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Spectrum and stability of a rigidly rotating compressible plasma

R. J. NIJBOER,¹ A. E. LIFSCHITZ²
and J. P. GOEDBLOED¹

¹FOM – Institute for Plasma Physics Rijnhuizen, Edisonbaan 14, Postbus 1207,
3430 BE Nieuwegein, The Netherlands

²Department of Mathematics, Statistics and Computer Science, University of Illinois,
Chicago, Illinois 60607, USA

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We consider the spectrum and stability of a compressible, rigidly rotating plasma column with constant magnetic pitch. It is found that when the pressure on axis is zero, a continuous spectrum arises, which may become unstable. When the pressure on axis is finite or a conducting core is included, the spectrum is discrete, but may still be unstable. The instability is due to the poloidal magnetic field and/or the rotation of the cylinder in combination with the density profile. It is found numerically that the pressure stabilizes both types of instabilities.

1. Introduction

Equilibrium flow occurs in both fusion and astrophysical plasmas, and has consequences for the behaviour of these plasmas. Therefore it is of importance to study the effect of flow on the stability of an equilibrium or, more generally, on the spectrum of the associated eigenvalue problem.

One of the earliest results on the effect of equilibrium flow on the spectrum is by Frieman and Rotenberg (1960). They showed that the eigenvalue problem becomes quadratic, which implies that the spectrum is no longer restricted to the axes in the complex plane. The spectrum of an incompressible, rigidly rotating plasma cylinder with constant magnetic pitch was calculated by Trehan (1959). However, since the eigenvalue problem for equilibria with flow is very complex in general, the calculation of the spectrum is very difficult. Therefore, most papers deal with stability conditions, like Howard and Gupta (1962) for an incompressible plasma. With the use of circle theorems, stability conditions for general plasma configurations were found by, for example, Barston (1977) and Hameiri (1981). They were applied to cylindrical geometries by Lucas (1981, 1986).

The effect of equilibrium flow on different types of modes has been studied for cylindrical geometries by a number of authors. For example, Suydam modes were considered both by Hameiri (1981) and by Bondeson *et al.* (1987). In astrophysics the stability of rotating jets was studied by, for example, Bodo *et al.* (1989) and Appl and Camenzind (1992) and the effect of compressibility on these jets was studied by Corbelli and Torricelli-Ciamponi (1989). Also, in astrophysics it was found that differentially rotating accretion discs may be unstable to the Balbus–Hawley instability (see Balbus and Hawley 1991).

The effect of a density profile on the stability of cylindrical incompressible plasmas inside an annulus was studied by Fung (1984). In that paper it was shown that radially increasing density profiles have a stabilizing effect, while radially decreasing density profiles may give rise to instabilities. It was also found that the magnetic field stabilizes these instabilities.

In this paper we discuss the effects of flow and compressibility on the spectrum of a rigidly rotating plasma with a constant-pitch magnetic field. This equilibrium has the advantage that one is able to further pursue the analytical study, which gives better insight in the behaviour of the spectrum. The effect of more realistic equilibria is then studied numerically. Furthermore, we restrict ourselves to cylindrically symmetric geometries. The effect of elliptical geometries on the spectrum is considered in Lifschitz (1997).

The structure of the paper is as follows. In Sec. 2 we state the general eigenvalue problem, both as a first-order and as a second-order system. In Sec. 3 we consider an incompressible rigidly rotating cylindrical plasma with constant magnetic pitch, and discuss its spectrum and stability. Then in Sec. 4 we consider compressible perturbations, and show that a vanishing pressure on axis may lead to a continuous spectrum. In Sec. 5 we consider the discrete spectrum, which occurs when the pressure on axis is finite. We give our conclusions in Sec. 6.

2. The eigenvalue problem

In this paper we consider the stability of rigidly rotating, constant magnetic-pitch plasma columns. In this section, however, we give the equations describing the stability of a general rotating plasma column. The rigidly rotating, constant-magnetic-pitch column is then a special case.

We assume that the equilibrium quantities depend only on the radius r , and denote them by ρ for the density, $\mathbf{v}$ for the velocity, $\mathbf{B}$ for the magnetic field and p for the pressure. The equilibrium should then satisfy the force-balance equation

$$(p + \frac{1}{2}B^2)' = \frac{\rho v_\theta^2 - B_\theta^2}{r}, \quad (2.1)$$

where $' \equiv d/dr$.

2.1. First-order system

Since we are interested in the stability of the plasma, we shall linearize the MHD equations about an equilibrium satisfying (2.1). Owing to the symmetry of the equilibrium profiles we can consider perturbations that have an $f(r)\exp[i(m\theta + kz - \omega t)]$ dependence. Then the equations describing the MHD stability of this plasma are (Hameiri 1981; Bondeson *et al.* 1987):

$$\frac{AS}{r} \begin{pmatrix} \chi \\ \Pi \end{pmatrix}' + \begin{pmatrix} C & D \\ E & -C \end{pmatrix} \begin{pmatrix} \chi \\ \Pi \end{pmatrix} = 0, \quad (2.2)$$

where $\chi \equiv r\xi_r$ is proportional to the radial component of the displacement vector ξ , and Π is the perturbation of the total pressure. The coefficients of (2.2)

are given by

$$A = \rho\tilde{\omega}^2 - F^2, \quad (2.3a)$$

$$S = (B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma pF^2, \quad (2.3b)$$

$$C = -[(B_\theta^2 - \rho v_\theta^2)\tilde{\omega}^2 + (B_\theta\tilde{\omega} + v_\theta F)^2]\frac{\rho^2\tilde{\omega}^2}{r^2} \\ + 2m(B_\theta F + \rho v_\theta\tilde{\omega})\frac{(B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma pF^2}{r^3}, \quad (2.3c)$$

$$D = \rho^2\tilde{\omega}^4 - \left(k^2 + \frac{m^2}{r^2}\right)[(B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma pF^2], \quad (2.3d)$$

$$E = -(\rho\tilde{\omega}^2 - F^2)[(B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma pF^2] \\ \times \left[\rho\tilde{\omega}^2 - F^2 + r\left(\frac{B_\theta^2 - \rho v_\theta^2}{r^2}\right)'\right]\frac{1}{r^2} \\ + 4(B_\theta F + \rho v_\theta\tilde{\omega})^2\frac{(B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma pF^2}{r^4} \\ - [(B_\theta^2 - \rho v_\theta^2)\rho\tilde{\omega}^2 + \rho(B_\theta\tilde{\omega} + v_\theta F)^2]^2\frac{1}{r^4}, \quad (2.3e)$$

where we have introduced the following notation for the Doppler-shifted frequency and the parallel wavenumber respectively:

$$\tilde{\omega} = \omega - \mathbf{k} \cdot \mathbf{v} = \omega - \frac{mv_\theta}{r} - kv_z, \\ F = \mathbf{k} \cdot \mathbf{B} = \frac{mB_\theta}{r} + kB_z.$$

Note that (2.2) has the same form as in the static case (Appert *et al.* 1974), and equilibrium flow only modifies the coefficients.

Equation (2.2) has to be supplied with boundary conditions. The boundary condition at the perfectly conducting wall located at $r = R$ has the form

$$\chi(R) = 0. \quad (2.4)$$

In most cases we also have a regularity condition at the axis, which implies that

$$\chi(0) = 0. \quad (2.5)$$

However, under very special circumstances (see Sec. 4), this condition should be dropped.

The perpendicular displacement $\Psi \equiv i(B_\theta\xi_z - B_z\xi_\theta)$ and parallel displacement $\Phi \equiv i(B_\theta\xi_\theta + B_z\xi_z)$ can be found when χ and Π are known, and are given by

$$\Psi = \frac{1}{A} \left[G\Pi - 2\frac{B_z}{r^2} (B_\theta F + \rho v_\theta\tilde{\omega}) \chi \right], \quad (2.6)$$

$$\Phi = \frac{1}{S} \left[-\gamma pF\Pi + \frac{\rho v_\theta^2 - 2B_\theta^2}{r^2} B^2 F \chi + 2(B^2 + \gamma p)\frac{B_\theta}{r^2} (B_\theta F + \rho v_\theta\tilde{\omega}) \chi \right], \quad (2.7)$$

where $G = kB_\theta - mB_z/r$ is the perpendicular wavenumber.

We see from (2.2), (2.6) and (2.7) that, apart from frequencies satisfying $A = 0$ and $S = 0$, there are no singularities. Hence we find the Alfvén continuum (given by $A = 0$) and the slow continuum (given by $S = 0$), and flow only gives a Doppler

shift to these continuous spectra. However, we shall show that, under some special conditions, there may be an extra continuous spectrum.

2.2. Second-order system

Instead of using the first-order system (2.2), it is often more convenient to use one second-order equation. Such an equation can be found by eliminating one of the dependent variables in favour of the other in (2.2). Here we give the second-order differential equation in terms of χ ,

$$\left(\frac{AS}{rD}\chi'\right)' + \left[U + \frac{V}{D} + \left(\frac{W}{D}\right)'\right]\chi = 0, \quad (2.8)$$

and, in terms of Π ,

$$\left(\frac{AS}{rE}\Pi'\right)' + \left[\frac{(U + H')D}{E} + \frac{V}{E} - \left(\frac{W - HD}{E}\right)'\right]\Pi = 0. \quad (2.9)$$

The coefficients are given by

$$H = \frac{2(B_\theta^2 - \rho v_\theta^2)}{r^2}, \quad (2.10a)$$

$$U = \frac{\rho\tilde{\omega}^2 - F^2}{r} - \left(\frac{B_\theta^2 - \rho v_\theta^2}{r^2}\right)', \quad (2.10b)$$

$$\begin{aligned} V = & -\left(k^2 + \frac{m^2}{r^2}\right) [(2B_\theta^2 - \rho v_\theta^2)^2 \rho\tilde{\omega}^2 - \rho^2 v_\theta^2 (2B_\theta\tilde{\omega} + v_\theta F)^2] \frac{1}{r^3} \\ & - 4(B_\theta F + \rho v_\theta\tilde{\omega})^2 \frac{\rho\tilde{\omega}^2 - k^2\gamma p}{r^3} \\ & + 4m(B_\theta F + \rho v_\theta\tilde{\omega})(2B_\theta^2 - \rho v_\theta^2) \frac{\rho\tilde{\omega}^2}{r^4}, \end{aligned} \quad (2.10c)$$

$$\begin{aligned} W = & 2[(B^2 + \gamma p)\rho\tilde{\omega}^2 - \gamma p F^2] \left[kB_\theta G + \frac{m}{r}\rho v_\theta\tilde{\omega} + \rho v_\theta^2 \left(k^2 + \frac{m^2}{r^2}\right) \right] \frac{1}{r^2} \\ & - \rho^2 v_\theta\tilde{\omega}^2 [2B_\theta F\tilde{\omega} + v_\theta(\rho\tilde{\omega}^2 + F^2)] \frac{1}{r^2}, \end{aligned} \quad (2.10d)$$

and, in deriving (2.8) and (2.9), we have made use of the following identities:

$$\begin{aligned} C^2 + DE &= -AS(V + UD + H'D)/r, \\ C &= W - HD, \\ U &= A/r - \frac{1}{2}H'. \end{aligned}$$

From the first-order system (2.2) we saw that the only singularities are given by $A = 0$ and $S = 0$. Hence the singularity $D = 0$ in (2.8) and its counterpart $E = 0$ in (2.9) are only apparent.

We supply (2.8) with boundary conditions (2.4) and (2.5). The corresponding boundary conditions for (2.9) can be found by expressing χ in terms of Π using (2.2). Then the expression for χ can be substituted into (2.4) and (2.5). This gives

$$\left[-\frac{AS}{Er}\Pi' + \frac{C}{E}\Pi\right]_{r=R} = 0, \quad (2.11)$$

$$\left[-\frac{AS}{Er}\Pi' + \frac{C}{E}\Pi \right]_{r=0} = 0. \quad (2.12)$$

Again, the condition (2.12) can be dropped under special circumstances.

Equations (2.8) and (2.9) are equivalent and, although (2.8) may seem the simplest, it turns out that for our choice of equilibria it is in fact (2.9) that gives the simplest description. In the following sections we shall therefore make use of (2.9) instead of (2.8).

3. Spectrum of an incompressible plasma

Before discussing the compressible case, we review the spectrum and stability of the incompressible plasma (Trehan 1959; Chandrasekhar 1961). We do so, first, because the structure of the spectrum has not been emphasized by previous authors and, secondly, because it will give a better insight in the effect of compressibility.

The rigidly rotating, constant-magnetic-pitch equilibrium with constant density is described by

$$\left. \begin{aligned} \rho &= \rho_0, \\ v_r &= 0, \quad v_\theta = \alpha_2 r, \quad v_z = \alpha_3, \\ B_r &= 0, \quad B_\theta = \beta_2 r, \quad B_z = \beta_3, \\ p &= p_0 + \frac{1}{2}(\rho_0 \alpha_2^2 - 2\beta_2^2)r^2, \end{aligned} \right\} \quad (3.1)$$

where $0 \leq r \leq R$. We now scale our problem with R as characteristic length, α_2 as characteristic frequency and ρ_0 as characteristic density. Then the equilibrium becomes

$$\left. \begin{aligned} \rho &= 1, \\ v_r &= 0, \quad v_\theta = r^*, \quad v_z = \alpha_3^*, \\ B_r &= 0, \quad B_\theta = M_p^{-1}r^*, \quad B_z = \beta_3^*, \\ p &= p_0^* + \frac{1}{2}(1 - 2M_p^{-2})r^{*2}, \end{aligned} \right\} \quad (3.2)$$

where $0 \leq r^* \leq 1$ and $M_p = \alpha_2/\beta_2$ is the poloidal Alfvén Mach number. With this equilibrium, $\tilde{\omega}^*$ and F^* become constants:

$$\tilde{\omega}^* = \omega^* - m - k\alpha_3^*, \quad F^* = mM_p^{-1} + k\beta_3^*.$$

Below, we suppress asterisks for the sake of clarity.

3.1. The eigenvalue problem

In order to find the equations describing incompressible perturbations, we, heuristically, take the limit as $\gamma \rightarrow \infty$. In this limit, and for the equilibrium (3.2), (2.9) yields

$$(\tilde{\omega}^2 - F^2) \left[\Pi'' + \frac{1}{r}\Pi' - \left(k^{*2} + \frac{m^2}{r^2} \right) \Pi \right] = 0, \quad (3.3)$$

where

$$k^{*2} = k^2(1 - 4\eta^2), \quad \eta = \frac{\tilde{\omega} + M_p^{-1}F}{\tilde{\omega}^2 - F^2}.$$

The corresponding boundary conditions are:

$$(\tilde{\omega}^2 - F^2)(-r\Pi' + 2m\eta\Pi) = 0, \quad (3.4)$$

for $r = 0$ and $r = 1$.

We note that the only effect of the axial velocity α_3 is a constant Doppler shift

of the spectrum. Therefore we shall take $\alpha_3 = 0$. The spectrum for $\alpha_3 \neq 0$ can then be found afterwards from the spectrum with $\alpha_3 = 0$ by giving the latter a constant Doppler shift.

When

$$\tilde{\omega}^2 = F^2, \quad (3.5)$$

any Π is a solution of (3.6), so that $\tilde{\omega}_{\pm} = \pm F$ are eigenvalues of infinite multiplicity representing the Alfvén/slow continuum. We assume below that $\tilde{\omega}$ does not belong to this continuum.

For $k = 0$, (3.3) becomes

$$(\tilde{\omega}^2 - m^2 M_p^{-2}) \left(\Pi'' + \frac{1}{r} \Pi' - \frac{m^2}{r^2} \Pi \right) = 0, \quad (3.6)$$

while the boundary conditions have the form (3.4). Equation (3.6) has solutions

$$\Pi_{\pm}(r) = r^{\pm m}, \quad (3.7)$$

which do not satisfy the boundary conditions (3.4). Hence $\tilde{\omega} = \pm m M_p^{-1}$ are the only spectral points, and the equilibrium is stable. This is in agreement with Fung (1984).

When $k \neq 0$, we write the solution of (3.3) in the form

$$\Pi = c_1 I_m(k^* r) + c_2 K_m(k^* r), \quad (3.8)$$

where I_m and K_m are modified Bessel functions. The boundary condition at $r = 0$ gives $c_2 = 0$, and the condition at $r = 1$ then gives:

$$k^* I'_m(k^*) \pm m[1 - (k^*/k)^2]^{1/2} I_m(k^*) = 0. \quad (3.9)$$

Zeros of this equation give the eigenvalues

$$\tilde{\omega}_i = \mu \lambda_i + \nu [\lambda_i^2 + \mu^2 M_p^{-2} F \lambda_i + F^2]^{1/2}, \quad \mu = \pm 1, \quad \nu = \pm 1, \quad (3.10)$$

where $\lambda_i = [1 - (k_i^*/k)^2]^{-1/2}$ and k_i^* is the i th zero of (3.9).

3.2. Description of the spectrum

The spectrum for incompressible perturbations is given by (3.10). Since μ and ν can be both 1 and -1 , there are four curves determining the spectrum for constant values of m and k . Two of these curves are always stable, and two may become unstable. Furthermore, we note that $0 \leq \lambda_i \leq 1$. As $\lambda_i \rightarrow 0$, two curves go to $\tilde{\omega} = +F$ and two curves go to $\tilde{\omega} = -F$, corresponding to the Alfvén/slow continuum point. As $\lambda_i \rightarrow 1$, we find

$$\tilde{\omega}_i \rightarrow \mu + \nu(1 + \mu^2 M_p^{-2} F + F^2)^{1/2}. \quad (3.11)$$

Instabilities occur when the expression in the square root in (3.10) becomes negative. This gives

$$(1 - M_p^{-2}) \lambda_i^2 + (M_p^{-1} \lambda_i + \mu F)^2 > 0 \quad (3.12)$$

as a condition for stability. So, if $M_p^2 > 1$, i.e. $v_\theta^2 > B_\theta^2$, then the equilibrium is always stable. Furthermore, if $M_p \rightarrow 0$ (no rotation) then there are always values of m and k such that the plasma is unstable, but if $M_p \rightarrow \infty$ (no poloidal magnetic field), the plasma is always stable. So we conclude that the instability is driven by the poloidal magnetic field and stabilized by the poloidal velocity.

Figure 1. The incompressible spectrum: the first 200 points of each branch of σ for $M_p = 0.156$, $\beta_3 = 0.192$, $m = 0$, $k = 50$ and $F = 9.58$.

When $\beta_3 = 0$, we find that the equilibrium is stable for $m = 0$ and $|m| \geq 2$. For $|m| = 1$ it is possible to have an instability. This is in agreement with Lucas (1986).

The eigenfunctions (3.8) have the same form as in the static case. In the static limit ($M_p \rightarrow 0$) we find that for $F \rightarrow 0$, $\omega_i \approx \nu(2\mu F/M_p)^{1/2}$, which is in agreement with equation (51) of Goedbloed (1971). However, a discussion of the different types of interchanges, as given there for the static case, lies beyond the scope of this paper.

In Fig. 1 we show the typical spectrum of an unstable, incompressible plasma for fixed values of m and k . For this we numerically found the first 200 zeros of (3.9) and inserted them into the expression for $\tilde{\omega}$. We also added the limit points $\tilde{\omega} = \pm F$ to this plot. In this figure, however, we have plotted $\sigma = -i\tilde{\omega}$ instead of $\tilde{\omega}$, as we shall do with all our spectra in this paper.

We see from fig. 1 that two branches start with $\text{Re}(\sigma) \neq 0$ and cluster towards $\sigma = \pm iF$, and two stable branches are completely on the imaginary axis, one above $\sigma = +iF$ and one below $\sigma = -iF$, and also cluster towards $\sigma = \pm iF$.

We clearly see from fig. 1 that the σ spectrum is symmetric with respect to the imaginary axis. However, the complete spectrum will also be symmetric with respect to the real axis when we take the spectrum corresponding to $(-m, -k)$ into account.

In fig. 2 we have plotted the complete unstable spectrum for $M_p = 0.156$, $\beta_3 = 0.192$ and $|k| = 50$. There are only eight values of m that give rise to instability. Since we only have four curves of eigenvalues, this means that every eigenvalue in the σ or $\tilde{\omega}$ plane is twofold degenerate. The upper curve corresponds to $(m, k) = (0, 50), (3, -50)$, the second to $(-2, 50), (1, -50)$, the third to $(-1, 50), (2, -50)$, and the lowest to $(-3, 50), (0, -50)$. For each of these pairs, the value of F is the same, giving rise to this degeneracy.

Since each curve corresponds to different values of m , each curve will have a different Doppler shift, so that the picture in the non-Doppler-shifted eigenvalue ω is changed. All curves with positive k are shifted upwards and all curves with negative k downwards by an amount m . This means that in ω the spectrum is no longer degenerate, but will still be symmetric with respect to the real and imaginary axes.

It is easy to show that each curve is part of a circle with origin $\tilde{\omega} = -M_p^{-1}F$

Figure 2. The complete unstable spectrum in the σ plane for $M_p = 0.156$, $\beta_3 = 0.192$, and $|k| = 50$.

and radius $R = |F|(M_p^{-2} - 1)^{1/2}$. This is in agreement with Lifschitz (1997). Furthermore, the number of unstable modes on each curve increases with $|k|$, as can be seen from the definition of λ_i . This may be explained by the fact that k scales the radius r . Note that the instability depends on k through F . So when we change k we should also change m in order to leave F invariant.

4. Spectrum of a compressible plasma: analytical results

In this section we shall consider the effect of compressibility on the spectrum and stability of a rigidly rotating, constant-magnetic-pitch equilibrium with constant density, described by (3.2). Unlike in the incompressible case, the equilibrium pressure now does play a role.

In order to obtain some analytical results in the compressible case, we simplify our equilibrium. That is, we take $p_0 = 0$ and $\beta_3 = 0$. Since the pressure cannot become negative, this gives the condition $M_p^2 \geq 2$. This means that these flows are necessarily super-Alfvénic. Note that, owing to our choice of p_0 the total pressure and the absolute temperature vanish on the axis, which is very special.

4.1. The eigenvalue problem

With the above choice for our equilibrium, (2.9) yields

$$(\tilde{\omega}^2 - m^2 M_p^{-2}) \left(\tilde{\omega}^2 - \frac{\varphi m^2}{1 + \varphi M_p^2} \right) \left[\Pi'' + \frac{1}{r} \Pi' - \left(k^{*2} + \frac{m^{*2}}{r^2} \right) \Pi \right] = 0, \quad (4.1)$$

where the following definitions are used:

$$\begin{aligned} m^{*2} &= m^2 - \frac{\tilde{\omega}^4 - (2\tilde{\omega} + m)^2}{(M_p^{-2} + \varphi)\tilde{\omega}^2 - \varphi m^2 M_p^{-2}}, \\ k^{*2} &= k^2 \left[1 + \frac{(1 - 2M_p^{-2})^2 \tilde{\omega}^2 - 4\varphi(\tilde{\omega} + m M_p^{-2}) - M_p^{-2} (2\tilde{\omega} + m)^2}{(\tilde{\omega}^2 - m^2 M_p^{-2})[(M_p^{-2} + \varphi)\tilde{\omega}^2 - \varphi m^2 M_p^{-2}]} \right], \\ \varphi &= \frac{1}{2}\gamma(1 - 2M_p^{-2}). \end{aligned}$$

The boundary condition at $r = 1$ has the form

$$-\frac{d\Pi}{dr}(1) + \Lambda\Pi(1) = 0, \quad (4.2)$$

where

$$\Lambda = \frac{2m(M_p^{-2}m + \tilde{\omega})}{\tilde{\omega}^2 - F^2} - \frac{[(M_p^{-2} - 1)\tilde{\omega}^2 + M_p^{-2}(\tilde{\omega} + m)^2]\tilde{\omega}^2}{(\tilde{\omega}^2 - F^2)[(M_p^{-2} + \varphi)\tilde{\omega}^2 - \varphi m^2 M_p^{-2}]}$$

Whether or not the boundary condition at $r = 0$ has to be imposed remains to be understood.

Note. In the incompressible limit, i.e. when $\gamma \rightarrow \infty$, we have $m^* \rightarrow m$ and $k^* \rightarrow k(1 - 4\eta^2)^{1/2}$, and we find the solution of the previous section. The condition on M_p is then lifted, since the pressure drops out of the problem, and thus p_0 can be chosen large enough to ensure a positive pressure everywhere in the domain.

As before, the problem (4.1), (4.2) has eigenvalues of infinite multiplicity, which are given by

$$\tilde{\omega}_{A,\pm} = \pm m M_p^{-1}, \quad \tilde{\omega}_{S,\pm} = \pm \left(\frac{\varphi m^2}{1 + \varphi M_p^2} \right)^{1/2}, \quad (4.3)$$

which represent the Alfvén and slow continua respectively. We assume below that $\tilde{\omega}$ does not belong to these continua.

4.2. Azimuthal modes

We start our investigation by considering azimuthal modes, i.e. $k = 0$. This means that our problem has become purely two-dimensional, describing motions in the (r, θ) plane. Equation (4.1) can then be written in the form

$$r \frac{d}{dr} \left(r \frac{d\Pi}{dr} \right) - m^{*2} \Pi = 0, \quad (4.4)$$

and the boundary condition simplifies accordingly. Temporarily, we consider m^* rather than $\tilde{\omega}$ as a spectral parameter. This equation must be considered in the space with inner product $\{\cdot, \cdot\}$ defined by

$$\{\Pi_1, \Pi_2\} = \int_0^1 \Pi_1 \bar{\Pi}_2 \frac{dr}{r}, \quad (4.5)$$

where the bar denotes the complex conjugate. Such a choice guarantees that the kinetic energy of the displacement ξ is finite provided that the norm of Π is finite. This can be shown by using (2.2), (2.6) and (2.7) for this particular case.

To understand the properties of the problem (4.4), (4.2), we introduce a new variable x such that

$$r = e^{-x}. \quad (4.6)$$

In terms of this new variable, (4.4) and the boundary condition (4.2) read

$$\frac{d^2 \Pi}{dx^2} - m^{*2} \Pi = 0, \quad (4.7)$$

$$\frac{d\Pi}{dx}(0) + \Lambda\Pi(0) = 0. \quad (4.8)$$

This problem is considered in the space of square-integrable functions with inner product

$$\{\Pi_1, \Pi_2\} = \int_0^\infty \Pi_1 \bar{\Pi}_2 dx. \quad (4.9)$$

This is a familiar problem from quantum mechanics, describing a particle moving on the positive semi-axis. It is well known (cf. e.g. Messiah 1961) that for such a problem the boundary condition at infinity is not needed and that its spectrum is continuous and coincides with the imaginary axis in the m^* complex plane.

Eigenfunctions of the problem (4.7) and (4.8) are

$$\Pi(x) = \cos(\kappa x) - \frac{\Lambda}{\kappa} \sin(\kappa x), \quad (4.10)$$

where $\kappa^2 = -m^{*2} > 0$. The value of κ may vary continuously, giving rise to a continuous spectrum. Note that the eigenfunctions (4.10) oscillate infinitely rapidly for $x \rightarrow \infty$, resulting in an infinite norm, $\|\Pi\|^2 = \{\Pi, \Pi\} = \infty$, as one can expect for eigenfunctions corresponding with eigenvalues belonging to the continuous spectrum.

The eigenfunctions of the problem (4.4), (4.2) are

$$\Pi(r) = \cos(\kappa \ln r) + \frac{\Lambda}{\kappa} \sin(\kappa \ln r), \quad (4.11)$$

and the corresponding displacement χ has the form

$$\chi(r) = \sin(\kappa \ln r). \quad (4.12)$$

The continuous spectrum originates from the fact that in the cylinder the origin is a stagnation point of both the magnetic field and the velocity field. Whenever the pressure also vanishes at the origin, the differential operator loses its elliptic character, and this gives rise to a continuous spectrum. This is reflected in S in (2.8) and (2.9). In this special case S has the form $(1 + \varphi M_p^2)(\tilde{\omega}^2 - \tilde{\omega}_{S,\pm}^2)r^2$, giving rise to the slow-continuum points $\tilde{\omega}_{S,\pm}$ and a new continuum related to $r = 0$. Although the latter continuous spectrum arises only in this special case, there is a discrete spectrum related to it for more general cases, as we show below and in the next section.

4.3. A plasma annulus

The continuous spectrum and its singular eigenfunctions may be avoided by including an infinitely conducting core inside the cylinder. Since $r = 0$ now lies outside our domain, we need a second boundary condition. This is provided by the inner wall at, say, $r = \varepsilon < 1$: $\chi(\varepsilon) = 0$. Then it can be shown that we still have solutions (4.11) and (4.12); see the appendix. However, the boundary condition at $r = \varepsilon$ now gives rise to discrete values of κ :

$$\kappa_n = \frac{n\pi}{\ln \varepsilon}, \quad n = 0, \pm 1, \pm 2, \dots \quad (4.13)$$

Since $r = 0$ now lies outside the domain, the corresponding eigenfunctions have finite norm. Note that in the limit as $\varepsilon \rightarrow 0$ we do not retrieve the continuous spectrum, because the discrete spectrum does not approximate the continuous curve uniformly in ε .

4.4. Non-azimuthal modes

For $k \neq 0$ we can write the spectral problem for m^* as

$$r \frac{d}{dr} \left(r \frac{d\Pi}{dr} \right) - (m^{*2} + k^{*2} r^2) \Pi = 0, \quad (4.14)$$

which we supply with boundary condition (4.2). Once again, this problem does not need a boundary condition at $r = 0$.

When written in terms of x , (4.14) assumes the form

$$\frac{d^2 \Pi}{dx^2} - (m^{*2} + k^{*2} e^{-2x}) \Pi = 0, \quad (4.15)$$

and the boundary condition has the form (4.8). It was shown by Lyantse (see Naimark 1967) that the decaying term in (4.15) does not affect its continuous spectrum, which still coincides with the imaginary axis in the m^* plane and is independent of k . At the same time, however, the discrete spectrum of the problem with a conducting core will depend on k , and therefore differ from the spectrum for $k = 0$.

4.5. Dispersion relation and spectrum

So far, we have only discussed the eigenvalue m^* , but we are actually interested in the eigenvalues $\tilde{\omega}$. They can be found from the relation $\kappa^2 = -m^{*2}$. To this end, we take κ to be a continuous parameter. The discrete spectrum may then be found by replacing κ with the appropriate discrete values κ_n .

From the relation $\kappa^2 = -m^{*2}$, we find the following dispersion relation for $\tilde{\omega}$:

$$\begin{aligned} \tilde{\omega}^4 - \left\{ 4 + (m^2 + \kappa^2) [M_p^{-2} + \tfrac{1}{2} \gamma (1 - 2M_p^{-2})] \right\} \tilde{\omega}^2 - 4m\tilde{\omega} \\ + \tfrac{1}{2} (m^2 + \kappa^2) \gamma (1 - 2M_p^{-2}) m^2 M_p^{-2} - m^2 = 0. \end{aligned} \quad (4.16)$$

We immediately note that since $M_p^2 \geq 2$, it follows that $m = 0$ only gives real $\tilde{\omega}$ solutions, and therefore is stable.

From the dispersion relation (4.16), we also see that if $(m, \tilde{\omega})$ is a solution then $(-m, -\tilde{\omega})$ and $(m, \tilde{\omega})$ are also solutions. We also note that the dispersion relation depends only on M_p^2 . Hence the sign of M_p has no influence on $\tilde{\omega}$.

In Fig. 3 we have plotted the two slow branches of the spectrum in the complex σ plane for three different values of the poloidal Alfvén Mach number and $\gamma = \frac{5}{3}$, $m = 1$ and $0 \leq \kappa \leq 10$. This was done by numerically solving the dispersion relation (4.16). The two fast branches turn out to be always stable, and are on the imaginary σ axis. The slow branches describe curves in the complex σ plane. For $M_p = \infty$ and $M_p = 8$ these start in the complex plane (unstable), but for $M_p = 1.89$ they start on the imaginary axis (stable).

As $\kappa \rightarrow \infty$, the slow branches go to the slow continuum point,

$$\tilde{\omega} \rightarrow \pm \left\{ \frac{\frac{1}{2} \gamma (M_p^2 - 2) m^2}{M_p^2 [1 + \frac{1}{2} \gamma (M_p^2 - 2)]} \right\}^{1/2},$$

and the fast branches go to $\pm \infty$. Hence for $M_p = \infty$ the slow branches go to $\tilde{\omega} = i\sigma = 0$ and are completely unstable; see the top part of Fig. 3.

The dispersion relation (4.16) is a fourth-order polynomial in $\tilde{\omega}$, and the stability of the equilibrium is now determined by the roots of this polynomial. Whenever it has a complex root with positive imaginary part, the equilibrium is unstable. The

Figure 3. The two branches of the slow spectrum in the complex σ plane for $M_p = \infty$ (top), $M_p = 8.00$ (middle) and $M_p = 1.89$ (bottom), $\gamma = \frac{5}{3}$, $m = 1$ and $0 \leq \kappa \leq 10$.

analytical expressions for the roots of this polynomial are quite cumbersome, and it is not easy to use them to decide when a root is real or complex. However, in order to find out if the plasma is stable or unstable, it is not necessary to calculate the roots explicitly. We can make use of a discriminant function Δ , which indicates the type of roots of a fourth-order polynomial (Burnside 1909).

A fourth-order polynomial can always be written in the form

$$z^4 + az^2 + bz + c = 0,$$

and in fact our dispersion relation is already in this form. We now define I and J by

$$I = \frac{a^2}{12} + c, \quad J = \frac{ac}{6} - \frac{b^2}{16} - \frac{a^3}{216}.$$

The discriminant function Δ is then defined by

$$\Delta = I^3 - 27J^2,$$

and the type of the roots of the polynomial is determined by the value of Δ . When $\Delta < 0$, there are two real and two imaginary roots. Since these imaginary roots must be conjugate to each other, this implies that the plasma will be unstable.

Figure 4. Stability diagram: (s) stable; (u) unstable. Left: $|m| = 1$; right: $|m| = 2$.

When $\Delta > 0$, there are four real or four imaginary roots. In our case, however, it can be shown that only four real roots can occur, and hence the plasma will now be stable. So the boundary between stable and unstable solutions is given by $\Delta = 0$.

In Fig. 4 we have plotted the stable (s) and unstable (u) areas in the (κ, M_p^{-2}) plane, making use of the discriminant function. We have again taken $\gamma = \frac{5}{3}$. Note that, since $M_p^2 \geq 2$, we have $0 \leq M_p^{-2} \leq 0.5$. For $|m| = 1$ there are two stable areas. This corresponds to the bottom part of Fig. 3, where we see that when κ increases, the slow branches are stable at first, then turn unstable, and finally become stable again. Furthermore, for every value of M_p there is a value of κ for which the equilibrium is unstable.

For $|m| = 2$ we find two regions of instability. This is also true for $|m| > 2$, where there are also two regions of instability, like $|m| = 2$. Moreover, the area of stability increases with $|m|$, and hence the plasma will be stable for large enough $|m|$.

We see from Fig. 4 that two-dimensional compressible perturbations can become unstable, in contrast to two-dimensional incompressible perturbations as we saw in the previous section. There are two mechanisms behind these instabilities. One type of instability is driven by the poloidal magnetic field, and corresponds to the unstable region near $M_p^{-2} = 0.5$ in Fig. 4. Rotation has a stabilizing effect on this type of instability, in the sense that fewer values for κ are unstable as M_p increases. The second type of instability is driven by the rotation. It corresponds to the unstable region near $M_p^{-2} = 0$ in Fig. 4. This type is due to the compressibility, since it was shown by Fung (1984) that a constant density profile is stable for incompressible perturbations. We now see from Fig. 4 that for $|m| = 1$ these instabilities evolve into each other as M_p changes. For $|m| \geq 2$, however, they are separated by a stable regime.

5. Spectrum of the compressible plasma: numerical result

In the previous section we discussed the continuous spectrum of a compressible-plasma cylinder and the discrete spectrum of a compressible-plasma annulus. Here we will discuss the *discrete spectrum* of a compressible-plasma cylinder, which arises *when the pressure on axis is finite*. In this case two boundary conditions are needed so that the spectrum is discrete. For small pressure this discrete spectrum proves to be close to the curves describing the continuous spectrum. Moreover, since the

Figure 5. Slow spectrum and first three unstable modes for $m = 1$, $p_0 = 0.001\,562\,5$, and $M_p = 8$.

singularity on the axis is now removed, the eigenfunctions are now regular. In fact, by using the Frobenius method, it can be shown that the eigenfunctions behave like

$$\left. \begin{aligned} \chi_1(r) &= r^{|m|-1+1} R_1(r), \\ \chi_2(r) &= r^{-|m|} R_2(r) + c \ln(r) \chi_1(r) \end{aligned} \right\} \quad (5.1)$$

as $r \rightarrow 0$. Here R_1 and R_2 are regular solutions, for which $R_i(0) = 1$. This is in agreement with Lifschitz (1989). The solution χ_2 should now be discarded because of the boundary condition at $r = 0$, and we are left with the regular solution χ_1 .

The behaviour as $r \rightarrow 0$ of the eigenfunctions is independent of p_0 . This means that in the limit as $p_0 \rightarrow 0$ we do not retrieve the singular functions and have no continuous spectrum. Therefore the continuous-spectrum case is a case on its own.

In order to study the discrete spectrum, we have performed some numerical calculations. For these, we used a finite-element code called LEDAFLOW, which is an extension with flow equilibria of the code LEDA (large-scale eigenvalue solver for the dissipative Alfvén spectrum) (Kerner *et al.* 1985). We shall give a full description of the code in another paper (Nijboer *et al.* 1997).

In LEDAFLOW the eight MHD equations for the velocity $\mathbf{v}$, the magnetic field $\mathbf{B}$, the density ρ and the temperature T are used. The gas pressure p is determined by the temperature and the density, and follows from the ideal gas law $p = \rho T$. The eight equations are linearized about an equilibrium, taking a $f(r) \exp(im\theta + ikz + \lambda t)$ dependence for the perturbed quantities. These perturbed quantities are then approximated on every grid point by finite elements, giving rise to a large eigenvalue system

$$\mathbf{A}\mathbf{w} = \lambda \mathbf{B}\mathbf{w}. \quad (5.2)$$

Finally, this system is solved by standard eigenvalue solvers.

Figure 6. The slow spectrum for $m = 1$, $M_p = 8$, and $p_0 = 0.015\,625$ (left) and $p_0 = 0.156\,25$ (right).

In this section we consider equilibria of the type (3.2) and take $p_0 \neq 0$. We shall take $\gamma = \frac{5}{3}$ and start by considering the two-dimensional case, meaning $k = 0$ and $\beta_3 = 0$. Note that when $k = 0$, the only effect of $\beta_3 \neq 0$ is a change of the pressure on axis, p_0 .

5.1. Finite pressure on axis

In Fig. 5 we show the slow spectrum in the complex σ plane and three eigenfunctions for $M_p = 8$ and $p_0 = 0.001\,562\,5$. The eigenvalue σ is related to λ and $\tilde{\omega}$ by $\sigma = i\mathbf{k} \cdot \mathbf{v}_0 + \lambda = -i\tilde{\omega}$, and the eigenfunction v_r is related to the eigenfunctions ξ_r and ξ_θ by $v_r = \sigma\xi_r + \xi_\theta$.

We clearly see from Fig. 5 that the spectrum is discrete and that the eigenfunctions are regular. Furthermore, comparing the spectrum with the middle figure of Fig. 3 shows that the eigenvalues are near the analytical curve.

The eigenfunctions correspond to the first three unstable modes, and we see that with each mode the number of zeros of the eigenfunction is increased by one. In the static case this property and the fact that the spectrum is restricted to the axes imply that the first unstable mode is the most unstable. In the case of flow equilibria this is no longer true, and we clearly see that in our case it is not the first but rather the second mode that is the most unstable. This is even clearer for the analytical curves of Fig. 3, where it is shown that the curve may even start from the stable axis.

In Fig. 6 we have plotted the slow spectrum for the same equilibrium as in Fig. 5, but we have changed the pressure on axis to be $0.015\,625$ (left) and $0.156\,25$ (right). The number of unstable modes decreases when p_0 increases, but the largest growth rate and even the shape of the curve on which these modes lie does not change

Figure 7. Plots of the growth rate of the most unstable mode-against M_p^{-1} for $m = 1$ (left), $m = 2$ (right) and $p_0 = 0.01$, (upper curves) and $p_0 = 0.1$ (lower curves).

much. The latter is not true in general, and may be due to the fact that p_0 is still relatively small.

5.2. Growth rates

In Fig. 7 we have plotted the growth rate of the most unstable mode ($= \max \text{Re}(\sigma)$) against the poloidal magnetic field M_p^{-1} . The left-hand part is for $m = 1$ and the right-hand part for $m = 2$. In both parts the upper curve is for $p_0 = 0.01$ and the lower one for $p_0 = 0.1$, so we see that the pressure on axis has a stabilizing effect. We also observe that this effect seems to be larger for large values of M_p^{-1} . However, we should keep in mind that our pressure profile is of the form

$$p(r) = p_0 + \frac{1}{2}(1 - 2M_p^{-2})r^2.$$

Hence, when we keep p_0 fixed, it is relatively large when M_p^{-1} is large and relatively small when M_p^{-1} is small. Thus we may expect a larger effect for large M_p^{-1} . Note that in Figs 5 and 6 the equilibrium is dominated by the velocity, and therefore p_0 will have less effect than for equilibria with a larger magnetic component.

Figure 7 is in good agreement with Fig. 4, although the finite value of p_0 and the fact that the spectrum is discrete enlarge the stable region in the case $m = 2$. For $m = 1$ we see that the equilibrium remains unstable for all values of M_p^{-1} when $p_0 = 0.01$, while it has a range of values of M_p^{-1} for which it is stable when $p_0 = 0.1$.

The fact that the growth rates for $m = 1$ and $p_0 = 0.01$ seem to be somewhat lower at $M_p^{-1} \approx 0.3$ and $M_p^{-1} \approx 0.65$ than one would expect from the trend of the curve is a consequence of the spectrum being discrete and the fact that these discrete modes lie on a curve. Therefore it is possible that two discrete modes lie on either side of the maximum-growth-rate point. Whenever M_p^{-1} is changed somewhat (increased or decreased!), this may result in an increment in the growth rate. This is what happens at $M_p^{-1} \approx 0.3$ and $M_p^{-1} \approx 0.65$, and therefore the growth rates are somewhat lower there.

We also conclude from Fig. 7 that there are two mechanisms for driving instabilities. One occurs for relatively high values of M_p^{-1} , and is therefore related to the poloidal magnetic field, and the other occurs for relatively small values of M_p^{-1} , and therefore corresponds to an instability driven by the rotation. It is good to note that rotation is quite important for both of these instabilities. One could, of course, construct an equilibrium without the poloidal velocity by taking p_0 large enough. However, it proves that such an equilibrium is stable.

Figure 8. Plots of the growth rate of the most unstable mode against ρ_2 for $p_0 = 1.1$, $m = 1$ and $M_p = \infty$ (upper curves) and $M_p = 10$ (lower curves).

5.3. Density stratification

The above-mentioned instabilities are also reflected in the following sufficient condition for stability, which was found by Lucas (1981):

$$r\alpha_2^2\rho' - \frac{(r^2B_\theta^2)'}{r^3} - \frac{r^2\rho^2\alpha_2^4}{\gamma p} \geq 0 \quad \text{on } [0, 1]. \quad (5.3)$$

Here the equilibrium density ρ and pressure p may depend on r , as does B_θ , a general poloidal magnetic field. The rotation rate α_2 is constant. Although this is a sufficient condition for stability, it reflects the fact that both the poloidal magnetic field and the rotation may give rise to instabilities. Furthermore, it shows that these instabilities may be stabilized by a density profile that increases with r , whereas a decreasing density profile may give rise to instabilities.

In order to study the effect of a varying density profile and to check Lucas's stability condition, we consider a parabolic density profile

$$\rho(r) = 1 + \rho_2 r^2, \quad (5.4)$$

which gives the following pressure profile from (2.1) for the rigidly rotating, constant-magnetic-pitch equilibrium

$$p(r) = p_0 + \frac{1}{2}(1 - 2M_p^{-2})r^2 + \frac{1}{4}\rho_2 r^4. \quad (5.5)$$

Here we shall only consider positive values for ρ_2 , since it is known that negative values of ρ_2 give rise to instabilities unless stabilized by the magnetic field (Fung 1984).

With the density profile (5.4) and pressure profile (5.5) the stability condition (5.3) becomes

$$r^2\rho_2 - 4M_p^{-2} - \frac{r^2\rho(r)^2}{\gamma p(r)} \geq 0 \quad \text{on } [0, 1], \quad (5.6)$$

and it can never be satisfied, because of the poloidal magnetic field. However, when $M_p = \infty$, (5.6) becomes

$$2\gamma p_0 - \frac{1}{\rho_2} + (\gamma - 2)r^2 + \frac{1}{2}(\gamma - 2)\rho_2 r^4 \geq 0, \quad \text{on } [0, 1], \quad (5.7)$$

and we see that for $\gamma < 2$ the equilibrium will be stable for ρ_2 between certain values, whereas it will be stable for large enough ρ_2 when $\gamma > 2$.

In Fig. 8 we have plotted the growth rate of the most unstable mode against ρ_2 .

Figure 9. Part of the spectrum for $m = 1$, $k = 10$, $M_p = 8$ and $p_0 = 0.001\,562\,5$

The upper curve is for $M_p = \infty$ and the lower one for $M_p = 10$. As before, we have taken $\gamma = \frac{5}{3}$.

We find from Lucas's condition that for $M_p = \infty$ the plasma is stable for approximately $0.30 \leq \rho_2 \leq 19.7$. This interval is in good agreement with Fig. 8, from which we also see that the plasma is unstable outside this interval. We note that the magnetic field decreases the growth rate, so that its effect is stabilizing. This is not in contradiction with Lucas's stability condition, since that is only a sufficient condition. We found that the equilibrium is stable for small enough values of M_p .

The magnetic field does not drive any instabilities in this case, which may be explained by the relatively large pressure. In Fig. 7 we saw that the pressure stabilizes the magnetic instability. This stabilizing effect of the pressure on the magnetic instability cannot, however, be found from the stability condition of Lucas.

Owing to the rotation and the compressibility, the plasma tends to have a higher density at the edge than at the centre of the cylinder. Therefore, if the density profile is too flat (small ρ_2), it becomes unstable. However, when the density profile is too steep (large ρ_2), the plasma turns out to be unstable too. The stabilizing effect of the magnetic field can be explained from the fact that the magnetic field lines are frozen into the plasma, and this gives the plasma an extra stiffness against these unstable density perturbations.

So we conclude that for compressible, azimuthal modes there are two types of instabilities. One is driven by the poloidal magnetic field, and is stabilized by rotation and pressure, and the other is driven by rotation and an unfavourable density profile, and is stabilized by the poloidal magnetic field and the pressure.

5.4. Non-azimuthal modes

When $k \neq 0$, it may be expected that, even for small values of p_0 , the discrete spectrum is not close to the curves of the continuous spectrum for $p_0 = 0$. This is

shown in Fig. 9, where we have plotted part of the spectrum for the same parameters as in Fig. 6 – only this time we have taken $k = 10$. We see that the maximum growth rate is lowered. Hence in this case k has a stabilizing effect. However, we also see that the number of unstable modes has increased. Furthermore, we note that for $k = 0$ there is only clustering of modes towards the slow continuum point from below, whereas for $k \neq 0$ there is also clustering from above. So a finite value of k may change the spectrum considerably. This will be especially true when the axial magnetic field is also taken into account.

6. Conclusions

In this paper we have considered the spectrum and stability of rigidly rotating, constant-magnetic-pitch equilibria that are cylindrically symmetric. For incompressible perturbations the spectrum has been calculated analytically, and we have shown some typical spectra.

For compressible perturbations we have found a new, possibly unstable, continuous spectrum whenever the equilibrium pressure vanishes on axis. Such an equilibrium is only possible when the poloidal Alfvén Mach number is larger than $\sqrt{2}$, which means that the equilibrium flow is super-Alfvénic. The behaviour of the continuous spectrum has been investigated, and we have shown that the curves describing the spectrum are completely on the stable axis, or start in the complex plane and then go to the stable axis. There is, however, one exception: for $|m| = 1$ and certain values of the poloidal Alfvén Mach number, the spectrum may start on the stable axis, become unstable, and then return to the stable axis.

This continuous spectrum cannot be found as a limit for vanishing pressure on axis or for a vanishing core at the centre of the cylinder. In both cases the spectrum is discrete. For azimuthal perturbations it lies near (respectively on) the curves of the continuous spectrum. Therefore this discrete spectrum may also be unstable. We have shown that there are two mechanisms for these instabilities. One is driven by the poloidal magnetic field, which is well known to give rise to instabilities, and is stabilized by rotation. The other is driven by the poloidal velocity and an unfavourable density profile, and is stabilized by the poloidal magnetic field. The pressure has a stabilizing effect on both these types of instabilities.

In order for the plasma to become unstable against azimuthal perturbations a number of ingredients are needed. First of all, compressibility is important, since incompressible, azimuthal perturbations are stable. Secondly, relatively high rotation speeds are needed in order to have a small pressure on axis, which will allow the instabilities to show up. Thirdly, the perturbations need to be non-axisymmetric, since perturbations that are both azimuthal and axisymmetric are always stable.

For non-azimuthal, compressible perturbations the spectrum changes considerably. This case has to be investigated more thoroughly.

Appendix A

When considering the continuous spectrum for a rigidly rotating, constant-magnetic-pitch plasma with constant density and vanishing pressure on axis, we may, for $k = 0$, also consider the problem in terms of χ . It can be shown that in this case

the same equation holds for Π and for χ :

$$L\chi = -m^{*2}\chi, \quad (\text{A } 1)$$

where we define the operator L by

$$L\chi = -r \frac{d}{dr} \left(r \frac{d\chi}{dr} \right).$$

This operator can be shown to be formally self-adjoint for the inner product (4.5). The norm of χ now becomes

$$\|\chi\|^2 = \{\chi, \chi\} = \int_0^1 |\chi|^2 \frac{dr}{r} = \int_0^1 |\xi_r|^2 r \, dr. \quad (\text{A } 2)$$

By applying the boundary condition $\chi(1) = 0$ and demanding regular solutions, i.e. $\{\chi, \chi\} < \infty$, the operator L can be extended to a self-adjoint operator $\hat{L}$. Since $\hat{L}$ is a positive operator, this means that its eigenvalues $-m^{*2}$ are real and positive.

When a conducting core is included, we should define the norm as

$$\|\chi\|^2 = \{\chi, \chi\} = \int_\varepsilon^1 |\chi|^2 \frac{dr}{r} = \int_\varepsilon^1 |\xi_r|^2 r \, dr, \quad (\text{A } 3)$$

where the inner boundary is located at $r = \varepsilon$. With this norm, demanding regular solutions, and applying the boundary conditions $\chi(\varepsilon) = 0$ and $\chi(1) = 0$, the operator can again be extended to a self-adjoint one for which the eigenvalues $-m^{*2}$ are again real and positive. However, because of the second boundary condition, the eigenvalues will now be discrete.

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