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Further counterexamples to a conjecture of Beilinson.

by

ROB DE JEU

Abstract

We give stronger counterexamples to a conjecture of Beilinson.

Key Words: K-theory, flat proper model, Beilinson conjecture

Mathematics Subject Classification 2000: Primary: 19B28, Secondary: 11R70, 19E08, 19F27

In [1, Conjecture 2.4.2.1] (see also [10, §3]) Beilinson posed the following conjecture.

Conjecture 1 *Let $X/\mathbb{Q}$ be a smooth, projective (but not necessarily geometrically irreducible) variety, and let $X/\mathbb{Z}$ be a flat and proper model of $X/\mathbb{Q}$. Then the image of the localization map*

$$K'_*(\mathcal{X}) \otimes \mathbb{Q} \rightarrow K'_*(X) \otimes \mathbb{Q} = K_*(X) \otimes \mathbb{Q}$$

is independent of $\mathcal{X}$.

The goal of this conjecture was to have a canonical subspace of $K_*(X) \otimes \mathbb{Q}$ that plays an important role in further conjectures by Beilinson; see [10, §5]. In [3] the author gave a counterexample to Conjecture 1, but a suitable canonical subspace of $K_*(X) \otimes \mathbb{Q}$ was constructed in a different way in [11, §1] using alterations, as we shall recall below. (In fact, in loc. cit. a canonical subspace is defined for motives with coefficients in a field of characteristic zero. We refer the reader to the original source for the corresponding details.)

More precisely, let X be as in Conjecture 1 and let $K_q(X) \otimes \mathbb{Q} = \bigoplus_n K_q^{(n)}(X)$ be the decomposition into Adams eigenspaces (see [6, Propositions 5 and 9]). Then the canonical subspace decomposes accordingly and it suffices to describe it for each $K_q^{(n)}(X)$. If $\mathcal{X}/\mathbb{Z}$ is a flat, projective model of $X/\mathbb{Q}$ that is regular then the desired subspace $K_q^{(n)}(\mathcal{X}/\mathbb{Z}) \subseteq K_q^{(n)}(X)$ is the image of the composition of the localization and projection maps,

$$K'_q(\mathcal{X}) \otimes \mathbb{Q} \rightarrow K'_q(X) \otimes \mathbb{Q} = K_q(X) \otimes \mathbb{Q} \rightarrow K_q^{(n)}(X),$$

this image being independent of $\mathcal{X}$ ([11, Theorem 1.1.6]; cf. [10, p. 13]). Such $\mathcal{X}$ is not known to exist in general, but by the theory of alterations (see [4] or [5])

there exists a regular $\mathcal{Y}$, projective and flat over $\mathbb{Z}$, for which its generic fibre Y admits a surjective, generically finite morphism $\phi : Y \rightarrow X$ (cf. [11, p. 475]). Then $K_q^{(n)}(X/\mathbb{Z})$ equals $\phi_*(K_q^{(n)}(Y/\mathbb{Z})) \subseteq K_q^{(n)}(X)$, which is independent of $\mathcal{Y}$ and ϕ ([11, Theorem 1.1.6]). Here ϕ_* is the composition

$$K_q^{(n)}(Y) \xrightarrow{\sim} \mathrm{Gr}_Y^n K_q(Y) \otimes \mathbb{Q} \rightarrow \mathrm{Gr}_Y^n K_q(X) \otimes \mathbb{Q} \xleftarrow{\sim} K_q^{(n)}(X),$$

with the isomorphisms coming from the Chern character (see [12, §1.5]) and the map in the centre from the (proper) pushforward

$$K_q(Y) \xrightarrow{\sim} K'_q(Y) \rightarrow K'_q(X) \xleftarrow{\sim} K_q(X)$$

(see [12, §1, (2.5)]), which induces a map $\mathrm{Gr}_Y^n K_q(Y) \otimes \mathbb{Q} \rightarrow \mathrm{Gr}_Y^n K_q(X) \otimes \mathbb{Q}$ since X and Y have the same dimension [12, §3, Theorem 1.1].

The counterexample to Conjecture 1 in [3] was for K_2 of certain elliptic curves $E/\mathbb{Q}$, and was related to the original discovery of the need for a canonical subspace in [2]. However, for an elliptic curve $E/\mathbb{Q}$ the rank of $K_2(E)/\text{torsion}$ is expected to be at most the number of primes of bad (or, more precisely, split multiplicative) reduction of E , plus 1. The goal of the present paper is to give an easier construction where, for fixed X , the image of $K'_1(\mathcal{X})$ in $K_1(X)$ is finitely generated but of arbitrarily large rank for suitably chosen flat and proper models $\mathcal{X}$. Clearly this shows that the image of $K'_1(\mathcal{X}) \otimes \mathbb{Q}$ in $K_1(X) \otimes \mathbb{Q}$ is not independent of $\mathcal{X}$, and can be arbitrarily big.

If $\mathcal{O}$ is the ring of algebraic integers in a fixed number field F and $R = \mathbb{Z} + N\mathcal{O}$ for some positive integer N then R is a subring of $\mathcal{O}$ and $R \otimes \mathbb{Q} \cong F$. Clearly R is a finite $\mathbb{Z}$ -algebra, so it is Noetherian and the map $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathbb{Z})$ is finite, hence proper. Also, $\mathrm{Spec}(R)$ is flat over $\mathrm{Spec}(\mathbb{Z})$ because R is a free $\mathbb{Z}$ -module. Therefore $\mathcal{X} = \mathrm{Spec}(R)$ is a flat and proper model over $\mathbb{Z}$ of $X = \mathrm{Spec}(F)$ over $\mathbb{Q}$, and we shall see that the rank of the image of $K'_1(\mathcal{X}) \rightarrow K_1(X)$, which depends on N , can be arbitrarily large if $F \neq \mathbb{Q}$.

The localization sequence for $\mathcal{O} \rightarrow F$ gives us the top row in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(\mathcal{O}) & \longrightarrow & K_1(F) & \longrightarrow & \coprod_{\mathcal{P}} K_0(\mathcal{O}/\mathcal{P}) & \longrightarrow & K_0(\mathcal{O}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K'_1(R) & \longrightarrow & K_1(F) & \longrightarrow & \coprod_{\mathcal{P}} K_0(R/\mathcal{P}) & \longrightarrow & K'_0(R) & \longrightarrow & \cdots \end{array}$$

The coproduct in that row is over all non-zero prime ideals $\mathcal{P}$ of $\mathcal{O}$, and we also used that $K'_*(\mathcal{O}) = K_*(\mathcal{O})$ since $\mathcal{O}$ is a regular ring (see [9, §4, Corollary 2]), as well as that $\mathcal{O}^* \cong K_1(\mathcal{O}) \rightarrow K_1(F) \cong F^*$ is injective (see [7, page 159]).

The localization sequence for $R \rightarrow F$ gives us the bottom row in this diagram, where the coproduct is over all non-zero prime ideals P of R , but we have to justify the zero on the left. For this we note that $\text{Spec}(\mathcal{O}) \rightarrow \text{Spec}(R)$ is also proper and preserves the codimension filtration. Therefore there is a pushforward that gives us a map from the localization sequence for $\mathcal{O}$ to the one for R . This gives us the commutative diagram as above, but with the zero in the top row replaced with $\coprod_{\mathcal{P}} K_1(\mathcal{O}/\mathcal{P})$ and the zero in the bottom row replaced with $\coprod_P K_1(R/P)$. However, the map $\coprod_{\mathcal{P}} K_1(\mathcal{O}/\mathcal{P}) \rightarrow \coprod_P K_1(R/P)$ is surjective because above each P in R there is a $\mathcal{P}$ in $\mathcal{O}$ and the map $K_1(\mathcal{O}/\mathcal{P}) \rightarrow K_1(R/P)$, corresponding to the norm map $(\mathcal{O}/\mathcal{P})^* \rightarrow (R/P)^*$, is surjective since the fields involved are finite. This, together with the injectivity of $K_1(\mathcal{O}) \rightarrow K_1(F)$, implies that $K'_1(R) \rightarrow K_1(F)$ is injective as well.

If we let $S = \mathcal{O}[\frac{1}{N}] = R[\frac{1}{N}]$, a regular ring, then the localization sequences and pushforward in this case yield the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(\mathcal{O}) & \longrightarrow & K_1(S) & \longrightarrow & \coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P}) & \longrightarrow & K_0(\mathcal{O}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K'_1(R) & \longrightarrow & K_1(S) & \longrightarrow & \coprod_{P|NR} K_0(R/P) & \longrightarrow & K'_0(R) & \longrightarrow & \cdots \end{array}$$

Here $K_1(\mathcal{O}) \rightarrow K_1(S)$ is injective because its composition with $K_1(S) \rightarrow K_1(F)$ is injective, and the injectivity of $K'_1(R) \rightarrow K_1(S)$ follows similarly. Since $K_1(S)$ is finitely generated, this implies that $K'_1(R)$ is also finitely generated.

From the last diagram we see that the cokernel of $K_1(S) \rightarrow \coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ injects into $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \text{Cl}(F)$ where $\text{Cl}(F)$ is the class group of F (see [7, Corollary 1.11]). Because $\text{Cl}(F)$ is the kernel of the composition of the localizations $K_0(\mathcal{O}) \rightarrow K_0(S) \rightarrow K_0(F) \cong \mathbb{Z}$, this cokernel is contained in $\text{Cl}(F)$, hence is finite. Also, if $\mathcal{P}$ is a non-zero prime ideal of $\mathcal{O}$ lying above a given non-zero prime ideal P of R then the map $K_0(\mathcal{O}/\mathcal{P}) \rightarrow K_0(R/P)$ is injective since it corresponds to viewing a finitely generated $\mathcal{O}/\mathcal{P}$ -vector space as a finitely generated R/P -vector space. Under the identifications $K_0(\mathcal{O}/\mathcal{P}) \cong \mathbb{Z}$ and $K_0(R/P) \cong \mathbb{Z}$ the image of $K_0(\mathcal{O}/\mathcal{P})$ in $K_0(R/P)$ is given by $[\mathcal{O}/\mathcal{P} : R/P] \cdot \mathbb{Z}$.

Let I_1 be the image of $K_1(S)$ in $\coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ and I_2 the image of $K_1(S)$ in $\coprod_{P|NR} K_0(R/P)$, so that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(\mathcal{O}) & \longrightarrow & K_1(S) & \longrightarrow & I_1 \longrightarrow 0 \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & K'_1(R) & \longrightarrow & K_1(S) & \longrightarrow & I_2 \longrightarrow 0. \end{array}$$

Note that $K_1(\mathcal{O}) \rightarrow K'_1(R)$ is injective, $I_1 \rightarrow I_2$ is surjective, and

$$\frac{K'_1(R)}{K_1(\mathcal{O})} \cong \text{Ker}(I_1 \rightarrow I_2).$$

Both I_1 and I_2 are free $\mathbb{Z}$ -modules, and, by our arguments above, I_1 has the same rank as $\coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ and I_2 has the same rank as $\coprod_{P|NR} K_0(R/P)$.

Let us now determine the rank of I_2 by determining the number of (non-zero) prime ideals of R that lie above $p\mathbb{Z}$ for each prime factor of N . Let P be a prime ideal of R lying above $p\mathbb{Z}$ where p is a prime number dividing N . If a is in $\mathcal{O}$, then $(Na)^2 = N(Na^2)$ lies in $pR \subseteq P$. So P contains $N\mathcal{O}$ and hence $p\mathbb{Z} + N\mathcal{O}$. Since this is a maximal ideal of R it must be equal to P , and P is unique. Hence the rank of I_2 is equal to the number of distinct prime numbers dividing N .

By applying a corollary of the Chebotarov density theorem (see [8, Corollary 6.5]) to the normal closure of $F/\mathbb{Q}$ we see that, for any $n \geq 1$, we can take n distinct prime numbers $p_1, \dots, p_n$ such that each $p_j\mathbb{Z}$ splits completely in $\mathcal{O}$. We let $N = p_1 \cdots p_n$ so that, with $d = [F:\mathbb{Q}]$, above each $p_j\mathbb{Z}$ there are d primes $\mathcal{P}$ of $\mathcal{O}$ but only one prime P of $R = \mathbb{Z} + N\mathcal{O}$. Then I_1 has rank nd , I_2 has rank n , and

$$\frac{K'_1(R)}{K_1(\mathcal{O})} \cong \text{Ker}(I_1 \rightarrow I_2) \cong \mathbb{Z}^{n(d-1)}$$

because $I_1 \rightarrow I_2$ is surjective. Since this holds for arbitrary $n \geq 1$ this means that the rank of $K'_1(R)$ can be arbitrarily large as long as $F \neq \mathbb{Q}$.

As a very explicit example let us take $F = \mathbb{Q}(i)$ so that $\mathcal{O} = \mathbb{Z}[i]$. Then each prime p congruent to 1 modulo 4 splits completely in $\mathbb{Z}[i]$. So for $N = p_1 \cdots p_n$ a product of $n \geq 1$ distinct such primes, and $R = \mathbb{Z} + N\mathbb{Z}[i] = \mathbb{Z}[Ni]$, we find that $K'_1(\mathbb{Z}[Ni])/K_1(\mathbb{Z}[i]) \cong \mathbb{Z}^n$, which implies that $K'_1(\mathbb{Z}[Ni]) \cong \mathbb{Z}^n \times \mathbb{Z}/4\mathbb{Z}$.

Remark 2 In the situation of a number field F with ring of algebraic integers $\mathcal{O}$ and $R = \mathbb{Z} + N\mathcal{O}$, the method we employed does not work for the image of $K_1(R)$ in $K_1(F) \cong F^*$ rather than that of $K'_1(R)$. In fact, for such R one cannot get any larger image than that of $K_1(\mathcal{O}) \cong \mathcal{O}^*$ because the localisation map $K_1(R) \rightarrow K_1(F)$ factors through $K_1(\mathcal{O}) \rightarrow K_1(F)$. And with $k = |(\mathcal{O}/N\mathcal{O})^*|$, if u is in $\mathcal{O}^*$ then u^k lies in $1 + N\mathcal{O} \subset R$ so that the image of $K_1(R)$ in $K_1(\mathcal{O})$ is of finite index. This shows that $K_1(R) \otimes \mathbb{Q}$ and $K_1(\mathcal{O}) \otimes \mathbb{Q}$ always have the same image in $K_1(F) \otimes \mathbb{Q}$.

It seems that, for general $X/\mathbb{Q}$ smooth and projective, and a flat and proper model $\mathcal{X}/\mathbb{Z}$ of $X/\mathbb{Q}$, the question if the image of $K_*(\mathcal{X}) \otimes \mathbb{Q} \rightarrow K_*(X) \otimes \mathbb{Q}$ is independent of $\mathcal{X}$ is open. Given the above this question seems more natural than Conjecture 1.

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