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Boer, D.; Mulders, P.J.G.; Pijlman, F.

published in

Nuclear Physics B
2003

DOI (link to publisher)

[10.1016/S0550-3213\(03\)00527-3](https://doi.org/10.1016/S0550-3213(03)00527-3)

document version

Publisher's PDF, also known as Version of record

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citation for published version (APA)

Boer, D., Mulders, P. J. G., & Pijlman, F. (2003). Universality of T-odd effects in single spin and azimuthal asymmetries. *Nuclear Physics B*, 667(1-2), 201-241. [https://doi.org/10.1016/S0550-3213\(03\)00527-3](https://doi.org/10.1016/S0550-3213(03)00527-3)

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Nuclear Physics B 667 (2003) 201–241

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Universality of T-odd effects in single spin and azimuthal asymmetries

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Received 7 March 2003; received in revised form 14 May 2003; accepted 11 June 2003

Abstract

We analyze the transverse-momentum-dependent distribution and fragmentation functions in space-like and time-like hard processes involving at least two hadrons, in particular, 1-particle inclusive lepton production, the Drell–Yan process and two-particle inclusive hadron production in electron–positron annihilation. As is well known, transverse momentum dependence allows for the appearance of unsuppressed single spin azimuthal asymmetries, such as Sivers and Collins asymmetries. Recently, Belitsky, Ji and Yuan obtained fully color-gauge-invariant expressions for the relevant matrix elements appearing in these asymmetries at leading order in an expansion in the inverse hard scale. We rederive these results and extend them to observables at the next order in this expansion. We observe that at leading order one retains a probability interpretation, contrary to a claim in the literature and show the direct relation between the Sivers effect in single spin asymmetries and the Qiu–Sterman mechanism. We also study fragmentation functions, where the process-dependent gauge link structure of the correlators is not the only source of T-odd observables and discuss the implications for universality.

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PACS: 13.60.Hb; 13.87.Fh; 13.88.+e

1. Introduction

The study of polarization and transverse-momentum-dependent distribution functions was initiated by Ralston and Soper [1]. Their study of the Drell–Yan process was performed at tree level and did not address the color gauge (non-)invariance of the distribution functions. At leading order (tree level) no single spin asymmetries were obtained in the

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Drell–Yan process (see also [2]). Sivers [3] proposed a specific non-trivial correlation involving polarization and transverse momentum, that would lead to unsuppressed single spin azimuthal asymmetries. For distribution functions such a correlation seemed to entail a violation of time-reversal invariance. Collins [4] showed that this was not the case for similar correlations in the fragmentation process. Nevertheless, phenomenological studies of the consequences of the Sivers effect were performed [5,6]. Recently, Brodsky, Hwang and Schmidt [7] (BHS) demonstrated in an explicit model calculation that the Sivers asymmetry can in principle arise, after which Collins [8] demonstrated that it is the presence of a path-ordered exponential in the definition of transverse-momentum-dependent distribution functions that allows for the Sivers effect without a violation of time-reversal invariance.

This generated renewed interest in the proper gauge-invariant definition of transverse-momentum-dependent correlators. The definitions of transverse-momentum-dependent parton densities of Refs. [9,10] did contain path-ordered exponentials (links) to ensure color gauge invariance, but these were not closed paths (each quark field has a straight link to infinity attached to it, but pieces at infinity are missing). If one includes such links by hand, it is no problem to consider also closed paths, but Efremov and Radyushkin [11] had demonstrated that the path of the link can be derived in transverse-momentum-integrated parton densities and this can also be done when the transverse momentum is not integrated over. In this way different processes can yield different paths [12], but no physical observable effects were expected from such links. However, until recently these derivations were incomplete, since the obtained paths were not closed. The missing piece would have to involve transverse gluon fields at lightcone infinity, which were thought not to affect physical matrix elements or at the very least lead to contributions suppressed compared to the leading order. Recently, the derivation of fully color-gauge-invariant matrix elements, with paths closed at light-cone infinity, was completed by Belitsky, Ji and Yuan [13,14]. They observed that transverse gluon fields do not always lead to suppression, contrary to common belief, formalizing the model results of BHS. The resulting fully color-gauge-invariant matrix elements strengthen the observation of Collins [8] that the presence of the link invalidated the earlier proof of the absence of the Sivers function due to time-reversal invariance.

With all these technical details clarified, the justification of the phenomenological studies of the Sivers (and similar) effects was provided. Next, however, the question of observable process-dependence arose. Collins [8] demonstrated that the Sivers asymmetry in (semi-inclusive) deep-inelastic scattering (DIS) and the Drell–Yan process must occur with opposite signs. This has been confirmed in the BHS model calculation [15], but still awaits experimental verification. It would be the first demonstration of an observable effect due to the presence of a path-ordered exponential in the hadron correlators and thereby would show the intrinsic non-locality of the operators occurring in these semi-inclusive processes.

Given this process dependence it is relevant to study which color-gauge-invariant distribution and fragmentation functions appear in different processes. In this paper we study color gauge invariance of transverse-momentum-dependent distribution and fragmentation functions appearing in hard processes, in particular in semi-inclusive deep-inelastic leptonproduction (SIDIS), the Drell–Yan process (DY) and e^+e^- annihilation.

Fig. 1. Diagrams contributing in inclusive deep-inelastic scattering.

We will employ a field theoretical approach to these hard processes and follow the notation and derivation of Ref. [12], now taking into account the additional contributions uncovered by Belitsky, Ji and Yuan [14]. Our analysis is different at several points, but we confirm their results. In addition, we obtain new results for the first sub-leading order results in an expansion in inverse powers of the hard scale. For instance, we demonstrate for the first time a direct relation between the Sivers effect in single spin asymmetries and the Qiu–Sterman mechanism. Also, we study transverse-momentum-dependent fragmentation functions, where the process dependence of the gauge link structure of the correlators is not simply an overall sign. Rather one finds that two different (but universal) matrix elements enter in different combinations.

In this paper ‘leading’ and ‘sub-leading’ always refer to the expansion in inverse powers of the hard scale. Perturbative QCD corrections beyond tree level (next-to-leading order in α_s) will need to be taken into account as well in further studies (see Refs. [4,10,16] for discussions of additional complications beyond tree level). In this paper we present a new way of isolating the leading and first sub-leading order parts of the cross sections in terms of correlators including the proper gauge links *before* evaluating them explicitly. These separate color-gauge-invariant expressions for each order have not been presented before. They facilitate the evaluation of asymmetries arising at a given order.

Now we will outline the more technical steps to be followed in this paper. In the field theoretical approach, expressions for the structure functions of inclusive deep-inelastic scattering (DIS) are obtained from diagrams as shown in Fig. 1 [17,18]. In these diagrams soft parts appear that represent matrix elements of the fields corresponding to the quark and gluon legs connecting the hard and soft parts of a diagram. The expressions for SIDIS structure functions are obtained from diagrams as shown in Fig. 2. Again soft parts represent specific matrix elements. In this paper we will only consider tree-level results, which means that if gluons appear, they are in essence legs of the soft parts. In other words, their (soft) couplings to the hard scattering part are included in the definition of the matrix elements. This approach requires a careful treatment of the diagrams involving quark–quark–gluon matrix elements such as those in Fig. 1(b) or Fig. 2(b)–(f). This is important in order to arrive at color-gauge-invariant matrix elements that form the universal quantities, the distribution and fragmentation functions, appearing in cross sections.

In Section 2 we outline the diagrammatic approach in a number of steps, using two complementary lightcone directions n_+ and n_- , which in the presence of a hard scale (in DIS or SIDIS, the photon momentum) are fixed by the hadron momenta. The simplest

Fig. 2. Diagrams contributing in 1-particle inclusive deep-inelastic scattering.

(handbag) diagrams in Figs. 1(a) and 2(a) only involve quark–quark matrix elements. In DIS the hadron momentum defines the lightcone direction n_+ and the nonlocality in the matrix elements is restricted along the lightcone direction n_- (for which $n_+ \cdot n_- = 1$). As is well known, diagrams as in Fig. 1(b) with any number of $A^+ = A \cdot n_-$ gluons yield the necessary gauge link connecting the two quark fields [11]. The nonlocal quark–quark operator combination with a gauge link can be expanded into a tower of local twist-two operators with different spins. Their matrix elements appear in the cross section as leading terms in an expansion in inverse powers of the hard scale. Diagrams with (transverse) A_T^α gluons or with A^- gluons appear in matrix elements of higher twist operators, which appear in the cross section in terms suppressed by inverse powers of the hard scale.

The situation in SIDIS (Fig. 2), discussed in Section 3, differs in a subtle way from that of DIS, because the nonlocality in the operator combinations is not restricted to the lightcone, but involves also transverse separations. The kinematics only constrain the nonlocality to the light *front*. In our analysis we first consider the A^+ gluon legs in diagrams

as in Fig. 2(b) and (f). These diagrams, as in DIS, will give rise to gauge links, but in this case connecting a quark field along the n_- direction to $\pm\infty$ (cf. [10,12]), where the sign depends on the type of process. By including diagrams of the type in Fig. 2(c) as well, one can absorb all A^+ gluons into the lower blob. Diagrams like Fig. 2(d) and (e) allow one to absorb all A^- gluons into the upper blob, resulting again in gauge links running along n_+ to infinity. Effectively one then considers only Fig. 2(a), but now with a Φ and Δ that contain the gauge links.

Diagrams with transverse A_T^α -legs, lead to quark–quark–gluon matrix elements, which will turn out to be suppressed, except for the boundary terms at lightcone infinity, recently discussed by Belitsky et al. [14]. We outline an alternative for this procedure and show that the latter appear when one expresses these fields in the appropriate field strength tensor $G^{+\alpha}$. The boundary terms that arise in this way combine into the transverse piece that completes the gauge link connecting the two quark fields (running via lightcone infinity). Upon integration over transverse momenta the result reduces to the correct gauge-invariant operator of ordinary inclusive DIS.

In Sections 4–6 a comparison is made between different processes involving at least two hadrons, in particular between 1-particle inclusive leptonproduction, the Drell–Yan process and two-particle inclusive hadron production in electron–positron annihilation. For instance, in Drell–Yan the links run in opposite directions along the n_- direction compared to SIDIS (as noticed in [8,12]). Because the two situations can be connected via a time-reversal operation, one can define T-even and T-odd functions that appear in the parametrization of the color-gauge-invariant matrix elements. For these functions factorization in principle should hold, although they appear with different signs in SIDIS and DY [8]. The T-odd functions appear in single spin asymmetries in these processes [3,4,6,19] or they appear in pairs in unpolarized azimuthal asymmetries [20–22]. In Section 7 we study the time-reversal properties of distribution and fragmentation functions and present explicit parametrizations.

2. Hadron tensor and correlators in SIDIS

The hadron tensor for 1-particle inclusive leptonproduction is given by

$$2M\mathcal{W}_{\mu\nu}^{(IH)}(q; P, S; P_h, S_h) = \frac{1}{(2\pi)^4} \int \frac{d^3 P_X}{(2\pi)^3 2P_X^0} (2\pi)^4 \delta^4(q + P - P_X - P_h) \times H_{\mu\nu}^{IH}(P_X; P, S; P_h, S_h), \quad (1)$$

with $H_{\mu\nu}$ being the product of current expectation values

$$H_{\mu\nu}^{(IH)}(P_X; P, S; P_h, S_h) = \langle P, S | J_\mu(0) | P_X; P_h, S_h \rangle \langle P_X; P_h, S_h | J_\nu(0) | P, S \rangle. \quad (2)$$

Due to the fact that the summation and integration over final states is not complete, prohibiting the formal use of the operator product expansion, we proceed along the lines of the diagrammatic approach of Refs. [1,17], based on nonlocal operators. The quark and gluon lines connected to the soft parts represent matrix elements of (nonlocal) quark and gluon operators.

The hadron tensor is calculated for current fragmentation in deep-inelastic scattering. In that case the exchanged momentum $-q^2 \equiv Q^2$ is large and one has for the target momentum P and the produced hadron momentum P_h the conditions that $P \cdot q$, $P_h \cdot q$ and $P \cdot P_h$ are large, of $\mathcal{O}(Q^2)$. One is able to make a systematic expansion in orders of $1/Q$, of which we will only consider the first two terms, $(1/Q)^0$ and $(1/Q)^1$. In this situation one uses the scaling variables

$$x_B = \frac{Q^2}{2P \cdot q} \approx -\frac{P_h \cdot q}{P_h \cdot P}, \quad (3)$$

$$z_h = \frac{P \cdot P_h}{P \cdot q} \approx -\frac{2P_h \cdot q}{Q^2}, \quad (4)$$

where the approximate sign indicates equalities up to $1/Q^2$ (mass) corrections. It is convenient to introduce lightlike vectors n_+ and n_- satisfying $n_+ \cdot n_- = 1$ along the hadron momenta writing

$$P^\mu = \frac{\xi}{\tilde{Q}\sqrt{2}} n_-^\mu + \frac{\tilde{Q}}{\xi\sqrt{2}} n_+^\mu, \quad (5)$$

$$P_h^\mu = \frac{\zeta}{\sqrt{2}} n_-^\mu + \frac{M_h^2}{\zeta\tilde{Q}\sqrt{2}} n_+^\mu, \quad (6)$$

$$q^\mu = \frac{\tilde{Q}}{\sqrt{2}} n_-^\mu - \frac{\tilde{Q}}{\sqrt{2}} n_+^\mu + q_T^\mu, \quad (7)$$

with $q_T^2 \equiv -Q_T^2$ and $\tilde{Q}^2 = Q^2 + Q_T^2$. These equations define the lightcone coordinates $a^\pm \equiv a \cdot n_\mp$ and the transverse projector $g_T^{\mu\nu} = g^{\mu\nu} - n_+^{(\mu} n_-^{\nu)}$. In our treatment of the 1-particle inclusive process we will consider $Q_T^2 \ll Q^2$, hence $\tilde{Q}^2 \approx Q^2$, while $\xi \approx x_B$ and $\zeta \approx z_h$ up to mass corrections of order $1/Q^2$. The vector $q_T^\mu \approx q^\mu + x P^\mu - P_h^\mu/z$ determines the off-collinearity in the process. In principle, mass corrections can straightforwardly be incorporated. Important to note is that the lightlike directions $n_\pm = n_\pm(P, P_h)$ are determined by the hadron momenta P and P_h .

From the diagrammatic expansion (see Fig. 2(a)–(e)) one obtains up to $\mathcal{O}(g)$,

$$\begin{aligned} & 2M\mathcal{W}_{\mu\nu}(q; P, S; P_h, S_h) \\ &= \int d^4p d^4k \delta^4(p+q-k) \left\{ \text{Tr}(\Phi(p)\gamma_\mu \Delta(k)\gamma_\nu) \right. \\ & \quad - \int d^4p_1 \text{Tr} \left(\gamma_\alpha \frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 + i\epsilon} \gamma_\nu \Phi_A^\alpha(p, p-p_1) \gamma_\mu \Delta(k) \right) \quad [\text{term 1}] \\ & \quad - \int d^4p_1 \text{Tr} \left(\gamma_\mu \frac{\not{k} - \not{p}_1 + m}{(k-p_1)^2 - m^2 - i\epsilon} \gamma_\alpha \Delta(k) \gamma_\nu \Phi_A^\alpha(p-p_1, p) \right) \quad [\text{term 2}] \\ & \quad - \int d^4k_1 \text{Tr} \left(\gamma_\nu \frac{\not{p} - \not{k}_1 + m}{(p-k_1)^2 - m^2 + i\epsilon} \gamma_\alpha \Phi(p) \gamma_\mu \Delta_A^\alpha(k-k_1, k) \right) \quad [\text{term 3}] \\ & \quad \left. - \int d^4k_1 \text{Tr} \left(\gamma_\alpha \frac{\not{p} - \not{k}_1 + m}{(p-k_1)^2 - m^2 - i\epsilon} \gamma_\mu \Delta_A^\alpha(k, k-k_1) \gamma_\nu \Phi(p) \right) \right\} \quad [\text{term 4}], \quad (8) \end{aligned}$$

Fig. 3. Soft parts representing the quark–quark and quark–quark–gluon matrix elements used in Eqs. (9)–(12).

where

$$\Phi_{ij}(p; P, S) = \int \frac{d^4\xi}{(2\pi)^4} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) \psi_i(\xi) | P, S \rangle, \quad (9)$$

$$\Delta_{ij}(k; P_h, S_h) = \sum_X \int \frac{d^4\xi}{(2\pi)^4} e^{ik \cdot \xi} \langle 0 | \psi_i(\xi) | P_h, X \rangle \langle P_h, X | \bar{\psi}_j(0) | 0 \rangle, \quad (10)$$

$$\begin{aligned} \Phi_{Aij}^\alpha(p, p - p_1; P, S) &= \int \frac{d^4\xi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\ &\quad \times \langle P, S | \bar{\psi}_j(0) g A^\alpha(\eta) \psi_i(\xi) | P, S \rangle, \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta_{Aij}^\alpha(k, k - k_1; P_h, S_h) &= \sum_X \int \frac{d^4\xi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} e^{ik \cdot \xi} e^{ik_1 \cdot (\eta - \xi)} \langle 0 | \psi_i(\xi) g A^\alpha(\eta) | P_h, X \rangle \\ &\quad \times \langle P_h, X | \bar{\psi}_j(0) | 0 \rangle, \end{aligned} \quad (12)$$

illustrated in Fig. 3. In the above expression we have omitted the contributions with the opposite direction on the fermion line. It adds to the result in Eq. (8) terms with $q \leftrightarrow -q$ and $\mu \leftrightarrow \nu$. In cross sections it will always lead to extending a sum over quarks and antiquarks.

The aim of the calculation is an expansion in powers of $1/Q$. For this a number of considerations are important. First, the matrix elements represented by blobs in the diagrammatic expansion should vanish fast enough when any of the products of momenta involved becomes large, e.g., the virtualities of the quarks or gluons. To be precise, in Fig. 3 the products $p^2 \sim p_1^2 \sim p \cdot p_1 \sim p \cdot P \sim p_1 \cdot P \sim P^2 = M^2 \ll Q^2$. With the choice of parametrization in Eqs. (5)–(7), this implies that for the momenta in Fig. 3(a) and (c) one has for the plus-components $p^+, p_1^+, P^+ \sim Q$, while the minus-components $p^-, p_1^-, P^- \sim 1/Q$. For the fragmentation parts (Fig. 3(b) and (d)) one has minus-components $k^-, k_1^-, P_h^- \sim Q$, while for the plus-components $k^+, k_1^+, P_h^+ \sim 1/Q$. The transverse momenta are of $\mathcal{O}(M)$. Introducing momentum fractions $x = p^+/P^+$ and $z = P_h^-/k^-$, writing

$$p^\mu = p^- n_-^\mu + x P^+ n_+^\mu + p_T^\mu, \quad (13)$$

$$k^\mu = \frac{P_h^-}{z} n_-^\mu + k^+ n_+^\mu + k_T^\mu, \quad (14)$$

one finds, when neglecting $\mathcal{O}(1/Q^2)$ contributions, that $\delta^4(p+q-k) \rightarrow \delta(x-x_B)\delta(z-z_h)\delta^2(p_T+q_T-k_T)$, thus identifying the scaling variables and momentum fractions, $x=x_B$ and $z=z_h$.

Thus, in a calculation up to $\mathcal{O}(1/Q^2)$, the integration over the minus-components of momenta in the matrix elements Φ and Φ_A can be performed, restricting them to the lightfront,

$$\begin{aligned} \Phi_{ij}(x, p_T) &= \int dp^- \Phi_{ij}(p; P, S) \\ &= \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) \psi_i(\xi) | P, S \rangle \Big|_{\xi^+=0}, \end{aligned} \quad (15)$$

while $\Phi_A^\alpha(p^+, p_T, p_1^+, p_{1T}) = \int dp^- dp_1^- \Phi_A^\alpha(p, p-p_1; P, S)$ involves two integrations over the minus-components of the parton momenta. We will occasionally also use the variable x_1 defined via $p_1^+ = x_1 P^+$. The integrations over minus-components are sufficient to render the time-ordering in these matrix elements superfluous, which can be proven completely analogous to the proof for the matrix elements in which also the integration over transverse momenta is performed, in that case restricting them to the lightcone [23,24]. In the matrix elements of the types Δ and Δ_A^α the integrations over the plus-components of the quark and gluon momenta can be performed, leading to lightfront correlation functions $\Delta(z, k_T)$,

$$\begin{aligned} \Delta_{ij}(z, k_T) &= \int dk^+ \Delta_{ij}(k; P_h) \\ &= \sum_X \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{ik \cdot \xi} \langle 0 | \psi_i(\xi) | P_h, X \rangle \langle P_h, X | \bar{\psi}_j(0) | 0 \rangle \Big|_{\xi^-=0}, \end{aligned} \quad (16)$$

and $\Delta_A^\alpha(k^-, k_T, k_1^-, k_{1T}) = \int dk^+ dk_1^+ \Delta_A^\alpha(k, k-k_1; P_h)$.

Matrix elements like $\Phi(x, p_T)$ have a particular Dirac structure, Lorentz structure, and canonical dimension, which must be visible in the parametrization of Φ through the dependence on non-integrated parton momenta and the hadron momentum and spin vectors. One deduces immediately the Dirac structure $\int dp^- \Phi \sim \not{n}_+ = \gamma^-$ giving a leading contribution and the Dirac structure involving the unit matrix in Dirac space requiring in addition a factor P^- in the parametrization, which will lead in the calculation of Eq. (8) to a suppression factor $1/Q$. The leading structure of $\int dp^- dp_1^- \Phi_A^\alpha$ matrix elements involving *two* integrations over minus components gives for $\int dp^- dp_1^- \Phi_A^+ \sim \gamma^-$, similar to $\int dp^- \Phi$, but for a transverse gluon one gets $\int dp^- dp_1^- \Phi_{A_T}^\alpha \sim P^- \gamma^-$ leading to a $1/Q$ suppression in the calculation of Eq. (8) (apart from the subtlety with the boundary terms, where the role of P^- is taken over by $\delta(p_1^+)$, to be elaborated upon below). Of course, in a parametrization of the latter matrix element also the transverse index must appear, e.g., a non-integrated parton transverse momentum p_T or the spin vector S_T in case of a transversely polarized hadron, but these are not relevant for an

expansion in powers of $1/Q$. The matrix element $\int dp^- dp_1^- \Phi_A^-$ will always appear suppressed by at least $(P^-)^2 \rightarrow 1/Q^2$.

For the fragmentation parts one has after integration over plus-components $\int dk^+ \Delta \sim \int dk^+ dk_1^+ \Delta_A^- \sim \gamma^+$, while $\int dk^+ dk_1^+ \Delta_{A_T}^\alpha \sim P_h^+ \gamma^+$, becoming suppressed by $1/Q$ (again apart from the subtlety with boundary terms). The matrix element $\int dk^+ dk_1^+ \Delta_A^+$ will always appear suppressed by at least $(P_h^+)^2 \rightarrow 1/Q^2$. The explicit parametrizations for the matrix elements in terms of distribution and fragmentation functions have been extensively discussed in many papers [2,6,19,20,25] and will be summarized in Section 7.

In order to find the leading contributions, we need in the calculation of Eq. (8) not only the first term (diagram in Fig. 2(a)), but also the terms involving Φ_A^+ and Δ_A^- (Fig. 2(b)–(e)) and even multiple-gluon matrix elements of the form Φ_{AA}^{++} (Fig. 2(f)), etc. Such a resummation of multiple-gluon matrix elements can be easily performed in DIS, where the integration over transverse momenta of partons can always be performed in addition to the minus-integration. The resummation leads to a modified first term in Eq. (8) with in the $\Phi(x) = \int dp^- d^2 p_T \Phi(p; P, S)$ matrix element the inclusion of a gauge link

$$\begin{aligned}
 U_{[a,\xi]}^- &= \mathcal{P} \exp \left(-ig \int_a^\xi d\zeta^- A^+(\zeta) \right) \Big|_{\zeta^+ = \xi^+ = a^+, \zeta_T = \xi_T = a_T} \\
 &= \sum_{N=0}^{\infty} (-ig)^N \int_{a^-}^{\xi^-} d\zeta_1^- A^+(\zeta_1) \cdots \int_{\zeta_{N-1}^-}^{\xi^-} d\zeta_N^- A^+(\zeta_N) \Big|_{\zeta_i^+ = \xi^+ = a^+, \zeta_{iT} = \xi_T = a_T}, \quad (17)
 \end{aligned}$$

connecting the quark fields, rendering the object color-gauge-invariant [11]. We will discuss the full procedure to obtain a color-gauge-invariant object in 1-particle inclusive leptonproduction in the next section, following in part Ref. [12] and recent work by Belitsky, Ji and Yuan [14].

3. Color gauge-invariance in SIDIS

In this section we will discuss the resummation of contributions in SIDIS coming from diagrams in Fig. 2(b)–(f).

At orders $(1/Q)^0$ (leading) and $(1/Q)^1$ (first sub-leading) the integrations over p^- , p_1^- , and k^+ in the corresponding soft parts can be performed. The result of the first term of four quark–quark–gluon contributions in Eq. (8) is

$$\begin{aligned}
 [\text{term } 1] &= - \int d^4 p d^4 k \delta^4(p + q - k) \\
 &\quad \times \left\{ \int d^4 p_1 \text{Tr} \left(\gamma_\alpha \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \gamma_\nu \Phi_A^\alpha(p, p - p_1) \gamma_\mu \Delta(k) \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
&= - \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) d p_1^+ d^2 p_{1T} \\
&\quad \times \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} \frac{d\eta^- d^2 \eta_T}{(2\pi)^3} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\
&\quad \times \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma_\alpha \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \\
&\quad \times \gamma_\nu g A^\alpha(\eta) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \eta^+ = 0}, \tag{18}
\end{aligned}$$

where Φ_A^α is made explicit (Eq. (11)) and the minus- and plus-integrations are performed. In the expression after the second equal sign, it is understood that in the integrand $p^- = p_1^- = k^+ = 0$ while $p^+ = x P^+$ and $k^- = P_h^-/z$.

Next, we will split off from the quark propagator those parts that are relevant at leading and first sub-leading order in $1/Q$. These parts depend on whether the index α of the gluon field is plus, transverse or minus. For the $1/Q$ order, we will restrict ourselves to obtaining a color-gauge-invariant expression for the hadron tensor integrated over the transverse momentum q_T of the photon. For this result one first needs to consider the leading order term unintegrated over q_T . The end results for the hadron tensor in several cases are summarized in the next section.

One finds for the quark propagator explicitly (with $k^- \approx q^- = Q/\sqrt{2}$)

$$\frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \approx \frac{(\not{k} + m) - \not{p}_1^+ - \not{p}_{1T}}{-p_1^+ Q\sqrt{2} + (k_T - p_{1T})^2 - m^2 + i\epsilon}. \tag{19}$$

Obvious contributions at leading order are, the $k^- \not{p}_-$ term of the quark propagator, which in combination with A^+ gluons leads to the link operator in the η^- direction. Less obvious are the contributions from fields that are independent of η^- which, as can be seen from Eq. (18), lead to a delta-function $\delta(p_1^+)$. In that case other leading contributions appear. In particular, a contribution coming from the last (transverse) term will lead to (leading) link contributions in the transverse direction.

Contributions at order $1/Q$ are coming from the $\not{p}_+$ term of the quark propagator in combination with the transverse gluons, and the transverse part of the quark propagator, $\not{p}_{1T}$, in combination with the A^+ gluons. These contributions can be combined into a color-gauge-invariant matrix element containing the field strength tensor. A summary is given in Table 1.

The leading contribution in Eq. (18) comes from Φ_A^+ . We use that $\gamma^-(\not{k} + m) = 2k^- - (\not{k} - m)\gamma^-$, the fact that $\Delta(k)(\not{k} - m) \sim \Delta_A \sim 1/Q$ (QCD equations of motion)

Table 1

The color-gauge-covariant objects into which the gluon fields in SIDIS are combined depending on the Dirac structure of specific terms in the hard quark propagator to which the gluon couples

	$\not{p}_+$	$\not{p}_-$	γ_T
A^+	–	U^-	$G^{+\alpha}$
A_T^α	$G^{+\alpha}$	–	U^T

to obtain

$$\begin{aligned}
& \Delta(k) \gamma^- \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} A^+(\eta) \\
& \approx -\Delta(k) \gamma^- \frac{\not{k} - \not{p}_1 + m}{2k^-(p_1^+ - i\epsilon)} A^+(\eta) \\
& \approx -\Delta(k) \frac{A^+(\eta)}{p_1^+ - i\epsilon} + \Delta(k) \frac{\gamma^- \not{p}_1 A^+(\eta)}{2k^-(p_1^+ - i\epsilon)} + \frac{\Delta(k)(\not{k} - m) \gamma^- A^+(\eta)}{2k^-(p_1^+ - i\epsilon)} \\
& \approx -\Delta(k) \frac{A^+(\eta)}{p_1^+ - i\epsilon} + \Delta(k) \frac{\gamma^- \not{p}_1 A^+(\eta)}{Q\sqrt{2} (p_1^+ - i\epsilon)} \quad (20)
\end{aligned}$$

with omitted parts being of $\mathcal{O}(1/Q^2)$. The first term inserted in Eq. (18) gives a leading contribution,

$$\begin{aligned}
[\text{term 1.1}] &= \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) d p_1^+ d^2 p_{1T} \\
&\quad \times \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} \frac{d\eta^- d^2 \eta_T}{(2\pi)^3} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\
&\quad \times \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma_\nu \frac{g A^+(\eta)}{p_1^+ - i\epsilon} \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \eta^+ = 0} \\
&= \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \\
&\quad \times \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma_\nu (-ig) \\
&\quad \times \int_{-\infty}^{\xi^-} d\eta^- A^+(\eta) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \eta^+ = 0; \eta_T = \xi_T}. \quad (21)
\end{aligned}$$

This is precisely the $\mathcal{O}(g)$ term in the expansion of $U_{[\infty, \xi]}^-$ multiplying $\psi(\xi)$. The result of the diagram in Fig. 2(f) with two A^+ -gluons gives the $\mathcal{O}(g^2)$ term, etc. From the second term in Eq. (18) (diagram in Fig. 2(c)) one obtains the $\mathcal{O}(g)$ term in the expansion of $U_{[0, \infty]}^-$ following $\psi(0)$.

The A^- -gluons in the other diagrams in Fig. 2(d) and (e) and corresponding higher orders can all be absorbed into link operators in modified soft parts of the form

$$\Phi_{ij}(x, p_T) \Rightarrow \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) U_{[0, \infty]}^- U_{[\infty, \xi]}^- \psi_i(\xi) | P, S \rangle \Big|_{\xi^+ = 0}, \quad (22)$$

$$\begin{aligned}
\Delta_{ij}(z, k_T) &\Rightarrow \sum_X \int \frac{d\xi^+ d^2 \xi_T}{(2\pi)^3} e^{ik \cdot \xi} \langle 0 | U_{[-\infty, \xi]}^+ \psi_i(\xi) | P_h, X \rangle \\
&\quad \times \langle P_h, X | \bar{\psi}_j(0) U_{[0, -\infty]}^+ | 0 \rangle \Big|_{\xi^- = 0}, \quad (23)
\end{aligned}$$

where $U_{[-\infty, \xi]}^+$ indicates a link along the lightcone plus-direction running from $-\infty$ to ξ^+ . These quantities, however, are not color-gauge-invariant, although we note that upon

integration over p_T and k_T one obtains color-gauge-invariant lightcone correlators $\Phi(x)$ and $\Delta(z)$, in which the two links merge into one connecting the lightlike separated points 0 and ξ . These are, e.g., important in q_T -integrated SIDIS cross sections at leading order. For the transverse-momentum-dependent functions, however, we are still missing a transverse piece that leads to color-gauge-invariant definitions. It has to come from transverse gluons, which are next to be investigated.

Since the dominant part of $\int dk^+ \Delta$ is proportional to γ^+ , one finds (naively) for the transverse gluons in term 1 (Eq. (18)),

$$\begin{aligned} \Delta(k) \gamma_\alpha \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} A_T^\alpha(\eta) &\approx \Delta(k) \gamma_\alpha \frac{p_1^+ \gamma^-}{2k^-(p_1^+ - i\epsilon)} A_T^\alpha(\eta) \\ &\approx -\Delta(k) \frac{\gamma^-}{Q\sqrt{2}} \gamma_\alpha A_T^\alpha(\eta). \end{aligned} \quad (24)$$

The (remaining) second term in Eq. (20) and the result of Eq. (24) give as $\mathcal{O}(1/Q)$ contribution in term 1,

$$\begin{aligned} [\text{term 1.2}] &= \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) dp_1^+ d^2 p_{1T} \\ &\quad \times \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} \frac{d\eta^- d^2 \eta_T}{(2\pi)^3} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\ &\quad \times \frac{1}{Q\sqrt{2}} \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma^- \\ &\quad \times \left(\gamma_\alpha A_T^\alpha(\eta) - \frac{\not{p}_{1T} A^+(\eta)}{(p_1^+ - i\epsilon)} \right) \gamma_\nu \psi(\xi) | P, S \rangle_{\xi^+ = \eta^+ = 0} \\ &= \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) d^2 p_{1T} \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \\ &\quad \times \frac{1}{Q\sqrt{2}} \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma^- \gamma_\alpha \\ &\quad \times \left(A_T^\alpha(\xi) - \int_{\infty}^{\xi^-} d\eta^- \partial_T^\alpha A^+(\eta^-) \right) \gamma_\nu \psi(\xi) | P, S \rangle_{\xi^+ = 0}. \end{aligned} \quad (25)$$

Including in addition all diagrams with longitudinal A^+ gluon fields, all colored fields become linked along the minus-direction, with the same link directions for Φ and Φ_A . Using the relation between $G^{+\alpha}$ and A_T^α , outlined in Appendix A including all minus links, we find (suppressing the links U^-)

$$A_T^\alpha(\xi) - \int_{\infty}^{\xi^-} d\eta^- (\partial_T^\alpha A^+(\eta^-)) = \int_{\infty}^{\xi^-} d\eta^- G^{+\alpha}(\eta) + A_T^\alpha(\infty^-), \quad (26)$$

with the points $\eta^- = (\eta^-, \xi^+, \xi_T)$ and $\infty^- = (\infty, \xi^+, \xi_T)$. The part of term 1.2 containing $G^{+\alpha}$ is

$$\begin{aligned}
 [\text{term 1.2a}] = & \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \\
 & \times \frac{1}{Q\sqrt{2}} \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma^- \gamma_\alpha \\
 & \times \int_{\infty}^{\xi^-} d\eta^- G^{+\alpha}(\eta) \gamma_\nu \psi(\xi) | P, S \rangle \Big|_{\eta^+ = \xi^+ = 0; \eta_T = \xi_T}. \quad (27)
 \end{aligned}$$

Upon integrating the hadron tensor over transverse momenta q_T , the above convolution factorizes and produces a color-gauge-invariant $\mathcal{O}(1/Q)$ term,

$$\begin{aligned}
 & \int d^2 q_T [\text{term 1.2a}] \\
 & = \text{Tr} \left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \int_{-\infty}^{\infty} dp_1^+ \frac{i}{p_1^+ - i\epsilon} \Phi_G^\alpha(p^+, p^+ - p_1^+) \gamma_\mu \Delta(z) \right), \quad (28)
 \end{aligned}$$

where (including link operators)

$$\begin{aligned}
 \Phi_{Gij}^\alpha(p^+, p^+ - p_1^+) = & \int \frac{d\xi^-}{2\pi} \frac{d\eta^-}{2\pi} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\
 & \times \langle P, S | \bar{\psi}_j(0) U_{[0, \eta]}^- g G^{+\alpha}(\eta) U_{[\eta, \xi]}^- \psi_i(\xi) | P, S \rangle \Big|_{LC}, \quad (29)
 \end{aligned}$$

also denoted $\Phi_G^\alpha(x, x - x_1)$ with $x_1 = p_1^+/P^+$, and with LC denoting $\{\xi^+ = \eta^+ = \xi_T = \eta_T = 0\}$.

We are left with a boundary term containing $A_T(\infty^-)$, which needs special care. The argument of the transverse field in the boundary term is fixed by the link direction in U^- . The consequence is that the η^- dependence disappears. We note that the integration over η^- thus can simply be performed, showing that one deals with a matrix element that is proportional to $\delta(p_1^+)$ in momentum space [26],

$$\begin{aligned}
 \delta(p_1^+) \Phi_{A(\infty)ij}^\alpha(p, p - p_1) \equiv & \int \frac{d^4 \xi}{(2\pi)^4} \frac{d^4 \eta}{(2\pi)^4} e^{ip \cdot \xi} e^{ip_1 \cdot (\eta - \xi)} \\
 & \times \langle P, S | \bar{\psi}_j(0) g A_T^\alpha(\infty, \eta^+, \eta_T) \psi_i(\xi) | P, S \rangle. \quad (30)
 \end{aligned}$$

Because $\delta(p_1^+) \sim 1/Q$ one finds that $\int dp^- dp_1^- \Phi_{A(\infty)}^\alpha \sim \gamma^+$, i.e., it is not suppressed. This means we have to revisit the approximations made to the fermion propagator for the boundary term. Going back to the starting point in Eq. (18) we obtain for the boundary

contribution after integration over η^- ,

$$\begin{aligned}
 [\text{term 1.2b}] = & - \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) d p_1^+ d^2 p_{1T} \\
 & \times \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} \frac{d^2 \eta_T}{(2\pi)^2} e^{ip \cdot \xi} e^{ip_{1T} \cdot (\eta_T - \xi_T)} \delta(p_1^+) \\
 & \times \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma_\alpha \\
 & \times \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \gamma_\nu g A_T^\alpha(\infty, \eta^+, \eta_T) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \eta^+ = 0}.
 \end{aligned} \tag{31}$$

After a substitution for A_T^α ,

$$g A_T^\alpha(\eta) = i \partial_\eta^\alpha (-ig) \int_{\infty_T}^{\eta_T} d\zeta_T \cdot A_T(\eta^-, \eta^+, \vec{\zeta}_T), \tag{32}$$

we do a partial integration. In the matrix element, we then encounter the following part, which we need to consider in the soft gluon limit ($p_1^+ = 0$) in which the denominator of the quark propagator can no longer be approximated as in Eqs. (20) or (24). Realizing that $p_1^+ \sim Q$ and $p_1^- \sim 1/Q$ we obtain

$$\begin{aligned}
 & \Delta(k) \not{p}_{1T} \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \delta(p_1^+) \\
 & \approx \Delta(k) \not{p}_1 \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \delta(p_1^+) \\
 & = \Delta(k) (-\not{k} + \not{p}_1 + m) \frac{\not{k} - \not{p}_1 + m}{(k - p_1)^2 - m^2 + i\epsilon} \delta(p_1^+) \\
 & = -\Delta(k) \delta(p_1^+).
 \end{aligned} \tag{33}$$

The result after integration over η_T , p_1^+ and p_{1T} is a term

$$\begin{aligned}
 [\text{term 1.2b}] = & \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \\
 & \times \langle P, S | \bar{\psi}(0) \gamma_\mu \Delta(z, k_T) \gamma_\nu (-ig) \\
 & \times \int_{\infty_T}^{\xi_T} d\zeta_T \cdot A_T(\infty, 0, \zeta_T) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = 0},
 \end{aligned} \tag{34}$$

which gives precisely the first term of the transverse link that is needed to modify Eq. (22) into a fully color-gauge-invariant matrix element. Note that we did not assume a specific pure gauge expression for the A_T field at $\zeta^- = \infty$. Furthermore, we did not neglect the quark masses in the quark propagator. For N transverse gluons we have to work out the boundary terms with more transverse gluons, for which we need the following relation that

Fig. 4. Link structure for $\Phi^{[+]}(x, p_T)$ (a) and (b) and for $\Phi(x)$ (c) after integration over transverse momenta.

also holds for nonabelian fields:

$$\begin{aligned}
 & \left((-ig)^M \int_{\infty}^{\eta} d\zeta_{1T} \cdot A_T(\zeta_1) \cdots \int_{\zeta_{M-1}}^{\eta} d\zeta_{MT} \cdot A_T(\zeta_M) \right) g A_T^{\alpha}(\eta) \\
 &= i \partial_{\eta}^{\alpha} \left((-ig)^{M+1} \int_{\infty}^{\eta} d\zeta_{1T} \cdot A_T(\zeta_1) \cdots \int_{\zeta_M}^{\eta} d\zeta_{M+1T} \cdot A_T(\zeta_{M+1}) \right). \quad (35)
 \end{aligned}$$

This indeed produces nicely the nested integrations needed in the path-ordered exponential and we find that the terms 1.1 and 1.2b, are taken care of by using in the $\mathcal{O}(g^0)$ part of Eq. (8) the color-gauge-invariant matrix element [14]

$$\begin{aligned}
 \Phi_{ij}^{[+]}(x, p_T) &= \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{ip \cdot \xi} \\
 &\times \langle P, S | \bar{\psi}_j(0) U_{[0, \infty]}^{-} U_{[0_T, \infty_T]}^T U_{[\infty_T, \xi_T]}^T U_{[\infty, \xi]}^{-} \psi_i(\xi) | P, S \rangle \Big|_{\xi^+ = 0}, \quad (36)
 \end{aligned}$$

with

$$\begin{aligned}
 U_{[a, \xi]}^T &= \mathcal{P} \exp \left(-ig \int d\zeta_T \cdot A_T(\zeta) \right) \Big|_{\zeta^+ = \xi^+ = a^+, \zeta^- = \xi^- = a^-} \\
 &= \sum_{N=0}^{\infty} (-ig)^N \int_{a_T}^{\xi_T} d\zeta_{1T} \cdot A_T(\zeta_1) \cdots \int_{\zeta_{N-1}}^{\xi} d\zeta_{NT} \cdot A_T(\zeta_N) \Big|_{\zeta_i^+ = \xi^+ = a^+, \zeta_i^- = \xi^- = a^-}. \quad (37)
 \end{aligned}$$

The $U_{[0_T, \infty_T]}^T$ -link of course comes from diagrams with transverse gluons like in Fig. 2(c) and higher orders. The link structure for the soft part describing the distribution of quarks in a hadron probed by a spacelike photon is illustrated in Fig. 4. The direction of the link, running to $+\infty$ along the minus direction is indicated via the superscript $[+]$ in Eq. (36). In other processes one will find that the link can also run in the opposite direction to $-\infty$ along the minus direction. This will be indicated with a superscript $[-]$.

Eq. (36) is an important expression, since in its full generality it allows for certain distribution functions, usually referred to as T-odd functions, that would be absent in case one ignores the gauge links (cf., e.g., [3,4,6]). Without the transverse gauge links, it may therefore seem that a choice of $A^+ = 0$ gauge would demonstrate the absence of such

T-odd functions. In the derivation of Eq. (36) no gauge was assumed (one can actually arrive at this result by first considering the $A^- = 0$ gauge as done in Ref. [12]), hence it should not be viewed as one out of many ways to “gauge-invariantize” the matrix element [27]. Up to the order we consider here, the result is derived rather than assumed.

We note that in leading results the color-gauge-invariant object $\Phi^{[+]}(x, p_T)$ contracted with γ^+ still is a semi-positive definite matrix in Dirac space, which is the basis for deriving positivity conditions, such as the Soffer bound [28] and many more [29]. Hence, we disagree with the statement “Structure functions are not parton probabilities” by Brodsky et al. [30]. To be precise, the distribution functions containing the transverse link are still probability densities.

As mentioned already, when one considers the q_T -integrated results one obtains the lightcone quark–quark correlations with the link in Eq. (17). The transverse link does not affect that result and one has $\Phi^{[+]}(x) = \Phi^{[-]}(x) = \Phi(x)$ (see Fig. 4(c)). If one looks at azimuthal asymmetries or weighted cross sections one needs to consider matrix elements weighted with transverse momentum. In those cases one explicitly needs to take into account the transverse part of the link. We define *transverse moments* $\Phi_\partial^{[\pm]\alpha}(x)$,

$$\Phi_\partial^{[\pm]\alpha}(x) \equiv \int d^2 p_T p_T^\alpha \Phi^{[\pm]}(x, p_T), \quad (38)$$

that in a straightforward way can be related to color-gauge-invariant quark–quark–gluon matrix elements Φ_G^α and Φ_D^α , the latter involving the covariant derivative,

$$\begin{aligned} & (\Phi_\partial^{[\pm]\alpha})_{ij}(x) \\ &= \int d^2 p_T \int \frac{d\xi^- d^2 \xi_\perp}{(2\pi)^3} e^{ip \cdot \xi} \\ & \quad \times \langle P, S | \bar{\psi}_j(0) U_{[0, \pm\infty]}^- U_{[0T, \pm\infty T]}^T i \partial_\xi^\alpha U_{[\pm\infty T, \xi T]}^T U_{[\pm\infty, \xi]}^- \psi_i(\xi) | P, S \rangle_{\xi^+ = 0} \\ &= \int \frac{d\xi^-}{(2\pi)} e^{ip \cdot \xi} \left\{ \langle P, S | \bar{\psi}_j(0) U_{[0, \xi]}^- i D_T^\alpha \psi_i(\xi) | P, S \rangle_{LC} \right. \\ & \quad \left. - \langle P, S | \bar{\psi}_j(0) U_{[0, \pm\infty]}^- \int_{\pm\infty}^{\xi^-} d\eta^- U_{[\pm\infty, \eta]}^- g G^{+\alpha}(\eta) U_{[\eta, \xi]}^- \psi_i(\xi) | P, S \rangle_{LC} \right\}, \quad (39) \end{aligned}$$

or

$$\Phi_\partial^{[\pm]\alpha}(x) = \Phi_D^\alpha(x) - \int_{-\infty}^{\infty} dp_1^+ \frac{i}{p_1^+ \mp i\epsilon} \Phi_G^\alpha(p^+, p^+ - p_1^+), \quad (40)$$

where

$$\begin{aligned} \Phi_{Dij}^\alpha(x) &= \int dp_1^+ \Phi_{Dij}^\alpha(p^+, p^+ - p_1^+) \\ &= \int \frac{d\xi^-}{2\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) U_{[0, \xi]}^- i D^\alpha(\xi) \psi_i(\xi) | P, S \rangle_{LC}. \quad (41) \end{aligned}$$

The important observation we want to make here is that the difference between correlation functions with links running to $\pm\infty$, respectively, is related to a quark–gluon correlator,

$$\Phi_{\partial}^{[+]\alpha}(x) - \Phi_{\partial}^{[-]\alpha}(x) = 2\pi\Phi_G^{\alpha}(x, x), \quad (42)$$

the latter being given the name *gluonic-pole* matrix element since it corresponds to the soft-gluon point $p_1^+ = 0$. Its consequences have been studied for several processes [26,31–35] and it is viewed as one of the possible mechanism to generate single spin asymmetries. We will comment on this further below, but already mention that the above relation between $\Phi_{\partial}^{[\pm]\alpha}(x)$ and $\Phi_G^{\alpha}(x, x)$ implies that the Sivers effect [3] is directly related to the Qiu–Sterman mechanism (the gluonic-pole matrix element), i.e., if one is nonzero, then the other also is. We will make this relation more specific below.

We further define

$$\Phi_{\partial}^{\alpha}(x) \equiv \frac{1}{2}(\Phi_{\partial}^{[+]\alpha}(x) + \Phi_{\partial}^{[-]\alpha}(x)), \quad (43)$$

$$\tilde{\Phi}_A^{\alpha}(x) = \text{PV} \int dx_1 \frac{1}{x_1} \Phi_G^{\alpha}(x, x - x_1), \quad (44)$$

where we use $\tilde{\Phi}_A^{\alpha}(x)$ to distinguish the function from the non-gauge-invariant $\Phi_A^{\alpha}(x)$. These definitions imply

$$\Phi_{\partial}^{[\pm]\alpha}(x) = \Phi_{\partial}^{\alpha}(x) \pm \pi\Phi_G^{\alpha}(x, x), \quad (45)$$

$$\Phi_{\partial}^{\alpha}(x) = \Phi_D^{\alpha}(x) - \tilde{\Phi}_A^{\alpha}(x). \quad (46)$$

The relations in Eqs. (44)–(46) are relations connecting color-gauge-invariant quantities. We will return to the above functions and their properties in Section 7.

We end this section by giving the remaining contributions in the q_T -integrated hadron tensor at order $1/Q$. We give a systematic summary of the SIDIS hadron tensor for several cases in the next section. We can use Eq. (40) to rewrite the twist-3 contribution in Eq. (28) obtained after q_T -integration into the form

$$\int d^2q_T [\text{term 1.2a}] = \text{Tr} \left(\frac{\gamma^- \gamma_{\alpha}}{Q\sqrt{2}} \gamma_{\nu} [\Phi_D^{\alpha}(x) - \Phi_{\partial}^{[+]\alpha}(x)] \gamma_{\mu} \Delta(z) \right). \quad (47)$$

The transverse gluons in the second term of Eq. (18) (diagram in Fig. 2(c)) produces besides the transverse link $U_{[0_T, \infty_T]}^T$, already absorbed into Eq. (36), also a twist-three piece. For this one needs matrix elements with interchanged arguments such as $\Phi_A(p - p_1, p)$. The resulting twist-3 term after integration over transverse momenta is

$$\begin{aligned} \int d^2q_T [\text{term 2.2a}] &= \text{Tr} \left(\gamma_{\nu} \int_{-\infty}^{\infty} dp_1^+ \frac{-i}{p_1^+ + i\epsilon} \Phi_G^{\alpha}(p^+ - p_1^+, p^+) \gamma_{\mu} \frac{\gamma_{\alpha} \gamma^-}{Q\sqrt{2}} \Delta(z) \right) \\ &= \text{Tr} \left(\gamma_{\nu} [\gamma_0 \Phi_D^{\alpha\dagger}(x) \gamma_0 - \Phi_{\partial}^{[+]\alpha}(x)] \gamma_{\mu} \frac{\gamma_{\alpha} \gamma^-}{Q\sqrt{2}} \Delta(z) \right), \end{aligned} \quad (48)$$

where we have used that $\Phi_{\partial}^{[+]\alpha}(x) = \gamma_0 \Phi_{\partial}^{[+]\alpha\dagger}(x) \gamma_0$. The third and fourth $\mathcal{O}(g)$ terms in Eq. (18) (diagrams in Fig. 2(d) and (e)) need the (direct and conjugate) quark propagators

Fig. 5. Link structure for $\Phi(x, p_T)$ (a) and (b) and $\Delta(z, k_T)$ (c) and (d).

(with $p^+ \approx -q^+ = Q/\sqrt{2}$)

$$\frac{\not{p} - \not{k}_1 + m}{(p - k_1)^2 - m^2 \pm i\epsilon} \approx \frac{(\not{p} + m) - \not{p}_- k_1^- - \not{k}_{1T}}{-k_1^- Q\sqrt{2} + (p_T - k_{1T})^2 - m^2 \pm i\epsilon}. \quad (49)$$

The calculation yields the links $U_{[-\infty, \xi]}^+$ and $U_{[0, -\infty]}^+$ running along the plus-direction in the matrix elements $\Delta(z, k_T)$ in Eq. (23) for the fragmentation part. Including the transverse gluons we get the fully color-gauge-invariant matrix element indicated as the spacelike fragmentation $\Delta^{[-]}(z, k_T)$ with the link as indicated in Fig. 5. Furthermore one obtains twist-3 contributions containing $\Delta_G^\alpha(k - k_1, k)$ and $\Delta_G^\alpha(k, k - k_1)$, which after integration over q_T yield

$$\begin{aligned} \int d^2 q_T [\text{term 3.2a}] &= \text{Tr} \left(\gamma_\mu \int_{-\infty}^{\infty} dk_1^- \frac{-i}{k_1^- - i\epsilon} \Delta_G^\alpha(k^- - k_1^-, k^-) \gamma_\nu \frac{\gamma_\alpha \gamma^+}{Q\sqrt{2}} \Phi(x) \right) \\ &= \text{Tr} \left(\gamma_\mu [\gamma_0 \Delta_D^{\alpha\dagger}(z) \gamma_0 - \Delta_\partial^{[-]\alpha}(z)] \gamma_\nu \frac{\gamma_\alpha \gamma^+}{Q\sqrt{2}} \Phi(x) \right), \end{aligned} \quad (50)$$

$$\begin{aligned} \int d^2 q_T [\text{term 4.2a}] &= \text{Tr} \left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \int_{-\infty}^{\infty} dk_1^- \frac{i}{k_1^- + i\epsilon} \Delta_G^\alpha(k^-, k^- - k_1^-) \gamma_\nu \Phi(x) \right) \\ &= \text{Tr} \left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu [\Delta_D^\alpha(z) - \Delta_\partial^{[-]\alpha}(z)] \gamma_\nu \Phi(x) \right). \end{aligned} \quad (51)$$

In deriving these results we stepped over some subtleties involving the inclusion of diagrams with crossed gluon lines and the use of the full QCD equations of motion, but for this we refer to Ref. [12].

4. SIDIS and DIS cross sections

The basic expression for $\mathcal{W}_{\mu\nu}(q; P, S; P_h, S_h)$ contains a convolution of the transverse-momentum-dependent functions. We have seen that upon integration over q_T ,

$$\int d^2 q_T d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \cdots = \int d^2 p_T d^2 k_T \cdots, \quad (52)$$

the integral can be deconvoluted. This is also true for azimuthal asymmetries constructed by weighting with q_T^α ,

$$\int d^2 q_T q_T^\alpha d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \cdots = \int d^2 p_T d^2 k_T (k_T^\alpha - p_T^\alpha) \cdots. \quad (53)$$

If one calculates the $\mathcal{O}(1/Q)$ result, one has to be careful, however. One cannot simply perform the integration over q_T in the hadron tensor, since the lepton tensor involves q . To proceed, one starts with a (Cartesian) set of vectors, starting with q defining in SIDIS a spacelike direction while the other external vectors are used to define orthogonal directions. In particular, it is convenient to start with q and $\tilde{P} = P - (P \cdot q/q^2)q$,

$$\hat{z}^\mu \equiv -\frac{q^\mu}{Q} = -\hat{q}^\mu, \quad (54)$$

$$\hat{t}^\mu = \frac{\tilde{P}^\mu}{\sqrt{\tilde{P}^2}} \approx \frac{q^\mu + 2x_B P^\mu}{Q}, \quad (55)$$

with $\hat{z}^2 = -1$ and $\hat{t}^2 = 1$. These two vectors can be used both for inclusive and semi-inclusive leptonproduction. The lepton tensor can be expressed in the vectors $\hat{t}$ and $\hat{z}$ and a perpendicular vector (equal for the lepton in initial and final state) which defines the azimuthal angle of the lepton scattering plane. From the Cartesian vectors two new lightlike vectors $n'_\pm = (\hat{t} \pm \hat{z})/\sqrt{2} = n'_\pm(P, q)$ can be constructed as well as a perpendicular tensor $g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu \hat{q}^\nu - \hat{t}^\mu \hat{t}^\nu = g^{\mu\nu} - n_+^{(\mu} n_-^{\nu)}$. Since the lightlike directions $n'_\pm$ are determined by P and q , instead of P and P_h , the momentum of the produced hadron P_h will have in general a nonvanishing perpendicular component enabling us to define a vector $X^\mu \equiv -P_{h\perp}^\mu/z_h$, which defines the azimuthal angle of the hadron production plane.

It is straightforward to see that up to $\mathcal{O}(1/Q^2)$ corrections, the previously defined set $\{n_+, n_-, q_T\}$ is related to the set $\{\hat{t}, \hat{z}, X\}$ or $\{n'_+, n'_-, X\}$ via

$$n_-^\mu \approx n'^\mu_- - 2 \frac{X^\mu}{Q\sqrt{2}}, \quad (56)$$

$$n_+^\mu \approx n'^\mu_+, \quad (57)$$

$$q_T^\mu \approx X^\mu - \sqrt{2} \frac{Q_T^2}{Q} n_+^\mu, \quad (58)$$

and $X^2 \approx -Q_T^2$. We note that the leptonic tensor is independent of X . The $1/Q$ term appearing on the righthandside of Eq. (58) is irrelevant in our calculations. Hence for experimental cross sections up to that order we can use for integrated and weighted cross sections the replacements $\int d^2 X \cdots \rightarrow \int d^2 q_T \cdots$ and $\int d^2 X X^\alpha \cdots \rightarrow \int d^2 q_T q_T^\alpha \cdots$. We

now consider separately integrated SIDIS, DIS and azimuthal asymmetries in SIDIS. We will also consider a few special cases. If one measures in the final state the jet-direction, this can be considered as a measurement of p_T , i.e., $q_T = -p_T$. This is referred to as JET SIDIS.

4.1. Integrated SIDIS cross section at leading order

We have seen that inclusion of appropriate quark–gluon matrix elements make the $\mathcal{O}(g^0)$ result in Eq. (8) color-gauge-invariant,

$$2M\mathcal{W}_{\mu\nu}^{(0)}(q; P, S; P_h, S_h) = \int d^2 p_T d^2 k_T \delta^2(p_T + q_T - k_T) \text{Tr}(\Phi^{[+]}(x, p_T) \gamma_\mu \Delta^{[-]}(z, k_T) \gamma_\nu). \quad (59)$$

At leading order the integration over transverse momenta of the produced hadrons simply can be performed and one obtains the basic result

$$\int d^2 X 2M\mathcal{W}_{\mu\nu}^{(0)}(q; P, S; P_h, S_h) = \text{Tr}(\Phi(x) \gamma_\mu \Delta(z) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} + \mathcal{O}(1/Q). \quad (60)$$

4.2. Azimuthal asymmetries in SIDIS at leading order

We consider here cross sections obtained after integration over X and explicit weighting with X^α . In practice, this means measurement of the azimuthal angle of the produced hadron and compare it with other azimuthal angles, such as that of the lepton scattering plane, the (transverse) spin of the target hadron or the (transverse) spin of the produced hadron. For our purposes it implies calculation of $\int d^2 X X^\alpha 2M\mathcal{W}_{\mu\nu}$, which at leading order gives

$$\begin{aligned} & \int d^2 X X^\alpha 2M\mathcal{W}_{\mu\nu}^{(0)}(q; P, S; P_h, S_h) \\ &= \text{Tr}(\Phi(x) \gamma_\mu \Delta_\partial^{[-]\alpha}(z) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} - \text{Tr}(\Phi_\partial^{[+]\alpha}(x) \gamma_\mu \Delta(z) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} \\ &+ \mathcal{O}(1/Q). \end{aligned} \quad (61)$$

In these cross sections one finds for instance the Collins and Sivers effects [5,6,36,37].

4.3. Integrated SIDIS cross section at $\mathcal{O}(1/Q)$

As outlined one must be careful in integrating over transverse momenta. At $\mathcal{O}(1/Q)$ the differences between using $n_\pm(P, P_h)$ or $n'_\pm(P, q)$ matter. In particular, the correlator $\Delta \propto \not{n}_-$ will lead to terms proportional to $\not{q}_T/Q\sqrt{2}$. To find the $\not{n}_-$ dependence in a Dirac space correlator, we use the projectors $P_\pm = \gamma^\mp \gamma^\pm/2 = \not{n}_\pm \not{n}_\mp/2$. We use that the leading term in Δ satisfies $P_- \Delta = \Delta P_+ = P_- \Delta P_+$ to write

$$\begin{aligned} \Delta(z, k_T) &= \frac{1}{4} (\not{n}_- \not{n}_+ \Delta(z, k_T) \not{n}_+ \not{n}_-) \\ &\approx \Delta(z, k_T) \Big|_{n_\pm \rightarrow n'_\pm} - \frac{1}{Q\sqrt{2}} (\not{q}_T \not{n}_+ \Delta(z, k_T) + \Delta(z, k_T) \not{n}_+ \not{q}_T), \end{aligned} \quad (62)$$

and we obtain the $1/Q$ contribution coming from Eq. (59),

$$\begin{aligned}
& \int d^2 X \, 2M \mathcal{W}_{\mu\nu}^{(0)}(q; P, S; P_h, S_h) \\
&= \mathcal{O}(1) \text{ result (Eq. (60))} \\
&= \frac{1}{Q\sqrt{2}} \int d^2 q_T \int d^2 p_T \, d^2 k_T \, \delta^2(p_T + q_T - k_T) \\
&\quad \times \left\{ \text{Tr}(\Phi^{[+]}(x, p_T) \gamma_\mu \not{q}_T \not{k}_+ \Delta^{[-]}(z, k_T) \gamma_\nu) \right. \\
&\quad \left. + \text{Tr}(\Phi^{[+]}(x, p_T) \gamma_\mu \Delta^{[-]}(z, k_T) \not{k}_+ \not{q}_T \gamma_\nu) \right\} \\
&= \mathcal{O}(1) \text{ result (Eq. (60))} \\
&\quad + \text{Tr} \left(\Phi_\partial^{[+]\alpha}(x) \gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta(z) \gamma_\nu \right) - \text{Tr} \left(\Phi(x) \gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta_\partial^{[-]\alpha}(z) \gamma_\nu \right) \\
&\quad + \text{Tr} \left(\Phi_\partial^{[+]\alpha}(x) \gamma_\mu \Delta(z) \frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \right) - \text{Tr} \left(\Phi(x) \gamma_\mu \Delta_\partial^{[-]\alpha}(z) \frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \right). \quad (63)
\end{aligned}$$

Including these $1/Q$ -terms and the four contributions from quark–gluon correlators with transverse gluons, one obtains the full integrated SIDIS cross section up to $\mathcal{O}(1/Q)$,

$$\begin{aligned}
& \int d^2 X \, 2M \mathcal{W}_{\mu\nu}^{(0+1)}(q; P, S; P_h, S_h) \\
&= \mathcal{O}(1) \text{ result (Eq. (60))} \\
&\quad + \text{Tr} \left(\Phi_\partial^{[+]\alpha}(x) \gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta(z) \gamma_\nu \right) - \text{Tr} \left(\Phi(x) \gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta_\partial^{[-]\alpha}(z) \gamma_\nu \right) \\
&\quad + \text{Tr} \left(\Phi_\partial^{[+]\alpha}(x) \gamma_\mu \Delta(z) \frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \right) - \text{Tr} \left(\Phi(x) \gamma_\mu \Delta_\partial^{[-]\alpha}(z) \frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \right) \\
&\quad + \text{Tr} \left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu [\Phi_D^\alpha(x) - \Phi_\partial^{[+]\alpha}(x)] \gamma_\mu \Delta(z) \right) \\
&\quad + \text{Tr} \left(\gamma_\nu [\gamma_0 \Phi_D^{\alpha\dagger}(x) \gamma_0 - \Phi_\partial^{[+]\alpha}(x)] \gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta(z) \right) \\
&\quad + \text{Tr} \left(\gamma_\mu [\gamma_0 \Delta_D^{\alpha\dagger}(z) \gamma_0 - \Delta_\partial^{[-]\alpha}(z)] \gamma_\nu \frac{\gamma_\alpha \gamma^+}{Q\sqrt{2}} \Phi(x) \right) \\
&\quad + \text{Tr} \left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu [\Delta_D^\alpha(z) - \Delta_\partial^{[-]\alpha}(z)] \gamma_\nu \Phi(x) \right) \\
&= \mathcal{O}(1) \text{ result (Eq. (60))} \\
&\quad + \text{Tr} \left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \Phi_D^\alpha(x) \gamma_\mu \Delta(z) \right) + \text{Tr} \left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \Delta_D^\alpha(z) \gamma_\nu \Phi(x) \right) \\
&\quad - \text{Tr} \left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \Delta_\partial^{[-]\alpha}(z) \gamma_\nu \Phi(x) \right) - \text{Tr} \left(\gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta_\partial^{[-]\alpha}(z) \gamma_\nu \Phi(x) \right) \\
&\quad + (\mu \leftrightarrow \nu)^*, \quad (64)
\end{aligned}$$

where the hermiticity properties of the various matrix elements have been used (see Section 7). We note that the $1/Q$ part in Eq. (63) is by itself not electromagnetically gauge-invariant, but together with the parts arising from quark–gluon matrix elements, leading to the result in Eq. (64), it is gauge-invariant.

4.4. Integrated DIS cross section at $\mathcal{O}(1/Q)$

The hadron tensor for this case is obtained by integrating the SIDIS result over z and using for the fragmentation part the free (massless) quark $\rightarrow$ quark result, $\Delta(z) = \not{n} \delta(1-z)$ and $\Delta_{\partial}^{[-]\alpha}(z) = \Delta_D^{\alpha}(z) = 0$. This gives the well-known result [17],

$$\begin{aligned} 2M\mathcal{W}_{\mu\nu}(q; P, S) &= \text{Tr}(\Phi(x)\gamma_{\mu}\gamma^{+}\gamma_{\nu}) \\ &+ \text{Tr}\left(\frac{\gamma^{-}\gamma_{\alpha}}{Q\sqrt{2}}\gamma_{\nu}\Phi_D^{\alpha}(x)\gamma_{\mu}\gamma^{+}\right) + \text{Tr}\left(\frac{\gamma^{-}\gamma_{\alpha}}{Q\sqrt{2}}\gamma_{\mu}\Phi_D^{\alpha}(x)\gamma_{\nu}\gamma^{+}\right)^{*}. \end{aligned} \quad (65)$$

4.5. Azimuthal asymmetries in JET SIDIS at leading order

Azimuthal asymmetries in JET DIS are obtained at measured $q_T = -P_{\text{jet}\perp}$ and with in addition still fixed P and q . We simply replace $\Delta(z, k_T) = \not{n} \delta(1-z)\delta^2(k_T)$. Hence, we start with

$$2M\mathcal{W}_{\mu\nu}^{(0)}(q; P, S; q_T) = \text{Tr}(\Phi^{[+]}(x, -q_T)\gamma_{\mu}\gamma^{+}\gamma_{\nu}). \quad (66)$$

In this situation one will have to be careful with Sudakov effects [16], but the following weighted azimuthal asymmetry is free of these effects,

$$\int d^2q_T q_T^{\alpha} 2M\mathcal{W}_{\mu\nu}^{(0)}(q; P, S; q_T) = -\text{Tr}(\Phi_{\partial}^{[+]\alpha}(x)\gamma_{\mu}\gamma^{+}\gamma_{\nu}). \quad (67)$$

This result is also a direct consequence of Eq. (61), taking for $\Delta^{[-]}(z, k_T)$ the quark $\rightarrow$ quark limit.

4.6. Other subleading cross sections

Azimuthal asymmetries at $\mathcal{O}(1/Q)$, azimuthal asymmetries involving higher weighting than with one power of the momentum X^{α} or SIDIS cross sections at $\mathcal{O}(1/Q^2)$, require considerably more theoretical efforts than the one presented above. Moreover, it might be impossible to unambiguously disentangle hadrons within a ‘jet’ from hadrons in other ‘jets’, since the ‘separation’ of jets is merely an $\mathcal{O}(Q^2)$ effect, meaning $P_{\text{jet}1} \cdot P_{\text{jet}2} \propto Q^2$. This also implies that a factorization proof most likely cannot be given, just like the failure of factorization in unpolarized processes at $1/Q^4$ [38]. As is well known, these difficulties do not appear for the inclusive DIS cross section involving just one (target) hadron which allows for a rigorous treatment at any order in powers of $1/Q$.

Fig. 6. Quark-quark (a) and one of the quark-quark-gluon (b) correlators in tree-level diagrams for Drell-Yan scattering.

5. The Drell-Yan cross sections

For Drell-Yan, one has a similar treatment as for leptonproduction. The calculation involves now two soft distribution parts and annihilation of a quark-antiquark pair into a gauge boson (we will only discuss the vector coupling here). The handbag diagram is given in Fig. 6(a) and an example of a diagram with an additional gluon in Fig. 6(b).

A full calculation at tree level including quark-gluon matrix elements as discussed for leptonproduction gives in this case

$$\begin{aligned}
 2M\mathcal{W}_{\mu\nu}(q; P_A, S_A; P_B, S_B) &= \int d^4p d^4k \delta^4(p+k-q) \left\{ \text{Tr}(\Phi(p)\gamma_\mu \bar{\Phi}(k)\gamma_\nu) \right. \\
 &\quad - \int d^4p_1 \text{Tr} \left(\gamma_\alpha \frac{-\not{k} - \not{p}_1 + m}{(k+p_1)^2 - m^2 + i\epsilon} \gamma_\nu \Phi_A^\alpha(p, p-p_1) \gamma_\mu \bar{\Phi}(k) \right) \\
 &\quad - \int d^4p_1 \text{Tr} \left(\gamma_\mu \frac{-\not{k} - \not{p}_1 + m}{(k+p_1)^2 - m^2 - i\epsilon} \gamma_\alpha \bar{\Phi}(k) \gamma_\nu \Phi_A^\alpha(p-p_1, p) \right) \\
 &\quad - \int d^4k_1 \text{Tr} \left(\gamma_\nu \frac{\not{p} + \not{k}_1 + m}{(p+k_1)^2 - m^2 + i\epsilon} \gamma_\alpha \Phi(p) \gamma_\mu \bar{\Phi}_A^\alpha(k-k_1, k) \right) \\
 &\quad \left. - \int d^4k_1 \text{Tr} \left(\gamma_\alpha \frac{\not{p} + \not{k}_1 + m}{(p+k_1)^2 - m^2 - i\epsilon} \gamma_\mu \bar{\Phi}_A^\alpha(k, k-k_1) \gamma_\nu \Phi(p) \right) \right\} + \dots, \quad (68)
 \end{aligned}$$

where $\Phi(p)$ and $\Phi_A(p, p-p_1)$ are the same as in leptonproduction, but the role of Δ and Δ_A is taken over by

$$\bar{\Phi}_{ij}(k; P_B, S_B) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ik \cdot \xi} \langle P_B, S_B | \psi_i(\xi) \bar{\psi}_j(0) | P_B, S_B \rangle, \quad (69)$$

$$\begin{aligned}
 \bar{\Phi}_{Aij}^\alpha(k, k-k_1; P_B, S_B) &= \int \frac{d^4\xi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} e^{-ik \cdot \xi} e^{-ik_1 \cdot (\eta - \xi)} \\
 &\quad \times \langle P_B, S_B | \psi_i(\xi) g A^\alpha(\eta) \bar{\psi}_j(0) | P_B, S_B \rangle \quad (70)
 \end{aligned}$$

(note that this implies $\bar{\Phi}_g^\alpha(x, k_T) = -k^\alpha \bar{\Phi}(x, k_T)$).

The important difference between DY and SIDIS turns out to be the direction of the links. The result for the quark propagator in a quark–gluon diagram as in Fig. 2(b), but then in the case of DY with a timelike outgoing photon (cf. Fig. 6(a)) yields a propagator

$$\frac{-\not{k} - \not{p}_1 + m}{(k + p_1)^2 - m^2 + i\epsilon} \approx \frac{-(\not{k} - m) - \not{p}_+ p_1^+ - \not{p}_{1T}}{p_1^+ Q\sqrt{2} + (k_T + p_{1T})^2 - m^2 + i\epsilon} \quad (71)$$

(since $k^- \approx q^- = Q/\sqrt{2}$). The difference with Eq. (19) is the sign with which p_1^+ appears in the denominator. For the A^+ -gluons this produces a link running along the minus direction to $-\infty$, i.e., one finds transverse-momentum-dependent distribution functions $\Phi^{[-]}(x_A, p_T)$, where $x_A \approx p^+/P_A^+ \approx q^+/P_A^+$. Also in the antiquark matrix element the link runs to $-\infty$ (along the plus direction with our choice of lightlike vectors), i.e., the A^- gluons lead to the matrix element $\bar{\Phi}^{[-]}(x_B, k_T)$, where $x_B \approx k^-/P_B^- \approx q^-/P_B^-$.

As in leptonproduction the tree-level calculation is most conveniently done with lightlike directions $n_\pm(P_A, P_B)$ defined via the hadron momenta. In order to perform the transverse integration at the level of the hadron tensor one introduces [2,39,40] a Cartesian set $\{\hat{t}, \hat{z}, X\}$ starting with $\hat{t} = q/Q$. A symmetric choice for $\hat{z}$ is $z = (x_A P_A - x_B P_B)/Q$ that defines the Collins–Soper frame. Using $n'_\pm = (\hat{t} \pm \hat{z})/\sqrt{2}$, one has

$$n_-^\mu \approx n'^\mu_- - \frac{q_T^\mu}{Q\sqrt{2}}, \quad (72)$$

$$n_+^\mu \approx n'^\mu_+ - \frac{q_T^\mu}{Q\sqrt{2}}, \quad (73)$$

while as in leptonproduction one finds that for the calculation of the cross section the orthogonal direction X differs from q_T only by the irrelevant $\mathcal{O}(1/Q)$ terms multiplying $n'_\pm$, i.e., for our purposes $X \approx q_T \approx q - x_A P_A - x_B P_B$.

5.1. Integrated DY cross section at leading order

As indicated, the inclusion of appropriate quark–gluon matrix elements make the $\mathcal{O}(g^0)$ result in Eq. (8) color-gauge-invariant leading to

$$\begin{aligned} \mathcal{W}_{\mu\nu}^{(0)}(q; P_A, S_A; P_B, S_B) \\ = \int d^2 p_T d^2 k_T \delta^2(p_T + k_T - q_T) \text{Tr}(\Phi^{[-]}(x_A, p_T) \gamma_\mu \bar{\Phi}^{[-]}(x_B, k_T) \gamma_\nu). \end{aligned} \quad (74)$$

This is analogous to the expression employed by Ralston and Soper [1], except that now the correlation functions are fully color-gauge-invariant. At leading order the integration over transverse momenta of the produced hadrons simply can be done and one obtains the basic result for the DY process,

$$\begin{aligned} \int d^2 X \mathcal{W}_{\mu\nu}^{(0)}(q; P_A, S_A; P_B, S_B) \\ = \text{Tr}(\Phi(x_A) \gamma_\mu \bar{\Phi}(x_B) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} + \mathcal{O}(1/Q). \end{aligned} \quad (75)$$

5.2. Azimuthal asymmetries in DY at leading order

We consider here cross sections obtained after integration over X and explicit weighting with X^α . In practice this means measurement of the azimuthal angle of the leptonic plane with respect to that of the hadron plane or relative to (transverse) spin azimuthal angles. For our purposes it implies calculation of $\int d^2X X^\alpha \mathcal{W}_{\mu\nu}$, which at leading order gives

$$\begin{aligned} & \int d^2X X^\alpha \mathcal{W}_{\mu\nu}^{(0)}(q; P_A, S_A; P_B, S_B) \\ &= -\text{Tr}(\Phi(x_A)\gamma_\mu \bar{\Phi}_\partial^{[-]\alpha}(x_B)\gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} + \text{Tr}(\Phi_\partial^{[-]\alpha}(x_A)\gamma_\mu \bar{\Phi}(x_B)\gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} \\ &+ \mathcal{O}(1/Q). \end{aligned} \quad (76)$$

5.3. Integrated DY cross section at $\mathcal{O}(1/Q)$

As outlined, above, one must be careful in the DY process in integrating over transverse momenta. In this case both correlators $\Phi \propto \not{n}_+$ and $\bar{\Phi} \propto \not{n}_-$ will lead to terms proportional to $\not{q}_T/Q\sqrt{2}$. To find the $\not{n}_-$ dependence in a Dirac space correlator, we again use the projectors $P_\pm$. We now get

$$\begin{aligned} \Phi(x_A, p_T) &= \frac{1}{4}(\not{n}_+\not{n}_- \Phi(x_A, p_T)\not{n}_-\not{n}_+) \\ &\approx \Phi(x_A, p_T) \Big|_{n_\pm \rightarrow n'_\pm} \\ &\quad - \frac{1}{2Q\sqrt{2}}(\not{q}_T\not{n}_- \Phi(x_A, p_T) + \Phi(x_A, p_T)\not{n}_-\not{q}_T), \end{aligned} \quad (77)$$

$$\begin{aligned} \bar{\Phi}(x_B, k_T) &= \frac{1}{4}(\not{n}_-\not{n}_+ \bar{\Phi}(x_B, k_T)\not{n}_+\not{n}_-) \\ &\approx \bar{\Phi}(x_B, k_T) \Big|_{n_\pm \rightarrow n'_\pm} \\ &\quad - \frac{1}{2Q\sqrt{2}}(\not{q}_T\not{n}_+ \bar{\Phi}(x_B, k_T) + \bar{\Phi}(x_B, k_T)\not{n}_+\not{q}_T). \end{aligned} \quad (78)$$

Combining this $1/Q$ contribution coming from Eq. (74) with the parts from the quark–gluon diagrams one obtains

$$\begin{aligned} & \int d^2X \mathcal{W}_{\mu\nu}^{(0+1)}(q; P_A, S_A; P_B, S_B) \\ &= \mathcal{O}(1) \text{ result (Eq. (75))} \\ &\quad - \text{Tr}\left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \Phi_D^\alpha(x_A) \gamma_\mu \bar{\Phi}(x_B)\right) + \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \bar{\Phi}_D^\alpha(x_B) \gamma_\nu \Phi(x_A)\right) \\ &\quad + \frac{1}{2} \text{Tr}\left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \Phi_\partial^{[-]\alpha}(x_A) \gamma_\mu \bar{\Phi}(x_B)\right) - \frac{1}{2} \text{Tr}\left(\gamma_\nu \frac{\gamma_\alpha \gamma^+}{Q\sqrt{2}} \Phi_\partial^{[-]\alpha}(x_A) \gamma_\mu \bar{\Phi}(x_B)\right) \\ &\quad - \frac{1}{2} \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \bar{\Phi}_\partial^{[-]\alpha}(x_B) \gamma_\nu \Phi(x_A)\right) + \frac{1}{2} \text{Tr}\left(\gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \bar{\Phi}_\partial^{[-]\alpha}(x_B) \gamma_\nu \Phi(x_A)\right) \\ &\quad + (\mu \leftrightarrow \nu)^*, \end{aligned} \quad (79)$$

Fig. 7. Quark–quark (a) and one of the quark–quark–gluon (b) correlators in tree-level diagrams for back-to-back jet production in electron–positron annihilation.

where the hermiticity properties of the various matrix elements have been used (see Section 7).

6. Back-to-back jet production in electron–positron annihilation

Also for 2-particle inclusive electron–positron annihilation we have a quite similar procedure. The calculation involves two soft fragmentation parts and the creation of a quark–antiquark pair. We will discuss only the case of creation from a (timelike) photon. The handbag diagram is given in Fig. 7(a) and an example of a diagram involving an additional gluon in Fig. 7(b).

The calculation of this tensor in a diagrammatic expansion proceeds as in the case of lepton production and gives

$$\begin{aligned}
 & \mathcal{W}_{\mu\nu}(q; P_1, S_1; P_2, S_2) \\
 &= \int d^4p d^4k \delta^4(p+k-q) \left\{ \text{Tr}(\bar{\Delta}(p)\gamma_\mu \Delta(k)\gamma_\nu) \right. \\
 & \quad - \int d^4p_1 \text{Tr} \left(\gamma_\alpha \frac{\not{k} + \not{p}_1 + m}{(k+p_1)^2 - m^2 + i\epsilon} \gamma_\nu \bar{\Delta}_A^\alpha(p, p-p_1) \gamma_\mu \Delta(k) \right) \\
 & \quad - \int d^4p_1 \text{Tr} \left(\gamma_\mu \frac{\not{k} + \not{p}_1 + m}{(k+p_1)^2 - m^2 - i\epsilon} \gamma_\alpha \Delta(k) \gamma_\nu \bar{\Delta}_A^\alpha(p-p_1, p) \right) \\
 & \quad - \int d^4k_1 \text{Tr} \left(\gamma_\nu \frac{-\not{p} - \not{k}_1 + m}{(p+k_1)^2 - m^2 + i\epsilon} \gamma_\alpha \bar{\Delta}(p) \gamma_\mu \Delta_A^\alpha(k-k_1, k) \right) \\
 & \quad \left. - \int d^4k_1 \text{Tr} \left(\gamma_\alpha \frac{-\not{p} - \not{k}_1 + m}{(p+k_1)^2 - m^2 - i\epsilon} \gamma_\mu \Delta_A^\alpha(k, k-k_1) \gamma_\nu \bar{\Delta}(p) \right) \right\} + \dots, \quad (80)
 \end{aligned}$$

where

$$\bar{\Delta}_{ij}(p; P, S) = \sum_X \int \frac{d^4\xi}{(2\pi)^4} e^{-ip \cdot \xi} \langle 0 | \bar{\psi}_j(0) | P_2, X \rangle \langle P_2, X | \psi_i(\xi) | 0 \rangle, \quad (81)$$

$$\begin{aligned} \bar{\Delta}_{Aij}^\alpha(p, p - p_1; P, S) &= \sum_X \int \frac{d^4\xi}{(2\pi)^4} \frac{d^4\eta}{(2\pi)^4} e^{-ip \cdot \xi} e^{-ip_1 \cdot (\eta - \xi)} \\ &\times \langle 0 | \bar{\psi}_j(0) g A^\alpha(\eta) | P_2, X \rangle \langle P_2, X | \psi_i(\xi) | 0 \rangle \end{aligned} \quad (82)$$

(note that $\bar{\Delta}_g^\alpha(z, p_T) = -p_T^\alpha \bar{\Delta}(z, p_T)$).

It turns out that the direction of the links in the fragmentation functions changes in going from SIDIS to the annihilation process, just as the direction of the links in distribution functions changed in going from SIDIS to DY. The A^- -gluons connected to hadron 1, produce a link running along the plus direction to $+\infty$, i.e., one finds transverse-momentum-dependent fragmentation functions $\Delta^{[+]}(z_1, k_T)$, where $z_1 \approx P_1^-/k^- \approx P_1^-/q^-$. Also in the antiquark matrix element the link runs to $+\infty$ (along minus direction with our choice of lightlike vectors), i.e., the inclusion of A^+ gluons lead to the matrix element $\bar{\Delta}^{[+]}(z_2, p_T)$, where $z_2 \approx P_2^+/p^+ \approx P_2^+/q^+$.

Again, to perform transverse integrations at the level of the hadron tensor one switches from $n_\pm(P_1, P_2)$ directions to directions $n'_\pm = (\hat{t} \pm \hat{z})/\sqrt{2}$ with $\hat{t} = q/Q$. In order to treat both 1-particle and 2-particle inclusive annihilation it is convenient [20] to fix $\hat{z}$ via one hadron momentum, for which we will choose P_2 , i.e., $\hat{z} = (q - 2P_2/z_2)/Q$. The relation between $n'_\pm$ and $n_\pm$, then is the same as in leptonproduction. The orthogonal direction determining the azimuthal angle of the hadron production plane, then is $X = -P_{1\perp}/z_1$ which for our purposes equals $X \approx q_T \approx q - P_1/z_1 - P_2/z_2$.

6.1. Integrated annihilation cross section at leading order

The inclusion of appropriate quark–gluon matrix elements make the $\mathcal{O}(g^0)$ result in Eq. (80) color-gauge-invariant,

$$\begin{aligned} \mathcal{W}_{\mu\nu}^{(0)}(q; P_1, S_1; P_2, S_2) \\ = \int d^2 p_T d^2 k_T \delta^2(p_T + k_T - q_T) \text{Tr}(\bar{\Delta}^{[+]}(z_2, p_T) \gamma_\mu \Delta^{[+]}(z_1, k_T) \gamma_\nu). \end{aligned} \quad (83)$$

At leading order the integration over transverse momenta of the produced hadrons can be performed and one obtains the basic result

$$\int d^2 X \mathcal{W}_{\mu\nu}^{(0)}(q; P_1, S_1; P_2, S_2) = \text{Tr}(\bar{\Delta}(z_2) \gamma_\mu \Delta(z_1) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} + \mathcal{O}(1/Q). \quad (84)$$

By taking the free (massless) quark result $\bar{\Delta}(z_2) = \not{n}_+ \delta(1 - z_2)$ we obtain the 1-particle inclusive result ($q_T = 0$),

$$W_{\mu\nu}^{(0)}(q; P_h, S_h) = \text{Tr}(\Delta(z_h) \gamma_\nu \gamma^- \gamma_\mu) + \mathcal{O}(1/Q). \quad (85)$$

6.2. Azimuthal asymmetries in the annihilation process at leading order

We consider here cross sections obtained after integration over X and explicit weighting with X^α , i.e., $\int d^2X X^\alpha \mathcal{W}_{\mu\nu}$, which at leading order gives

$$\begin{aligned} & \int d^2X X^\alpha \mathcal{W}_{\mu\nu}^{(0)}(q; P_1, S_1; P_2, S_2) \\ &= \text{Tr}(\bar{\Delta}(z_2) \gamma_\mu \Delta_\partial^{[+]\alpha}(z_1) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} - \text{Tr}(\bar{\Delta}_\partial^{[+]\alpha}(z_2) \gamma_\mu \Delta(z_1) \gamma_\nu) \Big|_{n_\pm \rightarrow n'_\pm} \\ &+ \mathcal{O}(1/Q). \end{aligned} \quad (86)$$

By taking $\bar{\Delta}(z_2) = \not{n}_+ \delta(1 - z_2)$ and $\bar{\Delta}_\partial^{[+]\alpha}(z_2) = 0$, we get the weighted $e^+e^- \rightarrow \text{jet}(P_j) + h(P_h) + X$ hadron tensor

$$\int d^2X X^\alpha \mathcal{W}_{\mu\nu}^{(0)}(q; P_h, S_h; P_j) = \text{Tr}(\Delta_\partial^{[+]\alpha}(z_h) \gamma_\nu \gamma^- \gamma_\mu) \Big|_{n_\pm \rightarrow n'_\pm} + \mathcal{O}(1/Q). \quad (87)$$

6.3. Integrated annihilation cross section at $\mathcal{O}(1/Q)$

As before, one must be careful in integrating over transverse momenta at $\mathcal{O}(1/Q)$. In particular the correlator $\Delta \propto \not{n}_-$ will lead to terms proportional to $\not{q}_T/Q\sqrt{2}$. We obtain

$$\begin{aligned} \Delta(z_1, k_T) &= \frac{1}{4}(\not{n}_- \not{n}_+ \Delta(z_1, k_T) \not{n}_+ \not{n}_-) \\ &\approx \Delta(z_1, k_T) \Big|_{n_\pm \rightarrow n'_\pm} - \frac{1}{Q\sqrt{2}}(\not{q}_T \not{n}_+ \Delta(z_1, k_T) + \Delta(z_1, k_T) \not{n}_+ \not{q}_T), \end{aligned} \quad (88)$$

which in analogy to leptonproduction leads to the full, integrated annihilation cross section up to $\mathcal{O}(1/Q)$,

$$\begin{aligned} & \int d^2X \mathcal{W}_{\mu\nu}^{(0+1)}(q; P_1, S_1; P_2, S_2) \\ &= \mathcal{O}(1) \text{ result (Eq. (84))} \\ &+ \text{Tr}\left(\frac{\gamma^- \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \bar{\Delta}_D^\alpha(z_2) \gamma_\mu \Delta(z_1)\right) - \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \Delta_D^\alpha(z_1) \gamma_\nu \bar{\Delta}(z_2)\right) \\ &+ \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \Delta_\partial^{[+]\alpha}(z_1) \gamma_\nu \bar{\Delta}(z_2)\right) - \text{Tr}\left(\gamma_\mu \frac{\gamma_\alpha \gamma^-}{Q\sqrt{2}} \Delta_\partial^{[+]\alpha}(z_1) \gamma_\nu \bar{\Delta}(z_2)\right) \\ &+ (\mu \leftrightarrow \nu)^*, \end{aligned} \quad (89)$$

where the hermiticity properties of the various matrix elements have been used (see Section 7).

With the choice of $\hat{t}$ and $\hat{z}$, the 1-particle inclusive result is found by taking $\Delta(z_1) = \not{n}_- \delta(1 - z_1)$, $\Delta_\partial^{[+]\alpha}(z_1) = \Delta_D^\alpha(z_1) = 0$. After a (for ease of comparison) change of plus $\rightarrow$ minus and taking the crossed (particle $\rightarrow$ antiparticle, $\mu \leftrightarrow \nu$) term we obtain

$$\begin{aligned} W_{\mu\nu}(q; P_h, S_h) &= \mathcal{O}(1) \text{ result (Eq. (85))} \\ &- \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\mu \Delta_D^\alpha(z_h) \gamma_\nu\right) - \text{Tr}\left(\frac{\gamma^+ \gamma_\alpha}{Q\sqrt{2}} \gamma_\nu \Delta_D^\alpha(z_h) \gamma_\mu\right)^*. \end{aligned} \quad (90)$$

7. Time-reversal properties

In order to discuss the possible relations between the expressions for the various processes given in the previous three sections, we first discuss the time-reversal properties of the correlation functions involved. After that we will discuss the parametrizations, such that we are able to address the Sivvers effect, the Collins effect and the Qiu–Sterman mechanism (gluonic poles), explicitly.

7.1. Distribution functions

Hermiticity, parity and time-reversal invariance yield conditions for the correlator Φ , that constrain its parametrization,

$$\Phi^\dagger(p; P, S) = \gamma_0 \Phi(p; P, S) \gamma_0 \quad [\text{Hermiticity}], \quad (91)$$

$$\Phi(p, P, S) = \gamma_0 \Phi(\bar{p}; \bar{P}, -\bar{S}) \gamma_0 \quad [\text{Parity}], \quad (92)$$

$$\Phi^*(p; P, S) = (-i\gamma_5 C) \Phi(\bar{p}; \bar{P}, \bar{S}) (-i\gamma_5 C) \quad [\text{Time reversal}], \quad (93)$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{p} = (p^0, -\mathbf{p})$. Similar conditions arise for the fragmentation matrix elements. Including link operators and for quark–gluon matrix elements slightly different conditions apply. For the gauge link one has

$$U_{[a,\xi]}^\dagger = U_{[\xi,a]}, \quad \mathcal{P}U_{[a,\xi]}\mathcal{P}^\dagger = U_{[\bar{a},\bar{\xi}]}, \quad \mathcal{T}U_{[a,\xi]}\mathcal{T}^\dagger = U_{[-\bar{a},-\bar{\xi}]}, \quad (94)$$

for which we used $A_\mu^\dagger = A_\mu$, $\mathcal{P}A_\mu(\xi)\mathcal{P}^\dagger = \bar{A}_\mu(\bar{\xi})$ and $\mathcal{T}A_\mu(\xi)\mathcal{T}^\dagger = \bar{A}_\mu(-\bar{\xi})$. This means that the space-reversed (time-reversed) correlation function has a different link structure, namely, a link running from $\bar{a}$ ($-\bar{a}$), respectively. However, if the common point is defined with respect to the two fields in the matrix element, no problem arises. For example, the straight line link with path $z^\mu(s) = (1-s)0^\mu + s\xi^\mu$ gives a path $\bar{z}^\mu$ after applying parity, but after a change of variables one ends up with the same path; similarly for time reversal.

For the transverse-momentum-dependent functions with links running along the minus direction connected at infinity, the situation is different. The point $\eta^- = \infty$ is defined by $\eta \cdot n_+ = \infty$, which after parity transforms into the point $\bar{\eta} \cdot \bar{n}_+ = \infty$, but after time-reversal transforms into the point $\bar{\eta} \cdot \bar{n}_+ = -\infty$. As a consequence, one finds that for the p^- -integrated functions

$$\Phi^{[+]\dagger}(x, p_T) = \gamma_0 \Phi^{[+]}(x, p_T) \gamma_0, \quad (95)$$

$$\Phi^{[+]}(x, p_T) = \gamma_0 \Phi^{[+]}(x, -p_T) \gamma_0, \quad (96)$$

$$\Phi^{[+]*}(x, p_T) = (-i\gamma_5 C) \Phi^{[-]}(x, -p_T) (-i\gamma_5 C), \quad (97)$$

where $\Phi^{[-]}$ is defined with the link running via $\eta^- = -\infty$, referred to as *timelike distribution* in Fig. 5. Concluding, in the parametrization of $\Phi^{[+]}(x, p_T)$ time reversal does not pose constraints. Application of this operation transforms $\Phi^{[-]}$ into $\Phi^{[+]}$ and vice versa. T-odd quantities will be defined as the ones that vanish when $\Phi^{[-]} = \Phi^{[+]}$.

Accounting for the transformation in Dirac space and the sign change of p_T , we define

$$\begin{aligned} 2\Phi^{[\text{T-even}]}(x, p_T) &\equiv \Phi^{[+]}(x, p_T) + (-i\gamma_5 C)\Phi^{[+]*}(x, -p_T)(-i\gamma_5 C) \\ &= \Phi^{[+]}(x, p_T) + \Phi^{[-]}(x, p_T), \end{aligned} \quad (98)$$

$$\begin{aligned} 2\Phi^{[\text{T-odd}]}(x, p_T) &\equiv \Phi^{[+]}(x, p_T) - (-i\gamma_5 C)\Phi^{[+]*}(x, -p_T)(-i\gamma_5 C) \\ &= \Phi^{[+]}(x, p_T) - \Phi^{[-]}(x, p_T). \end{aligned} \quad (99)$$

Note that the name ‘T-odd’ does *not* imply a violation of time-reversal invariance. For the integrated distributions the different links ($[\pm]$) merge into one, $\Phi(x) = \Phi^{[+]}(x) = \Phi^{[-]}(x)$ and $\Phi(x)$ is T-even. For transverse-momentum-dependent correlations the sum of $\Phi^{[+]}$ and $\Phi^{[-]}$, i.e., $\Phi_\theta^\alpha(x)$, is T-even, while the difference, related to $\Phi_G^\alpha(x, x)$ (see Eq. (42)), is T-odd. Summarizing, we have

$$\Phi^{[\pm]}(x, p_T) = \Phi^{[\text{T-even}]}(x, p_T) \pm \Phi^{[\text{T-odd}]}(x, p_T), \quad (100)$$

and for the integrated and weighted distribution correlators

$$\Phi(x) = \Phi^{[\text{T-even}]}(x), \quad (101)$$

$$\Phi_\theta^{[\pm]\alpha}(x) = \Phi_\theta^{\alpha[\text{T-even}]}(x) \pm \pi \Phi_G^{\alpha[\text{T-odd}]}(x, x). \quad (102)$$

These results imply that T-odd distribution functions, e.g., the Sivvers effect appearing in single spin azimuthal asymmetries in leptonproduction ($\Phi_\theta^{[+]\alpha}$) and in Drell–Yan scattering ($\Phi_\theta^{[-]\alpha}$) are opposite in sign [8]. As shown in Ref. [29] the T-odd functions can be considered as imaginary parts in a helicity matrix representation, leading to the representation in Fig. 8(a). The behavior under time reversal of gluonic matrix elements, like Φ_A , Φ_D and Φ_G , can also be studied separately. It turns out that the quantity $\Phi_D(x)$ is T-even, while $\Phi_G^\alpha(x, x)$ is T-odd. The latter can also be seen in $A^+ = 0$ gauge. Using relation (A.10) in the matrix elements in Eqs. (29) and (30) yields $2\pi \Phi_G^\alpha(x, x) = \Phi_{A(\infty)}^\alpha(x, x) - \Phi_{A(-\infty)}^\alpha(x, x)$, i.e., the gluonic pole matrix element is the difference of the boundary terms that transform into each other under time reversal.

7.2. Fragmentation functions

Constraints on the correlator Δ come from hermiticity, parity and time-reversal invariance. The essential difference with distribution functions is that time-reversal transforms the out-states in the definition of fragmentation matrix elements into in-states. Taking this into account and explicitly adding subscripts in and out, one obtains the conditions,

$$\Delta_{\text{out}}^{[-]\dagger}(z, k_T) = \gamma_0 \Delta_{\text{out}}^{[-]}(z, k_T) \gamma_0, \quad (103)$$

$$\Delta_{\text{out}}^{[-]}(z, k_T) = \gamma_0 \Delta_{\text{out}}^{[-]}(z, -k_T) \gamma_0, \quad (104)$$

$$\Delta_{\text{out}}^{[-]*}(z, k_T) = (-i\gamma_5 C) \Delta_{\text{in}}^{[+]}(z, -k_T) (-i\gamma_5 C), \quad (105)$$

where $\Delta^{[\pm]}$ are the spacelike/timelike fragmentation functions, illustrated in Fig. 5. Defining Δ_O and Δ_{FSI} as sum and differences of matrix elements with out and in-states

Table 2

Summary of time-reversal allowed (yes) and forbidden (no) parts in integrated or weighted distribution and fragmentation correlators

	T-even	T-odd		T-even	T-odd		T-even	T-odd		T-even	T-odd
Φ	yes	no	Δ	yes	yes	Δ_O	yes	no	Δ_{FSI}	no	yes
Φ_∂	yes	no	Δ_∂	yes	yes	$\Delta_{\partial O}$	yes	no	$\Delta_{\partial \text{FSI}}$	no	yes
Φ_D	yes	no	Δ_D	yes	yes	Δ_{DO}	yes	no	$\Delta_{D \text{FSI}}$	no	yes
$\tilde{\Phi}_A$	yes	no	$\tilde{\Delta}_A$	yes	yes	$\tilde{\Delta}_{AO}$	yes	no	$\tilde{\Delta}_{A \text{FSI}}$	no	yes
Φ_G	no	yes	Δ_G	yes	yes	Δ_{GO}	no	yes	$\Delta_{G \text{FSI}}$	yes	no

respectively, one has

$$4\Delta_O^{[\text{T-even}]}(z, k_T) = \Delta_{\text{out}}^{[+]}(z, k_T) + \Delta_{\text{out}}^{[-]}(z, k_T) + \Delta_{\text{in}}^{[+]}(z, k_T) + \Delta_{\text{in}}^{[-]}(z, k_T), \quad (106)$$

$$4\Delta_O^{[\text{T-odd}]}(z, k_T) = \Delta_{\text{out}}^{[+]}(z, k_T) - \Delta_{\text{out}}^{[-]}(z, k_T) + \Delta_{\text{in}}^{[+]}(z, k_T) - \Delta_{\text{in}}^{[-]}(z, k_T), \quad (107)$$

$$4\Delta_{\text{FSI}}^{[\text{T-odd}]}(z, k_T) = \Delta_{\text{out}}^{[+]}(z, k_T) + \Delta_{\text{out}}^{[-]}(z, k_T) - \Delta_{\text{in}}^{[+]}(z, k_T) - \Delta_{\text{in}}^{[-]}(z, k_T), \quad (108)$$

$$4\Delta_{\text{FSI}}^{[\text{T-even}]}(z, k_T) = \Delta_{\text{out}}^{[+]}(z, k_T) - \Delta_{\text{out}}^{[-]}(z, k_T) - \Delta_{\text{in}}^{[+]}(z, k_T) + \Delta_{\text{in}}^{[-]}(z, k_T), \quad (109)$$

which implies

$$\begin{aligned} \Delta_{\text{out}}^{[\pm]}(z, k_T) &= [\Delta_O^{[\text{T-even}]}(z, k_T) + \Delta_{\text{FSI}}^{[\text{T-odd}]}(z, k_T)] \\ &\quad \pm [\Delta_O^{[\text{T-odd}]}(z, k_T) + \Delta_{\text{FSI}}^{[\text{T-even}]}(z, k_T)], \end{aligned} \quad (110)$$

and for the integrated and weighted fragmentation functions

$$\Delta_{\text{out}}(z) = \Delta_O^{[\text{T-even}]}(z) + \Delta_{\text{FSI}}^{[\text{T-odd}]}(z), \quad (111)$$

$$\begin{aligned} \Delta_{\partial \text{out}}^{[\pm]\alpha}(z) &= \underbrace{[\Delta_{\partial O}^{\alpha[\text{T-even}]}(z) + \Delta_{\partial \text{FSI}}^{\alpha[\text{T-odd}]}(z)]}_{\Delta_{\partial \text{out}}^{\alpha}(z)} \\ &\quad \pm \underbrace{[\pi \Delta_{GO}^{\alpha[\text{T-odd}]}(z, z) + \pi \Delta_{G \text{FSI}}^{\alpha[\text{T-even}]}(z, z)]}_{\pi \Delta_{G \text{out}}^{\alpha}(z, z)}. \end{aligned} \quad (112)$$

The essential difference with the distribution functions is that for the fragmentation functions the differences between in and out states become relevant. These constitute final state interactions *within* the soft part (here labelled by ‘FSI’), decoupled from the quark and gluon operators that make the connection to the hard scattering part. The effects arising from the difference between $[+]$ and $[-]$ are labelled by a subscript ‘O’. Note that in the literature [7,15,30] this is also referred to as initial or final state interactions, depending on the process under consideration. The behavior of the various correlators is given in Table 2. Therefore, in contrast to the distribution functions, Δ_∂ and Δ_G contain T-even *and* T-odd parts. Also the correlators Δ_D and $\tilde{\Delta}_A$ contain T-even and T-odd parts.

At first sight Eqs. (110) and (112) seem to imply a breaking of universality. Particular transverse-momentum-dependent functions obtained as Dirac projections of $\Delta_{\text{out}}^{[\pm]}(z, k_T)$ give unrelated T-odd (and also T-even) results for the two possible link configurations, indicated with superscripts $[+]$ or $[-]$. In $\Delta_{\partial \text{out}}^{[\pm]\alpha}$ there arise T-odd parts from $\Delta_{\partial \text{FSI}}^{\alpha}$ and from $\pi \Delta_{GO}^{\alpha}$, the sign in front of the latter coupled to the link structure. Thus, the T-odd

Fig. 8. Illustration of the role of gauge links ([+] versus [−]) and the role of final states ([in] versus [out]) in T-odd effects for distribution and fragmentation functions.

(Collins) effects in pion leptonproduction and in electron–positron annihilation are a priori not identical. The fact that a certain azimuthal asymmetry arises from a combination of correlation functions (Δ_θ and Δ_G), however, need not imply a breaking of universality. It is quite similar to the q_T -integrated structure functions of different processes which involve different flavor weights. It seems possible that a factorization proof can be established for the correlation functions Δ_θ and Δ_G separately.

In Fig. 8 we illustrate the differences between $\Delta^{[\pm]}$ and those between the $\Delta_{\text{in/out}}$. For the former one can argue that forgetting about the effect from in- and out-states one has the same situation as for distribution functions where the T-odd parts can be considered as imaginary parts of helicity amplitudes (shown as the projections along the vertical axis in Fig. 8(a)). For $\Delta_{\text{in/out}}$ one can use the fact that in- and out-states can be obtained by different Möller operators, allowing a connection of Δ_{in} and Δ_{out} via a unitary operation, illustrated in Fig. 8(b). The combined effect is illustrated in Fig. 8(c). It shows that the T-odd parts of $\Delta_{\theta^{\text{out}}}^{[+]}$ in e^+e^- and $\Delta_{\theta^{\text{out}}}^{[-]}$ in SIDIS are in general not equal in magnitude.

Actually, not only the T-odd effects acquire contributions from both terms in $\Delta_\theta \pm \Delta_G$, but also the T-even effects, hence affecting all comparisons of azimuthal asymmetries in leptonproduction and electron–positron annihilation.

7.3. Parametrization of quark and quark–gluon correlation functions

Here we will discuss the parametrizations for transverse-momentum-dependent distribution functions and fragmentation functions. The parametrizations of $\Phi(x, p_T)$ and $\Delta(z, k_T)$ consistent with the conditions imposed by hermiticity and parity, including the parts proportional to $(M/P^+)^0$ and $(M/P^+)^1$ are given in Appendix B [6,19]. In principle, the functions could differ depending on the $\pm$ gauge link structure for both distribution and fragmentation functions and also on the O/FSI characterization for fragmentation functions.

We will first discuss the parametrizations for the transverse momentum integrated and then the once-weighted functions (transverse moments). As explained in Section 4.6, we do not address twice-weighted functions (relevant for k_T broadening and the average k_T^2 in jets) in view of potential problems with factorization.

The result for the correlator Φ after integration is

$$\Phi(x) = \frac{1}{2} \left\{ f_1(x) \not{n}_+ + S_L g_1(x) \gamma_5 \not{n}_+ + h_1(x) \frac{\gamma_5 [\not{s}_T, \not{n}_+]}{2} \right\} + \frac{M}{2P^+} \left\{ e(x) + g_T(x) \gamma_5 \not{s}_T + S_L h_L(x) \frac{\gamma_5 [\not{n}_+, \not{n}_-]}{2} \right\}. \quad (113)$$

For the distribution functions the T-odd part vanishes, i.e., $e_L(x) = f_T(x) = h(x) = 0$. For the parametrization of fragmentation functions one has both T-even and T-odd functions, i.e., the functions $E_L(z)$, $D_T(z)$ and $H(z)$ appear. Explicitly,

$$\Delta_{\text{out}}(z) = z D_1(z) \not{n}_- + S_{hL} z G_1(z) \gamma_5 \not{n}_+ + z H_1(z) \frac{\gamma_5 [\not{s}_{hT}, \not{n}_-]}{2} + \frac{M_h}{P_h^-} \left\{ z E(z) + z G_T(z) \gamma_5 \not{s}_{hT} + S_{hL} z H_L(z) \frac{\gamma_5 [\not{n}_-, \not{n}_+]}{2} - S_{hL} z E_L(z) i \gamma_5 + z D_T(z) \epsilon_T^{\rho\sigma} \gamma_\rho S_{hT\sigma} + z H(z) \frac{i [\not{n}_-, \not{n}_+]}{2} \right\}. \quad (114)$$

The T-even functions are ‘O’-type, the T-odd functions are ‘FSI’-type, since $\Delta_{\text{out}}^{[\text{T-even}]}(z) = \Delta_{\text{O}}^{[\text{T-even}]}(z)$ and $\Delta_{\text{out}}^{[\text{T-odd}]}(x) = \Delta_{\text{FSI}}^{[\text{T-odd}]}(z)$, while the link structure does not play a role. T-odd functions only appear in sub-leading parts of the cross sections.

Next we consider the p_T -weighted results referred to as transverse moments. Restricting ourselves to the leading $(M/P^+)^0$ part, we write for the correlators $\Phi_\theta^{\alpha[\pm]}$ (Eq. (38))

$$\Phi_\theta^{\alpha[\pm]}(x) = \frac{M}{2} \left\{ g_{1T}^{(1)[\pm]}(x) S_T^\alpha \gamma_5 \not{n}_+ - S_L h_{1L}^{\perp(1)[\pm]}(x) \frac{\gamma_5 [\gamma^\alpha, \not{n}_+]}{2} - f_{1T}^{\perp(1)[\pm]}(x) \epsilon_{\mu\nu\rho}^\alpha \gamma^\mu n_+^\nu S_T^\rho - h_1^{\perp(1)[\pm]}(x) \frac{i [\gamma^\alpha, \not{n}_+]}{2} \right\}, \quad (115)$$

where we have defined $p_T^2/2M^2$ -moments (transverse moments) as

$$g_{1T}^{(1)}(x) = \int d^2 p_T \frac{p_T^2}{2M^2} g_{1T}(x, p_T^2), \quad (116)$$

and similarly for the other functions.

For the transverse moments, the average of the $\pm$ correlators in Eq. (43) is T-even, i.e.,

$$\Phi_\theta^\alpha(x) = \frac{M}{2} \left\{ g_{1T}^{(1)}(x) S_T^\alpha \gamma_5 \not{n}_+ - S_L h_{1L}^{\perp(1)}(x) \frac{\gamma_5 [\gamma^\alpha, \not{n}_+]}{2} \right\}, \quad (117)$$

while the gluonic pole contribution $\Phi_G^\alpha(x, x)$ in Eq. (42) is T-odd. Writing

$$\pi \Phi_G^\alpha(x, x) = \frac{M}{2} \left\{ -\tilde{f}_{1T}^{\perp(1)}(x) \epsilon_{\mu\nu\rho}^\alpha \gamma^\mu n_+^\nu S_T^\rho - \tilde{h}_1^{\perp(1)}(x) \frac{i [\gamma^\alpha, \not{n}_+]}{2} \right\}, \quad (118)$$

we have the relations

$$g_{1T}^{(1)[\pm]}(x) = g_{1T}^{(1)}(x), \quad (119)$$

$$h_{1L}^{\perp(1)[\pm]}(x) = h_{1L}^{\perp(1)}(x), \quad (120)$$

$$f_{1T}^{\perp(1)[\pm]}(x) = \pm \tilde{f}_{1T}^{\perp(1)}(x), \quad (121)$$

$$h_1^{\perp(1)[\pm]}(x) = \pm \tilde{h}_1^{\perp(1)}(x). \quad (122)$$

These results show that azimuthal spin asymmetries involving the distribution functions $g_{1T}^{(1)}$ and $h_{1L}^{(1)}$ are process-independent, while those involving the functions $f_{1T}^{\perp(1)}$ and $h_1^{\perp(1)}$ change sign. Eq. (121) represents the explicit connection between the Sivers effect (l.h.s.) and the Qiu–Sterman effect (r.h.s., cf. Ref. [26]); Eq. (122) is its chiral-odd counterpart. We note that due to the presence of gluonic pole effects the evolution equations of $f_{1T}^{\perp(1)}$ and $h_1^{\perp(1)}$ [41] need to be reconsidered. These functions originate solely from gluonic pole effects.

For the fragmentation functions, not only the parametrization for $\Delta_{\theta}^{\alpha[\pm]}$ contains both T-even and T-odd parts, but also the average contains T-even and T-odd parts, i.e.,

$$\Delta_{\theta}^{\alpha}(z) = M_h \left\{ z G_{1T}^{(1)}(z) S_T^{\alpha} \gamma_5 \not{p}_- - S_{hL} z H_{1L}^{\perp(1)}(z) \frac{\gamma_5 [\gamma^{\alpha}, \not{p}_-]}{2} - z D_{1T}^{\perp(1)}(z) \epsilon^{\alpha}_{\mu\nu\rho} \gamma^{\mu} n_{-}^{\nu} S_{hT}^{\rho} - z H_1^{\perp(1)}(z) \frac{i[\gamma^{\alpha}, \not{p}_-]}{2} \right\}. \quad (123)$$

For the gluonic pole contribution one obtains

$$\pi \Delta_G^{\alpha}(z, z) = M_h \left\{ z \tilde{G}_{1T}^{(1)}(z) S_{hT}^{\alpha} \gamma_5 \not{p}_- - S_{hL} z \tilde{H}_{1L}^{\perp(1)}(z) \frac{\gamma_5 [\gamma^{\alpha}, \not{p}_+]}{2} - z \tilde{D}_{1T}^{\perp(1)}(z) \epsilon^{\alpha}_{\mu\nu\rho} \gamma^{\mu} n_{-}^{\nu} S_{hT}^{\rho} - z \tilde{H}_1^{\perp(1)}(z) \frac{i[\gamma^{\alpha}, \not{p}_-]}{2} \right\}, \quad (124)$$

and we obtain for the transverse moment in $\Delta_{\theta}^{\alpha[\pm]}$,

$$G_{1T}^{(1)[\pm]}(z) = G_{1T}^{(1)}(z) \pm \tilde{G}_{1T}^{(1)}(z), \quad (125)$$

$$H_{1L}^{\perp(1)[\pm]}(z) = H_{1L}^{\perp(1)}(z) \pm \tilde{H}_{1L}^{\perp(1)}(z), \quad (126)$$

$$D_{1T}^{\perp(1)[\pm]}(z) = D_{1T}^{\perp(1)}(z) \pm \tilde{D}_{1T}^{\perp(1)}(z), \quad (127)$$

$$H_1^{\perp(1)[\pm]}(z) = H_1^{\perp(1)}(z) \pm \tilde{H}_1^{\perp(1)}(z). \quad (128)$$

The occurrence of out-states in the fragmentation matrix elements is responsible for the appearance of the T-even functions $\tilde{G}_{1T}^{(1)}$ and $\tilde{H}_{1L}^{\perp(1)}$ and the T-odd functions $D_{1T}^{\perp(1)}$ and $H_1^{\perp(1)}$. We note that these results differ from Ref. [19]. For example, Eq. (128) shows the explicit forms of the Collins function in e^+e^- (plus sign) and SIDIS (minus sign), respectively. Furthermore, the above results imply that the evolution equations of $G_{1T}^{(1)}$, $H_{1L}^{\perp(1)}$, $D_{1T}^{\perp(1)}$ and $H_1^{\perp(1)}$ [41] also need to be reconsidered.

For functions weighted twice with a transverse momentum in $\Phi_{\partial\partial}^{\alpha\beta}$, one needs higher transverse moments of the functions, such as $h_{1T}^{\perp(2)}(x)$. Relations for these functions involve not only twist-three, but also twist-four parts in the correlators. This includes the simple p_T^2 average in $\Phi_{\partial^2}(x)$. As mentioned above, for these functions we do not expect a

simple process dependence as given for the once-weighted results, but this requires further investigation.

In Refs. [26,42] parametrizations have been given for two-argument quark–gluon correlations. As shown in Eq. (41) one finds after integration correlation functions with $D^\alpha \psi(x)$ at the same point, for which the QCD equations of motion can be used. These relate $\Phi_D(x)$ (see Eq. (41)) and also $\gamma_0 \Phi_D(x) \gamma_0$ to $\Phi(x)$; explicitly,

$$\begin{aligned} \Phi_D^\alpha(x) = \frac{M}{2} \left\{ \left(x g_T(x) - \frac{m}{M} h_1(x) \right) S_T^\alpha \gamma_5 \not{x}_+ + S_L \left(x h_L(x) - \frac{m}{M} g_1(x) \right) \frac{\gamma_5 [\gamma^\alpha, \not{x}_+]}{4} \right. \\ \left. - \left(x e(x) - \frac{m}{M} f_1(x) \right) \frac{[\gamma^\alpha, \not{x}_+]}{4} \right\}, \end{aligned} \quad (129)$$

which contains only T-even functions. For fragmentation functions one also finds T-odd functions,

$$\begin{aligned} \Delta_D^\alpha(z) = M_h \left\{ \left(G_T(z) - \frac{m}{M_h} z H_1(z) \right) S_{hT}^\alpha \gamma_5 \not{x}_- \right. \\ + S_{hL} \left(H_L(z) - \frac{m}{M_h} z G_1(z) \right) \frac{\gamma_5 [\gamma^\alpha, \not{x}_-]}{4} \\ - \left(E(z) - \frac{m}{M_h} z D_1(z) \right) \frac{[\gamma^\alpha, \not{x}_-]}{4} + D_T(z) \epsilon^\alpha_{\mu\nu\rho} \gamma^\mu n_-^\nu S_{hT}^\rho \\ \left. + H(z) \frac{i[\gamma^\alpha, \not{x}_-]}{4} + E_L(z) \frac{i\gamma_5 [\gamma^\alpha, \not{x}_-]}{4} \right\}. \end{aligned} \quad (130)$$

Besides the relations resulting from the equations of motion, also Lorentz invariance may lead to relations between correlators [19,41,43]. As can be seen from the explicit treatment in Ref. [44], these relations are derived from the Lorentz structure of non-integrated quark–quark correlators as in Eq. (9). At present it is not clear how matrix elements of A_T fields at infinity and hence the link structure play a role in these relations (see also Ref. [45]).

8. Summary and conclusions

In this paper we have analyzed transverse-momentum-dependent distribution and fragmentation functions appearing in several hard processes in which at least two hadrons are involved in initial and/or final state. In these processes one has besides a hard scale Q , a non-collinearity q_T which is characterized by a hadronic scale Q_T and an azimuthal angle. We have shown explicitly, using the results of Belitsky et al. [14] how quark–quark–gluon matrix elements lead to fully color-gauge-invariant definitions of the correlation functions that appear in leading and first sub-leading order in $1/Q$ in the hadron tensor of hard processes. The gluon fields appear in the gauge link connecting the quark fields. The transverse gluons needed in the gauge link for transverse-momentum-dependent functions involve gluon fields at lightlike infinity. The fact that the gluonic effects can be cast into (conjugate) links attached to the two (conjugate) quark fields still allows for an interpretation of these functions as probability densities.

The structure of the gauge links in hard processes is not always the same. In particular the gauge links in distribution functions in SIDIS and the DY process run in opposite lightlike directions indicated with indices ‘ $\pm$ ’. The two different correlators are connected via a time-reversal operation. Similarly, the gauge links in fragmentation functions in SIDIS and electron–positron annihilation run in opposite lightlike directions. At leading order, the difference between $\pm$ correlators vanishes upon integration over transverse momenta. In q_T -weighted cross sections, one finds correlators weighted with transverse momentum (transverse moments), which are dependent on the ($\pm$) link structure. The same quantities appear in subleading integrated cross sections. The difference between transverse moments with different ($\pm$) link structure corresponds to a (color-gauge-invariant) gluonic pole matrix element, the word pole referring to the fact that one deals with a zero-momentum gluon field. This matrix element appears in different processes as the Qiu–Sterman effect.

Considering the behavior under time reversal for distribution functions, it turns out that the gluonic pole contributions coincide with T-odd effects, leading to single spin asymmetries, like the Sivers effect. This establishes the direct connection between the Sivers effect and the Qiu–Sterman effect. Since for distribution functions gluonic poles are the only source of T-odd effects, one finds that these effects have opposite signs in SIDIS and the DY process. For fragmentation functions, T-odd effects leading to single spin asymmetries, like the Collins effect in pion leptonproduction, arise not only from the gluonic pole contribution, but also from final state interactions. The latter are purely soft interactions in the fragmentation part and has the same sign in different processes. Hence the T-odd effects in SIDIS and electron–positron annihilation are not connected by a simple sign relation. This is not a breaking of universality, but rather the appearance of different combinations of fragmentation functions in different processes. The T-odd effects will not only appear in Collins asymmetries in SIDIS or electron–positron annihilation, but likely also in other processes. We emphasize the importance of an analysis of the link structure in processes like $pp^\uparrow \rightarrow \pi X$. We note that also for distribution functions T-odd contributions with the same sign in different processes could arise if time reversal is realized in a nonstandard way [46]. This would spoil the simple sign switches in Eqs. (121) and (122).

Gluonic pole contributions appear in the azimuthal asymmetries, in the processes that we have considered, with a particular sign. This affects not only T-odd, but (in case of fragmentation) also T-even azimuthal asymmetries. The transverse moments with a different link structure differ by ‘effective’ twist-three functions. The consequences of these gluonic pole contributions on the evolution of the transverse moments have not been considered. Given the known operator structure of the contributions, however, such a study ought to be doable.

Although in trying to model distribution or fragmentation functions [44,47] one is always stuck with the problem of evolution, modelling has been proven useful to illustrate several effects [7,15,48,49], such as the sign change in the Sivers functions $f_{1T}^{\perp[\pm]}$ and the sign behavior for the Collins function $H_1^{\perp[\pm]}$. As seen from Eq. (128), the latter can have two contributions with different signs. It is not clear whether both contributions are present in the models studied.

Our general analysis of the various correlators in leading and subleading order single spin and azimuthal asymmetries in hard processes may help to analyze the results of the first generation of experiments that presently are being performed or planned by, for instance, HERMES (DESY), COMPASS (CERN), BELLE (KEK) and the results that are obtained by looking at existing LEP data. We hope to see the emergence of a coherent picture of these asymmetries.

Acknowledgements

We acknowledge discussions with A. Bacchetta, A. Belitsky, S.J. Brodsky, J.C. Collins, D.S. Hwang, P. Hoyer, X. Ji, A. Metz, and J.G. Milhano. The research of D.B. has been made possible by financial support from the Royal Netherlands Academy of Arts and Sciences.

Appendix A. Transverse gluon fields and the field strength tensor

In lightcone gauge $A^+ = 0$, the relation between A_T^α and $G^{+\alpha}$ becomes

$$[\partial_x^+, A_T^\alpha(x)] = G^{+\alpha}(x), \quad (\text{A.1})$$

which can be inverted to yield A_T^α in terms of a boundary term and an integral along the minus direction with $G^{+\alpha}$ in the integrand. Without gauge choice we have

$$igG^{+\alpha}(x) = [iD^+(x), iD^\alpha(x)] = [iD^+(x), gA_T^\alpha(x)] - ig[\partial_x^\alpha, A^+(x)] \quad (\text{A.2})$$

as our starting point. Next we multiply from left and right with link operators $U_{[a,x]}^-$ and $U_{[x,a]}^-$, respectively, built from A^+ fields, running along the minus direction. They are denoted by

$$U_{[a,x]}^- = \mathcal{P} \exp \left(-ig \int_{a^-}^{x^-} d\zeta^- A^+(\zeta^-, x^+, x_T) \right), \quad (\text{A.3})$$

and satisfy

$$\partial_x^+ U_{[a,x]}^- = U_{[a,x]}^- D^+(x). \quad (\text{A.4})$$

We then obtain

$$\begin{aligned} & igU_{[a,x]}^- G^{+\alpha}(x) U_{[x,a]}^- \\ &= [U_{[a,x]}^- iD^+(x) U_{[x,a]}^-, U_{[a,x]}^- gA_T^\alpha(x) U_{[x,a]}^-] - igU_{[a,x]}^- [\partial_x^\alpha, A^+(x)] U_{[x,a]}^- \\ &= [i\partial_x^+, U_{[a,x]}^-] gA_T^\alpha(x) U_{[x,a]}^- - igU_{[a,x]}^- [\partial_x^\alpha, A^+(x)] U_{[x,a]}^-. \end{aligned} \quad (\text{A.5})$$

Thus we find

$$[\partial_x^+, U_{[a,x]}^- A_T^\alpha(x) U_{[x,a]}^-] = U_{[a,x]}^- (G^{+\alpha}(x) + [\partial_x^\alpha, A^+(x)]) U_{[x,a]}^-, \quad (\text{A.6})$$

which is the relation needed to express the transverse gluon fields in terms of the field strength. In particular, one has

$$\begin{aligned} & U_{[\infty, \xi]}^- A_T^\alpha(\xi) U_{[\xi, \infty]}^- - A_T^\alpha(\infty^-) \\ &= \int_{\infty}^{\xi^-} d\eta^- U_{[\infty, \eta]}^- (G^{+\alpha}(\eta) + [\partial_T^\alpha, A^+(\eta^-)]) U_{[\eta, \infty]}^-, \end{aligned} \quad (\text{A.7})$$

or in $A^+ = 0$ gauge

$$A_T^\alpha(\xi) - A_T^\alpha(\infty^-) = \int_{\infty}^{\xi^-} d\eta^- G^{+\alpha}(\eta). \quad (\text{A.8})$$

By considering Eq. (A.7) for $\xi = -\infty^-$, one arrives at

$$\begin{aligned} & U_{[-\infty, \infty]}^- A_T^\alpha(\infty^-) - A_T^\alpha(-\infty^-) U_{[-\infty, \infty]}^- \\ &= \int_{-\infty}^{\infty} d\eta^- U_{[-\infty, \eta]}^- (G^{+\alpha}(\eta) + [\partial_T^\alpha, A^+(\eta^-)]) U_{[\eta, \infty]}^-, \end{aligned} \quad (\text{A.9})$$

or in $A^+ = 0$ gauge

$$A_T^\alpha(\infty^-) - A_T^\alpha(-\infty^-) = \int_{-\infty}^{\infty} d\eta^- G^{+\alpha}(\eta). \quad (\text{A.10})$$

Appendix B. Parametrizations of transverse-momentum-dependent functions

The parametrization of $\Phi(x, p_T)$ for a spin 0 or spin 1/2 target, consistent with the conditions imposed by hermiticity and parity, including the parts proportional to $(M/P^+)^0$ and $(M/P^+)^1$ is given by

$$\begin{aligned} \Phi(x, p_T) = \frac{1}{2} \Big\{ & f_1(x, p_T^2) \not{n}_+ + g_{1s}(x, p_T) \gamma_5 \not{n}_+ \\ & + h_{1T}(x, p_T^2) \frac{\gamma_5 [\not{p}_T, \not{n}_+]}{2} + h_{1s}^\perp(x, p_T) \frac{\gamma_5 [\not{p}_T, \not{n}_+]}{2M} \\ & + f_{1T}^\perp(x, p_T^2) \frac{\epsilon_{\mu\nu\rho\sigma} \gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} + h_1^\perp(x, p_T^2) \frac{i[\not{p}_T, \not{n}_+]}{2M} \Big\} \end{aligned}$$

$$\begin{aligned}
& + \frac{M}{2P^+} \left\{ e(x, \mathbf{p}_T^2) + f^\perp(x, \mathbf{p}_T^2) \frac{\not{p}_T}{M} + g'_T(x, \mathbf{p}_T^2) \gamma_5 \not{s}_T \right. \\
& \quad + g_s^\perp(x, \mathbf{p}_T) \frac{\gamma_5 \not{p}_T}{M} + h_T^\perp(x, \mathbf{p}_T^2) \frac{\gamma_5 [\not{s}_T, \not{p}_T]}{2M} \\
& \quad + h_s(x, \mathbf{p}_T) \frac{\gamma_5 [\not{p}_+, \not{p}_-]}{2} - f_T(x, \mathbf{p}_T^2) \epsilon_T^{\rho\sigma} \gamma_\rho S_{T\sigma} \\
& \quad - S_L f_L^\perp(x, \mathbf{p}_T^2) \frac{\epsilon_T^{\rho\sigma} \gamma_\rho p_{T\sigma}}{M} - e_s(x, \mathbf{p}_T) i \gamma_5 \\
& \quad \left. + h(x, \mathbf{p}_T^2) \frac{i[\not{p}_+, \not{p}_-]}{2} \right\}. \tag{B.1}
\end{aligned}$$

Here the spin vector is defined

$$S = S_L \left(\frac{P^+}{M} n_+ - \frac{P^-}{M} n_- \right) + S_T \tag{B.2}$$

($S = 0$ for spin 0), and we have used the shorthand notation

$$g_{1s}(x, \mathbf{p}_T) \equiv S_L g_{1L}(x, \mathbf{p}_T^2) + g_{1T}(x, \mathbf{p}_T^2) \frac{(\mathbf{p}_T \cdot \mathbf{S}_T)}{M}, \tag{B.3}$$

and similarly for $h_{1s}^\perp$, $g_s^\perp$ and h_s . We note that all noncontracted p_T -dependence (including appearance of dot products like $p_T \cdot S_T$) is treated explicitly, leaving functions depending on p_T^2 , which is important in distinguishing T-even and T-odd behavior. The structures multiplying the ‘twist-two’ functions f_1 , g_{1s} , h_{1T} , $h_{1s}^\perp$ or the ‘twist-three’ functions e , $f^\perp$, g'_T , $g_s^\perp$, $h_T^\perp$, h_s are T-even (satisfying Eq. (97)). The structures multiplying the ‘twist-two’ functions $f_{1T}^\perp$, $h_1^\perp$ or the ‘twist-three’ functions f_T , $f_L^\perp$, e_s , h are T-odd (satisfying Eq. (97) with an additional minus sign).

As notation in the parametrizations for fragmentation we employ capital letters, to be precise

$$\begin{aligned}
\Delta(z, k_T) = & z D_1(z, -z \mathbf{k}_T) \not{p}_- + z G_{1s}(z, -z \mathbf{k}_T) \gamma_5 \not{p}_- \\
& + z H_{1T}(z, -z \mathbf{k}_T) \frac{\gamma_5 [\not{s}_{hT}, \not{p}_-]}{2} + z H_{1s}^\perp(z, -z \mathbf{k}_T) \frac{\gamma_5 [\not{k}_T, \not{p}_-]}{2M_h} \\
& + z D_{1T}^\perp(z, -z \mathbf{k}_T) \frac{\epsilon_{\mu\nu\rho\sigma} \gamma^\mu n_-^\nu k_T^\rho S_{hT}^\sigma}{M_h} + z H_1^\perp(z, -z \mathbf{k}_T) \frac{i[\not{k}_T, \not{p}_-]}{2M_h} \\
& + \frac{M_h}{P_h^-} \left\{ z E(z, -z \mathbf{k}_T) + z D^\perp(z, -z \mathbf{k}_T) \frac{\not{k}_T}{M_h} + z G'_T(z, -z \mathbf{k}_T) \gamma_5 \not{s}_{hT} \right. \\
& \quad + z G_s^\perp(z, -z \mathbf{k}_T) \frac{\gamma_5 \not{k}_T}{M_h} + z H_T^\perp(z, -z \mathbf{k}_T) \frac{\gamma_5 [\not{s}_{hT}, \not{k}_T]}{2M_h} \\
& \quad + z H_s(z, -z \mathbf{k}_T) \frac{\gamma_5 [\not{p}_-, \not{p}_+]}{2} + z D_T(z, -z \mathbf{k}_T) \epsilon_T^{\rho\sigma} \gamma_\rho S_{hT\sigma} \\
& \quad + S_{hL} z D_L^\perp(z, -z \mathbf{k}_T) \frac{\epsilon_T^{\rho\sigma} \gamma_\rho k_{T\sigma}}{M_h} - z E_s(z, -z \mathbf{k}_T) i \gamma_5 \\
& \quad \left. + z H(z, -z \mathbf{k}_T) \frac{i[\not{p}_-, \not{p}_+]}{2} \right\}. \tag{B.4}
\end{aligned}$$

Here the spin vector is defined

$$S_h = S_{hL} \left(\frac{P_h^-}{M_h} n_- - \frac{P_h^+}{M} n_+ \right) + S_{hT} \quad (\text{B.5})$$

($S_h = 0$ for spin 0), and we have used the shorthand notation G_{1s} , etc.,

$$G_{1s}(z, -z\mathbf{k}_T) = S_{hL} G_{1L}(z, -z\mathbf{k}_T) + G_{1T}(z, -z\mathbf{k}_T) \frac{(\mathbf{k}_T \cdot \mathbf{S}_{hT})}{M_h}. \quad (\text{B.6})$$

The second argument of the fragmentation functions is chosen to be $k'_T = -z\mathbf{k}_T$, which is the transverse momentum of the produced hadron with respect to the quark in a frame in which the quark does not have transverse momentum. In fact, the functions only depend on k'^2_T . As for the distribution functions a division can be made into T-even functions (D_1 , G_{1s} , H_{1T} , $H_{1s}^\perp$, E , $D^\perp$, G_T' , $G_s^\perp$, $H_T^\perp$, H_s) and T-odd functions ($D_{1T}^\perp$, $H_1^\perp$, D_T , $D_L^\perp$, E_s , H).

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