Chapter 4
Completeness for a Class of Banach Space Operators

In this chapter we consider the completeness problem for a more general class of bounded linear operators than those considered in Chaps. 2 and 3. Moreover noncompact operators are included too, and we allow the underlying spaces to be complex Banach spaces.

4.1 A Special Class of Operators

In this section we describe the class of operators, let $q$ be a scalar entire function, and let $P$ be an operator-valued entire function, $P : \mathbb{C} \to \mathcal{L}(X, X)$, where $X$ is a Banach space. We say that an operator $T$ on $X$ is related to the pair of entire functions $\{q, P\}$ if $q$ is not zero in a neighbourhood of zero and for each $z \in \mathbb{C}$ the operator $I - zT$ is invertible whenever $q(z) \neq 0$ and in that case

$$q(z) \neq 0 \implies (I - zT)^{-1} = \frac{1}{q(z)}P(z).$$

(4.1)

Such an operator $T$ is uniquely determined by the functions $\{q, P\}$. Indeed, assume that $T_1$ on $X$ is also related to the functions $\{q, P\}$. Then, by (4.1), the functions $(I - zT)^{-1}$ and $(I - zT_1)^{-1}$ coincide in a neighbourhood of zero, which implies that $I - zT$ and $I - zT_1$ coincide in a neighbourhood of zero, and thus $T = T_1$. Since $q$ does not vanish identically, the fact that $T$ is related to the entire functions $\{q, P\}$ implies that the non-zero part of the spectrum of $T$ consists of isolated points only. Thus for each $0 \neq \lambda \in \sigma(T)$ the spectral projection $P_\lambda$ is well defined, and so is the linear space

$$\mathcal{M}_T := \text{span}\{\text{Im } P_\lambda \mid 0 \neq \lambda \in \sigma(T)\}.$$
The problem is to obtain necessary and sufficient guaranteeing $\mathcal{M}_T$ to be dense in $X$, or more generally, to find a natural closed linear subspace $L$ in $X$ such that $\mathcal{M}_T \oplus L$ is dense in $X$.

The converse in (4.1) does not have to be true. More precisely, if $I - zT$ is invertible, then it may happen that $q(z) = 0$. To see this, let $T$ be related to the pair of entire functions $\{q, P\}$. Choose $0 \neq z_0 \in \mathbb{C}$ such that $q(z_0) \neq 0$, and define

$$q_1(z) = (z - z_0)q(z) \quad \text{and} \quad P_1(z) = (z - z_0)P(z).$$

Then $q_1$ is a scalar entire function, and $q_1$ is not zero in a neighbourhood of zero. Now fix $0 \neq z \in \mathbb{C}$. Then, using (4.1), we obtain

$$q_1(z) \neq 0 \implies q(z) \neq 0 \implies (I - zT)^{-1} = \frac{1}{q(z)}P(z) = \frac{1}{q_1(z)}P_1(z).$$

It follows that $T$ is related to the entire functions $\{q_1, P_1\}$. Since $q(z_0) \neq 0$, we know that $I - z_0T$ is invertible. However, $q_1(z_0) = 0$. Thus the right hand side of (4.1) is satisfied for the pair $\{q_1, P_1\}$ but the left hand side is not.

We summarise this discussion in the following definition.

**Definition 4.1.1** We say that an operator $T$ on $X$ is determined by the (ordered pair of) entire functions $\{q, P\}$ if $q$ is not zero in a neighbourhood of zero and for each $z \in \mathbb{C}$ the operator $I - zT$ is invertible whenever $q(z) \neq 0$ and that (4.1) holds.

There are many examples of operators of the type described above. In fact, we already met a few examples. For instance, if $T = T_g$, where $T_g$ is the operator appearing in the final paragraph of Sect. 1.1 (see (1.40)), then we know from (2.34) in Lemma 2.3.1 that (4.1) holds with

$$q(z) = 1 - z \int_0^1 e^{zs}g(s)\,ds \quad \text{and} \quad P(z) = q(z)(I - zT)^{-1}.$$

Furthermore, if $T$ is Hilbert-Schmidt, then we know from Lemmas 3.2.3 and 3.2.2 that the operator $T$ is determined by the entire functions $\{q, P\}$ with

$$q(z) = \det_2(I - zT) \quad \text{and} \quad P(z) = (\det_2(I - zT))(I - zT)^{-1}.$$

Note that in both cases the pair $\{q, P\}$ is optimal.

To solve the problems referred to above we exploit the availability of detailed information about the asymptotic behaviour of entire functions using the Phragmén-Lindelöf indicator function and entire functions of completely regular growth. The theory of these functions is reviewed in Chap. 14. On the one hand this final chapter presents an overview of the classical theory of entire functions, while on the other hand a number of new results are presented which are relevant for the results and examples discussed in the present and later chapters.
In what follows we shall need the notion of a scalar entire function “dominating” a vector-valued entire function. See Definition 14.7.1 where this notion is defined for scalar functions.

**Definition 4.1.2** Let \( q \) be an entire function of finite non-zero order \( \rho \) and finite type, and let \( F : \mathbb{C} \rightarrow X \) be a vector-valued entire function of order at most \( \rho \) (see the final section of Chap. 14 for the definition and properties of vector-valued entire functions). We say that \( q \) dominates \( F \) if for every \( x^* \in X^* \) the function \( q \) dominates \( z \mapsto \langle x^*, F(z) \rangle \) in the sense of Definition 14.7.1.

We are now ready to formulate the main theorem of this chapter. For the notion of “completely regular growth” which plays an important role in the theorem, we refer to Definition 14.6.1.

**Theorem 4.1.3** Let \( T \) be a bounded linear operator on the Banach space \( X \), and assume that there exists a positive integer \( k \) such that

\[
X = \text{Im} T^k \oplus \text{Ker} T^k. \tag{4.2}
\]

Assume that \( T \) is determined by the pair of entire functions \( \{q, P\} \), where

1. the scalar entire function \( q \) is of finite non-zero order \( \rho \) and has infinitely many zeros,
2. the operator-valued entire function \( P(z) : X \rightarrow X \) is of order at most \( \rho \).

Moreover, let \( Y \) be a Banach space, and suppose that there is an operator-valued polynomial \( \Lambda : \mathbb{C} \rightarrow \mathcal{L}(X, Y) \) such that

\[
\Lambda(z) P(z) x = 0 \quad (\text{for all } z \in \mathbb{C}) \implies x = 0. \tag{4.3}
\]

Furthermore, suppose that there exist a \( \rho \)-admissible set of half-lines in the complex plane, \( \{\text{ray}(\theta_j; z_0, s_0) \mid j = 1, \ldots, \kappa\} \), and a non-negative integer \( m \) and a positive constant \( M \) such that \( I - zT \) is invertible and for each \( z \in \text{ray}(\theta_j; z_0, s_0) \) with \( j = 1, 2, \ldots, \kappa \), we have

\[
0 < |q(z)| \leq M(1 + |z|^m) \quad \text{and} \quad \|(I - zT)^{-1}\| \leq M(1 + |z|^m). \tag{4.4}
\]

Finally, suppose the entire function \( z \mapsto q(z + z_0) \) is of completely regular growth, and define

\[
\mathcal{F}_{T, \Lambda, z_0} := \{x \in X \mid q(z_0 + z) \text{ dominates } \Lambda(z_0 + z) P(z_0 + z) x\}. \tag{4.5}
\]

Then \( \mathcal{F}_{T, \Lambda, z_0} \) is a closed linear subspace of \( X \) and the closure of the generalised eigenspace of \( T \) has the following properties:

\[
\overline{\mathcal{M}_T \oplus \text{Ker} T^k} = \mathcal{F}_{T, \Lambda, z_0} \quad \text{and} \quad X = \overline{\mathcal{M}_T \oplus S_T}, \tag{4.6}
\]
\[
\mathcal{F}_{T,\Lambda,z_0} \cap S_T = \ker T^k, \tag{4.7}
\]
\[
\mathcal{F}_{T,\Lambda,z_0} \cap \text{Im} T^k = \overline{M_T}. \tag{4.8}
\]

where \( S_T \) is the closed linear subspace of \( X \) defined by

\[
S_T = \{ x \in X \mid z \mapsto (I - zT)^{-1}x \text{ is an entire function} \}.
\]

**Remark 4.1.4** If \( q \) in item (1) of the above theorem has only a finite number of zeros, then (4.1) implies that the non-zero part of the spectrum of \( T \) consists of a finite number of non-zero eigenvalues, \( \lambda_1, \ldots, \lambda_n \) say. The latter implies that

\[
\mathcal{M}_T = \bigoplus_{1 \leq j \leq n} \text{Im} P_{\lambda_j} \quad \text{and} \quad S_T = \bigcap_{1 \leq j \leq n} \ker P_{\lambda_j}.
\]

It follows that \( \mathcal{M}_T \) and \( S_T \) are closed subspaces of \( X \), and

\[
X = \mathcal{M}_T \oplus S_T.
\]

But in that case the completeness problem is not interesting. Both the first and second part of item (1) exclude that \( q \) is a polynomial. Finally, if \( q \) has a finite number of zeros, then this does not exclude that \( q \) can be of non-zero order. Indeed, assume \( q \) is a polynomial, and put \( q_1(z) = \exp(z)q(z) \). Then \( q_1 \) has a finite number of zeros, but the order of \( q_1 \) is one. Moreover,

\[
(I - zT)^{-1} = \frac{1}{q(z)} P(z) = \frac{1}{q_1(z)} P_1(z),
\]

where \( P_1(z) \) is the entire function \( P_1(z) := \exp(-z)P(z) \).

**Remark 4.1.5** If \( Y = X \) and \( \Lambda(z) \) is the identity operator on \( X = Y \) for all \( z \), then (4.3) is automatically fulfilled. Indeed, from condition (4.1) it follows that \( P(0) = q(0)I \) and \( q(0) \neq 0 \). But then \( 0 = P(0)x = q(0)x \) implies that \( x = 0 \), as desired.

**Remark 4.1.6** Since \( P(z) = q(z)(I - zT)^{-1} \) by (4.1), the inequalities in (4.4) imply that \( P(z) \) is also polynomially bounded on each ray \( (\theta_j; z_0, s_0), j = 1, 2, \ldots, \kappa \). Since \( \Lambda(z) \) is a polynomial, it follows that the entire function \( z \mapsto \Lambda(z)P(z) \) has the same property, that is, \( \Lambda(z)P(z) \) is also polynomially bounded on each ray \( (\theta_j; z_0, s_0), j = 1, 2, \ldots, \kappa \). The latter will play an important role when we prove Theorem 4.1.3 in Sect. 4.4.

In the more concrete settings treated in the next two chapters the choice of the entire functions \( q \) and \( P \) determining the operator \( T \) and the choice of the operator polynomial \( \Lambda(z) \) will be rather natural. See the remarks preceding Theorems 5.2.6 and 6.2.1.
Corollary 4.1.7 Let $T$ be a bounded linear operator on a complex Banach space $X$ satisfying all the assumptions of Theorem 4.1.3. Then $T$ has a complete span of eigenvectors and generalized eigenvectors if and only if $\mathcal{F}_{T, \Lambda, z_0} = X$, where $\mathcal{F}_{T, \Lambda, z_0}$ is defined by (4.5). Moreover, in that case

$$\mathcal{M}_T = \overline{\text{Im} T^k} \quad \text{and} \quad S_T = \ker T^k.$$  

Proof The “if and only if” statement follows from the first identity in (4.6). Furthermore, given $\mathcal{F}_{T, \Lambda, z_0} = X$, the first part of the final statement follows from the identity (4.8) while the second part follows from identity (4.7). \qed

Note that condition (4.2) is satisfied with $k = 1$ whenever $T$ is one-to-one and has dense range. Indeed, in that case we have $\ker T = \{0\}$ and (4.2) reduces to $\overline{\text{Im} T} = X$. Moreover, if $X = H$ is a Hilbert space, then (4.2) holds with $k = 1$ if $\ker T = \ker T^*$, and in that case the right hand side of (4.2) is an orthogonal direct sum decomposition. Furthermore, condition (4.2) implies that $\ker T^{k+1} = \ker T^k$ and $\overline{\text{Im} T^{k+1}} = \overline{\text{Im} T^k}$. In fact, the following proposition holds.

Proposition 4.1.8 Let $T$ be a bounded linear operator on a complex Banach space $X$. Assume that there exists a positive integer integer $k$ such that condition (4.2) is satisfied. Then

$$\ker T^k = \ker T^n \quad \text{and} \quad \overline{\text{Im} T^k} = \overline{\text{Im} T^n} \quad (n = k + 1, k + 2, \ldots).$$ (4.9)

In order to prove the above proposition we first derive an auxiliary result.

Lemma 4.1.9 If a bounded linear operator $T$ on a complex Banach space $X$ is one-to-one and has dense range, then $T^n$ has the same properties for each positive integer $n$.

Proof Assume $T$ is one-to-one and has dense range, and let $n$ be a positive integer. Then clearly $T^n$ is one-to-one. The fact that $T$ has dense range implies that $T^*$ on the Banach dual space $X^*$ is one-to-one, and hence $(T^*)^n$ is one-to-one too. But $(T^*)^n = (T^n)^*$, and therefore

$$\overline{\text{Im} T^n} = \perp \left(\left(\overline{\text{Im} T^n}\right)^\perp\right) = \perp \left(\ker (T^n)^*\right) = \perp \{0\} = X,$$

which completes the proof. \qed

Proof of Proposition 4.1.8 In the sequel $X_0 = \overline{\text{Im} T^k}$ and $T_0 = T|_{X_0}$. We split the proof into five parts.

Part 1. We prove the first part of (4.9). Let $n \geq k + 1$ be an integer. Since $\ker T^k \subset \ker T^n$ it suffices to show that $\ker T^n \subset \ker T^k$. Therefore take $x \in \ker T^n$, and put $y = T^{n-1}x$. Then $Ty = T^n x = 0$. Thus $y \in \ker T \subset \ker T^k$. Also, $y \in \overline{\text{Im} T^{n-1}} \subset \overline{\text{Im} T^k}$. But then
Part 2. The operator $T_0$ is one-to-one. Indeed, if $T_0 x_0 = 0$ for some $x_0 \in X_0$. Then $x_0 \in \text{Ker } T \subset \text{Ker } T^k$. Thus $x_0 \in X_0 \cap \text{Ker } T^k$. But the latter intersection consists of the zero vector only. Therefore $x_0 = 0$, and $T_0$ is one-to-one.

Part 3. We prove that $\text{Im } T^n \subset \overline{\text{Im } T^n_0}$ for each integer $n \geq k$. Take $y \in \text{Im } T^n$. Then $y = T^n x$ for some $x \in X_0 \oplus \text{Ker } T^k$. Thus there exist $u_j \in X_0$ and $v_j \in \text{Ker } T^k$, $j = 1, 2, \ldots$ such that $x = \lim_{j \to \infty} T^n (u_j + v_j)$. Since $n \geq k$, we have $T^n v_j = 0$ for each $j = 1, 2, \ldots$, and therefore

$$y = T^n x = \lim_{j \to \infty} T^n (u_j + v_j) = \lim_{j \to \infty} T^n u_j = \lim_{j \to \infty} T^n_0 u_j \in \overline{\text{Im } T^n_0}.$$ 

Hence $\text{Im } T^n \subset \overline{\text{Im } T^n_0}$.

Part 4. We prove that $T_0$ has dense range. From the previous step we know that $\text{Im } T^k \subset \overline{\text{Im } T^k_0}$, and thus $X_0 = \overline{\text{Im } T^k} \subset \overline{\text{Im } T^k_0} \subset X_0$. Therefore $\overline{\text{Im } T^k} = X_0$. On the other hand $\text{Im } T^k_0 \subset \text{Im } T_0$, and hence the range of $T_0$ is dense in $X_0$.

Part 5. We prove the second part of (4.9). Let $n \geq k$. We know that $T_0$ is one-to-one and has dense range (by Parts 2 and 4). Thus $\overline{\text{Im } T^n_0} = X_0$. From Part 3 we know that $\text{Im } T^n \subset \overline{\text{Im } T^n_0} \subset \overline{\text{Im } T^n}$. Thus $\overline{\text{Im } T^n} = \overline{\text{Im } T^n_0} = X_0 = \overline{\text{Im } T^k}$, as desired. \hfill \(\square\)

The remaining part of this chapter consists of six sections. The first presents some spectral preliminaries. The second shows that in order to prove Theorem 4.1.3 it suffices to consider the case when $z_0$ is zero. The third section then contains the proof of Theorem 4.1.3. The fourth presents an example illustrating the main theorem. Some additional remarks are presented in the fifth section. In the sixth section we revisit Theorem 3.4.1.

### 4.2 Spectral Preliminaries II

Before we prove Theorem 4.1.3, we first derive two lemmas. These two lemmas present some basic properties of sets of the form $\mathcal{F}_{T, \Lambda, z_0}$ (see formula (4.5)), using vector-valued versions of the lemmas presented in Sect. 14.7. What follows may be seen as an addition to the spectral results presented in Sect. 1.2.

**Lemma 4.2.1** Let $X$ be a complex Banach space, and let $T$ be a bounded linear operator on $X$ such that $T$ is determined by the entire functions $\{q, P\}$. Furthermore, let $Y$ be a Banach space, and let $\Lambda : \mathbb{C} \to \mathcal{L}(X, Y)$ be an operator-valued polynomial. Assume that
4.2 Spectral Preliminaries II

(1) $q$ is of finite non-zero order $\rho$ and of completely regular growth,
(2) $\Lambda(z)P(z)$ is of order at most $\rho$.

Put
\[
F := \{ x \in X \mid q(z) \text{ dominates } \lambda \}
\]  \hspace{1em} (4.10)

Then the set $F$ is a $T$-invariant linear subspace of $X$, and $x \in F$ if and only if $Tx \in F$. Furthermore, for $z \in \mathbb{C}$ we have
\[
q(z) \neq 0 \implies (I - zT)^{-1}F \subset F.
\]  \hspace{1em} (4.11)

Finally, if there exists a $\rho$-admissible set of half-lines in the complex plane,
\[
\{ \text{ray}(\theta_j; z_0, s_0) \mid j = 1, \ldots, \kappa \},
\]
and a non-negative integer $m$ such that for every $x \in X$ there is a constant $M = M(x)$, depending continuously on $x$, with
\[
\| \Lambda(z)P(z)x \|_Y \leq M(1 + |z|^m) \quad \text{for } z \in \text{ray}(\theta_j; z_0, s_0)
\]
and $j = 1, 2, \ldots, \kappa$, \hspace{1em} (4.12)

then $F$ is norm-closed.

**Proof** Fix $x_1 \in F$, $x_2 \in F$, and $y^* \in Y^*$. Define
\[
f_j(z) = \langle y^*, \Lambda(z)P(z)x_j \rangle, \quad z \in \mathbb{C} \quad (j = 1, 2).
\]

Put $f(z) = \langle y^*, \Lambda(z)P(z)(x_1 + \lambda x_2) \rangle$. Note that $f_1$, $f_2$, and $f$ are entire functions of order at most $\rho$, and $f(z) = f_1(z) + \lambda f_2(z)$. Furthermore, $q$ dominates $f_1$ and $f_2$. Since $q$ is of completely regular growth, $q$ is of normal type by definition. But then we can apply Lemma 14.7.2 to show that $f$ is dominated by $q$. This holds for each choice of $y^*$ in $X^*$. It follows that $x_1 + \lambda x_2 \in F$. Therefore $F$ is a linear subspace of $X$.

In order to show that $F$ is $T$-invariant, we first observe that
\[
(I - zT)^{-1}T = T(I - zT)^{-1} = \frac{1}{z}((I - zT)^{-1} - I),
\]
and therefore we can rewrite $z\Lambda(z)P(z)Tx$ as follows
\[
z\Lambda(z)P(z)Tx = zq(z)\Lambda(z)(I - zT)^{-1}Tx
= q(z)\Lambda(z)(I - zT)^{-1}x - q(z)\Lambda(z)x
= \Lambda(z)P(z)x - q(z)\Lambda(z)x.
\]  \hspace{1em} (4.13)
Since \( \Lambda(z) \) is an operator polynomial, it follows from Theorem 14.1.1 that the order of \( q \) equals the order of \( q \Lambda \). Therefore Lemma 14.7.4 yields that \( q(z) \Lambda(z)x \) for \( x \in X \). But then it follows from (4.13) that \( q(z) \) dominates \( z\Lambda(z)P(z)Tx \) if and only if \( q(z) \) dominates \( \Lambda(z)P(z)x \). Furthermore, again by Lemma 14.7.4, the function \( q(z) \) dominates \( z\Lambda(z)P(z)Tx \) if and only if \( q(z) \) dominates \( \Lambda(z)P(z)x \). This proves that \( x \in F \).

To prove (4.11), take \( z_1 \in \mathbb{C} \) and assume that \( q(z_1) \neq 0 \). Then \( I - z_1T \) is invertible. Define

\[
y := (I - z_1T)^{-1}(z - z_1)Tx = (z - z_1)(I - z_1T)^{-1}x.
\]

From what has been proved in the preceding paragraphs, it follows that \( x \in F \) if and only if \( (z - z_1)T = z_1T \) is invertible. Define

\[
y := (I - z_1T)^{-1}(z - z_1)Tx = (z - z_1)(I - z_1T)^{-1}x.
\]

From what has been proved in the preceding paragraphs, it follows that \( x \in F \) if and only if \( y \in F \). Since \( z_1 \in \mathbb{C} \) with \( q(z_1) \neq 0 \), we have for \( x \in F \) and for \( y \) given by (4.14)

\[
(I - zT)^{-1}x - (I - z_1T)^{-1}x = (I - zT)^{-1}y,
\]

and hence

\[
\Lambda(z)P(z)y = \Lambda(z)P(z)x - q(z)\Lambda(z)(I - z_1T)^{-1}x.
\]

From Lemma 14.7.4 it follows that \( q(z) \) dominates \( \Lambda(z)(I - z_1T)^{-1}x \). So we obtain from (4.15) that \( q(z) \) dominates \( \Lambda(z)P(z)x \) if and only if \( q(z) \) dominates \( \Lambda(z)P(z)y \), and this proves (4.11).

Finally, using (4.12), we prove that \( \mathcal{F} \) is closed. Let \( \{x_k\}_{k \geq 1} \) be a sequence in \( \mathcal{F} \) such that \( x_k \to x \) in \( X \) as \( k \to \infty \). We have to prove that \( x \in \mathcal{F} \). Fix \( y^* \in Y^* \). Define \( f_k : \mathbb{C} \to \mathbb{C} \) and \( f : \mathbb{C} \to \mathbb{C} \) by

\[
f(z; y^*) = \langle y^*, \Lambda(z)P(z)x \rangle \quad \text{and} \quad f_k(z; y^*) = \langle y^*, \Lambda(z)P(z)x_k \rangle, \quad k \geq 1.
\]

Obviously, \( f \) and \( f_k \) (\( k \geq 1 \)), are entire functions of order at most \( \rho \). According to our hypotheses (see (4.12) in particular) there exists a \( \rho \)-admissible set of half-lines in the complex plane, \( \{\text{ray}(\theta_j; z_0, s_0) \mid j = 1, \ldots, \kappa\} \), such that

\[
\|f_k(z)\| \leq M(1 + |z|^m) \quad \text{for} \ z \in \text{ray}(\theta_j; z_0, s_0)
\]

and \( j = 1, 2, \ldots, \kappa \).

The fact that \( M \) does not depend on \( k \) follows from \( x_k \to x \) as \( k \to \infty \) using that \( M = M(x) \) in (4.12) depends continuously on \( x \). Since \( f_k \to f \) pointwise we can apply Lemma 14.7.6 with \( q \) in place of \( g \) to show that \( q \) dominates \( f \). Thus \( q \) dominates the entire function \( z \mapsto \langle y^*, \Lambda(z)P(z)x \rangle \) for each \( y^* \in Y^* \). Hence \( x \in \mathcal{F} \). Thus \( \mathcal{F} \) is closed.
Lemma 4.2.2 Let $X$ be a complex Banach space, and let $T$ be a bounded linear operator on $X$ such that $T$ is determined by the entire functions \{q, P\}. Furthermore, let $Z$ be a Banach space, and let $\Lambda_* : \mathbb{C} \to \mathcal{L}(X^*, Z)$ be an operator-valued polynomial. Assume that

1. $q$ is of finite non-zero order $\rho$ and of completely regular growth,
2. $\Lambda_*(z) P(z)^*$ is of order at most $\rho$.

Put

$$F_* = \{ x^* \in X^* | q(z) \text{ dominates } \Lambda_*(z) P(z)^* x^* \}. \quad (4.16)$$

Then the set $F_*$ is a $T^*$-invariant linear subspace of $X^*$. Moreover, if there exists a $\rho$-admissible set of half-lines in the complex plane,

$$\{ \text{ray } (\theta_j; z_0, s_0) \mid j = 1, \ldots, \kappa \},$$

and a non-negative integer $m$ such that for every $x^* \in X^*$ there is a constant $M = M(x^*)$ with

$$\| \Lambda_*(z) P(z)^* x^* \|_Z \leq M(1 + |z|^m) \quad \text{for } z \in \text{ray } (\theta_j; z_0, s_0)$$

and $j = 1, 2, \ldots, \kappa$, \quad (4.17)

then the set $F_*$ is a weak*-closed linear subspace of $X^*$.

Proof Let $T^*$ denote the Banach adjoint of $T$ on the Banach dual space $X^*$. Then the resolvent operator $(I - zT^*)^{-1}$ admits the representation

$$(I - zT^*)^{-1} = \frac{1}{q(z)} P(z)^*. \quad (4.18)$$

The above representation for $(I - zT^*)^{-1}$ follows directly from the definition of the adjoint of a Banach space operator. Indeed, if $x^*$ is a continuous linear functional on $X$, then for each $x \in X$ we have

$$\langle (I - zT^*)^{-1} x^*, x \rangle = \langle x^*, (I - zT)^{-1} x \rangle = \frac{1}{q(z)} \langle x^*, P(z)x \rangle$$

$$= \frac{1}{q(z)} \langle P(z)^* x^*, x \rangle.$$

Thus we can apply Lemma 4.2.1 with $X := X^*$ and $T := T^*$ to obtain that the set $F_{T^*}$ is a $T^*$-invariant linear subspace of $X^*$. Note that $P(z)^*$ is of order at most $\rho$. Thus we can apply the first part of Lemma 4.2.1 with $X := X^*$ and $T := T^*$, with
$P(z)^*$ in place of $P(z)$, and with $\Lambda_*(z)$ in place of $\Lambda(z)$. This yields that $\mathcal{F}_*$ is a $T^*$-invariant linear subspace of $X^*$.

It remains to show $\mathcal{F}_*$ is weak*-closed. To do this, let $\{x_k^*\}_{k \geq 1}$ be a sequence in $\mathcal{F}_{T^*}$ such that $x_k^* \to x$ in $X$ as $k \to \infty$. We have to prove that $x^*$ belongs to $\mathcal{F}_{T^*}$ as well. For $y^* \in Y^*$ define $f_k : \mathbb{C} \to \mathbb{C}, k \geq 1,$ and $f : \mathbb{C} \to \mathbb{C},$ respectively, by

$$f(z; y^*) = \langle y^*, \Lambda_*(z)P(z)x^* \rangle \quad \text{and} \quad f_k(z; y^*) = \langle y^*, \Lambda_*(z)P(z)x_k^* \rangle, \quad k \geq 1.$$ 

First note that $f_k \to f$ pointwise and that the functions $f$ and $f_k$ are entire functions of order at most $\rho$ which are polynomially bounded on a $\rho$-admissible \{ray $(\theta_j; z_0, s_0) \mid j = 1, \ldots, \kappa$\} by (4.17). Since $\{x_k^*\}_{k \geq 1}$ is a sequence in $\mathcal{F}_{T^*}$, it follows that $q$ dominates $f_k$ for $k \geq 1$. Therefore an application of Lemma 14.7.6 implies that $q$ dominates $f$ for every $y^* \in Y^*$. This completes the proof that $\mathcal{F}_{T^*}$ is weak*-closed.

4.3 Theorem 4.1.3 Reduced to the Case When $z_0$ Is Zero

In this section we show that it suffices to prove Theorem 4.1.3 in case the point $z_0$ is zero. First note that if $z_0$ is zero, then we can rephrase Theorem 4.1.3 as follows.

**Theorem 4.3.1** Let $T$ be an bounded linear operator on the Banach space $X$, and assume that there exists a positive integer $k$ such that

$$X = \overline{\text{Im} T^k \oplus \text{Ker} T^k}. \quad (4.19)$$

Assume that $T$ is determined by the entire functions $\{q, P\}$, where

1. the scalar entire function $q$ is of finite non-zero order $\rho$, has infinite many zeros, and is of completely regular growth,
2. the operator-valued entire function $P(z) : X \to X$ is of order at most $\rho$.

Moreover, let $Y$ be a Banach space, and suppose that there is an operator-valued polynomial $\Lambda : \mathbb{C} \to \mathcal{L}(X, Y)$ such that

$$\Lambda(z)P(z)x = 0 \quad (\text{for all } z \in \mathbb{C}) \quad \implies \quad x = 0. \quad (4.20)$$

Furthermore, suppose that there exist a $\rho$-admissible set of half-lines in the complex plane, $\{\text{ray } (\theta_j; 0, s_0) \mid j = 1, \ldots, \kappa\}$, and a non-negative integer $m$ and a constant $M$ such that for each $z \in \text{ray } (\theta_j; 0, s_0), j = 1, 2, \ldots, \kappa$, we have

$$0 < |q(z)| \leq M(1 + |z|^m) \quad \text{and} \quad \| (I - zT)^{-1} \| \leq M(1 + |z|^m). \quad (4.21)$$
Finally, define
\[ F_{T, \Lambda} := \{ x \in X \mid q(z) \text{ dominates } \Lambda(z) P(z)x \} . \] (4.22)

Then \( F_{T, \Lambda} \) is closed and the closure of the generalised eigenspace of \( T \) has the following properties:
\[ M_T \oplus \ker T^k = F_{T, \Lambda} \quad \text{and} \quad X = \overline{M_T \oplus S_T}, \] (4.23)
\[ F_{T, \Lambda} \cap S_T = \ker T^k, \] (4.24)
\[ F_{T, \Lambda} \cap \text{Im } T^k = \overline{M_T}. \] (4.25)

Here as usual \( S_T = \{ x \in X \mid z \mapsto (I - zT)^{-1}x \text{ is entire} \} \).

Theorem 4.3.1 not only appears as a special case of Theorem 4.1.3. In fact our aim is to show that the converse statement, that is, Theorem 4.1.3 appears as a corollary of Theorem 4.3.1, also holds true. In other words we shall prove the following proposition.

**Proposition 4.3.2** In order to prove Theorem 4.1.3 it suffices to prove the theorem for the case when \( z_0 = 0 \).

It will be convenient first to prove the following auxiliary result.

**Lemma 4.3.3** Let \( T \) be a bounded operator on the Banach space \( X \) determined by the entire functions \( \{ q, P \} \). Assume \( I - z_0T \) is invertible. Define
\[ \tilde{T} = (I - z_0T)^{-1}T, \quad \tilde{q}(z) = q(z + z_0), \quad \tilde{P}(z) = P(z + z_0)(I - z_0T). \]

Then \( \tilde{T} \) is determined by the pair \( \{ \tilde{q}, \tilde{P} \} \). Furthermore, the following holds:
(a) \( \lambda \in \sigma(\tilde{T}) \setminus \{0\} \) if and only if
\[ 1 + \lambda z_0 \neq 0 \quad \text{and} \quad \frac{\lambda}{1 + \lambda z_0} \in \sigma(T) \setminus \{0\}; \] (4.26)
(b) \( \mu \in \sigma(T) \setminus \{0\} \) if and only if
\[ 1 - \mu z_0 \neq 0 \quad \text{and} \quad \frac{\mu}{1 - \mu z_0} \in \sigma(\tilde{T}) \setminus \{0\}. \] (4.27)

Moreover the map
\[ \lambda \mapsto \frac{\lambda}{1 + \lambda z_0}, \quad \lambda \in \sigma(\tilde{T}) \setminus \{0\} \] (4.28)
is well defined and maps $\sigma(\tilde{T}) \setminus \{0\}$ in a one-to-one way onto $\sigma(\tilde{T}) \setminus \{0\}$, and the inverse map is given by

$$
\mu \mapsto \frac{\mu}{1-\mu z_0}, \quad \mu \in \sigma(T) \setminus \{0\}.
$$

(4.29)

Furthermore, if $\lambda \in \sigma(\tilde{T}) \setminus \{0\}$ and $\mu \in \sigma(T) \setminus \{0\}$ and $\lambda = \mu (1-\mu z_0)^{-1}$, then the Riesz projection of $\tilde{T}$ at $\lambda$ and the Riesz projection of $T$ at $\mu$ coincide.

**Proof** The fact that $\tilde{T}$ is determined by the entire functions $\{\tilde{q}, \tilde{P}\}$ follows from the following calculation:

$$
\tilde{q}(z)(I-z\tilde{T}) = q(z+z_0) \left( I - z(I-z_0T)^{-1}T \right)^{-1}
= q(z+z_0) \left( (I-z_0T) - zT \right)^{-1} (I-z_0T)
= q(z+z_0) (I - (z+z_0)T)^{-1} (I-z_0T)
= P(z+z_0)(I-zT_0) = \tilde{P}(z).
$$

Since both $T$ and $\tilde{T}$ are determined by a pair of entire functions, both $\sigma(T) \setminus \{0\}$ and $\sigma(\tilde{T}) \setminus \{0\}$ are countable sets consisting of isolated points only. Next, observe that

$$
\lambda I - \tilde{T} = \lambda I - (I-z_0T)^{-1}T = (I-z_0T)^{-1}(\lambda I - \lambda z_0 T - T)
= (I-z_0T)^{-1}(\lambda I - (1+\lambda z_0)T).
$$

(4.30)

Note that the various operators appearing in (4.30) commute with each other.

**Proof of Item (a)** Let $\lambda \in \sigma(\tilde{T}) \setminus \{0\}$. If $1+\lambda z_0 = 0$, then the identity (4.30) reduces to $\lambda I - \tilde{T} = \lambda(I-I_0T)^{-1}$ which implies that $\lambda I - \tilde{T}$ is invertible. The latter contradicts the fact that $\lambda \in \sigma(\tilde{T})$. Thus $1+\lambda z_0 \neq 0$. But then (4.30) can be rewritten as

$$
\lambda I - \tilde{T} = (1+\lambda z_0)(I-z_0T)^{-1}\left(\frac{\lambda}{1+\lambda z_0}I-T \right).
$$

(4.31)

The left hand side of the above identity is not invertible by assumption. This can only happen if the operator $\lambda(1+\lambda z_0)^{-1}I-T$ in the right hand side of (4.31) is not invertible too. Hence we have proved the two statements in (4.26).

Next, we consider the reverse implication. So assume that the two parts of (4.26) hold true. First, note that the second part of (4.26) implies that $\lambda$ is not zero. Given the first part of (4.26), we can rewrite (4.30) as (4.31). In the present case we know that the operator $\lambda(1+\lambda z_0)^{-1}I-T$ in the right hand side of (4.31) is not invertible. This implies that $\lambda - \tilde{T}$ is also not invertible. Thus $\lambda \in \sigma(\tilde{T})$ as desired.
4.3 Theorem 4.1.3 Reduced to the Case When \( z_0 \) Is Zero

**Proof of Item (b)** Let \( \mu \in \sigma(T) \setminus \{0\}. \) If \( 1 - \mu z_0 = 0. \) Then \( \mu(I - z_0 T) = \mu I - T. \)

In that case, since \( I - z_0 T \) is invertible, \( \mu \neq 0 \) implies that \( \mu I - T \) is invertible. However, the latter contradicts the fact that \( \mu \in \sigma(T). \) Thus \( 1 - \mu z_0 \neq 0. \) Put

\[ \lambda := \frac{\mu}{1 - \mu z_0}. \] (4.32)

Since \( \mu \neq 0, \) the same holds true for \( \lambda. \) From the definition of \( \lambda \) in (4.32) it also follows that

\[ \lambda(1 - \mu z_0) = \mu \text{ and hence } \lambda = \mu(1 + \lambda z_0). \] (4.33)

As \( \lambda \neq 0, \) we also have \( (1 + \lambda z_0) \neq 0. \) Using (4.30) we see that

\[ \lambda I - \tilde{T} = (1 + \lambda z_0)(I - z_0 T)^{-1}\left(\frac{\lambda}{1 + \lambda z_0}I - T\right). \] (4.34)

According to the second part of (4.33) we have \( \lambda(1 + \lambda z_0)^{-1} = \mu. \) Moreover, \( \mu I - T \) is not invertible by assumption. Thus (4.34) shows that \( \lambda I - \tilde{T} \) is not invertible too.

We conclude that \( \lambda \in \sigma(\tilde{T}) \setminus \{0\}, \) and (4.27) is proved.

We continue with reverse implication. Since \( 1 - \mu z_0 \neq 0, \) we can define

\[ \lambda := \frac{\mu}{1 - \mu z_0}. \] (4.35)

According to the second part of (4.27), we have \( \lambda \neq 0. \) But then also \( \mu \neq 0 \) because of (4.35). From (4.35) it follows that \( \lambda - \lambda \mu z_0 = \mu, \) and hence \( \lambda = \mu(1 + \lambda z_0). \)

Since, \( \lambda \neq 0, \) we conclude that both \( \mu \) and \( 1 + \lambda z_0 \) are non-zero. Furthermore, from (4.30) we see that

\[ \lambda I - \tilde{T} = (1 + \lambda z_0)(I - z_0 T)^{-1}\left(\frac{\lambda}{1 + \lambda z_0}I - T\right) = (1 + \lambda z_0)(I - z_0 T)^{-1}(\mu I - T). \] (4.36)

From the second part of (4.27) and using (4.35), we know that \( \lambda I - \tilde{T} \) is not invertible, because of (4.36), and hence the same is true for \( \mu I - T. \) We proved that \( \mu \in \sigma(T) \setminus \{0\}, \) as desired.

**The Remaining Part of the Proof** We begin with an elementary observation. Let \( \lambda \) and \( \mu \) be complex numbers, and assume that both \( 1 + \lambda z_0 \) and \( 1 - \mu z_0 \) are non-zero. Then

\[ \mu = \frac{\lambda}{1 + \lambda z_0} \iff \lambda = \frac{\mu}{1 - \mu z_0}. \] (4.37)
Given (4.37) and items (a) and (b), it is straightforward to prove the one-to-one and onto property of the maps in (4.28) and (4.29).

Next, we prove the statement about the Riesz projections. Let $\lambda \in \sigma(\tilde{T}) \setminus \{0\}$ and $\mu \in \sigma(T) \setminus \{0\}$, and assume that $\lambda = \mu(1 - \mu z_0)^{-1}$. By $\tilde{P}$ we denote the Riesz projection corresponding to $\tilde{T}$ and $\lambda_0$, and $P$ denotes the Riesz projections corresponding to $T$ at $\mu_0$. We shall prove that $\tilde{P} = P$. In order to do this, let $\tilde{\Gamma}$ be a circle with centre $\lambda_0$, and let $\tilde{\Omega}$ denote the corresponding open disc. We choose the radius $\tilde{r}$ of the circle $\tilde{\Gamma}$ in such a way that

(i) there are no other spectral points of $\tilde{T}$ in $\tilde{\Omega} \cup \tilde{\Gamma}$;
(ii) $1 + \lambda z_0$ is non-zero for all $\lambda \in \tilde{\Omega} \cup \tilde{\Gamma}$.

From item (i) it follows that

$$
\tilde{P} = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\lambda I - \tilde{T})^{-1} d\lambda.
$$

Since $1 - \mu_0z_0$ is invertible, the same holds true for $1 - \mu z_0$ provided the distance $|\mu - \mu_0|$ is sufficiently small. This leads to a further restriction on the radius $\tilde{r}$ of the circle $\tilde{\Gamma}$. Put

$$
\Gamma = \{ \mu \mid \mu = \frac{\lambda}{1 + \lambda z_0}, \ \lambda \in \tilde{\Gamma} \},
$$
$$
\Omega = \{ \mu \mid \mu = \frac{\lambda}{1 + \lambda z_0}, \ \lambda \in \tilde{\Omega} \}.
$$

Now we put an additional constraint on the radius $\tilde{r}$ of $\tilde{\Gamma}$, namely in such a way that the following additional properties are satisfied:

(iii) there are no other spectral points of $T$ in $\Omega \cup \Gamma$;
(iv) $1 - \mu z_0$ is non-zero for all $\mu \in \Omega \cup \Gamma$.

Next note that

$$
1 + \lambda z_0 = 1 + \left( \frac{\mu}{1 - \mu z_0} \right) z_0 = \frac{1 - \mu z_0 + \mu z_0}{1 - \mu z_0} = \frac{1}{1 - \mu z_0}.
$$

Thus $1 + \lambda z_0 = (1 - \mu z_0)^{-1}$. Using the latter identity in (4.34) we see that

$$
(\lambda I - \tilde{T})^{-1} = (1 - \mu z_0)(I - z_0 T)(\mu - T)^{-1}.
$$

Next, using $\lambda = \mu(1 - \mu z_0)^{-1}$, we obtain that

$$
d \frac{\lambda}{d \mu} = \frac{(1 - \mu z_0) - \mu(-z_0)}{(1 - \mu z_0)^2} = \frac{1}{(1 - \mu z_0)^2}.
$$
It follows that
\[
\tilde{P} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \tilde{T})^{-1} d\lambda = (I - z_0 T)(\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(1 - \mu z_0)(\mu I - T)^{-1}} d\mu)
\]
\[= (I - z_0 T)(\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(1 - \mu z_0)(\mu I - T)^{-1}} d\mu)
\]
\[= (I - z_0 T)(I - z_0 T)^{-1} P = P.
\]
Here the one but last equality results from the spectral calculation theory (see, e.g. [32].) We conclude that \( \tilde{P} = P \), and the proof is complete. \( \Box \)

**Proof of Proposition 4.3.2** Let \( T \) satisfy all conditions in Theorem 4.1.3. In particular, we know that \( I - z_0 T \) is invertible. Define
\[
\tilde{T} = (I - z_0 T)^{-1} T, \quad \tilde{q}(z) = q(z + z_0), \quad \tilde{P}(z) = P(z + z_0)(I - z_0 T).
\]
The invertibility of \( I - z_0 T \) implies that condition (4.2) remains true if \( T \) is replace by \( \tilde{T} \). In fact, we have
\[
\text{Ker} \tilde{T}^j = \text{Ker} T^j \quad \text{and} \quad \text{Im} \tilde{T}^j = \text{Im} T^j \quad (j = 0, 1, 2, \ldots).
\]
(4.39)

It follows that (4.19) holds true if and only if (4.2) is satisfied.

From the preceding lemma we know that \( \tilde{T} \) is determined by the entire functions \( \{\tilde{q}, \tilde{P}\} \). If \( f \) is an entire function, then \( \tilde{f}(z) = f(z + z_0) \) is also entire. Moreover, \( f \) and \( \tilde{f} \) are of the same order (see Lemma 14.1.4) and have the same number of zeros. It follows that \( \tilde{q} \) is of finite non-zero order \( \rho \) and as for \( q(z) \) the function \( \tilde{q}(z) \) has infinitely many zeros. Also, as for \( P \), the entire function \( \tilde{P} \) is of order at most \( \rho \).

We also know (see the sentence after (4.4)) that \( \tilde{q} \) is of completely regular growth. Summarising we have

(1) \( \tilde{q} \) is of finite non-zero order \( \rho \), has infinitely many zeros, and is of completely regular growth,

(2) \( \tilde{P} \) is of order at most \( \rho \).

Next, define \( \tilde{\Lambda}(z) = \Lambda(z + z_0) \). Then \( \tilde{\Lambda} \) is an operator polynomial, \( \tilde{\Lambda} : \mathbb{C} \to \mathcal{L}(X, Y) \). Moreover,
\[
\tilde{\Lambda}(z)\tilde{P}(z)x = 0 \quad (\text{for all } \in \mathbb{C}) \implies
\]
\[\implies \Lambda(z + z_0)P(z + z_0)(I - z_0 T)x = 0 \quad (\text{for all } \in \mathbb{C})
\]
\[\implies \Lambda(z)P(z)(I - z_0 T)x = 0 \quad (\text{for all } \in \mathbb{C})
\]
\[\implies (I - z_0 T)x = 0 \implies x = 0.
\]
Hence (4.3) holds with $\tilde{\Lambda}$ and $\tilde{P}$ in place of $\Lambda$ and $P$, respectively. Next we consider the $\rho$-admissible set of half-lines

$$\{\text{ray } (\theta_j; 0, s_0) \mid j = 1, \ldots, \kappa\}. $$

From the first inequality in (4.4) we know that there exist a non-negative integer $m$ and a constant $M$ such that for each $z \in \text{ray } (\theta_j; 0, s_0)$, $j = 1, 2, \ldots, \kappa$, we have

$$0 < q(z) \leq M \left(1 + |z|^m\right), \quad (4.40)$$

$$\| (I - (z + z_0)^{-1} T)^{-1} \| \leq M \left(1 + |z + z_0|^m\right). \quad (4.41)$$

A straightforward calculation (considering two cases, $|z| \leq 1$ and $|z| > 1$) we see that there exists a constant $c > 0$ such that

$$1 + |z + z_0|^m \leq c \left(1 + |z|^m\right), \quad \text{for all } z \in \mathbb{C}. $$

Given the latter inequality, we obtain

$$0 < \tilde{q}(z) \leq M c \left(1 + |z|^m\right),$$

$$\| (I - z\tilde{T})^{-1} \| \leq M c \| I - z_0 T \| \left(1 + |z|^m\right).$$

Thus there exists a constant $\tilde{M}$ such that for each $z \in \text{ray } (\theta_j; 0, s_0)$, $j = 1, 2, \ldots, \kappa$, we have

$$0 < \tilde{q}(z) \leq \tilde{M} \left(1 + |z|^m\right) \quad \text{and} \quad \| (I - z\tilde{T})^{-1} \| \leq \tilde{M} \left(1 + |z|^m\right).$$

Hence (4.4) holds with $\tilde{q}$ in place of $q$, with $\tilde{T}$ in place of $T$, and with $\tilde{M}$ in place of $M$.

Next we consider the space

$$\mathcal{F}_{\tilde{T},\tilde{\Lambda},0} = \{x \in X \mid \tilde{q}(z) \text{ dominates } \tilde{\Lambda}(z)\tilde{P}(z)x\}. $$

We claim that $\mathcal{F}_{\tilde{T},\tilde{\Lambda},0} = \mathcal{F}_{T,\Lambda,z_0}$. To prove this note that (using the definitions of $\tilde{q}$, $\tilde{\Lambda}$, and $\tilde{P}$) we have

$$\mathcal{F}_{\tilde{T},\tilde{\Lambda},0} = \{x \in X \mid q(z + z_0) \text{ dominates } \Lambda(z + z_0)P(z + z_0)(I - z_0 T)x\}. $$
4.4 Proof of Theorem 4.1.3

Since \( P(z) = q(z)(I - zT)^{-1} \), it follows that for each \( z \in \mathbb{C} \) the operator \( P(z) \) commutes with the operator \( I - z_0T \). Thus

\[
F_{\tilde{T},\Lambda,z_0} = \{ x \in X \mid q(z + z_0) \text{ dominates } (I - z_0T)\Lambda(z + z_0)P(z + z_0)x \}
= \{ x \in X \mid q(z + z_0) \text{ dominates } \Lambda(z + z_0)P(z + z_0)x \}
= F_{T,\Lambda,z_0}.
\]

Finally, we prove that \( \overline{\mathcal{M}_T} = \overline{\mathcal{M}_{\tilde{T}}} \) and \( S_{\tilde{T}} = S_T \). The first identity follows from the last part of Lemma 4.3.3 which tells us that \( \mathcal{M}_{\tilde{T}} = \mathcal{M}_T \). To prove the second identity, recall that

\[
(I - z\tilde{T}) = I - z(I - z_0T)^{-1}T = (I - z_0T)^{-1}(I - (z + z_0)T).
\]

Thus

\[
(I - z\tilde{T})^{-1} = (I - (z + z_0)T)(I - z_0T) = (I - z_0T)(I - (z + z_0)T).
\]

It follows that \( (I - z\tilde{T})^{-1}x \) is entire if and only if \( (I - (z + z_0)T)x \) is entire. But the latter is equivalent to \( (I - zT)x \) being entire. Thus \( x \in S_{\tilde{T}} \) if and only if \( x \in S_T \), that is, \( S_{\tilde{T}} = S_T \).

We conclude that \( \tilde{T} \) satisfies all the conditions of Theorem 4.1.3 or, equivalently, \( \tilde{T} \) satisfies all the conditions of Theorem 4.3.1. Thus (4.23), (4.24), and (4.25) hold true. But then the identities proved in the preceding two paragraphs show that the identities (4.6), (4.7), and (4.8) in Theorem 4.1.3 hold true. Thus Theorem 4.1.3 is proved.

\[\Box\]

4.4 Proof of Theorem 4.1.3

From the previous section we know that in order to prove Theorem 4.1.3 it suffices to prove Theorem 4.3.1. Therefore, unless stated otherwise, we assume throughout this section that \( T \) is a bounded linear operator on the Banach space \( X \), and we assume that \( T \) is determined by the entire functions \( \{q, P\} \), with \( q \) and \( P \) having the following properties:

1. the scalar entire function \( q \) is of finite non-zero order \( \rho \), has infinitely many zeros, and is of completely regular growth,
2. the operator-valued entire function \( P(z) : X \to X \) is of order at most \( \rho \).
By $\mathcal{M}_T$ and $\mathcal{M}_{T^*}$ we denote the generalised eigenspace of $T$ and of its Banach adjoint $T^*$, respectively. Furthermore,

$$S_T = \{ x \in X \mid z \mapsto (I - zT)^{-1}x \text{ is entire} \},$$

$$S_{T^*} = \{ x^* \in X^* \mid z \mapsto (I - zT^*)^{-1}x^* \text{ is entire} \}.$$

In order to prove Theorem 4.3.1 we first derive the following intermediate result.

**Proposition 4.4.1** Let $T$ be a bounded linear operator on the Banach space $X$ such that (4.2) is satisfied and $T$ is determined by the entire functions $\{q, P\}$. Let $Y$ be a Banach space, and suppose that there exist operator-valued polynomials $\Lambda_L : \mathbb{C} \rightarrow \mathcal{L}(X, Y)$ and $\Lambda_R : \mathbb{C} \rightarrow \mathcal{L}(Y, X)$ such that $q$ and $P$ have the following properties:

1. The scalar entire function $q$ is of finite non-zero order $\rho$ and is of completely regular growth,
2. The operator-valued entire function $\Lambda_L(z)P(z) : X \rightarrow Y$ is of order at most $\rho$,
3. The operator-valued entire function $\Lambda_R(z^*)P(z^*)^* : X^* \rightarrow Y^*$ is of order at most $\rho$.

Furthermore suppose that

(a) if $\Lambda_L(z)P(z)x = 0$ for all $z \in \mathbb{C}$, then $x = 0$;
(b) if $\Lambda_R(z^*)P(z^*)^*x^* = 0$ for all $z \in \mathbb{C}$, then $x^* = 0$.

Define

$$F_T = \{ x \in X \mid q(z) \text{ dominates } \Lambda_L(z)P(z)x \},$$

$$F_{T^*} = \{ x^* \in X^* \mid q(z) \text{ dominates } \Lambda_R(z^*)P(z^*)^*x^* \}.$$

Finally, assume that

(c) $F_T$ is closed and $F_T \cap S_T = \text{Ker} T^k$.
(d) $F_{T^*}$ is weak* -closed, and $F_{T^*} \cap S_{T^*} = \text{Ker} (T^*)^k$.

Then the closure of the generalised eigenspace of $T$ satisfies the following identity

$$X = \overline{\mathcal{M}_T} \oplus \overline{S_T}. \quad (4.44)$$

Moreover, if $F_T = X$, then $S_T = \text{Ker} T^k$ and $X = \overline{\mathcal{M}_T} \oplus \text{Ker} T^k$.

To prove the above proposition it will be convenient to first derive the following lemmas.

**Lemma 4.4.2** Let $T$ be the operator given in the first paragraph of this section, and let $Y$ be a Banach space, and suppose that there exists an operator-valued
polynomials $\Lambda : \mathbb{C} \rightarrow \mathcal{L}(X, Y)$ such that $\Lambda(z)P(z)x = 0$ for all $z \in \mathbb{C}$ implies that $x = 0$. Then

$$\mathcal{M}_T \subset \mathcal{F}_\Lambda := \{x \in X \mid q(z) \text{ dominates } \Lambda(z)P(z)x\}. \quad (4.45)$$

**Proof** Take $0 \neq \lambda_0 \in \sigma(T)$, and let $P_{\lambda_0}$. Put $X_0 = \text{Im } P_{\lambda_0}$, let $I_0$ be the identity operator on $X_0$, and let $T_0$ be the restriction of $T$ to $X_0$. Note that $T_0 - \lambda I_0$ is nilpotent. It follows that for each $x_0 \in X_0$ we have

$$(I - zT)^{-1}x_0 = (I_0 - zT_0)^{-1}x = \left((1 - z\lambda_0)I_0 - z(T_0 - \lambda I_0)\right)^{-1}x_0$$

$$= \frac{1}{1 - z\lambda_0} \sum_{v=0}^{n_0} \left(\frac{z}{1 - z\lambda_0}\right)^v (T_0 - \lambda I_0)^v x_0.$$  

Here $n_0$ is the order of nilpotency of $T_0 - \lambda I_0$. The above calculation shows that $(x^*, (I - zT)^{-1}x_0)$ is a rational function for any vector $x^* \in X^*$. Since $\Lambda(z)$ is an operator polynomial, it follows that $(x^*, \Lambda(z)(I - zT)^{-1}x_0)$ is a rational function for any vector $x^* \in X^*$ as well.

Now fix $x^* \in X^*$ and $x_0 \in X_0$. Put

$$f_1(z) = \langle x^*, q(z)(I - zT)^{-1}x_0 \rangle = \langle x^*, \Lambda(z)P(z)x_0 \rangle \text{ and } f_2(z) = q(z).$$

Then $f_1$ is an entire function of order $\rho_1 \leq \rho$ and $f_2$ is an entire function of finite non-zero order $\rho$ which is of completely regular growth. Moreover, by the result of the previous paragraph, $f_1/f_2$ is a rational function. But then Lemma 14.7.3 tells us that $q = f_2$ dominates $f_1$. This holds for arbitrary $x^* \in X^*$, and thus $q(z)$ dominates $\Lambda(z)P(z)x_0$, that is, $x_0 \in \mathcal{F}_\Lambda$ defined by (4.45). Since $x_0$ is an arbitrary vector in $X_0$, we see that $\text{Im } P_{x_0} = X_0 \subset \mathcal{F}_\Lambda$. Using $\mathcal{M}_T = \bigoplus_{0 \neq \lambda \in \sigma(T)} \text{Im } P_{\lambda}$, we conclude that $\mathcal{M}_T \subset \mathcal{F}_\Lambda$. \hfill \square

**Corollary 4.4.3** Let $T$ be the operator given in the first paragraph of this section, and let $\Lambda_L : \mathbb{C} \rightarrow \mathcal{L}(X, Y)$ and $\Lambda_R : \mathbb{C} \rightarrow \mathcal{L}(Y, X)$ be operator polynomials such that conditions (a) and (b) in Proposition 4.4.1 are satisfied. Furthermore, let $\mathcal{F}_T$ and $\mathcal{F}_T^*$ be defined by (4.42) and (4.43), respectively. Then

$$\overline{\mathcal{M}_T} \subset \mathcal{F}_T \quad \text{and} \quad \overline{\mathcal{M}_T^*} \subset \mathcal{F}_T^*. \quad (4.46)$$

**Proof of Proposition 4.4.1** We split the proof into two parts.

**Part 1.** In this part we prove the identity (4.44). From Corollary 4.4.3 above we know that $\overline{\mathcal{M}_T} \subset \mathcal{F}_T$. But then we can use the second part of assumption (c) to show that $\overline{\mathcal{M}_T} \cap S_T \subset \text{Ker } T^k$. On the other hand, according to the first part of (1.47), we have $\overline{\mathcal{M}_T} \cap \text{Ker } T^k = \{0\}$. It follows that $\overline{\mathcal{M}_T} \cap S_T = \{0\}$. Thus in order to prove the identity (4.44) it remains to show that the annihilator of $\overline{\mathcal{M}_T} \oplus S_T$ consists of the zero vector only.
But

\[(\overline{\mathcal{M}_T \oplus S_T})^\perp = (\mathcal{M}_T \oplus S_T)^\perp = \mathcal{M}_T^\perp \cap S_T^\perp).\]

Thus we have to show that \(\mathcal{M}_T^\perp \cap S_T^\perp = \{0\}\). Now observe that \(\mathcal{M}_T^\perp = S_T^*\) by (1.55) and \(S_T^\perp = \overline{\mathcal{M}_T^*}\) by (1.56). Here \(\sim\) stands for weak*-closure. By the second inclusion in (4.46), and using the fact that \(\mathcal{F}_T^*\) is weak*-closed by the first part of assumption (d), we see that

\[\mathcal{M}_T^\perp \cap S_T^\perp = S_T^* \cap \overline{\mathcal{M}_T^*} \subset S_T^* \cap \mathcal{F}_T^* = (\text{Ker} T^*)^k,\]

where the last identity follows from the second part of assumption (d). We conclude that

\[S_T^* \cap \overline{\mathcal{M}_T^*} \subset (\text{Ker} T^*)^k.\]

Since (4.2) is satisfied, the second identity in (1.47) tells us that the linear space \(\overline{\mathcal{M}_T^*} \cap (\text{Ker} T^*)^k\) consists of the zero element only, and hence \(S_T^* \cap \overline{\mathcal{M}_T^*} = \{0\}\), as desired.

**Part 2.** In this part we assume \(\mathcal{F}_T = X\), and we prove the final statement of the proposition. Since \(\mathcal{F}_T = X\), we can use the second part of assumption (c), to show that

\[S_T = X \cap S_T = \mathcal{F}_T \cap S_T = \text{Ker} T^k.\]

Thus \(S_T = \text{Ker} T^k\), and (4.44) reduces to \(X = \overline{\mathcal{M}_T \oplus \text{Ker} T}\). This completes the proof. \(\square\)

As a next step towards the proof of Theorem 4.1.3 we derive the following additional lemma.

**Lemma 4.4.4** Let \(T\) be the operator given in the first paragraph of this section, and assume condition (4.2) is satisfied. Furthermore, let \(Y\) be a Banach space, and suppose that there is an operator-valued polynomial \(\Lambda : \mathbb{C} \rightarrow \mathcal{L}(X, Y)\) such that

\[\Lambda(z)P(z)x = 0 \quad (\text{for all } z \in \mathbb{C}) \implies x = 0. \quad (4.47)\]

Put

\[X_0 := \{x \in X \mid q(z) \text{ dominates } \Lambda(z)P(z)x\}.\]
and assume that $X_0$ is a norm-closed subspace of $X$. Furthermore, put $T_0 = T|_{X_0}$ and $P_0(z) = P(z)|_{X_0}$. Then $T_0$ is a bounded linear operator on the Banach space $X_0$.

\[ \text{Im} T_0^k \oplus \text{Ker} T_0^k = X_0. \quad (4.48) \]

Moreover, $P_0(z)$ is an entire function, its values are operators on $X_0$, and

\[ (I_0 - zT)^{-1} = \frac{1}{q(z)}P_0(z), \quad q(z) \neq 0. \quad (4.49) \]

Here $I_0$ denotes the identity operator on $X_0$. Finally, $S_{T_0} = S_T \cap X_0$.

**Proof** From Lemma 4.2.1 we know that $X_0$ is $T$ invariant, and $x \in X_0$ if and only if $Tx \in X_0$. Moreover, for $z \in \mathbb{C}$ we have

\[ q(z) \neq 0 \implies (I - zT)^{-1}X_0 \subset X_0. \quad (4.50) \]

These properties imply that $T_0$ is a well defined bounded linear operator on the Banach space $X_0$. Moreover, we have

\[ (I_0 - zT_0)^{-1} = (I - zT)^{-1}|_{X_0}. \quad (4.51) \]

Indeed, if $x \in X_0$ and $(I - zT)^{-1}x = y$, then it follows from (4.50) that $y \in X_0$. Therefore

\[ x = y - zTy = y - zT_0y = (I_0 - zT_0)y, \]

which proves (4.51).

Now assume that $q(z) \neq 0$. Then $I - zT$ is invertible, and we have $P(z)x_0 = q(z)(I - zT)^{-1}x_0 \in X_0$. Thus $P(z)$ maps $X_0$ into $X_0$, and $P_0(z) = P(z)|_{X_0}$ is a well defined entire function whose values are bounded linear operators on $X_0$. From (4.51) we obtain (4.49). Hence $T_0$ is determined by the pair of entire functions \{q, P_0\}.

From (4.46) we know that $\overline{M_T} \subset X_0$. Since $x \in \text{Ker} T^n$ implies that the function $(I - zT)^{-1}x$ is a polynomial, we also have $\text{Ker} T^n \subset X_0$ for each positive integer $n$. It follows that $M_{T_0} = M_T$ and $\text{Ker} T_0^k = \text{Ker} T^k$. Furthermore, we have

\[ \text{Im} T_0^k = \text{Im} T^k \cap X_0. \quad (4.52) \]

To see this, recall that $\text{Im} T_0^k \subset \text{Im} T^k$ and $\text{Im} T_0^k \subset X_0$. From these inclusions we conclude that $\text{Im} T_0^k \subset \text{Im} T^k \cap X_0$. On the other hand, if $y \in \text{Im} T^k \cap X_0$, then $y = T^k v \in X_0$ and, by induction, it follows from Lemma 4.2.1 that $v \in X_0$. Hence
$T^k v = T_0^k v \in \text{Im } T_0^k$. Therefore $y \in \text{Im } T_0^k \cap X_0$. This proves (4.52). Since $X_0$ is closed, we also have

$$\overline{\text{Im } T_0^k} = \overline{\text{Im } T^k \cap X_0}.$$ 

Next we prove (4.48). Recall that $X_0^* \simeq X^*/X_0^\perp$, and let us compute the annihilator:

$$\left( \overline{\text{Im } T_0^k} \oplus \text{Ker } T_0^k \right)^\perp = \left( \overline{\text{Im } T_0^k} \right)^\perp \cap (\text{Ker } T_0^k)^\perp = \left( (\text{Im } T_0^k)^\perp \oplus X_0^\perp \right) \cap (\text{Ker } T_0^k)^\perp = X_0^\perp.$$ (4.53)

To prove the final equality in (4.53), recall that $\text{Ker } T_0^k \subset X_0$. Hence $X_0^\perp \subset (\text{Ker } T_0^k)^\perp$, and we see that

$$X_0^\perp \subset \left( (\text{Im } T_0^k)^\perp \oplus X_0^\perp \right) \cap (\text{Ker } T_0^k)^\perp.$$ (4.54)

On the other hand, if $y \in \left( (\text{Im } T_0^k)^\perp \oplus X_0^\perp \right) \cap (\text{Ker } T_0^k)^\perp$, then $y = x + e$, where $x \in (\text{Im } T_0^k)^\perp$ and $e \in X_0^\perp$. But $X_0^\perp \subset (\text{Ker } T_0^k)^\perp$. So $x = y - e \in (\text{Ker } T_0^k)^\perp$. Therefore, $x \in (\text{Im } T_0^k)^\perp \cap (\text{Ker } T_0^k)^\perp$. By (4.2) the latter space consists of the zero vector only. Thus $x = 0$ and $y = e \in X_0^\perp$, and we proved that

$$\left( (\text{Im } T_0^k)^\perp \oplus X_0^\perp \right) \cap (\text{Ker } T_0^k)^\perp \subset X_0^\perp.$$ (4.55)

Together the two inclusions (4.54) and (4.55) yield (4.53). From (4.53) it follows that $\left( \overline{\text{Im } T_0^k} \oplus \text{Ker } T_0^k \right)^\perp = \{0\}$ in $X_0^*$ which proves (4.48). The final statement is a straightforward corollary of (4.51).

\vspace{1em}

**Proof of Theorem 4.1.3** Throughout the proof we assume that $z_0 = 0$, which we may do without loss of generality by Proposition 4.3.2. Furthermore, we use the following notation:

$$\mathcal{F}_T := \{ x \in X \mid q(z) \text{ dominates } \Lambda(z) P(z) x \},$$ \hspace{1em} (4.56)

$$\mathcal{F}_{T^*} := \{ x^* \in X^* \mid q(z) \text{ dominates } P(z)^* x^* \}.$$ \hspace{1em} (4.57)

From Lemma 4.2.1 we know that $\mathcal{F}_T$ is a closed linear subspace of $X$, and by Lemma 4.2.2 the set $\mathcal{F}_{T^*}$ is a weak*-closed linear subspace of $X^*$. We split the proof into three parts. The first two parts concern the inclusions

$$\mathcal{F}_T \cap \mathcal{S}_T = \text{Ker } T_0^k \quad \text{and} \quad \mathcal{F}_{T^*} \cap \mathcal{S}_{T^*} = \text{Ker } (T^*)_0^k,$$ \hspace{1em} (4.58)
which appear in conditions (c) and (d) in Proposition 4.4.1. In the third part we employ Lemma 4.4.4 to finish the proof.

**Part 1.** In this part we prove that \( \mathcal{F}_T \cap S_T = \text{Ker} \ T_k \). Assume \( x \in \text{Ker} \ T_k \). In that case \( (I - zT)^{-1} x \) is a polynomial in \( z \), and hence \( x \in S_T \). Next, note that

\[
\Lambda(z) P(z)x = q(z) \Lambda(z)(I - zT)^{-1}x.
\]

Since both \( (I - zT)^{-1} x \) and \( \Lambda(z) \) are polynomials, it follows that \( q(z) \) dominates \( \Lambda(z) P(z)x \), and hence \( x \in \mathcal{F}_T \). Thus \( \text{Ker} \ T_k \) is a subset of \( \mathcal{F}_T \cap S_T \).

To prove the reverse inclusion, suppose that \( x \in \mathcal{F}_T \cap S_T \). Since \( x \in S_T \), it follows that for arbitrary \( y^* \in Y^* \), the function

\[
\tilde{f}(z; y^*) := \langle y^*, \Lambda(z)(I - zT)^{-1}x \rangle
\]

is an entire function. Put

\[
f_1(z; y^*) = \langle y^*, \Lambda(z)P(z)x \rangle \quad \text{and} \quad f_2(z) = q(z).
\]

Then \( f_1 \) and \( f_2 \) are also entire functions, \( f_1 \) is of order at most \( \rho \), and \( \tilde{f} = f_1/f_2 \). Since \( x \in \mathcal{F}_T \), we know from (4.56) that \( q(z) \) dominates \( \Lambda(z) P(z)x \), and hence \( f_2 \) dominates \( f_1 \). By assumption, there exists a \( \rho \)-admissible set of half-lines \( \{ \text{ray} (\theta_j; 0, s_0) \mid j = 1, \ldots, \kappa \} \) such that the two inequalities in (4.4) of Theorem 4.1.3 are satisfied. Recall that \( \Lambda(z) \) is a polynomial and \( P(z) = q(z)(I - zT)^{-1} \). It follows (see Remark 4.1.6) that the two inequalities in (4.4) imply that the entire function \( f_1(z, y^*) \) is also polynomially bounded on the half-lines \( \text{ray} (\theta_j; 0, s_0), j = 1, \ldots, \kappa \).

Furthermore, since \( f_1 \) is entire, there also exists a constant \( M_1 = M_1(y^*) \) such that

\[
|f_1(z; y^*)| \leq M_1 \quad \text{for } |z| \leq s_0.
\]

Summarising we conclude that there exist a non-negative integer \( m \) and a constant \( M = M(y^*) \) such that

\[
|f_1(z; y^*)| \leq M(1 + |z|^m) \quad \text{for } z \in \text{ray} (\theta_j), \quad j = 1, 2, \ldots, \kappa.
\]

But then we can apply Lemma 14.7.5 with \( g = f_2 \), with \( f = f_1 \), and with \( p = \tilde{f} \). Since \( \tilde{f} = f_1/f_2 \) is entire we see that Lemma 14.7.5 tells us that the following two statements are true.

(a) If the order of \( f_1 \) is strictly less that \( \rho \), then \( f_2 \) has finitely many zeros.

(b) If the order of \( f_1 \) equals \( \rho \), then \( \tilde{f} = f_1/f_2 \) is a polynomial of degree at most \( m \).
Since by assumption $f_2 = q$ has infinitely many zeros, the order of $f_1$ cannot be strictly less than $\rho$. But then item (b) tells us that $\tilde{f}$ is a polynomial of degree at most $m$.

Next, let $x_{m+1} = T^{m+1}x$, and define

$$\tilde{g}(z; y^*) = \langle y^*, \Lambda(z)(I - zT)^{-1}x_{m+1} \rangle.$$ 

Since $x \in \mathcal{F}_T \cap S_T$ and the two spaces $\mathcal{F}_T$ and $S_T$ are $T$-invariant, we have that $x_{m+1} \in \mathcal{F}_T \cap S_T$, and it follows that $\tilde{g}$ is an entire function as well. From the identity

$$(I - zT)^{-1} T^k x = z^{-k} (I - zT)^{-1} x - \sum_{j=0}^{k-1} z^{-(k-j)} T^j x, \quad k \geq 1,$$

with $k = m + 1$, we find

$$\tilde{g}(z; y^*) = z^{-(m+1)} \langle y^*, \Lambda_L(z + z_0)(I - (z + z_0)T)^{-1}x \rangle + O(z^{-1})$$

$$= z^{-(m+1)} \tilde{f}(z; y^*) + O(z^{-1}).$$

This implies that the entire function $\tilde{g}$ tends to zero as $|z| \to \infty$, and thus $\tilde{g} = 0$. It follows that

$$\langle y^*, \Lambda(z) P(z)x_{m+1} \rangle = q(z) \langle y^*, \Lambda(z)(I - zT)^{-1}x_{m+1} \rangle = 0.$$ 

Since $y^* \in Y^*$ is arbitrary and $\Lambda(z)$ satisfies (4.3) in Theorem 4.1.3, we conclude that $x_{m+1} = 0$. Recall $x_{m+1} = T^{m+1}x$. So $T^{m+1}x = 0$ and $x \in \text{Ker } T^{m+1}$, but $\text{Ker } T^k = \text{Ker } T^n, n = k + 1, k + 2, \ldots$, by the first identity in (4.9). Hence $x \in \text{Ker } T^k$. This proves that $\mathcal{F}_T \cap S_T = \text{Ker } T^k$.

In Part 2 we prove that $\mathcal{F}_{T^*} \cap S_{T^*} = \text{Ker } (T^*^k)$. To do this we argue similarly as in Part 1. First we show that $\text{Ker } (T^*^k)$ is a subset of $\mathcal{F}_{T^*} \cap S_{T^*}$. Take $x^* \in \text{Ker } (T^*^k)$. Then $(I - zT^*)^{-1} x^*$ is a polynomial in $z$, and hence $x^* \in S_{T^*}$. Next, note that

$$P(z)^* x^* = q(z)(I - zT^*)^{-1} x^*.$$ 

Since $(I - zT^*)^{-1} x^*$ is a polynomial, it follows (use Lemma 14.7.4) that $q(z)$ dominates $P(z)^* x^*$, and hence $x^* \in \mathcal{F}_{T^*}$. Thus $\text{Ker } (T^*^k)$ is a subset of the space $\mathcal{F}_{T^*} \cap S_{T^*}$.

To prove the reverse inclusion suppose that $x^* \in \mathcal{F}_{T^*} \cap S_{T^*}$. The fact that $x^* \in S_{T^*}$ implies that for arbitrary $u \in X$, the function

$$\tilde{f}(z; u) := \langle x^*, (I - zT)^{-1} u \rangle = \langle (I - zT^*)^{-1} x^*, u \rangle$$
is an entire function. Define, as before

\[ f_1(z; u) = \langle x^*, P(z)u \rangle \quad \text{and} \quad f_2(z) = q(z). \]

Note that \( f_1 \) is an entire function of order at most \( \rho \), and \( \tilde{f} = f_1/f_2 \). By assumption, there exists a \( \rho \)-admissible set of half-lines \( \{ \text{ray} (\theta_j; 0, s_0) \mid j = 1, \ldots, \kappa \} \) such that the inequalities in (4.4) of Theorem 4.1.3 are satisfied. Since \( f_1 \) is entire, there exists a constant \( M_1 = M_1(u) \) such that

\[ |f_1(z; u)| \leq M_1 \quad \text{for} \ |z| \leq s_0. \]

Together with the estimate for \( (I - zT)^{-1} \) in (4.4) we see that there exist a non-negative integer \( m \) and a constant \( M \) such that

\[ |f_1(z; u)| \leq M(1 + |z|^m) \quad \text{for} \ z \in \text{ray} (\theta_j), \ j = 1, 2, \ldots, \kappa. \]

Since \( x^* \) also belongs to \( F_{T^*} \), the function \( q(z) \) dominates \( P(z)x^* \). But then \( f_2 \) dominates \( f_1 \), and we can apply Lemma 14.7.5 with \( g = f_2 \), \( f = f_1 \), and with \( p = \tilde{f} \). Since \( \tilde{f} = f_1/f_2 \) is entire, we see that Lemma 14.7.5 tells us that the following two statements are true.

(a) If the order of \( f_1 \) is strictly less than \( \rho \), then \( f_2 \) has finitely many zeros.

(b) If the order of \( f_1 \) equals \( \rho \), then \( \tilde{f} = f_1/f_2 \) is a polynomial of degree at most \( m \).

Since by assumption \( f_2 = q \) has infinitely many zeros, the order of \( f_1 \) cannot be strictly less than \( \rho \). But then item (b) tells us that \( \tilde{f} \) is a polynomial of degree at most \( m \). Put \( x^*_{m+1} = (T^*)^{m+1}x^* \) and define

\[ \tilde{g}(z; u) = \langle x^*_{m+1}, (I - zT)^{-1}u \rangle = \langle x^*, (I - zT)^{-1}T^{m+1}u \rangle. \]

Similarly as in Part 1, since \( x^* \in F_{T^*} \cap S_{T^*} \) and \( F_{T^*} \) and \( S_{T^*} \) are \( T^* \)-invariant, we have that \( x^*_{m+1} \in F_{T^*} \cap S_{T^*} \), and it follows that \( \tilde{g} \) is an entire function as well. From the identity (4.59) with \( k = m + 1 \), we find

\[ \tilde{g}(z; u) = z^{-(m+1)}\langle x^*, (I - zT)^{-1} \rangle + O(z^{-1}) \]

\[ = z^{-(m+1)} f(z; u) + O(z^{-1}). \]

This implies that the entire function \( \tilde{g} \) tends to zero as \( |z| \to \infty \), and thus \( \tilde{g} = 0 \). Since \( u \in X \) is arbitrary, this implies that \( x^*_{m+1} = 0 \). So \( (T^*)^{m+1}x^* = 0 \), but \( \text{Ker}(T^*)^k = \text{Ker}(T^*)^n, \ n = k + 1, k + 2, \ldots \), by the second identity in (4.9), we have \( x^* \in \text{Ker}(T^*)^k \). This shows that \( F_{T^*} \cap S_{T^*} \) is a subset of \( \text{Ker}(T^*)^k \).
Summary of what has been proved so far. The fact that conditions (c) and (d) in Proposition 4.4.1 have been proved, allows us to apply Proposition 4.4.1 with \( \Lambda_L = \Lambda \) and with \( \Lambda_R = I \), the identity operator on \( X \). Thus we may conclude that the closure of the generalised eigenspace of \( T \) satisfies the identity

\[
X = \overline{\mathcal{M}_T} \oplus S_T.
\]

Moreover, if \( \mathcal{F}_T = X \), then \( S_T = \text{Ker} T^k \) and \( X = \overline{\mathcal{M}_T} \oplus \text{Ker} T^k \). In particular, the second identity in (4.6) is proved, and the first identity in (4.6) is also proved provided we have \( \mathcal{F}_T = X \).

Part 3.

In this part we prove the first identity in (4.6) by reducing the problem to the case when \( \mathcal{F}_T = X \) using Lemma 4.4.4. As before we assume \( z_0 = 0 \) and \( \mathcal{F}_T \) is defined by (4.56).

As in Lemma 4.4.4 let

\[
X_0 = \mathcal{F}_T = \mathcal{F}_{T,\Lambda,0} = \{ x \in X \mid q(z) \text{ dominates } \Lambda(z) P(z)x \}.
\]

From the final part of Lemma 4.2.1 we know that \( X_0 = \mathcal{F}_T \) is norm-closed. This allows us to apply Lemma 4.4.4. Put

\[
T_0 = T|_{X_0} : X_0 \to X_0, \quad P_0(z) = P(z)|_{X_0} : X_0 \to X_0, \quad \Lambda_0(z) = \Lambda(z)|_{X_0} : X_0 \to Y.
\]

The operator \( T_0 \), the entire function \( P_0 \) and the operator polynomial satisfy all the conditions in Theorem 4.1.3 with \( T_0 \) in place of \( T \), with \( P_0(z) \) in place of \( P(z) \), and with \( \Lambda_0(z) \) in place of \( \Lambda(z) \). In this case

\[
\mathcal{F}_{T_0} = \{ x \in X_0 \mid q(z) \text{ dominates } \Lambda_0(z) P_0(z)x \} = \mathcal{F}_T = X_0.
\]

From what has been proved so far (see the summary paragraph above) it follows that

\[
X_0 = \overline{\mathcal{M}_{T_0}} \oplus \text{Ker} T^k_0 \quad \text{and} \quad S_{T_0} = \text{Ker} T^k_0.
\]

(4.60)

According to the first inclusion in (4.46) we have \( \overline{\mathcal{M}_T} \subset \mathcal{F}_T \), and hence

\[
\overline{\mathcal{M}_{T_0}} = \overline{\mathcal{M}_T}.
\]

Furthermore, from Part 1 of the proof we know that \( \mathcal{F}_T \cap S_T = \text{Ker} T^k \), i.e., (4.7) is proved. The identity \( \mathcal{F}_T \cap S_T = \text{Ker} T^k \) also implies that
Ker $T^k = \text{Ker} T_0^k$. It follows that the first identity in (4.60) can be rewritten as

$$\mathcal{F}_T = \overline{\mathcal{M}_T} \oplus \text{Ker} T^k,$$

which yields the first identity in (4.6). This completes the proof. \hfill \Box

Theorem 4.1.3 and Corollary 4.1.7 specified for the case when the entire function $q$ belongs to the Paley-Wiener class $\mathcal{PW}$ yields the following result.

**Theorem 4.4.5** Let $T$ be an bounded linear operator on the Banach space $X$, and assume that there exists a positive integer $k$ such that

$$X = \text{Im} T^k \oplus \text{Ker} T^k. \quad (4.61)$$

Assume that $T$ is determined by the entire functions $\{q, P\}$, where

1. the scalar entire function $q$ belongs to the Wiener class $\mathcal{PW}$ and has infinitely many zeros,

2. the operator-valued entire function $P(z) : X \to X$ is of order at most one.

Moreover, let $Y$ be a Banach space, and suppose that there is an operator-valued polynomials $\Lambda : \mathbb{C} \to \mathcal{L}(X, Y)$ such that

$$\Lambda(z)P(z)x = 0 \quad (\text{for all } z \in \mathbb{C}) \quad \implies \quad x = 0. \quad (4.62)$$

Furthermore, let $\ell = i\mathbb{R} + a$, where $a \in \mathbb{R}$, and assume that there exist a non-negative integer $m$ and constant $M$ such that

$$0 < |q(z)| \leq M(1 + |z|^m) \quad \text{and} \quad \|(I - zT)^{-1}\| \leq M(1 + |z|^m) \quad (\text{for all } z \in \ell)$$

Define

$$\mathcal{F}_{T,\Lambda,a} := \{ x \in X \mid q(z) \text{ dominates } \Lambda(z + a)P(z + a)x \}.$$ 

Then $\mathcal{F}_{T,\Lambda,a}$ is closed and the closure of the generalised eigenspace of $T$ has the following properties:

$$\overline{\mathcal{M}_T} \oplus \text{Ker} T^k = \mathcal{F}_{T,\Lambda,a} \quad \text{and} \quad X = \overline{\mathcal{M}_T} \oplus S_T,$$

$$\mathcal{F}_{T,\Lambda,a} \cap S_T = \text{Ker} T^k,$$

$$\mathcal{F}_{T,\Lambda,a} \cap \text{Im} T^k = \mathcal{M}_T.$$

Here as usual $S_T = \{ x \in X \mid z \mapsto (I - zT)^{-1}x \text{ is entire} \}$. In particular, $T$ has a complete span of eigenvectors and generalised eigenvectors if and only if $\mathcal{F}_{T,\Lambda,a} = X$. Moreover, in that case $S_T = \text{Ker} T^k.$
Theorem 4.4.5 can be used to study completeness of the integral operator $T_g$ given by (1.40) on various Banach spaces. For example, we can consider $T_g$ on $C[0, 1]$, the space of all complex-valued continuous functions endowed with the supremum norm. Note that from representation (2.32) for the resolvent operator of $T_g$, it follows that $(I - zT_g)^{-1}$ admits a representation (4.1) with

$$q(z) = 1 - z \int_0^1 e^{zs} g(s) \, ds.$$  

Using the estimates (2.36) and (2.37), we can verify that $T_g$ satisfies all other assumptions of Theorem 4.4.5. This allows us to conclude that Corollary 2.3.2 remains true in the Banach space setting with the Banach space $C[0, 1]$ instead of the Hilbert space $L^2[0, 1]$. For other examples of completeness theorems for operators of the type $T_g$, see Sects. 7.2 and 7.3, in particular, Theorems 7.2.1 and 7.3.2.

The next section provides an additional example of a different character.

4.5 An Additional Example

We shall illustrate Theorem 4.1.3 by studying completeness for integral operators of the form

$$(T_Ax)(t) = x(1) + A \int_0^t x(s) \, ds, \quad 0 \leq t \leq 1, \quad (4.63)$$

where $A$ is a singular $n \times n$ matrix and $T_A$ acts on $X = C([0, 1], \mathbb{C}^n)$, the Banach space of all $\mathbb{C}^n$-valued continuous functions on $[0, 1]$. We shall see that for this class of operators condition (4.2) is satisfied. The first lemma shows that the resolvent of the integral operator $T_A$ given by (4.63) admits a representation of the form (4.1).

**Lemma 4.5.1** Let $T_A$ be the operator given by (4.63). Put

$$q(z) = \det(I - ze^{zA}) \quad \text{and} \quad P(z) = q(z)(I - zT_A)^{-1}. \quad (4.64)$$

Then $I - zT_A$ is invertible if and only if $q(z) \neq 0$, and the function $P(z)$ is entire. In particular, $T_A$ is determined by the entire functions $\{q, P\}$.

**Proof** We first compute the resolvent of $T_A$. The calculation is similar to the one given in Sect. 2.3. Let $x \in C([0, 1], \mathbb{C}^n)$, and let $y \in C([0, 1], \mathbb{C}^n)$ be a solution of the equation $(I - zT_A)y = x$. Then $y$ satisfies the integral equation

$$x(t) = y(t) - z(T_Ay)(t) = y(t) - zy(1) - zA \int_0^t y(s) \, ds. \quad (4.65)$$
Assume for the moment that $x$ is continuously differentiable. Then $y$ is continuously differentiable too, and (4.65) is equivalent to the inhomogeneous differential equation

$$
\dot{y}(t) = zAy(t) + \dot{x}(t)
$$

(4.66)

with boundary condition

$$
y(0) - zy(1) = x(0).
$$

(4.67)

Using the variation-of-constants formula to solve (4.66) and (4.67), we obtain

$$
y(t) = e^{zAt}y(0) + \int_0^t e^{zA(t-s)}\dot{x}(s)\,ds \\
= e^{zAt}y(0) + x(t) - e^{zAt}x(0) + zA\int_0^t e^{zA(t-s)}x(s)\,ds.
$$

(4.68)

Now let us assume that $I - zT_A$ is not invertible. Since $T_A$ is compact, it follows that $I - zT_A$ is not one-to-one. Thus there exists $y \neq 0$ such that $(I - zT_A)y = 0$. But then using (4.68) with $x = 0$ we see that

$$
y(t) = e^{zAt}y(0), \quad 0 \leq t \leq 1.
$$

It follows, using (4.67) with $x = 0$, that

$$
\left(I - ze^{zA}\right)y(0) = y(0) - ze^{zA}y(0) = y(0) - zy(1) = 0.
$$

Thus $q(z) = 0$.

Conversely, let us assume that $q(z) = 0$. Then there exists a vector $y_0 \neq 0$ such that $(I - ze^{zA})y_0 = 0$. Define $y(t) = e^{zAt}y_0$. Then $y \neq 0$, and

$$
\dot{y}(t) = zAy(t) \quad \text{and} \quad y(0) - zy(1) = (I - ze^{zA})y_0 = 0.
$$

Using (4.65) and (4.66), it follows that $(I - zT_A)y = 0$, and hence $I - zT_A$ is not invertible. Summarising, we have proved that $I - zT_A$ is invertible if and only if $q(z) \neq 0$, and hence the pair $\{q, P\}$ is optimal (see Definition 4.1.1).

Now let us assume that $q(z) \neq 0$. Thus the matrix $I - ze^{zA}$ is invertible and

$$
(I - ze^{zA})^{-1} = \frac{1}{q(z)}\text{adj}(I - ze^{zA}),
$$

(4.69)
where $\text{adj } M$ denotes the adjugate of a square matrix $M$ (for the definition of the term adjugate see [46, page 20]). Let us solve the equation $(I - zT_A)y = x$. For $t = 1$ the identity (4.68) yields

$$y(1) = e^{zA}y(0) + x(1) - e^{zA}x(0) + zA \int_0^1 e^{zA(1-s)}x(s)\,ds. \quad (4.70)$$

If we substitute the above expression for $y(1)$ in the boundary condition (4.67) and solve for $y(0)$, we find

$$y(0) = x(0) + z\left(I - ze^{zA}\right)^{-1}\left[x(1) + zA \int_0^1 e^{zA(1-s)}x(s)\,ds\right].$$

Using this formula for $y(0)$ in (4.68) yields

$$y(t) = x(t) + ze^{zAt} \left(I - ze^{zA}\right)^{-1}\left[x(1) + zA \int_0^1 e^{zA(1-s)}x(s)\,ds + A \int_0^t e^{-zAs}x(s)\,ds\right]. \quad (4.71)$$

But then (4.69) shows that $P(z) = q(z)(I - zT_A)^{-1}$ is well defined and analytic in $z$. Thus $P(z)$ is an entire function, as desired. $\square$

In the next lemma condition (4.2) appears in a non-trivial way (with $k = 1$).

**Lemma 4.5.2** If $n = 2$ and $A$ is the diagonal matrix $A = \text{diag} \left[1, 0\right]$, then condition (4.2) in Theorem 4.1.3 is satisfied with $k = 2$.

**Proof** In this case $T_A$ is the block diagonal operator $T_A = \text{diag} \left[T_{A,1}, T_{A,2}\right]$, where $T_{A,1}$ is the integral operator on $C[0, 1]$ given by

$$(T_{A,1}f)(t) = f(1) + \int_0^t f(s)\,ds, \quad 0 \leq t \leq 1, \quad (4.72)$$

and $T_{A,2}$ is the projection on $C[0, 1]$ defined by $(T_{A,2}f)(t) = f(1)$ for each $0 \leq t \leq 1$. We claim that

$$\text{Im } T_A = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mid y_2(\cdot) = y_2(1) \right\}, \quad (4.73)$$

$$\text{Ker } T_A = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1(\cdot) = 0, \ x_2(1) = 0 \right\}. \quad (4.74)$$
Clearly the operator $T_{A,1}$ in (4.72) is one-to-one, and hence $\text{Ker } T_{A,1} = \{0\}$. The null space of $T_{A,2}$ consists of $f$ such that $f(1) = 0$. Together these results yield (4.74). To prove (4.73), put $f_n(t) = t^n$, $n = 0, 1, 2, \ldots$. Note that

$$(T_{A,1} f_n) = 1 + \int_0^t s^n \, ds = 1 + \frac{1}{n+1} t^{n+1} \to 1 \quad (n \to \infty),$$

and the convergence is uniform on $0 \leq t \leq 1$. Since $f_0(t) \equiv 1$, we see that $f_0 \in \text{Im } T_{A,1}$. Next observe that $(n+1)(T_{A,1} f_n - f_0) = f_{n+1}$, which shows that $f_{n+1} \in \text{Im } T_{A,1}$, $n = 0, 1, 2, \ldots$. Hence all polynomials belong to $\text{Im } T_{A,1}$. Thus $\text{Im } T_{A,1} = C[0, 1]$. Since $\text{Im } T_{A,2} = \text{span } \{f_0\}$, we obtain (4.73). From (4.73) and (4.74) we see that for $A = \text{diag } [1, 0]$ the condition (4.2) in Theorem 4.1.3 is satisfied with $k = 2$.

For the special case when $n = 2$ and $A$ is the diagonal matrix $A = \text{diag } [1, 0]$ the functions $q$ and $P$ in (4.64) are given by

$$q(z) = (1 - ze^z)(1 - z) \quad \text{and} \quad P(z) = \text{diag } [P_1(z), P_2(z)],$$

where

$$(P_1(z)x_1)(t) = q(z)x_1(t) + (1 - z)ze^z \left[ x_1(1) + z \int_0^1 e^{z(1-x)} x_1(x) \, dx + \int_0^1 e^{-z \sigma} x_1(\sigma) \, d\sigma \right] + \int_0^t e^{-z \sigma} x_1(\sigma) \, d\sigma,$$

$$(P_2(z)x_2)(t) = q(z)x_2(t) + z(1 - ze^z)x_2(1).$$

Note that $q(z)$ can be rewritten as

$$q(z) = (1 - z) + (z^2 - z)e^z = (1 - z) + (z^2 - z) \int_0^1 e^{z \sigma} \, d\eta_{\circ}(s),$$

where $\eta_{\circ}$ is the function of bounded variation given by $\eta_{\circ}(s) = 0$ for $0 \leq s < 1$ and $\eta_{\circ}(1) = 1$. But then we can apply Corollary 14.3.10 to show that the entire function $q$ belongs to the Paley-Wiener class $\mathcal{PW}$, and hence by Theorem 14.6.3 the function $q$ is of completely regular growth. By direct checking or using Proposition 7.2.4 one sees that $q$ has infinitely many zeros. Note that the function $P(z)$, acting on $C([0, 1], C^2)$, is an entire operator-valued function of order at most one. Furthermore, one checks that for this example the estimates in (4.4) are satisfied with

$$\theta_1 = \pi/2, \quad \theta_2 = 3\pi/2, \quad \text{and} \quad m = 2, \quad z_0 = 0, \quad s_0 = 1.$$
This allows us to apply Theorem 4.1.3 with $\Lambda(z)$ being the function identically equal to the identity operator on $X = C([0, 1], \mathbb{C}^2)$. The latter implies that (4.4) is satisfied trivially. Finally, it turns out that in this case the function $q(z)$ dominates $P(z)x$ for each $x \in C([0, 1], \mathbb{C}^2)$, that is, the space $\mathcal{F}_{T_A, \Lambda, 0}$ is equal to $C([0, 1], \mathbb{C}^2)$. Note that the definition of the space $\mathcal{F}_{T_A, \Lambda, 0}$ is given by (4.5). It follows that Theorem 4.1.3 yields the following corollary.

**Corollary 4.5.3** If $n = 2$ and $A$ is the diagonal matrix $A = \text{diag}[1, 0]$, then the operator $T_A$ has a complete span of eigenvectors and generalised eigenvectors. More precisely,

$$\mathcal{M}_T \oplus \ker T_A = C([0, 1], \mathbb{C}^2).$$

### 4.6 Some Additional Remarks

Note that Theorem 4.1.3 remains true if the estimate in item (ii) is replaced by a uniform estimate on the norm of the resolvent $(I - zT)^{-1}$. Moreover in that case the operator $T$ has a complete span of eigenvectors and generalised eigenvectors if and only if $T^*$ has a complete span of eigenvectors and generalised eigenvectors. More precisely the following theorem holds.

**Theorem 4.6.1** Let $X$ be a complex Banach space, and let $T$ be a bounded linear operator on $X$ satisfying the assumptions of Theorem 4.1.3. If, in addition, $\Lambda(z) = I$ for all $z \in \mathbb{C}$, then the two identities in (4.6) also hold for $T^*$ in place of $T$. In particular, in this case, $T$ has a complete span of eigenvectors and generalised eigenvectors if and only if $T^*$ has a complete span of eigenvectors and generalised eigenvectors.

**Corollary 4.6.2** Let $X$ be a complex Banach space, and let $T$ be a bounded linear operator on $X$ satisfying the assumptions of Theorem 4.1.3. If $X$ is reflexive, then $T$ has a complete span of eigenvectors and generalised eigenvectors if and only if $T^*$ has a complete span of eigenvectors and generalised eigenvectors.

**Corollary 4.6.3** Let $X$ be a complex Banach space, and let $T$ be a bounded linear operator on $X$ satisfying the assumptions of Theorem 4.1.3. Then $\overline{\mathcal{M}_T} = X^*$ implies that $\mathcal{M}_T = X$.

**Proof** From Part 3 of the proof of Theorem 4.1.3 we know that $S_T \cap \overline{\mathcal{M}_T} = \{0\}$. Thus $\overline{\mathcal{M}_T} = X^*$ implies that $S_T = \{0\}$. But then it follows from (1.55) that $\mathcal{M}_T = \{0\}$, and thus $\overline{\mathcal{M}_T} = X$. $\square$
4.7 Theorem 3.4.1 Revisited

We end this section with two “if and only if” versions of Theorem 3.4.1.

**Theorem 4.7.1** Let $H$ be a complex Hilbert space, and let $T$ be a compact operator on $H$ of order one such that $\text{Ker } T = \text{Ker } T^*$. Assume

(a) the function $\| (I - zT)^{-1} \|$ is polynomially bounded along the imaginary axis,
(b) the regularised determinant $\det_2(I - zT)$ is an entire function which is of exponential type and polynomially bounded along the imaginary axis.

Then $T$ has a complete span of eigenvectors and generalised eigenvectors if and only if

$$\limsup_{r \to \infty} \frac{1}{r} \log \| \det_2(I - rT)(I - rT)^{-1} \| \leq \limsup_{r \to \infty} \frac{1}{r} \log |\det_2(I - rT)|,$$  \hspace{1cm} (4.76)

$$\limsup_{r \to \infty} \frac{1}{r} \log \| \det_2(I + rT)(I + rT)^{-1} \| \leq \limsup_{r \to \infty} \frac{1}{r} \log |\det_2(I + rT)|.$$  \hspace{1cm} (4.77)

**Proof** We split the proof into two parts. In the first part we show that $T$ satisfies all conditions listed in Theorem 4.1.3. In the second part we apply Theorem 4.1.3 to get the desired completeness result.

**Part 1.** We shall apply Theorem 4.1.3 with $X = H$, with $Y = H$, and with $\Lambda$ being the identity operator on $H$. Since $H$ is a Hilbert space, the assumption $\text{Ker } T = \text{Ker } T^*$ implies that $H = \text{Ker } T \oplus \text{Im } T$. Hence (4.2) is satisfied with $k = 1$. Put

$$q(z) = \det_2(I - zT) \text{ and } P(z) = q(z)(I - zT)^{-1}.$$  

From Lemma 3.2.2 we know that $q$ is an entire function, and $I - zT$ is invertible if and only if $q(z) \neq 0$. Thus the operator $T$ is determined by the entire functions $\{q, P\}$. By assumption (see the first part of condition (b)) the function $q(z)$ is of exponential type (and thus its order $\rho$ is precisely one). Furthermore, by Theorem 3.3.1, the operator-valued function $P$ is entire and of order at most one. We conclude that items (1) and (2) in Theorem 4.1.3 are satisfied with $\rho = 1$. The fact that $\Lambda$ is assumed to be the identity operator on $H$, tells us that condition (4.3) is automatically fulfilled.

By assumption (see the second part of condition (b)) the function $q(z)$ is polynomially bounded along the imaginary axis, and the same holds true
for \(\|(I-zT)^{-1}\|\) by condition (a). Thus (4.4) is satisfied with \(\theta_1 = \pi/2, \theta_2 = (3\pi)/2, z_0 = 0\), and with some \(s_0 > 0\). We conclude that all the conditions in Theorem 4.1.3 are fulfilled, and we are ready to apply the theorem.

Part 2.

Put \(\mathcal{F}_T = \{x \in H \mid q(z) \text{ dominates } P(z)x\}\). From Theorem 4.1.3 we know that

\[
\mathcal{F}_T = \mathcal{M}_T \oplus \text{Ker } T \quad \text{and} \quad H = \mathcal{M}_T \oplus S_T.
\]

Hence in order to complete the theorem it remains to show that \(\mathcal{F}_T = H\) if and only if the two conditions (4.76) and (4.77) are satisfied.

Recall that \(q\) is of exponential type and polynomially bounded along the imaginary axis by condition (b). But then we can use Proposition 14.3.4 to conclude that \(q\) belongs to the class \(\mathcal{PW}\). The latter implies that (4.76) and (4.77) can be rewritten as

\[
\limsup_{r \to \infty} \frac{1}{r} \log \|P(r)\| \leq \limsup_{r \to \infty} \frac{1}{r} \log |q(r)| = h_q(0), \tag{4.78}
\]

\[
\limsup_{r \to \infty} \frac{1}{r} \log \|P(-r)\| \leq \limsup_{r \to \infty} \frac{1}{r} \log |q(-r)| = h_q(\pi). \tag{4.79}
\]

Here \(h_q\) is the Phragmén-Lindelöf indicator function which is defined by (14.55). Furthermore, since \(q \in \mathcal{PW}\), we can apply Theorem 14.6.3 to show that \(q\) is an entire function of completely regular growth and its indicator function \(h_q\) is given by

\[
h_q(\theta) = \begin{cases} 
  h_q(0) \cos \theta & \text{when } -\pi/2 \leq \theta \leq \pi/2, \\
  -h_q(\pi) \cos \theta & \text{when } \pi/2 \leq \theta \leq 3\pi/2.
\end{cases} \tag{4.80}
\]

Assume (4.78) and (4.79) are satisfied, and let us prove that \(\mathcal{F}_T = H\). Fix \(x \in H\). Recall that \(P(z)x = q(z)(I-zT)^{-1}x\) is an entire function of order at most one. Since \(q(z)\) and \((I-zT)^{-1}x\) are both polynomially bounded on the imaginary axis, the same holds true for \(P(z)x\). Put \(f(z) = \langle y^*, P(z)x \rangle\), where \(y^*\) is a continuous linear functional on \(H\). Note that the order of \(f\) is at most one. If the order of \(f\) is strictly less than one, then \(q\) dominates \(f\) by definition. If the order of \(f\) is one, then it follows from (4.78) and (4.79) that the lim sups

\[
\limsup_{r \to \infty} \frac{1}{r} \log |f(r)| \quad \text{and} \quad \limsup_{r \to \infty} \frac{1}{r} \log |f(-r)|
\]
exist. But then we can apply Proposition 14.4.7 to show that \( f \) belongs to the class \( \mathcal{PW} \). The latter implies that

\[
h_f(\theta) = \begin{cases} 
  h_f(0) \cos \theta & \text{ when } -\pi/2 \leq \theta \leq \pi/2, \\
  -h_f(\pi) \cos \theta & \text{ when } \pi/2 \leq \theta \leq 3\pi/2.
\end{cases}
\] (4.81)

Note that (4.80) and (4.81) show that \( h_f(\theta) \leq h_q(\theta) \) for \(-\pi/2 \leq \theta \leq 3\pi/2\) if and only if

\[
h_f(0) \leq h_q(0) \quad \text{ and } \quad h_f(\pi) \leq h_q(\pi).
\] (4.82)

Assume (4.78) and (4.79) are satisfied, then it follows from (4.82) that \( q \) dominates \( f \). Since \( y^* \) is an arbitrary continuous linear functional on \( H \) we obtain that \( q \) dominates \( P(z)x \), that is, \( x \in \mathcal{F}_T \). But \( x \) is an arbitrary vector in \( H \), and therefore \( H = \mathcal{F}_T \) as desired.

On the other hand if \( T \) has a complete span of eigenvectors and generalised eigenvectors, then \( H = \mathcal{F}_T \) and \( q \) dominates \( P(z)x \) for every \( x \in H \). So for an arbitrary continuous linear functional \( y^* \) on \( H \), we have \( h_f(\theta) \leq h_q(\theta) \) for \(-\pi/2 \leq \theta \leq 3\pi/2\) with \( f(z) = \langle y^*, P(z)x \rangle \), but then (4.82) holds as well. Thus, for every \( x \in H \) and \( y^* \) a continuous linear functional on \( H \) we obtain

\[
\limsup_{r \to \infty} \frac{1}{r} \log \langle y^*, P(r)x \rangle \leq h_q(0)
\]

\[
\limsup_{r \to \infty} \frac{1}{r} \log \langle y^*, P(-r)x \rangle \leq h_q(\pi).
\]

This shows that (4.78) and (4.79) are satisfied. \( \square \)

In the next theorem we reformulate the preceding theorem directly in terms of the distribution of the eigenvalues.

**Theorem 4.7.2** Let \( H \) be a complex Hilbert space, and let \( T \) be a compact operator on \( H \) of order one such that \( \ker T = \ker T^* \). Suppose that the function \( \| (I - zT)^{-1} \| \) is polynomially bounded along the imaginary axis. Furthermore, assume that the non-zero eigenvalues of \( T \), \( \lambda_j = \lambda_j(T) \), \( j = 1, 2, \ldots \), satisfy the following three conditions:

(1) \( \sum_{j=1}^{\infty} |\text{Re } \lambda_j| < \infty \);  
(2) \( \lim_{r \to \infty} \sum_{|\lambda_j| > 1/r} \lambda_j \) exists;  
(3) \( \lim_{r \to \infty} \frac{1}{r^2} \sum_{j \in I} \frac{1}{r} | \lambda_j | \geq 1/r < \infty \).

Then \( T \) has a complete span of eigenvectors and generalised eigenvectors if and only if conditions (4.76) and (4.77) are satisfied.
Proof From the assumption \( \text{Ker} \, T = \text{Ker} \, T^* \), it follows that \( H = \text{Ker} \, T \oplus \text{Im} \, T \). Hence we can apply Theorem 4.1.3 with \( k = 1 \). Let \( q(z) = \det_2(I - zT) \). From Lemma 14.8.13, it follows that \( q \) is an entire function of completely regular growth and that the indicator function of \( q(z) \) is given by
\[
h_q(\theta) = \begin{cases} 
  h_q(0) \cos \theta & \text{when } -\pi/2 \leq \theta \leq \pi/2, \\
  -h_q(\pi) \cos \theta & \text{when } \pi/2 \leq \theta \leq 3\pi/2.
\end{cases}
\]

Therefore, as in the proof of Theorem 4.7.1, the conditions (4.78) and (4.79) hold if and only if \( q \) dominates \( P(z)x \) for each \( x \in X \) with \( P(z)x = q(z)(I - zT)^{-1}x \). But then Theorem 4.1.3, with \( k = 1, \theta_1 = \pi/2 \) and \( \theta_2 = (3\pi)/2, z_0 = 0 \), and for some \( s_0 > 0 \), yields that \( H = \mathcal{F}_T \) if and only if (4.76) and (4.77) are satisfied. \( \square \)

Final Remark In the second part of this book, we introduce a number of classes of operators defined in terms of a characteristic matrix function to which Theorem 4.1.3 can be applied. Using the characteristic matrix function, the verification of the condition whether \( q \) dominates \( P(z)x \) used in the definition of \( \mathcal{F}_T \), can be reduced to a finite dimensional problem. Furthermore, the characterisation of \( M_T \), given by the identities in (4.6) reduces from a vector-valued characterisation to a scalar-valued characterisation easily verifiable in examples. This will drastically simplify the computations, see Chap. 7.