Characterizations of solutions for cooperative games with transferable utility

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Characterizations of Solutions for Cooperative Games with Transferable Utility

By
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Contents

Introduction

Game theory studies mathematical models of conflict and cooperation among rational decision-makers. Game theory provides general mathematical methods for analyzing situations in which two or more rational decision-makers make decisions that will influence one another's benefit. One of the milestones of the study of game theory is the publication of the fundamental book *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944, [113]). At present, game theory has been widely applied in biology, economics, computer science, political science, and many other disciplines.

Generally speaking, game theory is classified into two branches: cooperative game theory and non-cooperative game theory. The classification of these two branches depends on whether players are able to make a binding commitment. In cooperative game theory, players usually make a binding commitment and form a coalition. Cooperative game theory deals with how the benefits of cooperation are shared among all players. In non-cooperative game theory, players cannot make binding commitments and they make their independent decisions (strategies) in order to obtaining the best possible outcomes for themselves. Taking account of each other's strategies, non-cooperative game theory emphasizes how every player maximizes one's own payoff without considering the payoffs of other players. A more workable distinction between cooperative and non-cooperative games can be based on 'the modelling technique', that is, in a non-cooperative game players have explicit strategies and payoff functions, whereas a cooperative game is described by a characteristic function that assigns to each coalition the worth that they can reach by cooperating. The Nash program is an important research agenda initiated in Nash (1953, [83]) in order to bridge the gap between these two game theoretic branches. This thesis will be devoted to a study of cooperative game theory.

The implicit assumption in cooperative games is that players can cooperate to form coalitions and make binding agreements on how to distribute the benefits of these coalitions over the players. A cooperative game is more abstract than a non-cooperative game in the sense that strategies are not explicitly modelled. Cooperative games with transferable utility (for short, TU-games) are a special type of cooperative games in which it is assumed that the earnings of a coalition can be expressed by a single number. These numbers can be considered as a joint utility for the players in the coalitions. One may think of this number as an amount of money, which can be distributed among the players in any conceivable way if the coalition is formed. In TU-games, an important issue is to build a universal solution that describes the distribution of the benefits of cooperation among the players in a reasonable way for every TU-game. This thesis is devoted to the study to judge the fairness and reasonableness of such solutions.

Axiomatization is a vital approach to show the fairness and reasonableness of solutions. Axiomatizations of solutions for TU-games usually consist of two steps: Firstly, formulate desirable axioms of solutions, as properties; secondly, identify the solutions satisfying the axioms in various combinations. The plausibility of solutions is judged by investigating a set of axioms the solutions satisfy. This normative method for TU-games has been pioneered by Shapley (1953, [94]), who introduced and characterized the well-known Shapley value by efficiency, additivity, symmetry, and the null player property. Since then, various solutions have been axiomatically introduced and studied in the literature, such as the equal division value (van den Brink 2007, [104]), the solidarity value (Nowak and Radzik 1994, [85]), the proportional division value (for short, PD value) (Banker 1981, [5]), the proportional Shapley value (Béal et al. 2018, [9]), the equal allocation of non-separable contributions value (for short, EANSC value) (Moulin 1985, [79]), and the center-of-gravity of the imputation set value (for short, CIS value) (Driessen and Funaki 1991, [36]). This thesis mainly studies axiomatizations of such solutions for TU-games. An overview of each chapter of the thesis is given in the next section.

Overview of the thesis

This thesis focuses on the area of solutions for TU-games. The thesis consists of six chapters. Chapter 1 introduces fundamental terminologies and notations about TU-games that will be used in the subsequent chapters. Chapters 2-5 provide axiomatizations of a new solution and several classical solutions for TU-games. Chapter 6 turns to an application of TU-games, specifically pollution cost-sharing problems, and provides axiomatizations of two methods (that assign allocations to share the cost of cleaning up any polluted river).

In Chapter 2, we define and characterize a new solution for TU-games, namely the *average-surplus value*. Firstly, we introduce a new concept called *marginal surplus* to describe the contribution level of every player. Inspired by known procedures of the Shapley value and the solidarity value, the average-surplus value is determined by an underlying procedure of sharing this marginal surplus. Then, we characterize the average-surplus value by introducing the *A-null surplus player property* and the *revised balanced contributions property*. These axiomatizations follow the spirit of the known null player property (Shapley 1953, [94]), respectively the balanced contributions property (Myerson 1980, [82]), used to characterize the Shapley value. Next, inspired by the work of Hart and Mas-Colell (1989, [46]), we define a variation of the potential function, called the *AS-potential function*, and show that the adjusted marginal contributions vector of the AS-potential function coincides with the average-surplus value. Finally, we provide a non-cooperative game, namely the *punishment-compensation bidding mechanism*, and show that the outcome in every subgame perfect equilibrium of this mechanism coincides with the payoff assigned by the average-surplus value.

In the axiomatic approach to solutions for TU-games, consistency is a crucial characteristic of viable and stable solutions. A solution is *consistent* if it allocates the same payoff to players in the original game as in a modified game. A well-known kind of modified games are the *associated games*. In associated games, each coalition revalues its worth by claiming part of the surplus in the game that is left after this coalition and the players outside the coalition get some initial share in the total worth. Different associated consistency axioms are defined in terms of different associated games. In Chapter 3, we provide new axiomatic characterizations of the EANSC value and the CIS value by using new associated consistency axioms. To define an associated game, Xu et al. (2009, [117] and 2013, [119]) assumed that any coalition is formed by its members joining one by one and each coalition considers players in the coalition as isolated elements. That is, they adopted “individual self-evaluation" to reevaluate the worths of coalitions to define the *Sh-individual associated game* and the *C-individual associated game*. In Chapter 3, inspired by the work of Xu et al. (2009, [117] and 2013, [119]), we first introduce an alternative way to reevaluate the worth by considering players in the coalition as a whole. That is, we adopt the idea of “union self-evaluation" instead of “individual self-evaluation", to define the *E-union associated game* and the *C-union associated game*. Then, adopting E-union associated consistency and C-union associated consistency, we provide new axiomatizations of the EANSC value and the CIS value. Moreover, inspired by the works of Hwang et al. (2005, [53]) and Hwang (2015, [51]), we also propose two dynamic processes on the basis of the Sh/C-individual associated game and the E/C-union associated game respectively that lead to the CIS value and EANSC value, starting from an arbitrary efficient payoff vector. This follows from a more general result showing that these dynamic processes can lead to any solution satisfying the inessential game property and continuity.

In TU-games, the *excess* is a well-known concept that describes the *dissatisfaction* of coalitions with respect to a payoff vector. A well-known solution for TU-games that is based on excess is the *nucleolus*, introduced by Schmeidler (1969, [92]), which is obtained by lexicographically minimizing the maximal excess (dissatisfaction) of coalitions over the non-empty imputation set. Besides the excess criterion, Hou et al. (2018, [49]) proposed two other criteria to measure the dissatisfaction of coalitions with respect to a payoff vector. In Chapter 4, we will discuss new criteria from the perspective of satisfaction. We present axiomatic characterizations of the proportional division value (for short, PD value) and the proportional allocation of non-separable contribution value (for short, PANSC value) for TU-games. The PD value allocates the worth of the grand coalition (that is the set of all players in a TU-game) in proportion to the *individual worths* of players, and the PANSC value allocates the worth of the grand coalition in proportion to the *separable contributions* of players. Firstly, inspired by the work of Schmeidler (1969, [92]) for the nucleolus, we define two new criteria from the perspective of satisfaction: the *optimistic satisfaction* and *pessimistic satisfaction*. We show that the PD value and the PANSC value can be obtained by maximizing the minimal optimistic satisfaction and pessimistic satisfaction, respectively, in the lexicographic order over the non-empty pre-imputation set. Secondly, we introduce two new axioms: *equal minimal optimistic satisfaction* and *equal minimal pessimistic satisfaction*, inspired by the kernel concept (Maschler et al. 1971, [76]). We show that the PD (respectively PANSC) value is the only solution satisfying equal minimal optimistic (respectively pessimistic) satisfaction and efficiency. Thirdly, we also characterize the PD value and the PANSC value by introducing *optimistic associated consistency* and *pessimistic associated consistency*. Finally, we define the dual axioms of the optimistic associated consistency and pessimistic associated consistency axioms, and characterize these two proportional values on the basis of these dual axioms.

In Chapter 5, we study axiomatic foundations of the class of weighted division values. A weighted division value allocates the worth of the grand coalition in proportion to a weight vector of players in a TU-game. The best-known egalitarian solution in the class of weighted division values is the equal division value which allocates the worth of the grand coalition equally over all players. In van den Brink (2007, [104]), the equal division value is characterized by efficiency, additivity, the nullifying player property, and symmetry. Firstly, while keeping efficiency, additivity and the nullifying player property from this axiomatization of the equal division value, we consider relaxations of symmetry in line with Casajus (2019, [20]), specifically *sign symmetry* and *weak sign symmetry*, to characterize the class of (positively) weighted division values. Secondly, Béal et al. (2016, [8]) introduced three different axiomatizations of the class of weighted division values. The first axiomatization involves efficiency, linearity, the nullifying player property and the null player in a productive environment property. The second axiomatization involves efficiency, linearity and the non-negative player property. The third axiomatization involves efficiency, linearity and nullified solidarity. We show that the class of weighted division values can also be characterized by replacing linearity in the three axiomatizations of Béal et al. (2016, [8]) with additivity. Finally, we show how strengthening an axiom regarding null, non-negative, respectively nullified players in these three axiomatizations, provides three axiomatizations of the positively weighted division values.

In Chapter 6, we turn to an application, specifically pollution cost-sharing problems, and explore how to share the cost of cleaning up a polluted river using cooperative game theory. We study axiomatic foundations of two classes of cost-sharing methods for pollution cost-sharing problems. A pollution cost-sharing problem describes a situation where a group of agents are located along a polluted river and every agent must pay a certain cost for cleaning up the polluted river. Ni and Wang (2007, [84]) developed a model for the pollution cost-sharing problems and discussed the question of how to split the cost of cleaning up a river among agents situated along the river. They proposed two cost-sharing methods: the local responsibility sharing method (for short, LRS method) and the upstream equal sharing method (for short, UES method), and characterized these two methods by introducing efficiency, additivity, no blind cost, independence of upstream costs, and upstream symmetry. Following the model of Ni and Wang (2007, [84]), we define and characterize two new classes of cost sharing methods by introducing weaker versions of some of their axioms. Firstly, we propose a relaxation of independence of upstream costs, called *sign independence of upstream costs,* and show that the UES method is characterized by replacing independence of upstream costs appearing in Ni and Wang (2007, [84]) with this weaker sign independence of upstream costs. Secondly, we propose the classes of *equal upstream responsibility methods* (for short, EUR methods) and *weighted upstream sharing methods* (for short, WUS methods), which generalize the LRS method and the UES method. We provide two axiomatizations of the class of EUR methods, one using this weak independence axiom (*sign independence of upstream costs*) and one using a weak version of the no blind cost axiom (*weak no blind cost*). Meanwhile, we also provide two axiomatizations of the class of WUS methods by introducing two weak versions of upstream symmetry: *sign upstream symmetry* and *proportionality*. Finally, we define a *pollution cost-sharing game*, and show that the *compromise method*, which is the average of the LRS method and UES method, coincides with applying the Shapley value to this game. Moreover, we also show that the compromise method coincides with the Shapley value, nucleolus and τ-value of the dual of this game.

Chapter 1

Preliminaries

Game theory is a formal, mathematical discipline which studies situations of competition and cooperation among rational players. Based on whether the players are able to make a binding commitment, game theory is classified into two branches: non-cooperative game theory and cooperative game theory. Non-cooperative game theory concentrates on situations of conflict whereas cooperative game theory deals with situations in which players cooperate. In this thesis, we will focus on cooperative game theory. As is well known, cooperative game theory describes situations where players make a binding commitment and form a coalition with the aim of obtaining more benefits. When it is assumed that all players cooperate and achieve a specified amount of revenue, this triggers the question how the total revenue should be allocated over the players. The objective of cooperative game theory is to answer this question.

In this chapter, we introduce some relevant terminology that will be used throughout the thesis. In Section 1.1, we formally introduce cooperative games with transferable utility. In Section 1.2, we introduce several solutions of cooperative games with transferable utility. In Section 1.3, we review some axioms and axiomatizations of solutions.

1.1 Cooperative games with transferable utility

A situation in which a finite set of players can achieve a specified amount of worths by cooperating can be described as a cooperative game with transferable utility, or simply a TU-game. For example, cooperatives or platforms usually reach a cooperative agreement and form a coalition with the aim of obtaining more benefits. The objective of cooperative game theory is to deal with how the benefits of cooperation are shared among all players. Next, we formally introduce TU-games.

We first review some notations. Let $\mathbb{N}$ be the set of natural numbers. The notation $T \subseteq S$ means that T is a subset of S, and the notation $T \subsetneq S$ means that T is a proper subset of S. Let $\mathcal{U} \subseteq \mathbb{N}$ be a universe of potential players, and let $N \subseteq \mathcal{U}$ be a finite set of n players. The *power set* of N is denoted by 2^N. The *cardinality* of a finite set S is denoted by $|S|$ or, if there is no ambiguity, appropriate small letter s. Let $\mathbb{R}$ be the set of all real numbers. Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the sets of all non-negative real numbers and strictly positive real numbers, respectively. A *permutation* of N is a bijection $\pi : N \to \{1, 2, ..., n\}$ where $\pi(i) = k$ means that player i has the kth position. The set of all permutations of N is denoted by Π^N. For each $\pi \in \Pi^N$ and $i \in N$, let $P_{\pi,i} = \{j \in N | \pi(j) \leq \pi(i)\}$ denote the set of all predecessors of i (including player i) under the permutation π, and $S_{\pi,i} = \{j \in N | \pi(j) \geq \pi(i)\}$ denote the set of all successors of i (including player i) under the permutation π.

A *cooperative game with transferable utility*, or simply a *TU-game*, is a pair $\langle N, v\rangle$, where N is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning to each coalition $S \in 2^N$, the worth $v(S)$ with $v(\emptyset) = 0$. The set N is usually called the *grand coalition*, and a subset $S \subseteq N$ is called a *coalition*. The worth $v(S)$ is the reward that players in coalition S can obtain by cooperating. Denote the set of all TU-games on player set N by $\mathcal{G}^N$, and denote the set of all TU-games with a finite player set in $\mathcal{U}$ by $\mathcal{G}$.

Various subclasses of TU-games have been considered in the literature. Some definitions of specific types of TU-games that will be used in later chapters are as follows.

A TU-game $\langle N, v \rangle \in \mathcal{G}^N$ is

- *inessential* or *additive*, if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$.
- *almost inessential*, if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subsetneq N$.
- *zero-monotonic*, if $v(S) \geq v(T) + \sum_{i \in S \setminus T} v(\{i\})$ for all $T \subseteq S \subseteq N$.
- *zero-normalized*, if $v(\{i\}) = 0$ for all $i \in N$.
- *individually positive*, if $v(\{i\}) > 0$ for all $i \in N$.
- *individually negative*, if $v(\{i\}) < 0$ for all $i \in N$.
- *marginally positive*, if $v(N) - v(N \setminus \{i\}) > 0$ for all $i \in N$.
- *marginally negative*, if $v(N) - v(N \setminus \{i\}) < 0$ for all $i \in N$.
- *a null game*, if $v(S) = 0$ for all $S \subseteq N$.
- *a constant game*, if $v(S) = c$ for all $S \subseteq N$ and some $c \in \mathbb{R}$.

Denote the set of all individually positive (respectively negative) TU-games on player set N by $\mathcal{G}_+^N$ (respectively $\mathcal{G}_-^N$). Denote the set of all marginally positive (respectively negative) TU-games on player set N by $\mathcal{G}_\oplus^N$ (respectively $\mathcal{G}_\ominus^N$).

For every TU-game $\langle N, v \rangle \in \mathcal{G}^N$, its dual game $\langle N, v^d \rangle$ is given as follows. For all $S \subseteq N$,

$$v^d(S) = v(N) - v(N \setminus S), \tag{1.1}$$

where $v^d(S)$ is the marginal worth of coalition S with respect to N. Generally, the duality operator is not closed on subclasses of TU-games. For example, the dual of an individually positive (respectively negative) TU-game is a marginally positive (respectively negative) TU-game. Given $\mathcal{A} \subseteq \mathcal{G}$, let $\mathcal{A}^d$ be the set of duals of TU-games in $\mathcal{A}$. Duality can also be applied to solutions and axioms (see, Charnes et al. 1978, [26] and Oishi et al. 2016, [87]), which will be introduced later.

For all $T \subseteq N$ and $T \neq \emptyset$, the *standard game* associated to T, $\langle N, e_T \rangle \in \mathcal{G}^N$, is defined by

$$e_T(S) = \begin{cases} 1, & \text{if } S = T; \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that the family of standard games $\{\langle N, e_T \rangle | T \subseteq N, T \neq \emptyset\}$ forms a basis for $\mathcal{G}^N$, i.e., every TU-game $\langle N, v \rangle \in \mathcal{G}^N$ can be expressed by $v = \sum_{T \subseteq N, T \neq \emptyset} v(T) e_T$.

For all $T \subseteq N$ and $T \neq \emptyset$, the *unanimity game* associated to T, $\langle N, u_T \rangle$, is defined by

$$u_T(S) = \begin{cases} 1, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

The family of unanimity games $\{\langle N, u_T \rangle | T \subseteq N, T \neq \emptyset\}$ also forms a basis for $\mathcal{G}^N$, i.e., every TU-game $\langle N, v \rangle \in \mathcal{G}^N$ can be expressed by $v = \sum_{T \subseteq N, T \neq \emptyset} \Delta_v(T) u_T$, where $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{t-s} v(S)$ is the *Harsanyi dividend* (Harsanyi 1959, [45]) of coalition T in TU-game $\langle N, v \rangle$.

Nowak and Radzik (1994, [85]) also suggested a basis for $\mathcal{G}^N$, denoted by $\{\langle N, b_T \rangle | T \subseteq N, T \neq \emptyset\}$. For all $T \subseteq N$ and $T \neq \emptyset$, $\langle N, b_T \rangle$ is defined by

$$b_T(S) = \begin{cases} \frac{(s-t)!t!}{s!}, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases}$$

Every TU-game $\langle N, v \rangle \in \mathcal{G}^N$ can be expressed by $v = \sum_{T \subseteq N, T \neq \emptyset} A^v(T) b_T$, where $A^v(T) = \frac{1}{t} \sum_{j \in T} [v(T) - v(T \backslash \{j\})]$ is the *average marginal contribution* of coalition T in TU-game $\langle N, v \rangle$.

1.2 Solutions for TU-games

As mentioned in the previous section, a TU-game describes a situation where players can achieve a specified amount of benefits by cooperating. A central issue is how to find a method to distribute the benefits of cooperation among these players. The solution part of cooperative game theory deals with how the benefits of cooperation are shared among all players. A solution for TU-games is a function that assigns to every TU-game a vector with the same dimension as the size of the player set, where each component of the vector represents the payoff assigned to the corresponding player. Various solutions for TU-games have been considered in the literature. In this section, we recall several solutions for TU-games that will be discussed in the following chapters.

A *payoff vector* for TU-game $\langle N, v\rangle \in \mathcal{G}^N$ is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. For notational convenience, denote $\sum_{i\in S} x_i$ by $x(S)$, $S \subseteq N$. The *pre-imputation set* consists of all payoff vectors such that the worth of the grand coalition is fully shared among all players.

Definition 1.1. For all $\langle N, v\rangle \in \mathcal{G}^N$, the *pre-imputation set* of $\langle N, v\rangle$ is defined by

$$I^*(N, v) = \{x \in \mathbb{R}^N | x(N) = v(N)\}.$$

The *imputation set* consists of all payoff vectors in the pre-imputation set such that no player obtains less than his individual worth.

Definition 1.2. For all $\langle N, v\rangle \in \mathcal{G}^N$, the *imputation set* of $\langle N, v\rangle$ is defined by

$$I(N, v) = \{x \in I^*(N, v) | x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

The *core*, introduced by Gillies (1953, [37]), is one of the most well-known set solutions in TU-games.

Definition 1.3. For all $\langle N, v\rangle \in \mathcal{G}^N$, the *core* of $\langle N, v\rangle$ is defined by

$$C(N, v) = \{x \in \mathbb{R}^N | x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Obviously, $C(N, v) \subseteq I(N, v) \subseteq I^*(N, v)$.

The *nucleolus*, introduced by Schmeidler (1969, [92]), is obtained by minimizing the excesses of coalitions in the lexicographic order over the non-empty imputation set if this is non-empty. The *excess* of coalition $S \subseteq N$ with respect to the payoff vector x of the TU-game $\langle N, v\rangle$ is given by $e(S, x, v) = v(S) - x(S)$. This can be seen as a measure of dissatisfaction of the coalition since a positive (respectively negative) excess means that the coalition obtains less (respectively more) than its own worth. Let $\theta^v(x) = (\theta^v_l(x))_{l\in\{1,2,\cdots,2^n-1\}}$ be the (2^n-1)-tuple vector whose components are the excesses of all non-empty coalitions $S \subseteq N$ with respect to $x \in \mathbb{R}^N$ in non-increasing order, that is, $\theta^v_l(x) \geq \theta^v_{l+1}(x)$ for all $l \in \{1, 2, ..., 2^n - 2\}$. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $x, y \in \mathbb{R}^N$, we call $\theta^v(x) \leq_L \theta^v(y)$ if and only if $\theta^v(x) = \theta^v(y)$ or there exists $t \in \{1, 2, ..., 2^n - 2\}$ such that $\theta^v_k(x) = \theta^v_k(y)$ for all $k \in \{1, 2, ..., t\}$ and $\theta^v_{t+1}(x) < \theta^v_{t+1}(y)$.

Definition 1.4. For all $\langle N, v\rangle \in \mathcal{G}^N$, the *nucleolus* of $\langle N, v\rangle$ is defined by

$$\eta(N, v) = \{x \in I(N, v)|\theta^v(x) \leq_L \theta^v(y), \text{ for all } y \in I(N, v)\}.$$

Since it is known that $\eta(N, v)$ is a singleton, we identify the nucleolus by its unique element. That is, the nucleolus is the unique payoff vector in the imputation set that lexicographically minimizes the excesses.

A *solution* on a class of TU-games $\mathcal{A} \subseteq \mathcal{G}$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to every TU-game $\langle N, v\rangle \in \mathcal{A}$. Given $\mathcal{A} \subseteq \mathcal{G}$ and a solution φ on $\mathcal{A}$, its *dual solution* φ^d on $\mathcal{A}^d$ is defined as, for all $\langle N, v\rangle \in \mathcal{A}^d$, $\varphi^d(N, v) = \varphi(N, v^d)$, where the dual game $\langle N, v^d\rangle$ is defined by Eq.(1.1).

One of the most important solutions for TU-games is the Shapley value introduced by Shapley (1953, [94]). For all $S \subseteq N$ and $i \in S$, let $MC^v_i(S) = v(S) - v(S\backslash\{i\})$ be the *marginal contribution* of player i to coalition S in TU-game $\langle N, v\rangle \in \mathcal{G}^N$. The *Shapley value* assigns to every player the expectation of all his marginal contributions to all coalitions entering before him, assuming that all permutations in which the grand coalition can be formed occur with equal probability.

Definition 1.5. The *Shapley value* on $\mathcal{G}^N$ is defined by

$$Sh_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{(n-s)!(s-1)!}{n!} MC^v_i(S),$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

The Shapley value is based on the individual marginal contributions of a player to all coalitions including him. Nowak and Radzik (1994, [85]) defined the solidarity value by replacing the individual marginal contribution with the average of the marginal contributions of all players in the coalition. For all $S \subseteq N$, let $A^v(S) = \frac{1}{s}\sum_{j\in S}[v(S) - v(S\backslash\{j\})]$ be the *average marginal contribution* of coalition S in TU-game $\langle N, v\rangle$. The *solidarity value* assigns to every player the expectation of the average marginal contributions to all coalitions entering before him, assuming that all permutations in which the grand coalition can be formed occur with equal probability.

Definition 1.6. The *solidarity value* on $\mathcal{G}^N$ is defined by

$$Sol_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{(n-s)!(s-1)!}{n!} A^v(S),$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

Instead of taking into account all coalitions, as the Shapley value and the solidarity value do, there also exist several solutions that only concern some particular class of coalitions. Various examples of such solutions are the equal division value (van den Brink 2007, [104]), the center-of-gravity of the imputation set value (Driessen and Funaki 1991, [36]), and the equal allocation of non-separable contribution value (Moulin 1985, [79]). Formally, the definition of these solutions are as follows.

The *equal division value* distributes the worth of the grand coalition equally among all players.

Definition 1.7. The *equal division value* on $\mathcal{G}^N$ is defined by

$$ED_i(N, v) = \frac{v(N)}{n},$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

Let $\Delta^n_+ = \{\omega \in \mathbb{R}^n | \sum_{i\in N} \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N\}$ and $\Delta^n_{++} = \Delta^n_+ \cap \mathbb{R}_{++}$. For $\omega \in \Delta^n_+$, the *ω-weighted division value*, studied by Béal et al. (2016, [8]), distributes the worth of the grand coalition in proportion to the weight coefficient ω.

Definition 1.8. Let $\omega \in \Delta^n_+$. The *ω-weighted division value* on $\mathcal{G}^N$ is defined by

$$WD^{\omega}_i(N, v) = \omega_i v(N),$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

An ω-weighted division value WD^{ω} with $\omega \in \Delta^n_{++}$ is also called a *positively weighted division value*.

The *center-of-gravity of the imputation set value* (for short, *CIS value*), introduced by Driessen and Funaki (1991, [36]), also called the equal surplus division value, first assigns to every player his individual worth, and then distributes the remaining worth equally among all players.

Definition 1.9. The *CIS value* on $\mathcal{G}^N$ is defined by

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})],$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

The *equal allocation of non-separable contribution value* (for short, *EANSC value*), introduced by Moulin (1985, [79]), assigns to every player his separable cost, and then distributes the remainder, called *non-separable contribution*, equally among all players. The *separable contribution* (also called *separable cost* sometimes) is the marginal contribution of a player to the grand coalition. Formally, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, the *separable contribution* is given by $SC_i(N, v) = v(N) - v(N \backslash \{i\})$.

Definition 1.10. The *EANSC value* on $\mathcal{G}^N$ is defined by

$$EANSC_i(N, v) = SC_i(N, v) + \frac{1}{n}[v(N) - \sum_{j \in N} SC_j(N, v)],$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

Notice that the CIS value and the EANSC value are dual to each other.

The *τ-value*, introduced by Tijs (1981, [101]), is essentially a compromise value between an upper bound payoff vector and a lower bound payoff vector. For all $\langle N, v\rangle \in \mathcal{G}^N$, let $SC(N, v) = (SC_i(N, v))_{i \in N} \in \mathbb{R}^N$

be the vector whose coordinates are the marginal contribution of each player to the grand coalition, i.e., the separable contribution. When we consider this as a vector of upper bound payoffs, then the vector $m(N,v) = (m_i(N,v))_{i\in N} \in \mathbb{R}^N$ whose coordinates are given by $m_i(N,v) = \max_{S\subseteq N, S\ni i}\{v(S) - \sum_{j\in S\setminus\{i\}} SC_j(N,v)\}$ for all $i \in N$, can be seen as a lower bound payoff vector. These vectors can indeed be interpreted as upper and lower bound payoff vectors, if the TU-game $\langle N,v\rangle$ is quasi-balanced, meaning that (i) $m_i(N,v) \leq SC_i(N,v)$ for all $i \in N$, and (ii) $\sum_{i\in N} m_i(N,v) \leq v(N) \leq \sum_{i\in N} SC_i(N,v)$. Denote the set of all quasi-balanced TU-games on player set N by $\mathcal{QB}^N$.

Definition 1.11. The *τ-value* on $\mathcal{QB}^N$ is defined by

$$\tau(N,v) = a\cdot m(N,v) + (1-a)\cdot SC(N,v),$$

for all $\langle N,v\rangle \in \mathcal{QB}^N$, where $a \in [0,1]$ is such that $\sum_{i\in N} \tau_i(N,v) = v(N)$.

The *proportional division value* (for short, *PD value*), introduced by Banker (1981, [5]), allocates the worth of the grand coalition in proportion to the individual worths among all players.

Definition 1.12. The *PD value* on $\mathcal{G}_+^N$ (or $\mathcal{G}_-^N$) is defined by

$$PD_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N),$$

for all $\langle N,v\rangle \in \mathcal{G}_+^N$ (or $\mathcal{G}_-^N$) and $i \in N$.

Notice that the PD value is considered on the subclass of TU-games $\mathcal{G}_+^N \subseteq \mathcal{G}^N$ (or $\mathcal{G}_-^N \subseteq \mathcal{G}^N$).

The *proportional allocation of non-separable contribution value* (for short, *PANSC value*) allocates the worth of the grand coalition in proportion to the marginal contributions to the grand coalition among all players.

Definition 1.13. The *PANSC value* on $\mathcal{G}_\oplus^N$ (or $\mathcal{G}_\ominus^N$) is defined by

$$PANSC_i(N,v) = \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N),$$

for all $\langle N, v\rangle \in \mathcal{G}^N_\oplus$ (or $\mathcal{G}^N_\ominus$) and $i \in N$.

Notice that the PANSC value is considered on the subclass of TU-games $\mathcal{G}^N_\oplus \subseteq \mathcal{G}^N$ (or $\mathcal{G}^N_\ominus \subseteq \mathcal{G}^N$). The PD value and the PANSC value are dual to each other.[1]

1.3 Axioms of solutions

There is no consensus on which is the best solution for TU-games. The plausibility of solutions is often judged by investigating a set of axioms the solutions satisfy. In axiomatizations of solutions for TU-games, we associate a solution with a set of axioms. Hence, we can interpret a solution through the interpretation of axioms that characterize it. Next, we review some classical axioms and axiomatizations of solutions from the literature.

1.3.1 Basic axioms

For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, the TU-game $\langle N, av + bw\rangle$ is given by $(av + bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$. Players $i, j \in N$, $i \neq j$, are *symmetric* in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $\alpha \in \mathbb{R}^N$, the TU-game $\langle N, v + \alpha\rangle$ is given by $(v + \alpha)(S) = v(S) + \sum_{j \in S} \alpha_j$ for all $S \subseteq N$.

Let φ be a solution on $\mathcal{G}^N$. [2]

- **Efficiency.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- **Symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i, j \in N$, $i \neq j$, are symmetric players in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = \varphi_j(N, v)$.
- **Linearity.** For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, it holds that $\varphi(N, av + bw) = a\varphi(N, v) + b\varphi(N, w)$.

[1] In Chapter 4, we study the PD value and the PANSC value on the family of all individually positive TU-games and the family of all marginally positive TU-games respectively.

[2] The axioms in this thesis are also defined on subclasses $\mathcal{A} \subseteq \mathcal{G}$, depending on the specifications of the TU-games at work. Notice that for axioms that involve transformations of a TU-game (such as linearity, additivity, translation covariance and continuity), the statement in the axiom is required only if the transformed game also belongs to the class.

- **Additivity.** For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v+w) = \varphi(N, v) + \varphi(N, w)$.
- **Translation covariance.** For all $\langle N, v\rangle \in \mathcal{G}^N$ and $\alpha \in \mathbb{R}^N$, it holds that $\varphi(N, v+\alpha) = \varphi(N, v) + \alpha$.
- **Inessential game property.** For all inessential games $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi_i(N, v) = v(\{i\})$ for all $i \in N$.
- **Almost inessential game property.** For all almost inessential games $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi_i(N, v) = v(\{i\}) + a[v(N) - \sum_{j\in N} v(\{j\})]$ for all $i \in N$ and some $a \in [0, 1]$.
- **Continuity.** For every sequence of TU-games $\{\langle N, v^k\rangle\}_{k=1}^{\infty}$ and its limit TU-game $\langle N, \tilde{v}\rangle$ in $\mathcal{G}^N$ (i.e., $\lim_{k\to\infty} v^k(S) = \tilde{v}(S)$ for all $S \subseteq N$), it holds that the sequence of the solution outcomes $\{\varphi(N, v^k)\}_{k=1}^{\infty}$ converges to the payoff vector $\varphi(N, \tilde{v})$ (i.e. $\lim_{k\to\infty} \varphi(N, v^k) = \varphi(N, \tilde{v})$).

Efficiency requires that the worth of the grand coalition should be fully shared among all players.

Symmetry requires that, if two players contribute equally to all coalitions that do not include them, their payoff should be equal.

Linearity requires that, taking a linear combination of two TU-games, the solution assigns the payoff vector that is equal to the corresponding linear combination assigned by the solution to the two separate TU-games. In case of $a = b = 1$, linearity reduces to additivity.

Translation covariance requires that, taking an affine transformation of a TU-game, the solution assigns the payoff vector that is equal to the corresponding affine transformation assigned by the solution to the TU-game.

The inessential game property requires that, if forming any coalition can produce no additional benefit, all players only get their individual worths.

The almost inessential game property requires that, if forming any coalition, except for the grand coalition, can produce no additional benefit, all players get the sum of their individual worths and an equal part of the surplus from the formation of the grand coalition. It is trivial that the almost inessential game property implies the inessential game property.

Continuity requires that, if two TU-games are almost the same, then the payoff assigned to the two TU-games are almost the same.

Given any pair of axioms of TU-games, if whenever a solution satisfies one of these axioms, the dual of the solution satisfies the other, then these two axioms are dual to each other. An axiom is called self-dual if the dual of the axiom is itself. Obviously, efficiency, symmetry, linearity, translation covariance, the inessential game property, and continuity are self-dual axioms.

1.3.2 Axioms related to null, dummy, nullifying, A-null and non-negative players

Now we review several axioms stating how much should be assigned to a specific type of player.

Player $i \in N$ is a *null player* in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N\backslash\{i\}$. Player $i \in N$ is a *dummy player* in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$.

- **Null player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is a null player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = 0$.

- **Dummy player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is a dummy player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = v(\{i\})$.

The null player property requires that, if a player has no contribution to any coalition, then he should get zero. The dummy player property requires that, if a player's contribution to each coalition is equal to his individual worth, then he should get his individual worth. Shapley (1953, [94]) used efficiency, additivity, symmetry and the null player property (or the dummy player property) to characterize the Shapley value.

Theorem 1.14 (Shapley 1953, [94]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and the null player property (or the dummy player property) if and only if φ is the Shapley value.*

Player $i \in N$ is a *nullifying player* in $\langle N, v\rangle$ if $v(S) = 0$ for all $S \subseteq N$ and $S \ni i$.

- **Nullifying player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is a nullifying player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = 0$.

The nullifying player property, introduced by Deegan and Packel (1978, [30])[3], requires that a nullifying player should receive a zero payoff. In van den Brink (2007, [104]) the equal division value is characterized by efficiency, symmetry, additivity and the nullifying player property.

Theorem 1.15 (van den Brink 2007, [104]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and the nullifying player property if and only if φ is the equal division value.*

Player $i \in N$ is an *A-null player* in $\langle N, v\rangle$ if $A^v(S) = 0$ for all $S \subseteq N$ and $S \ni i$.

- **A-null player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is an A-null player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = 0$.

The A-null player property requires that, if a player is such that the average marginal contributions to all coalitions including him are equal to zero, then he should get zero. Using the A-null player property instead of the null player property, Nowak and Radzik (1994, [85]) characterized the solidarity value.

Theorem 1.16 (Nowak and Radzik 1994, [85]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry and the A-null player property if and only if φ is the solidarity value.*

Player $i \in N$ is a *non-negative player* in $\langle N, v\rangle$ if $v(S) \geq 0$ for all $S \subseteq N$ and $S \ni i$.

[3] Deegan and Packel (1978, [30]) refer to nullifying players as zero players and use this property to characterize their (non-efficient) Deegan-Packel value.

- **Non-negative player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is a non-negative player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) \geq 0$.

- **Null player in a productive environment property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ with $v(N) \geq 0$ whenever $i \in N$ is a null player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) \geq 0$.

The non-negative player property is introduced and used by Béal et al. (2016, [8]) to characterize the weighted division values, and it requires that for a given player, if the worths of all coalitions including him are non-negative, then he gets at least a zero payoff.

The null player in a productive environment property, introduced by Casajus and Huettner (2013, [22]), requires that if the grand coalition generates a non-negative worth, then a null player should not receive a negative payoff. Béal et al. (2016, [8]) used efficiency, linearity, the nullifying player property and the null player in a productive environment property to characterize the weighted division values.

Theorem 1.17 (Béal et al. 2016, [8]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity and the non-negative player property if and only if there exists a weight vector $\omega \in \Delta^n_+$ such that $\varphi = WD^\omega$.*

Theorem 1.18 (Béal et al. 2016, [8]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity, the nullifying player property and the null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta^n_+$ such that $\varphi = WD^\omega$.*

1.3.3 Axioms related to contributions

Axioms related to contributions refer to some specific changes on the payoffs for a pair of players related to particular modifications of a TU-game.

For all $\langle N, v\rangle \in \mathcal{G}^N$ and $S \subseteq N$, let $\langle S, v|_S\rangle$ denote the TU-game in which the domain of v is restricted from 2^N to 2^S, i.e. $v|_S(T) = v(T)$ for all $T \subseteq S$. For simplicity, $\langle S, v|_S\rangle$ is simply denoted by $\langle S, v\rangle$.

- **Balanced contributions.** For all $\langle N, v\rangle \in \mathcal{G}$ and $i, j \in N$, it holds that $\varphi_i(N, v) - \varphi_i(N\backslash\{j\}, v) = \varphi_j(N, v) - \varphi_j(N\backslash\{i\}, v)$.

Balanced contributions, introduced by Myerson (1980, [82]), requires that, for each pair of players, the influence of a player who leaves the grand coalition on the payoff of the other player is the same as the impact of the other player's departure on his payoff. Myerson [82] used balanced contributions and efficiency to characterize the Shapley value.

Theorem 1.19 (Myerson 1980, [82]). *A solution φ on $\mathcal{G}$ satisfies efficiency and balanced contributions if and only if φ is the Shapley value.*

Considering a different mutual effect for a pair of players, Xu et al. (2016, [115]) suggested a modification of balanced contributions, called quasi-balanced contributions.

- **Quasi-balanced contributions.** For all $\langle N, v\rangle \in \mathcal{G}$ and $i, j \in N$, it holds that $\varphi_i(N,v) - \varphi_i(N\backslash\{j\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{j\})] = \varphi_j(N,v) - \varphi_j(N\backslash\{i\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{i\})]$.

Given a pair of players $\{i, j\}$, the amount $\frac{1}{n}[v(N) - v(N\backslash\{j\})]$ is regarded as player i's loss because of player j's departure from the grand coalition, and correspondingly, the amount $\frac{1}{n}[v(N) - v(N\backslash\{i\})]$ is regarded as player j's loss because of player i's departure from the grand coalition. Quasi-balanced contributions reveals the degree of mutual effect for a pair of players when they leave the grand coalition separately. Xu et al. (2016, [115]) characterized the solidarity value by using efficiency and quasi-balanced contributions.

Theorem 1.20 (Xu et al. 2016, [115]). *A solution φ on $\mathcal{G}$ satisfies efficiency and quasi-balanced contributions if and only if φ is the solidarity value.*

For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, the TU-game $\langle N, v^{\mathbf{N}i}\rangle$ in which i is nullified, is defined by

$$v^{\mathbf{N}i}(S) = v(S\backslash\{i\}), \text{ for all } S \subseteq N. \tag{1.2}$$

Obviously, player i is a null player in $\langle N, v^{\mathbf{N}i}\rangle$.

- **Nullified solidarity.** For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N$, it holds that $\varphi_i(N,v) \geq \varphi_i(N, v^{\mathbf{N}i})$ implies $\varphi_j(N,v) \geq \varphi_j(N, v^{\mathbf{N}i})$.

Nullified solidarity, proposed by Béal et al. (2014, [7]), compares a TU-game before and after a specified player becomes a null player. Nullified solidarity requires uniformity in the direction of the payoff variation for all players in situations where the considered player is nullified.

Theorem 1.21 (Béal et al. 2016, [8]). *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity and nullified solidarity if and only if there exists a weight vector $\omega \in \Delta_+^n$ such that $\varphi = WD^\omega$.*

1.3.4 Axioms related to associated consistency

In the axiomatic approach to solutions for TU-games, consistency is a crucial characteristic of viable and stable solutions. A solution is *consistent* if it allocates the same payoff to players in the original game as in a modified game. There are two kinds of modified games in the existing literature, the *reduced game* and the *associated game*. *Reduced game consistency* and *associated consistency* are defined in terms of the reduced game and the associated game, respectively. Both types of consistency axioms require the payoffs of players to be invariant for certain changes in the game.

In this thesis, we focus on associated consistency. In associated games, the player set does not change, but coalitions revalue their worth by claiming part of the surplus in the game that is left after this coalition and the players outside the coalition are assigned some initial share in the total worth. In the following, we review several associated consistency axioms.

The concept of associated consistency was firstly introduced by Hamiache (2001, [42]) to characterize the Shapley value. Hamiache (2001, [42]) defined its associated games (called *Hamiache's associated game* in this thesis) as follows.

Definition 1.22 (Hamiache 2001, [42]). Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, *Hamiache's associated game* $\langle N, v^*_{\lambda,H}\rangle$ is defined by $v^*_{\lambda,H}(\emptyset) = 0$ and for all $S \subseteq N, S \neq \emptyset$,

$$v^*_{\lambda,H}(S) = v(S) + \lambda \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})].$$

Hamiache's associated game reflects an optimistic self-evaluation of worths of coalitions, where each coalition believes that it can obtain the appropriation of at least a part of the surplus $[v(S \cup \{j\}) - v(S) - v(\{j\})]$, $j \in N \backslash S$. Thus, coalition S reevaluates its worth, $v^*_{\lambda,H}(S)$, as the sum of its worth in the original game, $v(S)$, and a given percentage λ of all the possible surpluses, $\sum_{j \in N \backslash S}[v(S \cup \{j\}) - v(S) - v(\{j\})]$. Let $\lambda \in [0, 1]$.

- **Hamiache's associated consistency for λ.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v) = \varphi(N, v^*_{\lambda,H})$.

Theorem 1.23 (Hamiache 2001, [42])**.** *Let $0 < \lambda < \frac{2}{n}$. A solution φ on $\mathcal{G}^N$ satisfies Hamiache's associated consistency for λ, continuity and the inessential game property if and only if φ is the Shapley value.*

Xu et al. (2009, [117]) modified the definition of Hamiache's associated game from a pessimistic point of view. The new associated game is called the *Sh-individual associated game* in this thesis as follows.

Definition 1.24 (Xu et al. 2009, [117])**.** Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *Sh-individual associated game* $\langle N, v^*_{\lambda,Sh,I}\rangle$ is defined by, for all $S \subseteq N$,

$$v^*_{\lambda,Sh,I}(S) = v(S) - \lambda \sum_{j \in S}[v(S) - v(S \backslash \{j\}) - SC_j(N, v)].$$

The Sh-individual associated game reflects a pessimistic self-evaluation of worths of coalitions, and coalitions may be willing to allow the computation of their payments to be based on these pessimistic expectations. In the process of reevaluating worth, Xu et al. (2009, [117]) assumed that any coalition is formed by its members joining one by one. It will cause a loss of benefits $v(S) - v(S \backslash \{i\}) - SC_i(N, v)$ if player i leaves coalition S and takes away his separable contribution $SC_i(N, v)$ from the worth of coalition S. The worth of coalition S in the Sh-individual associated game differs from the initial worth, by taking into account the possible loss of benefits due to the departure of players in coalition S. In the Sh-individual associated game, each coalition S considers players in S as isolated elements. That is, Xu et al. (2009, [117]) adopt "individual self-evaluation" to reevaluate the worths of coalitions. Let $\lambda \in [0,1]$.

- **Sh-individual associated consistency for λ.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi(N,v) = \varphi(N, v^*_{\lambda,Sh,I})$.

Theorem 1.25 (Xu et al. 2009, [117])**.** *Let $0 < \lambda < \frac{2}{n}$. A solution φ on $\mathcal{G}^N$ satisfies Sh-individual associated consistency for λ, continuity and the inessential game property if and only if φ is the Shapley value.*

Instead of the separable contribution in the Sh-individual associated game, Xu et al. (2013, [119]) also defined a new associated game (called the *C-individual associated game* in this thesis) by using the individual worth. The C-individual associated game is given as follows.

Definition 1.26 (Xu et al. 2013, [119])**.** Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \le \lambda \le 1$, the *C-individual associated game* $\langle N, v^*_{\lambda,C,I}\rangle$ is defined by

$$v^*_{\lambda,C,I}(S) = \begin{cases} v(S) - \lambda \sum_{j\in S}[v(S) - v(S\backslash\{j\}) - v(\{j\})], & \text{if } S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

As mentioned, the worth of coalition S in the associated games differs from the initial worth, by taking into account the possible loss of benefits due to the departure of players in coalition S. Xu et al. (2013, [119]) also considered players in every coalition S as isolated elements, and adopt "individual self-evaluation" to reevaluate the worths of coalitions. Let $\lambda \in [0,1]$.

- **C-individual associated consistency for λ.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi(N,v) = \varphi(N, v^*_{\lambda,C,I})$.

Xu et al. (2013, [119]) characterized the CIS value by using C-individual associated consistency.

Theorem 1.27 (Xu et al. 2013, [119])**.** *Let $0 < \lambda < \frac{2}{n}$. A solution φ on $\mathcal{G}^N$ satisfies C-individual associated consistency for λ, continuity, the almost inessential game property and efficiency if and only if φ is the CIS value.*

Chapter 2

Characterizations of the average-surplus value

2.1 Introduction

As mentioned in Chapter 1, the Shapley value and the solidarity value are two of the most popular solutions for TU-games. The Shapley value assigns to every player the expectation of all his marginal contributions to all coalitions entering before him, assuming that all permutations in which the grand coalition can be formed occur with equal probability. The solidarity value assigns to every player the expectation of the average marginal contributions to all coalitions entering before him, assuming that all permutations in which the grand coalition can be formed occur with equal probability. Both the Shapley value and the solidarity value are defined in terms of marginal contribution that is a significant index to measure every player's contribution to cooperation.

In this chapter, which is based on Li et al. (2021, [73]), we introduce a new concept, called *marginal surplus*, to describe the contribution level of every player. Marginal surplus is defined by the difference between marginal contribution and individual worth, and it can be regarded as the net earning of the player joining a coalition. Compared with marginal contribution, marginal surplus puts more emphasis on the individual worth, but similarly, it is also an alternative index to measure every player's contribution to cooperation. Based on marginal surplus, we define a new solution for TU-games, called the *average-surplus value*. Inspired by the procedures of the Shapley value and the solidarity value, the average-surplus value is determined by an underlying procedure of sharing marginal surplus. Meanwhile, we also characterize the average-surplus value by three classical methods in cooperative game theory: axiomatization, the potential approach and mechanism design.

Our axiomatizations follow the spirit of the axioms used to characterize the Shapley value and the solidarity value mentioned in Chapter 1. There are various characterizations of the Shapley value and the solidarity value in the literature, e.g. see Casajus and Huettner (2014, [23, 24]), Chun (1991, [27]), Hamiache (2001, [42]), Kamijoab (2012, [60]), van den Brink (2002, [103]), Xu et al. (2008, [116]) and Young (1985, [121]). We focus on two representative characterizations of the Shapley value and the solidarity value, respectively. One is based on axioms related to null, respectively A-null players. Shapley (1953, [94]) characterized the Shapley value by using efficiency, additivity, symmetry and the null player property, while Nowak and Radzik (1994, [85]) replaced the null player property with the A-null player property to characterize the solidarity value. Another characterization of the average-surplus value is based on axioms related to contributions. Myerson (1980, [82]) proposed balanced contributions to characterize the Shapley value, while Xu et al. (2016, [115]) used a variation of this balanced contributions to characterize the solidarity value. In this chapter, we define two new axioms, the *A-null surplus player property* and *revised balanced contributions*. The A-null surplus player property requires that a player should obtain his individual worth if his average net earning to each coalition is equal to zero. Revised balanced contributions reveals a relation of the mutual effect for every pair of players on each other's payoff. We characterize the average-surplus value by these two axioms and several standard axioms.

The concept of *potential function* is firstly applied to TU-games by Hart and Mas-Colell (1989, [46]). Every TU-game is mapped into a real number by the potential function. Hart and Mas-Colell showed that the marginal contributions vector of a potential function coincides with the Shapley value. Subsequently, the potential approach is also used to implement various other solutions for TU-games, such as the semivalues (Carreras and Giménez 2011, [16]), the Banzhaf value (Dragan 1996, [33]), and the solidarity value (Xu et al. 2016, [115]). Monderer and Shapley (1996, [78]) also applied the potential approach to non-cooperative games. Non-cooperative potential functions have also been used to analyze solutions for TU-games in Monderer and Shapley (1996, [78]) and Qin (1996, [90]). In this chapter, inspired by the work of Hart and Mas-Colell (1989, [46]), we define a variation of the potential function, called the *AS-potential function*, and show that the adjusted marginal contributions vector of the AS-potential function coincides with the average-surplus value.

Finally, mechanism design can be seen as a part of the Nash program to bridge the gap between cooperative and non-cooperative game theory. It is a significant approach to characterize solutions for TU-games, and has been widely studied in the field of cooperative games. Various implementations of the Shapley value can be found in Gul (1989, [41]), Hart and Mas-Colell (1996, [47]) and Pérez-Castrillo and Wettstein (2001, [89]) and so on. Specifically, Pérez-Castrillo and Wettstein (2001, [89]) proposed a bidding mechanism that gives rise to the Shapley value as the result of equilibrium behavior. Inspired by the work of Pérez-Castrillo and Wettstein, several classical solutions for TU-games, including the discounted Shapley values (van den Brink and Funaki 2015, [107]), the egalitarian Shapley values (van den Brink et al. 2013, [108]) and the consensus values (Ju et al. 2007, [56], and Ju and Wettstein 2009, [58]), are also implemented as the payoff distribution in every subgame perfect equilibrium of a bidding mechanism. Moreover, Albizuri et al. (2015, [1]) and Ju et al. (2014, [57]) applied the approach to airport problems and queueing problems, respectively. A natural question concerning the average-surplus value is whether players can reach it through non-cooperative behavior. In this chapter, we provide a non-cooperative game, namely the *punishment-compensation bidding mechanism*. This mechanism will exert a punishment on a proposer whose offer is rejected, and every player except the proposer will receive a compensation for losses caused by the proposer's departure from the grand coalition. We show that the equilibrium outcome of this mechanism coincides with the payoff assigned by the average-surplus value.

The rest of this chapter is organized as follows. In Section 2.2, we recall procedural schemes of the Shapley value and the solidarity value. In Section 2.3, the average-surplus value is determined by an underlying procedure of sharing marginal surplus. In Section 2.4, we characterize the average-surplus value by introducing the A-null surplus player property and revised balanced contributions. In Section 2.5, we define a variation of the potential function, the AS-potential function, that is used to implement the average-surplus value. In Section 2.6, we provide a non-cooperative game, which outcome in every subgame perfect equilibrium coincides with the payoff assigned by the average-surplus value. Section 2.7 provides all proofs of this chapter. Section 2.8 concludes with a brief comparison.

2.2 Procedural schemes of the Shapley value and the solidarity value

In this section, we recall procedural schemes of the Shapley value and the solidarity value, by considering the assumption that the grand coalition N is gradually formed as players enter the game one by one.

Consider that players join the grand coalition N in a random order (permutation) $\pi \in \Pi^N$ and all orders are equally probable. Every joining player, $i \in N$, brings his marginal contribution, $v(P_{\pi,i}) - v(P_{\pi,i} \backslash \{i\})$, to the coalition of his predecessors, and then this marginal contribution is divided among all players in $P_{\pi,i}$ according to some fixed procedural scheme. In the procedural scheme of the Shapley value, every joining player obtains all of his marginal contribution and shares nothing with his predecessors, while in the procedural scheme of the solidarity value, his marginal contribution is equally shared among himself and all his predecessors. In this way, for every permutation $\pi \in \Pi^N$, the worth $v(N)$ of the grand coalition is distributed among all players. The expected payoff over all permutations is the Shapley value (the solidarity value, respectively) according to the procedural scheme of the Shapley value (the solidarity value, respectively).

Formally, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $\pi \in \Pi^N$, according to the procedural scheme of the Shapley value, the payoff of each player, $i \in N$, under π is given by

$$x_i^\pi(N,v) = v(P_{\pi,i}) - v(P_{\pi,i}\backslash\{i\}).$$

Then, the expected payoff over all permutations of each player, $i \in N$, can be expressed as

$$Sh_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi^N} x_i^\pi(N,v).$$

Similarly, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $\pi \in \Pi^N$, according to the procedural scheme of the solidarity value, the payoff of each player, $i \in N$, under π is given by

$$y_i^\pi(N,v) = \sum_{j\in S_{\pi,i}} \frac{1}{|P_{\pi,j}|}(v(P_{\pi,j}) - v(P_{\pi,j}\backslash\{j\})).$$

Then, the solidarity value of a player is the expected payoff over all permutations, that is, for all $i \in N$,

$$Sol_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi^N} y_i^\pi(N,v).$$

2.3 The average-surplus value

In this section, we define a new solution for TU-games in terms of marginal surplus, called the *average-surplus value*. Based on the procedures for the Shapley value and the solidarity value, the average-surplus value is determined by an underlying procedure of sharing marginal surplus.

For all $\langle N, v\rangle \in \mathcal{G}^N$, $S \subseteq N$ and $i \in S$, player i's *marginal surplus* to S, denoted by $MS_i^v(S)$, is defined by

$$MS_i^v(S) = v(S) - v(S\backslash\{i\}) - v(\{i\}).$$

$MS_i^v(S)$ can be interpreted as the net earning that player i brings when player i joins in coalition S. Obviously, marginal surplus is the difference between marginal contribution and individual worth. It is also an alternative index to measure the contribution level of each player. For all zero-monotonic TU-games, each player has a non-negative marginal surplus to every coalition containing himself.

Next, we introduce a procedural scheme of sharing marginal surplus. The procedural scheme consists of the following steps.

Step 1 Players join the grand coalition N in a random order (permutation) $\pi \in \Pi^N$.

Step 2 Every joining player, $i \in N$, obtains his individual worth $v(\{i\})$.

Step 3 The marginal surplus, $v(P_{\pi,i}) - v(P_{\pi,i} \setminus \{i\}) - v(\{i\})$, is equally shared among the players in $P_{\pi,i}$.

Step 4 A payoff vector $AS^{\pi} \in \mathbb{R}^N$ is obtained by Step 1 to Step 3.[1] Considering all orders with equal probability, the average-surplus value AS is the expected payoff vector over all orders.

We illustrate this procedural scheme with an example with three players. Consider a 3-person game $\langle N, v\rangle$ where $N = \{1, 2, 3\}$. Let $\pi_0 \in \Pi^N$ with $\pi_0(1) = 2$, $\pi_0(2) = 1$, $\pi_0(3) = 3$, that is, player 2 enters the game firstly, then player 1, who is followed by player 3. Every player's payoff under π_0 is shown in Table 2.1.[2]

TABLE 2.1 Player's payoff under π_0

Permutation	Player 1's payoff	Player 2's payoff	Player 3's payoff
Player 2	0	$v(2)$	0
Player 1	$v(1) + \frac{v(12)-v(2)-v(1)}{2}$	$\frac{v(12)-v(2)-v(1)}{2}$	0
Player 3	$\frac{v(123)-v(12)-v(3)}{3}$	$\frac{v(123)-v(12)-v(3)}{3}$	$v(3) + \frac{v(123)-v(12)-v(3)}{3}$

Thus, the player's payoff under π_0 is given by

$$AS_1^{\pi_0}(N, v) = v(1) + \frac{v(12) - v(2) - v(1)}{2} + \frac{v(123) - v(12) - v(3)}{3};$$

[1] In this way, for every order $\pi \in \Pi^N$ the worth $v(N)$ is distributed among all players.

[2] For simplicity, $v(\{1, 2, 3\})$ is simply rewritten as $v(123)$, $v(\{1, 2\})$ is simply rewritten as $v(12)$, etc.

$$AS_2^{\pi_0}(N,v) = v(2) + \frac{v(12)-v(2)-v(1)}{2} + \frac{v(123)-v(12)-v(3)}{3};$$
$$AS_3^{\pi_0}(N,v) = v(3) + \frac{v(123)-v(12)-v(3)}{3}.$$

According to Step 1 to Step 3, for all $\langle N,v\rangle \in \mathcal{G}^N$ and $\pi \in \Pi^N$, the payoff of a player, $i \in N$, under π can be expressed as

$$AS_i^{\pi}(N,v) = v(\{i\}) + \sum_{j\in S_{\pi,i}} \frac{1}{|P_{\pi,j}|}(v(P_{\pi,j}) - v(P_{\pi,j}\backslash\{j\}) - v(\{j\})).$$

Considering all orders with equal probability, the average-surplus value AS is defined as the expected payoff vector of AS^{π} over all permutations as follows.

Definition 2.1. The *average-surplus value* on $\mathcal{G}^N$ is defined by

$$AS_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi(N)} AS_i^{\pi}(N,v). \tag{2.1}$$

for all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$.

Alternatively, we give an equivalent definition of the average-surplus value. Let $\tilde{A}^v(S) = \frac{1}{s}\sum_{j\in S}[v(S) - v(S\backslash\{j\}) - v(\{j\})]$ be the *average marginal surplus* of coalition S in TU-game $\langle N,v\rangle$. The average-surplus value is given by

$$AS_i(N,v) = v(\{i\}) + \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}\tilde{A}^v(S), \tag{2.2}$$

for all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$.

The equivalence of Eq.(2.1) and Eq.(2.2) is easily verified, and we omit it.

For all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$, by Eq.(2.2), we have

$$AS_i(N,v)$$

$$\begin{aligned}
&= \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}\frac{1}{s}\sum_{j\in S}(v(S)-v(S\backslash\{j\})+v(\{i\})-v(\{j\})) \\
&= Sol_i(N,v) + \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}\frac{1}{s}\sum_{j\in S}(v(\{i\})-v(\{j\})) \\
&= Sol_i(N,v) + v(\{i\}) - Sol_i(N,v^0), \qquad (2.3)
\end{aligned}$$

where $\langle N, v^0\rangle$ is the additive game generated by the individual worths in $\langle N, v\rangle$, that is, $v^0(S) = \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$. Thus, the average-surplus value $AS(N, v)$ is the sum of the solidarity value $Sol(N, v)$ and the individual worth vector $(v(\{k\}))_{k\in N}$ minus the solidarity value $Sol(N, v^0)$ of the additive game $\langle N, v^0\rangle$.

Proposition 2.2. *For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, it holds that $AS_i(N, v) \geq Sol_i(N, v)$ if and only if $v(\{i\}) \geq \frac{1}{n}\sum_{k\in N} v(\{k\})$.*

The proof of Proposition 2.2 and of all other results in this chapter can be found in Section 2.7.

By Proposition 2.2, it holds that $AS_i(N, v) = Sol_i(N, v)$ if and only if $v(\{i\}) = \frac{1}{n}\sum_{k\in N} v(\{k\})$ for all $i \in N$. This implies the following corollary.

Corollary 2.3. *For each zero-normalized game $\langle N, v\rangle \in \mathcal{G}^N$, the average-surplus value coincides with the solidarity value, that is, $AS(N, v) = Sol(N, v)$.*

2.4 Axiomatizations of the average-surplus value

A major purpose of axiomatizing solutions in TU-games is to show the reasonability of solutions. In this section, we propose two new axioms, the *A-null surplus player property* and *revised balanced contributions*, to characterize the average-surplus value.

2.4.1 A-null surplus player property

As mentioned in Chapter 1, the null player property and the A-null player property are two classical axiom in TU-games. The null player property requires that a null player gets a zero payoff while the A-null player property requires that an A-null player gets a zero payoff. In this subsection, we define a new type of players, the *A-null surplus player*.

Definition 2.4. Player $i \in N$ is an *A-null surplus player* in $\langle N, v\rangle$ if $\tilde{A}^v(S) = 0$ for all $S \subseteq N$ and $S \ni i$.

- **A-null surplus player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ whenever $i \in N$ is an A-null surplus player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = v(\{i\})$.

The A-null surplus player property requires that, if a player is such that the average marginal surplus to all coalitions including him are equal to zero, then he only obtains his individual worth. It is straightforward to verify that the average-surplus value satisfies the A-null surplus player property.

To characterize the average-surplus value, we firstly define a new basis of the linear space $\mathcal{G}^N$, denoted by $\{\langle N, w_T\rangle | T \subseteq N, T \neq \emptyset\}$. Formally, for all $T \subseteq N$ with $t = 1$,

$$w_T(S) = \begin{cases} 1, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

and for all $T \subseteq N$ with $t \geq 2$,

$$w_T(S) = \begin{cases} \frac{(s-t)!t!}{s!}, & \text{if } S \supseteq T; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Lemma 2.5. *The family $\{\langle N, w_T\rangle | T \subseteq N, T \neq \emptyset\}$ of TU-games forms a basis of the linear space $\mathcal{G}^N$, that is, for each $\langle N, v\rangle \in \mathcal{G}^N$, there exists a set of real numbers $\{\lambda_T\}_{T \subseteq N, T \neq \emptyset}$ such that $v = \sum_{T \subseteq N, T \neq \emptyset} \lambda_T w_T$. Moreover, the coefficients $\{\lambda_T\}_{T \subseteq N, T \neq \emptyset}$ are given by*

$$\lambda_T = \begin{cases} v(T), & \text{if } t = 1; \\ \frac{1}{t}\sum_{j \in T}[v(T) - v(T \backslash \{j\}) - v(\{j\})], & \text{if } t \geq 2. \end{cases} \tag{2.6}$$

The following theorem gives a characterization of the average-surplus value using the A-null surplus player property.

Theorem 2.6. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, symmetry, additivity and the A-null surplus player property if and only if φ is the average-surplus value.*

Logical independence of the axioms used in Theorem 2.6 can be shown by the following alternative solutions.

(i) The solution φ, defined by $\varphi_i(N, v) = v(\{i\})$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms of Theorem 2.6 except efficiency.

(ii) The solution φ, defined by

$$\varphi_i(N, v) = v(\{i\}) + \sum_{S \subseteq N, S \ni i} \frac{s!(n-s)!}{n!} \frac{\omega_i}{\sum_{j \in S} \omega_j} \tilde{A}^v(S),$$

for all $\langle N, v\rangle \in \mathcal{G}^N$, $i \in N$ and $\omega \in \mathbb{R}^N_{++}$ such that $\omega_k \neq \omega_l$ for all $k, l \in N$, satisfies all axioms of Theorem 2.6 except symmetry.

(iii) The solution φ, defined by

$$\varphi_i(N, v) = v(\{i\}) + \sum_{S \subseteq N, S \ni i} \frac{s!(n-s)!}{n!} \frac{v(\{i\})^2 + 1}{\sum_{j \in S} v(\{j\})^2 + s} \tilde{A}^v(S)$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms of Theorem 2.6 except additivity.

(iv) The Shapley value satisfies all axioms of Theorem 2.6 except the A-null surplus player property.

The axiomatization of the average-surplus value in Theorem 2.6 is inspired by that of the Shapley value (Shapley 1953, [94]) and the solidarity value (Nowak and Radzik 1994, [85]). The commonness in the axiomatizations of the three solutions is that all three use efficiency, symmetry and additivity. The only difference is to determine which type of player gets zero or the individual worth. The average-surplus value adopts the A-null surplus player property, while the Shapley value and the solidarity value adopt the null player property and the A-null player property, respectively.

Besides these mentioned above, axioms based on special types of players are used frequently to characterize other solutions for TU-games (e.g. see, van den Brink 2007, [104], van den Brink et al. 2017, [106] and Wang et al. 2017, [114]).

2.4.2 Revised balanced contributions

Balanced contributions, introduced by Myerson (1980, [82]), requires that, for each pair of players, the influence of a player who leaves the grand coalition on the other player is the same as the impact of the other player's departure on him. Myerson (1980, [82]) used balanced contributions and efficiency to characterize the Shapley value. In this subsection, we propose a variation of balanced contributions, called *revised balanced contributions*, to characterize the average-surplus value.

Revised balanced contributions reveals a new relation of the mutual effect for each pair of players. For each pair of players $\{i, j\}$, it will lead to the loss of the marginal surplus $v(N) - v(N\backslash\{i\}) - v(\{i\})$ in the wake of the departure of player i from the grand coalition. This loss should be equally undertaken by all players in the grand coalition N. Thus, a loss $\frac{1}{n}(v(N) - v(N\backslash\{i\}) - v(\{i\}))$ for player j is caused by player i's departure, and a corresponding loss $\frac{1}{n}(v(N)-v(N\backslash\{j\})-v(\{j\}))$ for player i is caused by player j's departure. Formally, revised balanced contributions is defined as follows.

- **Revised balanced contributions.** For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N, i \neq j$, it holds that

$$
\begin{aligned}
&\varphi_i(N, v) - \varphi_i(N\backslash\{j\}, v) - \frac{1}{n}(v(N) - v(N\backslash\{j\}) - v(\{j\})) \\
=&\varphi_j(N, v) - \varphi_j(N\backslash\{i\}, v) - \frac{1}{n}(v(N) - v(N\backslash\{i\}) - v(\{i\})).
\end{aligned}
$$

Next, we use revised balanced contributions and efficiency to characterize the average-surplus value.

Theorem 2.7. *A solution φ on $\mathcal{G}$ satisfies revised balanced contributions and efficiency if and only if φ is the average-surplus value.*

Logical independence of the axioms used in Theorem 2.7 can be shown by the following alternative solutions.

(i) The solution φ, defined by $\varphi_i(N, v) = 0$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies revised balanced contributions but not efficiency.

(ii) The Shapley value satisfies efficiency but not revised balanced contributions.

2.5 A potential approach to the average-surplus value

The potential approach is firstly introduced for TU-games by Hart and Mas-Colell (1989, [46]) to characterize the Shapley value. Every TU-game is mapped into a real number by the potential function. The potential approach is also applied in non-cooperative games by Monderer and Shapley (1996, [78]) to analyze strategic form games.

In TU-games, a function $P : \mathcal{G} \to \mathbb{R}$ with $P(\emptyset, v) = 0$ is called a *potential function* if it satisfies, for all $\langle N, v\rangle \in \mathcal{G}$,

$$\sum_{i\in N} D_i P(N, v) = v(N),$$

where $D_i P(N, v) = P(N, v) - P(N\backslash\{i\}, v)$ represents the marginal contribution of player $i \in N$ to the potential function. Hart and Mas-Colell (1989, [46]) showed that the potential function is unique and the Shapley value coincides with the marginal contributions vector of the potential function.

Later, Xu et al. (2016, [115]) proposed an adjusted potential function to characterize the solidarity value, namely *A-potential function*. A function $P^* : \mathcal{G} \to \mathbb{R}$ with $P^*(\emptyset, v) = 0$ is called an *A-potential function* if it satisfies, for all $\langle N, v\rangle \in \mathcal{G}$,

$$\sum_{i\in N} D_i P^*(N, v) = v(N) - \frac{1}{n}\sum_{i\in N} v(N\backslash\{i\}),$$

where $D_iP^*(N,v) = P^*(N,v) - P^*(N\backslash\{i\},v)$ represents the marginal contribution of player $i \in N$ to the A-potential function.

Compared with the solidarity value, the average-surplus value lays emphasis on taking the influence of the individual worth into account. Correspondingly, we modify the A-potential function by adding the individual worth in order to characterize the average-surplus value. Formally, the revised potential function is defined as follows.

Definition 2.8. A function $\tilde{P} : \mathcal{G} \to \mathbb{R}$ with $\tilde{P}(\emptyset, v) = 0$ is called an *AS-potential function* if it satisfies, for all $\langle N, v\rangle \in \mathcal{G}$,

$$\sum_{i\in N} D_i\tilde{P}(N,v) = v(N) - \frac{1}{n}\sum_{i\in N}(v(N\backslash\{i\}) + v(\{i\})), \tag{2.7}$$

where $D_i\tilde{P}(N,v) = \tilde{P}(N,v) - \tilde{P}(N\backslash\{i\},v)$ represents the marginal contribution of player $i \in N$ to the AS-potential function.

Obviously, the only difference between the A-potential function and the AS-potential function is the individual worth, which is also the distinction between the solidarity value and the average-surplus value. Moreover, condition (2.7) can be represented as

$$\sum_{i\in N}\Big[D_i\tilde{P}(N,v) + v(\{i\}) + \frac{1}{n}(v(N\backslash\{i\}) - \sum_{j\in N\backslash\{i\}} v(\{j\}))\Big] = v(N). \tag{2.8}$$

Compared with the potential function P by Hart and Mas-Colell (1989, [46]), in Eq.(2.8), an adjustment item $v(\{i\})+\frac{1}{n}(v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}} v(\{j\}))$ is added to the marginal contribution $D_i\tilde{P}(N,v)$ for every player $i \in N$, in order to obtain the efficiency normalization condition.

According to the definition of the AS-potential function, for each subgame $\langle S, v\rangle$, we have

$$\sum_{i\in S} D_i\tilde{P}(S,v) = v(S) - \frac{1}{s}\sum_{i\in S}(v(S\backslash\{i\}) + v(\{i\})).$$

Then, we have

$$\sum_{i\in S}[\tilde{P}(S,v)-\tilde{P}(S\backslash\{i\},v)]=\frac{1}{s}\sum_{i\in S}(v(S)-v(S\backslash\{i\})-v(\{i\})).$$

Thus, it holds that

$$s\tilde{P}(S,v)-\sum_{i\in S}\tilde{P}(S\backslash\{i\},v)=\frac{1}{s}\sum_{i\in S}(v(S)-v(S\backslash\{i\})-v(\{i\})).$$

Therefore, an equivalent recursive definition of the AS-potential function can be given as follows,

$$\tilde{P}(S,v)=\frac{1}{s}\left[\sum_{i\in S}\tilde{P}(S\backslash\{i\},v)+\frac{1}{s}\sum_{i\in S}(v(S)-v(S\backslash\{i\})-v(\{i\}))\right], \quad (2.9)$$

for all $S\subseteq N$ with $\tilde{P}(\emptyset,v)=0$. Furthermore, we obtain the following proposition.

Proposition 2.9. *For all $\langle N,v\rangle\in\mathcal{G}^N$, it holds that*

$$\tilde{P}(N,v)=\frac{1}{n}v(N)+\sum_{S\subsetneq N}\frac{(s-1)!(n-s-1)!}{n!(s+1)}v(S)-\frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\sum_{i\in N}v(\{i\}). \quad (2.10)$$

The *adjusted marginal contributions vector* $A\tilde{P}$ is given as follows. For all $\langle N,v\rangle\in\mathcal{G}^N$ and $i\in N$,

$$A_i\tilde{P}(N,v)=D_i\tilde{P}(N,v)+v(\{i\})+\frac{1}{n}(v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}}v(\{j\})). \quad (2.11)$$

The marginal contributions vector $D_i\tilde{P}(N,v)$ is modified to the adjusted marginal contributions vector $A_i\tilde{P}(N,v)$ by two terms. The first term $v(\{i\})$ reflects that every player first gets his individual worth. The second term $\frac{1}{n}(v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}}v(\{j\}))$ reflects the feature of the average distribution of coalitional surplus. It turns out that the adjusted marginal contributions vector of the AS-potential function coincides with the average-surplus value.

Theorem 2.10. *There is a unique AS-potential function $\tilde{P}$ on $\mathcal{G}$. Moreover, the adjusted marginal contributions vector of the AS-potential function coincides with the average-surplus value, that is, $A\tilde{P}(N,v) = AS(N,v)$.*

According to Theorem 2.10, a recursive formula of the average-surplus value is obtained as follows.

Proposition 2.11. *For all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$, it holds that*

$$AS_i(N,v) = \frac{1}{n}\left[v(N) - \frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\}) - v(\{i\}))\right] + \frac{1}{n}\sum_{j\in N\backslash\{i\}} AS_i(N\backslash\{j\},v). \quad (2.12)$$

The recursive formulae of the Shapley value (Hart and Mas-Collel 1989, [46]) and the solidarity value (Xu et al. 2016, [115]) can be derived analogously, by replacing $\frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\}) - v(\{i\}))$ in Proposition 2.11 with $v(N\backslash\{i\})$ and $\frac{1}{n}\sum_{j\in N} v(N\backslash\{j\})$, respectively.

2.6 Punishment-compensation bidding mechanism

In this section, we embed a punishment-compensation principle into the bidding mechanism which is introduced by Pérez-Castrillo and Wettstein (2001, [89]). This *punishment-compensation bidding mechanism* is proposed to implement the average-surplus value, that is, the subgame perfect equilibrium (SPE) outcome of this mechanism coincides with the payoff assigned by the average-surplus value. Formally, the mechanism is defined recursively as follows.

Punishment-compensation bidding mechanism: Each player, $i \in N$, receives his individual worth $v(\{i\})$ when the player set contains only himself. When there is more than one player, the mechanism is defined recursively as follows. Suppose that the rules of the bidding mechanism have been known when there are at most $n-1$ players. Then, the bidding mechanism for a set of players $N = \{1, 2, ..., n\}$ proceeds as follows.

- Stage 1: Each player, $i \in N$, makes bids, $b^i_j \in \mathbb{R}$, for every player $j \in N\backslash\{i\}$. Let $B^i = \sum_{j\in N\backslash\{i\}}(b^i_j - b^j_i)$ be the net bid of player i. Choose a player $\alpha = \arg\max_{i\in N}\{B^i\}$ who has the maximal net bid to be the proposer. If there is more than one player with the maximal net bid, the proposer α is equally chosen among them. Once chosen, the proposer α must pay his bids, b^α_j, to every player $j \in N\backslash\{\alpha\}$.

- Stage 2: The proposer α makes offers, $x^\alpha_j \in \mathbb{R}$, to every player $j \in N\backslash\{\alpha\}$.

- Stage 3: Each player other than α sequentially decides whether or not to accept the offer. If all players accept the offer, then the offer is accepted, otherwise the offer is rejected.

  - If the offer is accepted, the game is over. Each player, $j \in N\backslash\{\alpha\}$ obtains payoff $b^\alpha_j + x^\alpha_j$, and the proposer α obtains payoff $v(N) - \sum_{j\in N\backslash\{\alpha\}}(b^\alpha_j + x^\alpha_j)$.
  - If the offer is rejected, the proposer α leaves from the grand coalition N, which leads to a loss $v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})$ since the proposer still has his own individual worth $v(\{\alpha\})$ after he leaves from the grand coalition. Then, the proposer α is punished by paying the amount, $\frac{1}{n}(v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\}))$, to every remaining player, $j \in N\backslash\{\alpha\}$. In other words, each player, $j \in N\backslash\{\alpha\}$, will receive a compensation, $\frac{1}{n}(v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\}))$, for losses caused by the proposer's departure from the grand coalition. As a result, the proposer α takes the payoff $[v(\{\alpha\}) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}(v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\}))]$. Then, all players other than α proceed to play the bidding mechanism where the set of players is $N\backslash\{\alpha\}$.

It turns out that for all zero-monotonic TU-games $\langle N, v\rangle \in \mathcal{G}^N$, the outcome in every SPE of the punishment-compensation bidding mechanism coincides with the payoff vector $AS(N, v)$ as prescribed by the average-surplus value.

Theorem 2.12. *For all zero-monotonic TU-games $\langle N, v\rangle \in \mathcal{G}^N$, the outcome in every SPE of the punishment-compensation bidding mechanism coincides with the payoff $AS(N,v)$ assigned by the average-surplus value.*

Comparing our bidding mechanism with Pérez-Castrillo and Wettstein's for the Shapley value (Pérez-Castrillo and Wettstein 2001, [89]), we conclude that there is no difference between the two bidding mechanisms at Stage 1 and Stage 2. The main difference is at Stage 3. When the proposer α's offer is rejected, each player $j \in N\backslash\{\alpha\}$ will receive a compensation $\frac{1}{n}(v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\}))$ to be paid by the proposer α in our bidding mechanism, while there is no compensation for each player $j \in N\backslash\{\alpha\}$ in Pérez-Castrillo and Wettstein's mechanism.

2.7 Proofs

Proof of Proposition 2.2. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, we have

$$
\begin{aligned}
&AS_i(N,v) - Sol_i(N,v) = v(\{i\}) - Sol_i(N,v^0)\\
=&v(\{i\}) - \left[\sum_{s=1}^{n} \frac{1}{ns} v(\{i\}) + \left(\frac{1}{n-1} - \frac{1}{n(n-1)}\sum_{s=1}^{n}\frac{1}{s}\right)\sum_{k\in N\backslash\{i\}} v(\{k\})\right]\\
=&\left(\frac{n}{n-1} - \sum_{s=1}^{n}\frac{1}{(n-1)s}\right)\left[v(\{i\}) - \frac{1}{n}\sum_{k\in N} v(\{k\})\right],
\end{aligned}
$$

where $\langle N, v^0\rangle$ is given by $v^0(S) = \sum_{i\in S} v(\{i\})$ for all $S \subseteq N$, and the first equation holds by Eq(2.3). Since $\frac{n}{n-1} - \sum_{s=1}^{n}\frac{1}{(n-1)s} > 0$ for all $n \geq 2$, then $AS_i(N,v) \geq Sol_i(N,v)$ if and only if $v(\{i\}) \geq \frac{1}{n}\sum_{k\in N} v(\{k\})$. □

Proof of Lemma 2.5. Firstly, we show that the family $\{\langle N, w_T\rangle | T \subseteq N, T \neq \emptyset\}$ of TU-games forms a basis of the linear space $\mathcal{G}^N$. Suppose that there exists a set of real numbers $\{\alpha_T\}_{T\subseteq N, T\neq\emptyset}$, not all of which are zero, satisfying $\sum_{T\subseteq N, T\neq\emptyset} \alpha_T w_T = \mathbf{0}$. Let T_0 be a minimum-size coalition such that

$\alpha_{T_0} \neq 0$. Then, we have

$$\sum_{T\subseteq N, T\neq\emptyset} \alpha_T w_T(T_0) = \alpha_{T_0} w_{T_0}(T_0) = \alpha_{T_0} \neq 0,$$

which is in contradiction with the hypothesis $\sum_{T\subseteq N,T\neq\emptyset} \alpha_T w_T = \mathbf{0}$. Thus, $\{\langle N, w_T\rangle | T \subseteq N, T \neq \emptyset\}$ is a basis of $\mathcal{G}^N$, that is, for each $\langle N, v\rangle \in \mathcal{G}^N$, there exists a set of real numbers $\{\lambda_T\}_{T\subseteq N,T\neq\emptyset}$ such that $v = \sum_{T\subseteq N,T\neq\emptyset} \lambda_T w_T$.

Next, we show that the coefficients $\{\lambda_T\}_{T\subseteq N,T\neq\emptyset}$ are given by Eq.(2.6). For each $\langle N, v\rangle \in \mathcal{G}^N$ and $S \subseteq N, S \neq \emptyset$, we have

$$\begin{aligned} v(S) = &\sum_{T\subseteq N,T\neq\emptyset} \lambda_T w_T(S) = \sum_{T\subseteq S,T\neq\emptyset} \lambda_T w_T(S) \\ = &\sum_{T\subseteq S,t=1} \lambda_T + \sum_{T\subseteq S,t\geq 2} \lambda_T \frac{(s-t)!t!}{s!}. \end{aligned}$$

Thus, a recursive formula of the coefficients is given by

$$\lambda_S = v(S) - \sum_{T\subsetneq S,t=1} \lambda_T - \sum_{T\subsetneq S,t\geq 2} \lambda_T \frac{(s-t)!t!}{s!}.$$

Obviously, for $s = 1$, $\lambda_S = v(S)$ and Eq.(2.6) holds. Suppose that Eq.(2.6) holds for all $t \leq s-1$ $(s \geq 2)$. Thus, we have

$$\begin{aligned} \lambda_S =& v(S) - \sum_{T\subsetneq S,t=1} \lambda_T - \sum_{T\subsetneq S,t\geq 2} \lambda_T \frac{(s-t)!t!}{s!} \\ =& v(S) - \sum_{j\in S} v(\{j\}) - \sum_{T\subsetneq S,t\geq 2} \sum_{j\in T} [v(T) - v(T\backslash\{j\}) - v(\{j\})] \frac{(s-t)!(t-1)!}{s!} \\ =& v(S) - \sum_{T\subsetneq S,t\geq 2} \sum_{j\in T} [v(T) - v(T\backslash\{j\})] \frac{(s-t)!(t-1)!}{s!} - \frac{2}{s} \sum_{j\in S} v(\{j\}) \\ =& v(S) - \frac{1}{s}[\sum_{T\subsetneq S,t=s-1} v(T) - \sum_{T\subsetneq S,t=1} v(T)] - \frac{2}{s} \sum_{j\in S} v(\{j\}) \\ =& \frac{1}{s} \sum_{j\in S} [v(S) - v(S\backslash\{j\}) - v(\{j\})]. \end{aligned}$$

Therefore, Eq.(2.6) holds for all $T \subseteq N, T \neq \emptyset$. □

Proof of Theorem 2.6. It is straightforward to verify that the average-surplus value satisfies efficiency, symmetry, additivity and the A-null surplus player property. It is left to show the uniqueness. Suppose that φ is a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. For all $T \subseteq N, T \neq \emptyset$ and $\alpha \in \mathbb{R}$, $\langle N, \alpha w_T \rangle$ is defined by Eq.(2.4) and Eq.(2.5). Let $i \in N \backslash T$, $S \subseteq N$ and $S \ni i$. If $t = 1$, we have

$$\begin{aligned}\tilde{A}^{\alpha w_T}(S) =& \frac{1}{s}\sum_{j\in S}(\alpha w_T(S) - \alpha w_T(S\backslash\{j\}) - \alpha w_T(\{j\}))\\ =& \frac{1}{s}\sum_{j\in S}(\alpha - \alpha - 0) = 0.\end{aligned}$$

If $t \geq 2$, we have

$$\begin{aligned}\tilde{A}^{\alpha w_T}(S) =& \frac{1}{s}\sum_{j\in S}(\alpha w_T(S) - \alpha w_T(S\backslash\{j\}) - \alpha w_T(\{j\}))\\ =& \alpha w_T(S) - \frac{1}{s}\sum_{j\in S}\alpha w_T(S\backslash\{j\}).\end{aligned}$$

We distinguish the following two different cases,

- if $S \not\supseteq T$, $\tilde{A}^{\alpha w_T}(S) = 0 - 0 = 0$.
- if $S \supseteq T$, we have

$$\begin{aligned}\tilde{A}^{\alpha w_T}(S) =& \alpha w_T(S) - \frac{1}{s}\sum_{j\in S}\alpha w_T(S\backslash\{j\})\\ =& \frac{\alpha(s-t)!t!}{s!} - \frac{\alpha}{s}(s-t)\frac{(s-1-t)!t!}{(s-1)!}\\ =& 0.\end{aligned}$$

Thus, for all $i \in N\backslash T$, i is an A-null surplus player in $\langle N, \alpha w_T \rangle$. By the A-null surplus player property, we have

$$\varphi_i(N, \alpha w_T) = \alpha w_T(\{i\}) = 0.$$

Moreover, for all $j, k \in T$, j and k are symmetric players in $\langle N, \alpha w_T \rangle$. By symmetry and efficiency, it holds that

$$\varphi_i(N, \alpha w_T) = \begin{cases} \alpha, & \text{if } i \in T \text{ and } t = 1; \\ \frac{\alpha(n-t)!(t-1)!}{n!}, & \text{if } i \in T \text{ and } t \geq 2. \end{cases}$$

By Lemma 2.5 and additivity, for all $\langle N, v \rangle \in \mathcal{G}^N$ and $i \in N$,

$$\varphi_i(N, v) = \varphi_i(N, \sum_{T \subseteq N, T \neq \emptyset} \lambda_T w_T) = \sum_{T \subseteq N, T \neq \emptyset} \varphi_i(N, \lambda_T w_T),$$

which implies, with the expression of $\varphi_i(N, \alpha w_T)$ above, that φ is uniquely determined. Since the average-surplus value satisfies these axioms, it is the unique solution on $\mathcal{G}^N$ satisfying efficiency, symmetry, additivity and the A-null surplus player property. □

Proof of Theorem 2.7. Firstly, we show the average-surplus value satisfies revised balanced contributions. For all $\langle N, v \rangle \in \mathcal{G}^N$ and $i, j \in N, i \neq j$, by Eq.(2.2), we have

$$\begin{aligned} & AS_i(N, v) - AS_i(N \backslash \{j\}, v) \\ = & \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} \tilde{A}^v(S) - \sum_{S \subseteq N \backslash \{j\}, S \ni i} \frac{(s-1)!(n-s-1)!}{(n-1)!} \tilde{A}^v(S) \\ = & \sum_{S \subseteq N, S \ni i,j} \frac{(s-1)!(n-s)!}{n!} \tilde{A}^v(S) - \sum_{S \subseteq N \backslash \{j\}, S \ni i} \frac{s!(n-s-1)!}{n!} \tilde{A}^v(S) \\ = & \sum_{S \subseteq N, S \ni i,j} \frac{(s-1)!(n-s)!}{n!} (\tilde{A}^v(S) - \tilde{A}^v(S \backslash \{j\})). \end{aligned}$$

Correspondingly, it holds that

$$AS_j(N, v) - AS_j(N \backslash \{i\}, v) = \sum_{S \subseteq N, S \ni i,j} \frac{(s-1)!(n-s)!}{n!} (\tilde{A}^v(S) - \tilde{A}^v(S \backslash \{i\})).$$

Thus, we have

$$AS_i(N, v) - AS_i(N \backslash \{j\}, v) - (AS_j(N, v) - AS_j(N \backslash \{i\}, v))$$

$$
\begin{aligned}
&= \sum_{S\subseteq N, S\ni i,j} \frac{(s-1)!(n-s)!}{n!}(\tilde{A}^v(S\setminus\{i\}) - \tilde{A}^v(S\setminus\{j\})) \\
&= \sum_{S\subseteq N, S\ni i,j} \frac{(s-1)!(n-s)!}{n!}[v(S\setminus\{i\}) - v(S\setminus\{j\}) + \frac{1}{s-1}(v(\{i\}) - v(\{j\}))] \\
&\quad - \sum_{S\subseteq N, S\ni i,j} \frac{(s-2)!(n-s)!}{n!} \sum_{k\in S\setminus\{i,j\}} (v(S\setminus\{i,k\}) - v(S\setminus\{j,k\})) \\
&= \sum_{S\subseteq N, S\ni i,j} \frac{(s-1)!(n-s)!}{n!}(v(S\setminus\{i\}) - v(S\setminus\{j\})) \\
&\quad - \sum_{S\subsetneq N, S\ni i,j} \frac{(s-1)!(n-s)!}{n!}(v(S\setminus\{i\}) - v(S\setminus\{j\})) \\
&\quad + \frac{n-2}{n(n-1)}(v(\{i\}) - v(\{j\})) + \frac{1}{n(n-1)}(v(\{i\}) - v(\{j\})) \\
&= \frac{1}{n}(v(N\setminus\{i\}) - v(N\setminus\{j\})) + \frac{1}{n}(v(\{i\}) - v(\{j\})).
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
&AS_i(N,v) - AS_i(N\setminus\{j\},v) - \frac{1}{n}(v(N) - v(N\setminus\{j\}) - v(\{j\})) \\
=&AS_j(N,v) - AS_j(N\setminus\{i\},v) - \frac{1}{n}(v(N) - v(N\setminus\{i\}) - v(\{i\})).
\end{aligned}
$$

Therefore, the average-surplus value satisfies revised balanced contributions.

Next, we show the uniqueness. Suppose that φ is a solution on $\mathcal{G}$ satisfying revised balanced contributions and efficiency. We show that $\varphi(T,v) = AS(T,v)$ for all $\langle T,v\rangle \in \mathcal{G}^T$ by induction on player set T. For all $\langle T,v\rangle \in \mathcal{G}^T$ with $|T| = 1$, efficiency implies $\varphi(T,v) = v(T) = AS(T,v)$. Let $\langle T,v\rangle \in \mathcal{G}^T$ with $|T| = 2$. Without loss of generality, suppose $T = \{i,j\}$. By revised balanced contributions, we have

$$
\begin{aligned}
&\varphi_i(T,v) - \varphi_i(\{i\},v) - \frac{1}{2}(v(T) - v(\{i\}) - v(\{j\})) \\
=&\varphi_j(T,v) - \varphi_j(\{j\},v) - \frac{1}{2}(v(T) - v(\{i\}) - v(\{j\})).
\end{aligned}
$$

By efficiency, it holds that $\varphi_i(\{i\}, v) = v(\{i\})$, $\varphi_j(\{j\}, v) = v(\{j\})$ and $\varphi_i(T, v) + \varphi_j(T, v) = v(\{i, j\})$. Thus,

$$\begin{aligned}\varphi_i(T, v) =& \frac{1}{2}(v(\{i, j\}) + v(\{i\}) - v(\{j\})) = AS_i(T, v);\\ \varphi_j(T, v) =& \frac{1}{2}(v(\{i, j\}) + v(\{j\}) - v(\{i\})) = AS_j(T, v).\end{aligned}$$

Proceeding by induction, suppose that it holds that $\varphi(T, v) = AS(T, v)$ for all TU-games $\langle T, v\rangle$ with $|T| < n$. Thus, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N$, we have

$$\varphi_i(N\backslash\{j\}, v) = AS_i(N\backslash\{j\}, v), \varphi_j(N\backslash\{i\}, v) = AS_j(N\backslash\{i\}, v).$$

By revised balanced contributions of φ and AS, we conclude that

$$\varphi_i(N, v) - AS_i(N, v) = \varphi_j(N, v) - AS_j(N, v)$$

Then, by fixing i and summing over $j \in N$, we have

$$\sum_{j\in N}(\varphi_i(N, v) - AS_i(N, v)) = \sum_{j\in N}(\varphi_j(N, v) - AS_j(N, v)).$$

Thus,

$$\begin{aligned}n(\varphi_i(N, v) - AS_i(N, v)) &= \sum_{j\in N}\varphi_j(N, v) - \sum_{j\in N}AS_j(N, v)\\ &= v(N) - v(N) = 0.\end{aligned}$$

Therefore, $\varphi_i(N, v) = AS_i(N, v)$ holds for all $i \in N$. □

Proof of Proposition 2.9. The result is proved by induction on the cardinality k of player set. For $k = 1$, it holds that $\tilde{P}(\{i\}, v) = 0$ for all $i \in N$. Proceeding by induction, suppose that Eq.(2.10) holds with $k = n - 1$. Then, it implies that

$$\tilde{P}(N\backslash\{i\}, v|_{N\backslash\{i\}}) = \frac{1}{n-1}v(N\backslash\{i\}) + \sum_{S\subsetneq N\backslash\{i\}}\frac{(s-1)!(n-s-1)!}{(n-1)!(s+1)}v(S)$$

$$- \frac{1}{n-1}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)\sum_{j\in N\backslash\{i\}} v(\{j\}).$$

For $k = n$, by Eq.(2.9), we have

$$\begin{aligned}
\tilde{P}(N,v) =& \frac{1}{n}\left[\sum_{i\in N}\tilde{P}(N\backslash\{i\},v) + \frac{1}{n}\sum_{i\in N}(v(N) - v(N\backslash\{i\}) - v(\{i\}))\right]\\
=& \frac{1}{n}\sum_{i\in N}\left[\frac{1}{n-1}v(N\backslash\{i\}) + \sum_{S\subsetneq N\backslash\{i\}}\frac{(s-1)!(n-s-1)!}{(n-1)!(s+1)}v(S)\right.\\
&\left. - \frac{1}{n-1}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)\sum_{j\in N\backslash\{i\}} v(\{j\})\right] + \frac{1}{n}v(N) - \frac{1}{n^2}\sum_{i\in N}(v(N\backslash\{i\}) + v(\{i\}))\\
=& \frac{1}{n^2(n-1)}\sum_{i\in N}v(N\backslash\{i\}) + \frac{1}{n}\sum_{i\in N}\sum_{S\subsetneq N\backslash\{i\}}\frac{(s-1)!(n-s-1)!}{(n-1)!(s+1)}v(S)\\
&- \frac{1}{n(n-1)}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)\sum_{i\in N}\sum_{j\in N\backslash\{i\}} v(\{j\}) + \frac{1}{n}v(N) - \frac{1}{n^2}\sum_{i\in N}v(\{i\})\\
=& \frac{1}{n}v(N) + \frac{1}{n^2(n-1)}\sum_{i\in N}v(N\backslash\{i\}) + \frac{1}{n}\sum_{s\leq n-2}\frac{(s-1)!(n-s)!}{(n-1)!(s+1)}v(S)\\
&- \frac{1}{n}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)\sum_{i\in N}v(\{i\}) - \frac{1}{n^2}\sum_{i\in N}v(\{i\})\\
=& \frac{1}{n}v(N) + \sum_{S\subsetneq N}\frac{(s-1)!(n-s-1)!}{n!(s+1)}v(S) - \frac{1}{n}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\sum_{i\in N}v(\{i\}).
\end{aligned}$$

Therefore, Eq.(2.10) holds for all $\langle N, v\rangle \in \mathcal{G}^N$. □

Proof of Theorem 2.10. It is straightforward to obtain existence and uniqueness of the AS-potential function by Eq.(2.10). It is left to show that $A\tilde{P}(N,v) = AS(N,v)$ for all $\langle N,v\rangle \in \mathcal{G}^N$. Therefore, it is sufficient to check that the axioms which uniquely determine the average-surplus value are also satisfied by $A\tilde{P}$.

Therefore, we prove that $A\tilde{P}$ satisfies efficiency, symmetry, additivity and the A-null surplus player property.

- **Efficiency**: By Eq.(2.7) and Eq.(2.11), we have

$$\begin{aligned}\sum_{i\in N} A_i\tilde{P}(N,v) =&\sum_{i\in N}[D_i\tilde{P}(N,v)+v(\{i\})+\frac{1}{n}(v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}}v(\{j\}))]\\ =&v(N)-\frac{1}{n}\sum_{i\in N}(v(N\backslash\{i\})+v(\{i\}))+\sum_{i\in N}v(\{i\})\\ &+\frac{1}{n}\sum_{i\in N}(v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}}v(\{j\}))\\ =&v(N).\end{aligned}$$

- **Additivity**: It is easy to check that the AS-potential function $\tilde{P}$ satisfies additivity by Eq.(2.10) , that is, $\tilde{P}(N,v+w)=\tilde{P}(N,v)+\tilde{P}(N,w)$ for all $\langle N,v\rangle,\langle N,w\rangle\in\mathcal{G}^N$. Thus, $D\tilde{P}(N,v)$ and $A\tilde{P}(N,v)$ also satisfy additivity by Definition 2.8 and Eq.(2.11).

- **Symmetry**: Let i and j be symmetric players. Firstly, we prove that $\tilde{P}(S\backslash\{i\},v)=\tilde{P}(S\backslash\{j\},v)$ for all $S\subseteq N, S\ni i,j$ by induction. It is obvious that $\tilde{P}(\{i\},v)=0=\tilde{P}(\{j\},v)$ for $S=\{i,j\}$. Proceeding by induction, suppose that the conclusion holds for $2\le|S|<n$. Then, for $|S|=n$, we have

$$\begin{aligned}&\tilde{P}(N\backslash\{i\},v)-\tilde{P}(N\backslash\{j\},v)\\ =&\frac{1}{n-1}\left[\sum_{k\in N\backslash\{i\}}\tilde{P}(N\backslash\{i,k\},v)+v(N\backslash\{i\})\right.\\ &\left.-\frac{1}{n-1}\sum_{k\in N\backslash\{i\}}(v(N\backslash\{i,k\})+v(\{k\}))\right]\\ &-\frac{1}{n-1}\left[\sum_{k\in N\backslash\{j\}}\tilde{P}(N\backslash\{j,k\},v)+v(N\backslash\{j\})\right.\\ &\left.-\frac{1}{n-1}\sum_{k\in N\backslash\{j\}}(v(N\backslash\{j,k\})+v(\{k\}))\right]\end{aligned}$$

$$=\frac{1}{n-1}\left[\sum_{k\in N\backslash\{i\}}\tilde{P}(N\backslash\{i,k\},v)-\sum_{k\in N\backslash\{j\}}\tilde{P}(N\backslash\{j,k\},v)\right]$$
$$=0.$$

Since $v(N\backslash\{i\})=v(N\backslash\{j\})$ and $v(\{i\})=v(\{j\})$,

$$\begin{aligned}
&A_i\tilde{P}(N,v)-A_j\tilde{P}(N,v)\\
=&D_i\tilde{P}(N,v)+v(\{i\})+\frac{1}{n}\left[v(N\backslash\{i\})-\sum_{k\in N\backslash\{i\}}v(\{k\})\right]\\
&-\left[D_j\tilde{P}(N,v)+v(\{j\})+\frac{1}{n}(v(N\backslash\{j\})-\sum_{k\in N\backslash\{j\}}v(\{k\}))\right]\\
=&\tilde{P}(N\backslash\{i\},v)-\tilde{P}(N\backslash\{j\},v)\\
=&0,
\end{aligned}$$

showing that $A\tilde{P}$ satisfies symmetry.

- **A-null surplus player property**: Let $i\in N$ be an A-null surplus player in $\langle N,v\rangle$, then $v(S)-\frac{1}{s}\sum_{j\in S}(v(S\backslash\{j\})+v(\{j\}))=0$ for all $S\subseteq N$ and $S\ni i$. Obviously, i is also an A-null surplus player in all subgames $\langle S,v\rangle$. Next, we prove that $A_i\tilde{P}(S,v)=v(\{i\})$ for all $\langle S,v\rangle$ by induction, in particular, $A_i\tilde{P}(N,v)=v(\{i\})$.

  It is trivial that $A_i\tilde{P}(\{i\},v)=v(\{i\})$ with $S=\{i\}$. Proceeding by induction, suppose that $A_i\tilde{P}(S,v)=v(\{i\})$ for all subgames with $|S|\leq n-1$, that is, for all $j\in N\backslash\{i\}$,

  $$D_i\tilde{P}(N\backslash\{j\},v)+v(\{i\})+\frac{1}{n-1}(v(N\backslash\{i,j\})-\sum_{k\in N\backslash\{i,j\}}v(\{k\}))=v(\{i\}).$$

  Thus, by Eq.(2.9), we have

  $$\begin{aligned}
  &nA_i\tilde{P}(N,v)\\
  =&n\tilde{P}(N,v)-n\tilde{P}(N\backslash\{i\},v)+nv(\{i\})+v(N\backslash\{i\})-\sum_{j\in N\backslash\{i\}}v(\{j\})
  \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in N} \tilde{P}(N\backslash\{j\},v) + v(N) - \frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\})) \\
&\quad - \sum_{j\in N\backslash\{i\}} \tilde{P}(N\backslash\{i,j\},v) + \frac{1}{n-1}\sum_{j\in N\backslash\{i\}} (v(N\backslash\{i,j\}) + v(\{j\})) \\
&\quad - v(N\backslash\{i\}) - \tilde{P}(N\backslash\{i\},v) + nv(\{i\}) + v(N\backslash\{i\}) - \sum_{j\in N\backslash\{i\}} v(\{j\}) \\
&= \sum_{j\in N\backslash\{i\}} D_i\tilde{P}(N\backslash\{j\},v) + \frac{1}{n-1}\sum_{j\in N\backslash\{i\}} v(N\backslash\{i,j\}) + nv(\{i\}) \\
&\quad - \frac{n-2}{n-1}\sum_{j\in N\backslash\{i\}} v(\{j\}) \\
&= \sum_{j\in N\backslash\{i\}} [D_i\tilde{P}(N\backslash\{j\},v) + v(\{i\}) + \frac{1}{n-1}(v(N\backslash\{i,j\}) \\
&\quad - \sum_{k\in N\backslash\{i,j\}} v(\{k\}))] + v(\{i\}) \\
&= nv(\{i\}).
\end{aligned}
$$

The proof is completed. □

Proof of Proposition 2.11. By Definition 2.8, Eq.(2.9) and Eq.(2.11), we have

$$
\begin{aligned}
A_i\tilde{P}(N,v) =& \tilde{P}(N,v) - \tilde{P}(N\backslash\{i\},v) + v(\{i\}) + \frac{1}{n}(v(N\backslash\{i\}) - \sum_{j\in N\backslash\{i\}} v(\{j\})) \\
=& \frac{1}{n}\left[v(N) - \frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\})) + v(N\backslash\{i\}) + nv(\{i\}) \right. \\
&\left. - \sum_{j\in N\backslash\{i\}} v(\{j\})\right] + \frac{1}{n}\sum_{j\in N\backslash\{i\}} \left[\tilde{P}(N\backslash\{j\},v) - \tilde{P}(N\backslash\{i\},v)\right] \\
=& \frac{1}{n}\left[v(N) - \frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\})) + v(N\backslash\{i\}) \right. \\
&\left. - \sum_{j\in N\backslash\{i\}} v(\{j\}) + nv(\{i\})\right] + \frac{1}{n}\sum_{j\in N\backslash\{i\}} \Big[A_i\tilde{P}(N\backslash\{j\},v)
\end{aligned}
$$

$$
\begin{aligned}
&-v(\{i\}) - A_j\tilde{P}(N\backslash\{i\},v) + v(\{j\})\Big] \\
=&\frac{1}{n}\left[v(N) - \frac{1}{n}\sum_{j\in N}(v(N\backslash\{j\}) + v(\{j\}) - v(\{i\}))\right] \\
&+\frac{1}{n}\sum_{j\in N\backslash\{i\}} A_i\tilde{P}(N\backslash\{j\},v).
\end{aligned}
$$

Thus, Eq.(2.12) holds for all $\langle N, v\rangle \in \mathcal{G}^N$ by Theorem 2.10. □

Proof of Theorem 2.12. This theorem is proved by induction on the cardinality k of player set. Obviously, the result holds for $k = 1$, since $AS_i(\{i\}, v) = v(\{i\})$ for every one-person TU-game $\langle\{i\}, v\rangle$. Proceeding by induction, suppose that the result holds for all $k < n$. We show that it also holds for $k = n$.

Let $N = \{1, 2, ..., n\}$. We now show that the average-surplus value is indeed a SPE outcome by considering the following strategies.

- At stage 1, each player, $i \in N$, makes bids $b^i_j = AS_j(N, v) - AS_j(N\backslash\{i\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{i\}) - v(\{i\})]$ for every player $j \in N\backslash\{i\}$.
- At stage 2, The proposer α makes offers $x^\alpha_j = AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ to every player $j \in N\backslash\{\alpha\}$.
- At stage 3, each player $j \in N\backslash\{\alpha\}$ will accept the offer if $x^\alpha_j \geq AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$, otherwise the offer is rejected.

It is obvious that the outcome of this strategy profile is the average-surplus value. We will verify that the strategies constitute a SPE. At stage 3, each player $j \in N\backslash\{\alpha\}$ obtains $AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ by the induction hypothesis, and the proposer α just receives $v(\{\alpha\}) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ in the case of rejection. Thus, the strategies are best responses at stage 3 and stage 2 as long as $v(N) - v(N\backslash\{\alpha\}) \geq v(\{\alpha\})$, which is the case for all zero-monotonic TU-games, since

$$\begin{aligned}
&v(N) - \sum_{j\in N\backslash\{\alpha\}} \left[AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]\right] \\
&\geq v(\{\alpha\}) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})] \\
\Leftrightarrow\ & v(N) - v(N\backslash\{\alpha\}) \geq v(\{\alpha\}).
\end{aligned}$$

At stage 1, by revised balanced contributions, for all $i \in N$, we have

$$\begin{aligned}
B^i =& \sum_{j\in N\backslash\{i\}} (b^i_j - b^j_i) \\
=& \sum_{j\in N\backslash\{i\}} \Big[AS_j(N, v) - AS_j(N\backslash\{i\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{i\}) - v(\{i\})] \\
& -[AS_i(N, v) - AS_i(N\backslash\{j\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{j\}) - v(\{j\})]]\Big] \\
=&0.
\end{aligned}$$

Therefore, if a player, $i \in N$, increases his total bid $\sum_{j\in N\backslash\{i\}} b^i_j$, he will become the proposer, but his payoff will decrease. If a player, $i \in N$, decreases his total bid $\sum_{j\in N\backslash\{i\}} b^i_j$, then his payoff is invariable since another player will be chosen as the proposer. Thus, this action is a best response at stage 1. Hence, the above strategy profile constitutes a SPE.

We now prove that any SPE yields the average-surplus value outcome by a series of claims.

Claim(a). In any SPE, at stage 3, every player $j \in N\backslash\{\alpha\}$ will accept the offer if $x^\alpha_j > AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$. The offer is rejected if there exists at least one player $j \in N\backslash\{\alpha\}$ such that $x^\alpha_j < AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$.

Note that in the case of rejection at stage 3, the payoff of each player, $j \in N\backslash\{\alpha\}$, is $AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ by the induction hypothesis. Thus, a player, $j \in N\backslash\{\alpha\}$, will accept the offer if $x^\alpha_j > AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ since he can improve his payoff. He will reject the offer if $x_j^\alpha < AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$. Claim (a) is proved by using the same argument for all $j \in N\backslash\{\alpha\}$.

Claim(b). If $v(N) > v(N\backslash\{\alpha\}) + v(\{\alpha\})$, the SPE strategies starting from stage 2 are as follows. At stage 2, the proposer α will offer $x_j^\alpha = AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ to each player $j \in N\backslash\{\alpha\}$; at stage 3, each player, $j \in N\backslash\{\alpha\}$, rejects any offer $x_j^\alpha < AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ and accepts the offer otherwise.

If $v(N) = v(N\backslash\{\alpha\}) + v(\{\alpha\})$, there exist SPE strategies besides the previous SPE strategies. At stage 2, the proposer α offers $x_j^\alpha \leq AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ to each player $j \in N\backslash\{\alpha\}$; at stage 3, each player, $j \in N\backslash\{\alpha\}$, rejects any offer $x_j^\alpha \leq AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ and accepts the offer otherwise.

We verify that these strategies constitute a SPE. Suppose that $v(N) > v(N\backslash\{\alpha\}) + v(\{\alpha\})$. In that case, the offer made by the proposer α is rejected and then the proposer α obtains $v(\{\alpha\}) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$, which cannot be part of a SPE, since the proposer α can improve his payoff by offering $AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})] + \varepsilon/(n-1)$ to each player $j \in N\backslash\{\alpha\}$ with $0 < \varepsilon < v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})$ so that the offer is accepted by claim (a). Therefore, $x_j^\alpha \geq AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ for all $j \in N\backslash\{\alpha\}$ in any SPE. However, an offer with $x_j^\alpha > AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ for some $j \in N\backslash\{\alpha\}$ cannot be part of a SPE. The reason is that the proposer α can improve his payoff by offering $AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})] + \varepsilon/(n-1)$ to each player $j \in N\backslash\{\alpha\}$ with $\varepsilon < x_j^\alpha - AS_j(N\backslash\{\alpha\}, v) - \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ and $\varepsilon > 0$. Hence, $x_j^\alpha = AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ for all $j \in N\backslash\{\alpha\}$, and acceptance of the offer implies that each player $j \in N\backslash\{\alpha\}$ accepts any offer $x_j^\alpha \geq AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$.

If $v(N) = v(N\backslash\{\alpha\}) + v(\{\alpha\})$, the proposer α offers at least $AS_j(N\backslash\{\alpha\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ to each player $j \in N\backslash\{\alpha\}$ so that the offer is accepted by the same argument in the previous case. The proposer gets $v(\{\alpha\}) - \sum_{j\in N\backslash\{\alpha\}} \frac{1}{n}[v(N) - v(N\backslash\{\alpha\}) - v(\{\alpha\})]$ in the case of rejection, which is identical to the payoff in the case of acceptance. Therefore, any offer that leads to a rejection also is a SPE.

Claim(c). In any SPE, the net bid $B^i = 0$ for all $i \in N$.

Let $\Lambda = \{i \in N | B^i = \max_{j\in N} B^j\}$. If $\Lambda = N$, the net bid $B^i = 0$ for all $i \in N$ due to $\sum_{i\in N} B^i = 0$. Otherwise, any $j \in \Lambda$ can improve his expected payoff by slightly changing his bids without altering the set Λ. Let $j \notin \Lambda$ and $i \in \Lambda$. Suppose that player i changes his strategy by making bids $b'^i_k = b^i_k + \delta$ for all $k \in \Lambda\backslash\{i\}$, $b'^i_j = b^i_j - |\Lambda|\delta$, and $b'^i_l = b^i_l$ for all $l \notin \Lambda$ and $l \neq j$. Then, the net bids are $B'^k = B^k - \delta$ for all $k \in \Lambda$; $B'^j = B^j + |\Lambda|\delta$; $B'^l = B^l$ for all $l \notin \Lambda$ and $l \neq j$. Because $B^l < B^i$ for all $l \notin \Lambda$, there must exist $\delta > 0$ such that $B^j + |\Lambda|\delta < B^i - \delta$ and $B'^l < B'^i = B'^k$ for all $k \in \Lambda$. Therefore, Λ remains unchanged, but player i's expected payoff increases.

Claim(d). In any SPE, the payoff of every player is invariable whoever is chosen as the proposer.

The net bids of all players are the same by claim (c). If a player would strictly prefer to be the proposer, he has to enhance his bids, which will result in a decrease of his payoff. If a player prefers that the proposer is one of the other players for sure, he needs to decrease his bids, which makes no difference to his payoff. Hence, every player is indifferent to whoever is chosen as the proposer.

Claim(e). In any SPE, the final payoff of every player coincides with the average-surplus value.

Firstly, if a player, $i \in N$, is the proposer, his final payoff is $y^i_i = v(N) - v(N\backslash\{i\}) - \sum_{j\in N\backslash\{i\}}[b^i_j + \frac{1}{n}[v(N) - v(N\backslash\{i\}) - v(\{i\})]]$. Then, if a player, $j \in N\backslash\{i\}$, is the proposer, player i's final payoff is $y^j_i = AS_i(N\backslash\{j\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{j\}) - v(\{j\})] + b^j_i$. Therefore, the sum of player i's payoff over all possible choices is as follows,

$$\begin{aligned}\sum_{j\in N} y_i^j =& v(N) - v(N\backslash\{i\}) - \sum_{j\in N\backslash\{i\}} \left[b_j^i + \frac{1}{n}[v(N) - v(N\backslash\{i\}) - v(\{i\})]\right] \\ &+ \sum_{j\in N\backslash\{i\}} \left[AS_i(N\backslash\{j\}, v) + \frac{1}{n}[v(N) - v(N\backslash\{j\}) - v(\{j\})] + b_i^j\right] \\ =& v(N) - \frac{1}{n}\sum_{j\in N}[v(N\backslash\{j\}) + v(\{j\})] + v(\{i\}) + \sum_{j\in N\backslash\{i\}} AS_i(N\backslash\{j\}, v) \\ =& nAS_i(N, v),\end{aligned}$$

where the last equality holds by Proposition 2.11. By claim (d), we have $y_i^j = y_i^k$ for all $j, k \in N$, and then we conclude that $y_i^j = AS_i(N, v)$ for all $j \in N$. Hence, the final payoff of every player coincides with the average-surplus value. □

2.8 Conclusions

In this chapter, we introduce a concept called *marginal surplus* to measure every player's contribution to cooperation in TU-games. Based on marginal surplus, we define a new solution for TU-games, the *average-surplus value*, which offers every player a weighted average of the average marginal surpluses to all coalitions including himself. We conclude the chapter by comparing the average-surplus value with some known solutions including the procedural values (Malawski 2013, [74]), the consensus value (Ju et al. 2007, [56]), the Shapley value and the solidarity value.

The procedural values (Malawski 2013, [74]) are implemented by a family of underlying procedures of sharing the marginal contributions to coalitions formed by players joining in random order. Different sharing rules lead to different values, including the Shapley value, the solidarity value, the egalitarian Shapley values (Joosten 1996, [55]) and so on. These procedures are based on marginal contribution, while in a similar way the average-surplus value is defined from the perspective of marginal surplus. In addition to the average-surplus value, the Shapley value and the consensus value are also determined by an underlying procedure of sharing marginal surplus. In the procedure of the Shapley value, every joining player receives his individual worth and all of his marginal surplus. In the procedure of the consensus value, every joining player receives his individual worth and half of his marginal surplus, and the remainder is equally distributed among his predecessors. Compared with the procedure in Section 2.3, the only difference is the proportion of sharing the marginal surplus between the joining player and his predecessors. Similar to the procedural values (Malawski 2013, [74]), we can also define a class of new “procedural" values on the basis of marginal surplus, which are determined by underlying procedures of sharing marginal surplus. Correspondingly, the average-surplus value, the Shapley value and the consensus value are three specific solutions of these “procedural" values.

Next, we make comparisons among these solutions from the point view of axiomatization. The consensus value and all procedural values are efficient, symmetric and additive, which are also satisfied by the average-surplus value. The consensus value is the unique solution satisfying the above three properties and the neutral dummy property, while the average-surplus value replaces the neutral dummy property (Ju et al. 2007, [56]) with the A-null surplus player property. Moreover, the bidding mechanisms to implement the consensus value and the average-surplus value are identical at Stage 1 and Stage 2. The main difference is in the case of the offer being rejected at Stage 3 (see, Ju et al. 2007, [56] for more details).

To conclude, we focus on analyzing how the Shapley value, the solidarity value and the average-surplus value change along with the individual worth. For all $\langle N, v\rangle \in \mathcal{G}^N$, let $\widetilde{v}(\{i\}) = v(\{i\}) + \Delta$ and $\widetilde{v}(S) = v(S)$ for all $S \subseteq N$ and $S \neq \{i\}$, then we have

$$\begin{aligned} AS_i(N,\widetilde{v}) - AS_i(N,v) &= \Delta; \\ Sh_i(N,\widetilde{v}) - Sh_i(N,v) &= \frac{1}{n}\Delta; \\ Sol_i(N,\widetilde{v}) - Sol_i(N,v) &= \frac{1}{n}\Delta. \end{aligned}$$

Obviously, the average-surplus value assigns more to a player who has a higher individual worth than the Shapley value and the solidarity value.

Chapter 3

Characterizations of the EANSC value and CIS value

3.1 Introduction

Consistency is a crucial characteristic of viable and stable solutions in the axiomatic approach to solutions for TU-games. A solution is consistent if it allocates the same payoff to players in the original game as in a modified game. There are two kinds of modified games in the existing literature, the reduced game and the associated game. Reduced game consistency and associated consistency are defined in terms of the reduced game and the associated game, respectively.

Reduced games consider situations where one or more players leave the game, and after an appropriate modification of the game, taking account of the effect of the leaving players on the worths that can be obtained by the remaining players, require the payoffs of the remaining players not to change. The concept of reduced game consistency, firstly proposed by Davis and Maschler (1965, [29]), has been used to characterize various solutions for TU-games, such as the Shapley value (Hart and Mas-Collel 1989, [46]), nucleolus (Snijders 1995, [95]), the efficient, symmetric and linear (ESL) values (Radzik and Driessen 2016, [91], Su, Driessen and Xu 2019, [98]) and so on. More results about reduced game consistency can be found in the survey paper by Driessen (1991, [34]).

In this chapter, which is based on Li et al. (2021, [67, 70]), we focus on associated games. In associated games, the player set does not change, but coalitions revalue their worths by claiming part of the surplus in the game that is left after this coalition and the players outside the coalition get some initial share in the total worth. An advantage of the associated consistency axioms is that no players leave or enter the game, and thus the player set does not change. Hamiache (2001, [42]) firstly introduced the concept of associated consistency to characterize the Shapley value. Subsequently, a matrix approach is applied to associated games to characterize the Shapley value in Xu et al. (2008, [116]) and Hamiache (2010, [43]). Driessen (2010, [35]) generalized Hamiache's associated game and characterized the class of the ESL values by a corresponding associated consistency. Hwang et al. (2006, [50] and 2017, [52]) showed that the EANSC value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to Hwang's associated game). Xu et al. (2015, [118]) gave comparable axiomatizations of the EANSC and the CIS values using associated consistency. Xu et al. (2013, [119]) showed that the CIS value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to the so-called C-individual associated game).

To define an associated game, Xu et al. (2009, [117] and 2013, [119]) assumed that any coalition is formed by its members joining one by one. They adopt "individual self-evaluation" to reevaluate the worths of coalitions. The worth of a coalition in the associated games differs from the initial worth, by taking into account the possible loss of benefits due to the departure of players in the coalition. Here, we introduce an alternative way to reevaluate the worth. Instead of considering the players in the coalition as isolated elements, we consider the players in the coalition as a whole. That is, we adopt "union self-evaluation" to reevaluate the worths of coalitions. In this chapter, under "union self-evaluation" instead of "individual self-evaluation", two alternative definitions of the associated games are constructed, namely the *E-union associated game* and the *C-union associated game*. We continue to develop the works of Xu et al. (2009, [117] and 2013, [119]). Firstly, we introduce the sequences of the E-union associated games and the C-union associated games and explore the convergence of the two sequences and their limit games by the matrix approach. Then, we characterize the EANSC value and the CIS value by associated consistency (with respect to the E-union associated game and the C-union associated game, respectively). Specifically, we show that the EANSC value is the unique solution satisfying E-union associated consistency, continuity, efficiency, symmetry and translation covariance, while the CIS value is the unique solution satisfying C-union associated consistency, continuity, efficiency, symmetry and translation covariance.

Besides axiomatization, we also consider dynamic processes derived from associated games in this chapter. Dynamic processes can be defined that lead the players to a specific solution, starting from an arbitrary Pareto-optimal payoff vector. Stearns (1968, [96]) firstly devised dynamic transfer schemes to implement a payoff vector which always converges to elements of bargaining sets, starting from an arbitrary Pareto-optimal payoff vector. Subsequently, Maschler and Owen (1989, [75]) applied the reduced game introduced by Hart and Mas-Colell (1989, [46]) to a dynamic process which leads to the Shapley value for hyperplane TU-games. Hwang et al. (2005, [53]) adopted Hamiache's associated game (2001, [42]) to provide a dynamic process leading to any solution satisfying both the inessential game property and continuity. Hwang (2015, [51]) also adopted the complement-associated game (2017, [52]) to provide a dynamic transfer scheme and proved the necessary convergence result. In this chapter, we continue and develop the works of Hwang et al. (2005, [53]) and Hwang (2015, [51]), and turn to different associated games, the *individual associated game* and the *union associated game*[1]. We propose two dynamic

[1]Both the Sh-individual associated game (mentioned in Chapter 1) and the C-individual associated game are simply called the individual associated game, and both the E-union associated game and the C-union associated game are simply called the union associated game.

processes on the basis of the individual associated game and the union associated game respectively that lead to the CIS value and EANSC value, starting from an arbitrary efficient payoff vector. This follows from a more general result showing that the dynamic processes can lead to any solution satisfying the inessential game property and continuity. Moreover, we also provide a dynamic transfer scheme that leads to any solution satisfying both the dummy player property and continuity.

The rest of this chapter is organized as follows. In Section 3.2, we define two different versions of the union associated games based on the idea of “union self-evaluation". In Section 3.3, we explore the convergence of the sequences of the union associated games by the matrix approach. In Section 3.4, we characterize the EANSC value and the CIS value by the union associated consistency axioms. In Section 3.5, we propose two dynamic processes on the basis of the individual associated game and the union associated game respectively that lead to the CIS value and EANSC value. Section 3.6 provides all proofs of this chapter. Section 3.7 concludes with a brief summary.

3.2 Union based associated games

In the framework of solution theory for TU-games, associated consistency is an important characteristic of viable and stable solutions. Associated consistency requires that the solution is invariant under the adaptation of the game into its associated game. As mentioned in Chapter 1, Xu et al. (2009, [117] and 2013, [119]) introduced the notion of the “individual associated game" to characterize the Shapley value and the CIS value by using two different associated consistency axioms. We review the two definitions of associated games as follows.

Definition 3.1 (Xu et al. 2009, [117]). Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *Sh-individual associated game* $\langle N, v^*_{\lambda,Sh,I}\rangle$ is defined by, for all $S \subseteq N$,

$$v^*_{\lambda,Sh,I}(S) = v(S) - \lambda \sum_{j\in S}[v(S) - v(S\backslash\{j\}) - SC_j(N, v)].$$

Definition 3.2 (Xu et al. 2013, [119]). Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *C-individual associated game* $\langle N, v^*_{\lambda,C,I}\rangle$ is defined by

$$v^*_{\lambda,C,I}(S) = \begin{cases} v(S) - \lambda \sum_{j\in S}[v(S) - v(S\backslash\{j\}) - v(\{j\})], & \text{if } S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

A common interpretation of the two associated games is as follows. For a given TU-game, coalitions may reevaluate their worths by taking into consideration the coalitions breaking down due to the departure of a player. Both associated games reflect a pessimistic self-evaluation of worths of coalitions. In the process of reevaluating worth, it is assumed that any coalition is formed as its members joining one by one. Thus, it will cause a loss of benefits $v(S)-v(S\backslash\{i\})-v(\{i\})$ according to the Sh-individual associated game (or $v(S) - v(S\backslash\{i\}) - SC_i(N, v)$ according to the C-individual associated game) derived from player i's leaving coalition S. In these associated games, the parameter λ is technical, and it can be interpreted as a percentage of all the possible losses.[2]

As mentioned, the worth of coalition S in the associated games differs from the initial worth, by taking into account the possible loss of benefits due to the departure of players in coalition S. In the associated games above, each coalition S considers players in S as isolated elements. That is, they adopt “individual self-evaluation" to reevaluate the worths of coalitions. The goal of this chapter is to see if we can get similar results if, instead of an “individual self-evaluation” approach, we take a “union self-evaluation” approach, where, instead of adding the individual effects of players in a coalition, we look at the impact when coalitions revealuate their worth as a whole. Similar as in the two associated games above, we reevaluate based on the separable contributions and individual worths, respectively. But now each coalition S considers itself as a whole, and it will suffer a loss of benefits $v(S) - \sum_{i\in S} SC_i(N, v)$ (respectively $v(S) - \sum_{i\in S} v(\{i\})$) due to the departure of players in coalition S. That

[2]In Hamiache’s associated game (Hamiache 2001, [42]), the parameter λ is interpreted as a percentage of all the possible surpluses.

is, we adopt “union self-evaluation" to reevaluate the worths of coalitions. Similar as above, two different versions of such “union associated games” can be defined as follows.

Definition 3.3. Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *E-union associated game* $\langle N, v^*_{\lambda,E,U}\rangle$ is defined by

$$v^*_{\lambda,E,U}(S) = \begin{cases} v(S) - \lambda\left[v(S) - \sum_{j\in S} SC_j(N,v)\right], & \text{if } S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

Definition 3.4. Given $\langle N, v\rangle \in \mathcal{G}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *C-union associated game* $\langle N, v^*_{\lambda,C,U}\rangle$ is defined by,

$$v^*_{\lambda,C,U}(S) = \begin{cases} v(S) - \lambda\left[v(S) - \sum_{j\in S} v(\{j\})\right], & \text{if } S \subsetneq N; \\ v(N), & \text{if } S = N. \end{cases}$$

Remark 3.1. For all $\langle N, v\rangle \in \mathcal{G}^N$ and its E-union associated game $\langle N, v^*_{\lambda,E,U}\rangle$, it holds that $v^*_{\lambda,E,U}(N) = v(N)$, and for all $i \in N$,

$$\begin{aligned} v^*_{\lambda,E,U}(N\backslash\{i\}) =& v(N \setminus \{i\}) - \lambda\left[v(N \setminus \{i\}) + SC_i(N,v) - \sum_{j\in N} SC_j(N,v)\right] \\ =& v(N\backslash\{i\}) - \lambda\left[v(N) - \sum_{j\in N} SC_j(N,v)\right]. \end{aligned}$$

Thus, it is easy to obtain that $EANSC(N, v^*_{\lambda,E,U}) = EANSC(N, v)$.

Remark 3.2. For all $\langle N, v\rangle \in \mathcal{G}^N$ and its C-union associated game $\langle N, v^*_{\lambda,C,U}\rangle$, it holds that $v^*_{\lambda,C,U}(N) = v(N)$ and $v^*_{\lambda,C,U}(\{i\}) = v(\{i\})$ for all $i \in N$. Thus, it is easy to see that $CIS(N, v^*_{\lambda,C,U}) = CIS(N, v)$.

3.3 Matrix approach and associated games

In this section, we consider the sequences of the E-union associated games and the C-union associated games respectively, where, starting with the original game, we take its associated game, the associated game of this associated game, etc. We show that these sequences converge to a special type of games. For all $\langle N, v\rangle \in \mathcal{G}^N$, the sequence of the E-union associated games, $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}^{\infty}_{m=0}$, is defined by $v^{0*}_{\lambda,E,U} = v$, and $v^{(m+1)*}_{\lambda,E,U} = (v^{m*}_{\lambda,E,U})^*_{\lambda,E,U}, m = 0, 1, \cdots$. Similarly, the sequence of the C-union associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}^{\infty}_{m=0}$, is defined by $v^{0*}_{\lambda,C,U} = v$, and $v^{(m+1)*}_{\lambda,C,U} = (v^{m*}_{\lambda,C,U})^*_{\lambda,C,U}, m = 0, 1, \cdots$. Next, we will explore the convergence of the two sequences and their limit games by the matrix approach.

The set $\mathcal{G}^N$ of all n-person TU-games with player set N is identified with the $(2^n - 1)$-dimensional vector space $\mathbb{R}^{2^n-1}$. The components of a (2^n-1)-dimensional vector represent the worths of the (2^n-1) non-empty coalitions in N. A linear solution for TU-games is a linear operator in the TU-games space $\mathcal{G}^N$ that can be represented as a matrix multiplication. Xu et al. (2008, [116]) introduced some concepts of coalitional matrices to analyze linear operators on $\mathcal{G}^N$ that will be used below. A matrix M is row-coalitional (column-coalitional) if the number of rows (columns) is $2^n - 1$ and each row (column) is indexed by a different non-empty coalition $S \subseteq N$. A (2^n-1)-dimensional vector x is row-inessential if $x_S = \sum_{i\in S} x_i$ for all non-empty coalitions $S \subseteq N$, and is almost-inessential if $x_S = \sum_{i\in S} x_i$ for all $S \subsetneq N$, where each component x_S of x is indexed by each non-empty coalition $S \subseteq N$. A $(2^n-1)\times m$ row-coalitional matrix M is row-inessential if the row of M indexed by the non-empty coalition $S \subseteq N$ is the sum of all rows of M indexed by $i \in S$, that is, $M_S = \sum_{i\in S} M_i$ for all non-empty coalitions $S \subseteq N$.

Linear solutions can be written as the product of an $n \times (2^n - 1)$-dimensional matrix M and the vector v representing the TU-game. Specifically, for all $\langle N, v\rangle \in \mathcal{G}^N$, the EANSC value can be rewritten in matrix form as

$$EANSC(N, v) = M^E v,$$

where $M^E = [M^E_{i,S}]_{i\in N, S\subseteq N, S\neq\emptyset}$ is a $n\times(2^n-1)$ column-coalitional matrix, which component $M^E_{i,S}$ is given by

$$M^E_{i,S} = \begin{cases} \frac{1}{n}, & \text{if } S = N; \\ -1+\frac{1}{n}, & \text{if } S = N\backslash\{i\}; \\ \frac{1}{n}, & \text{if } S = N\backslash\{j\},\ j\in N\backslash\{i\}; \\ 0, & \text{otherwise.} \end{cases}$$

Associated games can be expressed as a linear transformation of TU-games, which is written by multiplication of a $(2^n-1)\times(2^n-1)$-dimensional matrix M with the TU-game vector v. Specifically, for all $\langle N, v\rangle \in \mathcal{G}^N$, the E-union associated game $\langle N, v^*_{\lambda,E,U}\rangle$ can be rewritten in matrix form as

$$v^*_{\lambda,E,U} = M^{E,U}\cdot v,$$

where $M^{E,U} = [M^{E,U}_{S,T}]_{S,T\subseteq N, S,T\neq\emptyset}$ is a $(2^n-1)\times(2^n-1)$ row-coalitional matrix, which component $M^{E,U}_{S,T}$ is given by

$$M^{E,U}_{S,T} = \begin{cases} 1-\lambda, & \text{if } T = S \subsetneq N; \\ 1, & \text{if } T = S = N; \\ -\lambda, & \text{if } T = N\backslash\{k\},\ k\in S\subsetneq N; \\ s\lambda, & \text{if } T = N,\ S\subsetneq N; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.5. *For the row-coalitional matrix $M^{E,U}$, the following three statements hold.*

(i) 1 is an eigenvalue of $M^{E,U}$, the dimension of the corresponding eigenspace is equal to n and the corresponding eigenvectors are row-inessential.

(ii) $1-\lambda$ is an eigenvalue of $M^{E,U}$ and the dimension of the corresponding eigenspace is equal to 2^n-n-2.

(iii) $1-n\lambda$ is an eigenvalue of $M^{E,U}$ and the dimension of the corresponding eigenspace is equal to 1.

The proof of Lemma 3.5 and of all other results in this chapter can be found in Section 3.6.

Lemma 3.6 (Xu et al. 2008, [116]). *Let A be a matrix and M be a row-coalitional matrix.*

(i) If M is row-inessential, then the matrix MA is also row-inessential.

(ii) If A is an invertible matrix, then MA is row-inessential if and only if M is row-inessential.

(iii) If M is a row-inessential matrix, then the TU-game $\langle N, Mv\rangle$ is inessential.

Now, we state our first main result on the convergence of the sequence of E-union associated games.

Proposition 3.7. *Let $0 < \lambda < \frac{1}{n}$. Then for all $\langle N, v\rangle \in \mathcal{G}^N$, the sequence of the E-union associated games $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}_{m=0}^{\infty}$ converges, and its limit game $\langle N, \hat{v}\rangle$ is inessential.*

Next, we consider the sequence of C-union associated games. For all $\langle N, v\rangle \in \mathcal{G}^N$, the C-union associated game $\langle N, v^*_{\lambda,C,U}\rangle$ can be rewritten in matrix form as

$$v^*_{\lambda,C,U} = M^{C,U} \cdot v,$$

where $M^{C,U} = [M^{C,U}_{S,T}]_{S,T\subseteq N, S,T\neq\emptyset}$ is a $(2^n-1)\times(2^n-1)$ row-coalitional matrix, and its component $M^{C,U}_{S,T}$ is given by

$$M^{C,U}_{S,T} = \begin{cases} 1-\lambda, & \text{if } T = S \subsetneq N; \\ 1, & \text{if } T = S = N; \\ \lambda, & \text{if } T = \{k\},\ k \in S \subsetneq N; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.8. *Eigenvalues of the row-coalitional matrix $M^{C,U}$ are equal to 1 or $1-\lambda$. Moreover, the eigenspace corresponding to eigenvalue 1 has dimension $(n+1)$ (the only free variables are x_N and x_k, $k \in N$, and every eigenvector is almost-inessential); also, the eigenspace corresponding to eigenvalue $1-\lambda$ has dimension (2^n-n-2) (the only free variables are x_S, $2 \leq s \leq n-1$).*

Lemma 3.9 (Xu et al. 2013, [119]). *Let A be a matrix and M be a row-coalitional matrix.*

(i) If M is almost-inessential, then the matrix MA is also almost-inessential.

(ii) If A is an invertible matrix, then MA is almost-inessential if and only if M is almost-inessential.

(iii) If M is an almost-inessential matrix, then the TU-game $\langle N, Mv\rangle$ is almost-inessential.

Next, we state our result on the convergence of the sequence of C-union associated games.

Proposition 3.10. *Let $0 < \lambda < 1$. Then for all $\langle N, v\rangle \in \mathcal{G}^N$, the sequence of the C-union associated games $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$ converges, and its limit game $\langle N, \bar{v}\rangle$ is almost-inessential.*

Remark 3.3. As mentioned, the convergence of the sequences of the two union associated games and the limit games is revealed by using the matrix approach. An alternative approach to prove convergence of the sequences is as follows. Let us take the sequence of the C-union associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$, as an example. Given $\langle N, v\rangle \in \mathcal{G}^N$ and $S \subsetneq N$, the term $v^{m*}_{\lambda,C,U}(S)$ can be expressed as a linear combination of $v(S)$ and $v(\{i\})$ for all $i \in S$, that is,

$$v^{m*}_{\lambda,C,U}(S) = \alpha_m v(S) + \beta_m \sum_{j\in S} v(\{j\}),$$

where $\alpha_m \in \mathbb{R}$ and $\beta_m \in \mathbb{R}$. We can prove that the coefficients α_m and β_m satisfy the following recursive relationships and determine the coefficients α_m and β_m. We can obtain the following three facts: (a) the coefficients α_m and β_m satisfy the recursive relationships, $\alpha_{m+1} = (1-\lambda)\alpha_m$ and $\beta_{m+1} = (1-\lambda)\beta_m + \lambda$; (b) the coefficients α_m and β_m are given by $\alpha_m = (1-\lambda)^m$ and $\beta_m = 1-(1-\lambda)^m$ for all $m \geq 1$; (c) the sequence of the C-union associated games, $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$, converges and its limit game is almost-inessential. These results are coincident with the conclusions in Proposition 3.10.

3.4 Axiomatizations of the EANSC value and the CIS value

Hwang et al. (2006, [50] and 2017, [52]) showed that the EANSC value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to Hwang's associated game). Xu et al. (2013, [119]) showed that the CIS value is the unique solution satisfying continuity, efficiency, symmetry, translation covariance and associated consistency (with respect to the C-individual associated game). In Section 3.2, we introduced two new associated games, the E-union associated game and the C-union associated game, that are based on union self-evaluation. In this section, we will characterize the EANSC value and the CIS value by associated consistency with respect to the E-union associated game and the C-union associated game, respectively.

Formally, these two new associated consistency axioms are given as follows. Let $\lambda \in [0, 1]$.

- **E-union associated consistency for** λ**.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v) = \varphi(N, v^*_{\lambda,E,I})$.
- **C-union associated consistency for** λ For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v) = \varphi(N, v^*_{\lambda,C,I})$.

Associated consistency shows stability with respect to a specific way that coalitions reevaluate their worth when players in the coalition stop cooperation. If a solution violates associated consistency, then players might not respect the original compromise but revise the payoff distribution. E-union associated consistency, respectively, C-union associated consistency says that a solution gives the same payments to players in the original game as it does to players of the E-union associated game, respectively, the C-union associated game.

Next, we characterize the EANSC value and the CIS value by E-union associated consistency and C-union associated consistency, respectively.

Theorem 3.11. *Let $0 < \lambda < \frac{1}{n}$. A solution φ on $\mathcal{G}^N$ satisfies E-union associated consistency for λ, continuity and the inessential game property if and only if φ is the EANSC value.*

An alternative axiomatization is provided by replacing the inessential game property with efficiency, symmetry and translation covariance. It is well-known that, if a solution satisfies efficiency, symmetry and translation covariance, then it satisfies the inessential game property. Thus, we can draw the following conclusion directly.

Corollary 3.12. *Let $0 < \lambda < \frac{1}{n}$. A solution φ on $\mathcal{G}^N$ satisfies E-union associated consistency for λ, continuity, efficiency, symmetry and translation covariance if and only if φ is the EANSC value.*

Next, we give an axiomatization of the CIS value using C-union associated consistency. As mentioned in Section 3.3, the sequence of the E-union associated games converges to an inessential game, while the sequence of the C-union associated games converges to an almost-inessential game. Replacing in Theorem 3.11, E-union associated consistency with C-union associated consistency, and replacing the inessential game property with the almost inessential game property and efficiency, characterizes the CIS-value.

Theorem 3.13. *Let $0 < \lambda < 1$. A solution φ on $\mathcal{G}^N$ satisfies C-union associated consistency for λ, continuity, the almost inessential game property and efficiency if and only if φ is the CIS value.*

Similar as Corollary 3.12 for the EANSC value, since efficiency, symmetry and translation covariance of a solution, implies that it satisfies the almost inessential game property, another axiomatization of the CIS value can be obtained by replacing the almost inessential game property with efficiency, symmetry and translation covariance.

Corollary 3.14. *Let $0 < \lambda < 1$. A solution φ on $\mathcal{G}^N$ satisfies C-union associated consistency for λ, continuity, efficiency, symmetry and translation covariance if and only if φ is the CIS value.*

Remark 3.4. Associated consistency is a requirement of “stability" in the sense that it expresses how payoffs of players are invariant if the worth of coalitions are reevaluated because (the expectation that) some players might not cooperate. The EANSC value and the CIS value have been characterized by different associated consistency properties before in, e.g. Hwang et al. (2006, [50] and 2017, [52]) and Xu et al. (2013, [119]). Different associated games take a different angle in ‘revaluating’ the worth of coalitions. Some associated games in the literature (see, Hamiache 2001, [42] and Hwang et al. 2017, [54]) focus on reevaluating the worth of a coalition by considering what the coalition expects from the surplus it can obtain from cooperation with players outside the coalition. However, in the E-union associated game and the C-union associated game considered in this chapter, the worths of coalitions are reevaluated in view of expecting that some players inside the coalition might not fully contribute. Besides the difference between focussing on gains or losses, the associated games in this chapter take a union self-evaluation approach, while other associated games consider individual self-evaluation (Xu et al. 2009, [117] and Xu et al. 2009, [119]).

We also want to remark that the introduction of union-associated consistency greatly simplifies the proof of the axiomatizations of the EANSC value and CIS value, as can be seen in Section 3.6.

3.5 Dynamic transfer schemes derived from associated games

Hwang et al. (2005, [53]) adopted Hamiache’s associated game (Hamiache 2001, [42]) to provide a dynamic process leading to the Shapley value. More precisely, they proved that this dynamic process converges to any solution satisfying both the inessential game property and continuity, depending on the definition of the sequence of games.

Definition 3.15 (Hwang et al. 2005, [53]). Given $\langle N, v\rangle \in \mathcal{G}^N$, $x \in \mathbb{R}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *x-Hamiache’s associated game*

$\langle N, v^*_{\lambda,x,H}\rangle$ is defined by

$$v^*_{\lambda,x,H}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) + \lambda \sum_{j\in N\setminus S}[v(S\cup\{j\}) - v(S) - x_j], & \text{otherwise.} \end{cases}$$

The x-Hamiache's associated game is constructed by replacing the individual worths vector "$v(\{j\})$" in Hamiache's associated game by the payoff "x_j". The sequence of associated games, $\{\langle N, v^{m*}_{\lambda,x,H}\rangle\}_{m=0}^{\infty}$, is inductively defined by $v^{0*}_{\lambda,x,H} = v$ and $v^{(m+1)*}_{\lambda,x,H} = (v^{m*}_{\lambda,x,H})^*_{\lambda,x,H}, m = 0, 1, \cdots$. Based on the x-Hamiache's associated game, Hwang et al. (2005, [53]) introduced a dynamic process that converges to any solution satisfying both the inessential game property and continuity. Given $\langle N, v\rangle \in \mathcal{G}^N$, let the set of efficient payoff vectors $X(N, v)$ be given by $X(N, v) = \{x \in \mathbb{R}^N \mid \sum_{k\in N} x_k = v(N)\}$.

Theorem 3.16 (Hwang et al. 2005, [53]). *Let $0 < \lambda < \frac{2}{n}$. Given $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in X(N, v)$, the dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and*

$$x^m = x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x,H}) - \varphi(N, v^{m*}_{\lambda,x,H})],\ m \geq 1,$$

converges to $\varphi(N, v)$ if the solution φ satisfies the inessential game property and continuity.

3.5.1 Process based on the individual associated game

In this subsection, we continue and develop the work of Hwang et al. (2005, [53]), and turn to the individual associated game. Specifically, we propose a dynamic transfer scheme on the basis of the individual associated game to lead to any solution satisfying both the inessential game property and continuity.

Definition 3.17. Given $\langle N, v\rangle \in \mathcal{G}^N$, $x \in \mathbb{R}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the *x-individual associated game* $\langle N, v^*_{\lambda,x,I}\rangle$ is defined by

$$v^*_{\lambda,x,I}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) - \lambda \sum_{j\in S}[v(S) - v(S\setminus\{j\}) - x_j], & \text{otherwise.} \end{cases} \quad (3.1)$$

Note that the x-individual associated game is constructed by replacing "$SC_j(N,v)$" (or, $v(\{j\})$) in the Sh-individual (or, C-individual) associated game by "x_j". Then, the sequence of the x-individual associated games, $\{\langle N, v^{m*}_{\lambda,x,I}\rangle\}_{m=0}^{\infty}$, is inductively defined by $v^{0*}_{\lambda,x,I} = v$ and $v^{(m+1)*}_{\lambda,x,I} = (v^{m*}_{\lambda,x,I})^*_{\lambda,x,I}, m = 0,1,\cdots$. According to Eq.(3.1), the general representation of the m-fold x-individual associated game $\langle N, v^{m*}_{\lambda,x,I}\rangle$ can be written as

$$v^{m*}_{\lambda,x,I}(S) = \sum_{T\subseteq S} a^s_m(t)v(T) + b_m \sum_{j\in S} x_j, \tag{3.2}$$

for all $S \subseteq N$, where $a^s_m(t)$ and b_m are certain coefficients with respect to λ. It is straightforward to obtain that $b_0 = 0$, $a^s_0(s) = 1$ and $a^s_0(t) = 0$ for each $1 \le t < s$.

We now try to identify these coefficients.

Lemma 3.18. *The coefficients $a^s_m(t)$ and b_m of the representation (3.2) satisfy the following recursive formulas,*

(i) $b_{m+1} = (1-\lambda)b_m + \lambda$ *for each* $m \ge 0$.

(ii) $a^s_{m+1}(s) = (1-s\lambda)a^s_m(s)$ *for each* $s \ge 1$ *and* $m \ge 0$.

(iii) $a^s_{m+1}(t) = (1-s\lambda)a^s_m(t) + (s-t)\lambda a^{s-1}_m(t)$ *for each* $s \ge 1$, $1 \le t < s$ *and* $m \ge 0$.

(iv) $a^s_{m+1}(t) = (1-t\lambda)a^s_m(t) + (s-t)\lambda a^s_m(t+1)$ *for each* $s \ge 1$, $1 \le t < s$ *and* $m \ge 0$.

(v) $a^s_m(s-k) = \sum_{d=0}^{k}(-1)^{k-d}\left(\begin{array}{c} k \\ d \end{array}\right) a^{s-d}_m(s-d)$ *for each* $s \ge 1$, $0 \le k < s$ *and* $m \ge 0$.

Lemma 3.19. *The coefficients $a^s_m(t)$ and b_m of the representation (3.2) satisfy the following recursive formulas,*

(i) $b_m = 1-(1-\lambda)^m$ *for each* $m \ge 1$.

(ii) $a^s_m(s-k) = \sum_{d=0}^{k}(-1)^{k-d}\begin{pmatrix} k \\ d \end{pmatrix}[1-(s-d)\lambda]^m$ *for each* $s \geq 1$, $0 \leq k < s$ *and* $m \geq 1$.

Now, we show the convergence of the sequence of the x-individual associated games $\{\langle N, v^{m*}_{\lambda,x,I}\rangle\}_{m=0}^{\infty}$.

Lemma 3.20. *For each* $0 < \lambda < \frac{2}{n}$*, the sequence of the* x*-individual associated games,* $\{\langle N, v^{m*}_{\lambda,x,I}\rangle\}_{m=0}^{\infty}$*, converges to the limit game* $\langle N, \bar{v}_x\rangle$ *which is given by* $\bar{v}_x(S) = \sum_{j\in S} x_j$ *for all* $S \subseteq N$*.*

The sequence of the x-individual associated games, $\{\langle N, v^{m*}_{\lambda,x,I}\rangle\}_{m=0}^{\infty}$, denotes an iterative process of reevaluating worths of coalitions. Every coalition S updates its worth by assigning to the coalition its own worth minus a certain loss of benefit derived from the fact that any member $j \in S$ secedes from the coalition S. Lemma 3.20 shows that the sequence of the x-individual associated games converges to an inessential game that is described by the proposed payoff vector x. Next, we introduce a dynamic process that leads to any solution satisfying the inessential game property and continuity. Let φ be a solution satisfying both the inessential game property and continuity. Given $\langle N, v\rangle \in \mathcal{G}^N$, $x \in X(N,v)$ and $0 < \lambda < \frac{2}{n}$, we define a dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and

$$x^m = x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x,I}) - \varphi(N, v^{m*}_{\lambda,x,I})],\ m \geq 1. \tag{3.3}$$

This is similar to a dynamic process introduced by Hwang et al. (2015, [51] and 2005, [53]). They also proposed a dynamic sequence to any solution satisfying both the inessential game property and continuity on the basis of Hamiache's associated game (Hamiache 2001, [42]) and the complement-associated game of Hwang et al. (2017, [52]), respectively. In this subsection, we use the individual associated game to define a dynamic sequence. The dynamic sequence $\{x^t\}_{t=0}^{\infty}$ is like a reappraised process that leads to any solution satisfying both the inessential game property and continuity, starting from an arbitrary payoff vector $x \in X(N,v)$. Imagine a situation where there is an arbitrator and every player obeys the suggestion of the arbitrator. The arbitrator will lead the players to a reasonable allocation by using a fair rule.

Theorem 3.21. *Let* $0 < \lambda < \frac{2}{n}$*. Given* $\langle N, v\rangle \in \mathcal{G}^N$ *and* $x \in X(N, v)$*, the dynamic sequence* $\{x^m\}_{m=0}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [\varphi(N, v_{\lambda,x,I}^{(m-1)*}) - \varphi(N, v_{\lambda,x,I}^{m*})],\ m \geq 1,$$

converges to $\varphi(N, v)$ *if the solution* φ *satisfies the inessential game property and continuity.*

If a solution φ satisfies the dummy player property, then it satisfies the inessential game property. Thus, we can obtain the following corollary by Theorem 3.21.

Corollary 3.22. *Let* $0 < \lambda < \frac{2}{n}$*. Given* $\langle N, v\rangle \in \mathcal{G}^N$ *and* $x \in X(N, v)$*, the dynamic sequence* $\{x^m\}_{m=0}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [\varphi(N, v_{\lambda,x,I}^{(m-1)*}) - \varphi(N, v_{\lambda,x,I}^{m*})],\ m \geq 1,$$

converges to $\varphi(N, v)$ *if the solution* φ *satisfies the dummy player property and continuity.*

The following two corollaries follow from Theorem 3.21. They state that the EANSC value and the CIS value can be implemented by a dynamic process as above respectively, starting from an arbitrary efficient payoff vector.

Corollary 3.23. *Let* $0 < \lambda < \frac{2}{n}$*. Given* $\langle N, v\rangle \in \mathcal{G}^N$ *and* $x \in X(N, v)$*, the dynamic sequence* $\{x^m\}_{t=m}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [EANSC(N, v_{\lambda,x,I}^{(m-1)*}) - EANSC(N, v_{\lambda,x,I}^{m*})],\ m \geq 1,$$

converges to the EANSC value payoff vector $EANSC(N, v)$*.*

Corollary 3.24. *Let* $0 < \lambda < \frac{2}{n}$*. Given* $\langle N, v\rangle \in \mathcal{G}^N$ *and* $x \in X(N, v)$*, the dynamic sequence* $\{x^m\}_{t=m}^{\infty}$ *with* $x^0 = x$ *and*

$$x^m = x^{m-1} + [CIS(N, v^{(m-1)*}_{\lambda,x,I}) - CIS(N, v^{m*}_{\lambda,x,I})],\ m \geq 1,$$

converges to the CIS value payoff vector $CIS(N, v)$*.*

3.5.2 Process based on the union associated game

In this subsection, we turn to the union associated game and propose a dynamic process on the basis of the union associated game to lead to any solution satisfying both the inessential game property and continuity.

Definition 3.25. Given $\langle N, v\rangle \in \mathcal{G}^N$, $x \in \mathbb{R}^N$ and a real number λ, $0 \leq \lambda \leq 1$, the x*-union associated game* $\langle N, v^*_{\lambda,x,U}\rangle$ is defined by

$$v^*_{\lambda,x,U}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) - \lambda[v(S) - x(S)], & \text{otherwise.} \end{cases} \tag{3.4}$$

The x-union associated game is given by replacing "$\sum_{j\in S} SC_j(N, v)$" (or, $\sum_{j\in S} v(\{j\})$) in the E-union (or, C-union) associated game by the payoff "$x(S)$". The sequence of the x-union associated games, $\{\langle N, v^{m*}_{\lambda,x,U}\rangle\}_{m=0}^{\infty}$, is inductively defined by $v^{0*}_{\lambda,x,U} = v$, and $v^{(m+1)*}_{\lambda,x,U} = (v^{m*}_{\lambda,x,U})^*_{\lambda,x,U}, m = 0, 1, \cdots$. In view of the representation (3.4) of the x-union associated game, the general representation of the m-fold x-union associated game $\langle N, v^{m*}_{\lambda,x,U}\rangle$ can be written as

$$v^{m*}_{\lambda,x,U}(S) = c^s_m v(S) + d^s_m x(S) \tag{3.5}$$

for all $S \subseteq N$, where c^s_m and d^s_m are certain coefficients with respect to λ.

The next lemma identifies these coefficients.

Lemma 3.26. *The coefficients* c^s_m *and* d^s_m *in expression* (3.5) *of the* m*-fold* x*-union associated game* $\langle N, v^{m*}_{\lambda,x,U}\rangle$ *satisfy the following recursive formulas:*

$$c^s_m = (1-\lambda)^m \textit{ and } d^s_m = 1 - (1-\lambda)^m.$$

The next lemma shows that, updating the worths of coalitions by assigning to every coalition S its worth minus a fraction of its excess according to the proposed payoff vector x, converges to an inessential game that is described by the payoff vector x.

Lemma 3.27. *For all $\langle N, v\rangle \in \mathcal{G}^N$, $x \in X(N, v)$ and $0 < \lambda < 1$, the sequence of the x-union associated games $\{\langle N, v^{m*}_{\lambda,x,U}\rangle\}_{m=0}^{\infty}$ converges to the limit game $\langle N, \hat{v}_x\rangle$ which is given by $\hat{v}_x(S) = x(S)$ for all $S \subseteq N$.*

Let φ be a solution satisfying both the inessential game property and continuity. Given $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in X(N, v)$, we define a dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and

$$x^m = x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x,U}) - \varphi(N, v^{m*}_{\lambda,x,U})],\ m \geq 1. \tag{3.6}$$

Similar as before, we show that the dynamic sequence described by expression (3.6) converges to any solution satisfying both the inessential game property and continuity.

Theorem 3.28. *Let $0 < \lambda < 1$. Given $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in X(N, v)$, the dynamic sequence $\{x^m\}_{m=0}^{\infty}$ with $x^0 = x$ and x^m described by expression (3.6), converges to $\varphi(N, v)$ if the solution φ satisfies the inessential game property and continuity.*

3.6 Proofs

Proof of Lemma 3.5. (i) Let I be the $(2^n - 1) \times (2^n - 1)$ identity matrix. It is easy to verify that the last row of matrix $(M^{E,U} - I)$ is the zero vector. Thus, 1 is an eigenvalue of $M^{E,U}$. Let x be the corresponding eigenvector of eigenvalue 1 indexed by non-empty coalition $S \subseteq N$. Since $(M^{E,U} - I)x = \mathbf{0}$ and $0 < \lambda < 1$, it implies that

$$-x_S - \sum_{k\in S} x_{N\setminus\{k\}} + s x_N = 0, \tag{3.7}$$

for all $S \subsetneq N$. If $s = 1$, then we have $x_N = x_k + x_{N\setminus\{k\}}$ for all $k \in N$. Together with Eq.(3.7), we can obtain that $x_S = \sum_{k\in S} x_k$

for all $S \subseteq N$, $S \neq \emptyset$. Therefore, any eigenvector x corresponding to eigenvalue 1 is row-inessential and the dimension of the corresponding eigenspace is equal to n.

(ii) Let $A = M^{E,U} - (1-\lambda)I$. Denote the columns of matrix A by A_T, $T \subseteq N$ and $T \neq \emptyset$. It is easy to verify that all columns A_T with $1 \leq t \leq n-2$ are zero vectors. Thus, $1-\lambda$ is an eigenvalue of $M^{E,U}$. Let x be the corresponding eigenvector of eigenvalue $1-\lambda$ indexed by non-empty coalition $S \subseteq N$. Since $Ax = 0$ and $0 < \lambda < 1$, we have

$$sx_N - \sum_{k \in S} x_{N\setminus\{k\}} = 0, \tag{3.8}$$

for all $S \subsetneq N$. If $S = N$, we have $x_N = 0$. Together with Eq.(3.8), we can obtain that $\sum_{k\in S} x_{N\setminus\{k\}} = 0$ for all $S \subsetneq N$. Then, we have $x_N = 0$ and $x_{N\setminus\{k\}} = 0$ for all $k \in N$. Therefore, the variables x_S with $1 \leq s \leq n-2$ are free variables and the dimension of the corresponding eigenspace is equal to $2^n - n - 2$.

(iii) Let $x = [x_S]_{S\subseteq N, S\neq\emptyset}$ be a (2^n-1)-dimensional vector with $x_N = 0$ and $x_S = s$ for all $S \subsetneq N$. Denote the rows of matrix $M^{E,U}$ by $M^{E,U}_S$, $S \subseteq N, S \neq \emptyset$. Then, we have $M^{E,U}_N x = 0$, and for all $S \subsetneq N$,

$$M^{E,U}_S x = (1-\lambda)s - \lambda(n-1)s = (1-n\lambda)s.$$

Thus, we have $M^{E,U}x = (1-n\lambda)x$, and $1-n\lambda$ is an eigenvalue of $M^{E,U}$. Suppose the multiplicities of the eigenvalue $1-n\lambda$ equals to m. Then, we have

$$1 \leq m \leq 2^n - 1 - n - (2^n - n - 2) = 1,$$

which implies that $m = 1$. Therefore, $1, 1-\lambda, 1-n\lambda$ are all eigenvalues of $M^{E,U}$ since the sum of their dimensions of the corresponding eigenspace equals to $2^n - 1$, and $M^{E,U}$ is diagonalizable.

□

Proof of Proposition 3.7. By Lemma 3.5, the matrix $M^{E,U}$ is diagonalizable and $M^{E,U} = PD_\lambda P^{-1}$, where $D_\lambda = diag(1,...,1,1-\lambda,...,1-\lambda,1-n\lambda)$ and P consists of eigenvectors of $M^{E,U}$ corresponding to eigenvalues 1, $1-\lambda$ and $1-n\lambda$. Since $0 < \lambda < \frac{1}{n}$, then we have

$$\lim_{k\to\infty}(M^{E,U})^k = \lim_{k\to\infty} P(D_\lambda)^k P^{-1} = PDP^{-1},$$

where $D = diag(1,...,1,0,...,0)$. Then, we have $PD = [x^1, x^2, ..., x^n, \mathbf{0}, ..., \mathbf{0}]$, where the column vectors x^i, $i = 1,...,n$, are the corresponding eigenvectors of eigenvalue 1. Since x^i, $i = 1,...,n$, are row-inessential by Lemma 3.5, PD is also row-inessential. Thus, by Lemma 3.6, PDP^{-1} is also row-inessential, and the TU-game $\langle N, PDP^{-1}v\rangle$ is inessential. Since $\hat{v} = \lim_{k\to\infty}(M^{E,U})^k \cdot v = PDP^{-1}v$, the limit game $\langle N, \hat{v}\rangle$ is inessential. $\square$

Proof of Lemma 3.8. Let I be the $(2^n-1)\times(2^n-1)$ identity matrix. It is easy to verify that the last row of matrix $(M^{C,U} - I)$ is the zero vector. Thus, 1 is an eigenvalue of $M^{C,U}$. Let x be the eigenvector corresponding to eigenvalue 1 indexed by non-empty coalition $S \subseteq N$. Since $(M^{C,U} - I)x = \mathbf{0}$ and $0 < \lambda < 1$, this implies that $x_S = \sum_{k\in S} x_k$ for all $S \subsetneq N$. Thus, the only free variables are x_N and x_k, $k \in N$. Therefore, any eigenvector x corresponding to eigenvalue 1 is almost-inessential and the dimension of the corresponding eigenspace is equal to $n+1$.

Let $A = M^{C,U} - (1-\lambda)I$. Denote the columns of matrix A by A_T, $T \subseteq N, T \neq \emptyset$. It is easy to verify that all columns A_T with $2 \le t \le n-1$ are zero vectors. Thus, $1-\lambda$ is an eigenvalue of $M^{C,U}$. Let x be the corresponding eigenvector of eigenvalue $1-\lambda$ indexed by non-empty coalition $S \subseteq N$. Since $Ax = 0$ and $0 < \lambda < 1$, we have $x_N = 0$ and $x_k = 0$ for all $k \in N$. Therefore, the variables x_S with $2 \le s \le n-1$ are free variables and the dimension of the corresponding eigenspace is equal to $2^n - n - 2$. $\square$

Proof of Proposition 3.10. By Lemma 3.8, the matrix $M^{C,U}$ is diagonalizable and $M^{C,U} = PD_\lambda P^{-1}$, where $D_\lambda = diag(1,...,1,1-\lambda,...,1-\lambda)$ and P consists of eigenvectors of $M^{C,U}$ corresponding to eigenvalues 1 and

$1-\lambda$. Since $0<\lambda<1$, then we have

$$\lim_{k\to\infty}(M^{C,U})^k = \lim_{k\to\infty} P(D_\lambda)^k P^{-1} = PDP^{-1},$$

where $D = diag(1,...,1,0,...,0)$. Then, we have $PD = [x^1,...,x^{n+1},\mathbf{0},...,\mathbf{0}]$, where the column vectors x^i, $i=1,...,n,n+1$, are the eigenvectors corresponding to eigenvalue 1. Since x^i, $i=1,...,n,n+1$, are almost-inessential by Lemma 3.8, then PD is also almost-inessential. Thus, by Lemma 3.9, PDP^{-1} is also almost-inessential, and the TU-game $\langle N, PDP^{-1}v\rangle$ is almost-inessential. Since $\bar{v} = \lim_{k\to\infty}(M^{C,U})^k\cdot v = PDP^{-1}v$, the limit game $\langle N,\bar{v}\rangle$ is almost-inessential. □

Proof of Theorem 3.11. It is straightforward to verify that the EANSC value satisfies continuity and the inessential game property. E-union associated consistency follows from Remark 3.1. It is left to show the uniqueness.

Suppose that a solution φ on $\mathcal{G}^N$ satisfies E-union associated consistency, continuity and the inessential game property. For all $\langle N,v\rangle \in \mathcal{G}^N$, by Proposition 3.7, the sequence of repeated E-union associated games $\{\langle N, v^{m*}_{\lambda,E,U}\rangle\}_{m=0}^{\infty}$ converges to an inessential game $\langle N,\hat{v}\rangle$. By E-union associated consistency and continuity, we have

$$\varphi(N,v) = \varphi(N, v^{1*}_{\lambda,E,U}) = \varphi(N, v^{2*}_{\lambda,E,U}) = \cdots = \varphi(N,\hat{v}).$$

By the inessential game property, it holds that $\varphi_i(N,\hat{v}) = \hat{v}(\{i\})$ for all $i\in N$. From this, φ is uniquely determined by these three axioms. Therefore, $\varphi(N,v) = EANSC(N,v)$. □

Proof of Theorem 3.13. It is straightforward to verify that the CIS value satisfies continuity, the almost inessential game property and efficiency. C-union associated consistency follows from Remark 3.2. It is left to show the uniqueness.

Suppose that a solution φ on $\mathcal{G}^N$ satisfies C-union associated consistency, continuity, the almost inessential game property and efficiency. For all $\langle N,v\rangle \in \mathcal{G}^N$, by Proposition 3.10, the sequence of repeated C-union associated games $\{\langle N, v^{m*}_{\lambda,C,U}\rangle\}_{m=0}^{\infty}$ converges to an almost inessential game

$\langle N, \bar{v}\rangle$. By C-union associated consistency and continuity, we have

$$\varphi(N, v) = \varphi(N, v^{1*}_{\lambda,C,U}) = \varphi(N, v^{2*}_{\lambda,C,U}) = \cdots = \varphi(N, \bar{v}).$$

By the almost inessential game property and efficiency, it holds that $\varphi_i(N, \bar{v}) = \bar{v}(\{i\}) + \frac{1}{n}[\bar{v}(N) - \sum_{j\in N} \bar{v}(\{j\})]$ for all $i \in N$. From this, φ is uniquely determined by these four axioms. Therefore, $\varphi(N, v) = CIS(N, v)$. □

Proof of Lemma 3.18. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in \mathbb{R}^N$, by Eq.(3.2), the $(m+1)$-fold x-individual associated game $\langle N, v^{(m+1)*}_{\lambda,x,I}\rangle$ is given by

$$v^{(m+1)*}_{\lambda,x,I}(S) = \sum_{T\subseteq S} a^s_{m+1}(t)v(T) + b_{m+1}\sum_{j\in S} x_j, \tag{3.9}$$

for all $S \subseteq N$ and $m \geq 0$. On the one hand, combining Eq.(3.1) and Eq.(3.2), we have

$$\begin{aligned}
&v^{(m+1)*}_{\lambda,x,I}(S) = (v^{m*}_{\lambda,x,I})^*_{\lambda,x,I}(S)\\
=&(1 - s\lambda)v^{m*}_{\lambda,x,I}(S) + \lambda\sum_{j\in S} v^{m*}_{\lambda,x,I}(S\backslash\{j\}) + \lambda\sum_{j\in S} x_j\\
=&(1 - s\lambda)\left[\sum_{T\subseteq S} a^s_m(t)v(T) + b_m\sum_{j\in S} x_j\right]\\
&+ \lambda\sum_{j\in S}\left[\sum_{T\subseteq S\backslash\{j\}} a^{s-1}_m(t)v(T) + b_m\sum_{k\in S\backslash\{j\}} x_k\right] + \lambda\sum_{j\in S} x_j\\
=&(1 - s\lambda)a^s_m(s)v(S) + (1 - s\lambda)\sum_{T\subsetneq S} a^s_m(t)v(T) + (1 - s\lambda)b_m\sum_{j\in S} x_j\\
&+ \lambda\sum_{j\in S}\sum_{T\subseteq S\backslash\{j\}} a^{s-1}_m(t)v(T) + \lambda\sum_{j\in S} b_m\sum_{k\in S\backslash\{j\}} x_k + \lambda\sum_{j\in S} x_j\\
=&(1 - s\lambda)a^s_m(s)v(S) + \sum_{T\subsetneq S}\left[(1 - s\lambda)a^s_m(t) + (s - t)\lambda a^{s-1}_m(t)\right]v(T)\\
&+ [(1 - \lambda)b_m + \lambda]\sum_{j\in S} x_j,
\end{aligned} \tag{3.10}$$

for all $S \subseteq N$ and $m \geq 0$. On the other hand, combining Eq.(3.1) and Eq.(3.2), we have

$$\begin{aligned}
&v_{\lambda,x,I}^{(m+1)*}(S) = (v_{\lambda,x,I}^{*})_{\lambda,x,I}^{m*}(S) \\
=& \sum_{T\subseteq S} a_m^s(t) v_{\lambda,x,I}^{*}(T) + b_m \sum_{j\in S} x_j \\
=& \sum_{T\subseteq S} a_m^s(t) \left[(1-t\lambda)v(T) + \lambda \sum_{j\in T} v(T\backslash\{j\}) + \lambda \sum_{j\in T} x_j \right] + b_m \sum_{j\in S} x_j \\
=& \sum_{T\subseteq S} (1-t\lambda) a_m^s(t) v(T) + \lambda \sum_{T\subseteq S} a_m^s(t) \sum_{j\in T} v(T\backslash\{j\}) \\
&+ \lambda \sum_{T\subseteq S} a_m^s(t) \sum_{j\in T} x_j + b_m \sum_{j\in S} x_j \\
=&(1-s\lambda) a_m^s(s) v(S) + \sum_{T\subsetneq S} \left[(1-t\lambda) a_m^s(t) + (s-t)\lambda a_m^s(t+1) \right] v(T) \\
&+ \left[b_m + \lambda \sum_{t=1}^{s} \binom{s-1}{t-1} a_m^s(t) \right] \sum_{j\in S} x_j, \qquad (3.11)
\end{aligned}$$

for all $S \subseteq N$ and $m \geq 0$.

Comparing the coefficients obtained by Eq.(3.9)-Eq.(3.11), we have

$$\begin{aligned}
b_{m+1} =&(1-\lambda) b_m + \lambda, \\
a_{m+1}^s(s) =&(1-s\lambda) a_m^s(s), \\
a_{m+1}^s(t) =&(1-s\lambda) a_m^s(t) + (s-t)\lambda a_m^{s-1}(t), \\
a_{m+1}^s(t) =&(1-t\lambda) a_m^s(t) + (s-t)\lambda a_m^s(t+1),
\end{aligned}$$

for each $s \geq 1$, $1 \leq t < s$ and $m \geq 0$. Then, (i)-(iv) of this lemma hold.

Next, we will show (v) of this lemma, that is, for each $s \geq 1$, $0 \leq k < s$ and $m \geq 1$,

$$a_m^s(s-k) = \sum_{d=0}^{k} (-1)^{k-d} \binom{k}{d} a_m^{s-d}(s-d). \qquad (3.12)$$

The proof proceeds by induction on the number $k = s-(s-k)$, where $s \geq 1$

and $0 \leq k < s$. For $k = 0$, it reduces to the trivial equality $a_m^s(s) = a_m^s(s)$ for all $s \geq 1$. By (iii) and (iv) of this lemma, we have

$$(1 - s\lambda)a_m^s(t) + (s-t)\lambda a_m^{s-1}(t) = (1 - t\lambda)a_m^s(t) + (s-t)\lambda a_m^s(t+1).$$

Then, it holds that $a_m^s(t) = a_m^{s-1}(t) - a_m^s(t+1)$, specifically $a_m^s(s-1) = a_m^{s-1}(s-1) - a_m^s(s)$. Thus, Eq.(3.12) holds for $k = 1$ and all $s > 1$. Suppose that Eq.(3.12) holds on $k-1$ and all $s > k-1$. Then, we have

$$\begin{aligned} a_m^s(s-(k-1)) &= \sum_{d=0}^{k-1} (-1)^{k-d-1} \binom{k-1}{d} a_m^{s-d}(s-d), \\ a_m^{s-1}(s-1-(k-1)) &= \sum_{d=0}^{k-1} (-1)^{k-d-1} \binom{k-1}{d} a_m^{s-d-1}(s-d-1). \end{aligned}$$

Thus, for any $k > 1$, we have

$$\begin{aligned} & a_m^s(s-k) = a_m^{s-1}(s-1-(k-1)) - a_m^s(s-(k-1)) \\ = & \sum_{d=0}^{k-1} (-1)^{k-d-1} \binom{k-1}{d} a_m^{s-d-1}(s-d-1) \\ & - \sum_{d=0}^{k-1} (-1)^{k-d-1} \binom{k-1}{d} a_m^{s-d}(s-d) \\ = & \sum_{d=1}^{k} (-1)^{k-d} \binom{k-1}{d-1} a_m^{s-d}(s-d) + \sum_{d=0}^{k-1} (-1)^{k-d} \binom{k-1}{d} a_m^{s-d}(s-d) \\ = & \sum_{d=0}^{k} (-1)^{k-d} \binom{k}{d} a_m^{s-d}(s-d), \end{aligned}$$

where the last equality is due to the fact that $\binom{k-1}{d-1} + \binom{k-1}{d} = \binom{k}{d}$ for each $1 \leq d \leq k-1$. □

Proof of Lemma 3.19. (i) By (i) of Lemma 3.18, it holds that $b_m = (1-\lambda)b_{m-1} + \lambda$. Then, we have $b_m - 1 = (1-\lambda)(b_{m-1} - 1)$, and $\frac{b_m - 1}{b_{m-1} - 1} =$

$1-\lambda$. Obviously, the sequence $\{b_m - 1\}_{m=0}^{\infty}$ is a geometric progression, and the common ratio is $1-\lambda$. Thus, we can obtain

$$b_m = 1 + (1-\lambda)^m (b_0 - 1) = 1 - (1-\lambda)^m.$$

(ii) By (ii) of Lemma 3.18, we have $a_{m+1}^s(s) = (1-s\lambda)a_m^s(s)$ for $m \geq 1$ and $a_0^s(s) = 1$. Then, $a_m^s(s) = (1-s\lambda)^m$ for $m \geq 1$. Thus, by (v) of Lemma 3.18, we have

$$\begin{aligned} a_m^s(s-k) &= \sum_{d=0}^{k} (-1)^{k-d} \left(\begin{array}{c} k \\ d \end{array} \right) a_m^{s-d}(s-d) \\ &= \sum_{d=0}^{k} (-1)^{k-d} \left(\begin{array}{c} k \\ d \end{array} \right) [1-(s-d)\lambda]^m, \end{aligned}$$

for each $s \geq 1$, $0 \leq k < s$ and $m \geq 1$.

□

Proof of Lemma 3.20. Firstly, the sequence $\{b_m\}_{m=0}^{\infty}$ converges to 1, provided $0 < \lambda < 1$, since $b_m = 1 - (1-\lambda)^m$ for each $m \geq 1$ by (i) of Lemma 3.19. Secondly, by (ii) of Lemma 3.19, it is straightforward to obtain that

$$a_m^s(t) = \sum_{d=0}^{s-t} (-1)^{s-t-d} \left(\begin{array}{c} s-t \\ d \end{array} \right) [1-(s-d)\lambda]^m,$$

for each $1 \leq t \leq s$ and $m \geq 1$. Then, the sequence $\{a_m^s(t)\}_{m=0}^{\infty}$ converges to 0 if and only if $-1 < 1-(s-d)\lambda < 1$ for all $1 \leq s \leq n$ and $0 \leq d < s$. For each $0 < \lambda < \frac{2}{n}$, we have

$$-1 < 1 - n\lambda \leq 1-(s-d)\lambda < 1.$$

Thus, the sequence $\{a_m^s(t)\}_{m=0}^{\infty}$ converges to 0, provided $0 < \lambda < \frac{2}{n}$. Therefore, for each $0 < \lambda < \frac{2}{n}$, the limit game $\langle N, \bar{v}_x \rangle$ of the sequence

$\{\langle N, v^{m*}_{\lambda,x,I}\rangle\}_{m=0}^{\infty}$ is given by

$$\begin{aligned}\bar{v}_x(S) &= \lim_{m\to\infty}\left[\sum_{T\subseteq S} a^s_m(t)v(T) + b_m\sum_{j\in S} x_j\right] \\ &= \sum_{T\subseteq S}\lim_{m\to\infty} a^s_m(t)v(T) + \lim_{m\to\infty} b_m\sum_{j\in S} x_j \\ &= \sum_{j\in S} x_j,\end{aligned}$$

for all $S \subseteq N$. □

Proof of Theorem 3.21. Let φ be a solution satisfying both the inessential game property and continuity. Then, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in X(N, v)$, consider the dynamic sequence

$$x^m = x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x,I}) - \varphi(N, v^{m*}_{\lambda,x,I})].$$

By recursion, we have

$$\begin{aligned}x^m =& x^{m-1} + [\varphi(N, v^{(m-1)*}_{\lambda,x,I}) - \varphi(N, v^{m*}_{\lambda,x,I})] \\ =& x^{m-2} + [\varphi(N, v^{(m-2)*}_{\lambda,x,I}) - \varphi(N, v^{m*}_{\lambda,x,I})] \\ =& \cdots \\ =& x^0 + [\varphi(N, v^{0*}_{\lambda,x,I}) - \varphi(N, v^{m*}_{\lambda,x,I})] \\ =& x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x,I})].\end{aligned}$$

By Lemma 3.20, the inessential game property and continuity, we obtain that $\lim_{m\to\infty}\varphi(N, v^{m*}_{\lambda,x,I}) = \varphi(N, \bar{v}_x) = x$. Therefore,

$$\begin{aligned}\lim_{t\to\infty} x^t =& \lim_{t\to\infty}\{x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x,I})]\} \\ =& x + [\varphi(N, v) - x] = \varphi(N, v),\end{aligned}$$

which completes the proof. □

Proof of Lemma 3.26. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $S \subseteq N$, combining Eq.(3.4) and Eq.(3.5), we have

$$\begin{aligned} v_{\lambda,x,U}^{(m+1)*}(S) =&(v_{\lambda,x,U}^{m*})_{\lambda,x,U}^{*}(S) \\ =&(1-\lambda)v_{\lambda,x,U}^{m*}(S) + \lambda x(S) \\ =&(1-\lambda)[c_m^s v(S) + d_m^s x(S)] + \lambda x(S). \end{aligned}$$

Since this must hold for every $v(S)$, $S \subseteq N$, we have $c_{m+1}^s = (1-\lambda)c_m^s$ and $d_{m+1}^s = (1-\lambda)d_m^s + \lambda$, where $c_1^s = 1-\lambda$ and $d_1^s = \lambda$. From this, we have the following recursive formulas:

$$\frac{c_{m+1}^s}{c_m^s} = 1-\lambda \text{ and } \frac{d_{m+1}^s - 1}{d_m^s - 1} = 1-\lambda.$$

Therefore, the coefficients c_m^s and d_m^s of the m-fold x-union associated game $\langle N, v_{\lambda,x,U}^{m*}\rangle$ satisfy $c_m^s = (1-\lambda)^m$ and $d_m^s = 1-(1-\lambda)^m$, $m = 1, 2, \cdots$. □

Proof of Lemma 3.27. For all $\langle N, v\rangle \in \mathcal{G}^N$, $x \in X(N,v)$ and $0 < \lambda < 1$, by Lemma 3.26, we have

$$\begin{aligned} \lim_{m\to\infty} v_{\lambda,x,U}^{m*}(S) =& \lim_{m\to\infty}\{(1-\lambda)^m v(S) + [1-(1-\lambda)^m]x(S)\} \\ =&x(S), \end{aligned}$$

for all $S \subseteq N$. □

Proof of Theorem 3.28. Let φ be a solution satisfying both the inessential game property and continuity. Then, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in X(N,v)$, consider the dynamic sequence

$$x^m = x^{m-1} + [\varphi(N, v_{\lambda,x,U}^{(m-1)*}) - \varphi(N, v_{\lambda,x,U}^{m*})].$$

By recursion, we have

$$\begin{aligned} x^m =&x^{m-1} + [\varphi(N, v_{\lambda,x,U}^{(m-1)*}) - \varphi(N, v_{\lambda,x,U}^{m*})] \\ =&x^{m-2} + [\varphi(N, v_{\lambda,x,U}^{(m-2)*}) - \varphi(N, v_{\lambda,x,U}^{m*})] \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= x^0 + [\varphi(N, v^{0*}_{\lambda,x,U}) - \varphi(N, v^{m*}_{\lambda,x,U})] \\
&= x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x,U})].
\end{aligned}
$$

By Lemma 3.27, the inessential game property and continuity, we obtain that $\lim_{m\to\infty} \varphi(N, v^{m*}_{\lambda,x,U}) = \varphi(N, \hat{v}_x) = x$. Therefore,

$$
\begin{aligned}
\lim_{m\to\infty} x^m &= \lim_{m\to\infty} \{x + [\varphi(N, v) - \varphi(N, v^{m*}_{\lambda,x,U})]\} \\
&= x + [\varphi(N, v) - x] = \varphi(N, v),
\end{aligned}
$$

which completes the proof. □

3.7 Conclusions

The work in this chapter belongs to the growing literature on associated consistency. Different associated games take a different angle in revaluating the worth of coalitions. Some associated games in the literature (Xu et al. 2009, [117] and Xu et al. 2009, [119]) focus on reevaluating the worth of a coalition by considering “individual self-evaluation". In this chapter, we introduce an alternative way to reevaluate the worth. Instead of considering the players in the coalition as isolated elements, we consider the players in the coalition as a whole. We define two different associated games according to the idea of “union self-evaluation" instead of “individual self-evaluation", and provide new axiomatizations of the EANSC value and the CIS value using associated consistency. Moreover, we also propose dynamic processes on the basis of the “individual self-evaluation" associated games and the “union self-evaluation" associated games that lead to any solution satisfying both the inessential game property and continuity, starting from an arbitrary efficient payoff vector.

As introduced in Section 3.1, various associated consistency axioms are used frequently to characterize different solutions for TU-games. Recently, somewhat interesting, Hamiache and Navarro (2020, [44]) introduced an extension of Hamiache's associated game (Hamiache 2001, [42]) to characterize a solution for cooperative games with incomplete communication. This triggers the question whether union associated games proposed in this chapter or other associated games can be extended for characterizing solutions for cooperative games with incomplete communication. For future research, we intend to modify some existing associated games, such as the union associated games and the individual associated games, and then characterize solutions for cooperative games with incomplete communication by using the corresponding associated consistency axioms. Moreover, we also intend to apply Hamiache and Navarro's approach (2020, [44]) to cooperative games with coalition structure, and study axiomatizations of solutions for cooperative games with coalition structure by using associated consistency.

Chapter 4

Characterizations of the PD value and the PANSC value

4.1 Introduction

The proportionality principle is a relatively popular allocation criterion in many economic situations. It is a norm of distributed justice rooted in law and custom (Young 1994, [122]). Moulin's survey (Moulin 2002, [81]) of cost and surplus sharing opens by emphasizing the importance of the proportionality principle. As introduced in Chapter 1, the PD value and the PANSC value are defined based on the idea of proportionality. The *PD value* distributes the overall worth of the grand coalition in proportion to player's individual worth among all players. As the dual value of the PD value, the *PANSC value* distributes the overall worth in proportion to their marginal contributions with respect to the grand coalition. Moreover, some other proportional values have been studied in the literature, such as the proper Shapley value (van den Brink et al. 2015, [110] and 2020, [111]), the proportional value (Kamijo and Kongo 2015, [61] and Ortmann 2000, [88]), and the proportional Shapley value (Béal et al. 2018, [9] and Besner 2019, [13]). In this chapter, which is based on Li et al. (2020, [68]), we mainly study the PD value and the PANSC value. We propose an optimization approach to the PD value and the PANSC value, and give several new axiomatizations of these values.

As introduced in Chapter 1, the *excess* proposed by Schmeidler (1969, [92]) is an important criterion to describe the dissatisfaction of coalitions with respect to a payoff vector. A positive excess of a coalition with respect to a payoff vector represents the loss that the coalition suffers from the payoff vector compared to its worth. Several famous solutions for TU-games, such as the nucleolus (Schmeidler 1969, [92]), the core and the kernel (Davis and Maschler 1965, [29]), are defined on the basis of excess. In particular, the nucleolus is obtained by lexicographically minimizing the maximal excess of coalitions over the non-empty imputation set. Besides the excess criterion, Hou et al. (2018, [49]) proposed two other criteria to measure the dissatisfaction of coalitions with respect to a payoff vector. Alternatively, this chapter defines two new criteria from the perspective of satisfaction: the optimistic satisfaction and pessimistic satisfaction. The PD value and the PANSC value are obtained by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order, respectively.

Satisfaction is a significant criterion to measure the preference degree of coalitions for a payoff vector. Firstly, we define a family of *optimal satisfaction payoff vectors* (for short, *OS payoff vectors*). In particular, two specific optimal satisfaction payoff vectors, the *optimal optimistic satisfaction payoff vector* (for short, *OOS payoff vector*) and the *optimal pessimistic satisfaction payoff vector* (for short, *OPS payoff vector*), are considered in this chapter. The optimistic satisfaction and the pessimistic satisfaction are defined by the individual worth vector and the separable contribution vector from the viewpoints of optimism and pessimism respectively[1]. On the optimistic side, players always take the individual worth of themselves into

[1]There are two representative biases in social comparisons (Menon et al. 2009, [77]), a comparative optimism bias (i.e., a tendency for people to evaluate themselves in a more positive light) and a comparative pessimism bias (i.e., a tendency for people to evaluate themselves in a more negative light). In balanced TU-games, the individual worth vector is a lower bound of the core while the separable contribution vector is a upper bound of the core. Thus, the individual worth vector and the separable contribution vector can be viewed as the least potential payoff vector and the ideal payoff vector respectively.

consideration and think of the ratio between the payoff and the individual worth as a measure of satisfaction. The optimistic satisfaction of a coalition is defined by the ratio between the payoff of the coalition and the sum of the individual worths of the players in the coalition. Conversely, pessimists always take the ideal payoff of themselves into consideration. The pessimistic satisfaction of a coalition is the ratio between the payoff of the coalition and the sum of the separable contributions of the players in the coalition. The OOS payoff vector and the OPS payoff vector are obtained by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order over the non-empty pre-imputation set, respectively. Interestingly, the two payoff vectors are coincident with the PD value and the PANSC value, respectively.

Axiomatization is one of the main ways to characterize the reasonability of solutions in TU-games. Zou et al. (2021, [123]) proposed several axiomatizations of the PD value on the basis of equal treatment of equals, monotonicity and reduced game consistency. In this chapter, we characterize the PD value and the PANSC value by introducing the equal minimal satisfaction axioms and the associated consistency axioms.

Firstly, we define *equal minimal optimistic satisfaction* and *equal minimal pessimistic satisfaction*, inspired by the kernel concept (Maschler et al. 1971, [76]). Equal minimal optimistic satisfaction requires that, for a pair of players $\{i, j\}$ and a payoff vector x, the minimal optimistic satisfaction of coalitions containing i and not j with respect to x should equal that of coalitions containing j and not i under the optimistic satisfaction criterion, while equal minimal pessimistic satisfaction describes this situation under the pessimistic satisfaction criterion. Then, we show that the PD (respectively PANSC) value is the only solution satisfying equal minimal optimistic (respectively pessimistic) satisfaction and efficiency.

Associated consistency is quite popular in the literature on the axiomatization of solutions for TU-games, such as the Shapley value (Hamiache 2001, [42], Xu et al. 2008, [116]), the EANSC value and the CIS value (Xu et al. 2015, [118]), linear, symmetric values (Kleinberg 2018 [64]) and the core (Kong et al. [65]). A solution satisfies associated consistency with respect to a certain associated game if it allocates the same payoff to players in the original game as in the associated game. In this chapter, we propose *optimistic associated consistency* and *pessimistic associated consistency* to characterize the PD value and the PANSC value respectively. Moreover, we also study the dual axioms of the two associated consistency axioms.

The rest of this chapter is organized as follows. In Section 4.2, we determine the PD value and the PANSC value by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order respectively. In Section 4.3, we characterize the PD value and the PANSC value by introducing the equal minimal satisfaction axioms and the associated consistency axioms. Section 4.4 provides all proofs of this chapter. Section 4.5 concludes with a brief summary.

4.2 Satisfaction and the PD and PANSC values

It is well known that the *nucleolus* is obtained by minimizing the excesses of coalitions in the lexicographic order over the non-empty imputation set. The *excess* of coalition $S \subseteq N$ with respect to the payoff vector x of the TU-game $\langle N, v\rangle$ is given by $e(S, x, v) = v(S) - x(S)$. The excess $e(S, x, v)$ is usually used to measure the dissatisfaction degree of a coalition S with respect to x. The larger the excess $e(S, x, v)$ is, the more unsatisfied the coalition S feels with respect to x. Conversely, the larger the minus excess $-e(S, x, v)$ is, the more satisfied the coalition S feels with respect to x. In a sense, the minus excess is a measure of satisfaction, and $-e(S, x, v)$ can be seen as the satisfaction of the coalition S with respect to x. In this section, we introduce the family of the OS payoff vectors for TU-games from the perspective of satisfaction.

For all $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in \mathbb{R}^N$, let $\vartheta(v, x)$ be the $(2^n - 1)$-tuple vector whose components are the *satisfactions* of all non-empty coalitions $S \subseteq N$ with respect to x in non-decreasing order, that is, $\vartheta_t(v, x) \leq \vartheta_{t+1}(v, x)$ for all $t \in \{1, 2, \cdots, 2^n - 2\}$. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $x, y \in \mathbb{R}^N$, we denote $\vartheta(v, x) \geq_L \vartheta(v, y)$ if and only if $\vartheta(v, x) = \vartheta(v, y)$, or there exists $t \in \{1, 2, \cdots, 2^n - 2\}$ such that $\vartheta_l(v, x) = \vartheta_l(v, y)$ for all $l \in \{1, 2, \cdots, t - 1\}$ and $\vartheta_t(v, x) > \vartheta_t(v, y)$.

Definition 4.1. For all $\langle N, v\rangle \in \mathcal{G}^N$, the *optimal satisfaction payoff vector* (for short, *OS payoff vector*) $x^{os}(N, v)$ is the unique payoff vector y in the pre-imputation set satisfying $\vartheta(v, y) \geq_L \vartheta(v, x)$ for all $x \in I^*(N, v)$.

The OS payoff vector can be viewed as an optimal solution for an optimization problem aiming to maximize the minimal satisfaction with respect to the payoff vector over the pre-imputation set in the lexicographic order. This optimization problem indeed has a unique solution (that is the OS payoff vector) similar as the nucleolus (see, Definition 1.4 in Chapter 1) is a singleton. It is easy to verify that the OS payoff vector is the pre-nucleolus of a TU-game $\langle N, v\rangle$ when the satisfaction criterion is the minus excess $-e(S, x, v)$.

Next, we define two special satisfaction criteria, called the optimistic satisfaction and the pessimistic satisfaction[2].

Definition 4.2. For all $\langle N, v\rangle \in \mathcal{G}_+^N$ and $x \in \mathbb{R}^N$, the *optimistic satisfaction* of coalition $S \subseteq N$ with respect to the payoff vector x of the TU-game $\langle N, v\rangle$ is defined by

$$e^o(S, x, v) = \frac{x(S)}{\sum_{k\in S} v(\{k\})}.$$

Definition 4.3. For all $\langle N, v\rangle \in \mathcal{G}_\oplus^N$ and $x \in \mathbb{R}^N$, the *pessimistic satisfaction* of coalition $S \subseteq N$ with respect to the payoff vector x of the TU-game $\langle N, v\rangle$ is defined by

$$e^p(S, x, v) = \frac{x(S)}{\sum_{k\in S} SC_k(N, v)}.$$

These two special satisfaction criteria are defined from the viewpoint of optimism and pessimism respectively. Given $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in C(N, v)$, it holds that $v(\{i\}) \leq x_i \leq SC_i(N, v)$ for all $i \in N$. Thus, the vector $(v(\{k\}))_{k\in N}$ is a lower bound of the core while $(SC_k(N, v))_{k\in N}$ is a upper bound of the core of a TU-game $\langle N, v\rangle$. Moreover, in the work of Hou

[2]In the definition of the optimistic (pessimistic) satisfaction, the TU-game has to be restricted on the class of all individually positive (marginally positive) TU-games in order for the denominator not to be zero. Notice that all the conclusions in this chapter also apply to the family of all individually negative TU-games or the family of all marginally negative TU-games.

et al. (2018, [49]), the individual worth vector $(v(\{k\}))_{k\in N}$ is viewed as the least potential payoff vector while the separable contribution vector $(SC_k(N,v))_{k\in N}$ is viewed as the ideal payoff vector. Hence, on the optimistic side, the players can take the least potential payoffs of themselves into consideration and think of the ratio between the payoff of a coalition and their least potential payoff as a measure of satisfaction of the coalition. Conversely, pessimists can take the ideal payoff payoffs of themselves into consideration.

In the following, we define the OOS payoff vector and the OPS payoff vector in terms of the optimistic satisfaction and the pessimistic satisfaction respectively, and show that the OOS payoff vector and the OPS payoff vector coincide with the PD value and the PANSC value respectively.

4.2.1 Optimistic satisfaction and the PD value

For all $\langle N,v\rangle \in \mathcal{G}_+^N$ and $x \in \mathbb{R}^N$, let $\vartheta^o(v,x)$ be the (2^n-1)-tuple vector whose components are the *optimistic satisfactions* of all non-empty coalitions $S \subseteq N$ with respect to x in non-decreasing order. The OOS payoff vector is defined as follows.

Definition 4.4. For all $\langle N,v\rangle \in \mathcal{G}_+^N$, the *optimal optimistic satisfaction payoff vector* (for short, *OOS payoff vector*) $x^{oos}(N,v)$ is the unique payoff vector y in the pre-imputation set satisfying $\vartheta^o(v,y) \geq_L \vartheta^o(v,x)$ for all $x \in I^*(N,v)$.

Next, we show that the PD value assigns to every individually positive game the OOS payoff vector which is obtained by lexicographically maximizing the minimal optimistic satisfaction.

Lemma 4.5. *For all* $\langle N,v\rangle \in \mathcal{G}_+^N$ *and* $x \in \mathbb{R}^N$, *let* $l = \arg\min_{k\in N}\{e^o(\{k\},x,v)\}$. *Then, it holds that* $e^o(\{l\},x,v) = \min_{S\subseteq N, S\neq\emptyset}\{e^o(S,x,v)\}$.

The proof of Lemma 4.5 and of all other results in this chapter can be found in Section 4.4.

Lemma 4.6. *For all* $\langle N,v\rangle \in \mathcal{G}_+^N$ *and* $x \in \mathbb{R}^N$, *let* $l = \arg\min_{k\in N}\{e^o(\{k\},x,v)\}$. *If there exists* $m \in N$ *such that* $e^o(\{m\},x,v) > e^o(\{l\},x,v)$, *define a new* *payoff vector x^* given by*

$$x_k^* = \begin{cases} x_k, & \text{for } k \in N \setminus \{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot v(\{l\}) - x_l \cdot v(\{m\})}{v(\{l\}) + v(\{m\})}$. *Then the following five statements hold.*

(i) $e^o(S, x^*, v) = e^o(S, x, v)$ *for all* $S \subseteq N$ *and* $S \not\ni l, m$.

(ii) $e^o(S, x^*, v) = e^o(S, x, v)$ *for all* $S \subseteq N$ *and* $S \ni l, m$.

(iii) $e^o(S, x^*, v) > e^o(S, x, v)$ *for all* $S \subseteq N$, $S \ni l$ *and* $S \not\ni m$.

(iv) $e^o(S, x^*, v) > e^o(\{l\}, x, v)$ *for all* $S \subseteq N$, $S \not\ni l$ *and* $S \ni m$.

(v) $\vartheta^o(v, x^*) >_L \vartheta^o(v, x)$.

Theorem 4.7. *For all* $\langle N, v \rangle \in \mathcal{G}_+^N$, *the following two statements hold.*

(i) $\frac{x_i^{oos}(N,v)}{v(\{i\})} = \frac{x_j^{oos}(N,v)}{v(\{j\})}$ *for all* $i, j \in N$.

(ii) $x_i^{oos}(N, v) = PD_i(N, v)$ *for all* $i \in N$.

Obviously, if $\sum_{k \in N} v(\{k\}) \leq v(N)$, then the PD value satisfies individual rationality, that is, $PD_k(N, v) \geq v(\{k\})$ for all $k \in N$. The following corollary is immediate for the reason that $C(N, v) \neq \emptyset$ implies $\sum_{k \in N} v(\{k\}) \leq v(N)$.

Corollary 4.8. *For all* $\langle N, v \rangle \in \mathcal{G}_+^N$ *with* $C(N, v) \neq \emptyset$, *the PD value outcome satisfies individual rationality, that is,* $PD_k(N, v) \geq v(\{k\})$ *for all* $k \in N$.

4.2.2 Pessimistic satisfaction and the PANSC value

In this subsection, we define the OPS payoff vector by lexicographically maximizing the minimal pessimistic satisfaction, and show that the PANSC value assigns to every marginally positive game the OPS payoff vector.

For all $\langle N, v \rangle \in \mathcal{G}_\oplus^N$ and $x \in \mathbb{R}^N$, let $\vartheta^p(v, x)$ be the $(2^n - 1)$-tuple vector whose components are the *pessimistic satisfactions* of all non-empty coalitions $S \subseteq N$ with respect to x in non-decreasing order. The OPS payoff vector is defined as follows.

Definition 4.9. For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$, the *optimal pessimistic satisfaction payoff vector* (for short, *OPS payoff vector*) $x^{ops}(N,v)$ is the unique payoff vector y in the pre-imputation set satisfying $\vartheta^p(v,y) \geq_L \vartheta^p(v,x)$ for all $x \in I^*(N,v)$.

Next, we will verify that the PANSC value assigns to every marginally positive game the OPS payoff vector. The proofs of Lemma 4.10, Lemma 4.11 and Theorem 4.12 are similar to those of Lemma 4.5, Lemma 4.6 and Theorem 4.7, and are omitted here.

Lemma 4.10. *For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$ and $x \in \mathbb{R}^N$, let $l = \arg\min_{k\in N}\{e^p(\{k\}, x, v)\}$. Then, it holds that $e^p(\{l\}, x, v) = \min_{S\subseteq N, S\neq\emptyset}\{e^p(S, x, v)\}$.*

Lemma 4.11. *For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$ and $x \in \mathbb{R}^N$, let $l = \arg\min_{k\in N}\{e^p(\{k\}, x, v)\}$. If there exists $m \in N$ such that $e^p(\{m\}, x, v) > e^p(\{l\}, x, v)$, define a new payoff vector x^* given by*

$$x_k^* = \begin{cases} x_k, & \text{for } k \in N\backslash\{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot SC_l(N,v) - x_l \cdot SC_m(N,v)}{SC_l(N,v) + SC_m(N,v)}$. Then the following five statements hold.

(i) $e^p(S, x^*, v) = e^p(S, x, v)$ *for all* $S \subseteq N$ *and* $S \not\ni l, m$.

(ii) $e^p(S, x^*, v) = e^p(S, x, v)$ *for all* $S \subseteq N$ *and* $S \ni l, m$.

(iii) $e^p(S, x^*, v) > e^p(S, x, v)$ *for all* $S \subseteq N$, $S \ni l$ *and* $S \not\ni m$.

(iv) $e^p(S, x^*, v) > e^p(\{l\}, x, v)$ *for all* $S \subseteq N$, $S \not\ni l$ *and* $S \ni m$.

(v) $\vartheta^p(v, x^*) >_L \vartheta^p(v, x)$.

Theorem 4.12. *For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$, the following two statements hold.*

(i) $\frac{x_i^{ops}(N,v)}{SC_i(N,v)} = \frac{x_j^{ops}(N,v)}{SC_j(N,v)}$ *for all* $i, j \in N$.

(ii) $x_i^{ops}(N,v) = PANSC_i(N,v)$ *for all* $i \in N$.

By Theorem 4.12, it holds that $PANSC_k(N,v) \leq SC_k(N,v)$ for all $k \in N$ if $\sum_{k\in N} SC_k(N,v) \geq v(N)$. Then the following corollary is immediate for the reason that $C(N,v) \neq \emptyset$ implies $\sum_{k\in N} SC_k(N,v) \geq v(N)$.

Corollary 4.13. *For all* $\langle N,v\rangle \in \mathcal{G}_{\oplus}^N$ *with* $C(N,v) \neq \emptyset$*, the PANSC value outcome is bounded by the payoff vector* $(SC_k(N,v))_{k\in N}$*, that is,* $PANSC_k(N,v) \leq SC_k(N,v)$ *for all* $k \in N$.

4.3 Axiomatizations of the PD value and the PANSC value

A major purpose of axiomatizing solutions in TU-games is to show the reasonability of solutions. In this section, we characterize the PD value and the PANSC value by introducing the equal minimal optimistic/pessimistic satisfaction axioms, the optimistic/pessimistic associated consistency axioms and their dual axioms.

4.3.1 Equal minimal satisfaction axioms

In this subsection, we propose two axioms, *equal minimal optimistic satisfaction* and *equal minimal pessimistic satisfaction*, inspired by the kernel concept (Maschler et al. 1971, [76]). The PD value and the PANSC value are characterized by these two axioms with efficiency, respectively.

For all $\langle N,v\rangle \in \mathcal{G}_{+}^N$ and $x \in \mathbb{R}^N$, let $m_{ij}^o(v,x) = \min\{e^o(S,x,v) | S \subseteq N, i \in S, j \notin S\}$ be the minimal optimistic satisfaction of player $i \in N$ over player $j \in N\backslash\{i\}$ with respect to x. Similarly, for all $\langle N,v\rangle \in \mathcal{G}_{\oplus}^N$ and $x \in \mathbb{R}^N$, let $m_{ij}^p(v,x) = \min\{e^p(S,x,v) | S \subseteq N, i \in S, j \notin S\}$ be the minimal pessimistic satisfaction of player $i \in N$ over player $j \in N\backslash\{i\}$ with respect to x.

- **Equal minimal optimistic satisfaction.** For all $\langle N,v\rangle \in \mathcal{G}_{+}^N$ and $i,j \in N$, it holds that $m_{ij}^o(v,\varphi(N,v)) = m_{ji}^o(v,\varphi(N,v))$.

- **Equal minimal pessimistic satisfaction.** For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^{N}$ and $i, j \in N$, it holds that $m_{ij}^{p}(v, \varphi(N, v)) = m_{ji}^{p}(v, \varphi(N, v))$.

Equal minimal optimistic satisfaction requires that for all $i, j \in N$, the minimal optimistic satisfaction over all coalitions containing i and not j should equal that over all coalitions containing j and not i with respect to the solution outcome under the optimistic satisfaction criterion. On the contrary, equal minimal pessimistic satisfaction describes this situation under the pessimistic satisfaction criterion.

Proposition 4.14. *Equal minimal optimistic satisfaction and equal minimal pessimistic satisfaction are dual to each other.*

Next, we characterize the PD value and the PANSC value by using equal minimal optimistic satisfaction and equal minimal pessimistic satisfaction respectively.

Theorem 4.15. *A solution φ on $\mathcal{G}_{+}^{N}$ satisfies efficiency and equal minimal optimistic satisfaction if and only if φ is the PD value.*

In TU-games, the duality operator is a very useful tool to derive new axiomatizations of solutions. If there is an axiomatization of solution φ, then we can get one axiomatization of its dual solution φ^{d} by determining the dual axioms of the axioms which are included in the axiomatization of φ. Oishi et al. (2016, [87]) derived new axiomatizations of several classical solutions for TU-games by this duality theory. Since equal minimal optimistic satisfaction and equal minimal pessimistic satisfaction are dual to each other and efficiency is self-dual, we obtain the following theorem.

Theorem 4.16. *A solution φ on $\mathcal{G}_{\oplus}^{N}$ satisfies efficiency and equal minimal pessimistic satisfaction if and only if φ is the PANSC value.*

4.3.2 Associated consistency

Associated consistency is a significant characteristic of feasible and stable solutions. Associated consistency requires that the solution should be invariant when the TU-game changes into its associated game.

Throughout this subsection we deal with two types of associated games, the optimistic associated game and the pessimistic associated game. In these two associated games, every coalition reevaluates its own worth. Every coalition S just considers the players in $N\backslash S$ as individual elements. On the optimistic side, every coalition S always thinks that players in $N\backslash S$ should just receive their individual worth vector $(v\{k\})_{k\in N\backslash S}$. The amount $v(N) - v(S) - \sum_{k\in N\backslash S} v(\{k\})$ can be regarded as the optimistic surplus arising from mutual cooperation between S itself and all $j \in N\backslash S$. On the pessimistic side, every coalition S takes into consideration the separable contribution vector and thinks that players in $N\backslash S$ can obtain their the separable contribution $(SC_k(N,v))_{k\in N\backslash S}$. The amount $v(N) - v(S) - \sum_{k\in N\backslash S} SC_k(N,v)$ is considered as the pessimistic surplus. Every coalition S believes that the appropriation of at least a part of the surpluses is within reach. Thus, every coalition S reevaluates its own worth $v_{\lambda,O}(S)$ in the optimistic associated game as the sum of its initial worth $v(S)$ and a percentage $\lambda \in (0,1)$ of a part $\frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}$ of the optimistic surplus $v(N) - v(S) - \sum_{k\in N\backslash S} v(\{k\})$. Similarly, the pessimistic surplus is taken into account in the pessimistic associated game.

Definition 4.17. Given $\langle N, v\rangle \in \mathcal{G}^N_+$ with $v(N) > 0$, and a real number λ, $0 \le \lambda \le 1$, the *optimistic associated game* $\langle N, v_{\lambda,O}\rangle$ is defined by

$$v_{\lambda,O}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) + \lambda\frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[v(N) - v(S) - \sum_{k\in N\backslash S} v(\{k\})], & \text{otherwise.} \end{cases}$$

It is easy to see that $\langle N, v_{\lambda,O}\rangle \in \mathcal{G}^N_+$ may not be true if $\langle N, v\rangle \in \mathcal{G}^N_+$. The purpose of requiring $v(N) > 0$ is in order to ensure that $\langle N, v_{\lambda,O}\rangle \in \mathcal{G}^N_+$. For convenience, let $\mathcal{G}^N_{++} = \{\langle N, v\rangle \in \mathcal{G}^N_+ | v(N) > 0\}$. It is easy to obtain that $\langle N, v_{\lambda,O}\rangle \in \mathcal{G}^N_{++}$ if $\langle N, v\rangle \in \mathcal{G}^N_{++}$. Moveover, let $\mathcal{G}^N_{\oplus\oplus} = \{\langle N, v\rangle \in \mathcal{G}^N_{\oplus} | v(N) > 0\}$. Obviously, $\mathcal{G}^N_{++}$ and $\mathcal{G}^N_{\oplus\oplus}$ are dual to each other.

Definition 4.18. Given $\langle N, v\rangle \in \mathcal{G}^N_{\oplus}$ and a real number λ, $0 \le \lambda \le 1$, the *pessimistic associated game* $\langle N, v_{\lambda,P}\rangle$ is defined by

$$v_{\lambda,P}(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ v(S) + \lambda\frac{\sum_{k\in S} SC_k(N,v)}{\sum_{k\in N} SC_k(N,v)}[v(N) - v(S) - \sum_{k\in N\backslash S} SC_k(N,v)], & \text{otherwise.} \end{cases}$$

Obviously, $\langle N, v_{\lambda,P}\rangle \in \mathcal{G}_{\oplus}^N$ if $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$. Let $\lambda \in [0,1]$.

- **Optimistic associated consistency for** λ**.** For all $\langle N, v\rangle \in \mathcal{G}_{++}^N$, it holds that $\varphi(N, v) = \varphi(N, v_{\lambda,O})$.
- **Pessimistic associated consistency for** λ**.** For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$, it holds that $\varphi(N, v) = \varphi(N, v_{\lambda,P})$.

Let us consider the dual relation between these two associated consistency axioms.

Remark 4.1. Optimistic associated consistency and pessimistic associated consistency are not dual to each other.

Given $\langle N, v\rangle \in \mathcal{G}_{++}^N$ and its dual game $\langle N, v^d\rangle \in \mathcal{G}_{\oplus\oplus}^N$, we only need to verify whether $\langle N, (v^d)_{\lambda,O}\rangle$ is equal to $\langle N, (v_{\lambda,P})^d\rangle$, to determine the dual relation between optimistic associated consistency and pessimistic associated consistency.

In the following, we will characterize the PD value and the PANSC value by using optimistic associated consistency and pessimistic associated consistency, respectively.

For all $\langle N, v\rangle \in \mathcal{G}_{++}^N$, the sequence of optimistic associated games, $\{\langle N, v_{\lambda,O}^t\rangle\}_{t=0}^{\infty}$, is defined by $v_{\lambda,O}^0 = v$, and $v_{\lambda,O}^{t+1} = (v_{\lambda,O}^t)_{\lambda,O}$. The following lemma states the convergence of the sequence of optimistic associated games.

Lemma 4.19. *For all $\langle N, v\rangle \in \mathcal{G}_{++}^N$ and $0 < \lambda < 1$, the sequence of optimistic associated games $\{\langle N, v_{\lambda,O}^t\rangle\}_{t=0}^{\infty}$ converges, and its limit game $\langle N, \hat{v}\rangle$ is inessential.*

Theorem 4.20. *Let $\lambda \in (0,1)$. A solution φ on $\mathcal{G}_{++}^N$ satisfies optimistic associated consistency for λ, continuity and the inessential game property if and only if φ is the PD value.*

For all $\langle N, v\rangle \in \mathcal{G}_{\oplus}^N$, the sequence of pessimistic associated games, $\{\langle N, v_{\lambda,P}^t\rangle\}_{t=0}^{\infty}$, is defined by $v_{\lambda,P}^0 = v$, and $v_{\lambda,P}^{t+1} = (v_{\lambda,P}^t)_{\lambda,P}$. Next, we prove the convergence of the sequence of pessimistic associated games.

Lemma 4.21. *For all $\langle N, v\rangle \in \mathcal{G}^N_{\oplus}$ and $0 < \lambda < 1$, the sequence of pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}^{\infty}_{t=0}$ converges and its limit game $\langle N, \check{v}\rangle$ is the sum of an inessential game $\langle N, u\rangle$ and a constant game $\langle N, w\rangle$.*

Now, to characterize the PANSC value, we introduce a new axiom, *the constant inessential game property*. A TU-game $\langle N, v\rangle \in \mathcal{G}^N$ is a *constant inessential game* if there exists a constant $c \in \mathbb{R}$ such that $v(S) - c = \sum_{i\in S}(v(\{i\}) - c)$ for all $S \subseteq N$. This implies that a constant inessential game is the sum of an inessential game and a constant game. It is easy to see that the dual of a constant inessential game is an almost inessential game.

- **Constant inessential game property.** For all constant inessential games $\langle N, v\rangle \in \mathcal{G}^N_{\oplus}$, it holds that $\varphi_i(N, v) = \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N)$.

Theorem 4.22. *Let $\lambda \in (0, 1)$. A solution φ on $\mathcal{G}^N_{\oplus}$ satisfies pessimistic associated consistency for λ, continuity and the constant inessential game property if and only if φ is the PANSC value.*

4.3.3 Dual axioms of associated consistency

In Remark 4.1, we mentioned that optimistic associated consistency and pessimistic associated consistency are not dual to each other. Next, let us consider the dual axioms of optimistic associated consistency and pessimistic associated consistency.

Definition 4.23. Given $\langle N, v\rangle \in \mathcal{G}^N_{\oplus\oplus}$, and a real number λ, $0 \leq \lambda \leq 1$, the *dual optimistic associated game* $\langle N, v^*_{\lambda,O}\rangle$ is defined by[3]

$$v^*_{\lambda,O}(S) = \begin{cases} v(S) + \lambda \frac{\sum_{j\in N\setminus S} SC_j(N,v)}{\sum_{j\in N} SC_j(N,v)}[\sum_{j\in S} SC_j(N, v) - v(S)], & \text{if } S \subset N; \\ v(N), & \text{if } S = N. \end{cases}$$

[3]Notice that the dual optimistic associated game is not the dual of the optimistic associated game, that is, $\langle N, v^*_{\lambda,O}\rangle \neq \langle N, (v_{\lambda,O})^d\rangle$.

Definition 4.24. Given $\langle N, v\rangle \in \mathcal{G}_{+}^{N}$, and a real number λ, $0 \leq \lambda \leq 1$, the *dual pessimistic associated game* $\langle N, v^{*}_{\lambda,P}\rangle$ is defined by[4]

$$v^{*}_{\lambda,P}(S) = \begin{cases} v(S) + \lambda\frac{\sum_{k\in N\setminus S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[\sum_{k\in S} v(\{k\}) - v(S)], & \text{if } S \subset N; \\ v(N), & \text{if } S = N. \end{cases}$$

Obviously, $\langle N, v^{*}_{\lambda,O}\rangle \in \mathcal{G}_{\oplus\oplus}^{N}$ if $\langle N, v\rangle \in \mathcal{G}_{\oplus\oplus}^{N}$, and $\langle N, v^{*}_{\lambda,P}\rangle \in \mathcal{G}_{+}^{N}$ if $\langle N, v\rangle \in \mathcal{G}_{+}^{N}$. Let $\lambda \in [0, 1]$.

- **Dual optimistic associated consistency for λ.** For all $\langle N, v\rangle \in \mathcal{G}_{\oplus\oplus}^{N}$, it holds that $\varphi(N, v) = \varphi(N, v^{*}_{\lambda,O})$.
- **Dual pessimistic associated consistency for λ.** For all $\langle N, v\rangle \in \mathcal{G}_{+}^{N}$, it holds that $\varphi(N, v) = \varphi(N, v^{*}_{\lambda,P})$.

In the following, we will prove that optimistic (pessimistic) associated consistency and dual optimistic (pessimistic) associated consistency are dual to each other.

Lemma 4.25. *For all $\langle N, v\rangle \in \mathcal{G}_{++}^{N}$ and $\langle N, w\rangle \in \mathcal{G}_{\oplus}^{N}$, it holds that $(v_{\lambda,O})^{d} = (v^{d})^{*}_{\lambda,O}$ and $(w_{\lambda,P})^{d} = (w^{d})^{*}_{\lambda,P}$.*

Proposition 4.26. *Optimistic associated consistency and dual optimistic associated consistency are dual to each other.*

The proof of Proposition 4.26 is in Section 4.4, and the proof of Proposition 4.27 is similar to that of Proposition 4.26 and is left to readers.

Proposition 4.27. *Pessimistic associated consistency and dual pessimistic associated consistency are dual to each other.*

Next, let us identify the dual axioms of other axioms which are included in the axiomatizations of the PD value and the PANSC value appearing in Theorems 4.20 and 4.22. It is easy to verify that continuity and the inessential game property are self-dual. In the following, we define the dual axiom of constant inessential game property.

[4]Notice that the dual pessimistic associated game is not the dual of the pessimistic associated game, that is, $\langle N, v^{*}_{\lambda,P}\rangle \neq \langle N, (v_{\lambda,P})^{d}\rangle$.

- **Dual constant inessential game property.** For all almost inessential games $\langle N, v\rangle \in \mathcal{G}_{+}^{N}$ and $i \in N$, it holds that $\varphi_i(N, v) = \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} v(N)$.

Obviously, the constant inessential game property and the dual constant inessential game property are dual to each other. Thus, it is straightforward to obtain the following two theorems by the duality theory (Oishi et al. 2016, [87]).

Theorem 4.28. *Let $\lambda \in (0, 1)$. A solution φ on $\mathcal{G}_{\oplus\oplus}^{N}$ satisfies dual optimistic associated consistency for λ, continuity and the inessential game property if and only if φ is the PANSC value.*

Theorem 4.29. *Let $\lambda \in (0, 1)$. A solution φ on $\mathcal{G}_{+}^{N}$ satisfies dual pessimistic associated consistency for λ, continuity and the dual constant inessential game property if and only if φ is the PD value.*

4.4 Proofs

Proof of Lemma 4.5. For all $\langle N, v\rangle \in \mathcal{G}_{+}^{N}$ and $x \in \mathbb{R}^N$, let $p = e^o(\{l\}, x, v)$. Then, it holds that $x_k \geq p \cdot v(\{k\})$ for all $k \in N$. Thus, we have

$$\begin{aligned} e^o(\{l\}, x, v) &\geq \min_{S \subseteq N, S \neq \emptyset} \{e^o(S, x, v)\} = \min_{S \subseteq N, S \neq \emptyset} \{\frac{x(S)}{\sum_{k \in S} v(\{k\})}\} \\ &\geq \min_{S \subseteq N, S \neq \emptyset} \{\frac{p \cdot \sum_{k \in S} v(\{k\})}{\sum_{k \in S} v(\{k\})}\} = p = e^o(\{l\}, x, v). \end{aligned}$$

Therefore, all inequalities are equalities, showing that

$$e^o(\{l\}, x, v) = \min_{S \subseteq N, S \neq \emptyset} \{e^o(S, x, v)\},$$

which completes the proof. □

Proof of Lemma 4.6. (i) It is obvious that $e^o(S, x^*, v) = e^o(S, x, v)$ for all $S \subseteq N$ and $S \not\ni l, m$ because $x^*(S) = x(S)$ for all $S \subseteq N$ and $S \not\ni l, m$.

(ii) It is trivial that $e^o(S, x^*, v) = e^o(S, x, v)$ for all $S \subseteq N$ and $S \ni l, m$.

(iii) It is easy to obtain that $\triangle > 0$ since $e^o(\{m\},x,v) > e^o(\{l\},x,v)$. Then for all $S \subseteq N$, $S \ni l$ and $S \not\ni m$, we have

$$\begin{aligned} e^o(S,x^*,v) =& \frac{x^*(S)}{\sum_{k\in S} v(\{k\})} = \frac{x(S)+\triangle}{\sum_{k\in S} v(\{k\})} \\ >& \frac{x(S)}{\sum_{k\in S} v(\{k\})} = e^o(S,x,v). \end{aligned}$$

(iv) Since $e^o(\{m\},x,v) > e^o(\{l\},x,v)$, we have

$$\begin{aligned} e^o(\{m\},x^*,v) =& \frac{x^*_m}{v(\{m\})} = \frac{x_m - \triangle}{v(\{m\})} = \frac{x_m + x_l}{v(\{m\}) + v(\{l\})} \\ >& \frac{\frac{x_l}{v(\{l\})} v(\{m\}) + x_l}{v(\{m\}) + v(\{l\})} = \frac{x_l}{v(\{l\}))} = e^o(\{l\},x,v). \end{aligned}$$

For all $S \subseteq N$, $S \not\ni l$ and $S \ni m$, we have

$$\begin{aligned} e^o(S,x^*,v) =& \frac{x(S\backslash\{m\}) + x^*_m}{\sum_{k\in S\backslash\{m\}} v(\{k\}) + v(\{m\})} \\ >& \frac{\frac{x_l}{v(\{l\})}\sum_{k\in S\backslash\{m\}} v(\{k\}) + \frac{x_l}{v(\{l\})} v(\{m\})}{\sum_{k\in S\backslash\{m\}} v(\{k\}) + v(\{m\})} \\ =& \frac{x_l}{v(\{l\}))} = e^o(\{l\},x,v), \end{aligned}$$

where the second inequality holds due to the fact that $e^o(\{m\},x^*,v) > e^o(\{l\},x,v)$ and $e^o(S\backslash\{m\},x,v) \geq e^o(\{l\},x,v)$ by Lemma 4.5.

(v) It holds that $\vartheta(v,x^*) >_L \vartheta(v,x)$ by Statement (i)–(iv). □

Proof of Theorem 4.7. (i) For all $\langle N,v\rangle \in \mathcal{G}^N_+$, we prove that $\frac{x^{oos}_i(N,v)}{v(\{i\})} = \frac{x^{oos}_j(N,v)}{v(\{j\})}$ for all $i,j \in N$ by reductio. Suppose there exist $m,j \in N$ such that $\frac{x^{oos}_m(N,v)}{v(\{m\})} \neq \frac{x^{oos}_j(N,v)}{v(\{j\})}$. Without loss of generality, suppose that $\frac{x^{oos}_m(N,v)}{v(\{m\})} > \frac{x^{oos}_j(N,v)}{v(\{j\})}$. Let $y = x^{oos}(N,v)$ and $l = \arg\min_{k\in N}\{e^o(\{k\},y,v)\}$, then we have $e^o(\{m\},y,v) > e^o(\{l\},y,v)$. By Lemma 4.6, there exists $x \in I^*(N,v)$ such that $\vartheta^o(v,x) >_L \vartheta^o(v,y)$, which contradicts with $\vartheta^o(v,y) \geq_L \vartheta^o(v,x)$ for all $x \in I^*(N,v)$. Therefore, $\frac{x_i^{oos}(N,v)}{v(\{i\})} = \frac{x_j^{oos}(N,v)}{v(\{j\})}$ for all $i,j \in N$.

(ii) It is immediate to deduce that

$$x_i^{oos}(N,v) = PD_i(N,v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N),$$

by the statement (i) and efficiency. □

Proof of Proposition 4.14. Given a solution φ on $\mathcal{G}_+^N$, let φ^d be the dual of φ. It is sufficient to prove that φ satisfies equal minimal optimistic satisfaction if and only if φ^d satisfies equal minimal pessimistic satisfaction.

Suppose that φ satisfies equal minimal optimistic satisfaction. For all $\langle N,v\rangle \in \mathcal{G}_\oplus^N$ and its dual game $\langle N,v^d\rangle \in \mathcal{G}_+^N$, by equal minimal optimistic satisfaction, we have $m_{ij}^o(v^d,\varphi(N,v^d)) = m_{ji}^o(v^d,\varphi(N,v^d))$ for all $i,j \in N$, and then it holds that

$$\begin{aligned}&\min\{\frac{\sum_{k\in S}\varphi_k(N,v^d)}{\sum_{k\in S} v^d(\{k\})}|S\subseteq N, i\in S, j\notin S\}\\ =&\min\{\frac{\sum_{k\in S}\varphi_k(N,v^d)}{\sum_{k\in S} v^d(\{k\})}|S\subseteq N, j\in S, i\notin S\}.\end{aligned}$$

Then, it holds that

$$\begin{aligned}&\min\{\frac{\sum_{k\in S}\varphi_k^d(N,v)}{\sum_{k\in S} SC_k(N,v)}|S\subseteq N, i\in S, j\notin S\}\\ =&\min\{\frac{\sum_{k\in S}\varphi_k^d(N,v)}{\sum_{k\in S} SC_k(N,v)}|S\subseteq N, j\in S, i\notin S\}.\end{aligned}$$

Then, we have $m_{ij}^p(v,\varphi^d(N,v)) = m_{ji}^p(v,\varphi^d(N,v))$ for all $i,j \in N$. Therefore, φ^d satisfies equal minimal pessimistic satisfaction.

Similarly, we can prove that φ satisfies equal minimal optimistic satisfaction if φ^d satisfies equal minimal pessimistic satisfaction, which is similar to the above proof. Therefore, we can conclude that equal minimal optimistic satisfaction and equal minimal pessimistic satisfaction are dual to each other. □

Proof of Theorem 4.15. Firstly, it is easy to show that the PD value satisfies efficiency. Then, for all $\langle N, v\rangle \in \mathcal{G}_+^N$, let $x = PD(N, v)$. For all $S \subseteq N$, $S \neq \emptyset$, we have $e^o(S, x, v) = \frac{v(N)}{\sum_{k\in N} v(\{k\})}$. Therefore, for all $i, j \in N$, we have

$$m_{ij}^o(v, x) = \frac{v(N)}{\sum_{k\in N} v(\{k\})} = m_{ji}^o(v, x),$$

showing that the PD value satisfies equal minimal optimistic satisfaction. It is left to show the uniqueness.

Suppose that φ is a solution on $\mathcal{G}_+^N$ satisfying efficiency and equal minimal optimistic satisfaction. Now suppose that $\varphi(N, v) \neq PD(N, v)$. Then there must exist $i, j \in N$ such that $\varphi_i(N, v) > PD_i(N, v)$ and $\varphi_j(N, v) < PD_j(N, v)$ by efficiency. Let $l = \arg\min_{k\in N}\{e^o(\{k\}, \varphi(N, v), v)\}$. It holds that $e^o(\{l\}, \varphi(N, v), v) = \min_{S\subseteq N, S\neq\emptyset}\{e^o(S, \varphi(N, v), v)\}$ by Lemma 4.5. Then, we have

$$e^o(\{l\}, \varphi(N, v), v) \leq e^o(\{j\}, \varphi(N, v), v) < \frac{v(N)}{\sum_{k\in N} v(\{k\})} < e^o(\{i\}, \varphi(N, v), v).$$

Without loss of generality, let $S_0 \subseteq N\backslash\{l\}$ be a coalition containing i such that $m_{il}^o(v, \varphi(N, v)) = e^o(S_0, \varphi(N, v), v)$. Thus, we have

$$\begin{aligned} m_{il}^o(v, \varphi(N, v)) =& \frac{\sum_{k\in S_0\backslash\{i\}} \varphi_k(N, v) + \varphi_i(N, v)}{\sum_{k\in S_0\backslash\{i\}} v(\{k\}) + v(\{i\})} \\ >& e^o(\{l\}, \varphi(N, v), v) = m_{li}^o(v, \varphi(N, v)), \end{aligned}$$

where the inequality holds from the facts that $e^o(S_0\backslash\{i\}, \varphi(N, v), v) \geq e^o(\{l\}, \varphi(N, v), v)$ and $e^o(\{i\}, \varphi(N, v), v) > e^o(\{l\}, \varphi(N, v), v)$, and the last equality holds because $e^o(\{l\}, \varphi(N, v), v) = \min_{S\subseteq N, S\neq\emptyset}\{e^o(S, \varphi(N, v), v)\}$. But $m_{il}^o(v, \varphi(N, v)) > m_{li}^o(v, \varphi(N, v))$ contradicts with equal minimal optimistic satisfaction. Therefore, the PD value is the unique solution on $\mathcal{G}_+^N$ that satisfies efficiency and equal minimal optimistic satisfaction. □

Proof of Lemma 4.19. Firstly, for all $\langle N, v\rangle \in \mathcal{G}^N_{++}$ and $t \in \mathbb{N}$, we have $v^t_{\lambda,O}(N) = v(N)$. Next, we show the convergence of the sequence of repeated optimistic associated games in two cases.

Case 1: $S \subseteq N$ **and** $|S| = 1$**.** We first show that

$$v^t_{\lambda,O}(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N),$$

for all $i \in N$ and $t \in \mathbb{N}$ by induction on t. When $t = 1$, by Definition 4.17, we have

$$v^1_{\lambda,O}(\{i\}) = (1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N),$$

for all $i \in N$. Suppose that $v^{t-1}_{\lambda,O}(\{i\}) = (1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)$. Then, we have

$$\begin{aligned}
v^t_{\lambda,O}(\{i\}) &= (1-\lambda)v^{t-1}_{\lambda,O}(\{i\}) + \frac{\lambda v^{t-1}_{\lambda,O}(\{i\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})}v(N)\\
=&(1-\lambda)\left((1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\right)\\
&+ \frac{\lambda\left((1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\right)}{\sum_{j\in N}\left((1-\lambda)^{t-1}v(\{j\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{j\})}{\sum_{k\in N} v(\{k\})}v(N)\right)}v(N)\\
=&(1-\lambda)^t v(\{i\}) + (1-\lambda)[1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\\
&+ \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N)\\
=&(1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N).
\end{aligned}$$

Thus, it holds that, for all $t \in \mathbb{N}$,

$$v^t_{\lambda,O}(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}v(N). \tag{4.1}$$

Therefore, for all $0 < \lambda < 1$, we have

$$\hat{v}(\{i\}) = \lim_{t\to\infty} v^t_{\lambda,O}(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N).$$

Case 2: $S \subseteq N$ **and** $|S| \geq 2$**.** For convenience, let $\rho_S = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}$ and $\sigma = v(N) - \sum_{k\in N} v(\{k\})$. Next, we will show that

$$\begin{aligned} v^t_{\lambda,O}(S) =& (1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\ &+ \lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}], \end{aligned} \tag{4.2}$$

for all $S \subseteq N, |S| \geq 2$ and $t \in \mathbb{N}$ by induction on t. When $t = 1$, by Definition 4.17, it holds that $v^1_{\lambda,O}(S) = (1-\lambda\rho_S)v(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)$, and Equation (4.2) holds. Without loss of generality, suppose that Equation (4.2) holds at $t-1$. Then, by Definition 4.17 and Equation (4.1), we have

$$\begin{aligned} v^t_{\lambda,O}(S) =& v^{t-1}_{\lambda,O}(S) + \lambda\frac{\sum_{k\in S} v^{t-1}_{\lambda,O}(\{k\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})}[v(N) - v^{t-1}_{\lambda,O}(S) - \sum_{k\in N\setminus S} v^{t-1}_{\lambda,O}(\{k\})] \\ =& (1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S[v(N) - (1-\rho_S)\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})] \\ =& (1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\ =& (1-\lambda\rho_S)^t v(S) + (1-\lambda\rho_S)[1-(1-\lambda\rho_S)^{t-1}]\rho_S v(N) \\ &+ \lambda\rho_S\sigma(1-\rho_S)(1-\lambda\rho_S)[\sum_{m=1}^{t-1}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-1-m}] \\ &+ \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\ =& (1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\ &+ \lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}]. \end{aligned}$$

Thus, Equation (4.2) holds for all $S \subseteq N, |S| \geq 2$ and $t \in \mathbb{N}$.

Let $a_t = \sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}$. Since $0 < \lambda < 1$ and $0 < \rho_S < 1$, we have $t(1-\lambda)^{t-1} \leq a_t \leq t(1-\lambda\rho_S)^{t-1}$. Since $\lim_{t\to\infty} t(1-\lambda)^{t-1} = 0$ and $\lim_{t\to\infty} t(1-\lambda\rho_S)^{t-1} = 0$, then $\lim_{t\to\infty} a_t = 0$. Thus, we have

$$\hat{v}(S) = \lim_{t\to\infty} v^t_{\lambda,O}(S) = \rho_S v(N) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N).$$

Therefore, the sequence of optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges and its limit game $\langle N, \hat{v}\rangle$ is given by $\hat{v}(S) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $S \subseteq N$. □

Proof of Theorem 4.20. It is easy to verify that the PD value satisfies continuity and the inessential game property. By Definition 4.17, we have $v_{\lambda,O}(N) = v(N)$ and for all $i \in N$,

$$\begin{aligned} v_{\lambda,O}(\{i\}) =& v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}[v(N) - \sum_{k\in N} v(\{k\})] \\ =& (1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) > 0. \end{aligned}$$

Then we have, for all $i \in N$

$$PD_i(N, v_{\lambda,O}) = \frac{v_{\lambda,O}(\{i\})}{\sum_{k\in N} v_{\lambda,O}(\{k\})} v_{\lambda,O}(N) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) = PD_i(N, v).$$

Therefore, the PD value satisfies optimistic associated consistency. It is left to show the uniqueness.

Now suppose that a solution φ on $\mathcal{G}^N_{++}$ satisfies these three axioms. For all $\langle N, v\rangle \in \mathcal{G}^N_{++}$, by Lemma 4.19, the sequence of optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges to an inessential game $\langle N, \hat{v}\rangle$. Then, by optimistic associated consistency and continuity, it holds that

$$\varphi(N, v) = \varphi(N, v^1_{\lambda,O}) = \varphi(N, v^2_{\lambda,O}) = \cdots = \varphi(N, \hat{v}).$$

By the inessential game property, we have

$$\varphi_i(N, \hat{v}) = v(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N),$$

for all $i \in N$. Thus, $\varphi(N,v) = PD(N,v)$. $\square$

Proof of Lemma 4.21. For all $\langle N,v\rangle \in \mathcal{G}_{\oplus}^{N}$, it holds that $v_{\lambda,P}^{t}(N) = v(N)$ and $v_{\lambda,P}^{t}(N\backslash\{i\}) = v(N\backslash\{i\})$ for all $i \in N$ and $t \in \mathbb{N}$. Then, we can obtain that $SC_i(N, v_{\lambda,P}^{t}) = SC_i(N,v)$ for all $i \in N$ and $t \in \mathbb{N}$. For convenience, let $\tau_S = \frac{\sum_{j\in S} SC_j(N,v)}{\sum_{j\in N} SC_j(N,v)}$. Next, we will prove that

$$v_{\lambda,P}^{t}(S) = (1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - \sum_{j\in N\backslash S} SC_j(N,v)] \quad (4.3)$$

for all $S \subseteq N, S \neq \emptyset$ and $t \in \mathbb{N}$, by induction on t. When $t=1$, by Definition 4.18, we have $v_{\lambda,P}^{1}(S) = (1-\lambda\tau_S)v(S) + \lambda\tau_S[v(N) - \sum_{j\in N\backslash S} SC_j(N,v)]$, and Equation (4.3) holds. Suppose that Equation (4.3) holds at $t-1$. Then, by Definition 4.18, we have

$$\begin{aligned}
v_{\lambda,P}^{t}(S) =& v_{\lambda,P}^{t-1}(S) + \lambda\frac{\sum_{j\in S} SC_j(N, v_{\lambda,P}^{t-1})}{\sum_{j\in N} SC_j(N, v_{\lambda,P}^{t-1})}[v(N) - v_{\lambda,P}^{t-1}(S) - \sum_{j\in N\backslash S} SC_j(N, v_{\lambda,P}^{t-1})] \\
=&(1-\lambda\tau_S)v_{\lambda,P}^{t-1}(S) + \lambda\tau_S[v(N) - \sum_{j\in N\backslash S} SC_j(N,v)] \\
=&(1-\lambda\tau_S)\{(1-\lambda\tau_S)^{t-1}v(S) + [1-(1-\lambda\tau_S)^{t-1}][v(N) - \sum_{j\in N\backslash S} SC_j(N,v)]\} \\
&+ \lambda\tau_S[v(N) - \sum_{j\in N\backslash S} SC_j(N,v)] \\
=&(1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - \sum_{j\in N\backslash S} SC_j(N,v)]
\end{aligned}$$

Thus, Equation (4.3) holds for all $S \subseteq N, S \neq \emptyset$ and $t \in \mathbb{N}$.

Due to $0 < \lambda < 1$ and $0 < \tau_S < 1$, for all $S \subseteq N, S \neq \emptyset$, we have

$$\check{v}(S) = \lim_{t\to\infty} v_{\lambda,P}^{t}(S) = v(N) - \sum_{j\in N\backslash S} SC_j(N,v).$$

Let $u(S) = \sum_{j\in S} SC_j(N,v)$ and $w(S) = v(N) - \sum_{j\in N} SC_j(N,v)$ for all $S \subseteq N, S \neq \emptyset$. Obviously, $\langle N,u\rangle$ is an inessential game and $\langle N,w\rangle$ is a constant game. The limit game $\langle N,\check{v}\rangle$ is given by $\check{v}(S) = u(S) + w(S)$. $\square$

Proof of Theorem 4.22. It is easy to verify that the PANSC value satisfies continuity and the constant inessential game property. By Definition 4.18, we have $v_{\lambda,P}(N) = v(N)$ and $v_{\lambda,P}(N\backslash\{i\}) = v(N\backslash\{i\})$ for all $i \in N$. Then, for all $i \in N$, we have

$$\begin{aligned} PANSC_i(N, v_{\lambda,P}) =& \frac{SC_i(N, v_{\lambda,P})}{\sum_{j\in N} SC_j(N, v_{\lambda,P})} v_{\lambda,P}(N) \\ =& \frac{SC_i(N, v)}{\sum_{j\in N} SC_j(N, v)} v(N) = PANSC_i(N, v) \end{aligned}$$

Therefore, the PANSC value satisfies pessimistic associated consistency. It is left to show the uniqueness.

Suppose that a solution φ on $\mathcal{G}^N_{\oplus}$ satisfies pessimistic associated consistency, continuity, the inessential game property and proportional constant additivity. For all $\langle N, v\rangle \in \mathcal{G}^N_{\oplus}$, by Lemma 4.21, the sequence of pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}_{t=0}^{\infty}$ converges to a game $\langle N, \check{v}\rangle$ which is expressed as the sum of a constant game $\langle N, w\rangle$ and an inessential game $\langle N, u\rangle$, where $w(S) = v(N) - \sum_{j\in N} SC_j(N, v)$ and $u(S) = \sum_{j\in S} SC_j(N, v)$ for all $S \subseteq N, S \neq \emptyset$. Obviously, $\langle N, \check{v}\rangle$ is a constant inessential game. By continuity and pessimistic associated consistency, we have

$$\varphi(N, v) = \varphi(N, v^1_{\lambda,P}) = \varphi(N, v^2_{\lambda,P}) = \cdots = \varphi(N, \check{v}).$$

By the constant inessential game property, for all $i \in N$

$$\varphi_i(N, \check{v}) = \frac{SC_i(N, v)}{\sum_{j\in N} SC_j(N, v)} v(N).$$

Therefore, $\varphi(N, v) = \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N) = PANSC(N, v)$. □

Proof of Lemma 4.25. Firstly, by Definition 4.17 and Definition 4.23, for all $\langle N, v\rangle \in \mathcal{G}^N_{++}$ and $S \subset N$, we have

$$\begin{aligned} &(v_{\lambda,O})^d(S) = v_{\lambda,O}(N) - v_{\lambda,O}(N\backslash S) \\ =& v(N) - v(N\backslash S) - \lambda \frac{\sum_{k\in N\backslash S} v(\{k\})}{\sum_{k\in N} v(\{k\})} [v(N) - v(N\backslash S) - \sum_{k\in S} v(\{k\})] \end{aligned}$$

$$=v^d(S)+\lambda\frac{\sum_{k\in N\backslash S}SC_k(N,v^d)}{\sum_{k\in N}SC_k(N,v^d)}[\sum_{k\in S}SC_k(N,v^d)-v^d(S)]$$
$$=(v^d)^*_{\lambda,O}(S).$$

For $S=N$, we have $(v_{\lambda,O})^d(N)=v(N)=(v^d)^*_{\lambda,O}(N)$. Thus, it holds that $(v_{\lambda,O})^d=(v^d)^*_{\lambda,O}$.

Secondly, by Definition 4.18 and Definition 4.24, for all $\langle N,w\rangle\in\mathcal{G}^N_{\oplus}$, we have

$$(w_{\lambda,P})^d(S)=w_{\lambda,P}(N)-w_{\lambda,P}(N\backslash S)$$
$$=w(N)-w(N\backslash S)-\lambda\frac{\sum_{k\in N\backslash S}SC_k(N,w)}{\sum_{k\in N}SC_k(N,w)}[w(N)-w(N\backslash S)-\sum_{k\in S}SC_k(N,w)]$$
$$=w^d(S)+\lambda\frac{\sum_{k\in N\backslash S}w^d(\{k\})}{\sum_{k\in N}w^d(\{k\})}[\sum_{k\in S}w^d(\{k\})-w^d(S)]$$
$$=(w^d)^*_{\lambda,P}(S).$$

For $S=N$, we have $(w_{\lambda,P})^d(N)=w(N)=(w^d)^*_{\lambda,P}(N)$. Thus, it holds that $(w_{\lambda,P})^d=(w^d)^*_{\lambda,P}$. □

Proof of Proposition 4.26. Given a solution φ on $\mathcal{G}^N_{++}$, let φ^d be the dual of φ. We just prove that φ satisfies optimistic associated consistency if and only if φ^d satisfies dual optimistic associated consistency.

If φ satisfies optimistic associated consistency, for all $\langle N,v\rangle\in\mathcal{G}^N_{\oplus\oplus}$ and its dual game $\langle N,v^d\rangle\in\mathcal{G}^N_{++}$, we have

$$\varphi^d(N,v)=\varphi(N,v^d)=\varphi(N,(v^d)_{\lambda,O})=\varphi(N,(v^*_{\lambda,O})^d)=\varphi^d(N,v^*_{\lambda,O}),$$

where the third equation holds by Lemma 4.25. Thus, φ^d satisfies dual optimistic associated consistency.

If φ^d satisfies dual optimistic associated consistency, for all $\langle N,v\rangle\in\mathcal{G}^N_{++}$ and its dual game $\langle N,v^d\rangle\in\mathcal{G}^N_{\oplus\oplus}$, we have

$$\varphi(N,v)=\varphi^d(N,v^d)=\varphi^d(N,(v^d)^*_{\lambda,O})=\varphi^d(N,(v_{\lambda,O})^d)=\varphi(N,v_{\lambda,O}).$$

Then, φ satisfies optimistic associated consistency. □

4.5 Conclusions

In this chapter, we introduce the family of the OS payoff vectors from the perspective of satisfaction criteria. According to the optimistic satisfaction criterion and the pessimistic satisfaction criterion, the PD value and the PANSC value are determined by lexicographically maximizing the corresponding minimal satisfaction. Then, we characterize these two proportional values by introducing the equal minimal optimistic/pessimistic satisfaction axioms, the optimistic/pessimistic associated consistency axioms and their dual axioms. As two representative values of the proportionality principle, the PD value and the PANSC value are relatively fair and reasonable allocations applied in many economic situations. For instance, in China's bankruptcy law, the bankruptcy assets shall be distributed on a proportionality principle when it is insufficient to pay off all debts. The proportionality principle is deeply rooted in law and custom as a norm of distributive justice.

In the future, we will study other characterizations of the PD value and the PANSC value relying on some existing characterizations of classical solutions for TU-games. Coordinating the optimistic satisfaction and the pessimistic satisfaction, we may elicit combinations of the PD value and the PANSC value by an underlying neutral satisfaction criterion, and apply them to some real situations.

Chapter 5

Characterizations of the weighted division values

5.1 Introduction

As introduced in Chapter 1, the Shapley value (Shapley 1953, [94]) and the equal division value are basic solutions for TU-games. In order to interpret asymmetries among players beyond the game, Shapley (1953, [93]) proposed a weighted version of the Shapley value, namely the positively weighted Shapley values, where these asymmetries are modelled by strictly positive weights for the players. Subsequently, Kalai and Samet (1987, [59]) extended the positively weighted Shapley values by taking into account a weight system that allows for zero weights of the players. There exists a number of axiomatic foundations for the class of weighted Shapley values in the literature (see, e.g., Besner 2020, [14]; Calvo and Santos 2000, [15]; Casajus 2018, [18]; Casajus 2019, [20]; Casajus 2021, [21]; Chun 1991, [27]; Hart and Mas-Colell 1989, [46]; Kalai and Samet 1987, [59]; Nowak and Radzik 1995, [86]; Yokote 2015, [120]). Similar to the Shapley value, the equal division value is also generalized by considering asymmetries between players. Given non-negative exogenous player weights, the corresponding weighted division value allocates the worth of the grand coalition (consisting of all players) proportional to these weights. If all weights are positive, we call it a positively weighted division value. In a sense, the weighted division values generalize the equal division value as the weighted Shapley values generalize the Shapley value. In this chapter, which is based on Li et al. (2021, [72]), we refer to the weighted division values when the weight vector has non-negative coordinates, and to the positively weighted division values for the subclass of weighted division values with only strictly positive weights.

A major purpose of axiomatizing solutions in TU-games is to show the reasonability of solutions by using some desirable axioms. A well-known axiomatization of the Shapley value involves efficiency, additivity, the null player property and symmetry. Symmetry requires that equally productive players should get the same payoff. Casajus (2019, [20]) suggested a relaxation of symmetry, called sign symmetry. Sign symmetry is a considerable weakening of symmetry since, instead of requiring equal payoffs for equally productive players, it only requires the payoffs to have the same sign. Notice that, in case there are differences between players that are not modelled in the game, requiring equal payoffs (symmetry) might be too strong.[1] Sign symmetry allows players that are equally productive in the game model to get a different share in the worth to be allocated, but still requires that equally productive players either all get a positive, or all get a negative or all get a zero share in that worth. Though sign symmetry is a considerable weakening of symmetry, Casajus (2019, [20]) showed that replacing symmetry by sign symmetry in the original axiomatization of the Shapley value still characterizes this value.[2] In van den Brink (2007, [104]) an axiomatization of the equal division value by using efficiency, additivity, the nullifying player property and symmetry is proposed. This triggers

[1] For example, in river games (see Ambec and Sprumont (2002, [4])) it can be that two players are equally productive in the game, but have a different role in the sense that the upstream player has access to the water that flows on its territory, while the downstream player derives utility from it. Although both players are needed to generate worth, they clearly have a different role that is no longer visible in the game.

[2] Casajus and Yokote (2017, [25]) showed that the fairness, or differential marginality, axiom in the axiomatization of the Shapley value given by van den Brink (2002, [103]), respectively Casajus (2011, [17]), can be replaced by a weaker sign version to characterize the Shapley value.

the question whether sign symmetry can serve as a substitute for symmetry in this axiomatization of the equal division value. Though this is not possible in the sense that it does not characterize the equal division value, interestingly, we can characterize the class of positively weighted division values by replacing symmetry in van den Brink's axiomatization with sign symmetry. Furthermore, we also suggest a weak version of sign symmetry, called weak sign symmetry, that requires equally productive players' payoffs to not have opposite signs. We show that weak sign symmetry together with efficiency, additivity and the nullifying player property characterizes the class of weighted division values.

There exist several axiomatic characterizations for the class of weighted division values in the literature (see, e.g., Béal et al. 2016, [8]; Béal et al. 2015, [11]; Kongo 2019, [66] and van den Brink 2009, [105]). Béal et al. (2016, [8]) introduced three different axiomatizations of the class of weighted division values. The first axiomatization involves efficiency, linearity, the nullifying player property and the null player in a productive environment property. The second axiomatization involves efficiency, linearity and the non-negative player property. The third axiomatization involves efficiency, linearity and nullified solidarity. Two common axioms used in these axiomatizations are efficiency and linearity. In their concluding remarks, Béal et al. (2016, [8]) state the claim that linearity can not be weakened to additivity in these axiomatizations. In this chapter, we show that the class of weighted division values can also be characterized by replacing linearity in the axiomatizations of Béal et al. (2016, [8]) with additivity. Moreover, we provide a condition that allows us to use additivity instead of linearity. This condition, called standard positivity, says that all players earn a non-negative payoff in a non-negative scaled standard game.

Inspired by the work of Béal et al. (2016, [8]), we suggest stronger versions of the null player in a productive environment property, the non-negative player property and nullified solidarity, called the sign null player in a productive environment property, the sign non-negative player property and sign nullified solidarity, respectively. The sign null player in a productive environment property requires that a null player is rewarded (by a positive payoff) or punished (by a negative payoff) or gets zero depending on whether the worth of the grand coalition is positive or negative or zero. The sign non-negative player property requires that a non-negative player will get a positive payoff if the worth of the grand coalition is positive, and will get nothing otherwise. Sign nullified solidarity requires that the payoffs for all players change in the same direction in case a specified player becomes a null player. We show that the positively weighted division values can be characterized by using these stronger axioms instead of the corresponding axioms in the axiomatizations of Béal et al. (2016, [8]).

The rest of this chapter is organized as follows. In Section 5.2, we characterize the classes of weighted division values and positively weighted division values by using relaxations of symmetry. In Section 5.3, we replace linearity in the axiomatizations of Béal et al. (2016, [8]) with additivity to characterize the class of weighted division values, and provide three axiomatizations of the class of positively weighted division values by strengthening one of the axioms in each of these axiomatizations. In Section 5.4, we provide a discussion between linearity and additivity. Section 5.5 provides all proofs of this chapter. Section 5.6 concludes with a brief summary.

5.2 Relaxations of symmetry and the weighted division values

In van den Brink (2007, [104]), the equal division value is characterized by efficiency, symmetry, additivity and the nullifying player property. It is clear that the weighted division values, except the equal division value, fail symmetry. Casajus (2018, [19]) introduced a relaxation of symmetry called sign symmetry, and showed that replacing symmetry by sign symmetry in the original axiomatization of the Shapley value still characterizes the Shapley value. Sign symmetry is a qualitative version of symmetry that is weaker than symmetry. Instead of equating payoffs for symmetric players, it just fixes a common reference point, the zero utility, and requires that symmetric players are either rewarded simultaneously (positive payoff) or punished simultaneously (negative payoff) or all get a zero payoff. Recall the sign function, sign: $\mathbb{R} \to \{-1, 0, 1\}$ given by $\text{sign}(t) = 1$ for $t > 0$, $\text{sign}(0) = 0$, and $\text{sign}(t) = -1$ for $t < 0$.

- **Sign symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i, j \in N$ are symmetric players in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(\varphi_j(N, v))$.

Sign symmetry is a considerable weakening of symmetry. One easily checks that the positively weighted division values satisfy sign symmetry. Next, we provide a characterization of the class of positively weighted division values by using sign symmetry instead of symmetry appearing in van den Brink's characterization (van den Brink 2007, [104]) for the equal division value.

Theorem 5.1. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and sign symmetry if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

The proof of Theorem 5.1 and of all other results in this chapter can be found in Section 5.5.

Logical independence of the axioms used in Theorem 5.1 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N, v) = 0$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by

$$\varphi_i(N, v) = \begin{cases} \frac{v(\{i\})^2}{\sum_{j\in N} v(\{j\})^2} v(N), & \text{if } \sum_{j\in N} v(\{j\})^2 \neq 0; \\ \frac{1}{n} v(N), & \text{if } \sum_{j\in N} v(\{j\})^2 = 0, \end{cases} \tag{5.1}$$

for all $\langle N, v\rangle \in \mathcal{G}^N$, satisfies all axioms except additivity.

(iii) The Shapley value Sh (Shapley 1953, [94], see Chapter 1) satisfies all axioms except the nullifying player property.

(iv) The weighted division values that are not positively weighted division values satisfy all axioms except sign symmetry.

The fourth example shows that, although the positively weighted division values satisfy sign symmetry, the other weighted division values fail sign symmetry. Next, we introduce a further relaxation of (sign) symmetry, called weak sign symmetry, that is satisfied by all weighted division values.[3]

- **Weak sign symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i, j \in N$ are symmetric players in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) > 0$ implies $\varphi_j(N, v) \geq 0$.

Since players i and j are interchangeable, the contraposition of the implication in weak sign symmetry entails that $\varphi_i(N, v) < 0$ implies $\varphi_j(N, v) \leq 0$. Weak sign symmetry relaxes sign symmetry: instead of requiring equal signs, it only rules out opposite signs. Weakening sign symmetry in this way in Theorem 5.1, we obtain a characterization of the class of all weighted division values.

Theorem 5.2. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and weak sign symmetry if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

It can be seen in Section 5.5 that the proof of Theorem 5.2 is almost the same as that of Theorem 5.1, where weak sign symmetry implies the same conclusions as sign symmetry, except that weak sign symmetry does not imply the weights (i.e. payoffs in $\langle N, e_N\rangle$) to be strictly positive.

Logical independence of the axioms in Theorem 5.2 can be shown by the same alternative solutions (i), (ii) and (iii) as those used to show logical independence of the axioms in Theorem 5.1, and replacing alternative solution (iv) by WD^ω with $\omega \in \{\omega \in \mathbb{R}^N | \sum_{i\in N} \omega_i = 1$ and $\omega_i < 0$ for at least one $i \in N\}$.

[3]This is different than weak sign symmetry as defined in Casajus (2019, [20]) to characterize the class of weighted Shapley values (Shapley 1953, [93]), which requires that the payoffs of mutually dependent players have the same sign.

5.3 Axiomatizations using null player related axioms

In Chapter 1, we revisited three axiomatizations of the class of weighted division values (see Theorem 1.17, 1.18, 1.21) proposed by Béal et al. (2016, [8]). In this section, we show that the class of weighted division values can also be characterized by replacing linearity in the axiomatizations of Béal et al. (2016, [8]) with the weaker additivity. Moreover, we also characterize the class of positively weighted division values by introducing stronger versions of the null player in a productive environment property, the non-negative player property and nullified solidarity.

5.3.1 Sign null player in a productive environment property

Firstly, we give an alternative axiomatization of the class of weighted division values by replacing linearity with additivity in Theorem 1.18.

Theorem 5.3. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and the null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms used in Theorem 5.3 can be shown by the alternative solutions mentioned in Béal et al. (2016, [8]) to show logical independence of the axioms in Theorem 1.18, since their example that is used to show that linearity is independent of the other axioms also does not satisfy additivity.

In Theorem 5.3, the null player in a productive environment property is used to characterize the class of weighted division values. In a sense, the null player in a productive environment property is not strong enough to generate only positively weighted division values. Next, we strengthen the null player in a productive environment property to characterize the class of positively weighted division values.[4]

[4]Notice that the 'sign' axioms in this section strengthen known null player related axioms, while sign symmetry weakened symmetry.

- **Sign null player in a productive environment property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a null player in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(v(N))$.

It is clear that the sign null player in a productive environment property is stronger than the null player in a productive environment property. Instead of only the non-negativity restrictions, the sign null player in a productive environment property requires that a null player is rewarded (positive payoff) or punished (negative payoff) or gets zero depending on whether the worth of the grand coalition is positive or negative or zero. One easily checks that the positively weighted division values satisfy the sign null player in a productive environment property, but the other weighted division values do not. Next, we provide a characterization of the class of positively weighted division values by replacing the null player in a productive environment property in Theorem 5.3 with the sign null player in a productive environment property.

Theorem 5.4. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and the sign null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms used in Theorem 5.4 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N, v) = v(N)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by Eq.(5.1), satisfies all axioms except additivity.

(iii) For $\alpha \in [0, 1]$, the corresponding α-egalitarian Shapley value Sh^α, introduced by Joosten (1996, [55]), is defined by

$$Sh_i^\alpha(N, v) = \alpha Sh_i(N, v) + (1-\alpha)\frac{v(N)}{n}, \tag{5.2}$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$. The α-egalitarian Shapley values with $0 < \alpha < 1$ satisfy all axioms except the nullifying player property.

(iv) The weighted division values that are not positively weighted division values satisfy all axioms except the sign null player in a productive environment property.

5.3.2 Sign non-negative player property

Similar as in the previous subsection, we first give an alternative axiomatization of the class of weighted division values by replacing linearity in Theorem 1.17 with additivity, and after that strengthen one of the axioms (related to null players) to obtain the positively weighted division values.

Theorem 5.5. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and the non-negative player property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms can be shown using the same alternative solutions used to show logical independence of the axioms in Theorem 4 (that is Theorem 1.17 in Chapter 1) in Béal et al. (2016, [8]), since their example that does not satisfy linearity, also does not satisfy additivity.

The non-negative player property is not strong enough to generate only positively weighted division values. To characterize the class of positively weighted division values, we strengthen the non-negative player property requiring the payoff of a player to be positive (respectively zero) if the worth of the grand coalition is positive (respectively zero).

- **Sign non-negative player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a non-negative player in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(v(N))$.

One easily checks that the positively weighted division values satisfy the sign non-negative player property. Next, we provide a characterization of the class of positively weighted division values by using the sign non-negative player property.

Theorem 5.6. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and the sign non-negative player property if and only if there exists a weight vector $\omega \in \Delta_{++}^N$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms used in Theorem 5.6 can be shown by the same alternative solutions (i), (ii) and (iii) (or (iv)) showing logical independence of the axioms in Theorem 5.4, where the first two examples also satisfy the stronger sign non-negative player property.

5.3.3 Sign nullified solidarity

Next, we replace linearity in Theorem 1.21 with additivity to give an alternative axiomatization of the class of weighted division values.

Theorem 5.7. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and nullified solidarity if and only if there exists a weight vector $\omega \in \Delta^N_+$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms used in Theorem 5.7 can be shown by the same alternative solutions as used to show logical independence of the axioms in Theorem 1.21, in Béal et al. (2016, [8]).

Finally, we strengthen nullified solidarity to characterize the class of positively weighted division values in a similar way as the sign non-negative player property is stronger than the non-negative player property.

- **Sign nullified solidarity.** For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N$, it holds that sign$(\varphi_i(N, v) - \varphi_i(N, v^{\mathbf{N}i}))$ = sign$(\varphi_j(N, v) - \varphi_j(N, v^{\mathbf{N}i}))$.

Next, we provide a characterization of the class of positively weighted division values by using sign nullified solidarity.

Theorem 5.8. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and sign nullified solidarity if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

Logical independence of the axioms used in Theorem 5.8 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N, v) = v(N)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by $\varphi_i(N, v) = ED_i(N, v) + t_i$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, where $t \in \mathbb{R}^N$ is such that $\sum_{i\in N} t_i = 0$ and $t_i \neq 0$ for some $i \in N$, satisfies all axioms except additivity.

(iii) The α-egalitarian Shapley value Sh^α, defined by Eq.(5.2), satisfies all axioms except sign nullified solidarity.

5.4 Discussion between linearity and additivity

In the previous axiomatizations of the class of weighted division values, we use additivity instead of linearity. This triggers the question under what conditions additivity can replace linearity. In this section, we define a new axiom, called standard positivity, which provides a condition that allows us to use additivity instead of linearity.

- **Standard positivity.** For all $T \subseteq N$, $T \neq \emptyset$ and $c \in \mathbb{R}_+$, it holds that $\varphi_i(N, ce_T) \geq 0$ for all $i \in N$.

Standard positivity is a weak version of the non-negative player property, and requires that each player obtains a non-negative payoff in every non-negative scalar multiplication a standard game.

Proposition 5.9. *If solution φ on $\mathcal{G}^N$ satisfies additivity and standard positivity, then φ satisfies linearity.*

By the above proposition, linearity can be replaced by additivity in an axiomatization that includes axioms that imply standard positivity. The following corollary, which is a straightforward conclusion from the proofs of Theorem 5.1, 5.3, 5.5 and 5.7, states that the axioms in the axiomatizations of the class of weighted division values provided in this chapter imply standard positivity.

Corollary 5.10. *If solution φ on $\mathcal{G}^N$ satisfies one of the following four conditions:*

(i) efficiency, the nullifying player property and sign symmetry;

(ii) efficiency, additivity, the nullifying player property and the null player in a productive environment property;

(iii) the non-negative player property;

(iv) efficiency, additivity and nullified solidarity;

then, φ satisfies standard positivity.

Remark 5.1. In this corollary, the axioms mentioned in (i) and (iii) appear in Theorem 5.1 and 5.5 respectively. Notice that additivity does not appear in (i) and (iii), but appears in Theorem 5.1 and 5.5. The axioms mentioned in (ii) and (iv) are the same as those in Theorem 5.3 and 5.7 respectively.

5.5 Proofs

Proof of Theorem 5.1. It is straightforward to verify that the positively weighted division values satisfy efficiency, sign symmetry, additivity and the nullifying player property. It is left to show that the axioms determine that φ is a positively weighted division value.

Let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. We will show that for some weight vector $\omega \in \Delta^N_{++}$, $\varphi = WD^\omega$. Let $c \in \mathbb{R}$. Firstly, for the null game $\langle N, \mathbf{0}\rangle$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$, efficiency and sign symmetry imply that $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$.[5]

Secondly, we consider $\langle N, ce_T\rangle$ for $\emptyset \neq T \subsetneq N$. By the nullifying player property, we have $\varphi_i(N, ce_T) = 0$ for all $i \in N \backslash T$. Then, by efficiency, we have $\sum_{i\in T} \varphi_i(N, ce_T) = 0$. Since players $i, j \in T$ are symmetric players in $\langle N, ce_T\rangle$, by sign symmetry, we have $\varphi_i(N, ce_T) = 0$ for all $i \in T$. Thus, $\varphi_i(N, ce_T) = 0$ for all $\emptyset \neq T \subsetneq N$ and $i \in N$.

Thirdly, we consider $\langle N, ce_N\rangle$. Set $\omega_i = \varphi_i(N, e_N)$ for all $i \in N$. By efficiency and sign symmetry, we have $\sum_{i\in N} \omega_i = 1$ and $\omega_i > 0$ for all $i \in N$, showing that $\omega \in \Delta^N_{++}$. Now, we show that $\varphi(N, ce_N) = c\varphi(N, e_N)$. Choose two sequences of rationals $\{r_k\}_{k=1}^\infty$ and $\{s_k\}_{k=1}^\infty$ which converge to

[5] Notice that this also follows from additivity since $\varphi_i(N, \mathbf{0}) + \varphi_i(N, \mathbf{0}) = \varphi_i(N, \mathbf{0})$ implies that $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$.

c from above and below, respectively. We obtain that, for all $i \in N$ and $k \in \mathbb{N}$,

$$\begin{aligned} \varphi_i(N, r_k e_N) - \varphi_i(N, c e_N) =& \varphi_i(N, (r_k - c) e_N) \geq 0, \text{ and} \\ \varphi_i(N, c e_N) - \varphi_i(N, s_k e_N) =& \varphi_i(N, (c - s_k) e_N) \geq 0, \end{aligned} \tag{5.3}$$

where in both cases the equality follows from additivity and the inequality follows from efficiency and sign symmetry. Notice that additivity[6] also implies that for all $i \in N$, $\varphi_i(N, r_k e_N) - \varphi_i(N, s_k e_N) = \varphi_i(N, (r_k - s_k) e_N) = (r_k - s_k)\varphi_i(N, e_N) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$. Then, we have $\varphi_i(N, r_k e_N) - \varphi_i(N, c e_N) + \varphi_i(N, c e_N) - \varphi_i(N, s_k e_N) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_N) - \varphi_i(N, c e_N) \geq 0$ and $\varphi_i(N, c e_N) - \varphi_i(N, s_k e_N) \geq 0$ by Eq.(5.3), this implies that $\varphi(N, r_k e_N) \to \varphi(N, c e_N)$ as $k \to \infty$. Since $\varphi(N, r_k e_N) = r_k \varphi(N, e_N) \to c\varphi(N, e_N)$ (where the equality follows by additivity, see Footnote 6) and $\varphi(N, r_k e_N) \to \varphi(N, c e_N)$ as $k \to \infty$, we have that $\varphi(N, c e_N) = c\varphi(N, e_N)$ for any scalar $c \in \mathbb{R}$.

Therefore, for all $\langle N, v \rangle \in \mathcal{G}^N$ and $i \in N$, with additivity it holds that

$$\begin{aligned} \varphi_i(N, v) =& \sum_{T \subseteq N} \varphi_i(N, v(T) e_T) = \varphi_i(N, v(N) e_N) \\ =& v(N)\varphi_i(N, e_N) = \omega_i v(N). \end{aligned}$$

The proof is completed. □

Proof of Theorem 5.2. It is clear that all the weighted division values satisfy efficiency, weak sign symmetry, additivity and the nullifying player property. To show that the axioms determine that the solution is a weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. We will show that for some weight vector $\omega \in \Delta_+^N$, $\varphi = WD^\omega$. By the nullifying player property (or additivity, see Footnote 5), we have $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$, where $\langle N, \mathbf{0} \rangle$ is the null game given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. Let $c \in \mathbb{R}$. Similar as in the proof of Theorem 5.1, for

[6] Given any rational r_k, there must exist two integers $a, b \in \mathbb{N}$, $a \neq 0$, such that $r_k = \frac{b}{a}$. Then, by additivity, we have $\varphi(N, r_k e_N) = \varphi(N, \frac{b}{a} e_N) = b\varphi(N, \frac{1}{a} e_N) = \frac{b}{a} \cdot a\varphi(N, \frac{1}{a} e_N) = \frac{b}{a}\varphi(N, \frac{a}{a} e_N) = r_k \varphi(N, e_N)$.

$\emptyset \neq T \subsetneq N$, (i) by the nullifying player property, $\varphi_i(N, ce_T) = 0$ for all $i \in N \backslash T$, and (ii) by efficiency and weak sign symmetry, $\varphi_i(N, ce_T) = 0$ for all $i \in T$. Set $\omega_i = \varphi_i(N, e_N)$ for all $i \in N$. By efficiency and weak sign symmetry, we have $\sum_{i \in N} \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N$, showing that $\omega \in \Delta_+^N$. Again similar as in the proof of Theorem 5.1, by efficiency, weak sign symmetry and additivity, we can also show that $\varphi(N, ce_N) = c\varphi(N, e_N)$. Finally, by additivity, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, we have $\varphi_i(N, v) = \sum_{T \subseteq N} \varphi_i(N, v(T)e_T) = \omega_i v(N)$. □

Proof of Theorem 5.3. Since linearity implies additivity, by Theorem 1.18, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity, the nullifying player property and the null player in a productive environment property. Also, by Theorem 1.18, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. By the nullifying player property this is obviously satisfied for $c = 0$. Notice that

$$\varphi(N, -cv) = \varphi(N, \mathbf{0}) - \varphi(N, cv) = -\varphi(N, cv),$$

where the first equality follows from additivity and the second equality follows from the nullifying player property. Then it suffices to show that $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and positive scalar $c \in \mathbb{R}_{++}$.

Let $c \in \mathbb{R}_{++}$. Firstly, we consider $\langle N, ce_T\rangle$ for $\emptyset \neq T \subsetneq N$. By the nullifying player property, we have $\varphi_i(N, ce_T) = 0$ for all $i \in N \backslash T$. Now we show that $\varphi_i(N, ce_T) = 0$ for all $i \in T$. If $T = \{i\}$, then by the nullifying player property we have $\varphi_j(N, ce_{\{i\}}) = 0$ for all $j \in N \setminus \{i\}$, and consequently by efficiency $\varphi_i(N, ce_{\{i\}}) = 0$. Now, suppose that $T \supsetneq \{i\}$. Set $w_T^i = e_T + e_{T\backslash\{i\}}$.[7] Since player i is a null player in $\langle N, cw_T^i\rangle$ and $cw_T^i(N) = 0$, we have

$$\varphi_i(N, ce_T) = \varphi_i(N, cw_T^i) - \varphi_i(N, ce_{T\backslash\{i\}}) \geq -\varphi_i(N, ce_{T\backslash\{i\}}),$$

[7] These are the same games that are used by Béal et al. (2016, [8]).

where the equality follows from additivity and the inequality from the null player in a productive environment property. By the nullifying player property, $\varphi_i(N, ce_{T\backslash\{i\}}) = 0$, and thus $\varphi_i(N, ce_T) \geq 0$ for all $i \in T$. Since we already showed that $\varphi_i(N, ce_T) = 0$ for all $i \in N\backslash T$, and $ce_T(N) = 0$ for $T \subsetneq N$, efficiency implies that $\varphi_i(N, ce_T) = 0$ for all $i \in T$.

Secondly, we show that $\varphi(N, ce_N) = c\varphi(N, e_N)$ for any scalar $c \in \mathbb{R}_{++}$. For all $i \in N$, we have

$$\varphi_i(N, ce_N) = \varphi_i(N, ce_N) + \varphi_i(N, ce_{N\backslash\{i\}}) = \varphi_i(N, ce_N + ce_{N\backslash\{i\}}) \geq 0, \tag{5.4}$$

where the first equality follows from the nullifying player property, the second equality follows from additivity, and the inequality follows from the null player in a productive environment property. Similar as in the proof of Theorem 5.1, choose two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively. By additivity, efficiency and Eq.(5.4), we obtain the same inequalities (5.3) as in the proof of Theorem 5.1, that is, for all $i \in N$ and $k \in \mathbb{N}$,

$$\begin{aligned} \varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) =& \varphi_i(N, (r_k - c)e_N) \geq 0, \text{ and} \\ \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) =& \varphi_i(N, (c - s_k)e_N) \geq 0. \end{aligned}$$

Similar as in the proof of Theorem 5.1[8], it follows that $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$, which concludes the proof. □

Proof of Theorem 5.4. It is straightforward to verify that the positively weighted division values satisfy the sign null player in a productive environment property.

[8] This is shown identical as in the proof of Theorem 5.1 as follows: For all $i \in N$, $\varphi_i(N, r_k e_N) - \varphi_i(N, s_k e_N) = \varphi_i(N, (r_k - s_k)e_N) = (r_k - s_k)\varphi_i(N, e_N) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$, and additivity. Then $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) + \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) \geq 0$ and $\varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \geq 0$, this implies that $\varphi(N, r_k e_N) \to \varphi(N, ce_N)$ and $\varphi(N, r_k e_N) = r_k\varphi(N, e_N) \to c\varphi(N, e_N)$ as $k \to \infty$, which proves that $\varphi(N, ce_N) = c\varphi(N, e_N)$ for any scalar $c \in \mathbb{R}_{++}$. Hence, by additivity, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$, which concludes the proof.

To show that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. Since the sign null player in a productive environment property is stronger than the null player in a productive environment property, by Theorem 5.3, φ is a weighted division value WD^ω for some $\omega \in \Delta_+^N$. We are left to prove that $\omega_i > 0$ for all $i \in N$. For all $i \in N$, let $\omega_i = \varphi_i(N, e_N)$. Then, for all $i \in N$, we have

$$\omega_i = \varphi_i(N, e_N) = \varphi_i(N, e_N) + \varphi_i(N, e_{N\setminus\{i\}}) = \varphi_i(N, e_N + e_{N\setminus\{i\}}) > 0,$$

where the second equality follows from the nullifying player property, the third equality follows from additivity, and the inequality follows from the sign null player in a productive environment property. Thus, $\omega \in \Delta_{++}^N$. $\square$

Proof of Theorem 5.5. By Theorem 1.17, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity and the non-negative player property. Similar as before, by Theorem 1.17, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$.

Similar as in previous proofs, choose two sequences of rationals $\{r_k\}_{k=1}^\infty$ and $\{s_k\}_{k=1}^\infty$ which converge to c from above and below, respectively. By additivity and the non-negative player property, we obtain that, for all $S \subseteq N$, $i \in N$ and $k \in \mathbb{N}$,

$$\begin{aligned}\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) =& \varphi_i(N, (r_k - c)e_S) \geq 0, \text{ and} \\ \varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) =& \varphi_i(N, (c - s_k)e_S) \geq 0.\end{aligned}$$

Now, we have $\varphi_i(N, r_k e_S) - \varphi_i(N, s_k e_S) = \varphi_i(N, (r_k - s_k)e_S) = (r_k - s_k)\varphi_i(N, e_S) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$, and additivity. Then $\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) + \varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) \geq 0$ and $\varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) \geq 0$, this implies that $\varphi(N, r_k e_S) \to \varphi(N, ce_S)$ and $\varphi(N, r_k e_S) = r_k\varphi(N, e_S) \to c\varphi(N, e_S)$ as $k \to \infty$, which proves that $\varphi(N, ce_S) = c\varphi(N, e_S)$ for any scalar $c \in \mathbb{R}$. Hence, similar as in previous proofs by additivity, $\varphi(N, cv) =$

$c\varphi(N,v)$ for all $\langle N,v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$, which concludes the proof. □

Proof of Theorem 5.6. It is straightforward to verify that the positively weighted division values satisfy the sign non-negative player property.

To show that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the three mentioned axioms. Since the sign non-negative player property is stronger than the non-negative player property, by Theorem 5.5, φ is a weighted division value WD^ω for some $\omega \in \Delta_+^N$ and, by its proof, $\omega_i = \varphi_i(N, e_N)$. We are left to prove that $\omega_i > 0$ for all $i \in N$. By the sign non-negative player property, we have $\omega_i = \varphi_i(N, e_N) > 0$ for all $i \in N$, showing that $\omega \in \Delta_{++}^N$. □

Proof of Theorem 5.7. By Theorem 1.21, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity and nullified solidarity. Similar as before, by Theorem 1.21, it suffices to show that φ is homogeneous, that is, $\varphi(N,cv) = c\varphi(N,v)$ for all $\langle N,v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. By additivity, this is satisfied for $c = 0$, since $\varphi_i(N,\mathbf{0}) = 0$ for all $i \in N$ (See Footnote 5). Also by additivity, we have

$$\varphi(N,-cv) = \varphi(N,\mathbf{0}) - \varphi(N,cv) = -\varphi(N,cv). \tag{5.5}$$

Then it suffices to show that $\varphi(N,cv) = c\varphi(N,v)$ for all $\langle N,v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$.

Let $c \in \mathbb{R}_{++}$. For all $S \subseteq N$ and $i \in S$, $(ce_S)^{\mathbf{N}i} = \mathbf{0}$, where $\langle N,(ce_S)^{\mathbf{N}i}\rangle$ is the TU-game where player i is nullified as defined by Eq.(1.2). Then, we have $\varphi_j(N,(ce_S)^{\mathbf{N}i}) = \varphi_j(N,\mathbf{0}) = 0$ for all $j \in N$. Now we show that $\varphi_i(N,ce_S) \geq 0$ for all $S \subseteq N$ and $i \in S$. On the contrary, suppose that there are $S \subseteq N$ and $i \in S$ such that $\varphi_i(N,ce_S) < 0$. By Eq.(5.5), we then have $\varphi_i(N,-ce_S) > 0$, and thus by $\varphi_i(N,(ce_S)^{\mathbf{N}i}) = \varphi_i(N,\mathbf{0}) = 0$, we have $\varphi_i(N,-ce_S) > \varphi_i(N,(-ce_S)^{\mathbf{N}i})$. Then, by nullified solidarity, $\varphi_j(N,-ce_S) \geq \varphi_j(N,(-ce_S)^{\mathbf{N}i}) = 0$ for all $j \in N$. Thus, we obtain

$\sum_{j\in N}\varphi_j(N,-ce_S)>0$, which is in contradiction with the fact that φ satisfies efficiency. So, we conclude that, $\varphi_i(N,ce_S)\geq 0$ for all $S\subseteq N$ and $i\in S$. Therefore, by nullified solidarity, $\varphi_i(N,ce_S)\geq 0=\varphi_i(N,(ce_S)^{\mathbf{N}i})$ implies that $\varphi_j(N,ce_S)\geq\varphi_j(N,(ce_S)^{\mathbf{N}i})=0$ for all $S\subseteq N$, $i\in S$ and $j\in N$. That is, $\varphi_j(N,ce_S)\geq 0$ for all $j\in N$. Similar as in the proof of Theorem 5.5, choosing two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively, we can prove that $\varphi(N,ce_S)=c\varphi(N,e_S)$ for all $S\subseteq N$ and scalar $c\in\mathbb{R}_{++}$. Hence, by additivity, $\varphi(N,cv)=c\varphi(N,v)$ for all $\langle N,v\rangle\in\mathcal{G}^N$ and scalar $c\in\mathbb{R}_{++}$, which concludes the proof. □

Proof of Theorem 5.8. It is straightforward to verify that the positively weighted division values satisfy sign nullified solidarity.

To prove that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the three mentioned axioms. Since sign nullified solidarity is stronger than nullified solidarity, by Theorem 1.21, φ is a weighted division value WD^{ω} for some $\omega\in\Delta_+^N$. We are left to prove that $\omega_i>0$ for all $i\in N$. Since $\varphi_i(N,(ce_N)^{\mathbf{N}i})=\varphi_i(N,\mathbf{0})=0$ for all $i\in N$, by sign nullified solidarity, we have $\text{sign}(\varphi_i(N,e_N))=\text{sign}(\varphi_j(N,e_N))$ for all $i,j\in N$. Thus, by efficiency, we have $\omega_i=\varphi_i(N,e_N)>0$ for all $i\in N$, showing that $\omega\in\Delta_{++}^N$. □

Proof of Proposition 5.9. Since φ is assumed to be additive, we only prove that φ is homogeneous, that is, $\varphi(N,cv)=c\varphi(N,v)$ for all $\langle N,v\rangle\in\mathcal{G}^N$ and $c\in\mathbb{R}$. Firstly, by additivity, it holds that $\varphi_i(N,\mathbf{0})=0$ for all $i\in N$. Then, we have $\varphi(N,-cv)=\varphi(N,\mathbf{0})-\varphi(N,cv)=-\varphi(N,cv)$ by again applying additivity. Thus, it suffices to show that $\varphi(N,cv)=c\varphi(N,v)$ for all $\langle N,v\rangle\in\mathcal{G}^N$ and positive scalar $c\in\mathbb{R}_{++}$.

Let $c\in\mathbb{R}_{++}$ and $\emptyset\neq T\subseteq N$. Choose two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively. We obtain that, for all $i\in N$ and $k\in\mathbb{N}$,

$$\begin{aligned}\varphi_i(N,r_ke_T)-\varphi_i(N,ce_T)=&\varphi_i(N,(r_k-c)e_T)\geq 0,\ \text{and}\\ \varphi_i(N,ce_T)-\varphi_i(N,s_ke_T)=&\varphi_i(N,(c-s_k)e_T)\geq 0,\end{aligned}$$

where in both cases the equality follows from additivity and the inequality follows from standard positivity. Then, similar as in the proof of Theorem 5.1, we can prove that $\varphi(N, ce_T) = c\varphi(N, e_T)$ for all $c \in \mathbb{R}_{++}$. Hence, by additivity, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $c \in \mathbb{R}_{++}$. □

5.6 Conclusions

In this chapter, we focus on studying axiomatic characterizations of the classes of weighted division values and positively weighted division values. We show that relaxing symmetry in van den Brink's characterization (van den Brink 2007, [104]) for the equal division value, by replacing it with sign symmetry of Casajus (2019, [20]), gives a characterization of the class of positively weighted division values. Then, a weaker version of sign symmetry allows to characterize the class of all weighted division values. Moreover, we show that the class of weighted division values can also be characterized by replacing linearity in three axiomatizations of Béal et al. (2016) with additivity. Meanwhile, we show how strengthening an axiom regarding null, non-negative, respectively nullified players in these three axiomatizations, provides three axiomatizations of the class of positively weighted division values.

The weighted division values constitute an interesting class of solutions for at least two reasons. Firstly, the weighted division principle is often considered intuitive in various applications of TU-games. It is desirable to have the option of treating players differently to reflect exogenous characteristics, such as technical skill, power, income or health status. This can be achieved by incorporating exogenous weights into the construction of a solution. Secondly, proportional (weighted) division methods are very often employed in many applications, such as bankruptcy problems (Thomson 2003, [100]), cost allocation problems (Tijs and Driessen 1986, [102]), surplus-sharing problems (Moulin 1991, [80]) and so on.

Somewhat surprising, whereas relaxing symmetry by sign symmetry in the traditional axiomatization of the Shapley value still gives the same

Shapley value in Casajus (2019, [20]), applying this relaxation in the axiomatization of the equal division value results in a class of solutions, specifically the weighted division values. Li et al. (2022, [69]) showed that Casajus (2019, [20]) result can be generalized to a subfamily of efficient, symmetric and linear values (for short, ESL values), in the sense that relaxing symmetry into sign symmetry in a specific axiomatization of such an ESL value still characterizes that ESL value.

Casajus (2018, [18]) replaces symmetry by sign symmetry in Young's axiomatization (Young 1985, [121]) of the Shapley value. In van den Brink (2007, [104]) another characterization of the equal division value is proposed by using efficiency, symmetry and coalitional monotonicity. In view of the former results, the question naturally arises whether the class of weighted division values can be characterized by efficiency, coalitional monotonicity and sign symmetry or weak sign symmetry.

Chapter 6

Sharing the cost of cleaning up a polluted river

6.1 Introduction

In the previous chapters we focused on characterizations of solutions for TU-games. In this chapter, which is based on Li et al. (2021, [71]), we turn to pollution cost-sharing problems, and explore how to share the cost of cleaning up a polluted river using cooperative game theory.

River (water) allocation among agents has emerged as one of the areas with exceptional interest for researchers due to its indispensable benefits to inhabitants of coastal communities. About 200 rivers are flowing through different countries in the world (see Ambec and Sprumont 2002, [4] and Barrett 1994, [6]). On the one hand, these water resources cater to people's daily routines and industrial productions. On the other hand, waste generated by domestic chores and production activities pollutes the sources of water, and this is harmful to humans, plants, and animals. In recent years, due to population growth and rapid industrialization, human's demand for (clean) water resources as well as the degree of water pollution are constantly increasing. This makes many countries and regions face water shortage. Considering this, reasonable allocation and utilization of water resources and efficient water pollution management would be effective measures to deal with this problem. Hence, the following questions need to be tackled: (1) How do inhabitants of coastal communities allocate the water resources? (2) How do inhabitants of coastal communities share the cost of cleaning up a polluted river? Harnessing both the burden of responsibility and the relieve that these water resources provide has become an issue of great importance recently and has presented a vital aspect that could make or mar societal development and peaceful coexistence in the riverine communities.

On the beneficial side, Ambec and Sprumont (2002, [4]) were the first to model a situation where a group of agents located along a river share its resources from a cooperative game-theoretic viewpoint, and studied how the water should be allocated among agents. They proposed the downstream incremental method in terms of the two main doctrines of Absolute Territorial Sovereignty (Godana 1985, [38]) and Unlimited Territorial Integrity (Kilgour and Dinar 1995, [63]) (for short, ATS and UTI, respectively) in international disputes. Ambec and Ehlers (2008, [3]) extended the model of sharing a river and considered the problem of efficiently sharing water from a river among a group of satiable agents. Gudmundsson et al. (2018, [40]) focused on implementing efficient outcomes of the river sharing problem by non-cooperative bargaining. Just recently, Steinmann and Winkler (2019, [97]) dug further in studying a river sharing model with downstream externalities. More results about the river sharing problem can be found in several survey papers, see e.g. Béal et al. (2013, [10]) and Beard (2011, [12]).

On the responsibility side, Ni and Wang (2007, [84]) first developed a model for the pollution cost-sharing problems and discussed the question of how to split the cost of cleaning up a river among agents situated along the river. They proposed two methods: the local responsibility sharing method (for short, LRS method) and the upstream equal sharing method (for short, UES method) by resorting to the two main doctrines of ATS and UTI in international disputes. The LRS method charges an agent the full cost of cleaning up the segment in which the agent is located, that is, every agent should take full responsibility for cleaning up its own area. In contrast, the UES method forces an agent and all its upstream counterparts to take the same responsibility for cleaning up its own area. To be precise, the UES method allocates the cost of cleaning up a segment equally among the agent in that segment and all its upstream counterparts.

The completeness of the above approach has been questioned by some authors. Alcalde-Unzu et al. (2015, [2]) proposed an alternative method which takes into account the transfer rate of waste from one segment to another. Sun et al. (2019, [99]) extended the approach by introducing the α-responsibility method which is the corresponding convex combination of the LRS method and the UES method and implemented this allocation method by a dynamic procedure. Gómez-Rúa (2013, [39]) proposed a family of rules by taking into account the different factors that influence the quality of the water. More recent research on the pollution cost-sharing problem can be found in the literature, see e.g. Hou et al. (2019, [48]) and van den Brink et al. (2018, [109]).

Considering the LRS and UES methods of Ni and Wang (2007, [84]), both the method where each agent takes full responsibility for cleaning up its own area and the method where each agent shares the responsibility equally with all its upstream counterparts, are debatable. The first method does not take into consideration that the pollutants of a river flow from upstream to downstream. The second method implicitly assumes that the agent in a segment and all its upstream counterparts have the same degree of responsibility for cleaning up the segment. In this chapter, we attempt to tackle the second question posed above by focusing on the responsibility of sharing the cost of cleaning up a polluted river. Inspired by the work of Ni and Wang (2007, [84]), we investigate two new classes of methods: the class of equal upstream responsibility methods (for short, EUR methods) and the class of weighted upstream sharing methods (for short, WUS methods). The EUR methods first assign to each agent a fraction of the cost of cleaning up the segment in which the agent is located, and then the remaining cost is distributed equally among its upstream counterparts. This fraction can be interpreted as an agent's responsibility level in its own pollution cost. The WUS methods require that each agent should pay for the amount of the cost of cleaning up its own segment and all its downstream segments in proportion to certain individual weights. These two classes of methods can be regarded as generalizations of the LRS method and the UES method.

Axiomatization is a common way to characterize the fairness and reasonability of methods for cost sharing problems, specifically for pollution cost sharing problems. Some standard properties have been applied to characterize the LRS method and the UES method, such as efficiency, additivity, independence of upstream costs, upstream symmetry and no blind cost, which are found in the literature, see e.g. Dong et al. (2012 [32]), Ni and Wang (2007, [84]), and Sun et al. (2019, [99]). In this chapter, we first characterize the UES method by introducing a relaxation of independence of upstream costs, called sign independence of upstream costs. Then, we define weak upstream symmetry and weak no blind cost by weakening these standard properties to give two axiomatizations of the class of EUR methods. Furthermore, we also provide two axiomatizations of the class of WUS methods by introducing two weak versions of upstream symmetry: sign upstream symmetry and proportionality. Finally, we analyze the problem from a cooperative game-theoretic viewpoint and define a (cooperative) pollution cost-sharing game. We define the compromise method that is the average of the LRS method and UES method, and show that the Shapley value of this pollution cost-sharing game is equal to the compromise method. Meanwhile, we also show the compromise method coincides with the Shapley value, nucleolus and τ-value of the dual of this pollution cost-sharing game.

The rest of this chapter is organized as follows. In Section 6.2, we introduce pollution cost-sharing problems. In Section 6.3, we characterize the UES method by relaxing independence of upstream costs. In Section 6.4, we define the classes of EUR methods and WUS methods, and provide several characterizations of these methods. In Section 6.5, we define the pollution cost-sharing game and show that the Shapley value of this game coincides with the compromise method. Section 6.6 provides all proofs of this chapter. Section 6.7 concludes with a summary.

6.2 Pollution cost-sharing problems

Consider a river which is divided into n segments from upstream to downstream. There are n agents (or countries) located along the river, and each agent is located in one of these segments indexed by a given order $i = 1, 2, \cdots, n$ from upstream to downstream. These agents generate a certain amount of pollutants, destroying the ecosystem of the river and influencing the quality of the waterbody. In order to guarantee the water quality, every agent has to clean up the polluted river at its segment to a certain level. To this end, the environmental authority sets a standard of the degree of pollution in every segment along the river, which requires agents paying the cost c_i to clean up the pollutants at segment i, so that the water quality is up to the environmental standard. The central issue is how to allocate the total costs, $\sum_{i\in N} c_i$, among the n agents. Ni and Wang (2007, [84]) firstly modeled this practical problem, called the pollution cost-sharing problem.

Formally, a *pollution cost-sharing problem* is a pair (N, c), where $N = \{1, \cdots, n\}$ is a finite set of agents and $c = (c_1, \cdots, c_n) \in \mathbb{R}^N_+$ is the pollution cost vector. The component c_i represents the cost of cleaning up the pollutants at segment i. For all $i, j \in N$, $i < j$ means that i is upstream from j. Denote the class of all pollution cost-sharing problems on agent set $N = \{1, \cdots, n\}$ by $\mathcal{P}^N$. A *cost share vector* for pollution cost-sharing problem $(N, c) \in \mathcal{P}^N$ is an n-dimensional vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^N_+$ whose component $x_i \geq 0$ represents the cost share allocated to agent i. A *method* on $\mathcal{P}^N$ is a map $\psi : \mathcal{P}^N \to \mathbb{R}^N_+$ that assigns a non-negative cost share vector $\psi(N, c)$ to every problem $(N, c) \in \mathcal{P}^N$.

Ni and Wang (2007, [84]) proposed two methods, the *local responsibility sharing method* (for short, *LRS method*) and the *upstream equal sharing method* (for short, *UES method*).[1] The *LRS method* on $\mathcal{P}^N$ is defined by $LRS_i(N, c) = c_i$ for all $(N, c) \in \mathcal{P}^N$ and $i \in N$. The LRS method assigns

[1] For the more general multiple spring rivers, Dong, Ni and Wang (2012, [32]) also introduced the *downstream equal sharing method* (for short, *DES method*) that allocates the cost of a segment equally among this segment and each of its downstream segments. The *DES method* on $\mathcal{P}^N$ is defined by $DES_i(N, c) = \sum_{k=1}^{i} \frac{1}{n-k+1} c_k$ for all $(N, c) \in \mathcal{P}^N$ and $i \in N$.

to each agent the full cost of cleaning up the segment where the agent is located. The *UES method* on $\mathcal{P}^N$ is defined by $UES_i(N,c) = \sum_{k=i}^{n} \frac{1}{k} c_k$ for all $(N,c) \in \mathcal{P}^N$ and $i \in N$. The UES method distributes the cost of cleaning up each segment equally among the agent in that segment and all agents situated upstream from it.

Now we recall some standard axioms, proposed by Ni and Wang (2007, [84]).

- **Efficiency.** For all $(N,c) \in \mathcal{P}^N$, it holds that $\sum_{i\in N} \psi_i(N,c) = \sum_{i\in N} c_i$.
- **Additivity.** For all $(N,c^1),(N,c^2) \in \mathcal{P}^N$, it holds that $\psi(N,c^1+c^2) = \psi(N,c^1) + \psi(N,c^2)$.
- **No blind cost.** For all $(N,c) \in \mathcal{P}^N$ and $i \in N$ such that $c_i = 0$, it holds that $\psi_i(N,c) = 0$.
- **Independence of upstream costs.** For all $(N,c^1),(N,c^2) \in \mathcal{P}^N$ and $i \in N$ such that $c_j^1 = c_j^2$ for all $j > i$, it holds that $\psi_j(N,c^1) = \psi_j(N,c^2)$ for all $j > i$.
- **Upstream symmetry.** For all $(N,c) \in \mathcal{P}^N$ and $i \in N$ such that $c_j = 0$ for all $j \in N \backslash \{i\}$, it holds that $\psi_l(N,c) = \psi_k(N,c)$ for all $l,k \leq i$.

Efficiency requires that all costs should be fully shared among all agents. Consider a situation where every agent $i \in N$ has two divisions of costs, c_i^1, c_i^2. Additivity says that, considering the sum of two polluted river problems where the cost for each segment equals the sum of the cost in the two separate problems, the associated cost allocation is equal to the sum of the cost allocation vectors assigned to the two separate problems. No blind cost says that, if the segment where an agent is located incurs no pollution cost, the agent should bear no cost. Independence of upstream costs says that an agent's cost share only depends on all costs of cleaning up its segment and all its downstream segments, but not on cost of cleaning up its upstream segments. Upstream symmetry requires that, given an agent $i \in N$, it and all its upstream agents have equal responsibilities for cleaning up its segment if other agents except agent i have no cleaning cost in their local segments.

Ni and Wang (2007, [84]) characterized the LRS method and the UES method by these above axioms.

Theorem 6.1 (Ni and Wang 2007, [84]). *(i) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity and no blind cost if and only if ψ is the LRS method. (ii) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, independence of upstream costs and upstream symmetry if and only if ψ is the UES method.*

Ni and Wang (2007, [84]) defined two different TU-games with respect to pollution cost-sharing problems. For convenience, the two TU-games are called the LRS game and the UES game in this chapter. The *LRS game* $\langle N, v^L \rangle$ is defined by $v^L(S) = \sum_{i \in S} c_i$ for all $S \subseteq N, S \neq \emptyset$, with $v^L(\emptyset) = 0$. The *UES game* $\langle N, v^U \rangle$ is defined by $v^U(S) = \sum_{i=\min_{j \in S}\{j\}}^{n} c_i$ for all $S \subseteq N, S \neq \emptyset$, with $v^U(\emptyset) = 0$. Ni and Wang (2007, [84]) showed that the cost allocations according to the LRS method and the UES method coincide with the Shapley value of the LRS game and the UES game, respectively.[2]

6.3 Sign independence of upstream costs for the UES method

As mentioned in Section 6.2, Ni and Wang (2007, [84]) characterized the UES method by efficiency, additivity, independence of upstream costs and upstream symmetry. In this characterization, independence of upstream costs reflects that an agent's utility share only depends on all costs of cleaning up its segment and all its downstream segments. Yet, independence of upstream costs involves the comparison of utilities, which is often criticized from the viewpoint of utility theory (Kaneko and Wooders 2004, [62]). In this section, we propose a relaxation of independence of upstream costs, called sign independence of upstream costs, that avoids such comparisons. Recall the sign function, sign: $\mathbb{R} \to \{-1, 0, 1\}$ given by $\text{sign}(t) = 1$ for $t > 0$, $\text{sign}(0) = 0$, and $\text{sign}(t) = -1$ for $t < 0$.

[2] In van den Brink, He and Huang (2018, [109]) it is shown that the UES method coincides with the permission value of a game with a permission structure where the game is the LRS game $\langle N, v^L \rangle$ and the linear order of the players is determined by the flow of the river.

- **Sign independence of upstream costs.** For all $(N, c^1), (N, c^2) \in \mathcal{P}^N$ and $i \in N$ such that $c^1_j = c^2_j$ for all $j > i$, it holds that $\text{sign}(\psi_j(N, c^1)) = \text{sign}(\psi_j(N, c^2))$ for all $j > i$.

Sign independence of upstream costs is a qualitative version of independence of upstream costs that relaxes independence of upstream costs.[3] Instead of equating cost shares in general, it just fixes a common reference point, the zero utility, and requires that, when all costs of agents downstream of i are the same in two cost vectors, then every agent downstream of i contributes or does not contribute in both vectors. We remark that sign independence of upstream costs is a considerable weakening of independence of upstream costs. Whereas independence of upstream costs requires complete independence of the contributions of an agent when costs of upstream agents change, the weaker sign independence of upstream costs allows that the contribution of an agent also changes when upstream costs change, and it might even have a different effect for different (downstream) agents. Since the UES method satisfies independence of upstream costs, it follows immediately that the UES method satisfies sign independence of upstream costs. We can characterize the UES method by replacing independence of upstream costs in Theorem 6.1 by the weaker sign independence of upstream costs.

Theorem 6.2. *A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, sign independence of upstream costs and upstream symmetry if and only if ψ is the UES method.*

The proof of Theorem 6.2 and of all other results in this chapter can be found in Section 6.6.

Remark 6.1. It can be seen from the proof of Theorem 6.2 that, a method that satisfies efficiency, additivity and sign independence of upstream costs, must satisfy independence of upstream costs. For this implication we do not need upstream symmetry, but we need it to apply Theorem 6.1 to characterize the UES method.

[3] As mentioned in Chapter 5, Casajus (2019, [20]) introduces a qualitative weaker version of symmetry for TU-games, where payoffs of symmetric players are required to have the same sign instead of the usual stronger requirement that these payoffs should be equal.

6.4 Generalizations of the LRS method and the UES method

In Ni and Wang (2007, [84]), the LRS method forces an agent to take full responsibility for cleaning up its segment, while the UES method assumes that the agent in a segment and all its upstream counterparts have the same degree of responsibility for cleaning up the segment. However, it is not obvious why an agent and all its upstream counterparts should be held equally responsible for cleaning up its segment. In this section, we define and characterize two different classes of methods that allow for different responsibilities of an agent and its upstream counterparts in cleaning up this agent's territory: the class of *equal upstream responsibility methods* (for short, *EUR methods*) and the class of *weighted upstream sharing methods* (for short, *WUS methods*).

6.4.1 Equal upstream responsibility methods

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^N_+$ with $\alpha_1 = 1$ and $0 \leq \alpha_i \leq 1$ for all $i \in N \setminus \{1\}$, be the responsibility level vector, whose component α_i means that agent i should pay for an α_i fraction of the cost of cleaning up its own segment. In particular, agent 1 has to take full responsibility for cleaning up its segment since agent 1 has no upstream agent, that is, $\alpha_1 = 1$. Let $A^N = \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^N_+ \mid \alpha_1 = 1 \text{ and } 0 \leq \alpha_i \leq 1 \text{ for all } i \in N \setminus \{1\}\}$ be the set of all such responsibility vectors. According to the responsibility level vector α, the α-equal upstream responsibility method (for short, α-EUR method) is defined as follows.

Definition 6.3. Let $\alpha \in A^N$. The α-EUR method on $\mathcal{P}^N$ is defined by[4]

$$EUR_i^{\alpha}(N, c) = \alpha_i c_i + \sum_{k=i+1}^{n} \frac{1 - \alpha_k}{k - 1} c_k,$$

for all $(N, c) \in \mathcal{P}^N$ and $i \in N$.

[4] We take the sum $\sum_{i=n+1}^{n} \ldots$ to be equal to 0.

The methods $EUR^{\alpha}, \alpha \in A^N$, are called EUR methods. The α-EUR method requires that each agent i pays a fraction α_i of the cost of cleaning up its own segment, and the remaining cost is equally allocated among all agents situated upstream from it.

Remark 6.2. In particular, in the case that $\alpha_i = 1$ for all $i \in N$, then the α-EUR method coincides with the LRS method. In the case that $\alpha_i = \frac{1}{i}$ for all $i \in N$, then the α-EUR method coincides with the UES method.

Remark 6.3. Generally, if we treat the α-EUR method as an allocation with $\alpha_i = (1 - \frac{1}{i})b + \frac{1}{i}$ for all $i \in N$ and some $b \in \mathbb{R}$, then the α-EUR method can be represented as a convex combination of the LRS method and the UES method, which is proposed by Sun et al. (2019, [99]), that is, for all $(N, c) \in \mathcal{P}^N$,

$$EUR^{\alpha}(N, c) = bLRS(N, c) + (1 - b)UES(N, c).$$

It is clear that the EUR methods fail no blind cost and upstream symmetry. To characterize the class of EUR methods, we introduce the following relaxations of these two axioms.

- **Weak no blind cost.** For all $(N, c) \in \mathcal{P}^N$ and $i \in N$ such that $c_j = 0$ for all $j \geq i$, it holds that $\psi_i(N, c) = 0$.
- **Weak upstream symmetry.** For all $(N, c) \in \mathcal{P}^N$ and $i \in N$ such that $c_j = 0$ for all $j \in N \backslash \{i\}$, it holds that $\psi_l(N, c) = \psi_k(N, c)$ for all $l, k < i$.

Weak no blind cost says that, if an agent and its downstream agents have no cleaning cost in their local segments, then it does not have to contribute anything. Weak upstream symmetry requires that, given an agent $i \in N$, all its upstream counterparts share the same cost if other agents except agent i have no cleaning cost in their local segments. One easily checks that the EUR methods satisfy weak no blind cost and weak upstream symmetry. We remark that these are also considerable relaxations of the classical axioms. Weak no blind costs allows agents to share in costs of other segments in case there is pollution to be cleaned downstream of this agent even if the cost of their own segments is zero. Although weak upstream symmetry reflects equal responsibility of upstream agents in case there is only one agent with positive pollution cost, it does not imply any sharing of the responsibility between this positive cost agent and its upstream agents.

Next, we give characterizations of the class of EUR methods in terms of weak no blind cost and weak upstream symmetry.

Theorem 6.4. *(i) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, sign independence of upstream costs and weak upstream symmetry if and only if there exists a responsibility level vector $\alpha \in A^N$ such that $\psi = EUR^{\alpha}$.*

(ii) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, weak no blind cost and weak upstream symmetry if and only if there exists a responsibility level vector $\alpha \in A^N$ such that $\psi = EUR^{\alpha}$.

6.4.2 Weighted upstream sharing methods

Let $\omega \in \mathbb{R}^N_{++}$ be a weight vector, whose component ω_i is the exogenous weight of agent i. In pollution cost-sharing problems, this weight can be determined in several ways, for example by the size of the populations living in each segment, the number of the factories generating pollutants, or other measurable pollution indicators. Given the weight vector ω, the ω-weighted upstream sharing method (for short, ω-WUS method) is defined as follows.

Definition 6.5. Let $\omega \in \mathbb{R}^N_{++}$. The ω-WUS method on $\mathcal{P}^N$ is defined by

$$WUS^{\omega}_i(N,c) = \sum_{k=i}^{n} \frac{\omega_i}{\sum_{j=1}^{k} \omega_j} c_k,$$

for all $(N,c) \in \mathcal{P}^N$ and $i \in N$.

The methods $WUS^{\omega}, \omega \in \mathbb{R}^N_{++}$, are called WUS methods. The ω-WUS method requires that each agent should pay for the amount of the cost of cleaning up its own segment and all its downstream segments in proportion to the weights given by ω.

Remark 6.4. If all agents have the same weight coefficient (that is, $\omega_i = t$ for all $i \in N$ and some $t \in \mathbb{R}_{++}$), then the ω-WUS method coincides with the UES method. Besides the UES method, no other EUR method (see Section 6.4.1) is a WUS method.

Remark 6.5. As mentioned in Section 6.2, Ni and Wang (2007, [84]) defined the UES game and showed that the UES method coincides with the Shapley value of that game. We remark that the class of WUS methods coincides with the class of weighted Shapley values (Kalai and Samet 1987, [59]) of the dual of the UES game.

It is clear that the UES method is the only WUS method that satisfies upstream symmetry. To characterize the class of WUS methods, we introduce the following two relaxations of upstream symmetry.

- **Sign upstream symmetry.** For all $(N, c) \in \mathcal{P}^N$ and $i \in N$ such that $c_j = 0$ for all $j \in N \backslash \{i\}$, it holds that sign$(\psi_l(N, c)) =$ sign$(\psi_k(N, c))$ for all $l, k \leq i$.

- **Proportionality.** For all $(N, c^1), (N, c^2) \in \mathcal{P}^N$ and $i \in N \backslash \{n\}$ such that $c_k^1 = 0$ for all $k \in N \backslash \{i\}$ and $c_k^2 = 0$ for all $k \in N \backslash \{i+1\}$, it holds that $\psi_l(N, c^1)\psi_k(N, c^2) = \psi_k(N, c^1)\psi_l(N, c^2)$ for all $l, k \leq i$.

Sign upstream symmetry is another qualitative version of upstream symmetry that is weaker than upstream symmetry. We remark that sign upstream symmetry relaxes upstream symmetry similar as sign symmetry relaxes symmetry by Casajus (2019, [20]) (see, Chapter 5). Instead of equating cost shares for a given agent i and all its upstream agents, when other agents except agent i have no cleaning cost in their local segments, sign upstream symmetry requires that i and all its upstream agents either contribute simultaneously or do not contribute simultaneously. Notice that there is no logical relation between sign upstream symmetry and weak upstream symmetry defined in the previous subsection.

Proportionality is also a weaker axiom than upstream symmetry. It requires that, an agent $i \in N$ and all its upstream agents contribute in the same proportion in two problems such that the unique agent with a positive cleaning cost in each of the two problems is an agent downstream of i, respectively, the downstream neighbour of this agent. This can be interpreted as a change where two downstream neighbours of i decide to transfer the cleaning cost among each other, for example because one can clean up the river cheaper than the other. We remark that sign upstream symmetry and proportionality are two different considerable weakenings of upstream symmetry. One easily checks that the WUS methods satisfy sign upstream symmetry and proportionality. In the following, we will characterize the class of WUS methods by using sign upstream symmetry and proportionality.

Theorem 6.6. *(i) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, sign independence of upstream costs, sign upstream symmetry and proportionality if and only if there exists a weight vector $\omega \in \mathbb{R}^N_{++}$ such that $\psi = WUS^\omega$.*

(ii) A method ψ on $\mathcal{P}^N$ satisfies efficiency, additivity, weak no blind cost, sign upstream symmetry and proportionality if and only if there exists a weight vector $\omega \in \mathbb{R}^N_{++}$ such that $\psi = WUS^\omega$.

6.5 Pollution cost-sharing games

As mentioned in Section 6.2, Ni and Wang (2007, [84]) proposed the LRS game and the UES game which Shapley values coincide with the LRS, respectively UES methods. In the LRS game, each coalition S is responsible only for the pollutant-cleaning costs in its own segments, and the total responsibility of the coalition S is simply the sum of its agents' local responsibilities, that is, $v^L(S) = \sum_{i \in S} c_i$. In the UES game, each coalition S takes the responsibility not only for the pollutant-cleaning costs in its own segment but also for all the costs in its downstream segments, that is, $v^U(S) = \sum_{i=\min_{j \in S}\{j\}}^{n} c_i$. In this section, we define a new TU-game with respect to the pollution cost-sharing problem which combines the characteristics of the LRS game and the UES game.

For all $i \in N$ and $S \subseteq N$, let $\bar{P}_i(S) = \{j \in S | j \leq i\}$ be the set consisting of agent i and all its upstream agents in coalition S. Denote the cardinality of $\bar{P}_i(S)$ by $|\bar{P}_i(S)|$. Obviously, $|\bar{P}_i(N)| = i$. We define the following cost game where every coalition takes the responsibility for the pollutant-cleaning costs of all segments in the coalition plus a certain part of the pollutant-cleaning costs of other segments outside the coalition. The part of the pollutant-cleaning costs of other segments outside the coalition depends the location and responsibility of the coalition of agents.

Definition 6.7. For all $(N, c) \in \mathcal{P}^N$, the pollution cost-sharing game $\langle N, v^c \rangle$ is given by

$$v^c(S) = \sum_{i \in S} c_i + \sum_{i \in N \setminus S} \frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i, \text{ for all } S \subseteq N, S \neq \emptyset,$$

and $v^c(\emptyset) = 0$.

For all $S \subseteq N$, the first part, $\sum_{i \in S} c_i$, is the pollutant-cleaning costs of all segments in the coalition S. For all $S \subseteq N$ and $i \in N \setminus S$, $\frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|}$ is the fraction of the number of agent i's upstream agents in coalition S. Then, $\frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i$ can be regarded as the proportional share of coalition S in the cost of cleaning up i's segment. Generally speaking, every coalition is assigned an extra share in the cost of cleaning each segment downstream of the coalition which is proportional to the number of upstream agents that belong to the coalition. Thus, the total responsibility of coalition S is $\sum_{i \in S} c_i + \sum_{i \in N \setminus S} \frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i$. Notice that in this game multiple agents/coalitions take responsibility for the cleaning cost in a segment. This is because $v^c(S)$ is a pessimistic measure of coalition S's responsibility. It takes partly responsibility for its downstream agents, although the downstream agents also take account of the possibility that they need to take full responsibility for their own cost in their 'worst case scenario'.

The pollution cost-sharing game can be regarded as a compromise between the LRS game and the UES game, taking into account the local responsibility principle (implied by ATS theory (Godana 1985, [38])) and the upstream responsibility principle (implied by the UTI theory (Kilgour and Dinar 1995, [63])) in International Water Law. The ATS theory says that a country has absolute sovereignty over the area of any river basin on its territory, which can be interpreted as that the responsibility for the costs of cleaning river pollutants in a segment should be assigned to the agent located in that segment. The UTI theory says that upstream countries should not change the natural flow of the water at the expense of its downstream countries, which can be interpreted as giving an agent the rights to ask all its upstream agents to pay the pollutant-cleaning costs at its segment. Considering both principles, in the pollution cost-sharing game, an upstream coalition bears not only its own pollutant-cleaning costs, but also some responsibilities for all downstream pollutant-cleaning costs, which here we assume to be proportional to the membership of the coalition. Notice that this is just one way to reflect the UTI principle, and thus how to reflect the responsibilities of upstream countries to contribute to the cleaning cost at a segment. To make a comparison with the LRS and UES games, notice that these can be written as $v^L(S) = \sum_{i \in S} c_i + \sum_{i \in N \setminus S} 0$, respectively $v^U(S) = \sum_{i \in S} c_i + \sum_{i \in N \setminus S} \text{sign}(|\overline{P}_i(S)|) c_i$ for all $S \subseteq N$. Obviously, $v^L(S) \leq v^c(S) \leq v^U(S)$ for all $S \subseteq N$.

Next, we show that applying the Shapley value to the pollution cost-sharing game $\langle N, v^c \rangle$ coincides with taking the average of the LRS method and the UES method. We refer to this method as the compromise method in this chapter.

Definition 6.8. The compromise method is defined by

$$\psi^{co}(N, c) = \frac{1}{2} LRS(N, c) + \frac{1}{2} UES(N, c),$$

for all $(N, c) \in \mathcal{P}^N$.

The compromise method reflects a compromise between the two polar opinions of ATS (reflected by the LRS method) and UTI (reflected by the UES method).

Theorem 6.9. *The method that applies the Shapley value to the pollution cost-sharing game $\langle N, v^c \rangle$ is equal to the compromise method: $Sh(N, v^c) = \psi^{co}(N, c)$ for all $(N, c) \in \mathcal{P}^N$.*

Next, we consider the dual game of the pollution cost-sharing game. In TU-games, it will be useful to think of the dual game of a cost game as a profit game and vice versa. By analyzing the dual game of the pollution cost-sharing game, we provide interesting results on the calculation of the nucleolus, Shapley value, and τ-value of the dual game. For every TU-game $\langle N, v\rangle \in \mathcal{G}^N$, its dual game $\langle N, v^d\rangle$ is given as follows: for all $S \subseteq N$, $v^d(S) = v(N) - v(N\backslash S)$. Given the pollution cost-sharing game $\langle N, v^c\rangle$, its dual game is called the dual pollution cost-sharing game $\langle N, v^{cd}\rangle$, in the following. By Definition 6.7, the dual pollution cost-sharing game $\langle N, v^{cd}\rangle$ is given by $v^{cd}(\emptyset) = 0$ and, for all $S \subseteq N$, $S \neq \emptyset$,[5]

$$v^{cd}(S) = v^c(N) - v^c(N\backslash S) = \sum_{i\in S} \frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i. \tag{6.1}$$

The dual pollution cost-sharing game can be regarded as a profit game. Since $\sum_{i\in S} LRS_i(N,c) = \sum_{i\in S} c_i \geq v^{cd}(S)$ for all $S \subseteq N$, the dual pollution cost-sharing game $\langle N, v^{cd}\rangle$ has a non-empty core. Then, the dual pollution cost-sharing game $\langle N, v^{cd}\rangle$ is quasi-balanced and has a non-empty imputation set. The dual pollution cost-sharing game $\langle N, v^{cd}\rangle$ can be rewritten as

$$v^{cd}(S) = w^c(S) + u^c(S) \tag{6.2}$$

for all $S \subseteq N$, where $\langle N, w^c\rangle$ is given by

$$w^c(S) = \sum_{i\in S} \frac{|\bar{P}_i(S)| - 1}{|\bar{P}_i(N)|} c_i \tag{6.3}$$

and $\langle N, u^c\rangle$ is given by

$$u^c(S) = \sum_{i\in S} \frac{1}{|\bar{P}_i(N)|} c_i \tag{6.4}$$

Now we recall two subclasses of TU-games. A TU-game $\langle N, v\rangle \in \mathcal{G}^N$ is an

[5]This follows since $v^{cd}(S) = v^c(N) - v^c(N\backslash S) = \sum_{i\in N} c_i - \sum_{i\in N\backslash S} c_i - \sum_{i\in S} \frac{|\bar{P}_i(N\backslash S)|}{|\bar{P}_i(N)|} c_i = \sum_{i\in S} c_i - \sum_{i\in S} \frac{|\bar{P}_i(N)| - |\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i = \sum_{i\in S}(1 - \frac{|\bar{P}_i(N)| - |\bar{P}_i(S)|}{|\bar{P}_i(N)|}) c_i = \sum_{i\in S} \frac{|\bar{P}_i(S)|}{|\bar{P}_i(N)|} c_i$.

additive game if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. A TU-game $\langle N, v\rangle \in \mathcal{G}^N$ is a *2-additive game* if it satisfies that (i) for each $i \in N$, $v(\{i\}) = 0$, and (ii) for each $S \subseteq N$ with $s \geq 2$, $v(S) = \sum_{T \subseteq S, t=2} v(T)$. Next, we show that game $\langle N, w^c\rangle$ defined by Eq.(6.3) is a 2-additive game, meaning that the worth of every singleton coalition is zero, and the worth of a coalition with two or more players equals the sum of the worths of its two-player subcoalitions.

Lemma 6.10. *For all $(N, c) \in \mathcal{P}^N$, the game $\langle N, w^c\rangle$ defined by Eq.(6.3) is a 2-additive game, that is, for all $i \in N$, $w^c(\{i\}) = 0$, and for all $S \subseteq N$ with $s \geq 2$,*

$$w^c(S) = \sum_{T \subseteq S, t=2} w^c(T).$$

Example 6.1. Consider a problem (N, c) where $N = \{1, 2, 3, 4\}$ and $c = (c_1, c_2, c_3, c_4)$. Then, for the game $\langle N, w^c\rangle$, the worth of the two player coalitions are given as follows

$$w^c(\{1,2\}) = \frac{1}{2}c_2, \quad w^c(\{1,3\}) = \frac{1}{3}c_3, \; w^c(\{1,4\}) = \frac{1}{4}c_4,$$
$$w^c(\{2,3\}) = \frac{1}{3}c_3, \; w^c(\{2,4\}) = \frac{1}{4}c_4, \; w^c(\{3,4\}) = \frac{1}{4}c_4.$$

The worth of coalitions with more than two players can be expressed as follows.

$$\begin{aligned}
w^c(\{1,2,3\}) =& \frac{1}{2}c_2 + \frac{2}{3}c_3 = w^c(\{1,2\}) + w^c(\{1,3\}) + w^c(\{2,3\}), \\
w^c(\{1,2,4\}) =& \frac{1}{2}c_2 + \frac{2}{4}c_4 = w^c(\{1,2\}) + w^c(\{1,4\}) + w^c(\{2,4\}), \\
w^c(\{1,3,4\}) =& \frac{1}{3}c_3 + \frac{2}{4}c_4 = w^c(\{1,3\}) + w^c(\{1,4\}) + w^c(\{3,4\}), \\
w^c(\{2,3,4\}) =& \frac{1}{3}c_3 + \frac{2}{4}c_4 = w^c(\{2,3\}) + w^c(\{2,4\}) + w^c(\{3,4\}), \\
w^c(\{1,2,3,4\}) =& \frac{1}{2}c_2 + \frac{2}{3}c_3 + \frac{3}{4}c_4 = w^c(\{1,2\}) + w^c(\{1,3\}) + w^c(\{1,4\}) \\
& + w^c(\{2,3\}) + w^c(\{2,4\}) + w^c(\{3,4\}).
\end{aligned}$$

From van den Nouweland et al. (1996, [112]), Chun and Hokari (2007,

[28]) and Deng and Papadimitriou (1994, [31]), it follows that the Shapley value, the nucleolus and the τ-value coincide for 2-additive games, and thus are equal for the game $\langle N, w^c \rangle$. Moreover, from van den Nouweland et al. (1996, [112]) it follows that these three solutions coincide for every game that is the sum of an additive and a 2-additive game, and thus, since $\langle N, u^c \rangle$ is an additive game, we have the following corollary.

Corollary 6.11. *The Shapley value of the dual pollution cost-sharing game $\langle N, v^{cd} \rangle$ defined by Eq.(6.1) coincides with the nucleolus and the τ-value of this game.*

By self-duality[6] of the Shapley value and Theorem 6.9, we then obtain that applying the Shapley value, the nucleolus and the τ-value to the dual pollution cost-sharing game $\langle N, v^{cd} \rangle$ coincides with taking the compromise method.

Corollary 6.12. *The method that applies the Shapley value, the nucleolus and the τ-value to the dual pollution cost-sharing game $\langle N, v^{cd} \rangle$ for all $(N, c) \in \mathcal{P}^N$, is equal to the compromise method.*

6.6 Proofs

Proof of Theorem 6.2. Since sign independence of upstream costs is weaker than independence of upstream costs, by Theorem 6.1.(ii) it suffices to show that efficiency, additivity, sign independence of upstream costs and upstream symmetry imply independence of upstream costs. Suppose that ψ is a method satisfying efficiency, additivity, sign independence of upstream costs and upstream symmetry. Let $(N, c^1), (N, c^2) \in \mathcal{P}^N$ and $i \in N$ be such that $c_j^1 = c_j^2$ for all $j > i$. For all $k \in N$, let (N, e^k) be defined by $e_k^k = 1$ and $e_l^k = 0$ for all $l \in N \backslash \{k\}$. Set $(N, c^0) \in \mathcal{P}^N$ with $c_k^0 = 0$ for all $k \in N$. It is straightforward to obtain that $\psi_k(N, c^0) = 0$ for all $k \in N$ by efficiency and

[6] A solution φ satisfies self-duality, if for every $\langle N, v \rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v) = \varphi(N, v^d)$ where $\langle N, v^d \rangle$ is the dual game of $\langle N, v \rangle$.

$\psi(N, c^0) \in \mathbb{R}^N_+$. Then, for all $j > i$, we have

$$\psi_j(N, c^1) = \psi_j(N, c^1 - \sum_{k>i} c^1_k e^k) + \sum_{k>i} \psi_j(N, c^1_k e^k) = \sum_{k>i} \psi_j(N, c^1_k e^k),$$

where the first equation holds by additivity and the second equation holds from the fact that sign$(\psi_j(N, c^1 - \sum_{k>i} c^1_k e^k)) = $ sign$(\psi_j(N, c^0)) = 0$ for all $j > i$ by sign independence of upstream costs. Similarly, for all $j > i$, it holds that

$$\psi_j(N, c^2) = \psi_j(N, c^2 - \sum_{k>i} c^2_k e^k) + \sum_{k>i} \psi_j(N, c^2_k e^k) = \sum_{k>i} \psi_j(N, c^2_k e^k).$$

Thus, we obtain $\psi_j(N, c^1) = \psi_j(N, c^2)$ for all $j > i$, which concludes the proof. □

Proof of Theorem 6.4. It is straightforward to verify that the EUR methods satisfies efficiency, additivity, sign independence of upstream costs, weak upstream symmetry and weak no blind cost. It is left to show that the axioms are sufficient for uniqueness.

(i) Let ψ be a method on $\mathcal{P}^N$ satisfying efficiency, additivity, sign independence of upstream costs and weak upstream symmetry. We will show that for some responsibility level vector α, $\psi = EUR^\alpha$. Similar as before, for all $k \in N$, (N, e^k) is given by $e^k_k = 1$ and $e^k_l = 0$ for all $l \in N \backslash \{k\}$. Set $\alpha_k = \psi_k(N, e^k)$ for all $k \in N$. Let $(N, c^0) \in \mathcal{P}^N$ with $c^0_k = 0$ for all $k \in N$. It is straightforward to obtain that $\psi_k(N, c^0) = 0$ for all $k \in N$ by efficiency and $\psi(N, c^0) \in \mathbb{R}^N_+$. Then, for all $i > k$, by sign independence of upstream costs, we have sign$(\psi_i(N, e^k)) = $ sign$(\psi_i(N, c^0)) = 0$. By efficiency and weak upstream symmetry, we obtain

$$\psi_i(N, e^k) = \begin{cases} 0, & \text{if } i > k; \\ \alpha_k, & \text{if } i = k; \\ \frac{1-\alpha_k}{k-1}, & \text{if } i < k. \end{cases} \tag{6.5}$$

Next, we show that ψ is homogeneous, that is, $\psi(N,tc) = t\psi(N,c)$ for all $(N,c) \in \mathcal{P}^N$ and scalar $t \in \mathbb{R}_+$. To show homogeneity for all $t \in \mathbb{R}_+$, choose two sequences of non-negative rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to t from above and below, respectively. By additivity, we obtain that, for all $i \in N$ and for all $k = 1, \ldots, \infty$,

$$\begin{aligned} &\psi_i(N, r_k c) - \psi_i(N, tc) = \psi_i(N, (r_k - t)c) \geq 0, \text{ and} \\ &\psi_i(N, tc) - \psi_i(N, s_k c) = \psi_i(N, (t - s_k)c) \geq 0. \end{aligned} \tag{6.6}$$

Notice that, for all $i \in N$, $\psi_i(N, r_k c) - \psi_i(N, s_k c) = \psi_i(N, (r_k - s_k)c) = (r_k - s_k)\psi_i(N,c) \to 0$ as $k \to \infty$,[7] since $(r_k - s_k) \to 0$ as $k \to \infty$. Then $\psi_i(N, r_k c) - \psi_i(N, tc) + \psi_i(N, tc) - \psi_i(N, s_k c) \to 0$ as $k \to \infty$. Since, both $\psi_i(N, r_k c) - \psi_i(N, tc) \geq 0$ and $\psi_i(N, tc) - \psi_i(N, s_k c) \geq 0$ by Eq.(6.6), this implies that $\psi(N, r_k c) \to \psi(N, tc)$ and $\psi(N, r_k c) = r_k \psi(N,c) \to t\psi(N,c)$ as $k \to \infty$, which proves the homogeneity of ψ. Thus, ψ is a linear map on $\mathcal{P}^N$. Therefore, for all $(N,c) \in \mathcal{P}^N$ and $i \in N$, it holds that

$$\begin{aligned} \psi_i(N,c) =& \psi_i(N, \sum_{k \in N} c_k e^k) = \sum_{k \in N} c_k \psi_i(N, e^k) \\ =& \alpha_i c_i + \sum_{k=i+1}^{n} \frac{1 - \alpha_k}{k - 1} c_k = EUR_i^{\alpha}(N,c), \end{aligned} \tag{6.7}$$

where the third equality follows from Eq.(6.5).

Notice that $\alpha_k = \psi_k(N, e^k)$ for all $k \in N$, efficiency and $\psi(N, e^k) \in \mathbb{R}_+^N$, implies that $0 \leq \alpha_i \leq 1$ for all $i \in N$. Moreover, similar as above, for all $i > 1$, by sign independence of upstream costs, we have sign$(\psi_i(N, e^1))$ = sign$(\psi_i(N, c^0)) = 0$, and thus by efficiency, $\alpha_1 = \psi_1(N, e^1) = 1$, showing that $\alpha \in A^N$.

(ii) Let ψ be a method on $\mathcal{P}^N$ satisfying efficiency, additivity, weak no blind cost and weak upstream symmetry. We can obtain that ψ is a

[7] Given any rational r_k, there must exist two integers $a, b \in \mathbb{N}$, $a \neq 0$, such that $r_k = \frac{b}{a}$. Then by additivity, we have $\psi(N, r_k c) = \psi(N, \frac{b}{a}c) = b\psi(N, \frac{1}{a}c) = \frac{b}{a} \cdot a\psi(N, \frac{1}{a}c) = \frac{b}{a}\psi(N, \frac{a}{a}c) = r_k \psi(N,c)$.

linear map on $\mathcal{P}^N$ similar as in the proof of part (i). Similar as in the proof of part (i), set $\alpha_k = \psi_k(N, e^k)$ for all $k \in N$. Then, for all $i > k$, by weak no blind cost, we have $\psi_i(N, e^k) = 0$. Together with efficiency and weak upstream symmetry, we again obtain Eq.(6.5) and, since ψ is a linear map on $\mathcal{P}^N$, then for all $(N, c) \in \mathcal{P}^N$ and $i \in N$, we obtain Eq.(6.7) similar as in the proof of part (i). Similar as in the proof of part (i), $\alpha_k = \psi_k(N, e^k)$ for all $k \in N$, efficiency and $\psi(N, e^k) \in \mathbb{R}^N_+$, imply that $0 \leq \alpha_i \leq 1$ for all $i \in N$. Now, weak no blind cost implies that $\psi_i(N, e^1) = 0$ for all $i > 1$. Thus by efficiency, $\alpha_1 = \psi_1(N, e^1) = 1$, showing that $\alpha \in A^N$. This concludes the proof. □

Proof of Theorem 6.6. It is straightforward to verify that the WUS methods satisfy efficiency, additivity, sign independence of upstream costs, sign upstream symmetry, proportionality and weak no blind cost. It is left to show that the axioms are sufficient for uniqueness.

(i) Let ψ be a method on $\mathcal{P}^N$ satisfying efficiency, additivity, sign independence of upstream costs, sign upstream symmetry and proportionality. We will show that for some weight vector $\omega \in \mathbb{R}^N_{++}$, $\psi = WUS^\omega$. For all $k \in N$, (N, e^k) is again given by $e^k_k = 1$ and $e^k_l = 0$ for all $l \in N \backslash \{k\}$. Set $\omega = \psi(N, e^n)$. By efficiency and sign upstream symmetry, we have $\sum_{i \in N} \omega_i = 1$ and $\omega_i > 0$ for all $i \in N$, showing that $\omega \in \mathbb{R}^N_{++}$.

Next, we show that $\psi_i(N, te^k) = 0$ for all $i > k$ and $t \in \mathbb{R}_+$. Let $(N, c^0) \in \mathcal{P}^N$ with $c^0_k = 0$ for all $k \in N$. By additivity,[8] $\psi_k(N, c^0) = 0$ for all $k \in N$. Then, for all $i > k$, by sign independence of upstream costs, we have sign$(\psi_i(N, te^k)) =$ sign$(\psi_i(N, c^0)) = 0$, showing that $\psi_i(N, te^k) = 0$.

[8] This follows since, by additivity it holds that $\psi_i(N, c^0) + \psi_i(N, c^0) = \psi_i(N, c^0)$ for all $i \in N$.

Now we show that $\psi_i(N, te^k) = \frac{\omega_i}{\sum_{j=1}^k \omega_j} t$ for all $i \leq k$. By proportionality, for all $i, j \leq k$, we have

$$\psi_i(N, te^k)\psi_j(N, e^{k+1}) = \psi_i(N, e^{k+1})\psi_j(N, te^k).$$

Then, by fixing i and summing over $j \leq k$, we have

$$\psi_i(N, te^k)\sum_{j=1}^{k}\psi_j(N, e^{k+1}) = \psi_i(N, e^{k+1})\sum_{j=1}^{k}\psi_j(N, te^k). \tag{6.8}$$

If $k = n-1$, by efficiency, for all $i \leq n-1$, we have

$$\psi_i(N, te^{n-1}) = \frac{\psi_i(N, e^n)}{\sum_{j=1}^{n-1}\psi_j(N, e^n)}\sum_{j=1}^{n-1}\psi_j(N, te^{n-1}) = \frac{\omega_i}{\sum_{j=1}^{n-1}\omega_j}t.$$

Proceeding by induction, suppose that it holds that $\psi_i(N, te^{k+1}) = \frac{\omega_i}{\sum_{j=1}^{k+1}\omega_j}t$ for some $k < n-1$. Then, by Eq.(6.8), for $i \leq k$, we have

$$\begin{aligned}\psi_i(N, te^k) =& \frac{\psi_i(N, e^{k+1})}{\sum_{j=1}^{k}\psi_j(N, e^{k+1})}\sum_{j=1}^{k}\psi_j(N, te^k)\\ =& \frac{\frac{\omega_i}{\sum_{j=1}^{k+1}\omega_j}t}{\sum_{j=1}^{k}\frac{\omega_j}{\sum_{h=1}^{k+1}\omega_h}t}t = \frac{\omega_i}{\sum_{j=1}^{k}\omega_j}t.\end{aligned}$$

Thus, for all $k \in N$ and $i \leq k$, $\psi_i(N, te^k) = \frac{\omega_i}{\sum_{j=1}^k \omega_j}t$. Finally, by additivity, for all $(N, c) \in \mathcal{P}^N$ and $i \in N$, it holds that

$$\begin{aligned}\psi_i(N, c) =& \psi_i(N, \sum_{k\in N} c_k e^k) = \sum_{k\in N}\psi_i(N, c_k e^k)\\ =& \sum_{k=i}^{n}\frac{\omega_i}{\sum_{j=1}^{k}\omega_j}c_k = WUS_i^{\omega}(N, c).\end{aligned} \tag{6.9}$$

(ii) Let ψ be a method on $\mathcal{P}^N$ satisfying efficiency, additivity, sign upstream symmetry, proportionality and weak no blind cost. Similar as in the proof of part (i), set $\omega = \psi(N, e^n)$. Then, for all $i > k$, by weak no blind cost, we have $\psi_i(N, te^k) = 0$. Together with efficiency, sign upstream symmetry and proportionality, we obtain that $\psi_i(N, te^k) = \frac{\omega_i}{\sum_{j=1}^{k} \omega_j} t$ for all $i \leq k$, similar as in the proof of part (i). Then, by additivity, for all $(N, c) \in \mathcal{P}^N$ and $i \in N$, we obtain Eq.(6.9) similar as in the proof of part (i). □

Proof of Theorem 6.9. Since the Shapley value can be represented by allocating the Harsanyi dividends (Harsanyi 1959, [45]) equally over the players in the corresponding unanimity coalition,[9] we first calculate the Harsanyi dividends of all coalitions in the pollution cost-sharing game $\langle N, v^c \rangle$.

If $S = \{i\}$, then $\Delta_{v^c}(\{i\}) = v^c(\{i\}) = c_i + \sum_{j>i} \frac{1}{j} c_j$.

If $S = \{i, j\}$ with $i < j$, then

$$\begin{aligned}
\Delta_{v^c}(\{i,j\}) =& v^c(\{i,j\}) - v^c(\{i\}) - v^c(\{j\}) \\
=& c_i + c_j + \sum_{k>i, k<j} \frac{1}{k} c_k + \sum_{k>j} \frac{2}{k} c_k - (c_i + \sum_{k>i} \frac{1}{k} c_k) - (c_j + \sum_{k>j} \frac{1}{k} c_k) \\
=& \sum_{k>i, k<j} \frac{1}{k} c_k + \sum_{k>j} \frac{2}{k} c_k - \sum_{k>i, k\leq j} \frac{1}{k} c_k - \sum_{k>j} \frac{2}{k} c_k \\
=& -\frac{1}{j} c_j.
\end{aligned}$$

Next, we show that $\Delta_{v^c}(S) = 0$ for all $S \subseteq N$ with $s \geq 3$. For all $S \subseteq N$ with $s \geq 3$, we have

$$\begin{aligned}
&\sum_{i \in S} \Delta_{v^c}(\{i\}) + \sum_{T \subseteq S, t=2} \Delta_{v^c}(T) \\
=& \sum_{i \in S} (c_i + \sum_{j>i} \frac{1}{j} c_j) - \sum_{i \in S} \sum_{j \in S, j>i} \frac{1}{j} c_j \\
=& \sum_{i \in S} c_i + \sum_{i \in S} \sum_{j \in N \setminus S, j>i} \frac{1}{j} c_j
\end{aligned}$$

[9] For all $\langle N, v \rangle \in \mathcal{G}^N$ and $i \in N$, the Shapley value is given by $Sh_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{s}$, where $\Delta_v(S) = v(S) - \sum_{T \subsetneq S, T \neq \emptyset} \Delta_v(T)$ is the Harsanyi dividend of coalition S in $\langle N, v \rangle$.

$$
\begin{aligned}
&= \sum_{i \in S} c_i + \sum_{j \in N \setminus S} \sum_{i \in S, i<j} \frac{1}{j} c_j \\
&= \sum_{i \in S} c_i + \sum_{j \in N \setminus S} \frac{|\bar{P}_j(S)|}{|\bar{P}_j(N)|} c_j = v^c(S). \qquad (6.10)
\end{aligned}
$$

Then for all $S \subseteq N$ with $s = 3$, $\Delta_{v^c}(S) = v^c(S) - \sum_{T \subsetneq S, T \neq \emptyset} \Delta_{v^c}(T) = 0$ by Eq.(6.10). Proceeding by induction, suppose that $\Delta_{v^c}(S) = 0$ for all $S \subseteq N$ with $k \geq s \geq 3$. Then, for all $S \subseteq N$ with $s = k+1$, we have

$$
\Delta_{v^c}(S) = v^c(S) - \sum_{T \subsetneq S, T \neq \emptyset} \Delta_{v^c}(T) = v^c(S) - \sum_{T \subsetneq S, 1 \leq t \leq 2} \Delta_{v^c}(T) = 0,
$$

showing that $\Delta_{v^c}(S) = 0$ for all $S \subseteq N$ with $s \geq 3$.

Therefore, for all $i \in N$, we have

$$
\begin{aligned}
Sh_i(N, v^c) = &\sum_{S \subseteq N, S \ni i} \frac{\Delta_{v^c}(S)}{s} = \Delta_{v^c}(\{i\}) + \sum_{S \subseteq N, s=2} \frac{\Delta_{v^c}(S)}{2} \\
= &c_i + \sum_{j>i} \frac{1}{j} c_j - \sum_{j<i} \frac{1}{2i} c_i - \sum_{j>i} \frac{1}{2j} c_j \\
= &\frac{1}{2} c_i + \sum_{j \geq i} \frac{1}{2j} c_j = \frac{1}{2} LRS_i(N, c) + \frac{1}{2} UES_i(N, c) = \psi_i^{co}(N, c),
\end{aligned}
$$

which completes the proof. □

Proof of Lemma 6.10. By Eq.(6.3), it is straightforward to obtain that $w^c(\{i\}) = 0$ for all $i \in N$. Moreover, TU-game $\langle N, w^c \rangle$ defined by Eq.(6.3) can be rewritten, for all $S \subseteq N$ with $s \geq 2$, as

$$
w^c(S) = \sum_{i \in S} \sum_{j \in S, j<i} \frac{1}{i} c_i
$$

Since it is straightforward that $w^c(\{i, j\}) = \frac{1}{i} c_i$ for all $i, j \in N$ with $j < i$, we have

$$
w^c(S) = \sum_{i \in S} \sum_{j \in S, j<i} w^c(\{i, j\})
$$

Therefore, it holds that $w^c(S) = \sum_{T \subseteq S, t=2} w^c(T)$ for all $S \subseteq N$ with $s \geq 2$. $\square$

6.7 Conclusions

We introduce and study two classes of cost-sharing methods for cleaning up a polluted river by considering every agent's responsibility for its own area. We propose the class of EUR methods and the class of WUS methods, and give several axiomatizations for these methods. The axiomatizations are based on weaker versions of independence of upstream costs, upstream symmetry and no blind cost, that are used by Ni and Wang (2007, [84]) to characterize the known UES and LRS methods. We also show that the UES method can be characterized by replacing independence of upstream costs by this weaker independence of upstream costs axiom, called sign independence of upstream costs. We remark that sign independence of upstream costs and sign upstream symmetry relax independence of upstream costs, respectively upstream symmetry in this chapter, similar as sign symmetry relaxes symmetry in Chapter 5. Except for these sign axioms, in Chapter 5, we also introduce the sign null player in a productive environment property, the sign non-negative player property and sign nullified solidarity by strengthening the null player in a productive environment property, the non-negative player property and nullified solidarity.

The axioms and the methods stated in this chapter are summarized in Table 6.1. In this table, '✓' has the meaning that the methods satisfy the axioms. There are some logical relations between the axioms in this table. Sign independence of upstream costs is a qualitative/weaker version of independence of upstream costs. Weak no blind cost is a weaker version of no blind cost. Weak upstream symmetry, sign upstream symmetry and proportionality are three different weaker versions of upstream symmetry, and there is no logical relation among weak upstream symmetry, sign upstream symmetry and proportionality.

Moreover, we define a corresponding pollution cost-sharing game, and interestingly, the Shapley value of this game gives a compromise in the

Axioms \ Methods	LRS	UES	Compromise	EUR	WUS
Efficiency	✓	✓	✓	✓	✓
Additivity	✓	✓	✓	✓	✓
No blind cost	✓				
Independence of upstream costs	✓	✓	✓	✓	✓
Sign independence of upstream costs	✓	✓	✓	✓	✓
Upstream symmetry		✓			
Weak no blind cost	✓	✓	✓	✓	✓
Weak upstream symmetry	✓	✓	✓	✓	
Sign upstream symmetry		✓	✓		✓
Proportionality		✓			✓

TABLE 6.1 Axioms of the methods for pollution cost-sharing problems

sense that it boils down to taking the average of the LRS method and the UES method. We refer to this method as the compromise method. We also show that the dual game of the pollution cost-sharing game is the sum of a 2-additive game and an additive game, implying that the compromise method coincides also with the nucleolus and τ-value of the dual pollution cost-sharing game.

In future research, we will apply these methods to the more general polluted river network model introduced by Dong et al. (2012, [32]) and generalize the classes of EUR methods and WUS methods for more general models. These more general models can include general games with a permission structure, or subclasses such as permission tree games or peer group games. Since different methods boil down to applying the Shapley value to different games, we also plan to characterize different pollution cost allocation games. For example, in Section 6.5, we saw that the LRS, UES and compromise methods all assign to a coalition the full costs of the agents in the coalition, but different shares in the costs of the agents downstream of the coalition.

Summary

This thesis consists of six chapters on cooperative game theory and its application. Except Chapter 1, which is an introductory chapter, each of the other five chapters contains original results. Our thesis can be divided into two parts: theoretic issues on characterizations of solutions for TU-games (Chapters 2-5) and an application on characterizations of cost-sharing methods for pollution cost-sharing problems (Chapter 6).

Chapter 2 introduces and studies a new solution for TU-games, the *average-surplus value*, which offers every player a weighted average of the average marginal surpluses to all coalitions including himself. Firstly, inspired by the axiomatizations of the Shapley value (Shapley 1953, [94] and Myerson 1980, [82]), we define two new axioms: the *A-null surplus player property* and *revised balanced contributions*, to characterize the average-surplus value. Secondly, inspired by the work of Hart and Mas-Colell (1989, [46]), we define the *AS-potential function*, and show that the adjusted marginal contributions vector of the AS-potential function coincides with the average-surplus value. Finally, we provide a non-cooperative game, namely the *punishment-compensation bidding mechanism*, to implement the average-surplus value.

Chapter 3 studies axiomatic characterizations of the equal allocation of non-separable contributions (EANSC) value and the center-of-gravity of the imputation set (CIS) value. Firstly, we construct two different associated games: the *E-union associated game* and the *C-union associated game*, by the idea of “union self-evaluation" that is an alternative way to reevaluate the worth. Then, we characterize the EANSC value and the CIS value using

new associated consistency axioms: *E-union associated consistency* and *C-union associated consistency*. Finally, we propose two dynamic processes on the basis of associated games that lead to any solution satisfying both the inessential game property and continuity, starting from an arbitrary efficient payoff vector.

Chapter 4 studies axiomatic characterizations of the proportional division (PD) value and the proportional allocation of non-separable contribution (PANSC) value. Firstly, we introduce the concepts of *optimistic satisfaction* and *pessimistic satisfaction*, and show that the PD value (respectively, PANSC value) can be obtained by lexicographically maximizing the minimal optimistic satisfaction (respectively, pessimistic satisfaction) over the non-empty pre-imputation set. Then, we characterize the PD value and the PANSC value by introducing the *equal minimal optimistic/pessimistic satisfaction* axioms, the *optimistic/pessimistic associated consistency* axioms and their dual axioms.

Chapter 5 studies axiomatic characterizations of the family of weighted division values. Firstly, we characterize the family of (positively) weighted division values by replacing symmetry in van den Brink's axiomatization (van den Brink 2007, [104]) with relaxations of symmetry. Then, we replace linearity in three axiomatizations of Béal et al. (2016, [8]) with additivity to characterize the family of weighted division values. Finally, we strengthen an axiom regarding null, non-negative, respectively nullified players in these three axiomatizations, to characterize the positively weighted division values.

Chapter 6 introduces and studies two new classes of cost-sharing methods for pollution cost-sharing problems. Firstly, we provide a characterization of the upstream equal sharing (UES) method by relaxing independence of upstream costs appearing in Ni and Wang (2007, [84]). Then, we define and characterize the classes of *equal upstream responsibility (EUR) methods* and *weighted upstream sharing (WUS) methods* by using some weak versions of these axioms in Ni and Wang (2007, [84]). Finally, we define the *pollution cost-sharing game* and show that the Shapley value of this game coincides with the *compromise method* which is the average of the LRS method and UES method.

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Wenzhong Li
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About the Author

Wenzhong Li was born on July 22, 1991 solar calendar, in Xiangcheng City of Henan Province, P. R. China. From 1999 until 2011, Wenzhong attended primary and secondary school in his hometown. In September 2011, he became a student of the Department of Applied Mathematics at China University of Mining and Technology in Xuzhou.

After receiving his Bachelor degree in June 2015, he started his mathematics study at Northwestern Polytechnical University and became a graduate student. He specialized in operations research, in particular, cooperative games, and subsequently, he completed his Master's thesis, entitled 'The axiomatizations of solutions and their applications in cooperative games', under the supervision of Prof. Dr. Genjiu Xu. In March 2018, he graduated with honors and received his Master degree. In March 2018, he started as a Ph.D. student of the Department of Applied Mathematics at Northwestern Polytechnical University, under the supervision of Prof. Dr. Genjiu Xu. During December 2019 to March 2020, he visited Vrije Universiteit Amsterdam as a visiting student, under the supervision of Prof. Dr. René van den Brink.

Starting from April 2021, he again visited Vrije Universiteit Amsterdam as a joint PhD student to perform research on solutions for cooperative games and their application under the supervision of Prof. Dr. René van den Brink. The research has been sponsored by the China Scholarship Council.