, c_* \in \mathbb{R}$ such that $u(x, t) \equiv (u_{*1}, u_{*2})$ is an equilibrium of (6.1) for $(v, b, c, \varepsilon) = (v_*, b_*, c_*, 0)$ with $1 - (u_{*1})^2 = b_* \in (0, 1)$.

In order to cast the problem in the framework of the previous sections, we set $u = (u_1, u_2)$, $\mu = (b, c)$, apply the change of variables $u = u_* + \tilde{u}$, $v = v_* + \tilde{v}$, $\mu = \mu_* + \tilde{\mu}$, and obtain, after dropping tildes and setting $a = 1 - u_{*1}^2 = b_*$ for notational convenience,

$$(6.2) \quad \begin{aligned} \dot{u} &= Lu + R(u, v, \mu, \varepsilon), \\ \dot{v} &= G(u, v, \mu, \varepsilon), \end{aligned}$$

where

$$(6.3) \quad L = \begin{pmatrix} \partial_x^2 + a \text{id} & -\text{id} \\ \text{id} & -a \text{id} \end{pmatrix}, \quad R(u, v, \mu, \varepsilon) = o(\|u\|), \quad G(u, v, \mu, \varepsilon) = \varepsilon,$$

and R is the remainder, linear in v and μ , and independent of ε .

System (6.2) has a homogeneous equilibrium $u = 0$ at $(v, \mu, \varepsilon) = (0, 0, 0)$ and, as we shall see below, Hypothesis 6.4 ensures that $u = 0$, considered as an equilibrium of the fast subsystem, undergoes a Hopf bifurcation for these parameter values. We now proceed to check the hypotheses of Theorem 4.5.

6.2.1. Choice of function spaces. We begin by selecting function spaces for the problem. In particular, we set

$$Z = H_{\text{per}}^2(0, 2\pi) \times L_{\text{per}}^2(0, 2\pi), \quad Y = X = L^2(0, 2\pi) \times L^2(0, 2\pi),$$

where $L^2(0, 2\pi)$ and $H_{\text{per}}^2(0, 2\pi)$ are standard Sobolev spaces endowed with the inner-product norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^2}$, respectively. We endow X and Z with norms

$$\|(u_1, u_2)\|_Z = (\|u_1\|_{H^2}^2 + \|u_2\|_{L^2}^2)^{1/2}, \quad \|(u_1, u_2)\|_X = (\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2)^{1/2}.$$

However, since u_1, u_2 are 2π -periodic functions defined on $(0, 2\pi)$, it will be convenient to write them in terms of their Fourier series,

$$u_i(x) = \sum_{n \in \mathbb{Z}} u_{i,n} \psi_n(x), \quad i = 1, 2, \quad \psi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{Z},$$

and use for X, Z norms which expose the respective Fourier coefficients [1]

$$\|(u_1, u_2)\|_X = \sum_{n \in \mathbb{Z}} \|(u_{1,n}, u_{2,n})\|_2^2, \quad \|(u_1, u_2)\|_Z \sim \sum_{n \in \mathbb{Z}} n^4 |u_{1,n}|^2 + |u_{2,n}|^2.$$

In passing we note that the latter norm is not equal, but only equivalent, to $\|\cdot\|_Z$.

6.2.2. Checking Hypothesis 4.2. To check Hypothesis 4.2(i) we prove that L is a continuous linear operator from Z to X . We fix $u \in Z$, let $f = Lu$, and obtain the following bound using the Cauchy-Schwarz inequality:

$$\begin{aligned} \|f_1\|_{L^2}^2 &\leq (\|\partial_x^2 u_1\|_{L^2} + a\|u_1\|_{L^2} + \|u_2\|_{L^2})^2 \\ &\leq (2 + a^2)(\|\partial_x^2 u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2) := K_1^2 \|u\|_Z^2. \end{aligned}$$

Similarly we find $\|f_2\|_{L^2}^2 \leq K_2^2 \|u\|_Z^2$, where $K_2^2 = 1 + a^2$. Combining the bounds we obtain $\|Lu\|_X \leq K \|u\|_Z$, where $K = \sqrt{\max(K_1, K_2)}$. This proves $L \in \mathcal{L}(Z, X)$.

Further, R is a cubic polynomial in X with $R(0, 0, 0, 0) = 0$ and $D_u R(0, 0, 0, 0) = 0$. Therefore, owing to the fact that Z is a Banach algebra, Hypothesis 4.2(ii) holds for any integer k .

6.2.3. Checking Hypothesis 4.3. To check this hypothesis, we determine the resolvent set of L , as in the following statement.

PROPOSITION 6.5. *Let $D_n(\lambda) = \lambda^2 + n^2\lambda + a(n^2 - a) + 1$ for $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$. The resolvent set of the operator $L: Z \rightarrow X$ defined in (6.3) is given by*

$$\rho(L) = \{\lambda \in \mathbb{C} : \lambda \neq -a, D_n(\lambda) \neq 0 \text{ for all } n \in \mathbb{Z}\}.$$

Proof. Let $E = \{\lambda \in \mathbb{C} : \lambda \neq -a, D_n(\lambda) \neq 0 \text{ for all } n \in \mathbb{Z}\}$. We show that $E \subseteq \rho(L)$ and $E \supseteq \rho(L)$, hence, $\rho(L) = E$.

Step 1: $E \subseteq \rho(L)$. We prove that $\lambda \neq -a$ and $D_n(\lambda) \neq 0$ for all $n \in \mathbb{Z}$ imply $\lambda \in \rho(L)$, that is, for all $f \in X$, there exist a constant $K(\lambda) > 0$ and a unique $u \in Z$ satisfying $(\lambda - L)u = f$ and $\|u\|_Z \leq K(\lambda)\|f\|_X$, hence, $(\lambda - L)^{-1} \in \mathcal{L}(X, Z)$. We fix $f \in X$, and we note that $(\lambda - L)u = f$ if and only if

$$(6.4) \quad \begin{pmatrix} \lambda + n^2 - a & 1 \\ -1 & \lambda + a \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix} \quad \text{for all } n \in \mathbb{Z}.$$

Since $D_n(\lambda) \neq 0$ is the determinant of the matrix above, the system has a unique solution for all $n \in \mathbb{Z}$. We now proceed to show that $u = \sum_{n \in \mathbb{Z}} u_n \psi_n(x) \in Z$. From the system above we get

$$(6.5) \quad \begin{aligned} D_n(\lambda)u_{1,n} &= (\lambda + a)f_{1,n} - f_{2,n}, \\ (\lambda + a)u_{2,n} &= (u_{1,n} + f_{2,n}). \end{aligned}$$

The first equation leads to the bound $|D_n(\lambda)u_{1,n}|^2 \leq (1 + |\lambda + a|^2)\|f_n\|_2^2$. We note that $D_n(\lambda) \sim n^2$ as $n \rightarrow \infty$: there exist an integer $n_0(\lambda)$ and a positive real constant $C(\lambda)$ such that $C(\lambda)|D_n(\lambda)| \geq n^2$ for all $n > n_0(\lambda)$, from which we deduce

$$n^4|u_{1,n}|^2 \leq C(\lambda)^{-2}(1 + |\lambda + a|^2)\|f_n\|_2^2 = K_1(\lambda)\|f_n\|_2^2 \quad \text{for all } n > n_0(\lambda).$$

Since $\lambda \neq -a$, the second equation in (6.5) gives the following bound,

$$\begin{aligned} |u_{2,n}|^2 &\leq 2|\lambda + a|^{-2}(|u_{1,n}|^2 + |f_{2,n}|^2) \\ &\leq 2|\lambda + a|^{-2}(K_1(\lambda) + 1)\|f_n\|_2^2 \\ &= K_2(\lambda)\|f_n\|_2^2 \quad \text{for all } n > n_0(\lambda); \end{aligned}$$

therefore, $n^4|u_{1,n}|^2 + |u_{2,n}|^2 \leq \max(K_1(\lambda), K_2(\lambda))\|f_n\|_2^2$ for all $n > N_1(\lambda)$ and owing to $f \in X$, we conclude that there exists $K(\lambda)$ such that $\|u\|_Z \leq K(\lambda)\|f\|_X$. For all n , u_n is the unique solution to (6.4), hence, the uniqueness of the Fourier series implies uniqueness of u .

Step 2: $E \supseteq \rho(L)$. Equivalently, we prove $\mathbb{C} \setminus E \subseteq \mathbb{C} \setminus \rho(L) = \sigma(L)$. If there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$ such that $D_n(\lambda) = 0$, then (6.4) does not have a solution, hence, $\lambda \in \sigma(L)$. If $\lambda = -a$, then $(\lambda - L)u = f$ does not have a solution in Z for all $f \in X$, because $\lambda = -a$ implies $u_1 = -f_2 \in L^2(0, 2\pi)$, hence, $u_1 \notin H_{\text{per}}^2(0, 2\pi)$ in general. We therefore have $-a \in \sigma(L)$. $\square$

Proposition 6.5 and Hypothesis 6.4 imply

$$\sigma(L) = \left\{ \pm i\sqrt{1 - a^2}, -a \right\} \cup \left\{ \frac{-n^2 + \sqrt{(n^2 - 2a)^2 - 4}}{2} : n \in \mathbb{Z}_{\neq 0} \right\},$$

therefore, the homogeneous equilibrium $u \equiv 0 \in X$ of the fast subsystem of (6.2) undergoes a Hopf bifurcation at $(v, b, c) = (0, 0, 0)$ with Hopf frequency $\omega_* = \sqrt{1 - a^2} = \sqrt{1 - (b^*)^2}$. It can be shown (we omit this calculation for brevity)

$$\text{Re } \sigma_s(L) \subset (-\infty, -a/2), \quad \sigma_c(L) = \{\pm i\omega_*\}, \quad \sigma_u(L) = \{\emptyset\},$$

therefore, Hypothesis 4.3 holds with $n = 2$ and $\gamma = a/2$.

6.2.4. Checking Hypothesis 4.4. Checking this hypothesis directly may be challenging in applications. However, in a wide class of problems, this hypothesis can be replaced by some other conditions, which are easier to check. In particular, since $X = Y$ and X, Y, Z are Hilbert spaces, it suffices to prove the following condition [44, section 2.2.3, Theorem 2.20].

PROPOSITION 6.6. *Let $L \in \mathcal{L}(Z, X)$ be the linear operator defined in (6.3). There exist $\omega_0, K_0 > 0$ such that, for all $\omega \in \mathbb{R}$ with $|\omega| > \omega_0$, $i\omega \in \rho(L)$ and*

$$(6.6) \quad \|(i\omega - L)^{-1}\|_{\mathcal{L}(X, X)} \leq K_0/|\omega|.$$

Proof. We note that $i\omega \in \sigma(L)$ if and only if $|\omega| = \sqrt{4 - (n^2 - 2a)^2}/2 < 1$, hence, $|\omega| > 1$ implies $i\omega \in \rho(L)$. Henceforth we set $\omega_1 = 1$ for notational convenience. We now fix ω with $|\omega| > \omega_1$, $f \in X$, and prove that there exist K_0, ω_0 such that

$$(6.7) \quad \|(i\omega - L)^{-1}f\|_X \leq K_0 \|f\|_X / |\omega|$$

which implies (6.6). Since $i\omega \in \rho(L)$, there exists a unique $u \in Z$ such that $u = (i\omega - L)^{-1}f$, hence,

$$(6.8) \quad -\partial_x^2 u_1 + \left(i\omega - a + \frac{1}{i\omega + b}\right)u_1 = f_1 - \frac{f_2}{i\omega + b}, \quad u_2 = \frac{u_1 + f_2}{i\omega + b}.$$

We use Lemma 6.1 to bound u_1 : the operator ∂_x^2 in (6.8) maps from $H_{\text{per}}^2((0, 2\pi), \mathbb{C})$ to $L^2((0, 2\pi), \mathbb{C})$, and

$$\left(i\omega - a + \frac{1}{i\omega + b}\right) \in \rho(\partial_x^2), \quad f_1 - \frac{f_2}{i\omega + b} \in L^2((0, 2\pi), \mathbb{C});$$

therefore, we obtain

$$|\omega| \left(1 - \frac{1}{\omega_1^2 + b^2}\right) \|u_1\|_{L^2} \leq \left\|f_1 - \frac{f_2}{i\omega + b}\right\|_{L^2} \leq \left(1 + \frac{1}{\sqrt{\omega_1^2 + b^2}}\right) \|f\|_X.$$

We conclude that there exist constants $K_2, \omega_2 = 1 + \omega_1 > 0$ such that for all $\omega \in \mathbb{R}$ with $|\omega| > \max(\omega_1, \omega_2)$

$$\|u_1\|_{L^2} \leq \frac{K_2}{|\omega|} \|f\|_X.$$

Taking the norm of the second equation in (6.8), we arrive at the following bound, valid for all ω with $|\omega| > \max(\omega_1, \omega_2)$:

$$\|u_2\|_{L^2} \leq \frac{1}{\sqrt{\omega^2 + b^2}} \left(1 + \frac{K_2}{|\omega_2|}\right) \|f\|_X = \frac{1}{|\omega|} \frac{|\omega| + K_2}{\sqrt{\omega^2 + b^2}} \|f\|_X,$$

and we conclude that there exist $K_3, \omega_3 > 0$ such that, for all $\omega \in \mathbb{R}$ with $|\omega| > \max(\omega_1, \omega_2, \omega_3)$

$$\|u_2\|_{L^2} \leq \frac{K_3}{|\omega|} \|f\|_X.$$

Using the bounds found for $\|u_1\|_{L^2}$ and $\|u_2\|_{L^2}$ we find, for all $\omega \in \mathbb{R}$ with $|\omega| > \max(\omega_1, \omega_2, \omega_3)$,

$$\begin{aligned} \|(i\omega - L)^{-1}f\|_X &= (\|u_1\|_{L^2} + \|u_2\|_{L^2})^{1/2} \\ &\leq \frac{\sqrt{K_2^2 + K_3^2}}{|\omega|} \|f\|_X. \end{aligned}$$

Therefore (6.7) holds and, hence, (6.8) holds with $\omega_0 = \max(\omega_1, \omega_2, \omega_3)$ and $K_0 = \sqrt{K_2^2 + K_3^2}$. $\square$

6.2.5. System of 2-fast, 1-slow variables. The sections above show that the hypotheses of Lemma 5.6 hold for the FitzHugh–Nagumo reaction-diffusion system (6.1). This leads to the finite-dimensional Hopf normal form (5.4) for which the function G in the slow equation is the identity. We obtain the system

$$(6.9) \quad \begin{aligned} \dot{A} &= i\omega A + \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + \delta\varepsilon^2 + \text{h.o.t.}, \\ \dot{B} &= \varepsilon, \end{aligned}$$

where A and B are one-dimensional complex and real variables, respectively. One can then apply classical results on slow passage through a Hopf bifurcation to this system. Such results were first unveiled by Shishkova [72] and then proved within a general framework by Neishtadt [64, 65, 66]; see also [68, 6, 48] for an independent treatment. We therefore have a theoretical local explanation for the numerical results shown in Figure 3. We note that it is straightforward to apply the theory above to any reaction-diffusion system of PDEs, for which we expect to find generically a slow passage through Hopf bifurcations. Other types of boundary conditions can also be easily included by changing the underlying function spaces.

6.3. Slow passage through Turing in a nonlocal reaction-diffusion equation. We have presented numerical results showing a slow passage through a Turing bifurcation for a system of reaction-diffusion equations, and for a one-component nonlocal reaction-diffusion equation model. In this section, we present in detail only the nonlocal case, which is less well studied. The computations for the other case follow in a similar way, and we omit them here.

We recall the model under consideration,

$$(6.10) \quad \begin{aligned} \partial_t u(x, t) &= d \partial_x^2 u(x, t) + v(t) u(x, t) - u(x, t) \int_{\Omega} w(x - y) u(y, t) dy, \\ \dot{v} &= \varepsilon \end{aligned}$$

for $(x, t) \in \Omega \times \mathbb{R}_{>0}$ with $\Omega = \mathbb{R}/2l\mathbb{Z}$, which implies periodic boundary conditions

$$u(-l, t) = u(l, t), \quad t \in \mathbb{R}_{>0}.$$

This model is obtained from (2.6) by setting $v \mapsto v - b$, which slightly simplifies the notation. We make some preliminary assumptions on the kernel and on the existence of a homogeneous equilibrium.

Hypothesis 6.7. The interaction kernel $w \in L^1(\Omega)$ is an even $2l$ -periodic function with Fourier coefficients $(w_i)_{i \in \mathbb{Z}}$, $w_0 = 1$. Further, there exist $u_*, v_*, d_* \in \mathbb{R}_{>0}$ and $n_* \in \mathbb{Z}_{\neq 0}$ such that $u(x, t) \equiv u_*$ is an equilibrium of (6.10) for $(v, d, \varepsilon) = (v_*, d_*, 0)$ with $u_* = v_*$, $d_* = -v_* w_{n_*} l^2 / (n_* \pi)^2$.

As we shall see below, the requirement on d_* guarantees that u_* undergoes a Turing-like bifurcation to a mode with wavelength $\pi/(ln_*)$. A change of variables, similar to the one used in other examples, leads to the system

$$(6.11) \quad \begin{aligned} \partial_t u &= (d_* \partial_x^2 u - u_* W) u + (d \partial_x^2 + v) u + u_* v - u W u, \\ \dot{v} &= \varepsilon, \end{aligned}$$

where we dropped the tildes and set $(Wu)(x) = \int_{\Omega} w(x - y) u(y) dy$. The system above is in the form (6.2) with

$$L = d_* \partial_x^2 - u_* W, \quad R(u, v, \mu, \varepsilon) = (\mu \partial_x^2 + v) u + u_* v - u W u, \quad G(u, v, \varepsilon) = \varepsilon,$$

where $\mu = d$.

6.3.1. Choice of function spaces. In this problem, we choose $Z = H_{\text{per}}^2(\Omega)$ and $Y = X = L_{\text{per}}^2(\Omega)$. We note that $X = Y$ is a consequence of the fact that the linear operator in the original problem (6.10) contains a Laplacian and is parameter dependent.

6.3.2. Checking Hypothesis 4.2. Since $w \in L^1(\Omega)$ by Hypothesis 6.7, the Young's convolution theorem [17, Theorem 4.15] gives

$$\|Wu\|_X \leq \|w\|_{L^1} \|u\|_X \quad \text{for all } u \in X, \quad \|Wu\|_Z \leq \|w\|_{L^1} \|u\|_Z \quad \text{for all } u \in Z.$$

We conclude that, since ∂_x^2 and W are continuous operators from Z to X , then their linear combination L is in $\mathcal{L}(Z, X)$. As for the nonlinear function R , we note that the mapping $u \mapsto uWu$, seen as a nonlinear operator from Z to Z , is the composition of W and a product in the Banach algebra Z . Since $W \in \mathcal{L}(Z)$, we conclude that $R \in C^k(Z \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, X)$ for any integer k .

6.3.3. Checking Hypothesis 4.3. We now study the spectrum of L which is a closed operator in X with domain Z . Since Z is compactly embedded in X by the Sobolev embedding theorem (see [17, Theorem 8.8]), it follows that L has a compact resolvent. Thus, the spectrum of L consists of at most a countable sequence of isolated eigenvalues with finite algebraic multiplicity (see [55, Theorem III.6.29]) which can accumulate at 0 or $\pm\infty$. For the problem under consideration we can directly compute this sequence, and obtain

$$\sigma(L) = \{-d_*(n\pi/l)^2 - u_*w_n : n \in \mathbb{Z}\}.$$

Since $w \in L^1(\Omega)$, the Riemann–Lebesgue lemma [71, Chapter 6] implies $w_n \rightarrow 0$ as $n \rightarrow \infty$, hence, the spectrum accumulates at infinity. Hypothesis 6.7 guarantees that

$$\sigma_c(L) = \{0\}, \quad \sigma_u(L) \cup \sigma_s(L) = \{-d_*(n\pi/l)^2 - u_*w_n : n \in \mathbb{Z}_{\neq n_*}\},$$

and we have a spectral gap for L .

6.3.4. Checking Hypothesis 4.4. Since $Y = X$, we proceed as in subsection 6.2.4, and we prove a result analogous to Proposition 6.6. We write $L = L_1 + L_2$, where $L_1 = d_*\partial_x^2$ and $L_2 = -u_*W$. We note that $\sigma(L_1)$ is purely real, hence, $i\omega \in \rho(L_1)$. We apply in sequence Lemmas 6.1 and 6.3, and conclude that there exist positive constants ω_0, K_0 such that $\|(i\omega - L)^{-1}\|_{\mathcal{L}(X)} \leq K_0/|\omega|$ for all $\omega \in \mathbb{R}$ with $|\omega| > \omega_0$, hence Hypothesis 4.4 holds.

6.3.5. Reduction to 1-slow, 1-fast system. We note that Hypothesis 5.8 holds with T_φ and S representing translations by $\varphi \in \mathbb{R}/2l\mathbb{Z}$ and reflections about the $x = 0$ axis, respectively, that is

$$(T_\varphi u)(x) = u(x - \varphi), \quad (Su)(x) = u(-x), \quad x, \varphi \in \mathbb{R}/2l\mathbb{Z}.$$

We can then apply Lemma 5.9 and obtain, in a neighborhood of the origin, a 1-slow, 1-fast system

$$(6.12) \quad \begin{aligned} \dot{A} &= \alpha(B, \mu, \varepsilon)A + \beta A|A|^2 + O((|B| + |\mu| + \varepsilon + |A|^2)^2), \\ \dot{B} &= \varepsilon, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. The reduction above leads to studying an ODE with 1-fast and m -slow variables. We have not computed the coefficients α and β explicitly, but the numerical bifurcation analysis performed in subsection 2.3.1 indicates a pitchfork bifurcation for $\varepsilon = 0$ (a Turing bifurcation of the nonlocal model). The results in [57] apply to (6.12) in this case: when $\varepsilon \neq 0$ we expect a slow passage through the pitchfork bifurcation; this confirms what has been found numerically in Figure 5, namely, the dynamics stays close to the branch of homogeneous steady states, past the Turing bifurcation. Similar considerations apply to the sub- and super-critical Turing bifurcations found in the local model presented in subsection 2.3.2.

6.4. Spatio-temporal canards in a neural-field model. In this section we study center-manifold reductions for the nonlocal neural-field problem (2.1) where Ω is a compact subset of $\mathbb{R}^d$. We introduce the integral operators

$$W: u \mapsto \int_{\Omega} w(\cdot, y)u(y) d\rho(y), \quad Q: (u, v) \mapsto \int_{\Omega} \theta(u, v)(x) d\rho(x),$$

where the integrals are over Ω , and the Nemytskii operator $N: (u, v) \mapsto \theta(u(x), v)$, and rewrite the original model as

$$\begin{aligned} \dot{u} &= -u + WN(u, v_1), \\ \dot{v}_1 &= \varepsilon(v_2 + cQ(u, v_1)), \\ \dot{v}_2 &= \varepsilon(-v_1 + a + bQ(u, v_1)). \end{aligned}$$

We make the following preliminary assumption.

Hypothesis 6.8. The firing rate function θ is in $C_b^k(\mathbb{R}^2)$, the space of bounded functions defined on $\mathbb{R}^2$ with continuous bounded derivative up to order k , for some $k \geq 2$. The synaptic kernel w is in $C(\Omega^2)$. There exists $(u_*, v_{*1}) \in C(\Omega) \times \mathbb{R}$ such that $u_* = WN(u_*, v_{*1})$. In addition, 1 is a simple eigenvalue of $WD_u N(u_*, v_{*1})$, where $D_u N(u_*, v_{*1}) : u \mapsto \partial_u \theta(u_*, v_{*1})u$ is regarded as an operator from $C(\Omega)$ to itself.

The setting of this example is different from the previous ones. First, we note that in this case u_* is not a homogeneous equilibrium of the model, and indeed canards were found in [3] close to saddle-node bifurcations of heterogeneous equilibria. Second, in this problem, we aim to provide an example of an evolution equation on a Banach space as opposed to a Hilbert space (which is why we demand $u_* \in C(\Omega)$). This functional setting is more challenging, because we cannot use resolvent estimates as in Propositions 6.5 and 6.6, and instead Hypothesis 4.4 must be checked directly. A Hilbert-space setting is, however, possible, and relies on results in [63, 44].

After setting $u = u_* + \tilde{u}$, $v_1 = v_{*1} + \tilde{v}_1$, and dropping tildes, we obtain

$$\begin{aligned} \dot{u} &= -u + WD_u N(u_*, v_{*1})u + R(u, v_1), \\ \dot{v}_1 &= \varepsilon(v_2 + cQ(u_* + u, v_{*1} + v_1)), \\ \dot{v}_2 &= \varepsilon(-v_{*1} - v_1 + a + bQ(u_* + u, v_{*1} + v_1)), \end{aligned} \tag{6.13}$$

which fits in our framework with $\mu = (a, b, c)$,

$$L: u \mapsto -u + WD_u N(u_*, v_{*1})u, \tag{6.14}$$

and nonlinear terms R, G given by

$$\begin{aligned} R(u, v_1) &= v_1 \int_{\Omega} w(x, y) \partial_{v_1} \theta(u_*(y), v_{*1}) dy + o(|u| + |v_1|), \\ G(u, v, \mu) &= (v_2 + \mu_3 Q(u_* + u, v_{*1} + v_1), -v_{*1} - v_1 + \mu_1 + \mu_2 Q(u_* + u, v_{*1} + v_1)). \end{aligned}$$

6.4.1. Function spaces and Hypothesis 4.2. For this example we choose $Z = Y = X = C(\Omega)$. For all $u \in Z$ we have, owing to Hypothesis 6.8,

$$\|Lu\|_X = \|Lu\|_{\infty} \leq (1 + \text{meas}(\Omega) \|w\|_{\infty} \|\partial_u \theta\|_{\infty}) \|u\|_{\infty} := K \|u\|_Z,$$

where the infinity norms are on $C(\Omega)$ or $C(\Omega^2)$, as appropriate. Therefore $L \in \mathcal{L}(Z, X)$. The nonlinear terms R satisfy Hypothesis 4.2.

6.4.2. Checking Hypothesis 4.3.

We first recall the following result.

LEMMA 6.9. *If Hypothesis 6.8 holds the operator $WD_uN \in \mathcal{L}(X)$ is compact.*

Proof. As $\bar{\Omega}$ is compact, the Weierstrass theorem provides a sequence of polynomials $(w_k)_k$ which tends to w in $C(\bar{\Omega}^2)$, and the same is true for the associated integral operators $(W_k)_k$ in $\mathcal{L}(X)$. These operators have finite-dimensional range and are thus compact operators. The set of compact operators is closed in $\mathcal{L}(X)$ (see [17]) hence leading to the conclusion that W is compact. As D_uN is a continuous operator on X , WD_uN is compact. $\square$

It follows that the spectrum $\sigma(WD_uN)$ of WD_uN contains 0, and $\sigma(WD_uN) \setminus \{0\}$ consists at most of a countable sequence of isolated eigenvalues with finite algebraic multiplicity which may accumulate at 0. Since $L = -\text{id}_X + WD_uN$ and Hypothesis 6.8 holds, Hypothesis 4.3(i) is verified, and there exists a spectral gap $\gamma > 0$ which may be chosen such that Hypothesis 4.3(ii) is satisfied. In addition, the structure of the spectrum of WD_uN implies that $\sigma_u(L)$ is finite and X_u is finite dimensional.

6.4.3. Checking Hypothesis 4.4. As anticipated above, since we are dealing with a quasilinear formulation ($Y = X$) on a Banach space, we cannot make use of resolvent estimates and we must verify Hypothesis 4.4 directly (see also [51] for a similar strategy). In the following, we write $T(t) = e^{tL} \in \mathcal{L}(X)$ the semigroup generated by L . We initially derive the following useful estimates.

PROPOSITION 6.10. *Assume Hypothesis 6.8, and let L be the operator defined in (6.14), and let P_u, P_s be the associated projectors on X_u, X_s , respectively. There exists a constant $M > 0$ such that*

$$(6.15) \quad \|T(t)P_u\| \leq Me^{\beta_u t} \quad \text{for all } t \leq 0, \beta_u \in \left(0, \min_{\lambda \in \sigma_u} \text{Re } \lambda\right),$$

$$(6.16) \quad \|T(t)P_s\| \leq Me^{-\beta_s t} \quad \text{for all } t \geq 0, \beta_s \in \left(0, -\sup_{\lambda \in \sigma_s} \text{Re } \lambda\right).$$

Proof. The space X_u is finite dimensional (see subsection 6.4.2), hence, $T(t)P_u$ has finite-dimensional range, and a Dunford decomposition yields (6.15). The operator $T(t)P_s$ is bounded on X . As a consequence, the spectral mapping theorem [34, section VII.3.6, Theorem 11] yields $\sigma(T(t)P_s) \setminus \{0\} = e^{t\sigma_s(L)}$. Finally, an application of the Gelfand spectral radius theorem [34] yields (6.16). Note that Hypothesis 6.8 provides the existence of β_u, β_s . $\square$

PROPOSITION 6.11. *Assume the hypotheses of Proposition 6.10. Then Hypothesis 4.4 holds with*

$$(K_{su}f)(t) = - \int_t^\infty T(t-\tau)P_u f(\tau) d\tau + \int_{-\infty}^t T(t-\tau)P_s f(\tau) d\tau.$$

Proof. Hypothesis 4.4 is stated in terms of the constant $\gamma > 0$, fixed as in Hypothesis 4.3(ii). The constant can be chosen to be $\gamma = \min(\beta_s, \beta_u)/2$, where β_u, β_s are in the range specified in Proposition 6.10. Henceforth we fix $\eta \in [0, \gamma]$, $f \in C_\eta(\mathbb{R}, Y_{su})$, let $u = K_{su}f$, and we prove that Hypothesis 4.4 holds for u , using 4 steps.

Claim 1: u is exponentially bounded. Owing to the standard properties of the projectors P_s, P_u , we have $u(t) \in Z_{su}$ for all $t \in \mathbb{R}$. We set $u = u_s + u_u$ with

$$u_s(t) = \int_{-\infty}^t T(t-\tau)P_s f(\tau) d\tau, \quad u_u(t) = - \int_t^\infty T(t-\tau)P_u f(\tau) d\tau,$$

and use (6.16) to derive the following bound,

$$\begin{aligned}\|u_u(t)\|_Z &\leq M \|f\|_\eta \int_t^\infty e^{\beta_u(t-\tau)+\eta|\tau|} d\tau \\ &\leq M \|f\|_\eta e^{\eta|t|} \int_0^\infty e^{(\tau-\beta_u)\tau} d\tau = \frac{M}{\beta_u - \eta} \|f\|_\eta e^{\eta|t|}.\end{aligned}$$

A similar bound can be found for u_s , and we obtain

$$\|u(t)\|_Z \leq \left(\frac{1}{\beta_u - \eta} + \frac{1}{\beta_s - \eta} \right) M \|f\|_\eta e^{\eta|t|}.$$

Claim 2: u is in $C_\eta(\mathbb{R}, Z_{su})$. The above estimate, the continuity of $f: \mathbb{R} \rightarrow X$, and the dominated convergence theorem imply that $u: \mathbb{R} \rightarrow X$ is continuous and $u \in C_\eta(\mathbb{R}, Z_{su})$. In passing we note that the estimate above also implies $\|K_{su}\| \leq \Pi(\eta) = M/(\beta_u - \eta) + M/(\beta_s - \eta)$.

Claim 3: u is differentiable in X and solves (4.2). We treat the term $u_s(t)$ for some fixed $t \in \mathbb{R}$, as the result for $u_u(t)$ follows in a similar way. Using a Taylor expansion of the exponential $T_s(\tau) = e^{L_s \tau}$ we obtain, for sufficiently small ε ,

$$\begin{aligned}u_s(t+\varepsilon) - u_s(t) &= \int_{-\infty}^t [T_s(t-\tau+\varepsilon) - T_s(t-\tau)] f(\tau) d\tau + \int_t^{t+\varepsilon} T_s(t-\tau+\varepsilon) f(\tau) d\tau \\ &= \varepsilon L_s \int_{-\infty}^t T_s(t-\tau) f(\tau) d\tau + T_s(\varepsilon) \int_t^{t+\varepsilon} T_s(t-\tau) f(\tau) d\tau + O(\varepsilon^2) \\ &= \varepsilon L_s \int_{-\infty}^t T_s(t-\tau) f(\tau) d\tau + \varepsilon P_s f(t) + O(\varepsilon^2).\end{aligned}$$

Hence, u_s is differentiable at t and $\dot{u}_s(t) = P_s f(t) + L_s u_s(t)$. It follows that u_s is C^1 in X . The claim is proved using a similar argument for u_u .

Claim 4: u is unique. Assume there are two solutions $u, \tilde{u}$ to (4.2). Then $\nu = u - \tilde{u}$ solves $\dot{\nu} = L_{su} \nu$ and belongs to $C_\eta(\mathbb{R}, X_{su})$. This implies $\nu(0) \in X_s \cap X_u = \{0\}$, hence, $\nu(t) \equiv 0$ and the uniqueness of u . $\square$

6.4.4. Reduction to 1-fast 2-slow system. We now apply Lemma 5.3 to (6.13). We note that $\gamma(\mu, \varepsilon) = 0$ because R is independent of (μ, ε) ; in addition, R does not depend on v_2 , hence, $\alpha(B)$ and higher order terms depend solely on B_1 . In a neighborhood of the origin we obtain the reduction

$$\begin{aligned}(6.17) \quad \dot{A} &= \alpha B_1 + \beta A^2 + \omega(A, B_1, \mu, \varepsilon), \\ \dot{B}_1 &= \varepsilon (B_2 + \mu_3 H(A, B_1, B_2, \mu, \varepsilon)), \\ \dot{B}_2 &= \varepsilon (-v_{*1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, \varepsilon)),\end{aligned}$$

where $H: \mathbb{R}^3 \times \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(6.18) \quad H(A, B_1, B_2, \mu, \varepsilon) = Q(u_* + A\zeta + \Psi(A, (B_1, B_2), \mu, \varepsilon), v_{*1} + B_1)$$

and

$$\omega(A, B_1, \mu, \varepsilon) = O(A^2(|B_1| + |\mu| + |\varepsilon|) + (|B_1| + |\mu| + |\varepsilon|)^2).$$

LEMMA 6.12. Assume $\alpha, \beta \neq 0$, and let O_μ be defined as in Lemma 5.3. There exists an open subset $U_\mu \subset O_\mu$ and a function $\xi: U_\mu \rightarrow \mathbb{R}$ such that system (6.17) admits a folded singularity at $(A, B_1, B_2, \mu) = (0, 0, \xi(\mu), \mu)$ for all $\mu \in U_\mu$. The folded singularity is

- a folded saddle if $J_{12}J_{21} > 0$;
- a folded node if $J_{11} < 0$ and $J_{11}^2 + 4J_{12}J_{21} > 0$;
- a folded-saddle node if $J_{12}J_{21} = 0$,

where

$$\begin{aligned} J_{11} &= \mu_3\beta/\alpha[\partial_A H(0, 0, \xi(\mu), \mu, 0) + \partial_{B_1} H(0, 0, \xi(\mu), \mu, 0)\partial_A \eta(0, \mu)], \\ J_{12} &= \beta/\alpha[1 + \mu_3\partial_{B_2} H(0, 0, \xi(\mu), \mu, 0)], \\ J_{21} &= -2(-v_{*1} + \mu_1 + \mu_2 H(0, 0, \xi(\mu), \mu, 0)). \end{aligned}$$

Proof. By setting $\tilde{t} = t/\beta$, $\tilde{\varepsilon} = \varepsilon/\beta$, $\kappa = \beta/\alpha$ (and dropping the tilde in ε) we cast system (6.17) as

$$\begin{aligned} \dot{A} &= \kappa B_1 + A^2 + \omega(A, B_1, \mu, \varepsilon), \\ \dot{B}_1 &= \varepsilon(B_2 + \mu_3 H(A, B_1, B_2, \mu, \varepsilon)), \\ \dot{B}_2 &= \varepsilon(-v_{*1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, \varepsilon)). \end{aligned} \quad (6.19)$$

After passing to the slow time $\tau = \varepsilon t$ in (6.19) we obtain, at $\varepsilon = 0$, the slow subsystem

$$\begin{aligned} 0 &= \kappa B_1 + A^2 + \omega(A, B_1, \mu, 0), \\ B_1' &= (B_2 + \mu_3 H(A, B_1, B_2, \mu, 0)), \\ B_2' &= (-v_{*1} - B_1 + \mu_1 + \mu_2 H(A, B_1, B_2, \mu, 0)). \end{aligned} \quad (6.20)$$

The critical manifold is a graph over A in the neighborhood of the origin; the implicit function theorem guarantees the existence of a subset $V_A \times V_\mu$ of $O_A \times O_\mu$, and a unique function $\eta: V_A \times V_\mu \rightarrow \mathbb{R}$ such that

$$0 = \kappa\eta(A, \mu) + A^2 + \omega(A, \eta(A, \mu), \mu, 0), \quad (A, \mu) \in V_A \times V_\mu.$$

The desingularized reduced system associated with (6.20) is given by

$$\begin{aligned} \dot{A} &= [\kappa + \partial_{B_1} \omega(A, \eta(A, \mu), \mu, 0)][B_2 + \mu_3 H(A, \eta(A, \mu), B_2, \mu, 0)], \\ \dot{B}_2 &= [-2A - \partial_A \omega(A, \eta(A, \mu), \mu, 0)][-v_{*1} - B_1 + \mu_1 + \mu_2 H(A, \eta(A, \mu), B_2, \mu, 0)]. \end{aligned} \quad (6.21)$$

Since by the implicit function theorem $\kappa + \partial_{B_1} \omega(A, \eta(A, \mu), \mu, 0)$ does not vanish in $V_A \times V_\mu$, then a folded singularity of (6.17) is an equilibrium of (6.21) satisfying

$$\begin{aligned} 0 &= B_2 + \mu_3 H(A, \eta(A, \mu), B_2, \mu, 0), \\ 0 &= -2A - \partial_A \omega(A, \eta(A, \mu), \mu, 0). \end{aligned}$$

The second equation holds for $A = 0$. A further application of the implicit function theorem to the first equation with $A = 0$ then yields the existence of $U_\mu \subset V_\mu$, a function $\xi: U_\mu \rightarrow \mathbb{R}$ such that $(A, B_1, B_2) = (0, 0, \xi(\mu))$ is a folded singularity. The statement follows from the fact that the Jacobian of (6.21) at the folded singularity is the matrix

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}. \quad \square$$

The previous lemma implies that solutions to the original neural-field model near saddle-node bifurcation points of the fast subsystem, which are patterned states, are expected to display canard segments. The existence of these structures was predicted

analytically in [3] in the case of a Heaviside firing rate θ and is therefore valid for generic smooth, bounded firing rates. Simulations have been reproduced from that paper in Figures 1 and 2. We note that, while Lemma 6.12 does not directly apply when Ω is a ring or a sphere, similar dynamics is observable on neural-field models with a heterogeneous external input, which are not equivariant with respect to the action of a Lie group, and to which Lemma 6.12 applies.

6.5. Delay-differential equation. We now present results for the DDE (2.4). The treatment of this case is separate from the others, as it requires different technical tools. We provide pointers to specialized literature on the topic further below in this section.

We denote by u_* an equilibrium of the fast subsystem associated with (2.4), set $u = u_* + x$, and obtain

$$(6.22) \quad \begin{aligned} \dot{x}(t) &= vx(t) - x(t - \tau) + u_*^3 - (u_* + x(t))^3, & t \geq 0, \\ x(t) &= \varphi(t), & t \in [-\tau, 0]. \end{aligned}$$

The analysis of DDEs is technical and relies on tools from functional analysis and semigroup theory [36]. In this section, we aim to provide a self-contained treatment for the analysis of (6.22), and we refer the reader to the textbooks [43, 31] for an exhaustive treatment of the subject.

To make sense of the right-hand side of (6.22), x must be known on a time interval of length τ , typically a *history* time interval $[t - \tau, t]$. This implies that the state space of (6.22) is infinite dimensional and included in some function space from $[-\tau, 0]$ to $\mathbb{R}$. The choice of this function space affects the analysis of (6.22). For example, if we were to choose $X = C^0([-\tau, 0], \mathbb{R})$, it would appear that the operator L in (4.1) encodes the DDE in its domain Z (see [43] and Remark 6.13). We therefore would not apply the center-manifold results presented in the previous sections, as the space Z would depend on the parameter v .

The standard formalism aimed to tackle this difficulty is that of the *sun-star* calculus [30, 31] which seeks solutions in a larger state space of less regular functions. This theory can also be applied to center-manifold reductions [31]. Here we present a self-contained example that does not require the additional technical knowledge of sun-star calculus. We proceed as follows:

1. We select the state spaces Z, Y, X as suggested by the sun-star theory and check that they are suitable for our problem (see also [10] for a way to bypass the use of sun-star calculus).
2. We do not apply the center manifold reduction exposed in [30, 31] because that approach looks for integral solutions to the problem and thus solves (4.2) in $C_\eta(\mathbb{R}, X)$ instead of $C_\eta(\mathbb{R}, Z)$, which is required by our formalism. It is possible to adapt the proof in [30, 31] to fit our framework, but we choose a direct and self-contained approach. We highlight that we will exploit (in Lemma 6.17) the fact that the nonlinearity R has a finite-dimensional range, as it is done in the sun-star references given above.

Before proceeding we introduce, for any $t \geq 0$ the *history function* $w_t : [-\tau, 0] \rightarrow X$ of $w : [-\tau, \infty) \rightarrow X$ defined by $w_t(\theta) := w(t + \theta)$.

6.5.1. Function spaces and Hypothesis 4.2. A first step in the presentation is to rewrite (6.22) as a Cauchy problem. We chose a Hilbert space setting with $X = Y = \mathbb{R} \times L^2((-\tau, 0), \mathbb{R})$ and we denote by π_1, π_2 the canonical projections from X onto $\mathbb{R}$ and $L^2((-\tau, 0), \mathbb{R})$, respectively. We set $Z = \{u \in \mathbb{R} \times W^{1,2}((-\tau, 0), \mathbb{R}) \mid (\pi_2 u)(0) =$

$\pi_1 u\}$ and note that Z is continuously embedded in X . This allows us to define the following operators (see [10]):

$$(6.23) \quad L = \begin{pmatrix} v & \Phi \\ 0 & \frac{d}{d\theta} \end{pmatrix} \in \mathcal{L}(Z, X), \quad R(u) = \begin{pmatrix} u_*^3 - (u_* + \pi_1(u))^3 \\ 0 \end{pmatrix},$$

where $\Phi = u_2 \rightarrow -u_2(-\tau) \in \mathcal{L}(W^{1,2}(-\tau, 0; \mathbb{R}), \mathbb{R})$. Note that $R(u) \notin Z$ for $u \in Z$. However, $R \in C^\infty(Z, X)$ because Z is a Banach algebra.

In conclusion, we consider the Cauchy problem

$$\dot{u} = Lu + R(u)$$

with initial condition in Z . Although it may appear mysterious at first, one can check, using the first component, that (6.22) is indeed encoded by the above Cauchy problem and this provides Hypothesis 4.2.

Remark 6.13. A possible alternative choice [43, 31] is to define

$$X = C^0([-\tau, 0], \mathbb{R}), \quad Z = \left\{ \varphi \in C^1([-\tau, 0], \mathbb{R}) \mid \frac{d}{d\theta} \varphi(0) = v\varphi(0) - \varphi(-\tau) \right\},$$

and then

$$L : Z \rightarrow X, \quad u \mapsto \frac{d}{d\theta} u.$$

With this choice Z depends on the parameter v , which is not suitable for our analysis because the spaces Z, Y, X cannot depend on a varying parameter. One then can use the sun-star formalism [30, 31, 36], akin to a double dual operation, to rewrite the problem in the space $X = \mathbb{R} \times L^\infty((-\tau, 0), \mathbb{R})$ obtaining a form similar to (6.23).

6.5.2. Checking Hypothesis 4.3. We now check hypotheses on the spectrum of L in (6.23), which is characterized in the following lemma.

LEMMA 6.14. *Fix $\tau > 0$, $v \in \mathbb{R}$. The spectrum of L is composed solely of eigenvalues and is given by $\sigma(L) = \{\lambda \in \mathbb{C} : v - e^{-\lambda\tau} - \lambda = 0\}$. An eigenvector associated with the eigenvalue $\lambda \in \sigma(L)$ is $(1, e^{\lambda\cdot}) \in Z$.*

Proof. In this proof, we write $L^2 = L^2((-\tau, 0), \mathbb{R})$ and $W^{1,2} = W^{1,2}((-\tau, 0), \mathbb{R})$ to simplify notations. Let $E = \{\lambda \in \mathbb{C} : v - e^{-\lambda\tau} - \lambda \neq 0\}$. We prove that $E \supseteq \rho(L)$ by showing that $\mathbb{C} \setminus E \subset \sigma(L)$. For each λ solution of $v - e^{-\lambda\tau} - \lambda = 0$, $(1, e^{\lambda\cdot}) \in Z$ is an eigenvector showing that $\lambda - L$ is not invertible.

Conversely, we consider $\lambda \in E$. We show that the equation $Lu = \lambda u + w$ has a unique solution $u \in Z$ for any $w \in X$. Writing $u = (u_1, u_2)$, one finds

$$\frac{d}{d\theta} u_2 = \lambda u_2 + w_2, \quad u_2(\theta) = e^{\lambda\theta} u_1 + \int_0^\theta e^{\lambda(\theta-s)} w_2(s) ds.$$

This implies that u_2 is of Sobolev regularity. One also has $v u_1 - u_2(-\tau) = \lambda u_1 + w_1$ which gives

$$(v - e^{-\lambda\tau} - \lambda) u_1 = w_1 + \int_0^{-\tau} e^{\lambda(-\tau-s)} w_2(s) ds.$$

This equation has a unique solution $u_1 \in \mathbb{R}$ because $\lambda \in E$ and this provides the unique solution to $Lu = \lambda u + w$. Finally, $\|u\|_Z = |u_1| + \|u_2\|_{W^{1,2}}$. From the Cauchy-Schwarz inequality

$$|u_1| \leq \frac{1}{|v - e^{-\lambda\tau} - \lambda|} (|w_1| + \|e^{\operatorname{Re} \lambda \cdot}\|_{L^2} \|w_2\|_{L^2}) = O(\|w\|_X).$$

From the definition of u_2 , we find

$$\|u_2\|_{L^2} \leq \|e^{\lambda \cdot}\|_{\infty} (|u_1| + \sqrt{\tau}\|w_2\|_{L^2}) = O(\|w\|_X).$$

Finally,

$$\left\| \frac{d}{d\theta} u_2 \right\|_{L^2} \leq |\lambda| \cdot \|u_2\|_{L^2} + \|w_2\|_{L^2} = O(\|w\|_X).$$

This implies that $\|u\|_{W^{1,2}} = O(\|w\|_X)$. We have shown that

$$\sigma(L) = \{\lambda \in \mathbb{C} : v - e^{-\lambda\tau} - \lambda = 0\}.$$

We note that the spectrum is composed of eigenvalues λ because the vector $(1, e^{\lambda \cdot}) \in Z$ is an associated eigenvector. $\square$

The spectrum of L is composed of the zeros of a holomorphic function, hence, the (at most countable) spectral elements are isolated. Therefore Hypothesis 4.3 holds and σ_u is finite. As a consequence X_u is finite dimensional. Note that the Dunford projectors P_u, P_s are easily expressed using the expression of the eigenvectors and the scalar product on X .

Remark 6.15. Using the different branches $(W_k)_{k \in \mathbb{Z}}$ of the Lambert function W (see [27]) which satisfies $W(z)e^{W(z)} = z$, $z \in \mathbb{C}$, it is possible to compute all the eigenvalues

$$\lambda_k = v + \frac{1}{\tau} W_k(-\tau e^{-\tau v}), k \in \mathbb{Z}.$$

6.5.3. Checking Hypothesis 4.4. We now check the last hypothesis required to have a center manifold, namely, Hypothesis 4.4 which is more challenging than the previous steps. The case we study is a nontrivial example for which the sufficient conditions stated in [44, section 2.2.3] do not hold, whereas Hypothesis 4.4 is satisfied. As for the neural-field example, we check directly that the solution to (4.2) is

$$(K_{su}f)(t) = - \int_t^\infty T(t-r)P_u f(r)dr + \int_{-\infty}^t T(t-r)P_s f(r)dr,$$

where $(T(t))$ is the (strongly continuous) semigroup of solutions generated by L . This semigroup can be directly expressed using the method of steps. We first give the expression of $(T(t))$ which allows us to properly define K_{su} .

LEMMA 6.16. *Let $(x, \varphi) \in X$. For each $t \in [0, \tau]$ we have*

$$(6.24) \quad \begin{aligned} \pi_1(T(t)(x, \varphi)) &= e^{vt}x - \int_0^t e^{v(t-s)}\varphi(s-\tau)ds := u_1(t), \\ \pi_2(T(t)(x, \varphi)) &= (u_1)_t, \end{aligned}$$

where $(u_1)_t$ is the history function of u_1 . In addition, $\text{range}(T(\tau)) \subset Z$.

Proof. Let us compute the linear flow $u(t) = T(t)(x, \varphi)$ for $(x, \varphi) \in X$. We write $u_i = \pi_i(u)$, for $i = 1, 2$. We start with $(x, \varphi) \in D(L) = Z$. Proposition 3.9 in [10] shows that u_1 is solution of

$$\begin{aligned} \dot{u}_1(t) &= vu_1(t) + \Phi((u_1)_t), \quad t \geq 0, \\ u_1(0) &= x, \\ u_1(\theta) &= \varphi(\theta), \quad t \in [-\tau, 0), \end{aligned}$$

and $u_2(t) = (u_1)_t$. We have $\dot{u}_1(t) = vu_1(t) - u_1(t - \tau)$ and for $t \in [0, \tau]$, we get

$$(6.25) \quad u_1(t) = e^{vt}x - \int_0^t e^{v(t-s)}\varphi(s - \tau)ds.$$

We thus have the expression $u(t) = T(t)(x, \varphi)$ for $t \in [0, \tau]$ on Z . Further, as Z is dense in X , we can consider a sequence such that $Z \ni (x_n, \varphi_n) \rightarrow (x, \varphi) \in X$. The above analytical expression shows that $\pi_1(T(t)(x_n, \varphi_n))$ converges to the right-hand side of (6.25) in X , namely,

$$\pi_1(T(t)(x, \varphi)) = e^{vt}x - \int_0^t e^{v(t-s)}\varphi(s - \tau)ds.$$

Proposition 3.11 in [10] shows that the second component $\pi_2(T(t)(x, \varphi))$ is actually $(u_1)_t$. We thus have found the expression of $T(t)$ for $t \in [0, \tau]$ in the whole space X . The last statement is straightforward to check, using an approach similar to the proof of Lemma 6.14. $\square$

Let us consider a general $t = n\tau + s$ with $s \in [0, \tau)$, the semigroup property gives $T(t) = T(\tau)^n T(s)$ and we thus have the expression of $T(t)$ for any $t \geq 0$. We also assume that Proposition 6.10 holds for the above semigroup; this is usually proved [80, 43] by showing that T is eventually norm continuous. Therefore, we have everything at hand to define $K_{su}f$ as above.

In order to check Hypothesis 4.4, we need to

1. show that $K_{su} \in \mathcal{L}(C_\eta(\mathbb{R}, X_{su}), C_\eta(\mathbb{R}, Z_{su}))$;
2. show that $K_{su}f \in C^1(\mathbb{R}, X)$;
3. show that equality (4.2) holds in X for all $t \in I$.

We will not complete all the steps above, as this is laborious and parallels the proof of [80, Proposition C4]. Instead, we underline some salient points which are different from the previous examples, the most notable one being the gain of regularity of the solution, that is, $K_{su}f$ belongs to Z , and is differentiable in X .

An additional technical point which proves useful is the following. From the proof of the center-manifold theorem in [44], it can be noted that the linear operator K_{su} is always applied to vectors such as $P_{su}R(u)$, $P_{su}DR(u)$, $\dots$. Given the particular form of these vectors $P_{su}(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix})$ for some $\alpha \in \mathbb{R}$ stemming from (6.23), we only have to solve (4.2) for functions f which have the same shape, i.e., that belong to the specific linear subspace $P_{su}(\mathbb{R} \times \{0\})$ of X .

We would wish to proceed as in the proof of Proposition 6.11, but we must adapt it because the spaces Z, X are different in the present case and this requires us to show the gain of regularity. This gain of regularity from X to Z is provided below. For the continuity of K_{su} from $C_\eta(\mathbb{R}, X_{su})$ in $C_\eta(\mathbb{R}, Z_{su})$, we refer to [80, Proposition C4].

LEMMA 6.17. *Let $f \in C_\eta(\mathbb{R}, \mathbb{R})$ and consider the stable component of K_{su} :*

$$u(t) = \int_{-\infty}^t T(t-r)P_s(f(r), 0)dr.$$

Then for all $t \in \mathbb{R}$, $u(t) \in Z$, u is differentiable in X , and $\dot{u}(t) = Lu(t) + P_s(f(t), 0)$.

Proof. Proposition 6.10 allows us to give a meaning to the expression of u by showing that it is well defined and finite. Here we first show that $u(t) \in Z$. Using a

change of variables, we find

$$u(t) = \int_{t-\tau}^t T(t-r)P_s(f(r), 0) dr + T(\tau) \int_{-\infty}^t T(t-r)P_s(f(r-\tau), 0) dr.$$

By Lemma 6.16, the second term is in Z , so we therefore focus on the first term, which we denote by u^1 . We fix $r \in \mathbb{R}$ and introduce

$$h(t, r) = \begin{cases} \pi_1 T(t)P_s(f(r), 0) & \text{if } t \geq 0, \\ f_2(r)(t) = 0 & \text{if } t \in [-\tau, 0). \end{cases}$$

We know from the previous lemma Lemma 6.16 that

$$\pi_2 T(t)P_s(f(r), 0)(\theta) = h(t + \theta, r) \quad \text{for all } \theta \in [-\tau, 0].$$

Owing to Lemma 6.16, the only point left to show is $\pi_2(u^1(t)) \in W^{1,2}((-\tau, 0), \mathbb{R}, \mathbb{R})$. For all $\theta \in [-\tau, 0]$ we have

$$\pi_2(u(t))(\theta) = \int_{t-\tau}^{t+\theta} h(t-r+\theta, r) dr = \int_{t-\tau}^{t+\theta} e^{v(t-r+\theta)} f(r) dr.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, it follows that $\pi_2(u^1(t)) \in W^{1,2}((-\tau, 0), \mathbb{R}, \mathbb{R})$ and $u(t) \in Z$.

Finally, we address the differentiability of the solution. For a fixed $t \in \mathbb{R}$ and ε , one finds

$$\frac{1}{\varepsilon}(u(t+\varepsilon) - u(t)) = \frac{1}{\varepsilon}(T(\varepsilon)u(t) - u(t)) + \frac{1}{\varepsilon} \int_t^{t+\varepsilon} T(t+\varepsilon-r)P_s(f(r), 0) dr.$$

By continuity of f , the last term converges to $P_s(f(r), 0)$ in X as ε tends to zero. As $u(t) \in Z$ which is the domain of L ,

$$\frac{T(\varepsilon)u(t) - u(t)}{\varepsilon} \rightarrow Lu(t) \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, u is differentiable in X and satisfies (4.2) in X . $\square$

6.5.4. Slow passage through Hopf bifurcation. Looking for purely imaginary eigenvalues $\pm i\omega$ to the equation $v - i\omega = e^{-i\omega\tau}$, one finds that this requires $|v| \leq 1$ by taking the real part. By taking the norm, we find $\omega = \pm\sqrt{1-v^2}$. Then the Hopf bifurcation point v_H is a solution of $v_H = \cos(\tau\sqrt{1-v_H^2})$. We now consider such values of v for which $\sigma_c = \{\pm i\omega\}$ with $\omega > 0$. We can henceforth proceed as in section 6.2.5 because the normal form and the slow system are identical to the present one.

7. Conclusions. In this paper, we have provided new local results on the outstanding problem of proving the existence of canard solutions in infinite-dimensional slow-fast dynamical systems as well as delayed bifurcation scenarios in this context. Namely, we have addressed the general case of systems with m -slow variables and infinitely many fast variables, that is, systems for which the fast component lives in a Banach space. In this general context, we have proven center-manifold reductions near points where normal hyperbolicity was lost as these correspond to where canard dynamics can emerge. This effectively enabled us to find local coordinates in which the original infinite-dimensional problem reduces to an m -slow/ n -fast system, where

standard results from canard theory and delayed bifurcations of ODEs apply. Therefore, we have obtained the existence of local canard segments as well as slow passages through bifurcations in the general framework of dynamical systems with m -slow and infinitely many fast variables. We then gave a complete rigorous description of such results near a fold bifurcation of the original fast subsystem. We also provided the main steps of the proofs in other cases, like in the slow passage through a Turing bifurcation, which had not been analyzed before. In the case of the slow passage through a Hopf bifurcation we could only show a “short” delay since analyticity of the original problem is lost by the center-manifold reduction. Hence, the proof of a “long” delay is an open problem. Finally, we brought new results along similar lines in slow-fast DDEs. Every case that has been covered theoretically was also accompanied by a computational example. Future work will include connecting our local results to global ones, in order to be able to fully describe canard solutions, in particular, canard cycles, in infinite-dimensional slow-fast dynamical systems, first with finitely many slow variables and then in systems where both fast and slow variables are infinite dimensional, which is a substantially more difficult problem. Yet the present work constitutes a necessary rigorous initial first step towards achieving this research program.

Appendix A. Slow-fast concepts for ODEs. Here we recall basic notions about slow-fast (finite-dimensional) dynamical systems and elements of GSPT [39, 53, 59, 85].

A.1. The standard GSPT setting. For ease of reference, we recall the singular perturbation problem (1.1),

$$(A.1) \quad \begin{aligned} \dot{u} &= F(u, v, \mu, \varepsilon), \\ \dot{v} &= \varepsilon G(u, v, \mu, \varepsilon), \end{aligned}$$

where $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, F and G are smooth functions on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$, and $0 < \varepsilon \ll 1$ is the timescale separation parameter. The overdot ($\dot{\cdot} = d/dt$) refers to the fast timescale t and, hence, we refer to (A.1) as the fast system. Alternatively, we may introduce the time transformation $d\tau = \varepsilon dt$ which transforms (A.1) to the equivalent *slow system*

$$(A.2) \quad \begin{aligned} \varepsilon u' &= F(u, v, \mu, \varepsilon), \\ v' &= G(u, v, \mu, \varepsilon), \end{aligned}$$

where prime ($' = d/d\tau$) refers now to the slow timescale τ . System (A.1), respectively, (A.2), is topologically equivalent and solutions often consist of a mix of slow and fast segments reflecting the dominance of one timescale or the other. As $\varepsilon \rightarrow 0$, the trajectories of (A.1) converge during fast segments to solutions of the n -dimensional layer problem

$$(A.3) \quad \begin{aligned} \dot{u} &= F(u, v, \mu, 0), \\ \dot{v} &= 0, \end{aligned}$$

while during slow segments, trajectories of (A.2) converge to solutions of

$$(A.4) \quad \begin{aligned} 0 &= F(u, v, \mu, 0), \\ v' &= G(u, v, \mu, 0), \end{aligned}$$

which is an m -dimensional differential-algebraic problem called the *reduced problem*. GSPT uses these lower-dimensional subsystems (A.4) and (A.3) to predict the dynamics of the full $(n + m)$ -dimensional system (A.2) or (A.1) for $\varepsilon > 0$.

DEFINITION A.1. *The set*

$$(A.5) \quad S := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m \mid F(u, v, \mu, 0) = 0\}$$

is the set of equilibria of (A.3). In general, this set S defines a regular m -dimensional differentiable manifold, i.e., the Jacobian $D_{(u,v)}F$ evaluated along S has full (row) rank n . We refer to S as the critical manifold.

Remark A.2. In general, the set of singularities S could be a union of disjoint manifolds or a union of manifolds intersecting along lower-dimensional submanifolds. The theory exists for these cases as well.

The existence of an m -dimensional manifold S in (A.3) implies that its Jacobian evaluated along S has at least m zero eigenvalues corresponding to the m -dimensional tangent space $T_{(u,v)}S$ of S at each point $(u, v) \in S$. We call these m eigenvalues *trivial* associated with the $\dot{v} = 0$ subsystem in (A.3). The remaining n eigenvalues are called *nontrivial* associated with the n -dimensional submatrix $D_u F|_S$.

A.2. Normal hyperbolicity.

DEFINITION A.3. *A subset $S_h \subseteq S$ is called normally hyperbolic if all $(u, v) \in S_h$ are hyperbolic equilibria of the layer problem (A.3), i.e., the nontrivial eigenvalues have all nonzero real part.*

- *We call a normally hyperbolic subset $S_a \subseteq S$ attracting if all nontrivial eigenvalues have negative real parts for $(u, v) \in S_a$.*
- *$S_r \subseteq S$ is called repelling if all nontrivial eigenvalues have positive real parts for $(u, v) \in S_r$.*
- *If $S_s \subseteq S$ is normally hyperbolic and neither attracting nor repelling we say it is of saddle type.*

Normal hyperbolicity induces a (unique) splitting of the corresponding tangent space along S , i.e.,

$$(A.6) \quad T_{(u,v)}\mathbb{R}^{n+m} = T_{(u,v)}S \oplus \mathcal{N}_{(u,v)} \quad \forall (u, v) \in S_h,$$

where $T_{(u,v)}S$ and $\mathcal{N}_{(u,v)}$ are invariant subspaces under the Jacobian

$$J = \begin{pmatrix} D_u F & D_v F \\ 0 & 0 \end{pmatrix}$$

of (A.3). More precisely, $T_{(u,v)}S$ is in the kernel of J and, hence, corresponds to the m trivial eigenvalues. This induces a linear map on the n -dimensional quotient space $\mathcal{N}_{(u,v)} = T_{(u,v)}\mathbb{R}^{n+m}/T_{(u,v)}S$ corresponding to the n nontrivial eigenvalues of J . The disjoint union of all $T_{(u,v)}S$, $TS = \cup T_{(u,v)}S$, forms the *tangent bundle* along S , while the disjoint union of all $\mathcal{N}_{(u,v)}$, $\mathcal{N} = \cup \mathcal{N}_{(u,v)}S$, forms the corresponding *fast fiber bundle*.

The reduced problem (A.4) is a differential algebraic problem and describes the evolution of the slow variables v constrained to the critical manifold S . As a consequence, S defines an interface between the two subsystems (A.3) and (A.4). In the case of normal hyperbolicity, the reduced vector field can be defined as a dynamical

system by appealing to the (unique) splitting (A.6) which defines a unique projection operator

$$\Pi^{S_h} : T\mathbb{R}^{n+m}|_{S_h} = TS_h \oplus \mathcal{N} \rightarrow TS_h,$$

a map from $T\mathbb{R}^{n+m}$ onto the base space TS_h along $\mathcal{N}$. This projection operator can be calculated explicitly, i.e., the reduced problem (A.4) along S is given by the following dynamical system:

$$(A.7) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \Pi^{S_h} \begin{pmatrix} F_1(u, v, \mu, 0) \\ G(u, v, \mu, 0) \end{pmatrix} = \begin{pmatrix} 0 & -(D_u F)^{-1} D_v F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_1(u, v, \mu, 0) \\ G(u, v, \mu, 0) \end{pmatrix} \\ = \begin{pmatrix} -(D_u F)^{-1} D_v F G(u, v, \mu, 0) \\ G(u, v, \mu, 0) \end{pmatrix},$$

where $F_1(u, v, \mu, 0)$ is the $O(\varepsilon)$ term in the power series expansion of the first component $F(u, v, \mu, \varepsilon)$ with respect to ε of the vector field (A.1); $G(u, v, \mu, 0)$ is the corresponding other (leading order) slow component. Note that $D_u F$ is a regular square matrix due to normal hyperbolicity. This regularity implies via the *implicit function theorem* that the critical manifold has a graph representation $S = \{(u, v) \in \mathbb{R}^{n+m} : u = h(v)\}$. Hence, in the case of normal hyperbolicity, the reduced problem (A.4) can be studied in this coordinate chart v given by

$$(A.8) \quad \dot{v} = G(h(v), v, \mu, 0).$$

The notion of normal hyperbolicity is central to the study of systems of the form (A.1) as key assumption for the persistence of segments of S , which are invariant for $\varepsilon = 0$, as local invariant *slow manifold* S_ε for small enough $\varepsilon > 0$. These major results have been obtained in the mid-1970s by Fenichel [39].

A.3. Loss of normal hyperbolicity. Geometrically, loss of normal hyperbolicity occurs (generically) along codimension-one submanifold(s) of S where nontrivial eigenvalue(s) of the layer problem crosses the imaginary axis. Within this set $S \setminus S_h$, we distinguish two subsets $\mathcal{AH}$ and $\mathcal{F}$.

DEFINITION A.4. Consider the two cases of loss of normal hyperbolicity associated with the two generic codimension-one bifurcations in the layer problem (A.3):

- the set $\mathcal{AH}$ of Andronov–Hopf points associated with the crossing of a pair of complex conjugate eigenvalues (with nonzero imaginary part);
- the set $\mathcal{F}$ of folds or saddle-node points associated with the crossing of a real eigenvalue.

The layer problem is thus considered a bifurcation problem; see, e.g., [60]. The main question is what happens to this bifurcation structure as the singular bifurcation parameter $0 < \varepsilon \ll 1$ is turned on, i.e., when the “bifurcation parameter” v starts to evolve slowly. To answer this question, we need to understand the reduced problem in a neighborhood of these codimension-one subsets.

In the case $\mathcal{AH}$, we note that $D_u F$ is still a regular square matrix, i.e., $\det(D_u F) \neq 0$ everywhere along S including $\mathcal{AH}$. Hence the reduced problem can still be studied in the slow coordinate chart given by (A.8). In general, the reduced vector field obeys $G(h(v), v, \mu, 0) \neq 0$ along $\mathcal{AH} \subset S$. Thus the reduced flow crosses from one normally hyperbolic branch of S via $\mathcal{AH} \subset S$ to another normally hyperbolic branch of S . Assuming this reduced flow is from an attracting branch S_a to a repelling branch $S_{r/s}$ this leads to the phenomenon of delayed loss of stability through an Andronov–Hopf bifurcation; see [65, 66, 46] for details.

In the case $\mathcal{F}$, the matrix $D_u F$ is singular, i.e., $\det(D_u F) = 0$ along $\mathcal{F}$. So, we cannot use (A.8) to describe the reduced flow. While system (A.7) is also not well defined near fold(s) $\mathcal{F} \subset S$, we can make an equivalence transformation as follows: first, we use the identity $(D_u F)^{-1} = \text{adj}(D_u F)/(\det D_u F)$, where $\text{adj}(D_u F)$ denotes the adjoint (or adjugate) of the matrix $D_u F$, i.e., the transpose of the cofactor matrix, which gives

$$(A.9) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\det(D_u F)} \text{adj}(D_u F) D_v F G(u, v, \mu, 0) \\ G(u, v, \mu, 0) \end{pmatrix}.$$

This isolates the singularity into the scalar function $\det(D_u F)$. We note that the matrix $\text{adj}(D_u F)$ is still well defined and of rank one, when $D_u F$ has rank deficiency of one (along $\mathcal{F}$). Now, we can remove the scalar singularity (pole) by the time transformation

$$d\tau = -\det(D_u F) d\tau_1$$

which gives the corresponding desingularized system

$$(A.10) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \text{adj}(D_u F) D_v F G(u, v, \mu, 0) \\ -\det(D_u F) G(u, v, \mu, 0) \end{pmatrix},$$

where with a slight abuse of notation the overdot ($\dot{} = d/d\tau_1$) refers now to the new slow time τ_1 . Importantly, (A.10) represents a dynamical system without singularities that can be analyzed by standard dynamical systems tools even in a neighborhood of $\mathcal{F} \subset S$. Note that systems (A.9) and (A.10) are equivalent on normally hyperbolic branches where $\det(D_u F) > 0$, and these systems are also equivalent up to an orientation change on normally hyperbolic branches where $\det(D_u F) < 0$.

DEFINITION A.5. Any point $(u, v) \in \mathcal{F}$ where

$$(A.11) \quad \text{adj}(D_u F) D_v F G(u, v, \mu, 0) \neq 0$$

is called a regular jump point.

Note, solutions of the reduced problem (A.9) in a neighborhood of $\mathcal{F}$ approach regular jump points (A.11) in forward or backward time and they cease to exist at regular jump points due to a finite time blowup, i.e., the reduced problem (A.9) has a pole at jump points. This finite time blowup of the reduced flow at a jump point happens exactly in the eigendirection of the defect of the layer problem which defines the locus (and direction) where a switch from slow to fast dynamics (or vice versa) in the full system is possible. This, together with an adequate global return mechanism via the layer problem may provide the seed for a singular relaxation cycle, a concatenation of slow and fast orbit segments that form a loop. For persistence results of such relaxation cycles under small perturbations $\varepsilon \ll 1$ we refer the reader to, e.g., [58, 75].

DEFINITION A.6. Any point $(u, v) \in \mathcal{F}$ where

$$(A.12) \quad \text{adj}(D_u F) D_v F G(u, v, \mu, 0) = 0$$

is called a folded singularity.

Folded singularities are equilibria of the desingularized problem (A.10) but not necessarily of the corresponding reduced problem (A.9) itself. At such a folded singularity, the vector field of (A.9) contains indeterminate forms which could prevent a

finite time blowup of certain solutions of the reduced problem approaching a folded singularity. Such special solutions of the reduced problem that are able to pass in finite time through such a folded singularity from one branch of S to another are called *canards*, and they play an important role in understanding the genesis of relaxation oscillations as well as in creating more complex oscillatory patterns in singular perturbation problems. The terminology “canard” was introduced in [12], and we refer the interest reader to important persistence results in, e.g., [33, 58, 11, 74, 83, 84].

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