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VRIJE UNIVERSITEIT
EKONOMISCHE FAKULTEIT
AMSTERDAM

QUALITATIVE STRUCTURE ANALYSIS OF
COMPLEX SYSTEMS

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1. Structure of Models

In recent years, the analysis of the causal structure of models has demonstrated a revival. Causality patterns do not only assume a functional relationship among variables, but also a specification of the nature and direction of impacts (leading to testable hypotheses). Lazarsfeld (1954), Wold (1954), and Leitner and Wohlschlägl (1980) have specified some conditions for a causality relationship (see also Blommestein and Nijkamp, 1981):

- causal order (including predictability).
- association (including testability).
- lack of spuriousness.

It should be emphasized that causality is only a feature of a model. There are many ways to study the causality patterns of a model, for instance, by analyzing the structure of the impact or transition matrix in terms of triangularity and diagonality properties (see Rietveld, 1981).

Suppose the following model:

$$\underline{f}(\underline{x}, \underline{z}) = \underline{0} \quad (1)$$

where $\underline{x}$ is a vector of endogenous variables x_i ($i = 1, \dots, I$), $\underline{z}$ a vector of exogenous variables z_j ($j = 1, \dots, J$), and $\underline{f}$ a set of implicit vector functions f_i ($i = 1, \dots, I$). Then a causal structure of model (1) is already given if a set of hypotheses on the structure of the relationships f_i can be generated that define a partition $\underline{y} = \underline{x} \cup \underline{z}$ and a set of binary relationships of the form $f_i \subset y_i$ (the symbol $\subset$ stands for: "contains") (see Gilly, 1980). So far, we have paid no attention to the level of measurement of the variables included in (1). In terms of the analysis of the structure of a model, the following measurement levels may be distinguished:

- a binary relationship, indicating whether or not a certain variable has an impact on another variable; this can be studied by means of graph theory.
- a directional relationship, indicating the qualitative direction of the impact of the one variable upon the other one, for instance, a positive, zero, or negative impact; this can be further analyzed by means of the calculus of qualitative relations (see Greenberg and Maybee, 1981).
- a ordinal relationship, indicating the order of magnitude of impacts in terms of rank orders, this can be studied by means of ordinal econometrics (see Nijkamp and Rietveld, 1982).
- a cardinal relationship, indicating the quantitative impacts between variables; this can be studied by means of conventional regression models.

In the context of the present paper, the attention will mainly be focused on the sign problems associated with qualitative causality analysis implied by qualitative calculus.

2. Qualitative Calculus

The analysis of qualitative relations (or qualitative calculus) has been developed to analyse the relationships between variables in a qualitative way, i.e., when the sign of the impact on some dependent variable is obtained from prior knowledge concerning the signs of the structural parameters. This type of information is especially relevant in economics, ecology and chemistry (see among others Jeffries, 1974; Lancaster, 1962; Pimm, 1982). Voogd (1982) has given a survey of qualitative calculus for spatial impact analysis. Qualitative calculus comes about when the information available has a qualitative character without knowledge of the magnitude of the variables. Suppose for example the following system of linear differential equations:

$$\dot{y} = \frac{dy}{dt} = Ay \quad (2)$$

with y a $(n \times 1)$ - vector and A a squared matrix of order n with elements a_{ij} containing qualitative information, viz. a positive, a negative or a zero sign, depending on the effect of variable j on the rate of change of variable i . We should distinguish between zero as a number and zero as a structure. In terms of qualitative calculus a zero sign is not really a number but represents a structure related to the impacts between variables. One main topic of qualitative calculus is the so called sign-solvability problem, which will be discussed below. When we consider a set of linear equations, expressed in matrix form by

$$Ax = b, \quad (3)$$

in which vector b and matrix A contain qualitative information, the sign-solvability problem gives sufficient as well as necessary conditions for solving (3) for variable x . The solution of (3) is (provided A non-singular:

$$x = A^{-1}b \quad (4)$$

Equation (3) can easily be rewritten into:

$$Cy = 0 \quad (5)$$

where

$$C = [A \mid -b] \quad (6)$$

and

$$y = \begin{bmatrix} x \\ \hline 1 \end{bmatrix} \quad (7)$$

Lancaster (1962) discussed three types of re-arrangements of the matrix C which do not have any consequence for the sign of the solution vector y :

- (1) interchangement of any two rows of C , so that the equations are re-ordered.
- (2) interchangement of any two columns of C , so that the variables are reordered.
- (3) reversement of the signs of any row of C , so that the equation is multiplied by -1 .

When the signs of any column of C are reversed, variable y_i is replaced by $-y_i$. Sufficient conditions for a complete sign-solvability of matrix C are given by Lancaster, viz. when the matrix C of order $n \times (n + 1)$ has the following shape (making use of the above mentioned row and column operations):

$$\text{sign } C = \begin{bmatrix} - & + & \dots & \dots & + \\ 0 & - & + & & \\ 0 & 0 & - & + & \\ . & & \text{---} & & \\ . & & & \text{---} & \\ . & & & & \text{---} \\ 0 & \dots & 0 & - & + \end{bmatrix}$$

Necessary conditions for full sign-solvability which are also sufficient make use of graph theory, especially signed directed graphs. A directed graph (or digraph) is defined as a finite non-empty binary relation on a set of points. A signed digraph is obtained from a digraph by giving signs to the lines. Then equation (3) is fully sign-solvable if and only if it can be put into the form (see also Bassett et al, 1968; Voogd, 1982):

$$Fx = h \quad (8)$$

where F is a matrix of order $n \times n$ with elements f_{ij} , $i, j = 1, \dots, n$ and

- (1) $f_{ii} < 0$, $i=1,2,\dots,n$
- (2) all cycles in F of length greater than one are non-positive. The matrix F has to be non-singular and the terms of this matrix must have the same sign when computing the determinant of F .
- (3) $h < 0$
- (4) if $h_k \neq 0$, then every path $i \rightarrow k$ is non-negative for every i ; $i=1,\dots,n$; $i \neq k$.

Consider for example the following set of equations, for which all of these four conditions hold.

$$\begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ - & 0 & - & + \\ 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ 0 \\ - \end{bmatrix}$$

The qualitative matrix Q , with elements like (6), from which the graph of this model can be determined becomes:

$$Q = \begin{bmatrix} - & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & + \\ - & 0 & - & + & 0 \\ 0 & 0 & 0 & - & + \end{bmatrix}$$

The corresponding graph representation is given in figure 1.

Figure 1. Graph Representation of Qualitative Matrix Q.

The matrix F is non-singular and the model (8) is fully sign-solvable. Then the solution $F^{-1}h$ becomes:

$$x = F^{-1}h = \begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ + & 0 & - & - \\ 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} 0 \\ - \\ 0 \\ - \end{bmatrix} = \begin{bmatrix} 0 \\ + \\ + \\ + \end{bmatrix}$$

While quantitative models encompass information about the direction as well as the size of the changes, qualitative calculus only gives information about the direction of the changes of variables. When a qualitative model is not fully sign-solvable we need additional information to obtain solvable results.

In addition to the sign-solvability, different types of stability characteristics are also studied in qualitative calculus. Because of its importance in studying large-scale systems that are highly interacted, the stability properties will be discussed in a separate section.

3. Stability.

3.1. Introduction.

In this section, the problem of stability will first be discussed in a general sense. Then the implications for qualitative calculus will be outlined in section 5. Research in steady-state situations and in stability conditions

take place in many disciplines (see inter alia problems of stable population growth in demography and biology).

Stability may occur in a local and a global way. They differ from each other by the level of disturbances from a steady-state or equilibrium situation. We speak of local stability when, with small disturbances from the equilibrium situation, the size of the population variables return to the equilibrium situation. But a situation of global stability takes place, if the steady-state solution will be retained for all possible levels of population growth (see also Pimm, 1982).

The stability conditions are developed for models with either difference equations or differential equations, i.e. variables which follow discrete or continuous changes in time, respectively. In the following sections, only problems of local stability will be dealt with, as they are particularly relevant in the framework of qualitative analysis.

Section 3.2 to section 4.3 follow a quantitative approach and contain introductory elements for the qualitative approach in section 5.

3.2. Local Stability and Difference Equations.

Suppose a simple case with two population variables and a discrete changing time parameter, described by means of a linear first-order difference equation. With initial values $x_{1,0}$ and $x_{2,0}$, the model becomes:

$$\begin{aligned}x_{1,t+1} &= c_{11}x_{1,t} + c_{12}x_{2,t} \\x_{2,t+1} &= c_{21}x_{1,t} + c_{22}x_{2,t}\end{aligned} \quad t = 0, 1, \dots \quad (9)$$

or, equivalently, in matrix form:

$$x_{t+1} = C x_t \quad (10)$$

This system is called stable when $x_{t+1} - x_t$ becomes asymptotically zero, which is equivalent to:

$$\lim_{t \rightarrow \infty} C^t = 0 \quad (11)$$

Condition (11) holds if and only if all eigenvalues λ_i , $i=1,2$ of the matrix C are smaller than one in absolute value or, equivalently, if the sum of squared real and imaginary parts are smaller than one, viz.,

$$[\operatorname{Re}(\lambda_i)]^2 + [\operatorname{Im}(\lambda_i)]^2 < 1 \quad i=1,2 \quad (12)$$

The von Neumann-series, defined by $\sum_{t=0}^{\infty} C^t$, is equal to $(I-C)^{-1}$ when the stability condition holds. For a matrix C with non-negative elements, a sufficient and necessary condition for stability is the leading minors of the matrix $(I-C)$ to be positive.

3.3. Local Stability and Differential Equations.

The predator-prey model is a widely used type of modeling in population biology where the growth rate of prey is affected by predator (see also Goh and Jennings, 1977; Jeffries, 1979; Pimm, 1982; Wilson, 1981). It has been developed separately by Lotka and Volterra earlier this century. In the two species case with only one prey and predator, represented by X_1 and X_2 respectively, a typical Lotka- Volterra differential model becomes:

$$\begin{aligned} \dot{X}_1 &= \frac{dX_1}{dt} = X_1 (b_1 - a_{11} X_1 - a_{12} X_2) \\ \dot{X}_2 &= \frac{dX_2}{dt} = X_2 (-b_2 + a_{21} X_1) \end{aligned} \quad (13)$$

The coefficients $b_1, b_2, a_{11}, a_{12}, a_{21}, a_{22}$ are assumed to be positive and the connection between the species is represented by the coefficients $a_{ij}, i,j=1,2$.

Equilibrium solutions are such that the number of species do not change with time, or $\dot{X}_1$ and $\dot{X}_2$ are both equal to zero. Model (13) has two equilibrium solutions: a trivial solution, where both species are equal to zero, is irrelevant for further analysis and the non-trivial solution which is equal to:

$$X_1^* = \frac{b_2}{a_{21}} \quad \text{and} \quad X_2^* = \frac{b_1}{a_{12}} - \frac{a_{11}}{a_{12}} X_1^* \quad (14)$$

While studying local stability a model like (13) is normally linearized for computational reasons by means of a Taylor's series approximation method. Linearization is permitted because only small perturbations close to the equilibrium are dealt with.

When a non-linear function f , like equation (13), is approximated by a linear function g and is evaluated at the equilibrium points

X_1^* and X_2^* , we get respectively:

$$g(\dot{X}_1) \approx f(X_1^*) + (X_1 - X_1^*) (b_1 - 2a_{11} X_1^* - a_{12} X_2^*)$$

and (15)

$$g(\dot{X}_2) \approx f(X_2^*) + (X_2 - X_2^*) (-a_{12} X_1^*)$$

The approximations hold best when X_1 and X_2 are close to X_1^* and X_2^* respectively. This condition holds because local stability around the equilibrium points is studied. The two parts of equation (15) can be added to obtain the equation in a two-dimensional space:

$$\dot{X}_1 = (X_1 - X_1^*) (-a_{11} X_1^*) + (X_2 - X_2^*) (-a_{12} X_1^*) \quad (16)$$

A similar operation is applied to $\dot{X}_2$ of equation (13) to obtain:

$$\dot{X}_2 = (X_1 - X_1^*) (a_{21} X_2^*) \quad (17)$$

The deviation from the equilibrium value $X_i - X_i^*$ will be denoted as x_i . Since $\dot{X}_i = \dot{x}_i$, we can now rewrite (16) and (17), with newly defined constants c_{ij} , in the following way:

$$\begin{aligned}\dot{x}_1 &= \dot{X}_1 = c_{11} x_1 + c_{12} x_2 \\ \dot{x}_2 &= \dot{X}_2 = c_{21} x_1\end{aligned}\tag{18}$$

The problem of local stability of model (18) is equivalent to the question whether perturbations $X_i - X_i^* = x_i$, $i=1,2$, become zero with increasing time. The Lotka-Volterra equations (12) cannot be solved analytically because of their non-linearity, but solutions can be found for the linearized approximations. The solutions of (17) depending on time t become:

$$\begin{aligned}\dot{x}_1 &= p_1 m_{11} \exp(\lambda_1 t) + p_2 m_{12} \exp(\lambda_2 t) \\ \dot{x}_2 &= p_1 m_{21} \exp(\lambda_1 t) + p_2 m_{22} \exp(\lambda_2 t)\end{aligned}\tag{19}$$

p_i, λ_i and m_{ij} are all constants. Stability of (19) depends on the sign of the largest eigenvalue λ_1 . Stability occurs with increasing time when both (real parts of the) eigenvalues are negative; when either of them is positive the system is unstable.

If the largest eigenvalue is negative the system is locally stable; in that case we do not know how fast an equilibrium solution is achieved after perturbations.

Equation (19) indicates how fast the stable situation is reattained: when the largest eigenvalue is very large in negative value both $\dot{x}_1$ and $\dot{x}_2$ become zero fastly. For that reason the return time is defined as minus the reciprocal of the real part of the largest eigenvalue (see also Pimm, 1982), i.e.:

$$\text{Return time} = - \frac{1}{\text{Re}(\lambda_{\max})} \quad \text{when } \lambda_{\max} < 0\tag{20}$$

4. Degree of Stability in Linear Differential Equations.

4.1. Introduction.

There are two different time-parameters in the analyses of stability. On the one hand we have the differential equation models with populations that follow a continuously changing pattern through time. The difference equation models consider, in addition to the differential models, distinct generations. The differential equation models, just like the Lotka-Volterra models discussed above, are based on the following assumptions which may lead to inconsistencies in practical applications (see also Goh and Jennings, 1977):

- a) the members of each population have identical properties.
- b) each population variable reacts immediately to changes in the population variables.
- c) the density of each variable changes continuously at all levels of the variables.

A set of linear differential equations like equation (2) is locally stable (also called stable in the Liapunov sense), if and only if the eigenvalues of the matrix A have negative real parts. A set of linear difference equations is called locally stable if the sum of squared real and imaginary parts of all eigenvalues is less than one. Conditions for stability are developed in this section for linear differential equations, while a set of linear difference equations can be interpreted as a discrete approximation of a set of linear difference equations.

The eigenvalues and eigenvectors of the matrix A from formula (2) are defined by:

$$AU = UD_{\lambda} \quad (21)$$

and

$$VA = D_{\lambda} V \quad (22)$$

where the columns of U and the rows of V are the matrices of right and left eigenvectors of A (with $V = U^{-1}$) and the elements of the diagonal matrix D_{λ} are the eigenvalues of A. Conditions for stability are:

$$a_{ii} < 0 \text{ for all } i, \text{ and } a_{ii} < 0 \text{ for at least one } i \quad (23)$$

and

$$|a_{ii}| \geq \left| \sum_{j \neq i} a_{ji} \right| \text{ for all } i \quad (24)$$

In addition to stability as such, Halfon (1976) has also paid attention to the degree of stability of a system, based on sensitivities of the eigenvalues. The eigenvalue sensitivity as well as the relative sensitivity is defined as:

$$\text{sensitivity} = \frac{\partial \lambda_m}{\partial a_{ij}} = v_{mi} u_{jm} \quad (25)$$

and:

$$\text{relative sensitivity} = \frac{\partial \lambda_m}{\partial a_{ij}} \Delta a_{ij} \frac{100}{|\lambda_m|} , \quad (26)$$

with λ_m the real part of the eigenvalue (see also Halfon, 1976 for a proof).

4.2. Numerical Illustration

Local stability and sensitivity of parameters is discussed by means of the following six differential equation models. The differential equations consist of two sets of three interrelated models (see also Parker, 1975), with equilibrium solutions between brackets:

$$\begin{cases} \dot{x}_1 = (2x_2 - 1)x_1 \\ \dot{x}_2 = -\frac{1}{2}(2x_2 - 1)x_1 \end{cases} \quad [(x_1^*, x_2^*) = (x_1^*, \frac{1}{2}) = (1, \frac{1}{2})] \quad (27a)$$

$$\begin{cases} \dot{x}_1 = (2x_2 - 1)x_1 \\ \dot{x}_2 = x_3 - x_1x_2 \\ \dot{x}_3 = \frac{1}{2}x_1 - x_3 \end{cases} \quad [(x_1^*, x_2^*, x_3^*) = (1, \frac{1}{2}, 0)] \quad (27b)$$

$$\begin{cases} \dot{x}_1 = (2x_2 - 1 - x_4)x_1 \\ \dot{x}_2 = (\frac{x_3}{x_2} - x_1)x_2 \\ \dot{x}_3 = (\frac{1}{2}\frac{x_1}{x_3} - 1 + \frac{x_1x_4}{4x_3} + \frac{1}{2}\frac{x_4}{x_3})x_3 \\ \dot{x}_4 = (\frac{1}{2}x_1 - 1)x_4 \end{cases} \quad [(x_1^*, x_2^*, x_3^*, x_4^*) = (2, \frac{1}{2}, 1, 0)] \quad (27c)$$

$$\begin{cases} \dot{x}_1 = (\frac{2x_2}{\frac{1}{2} + x_2} - 1)x_1 \\ \dot{x}_2 = -\frac{1}{2}(\frac{2x_2}{\frac{1}{2} + x_2} - 1)x_1 \end{cases} \quad [(x_1^*, x_2^*) = (0, \frac{1}{2})] \quad (28a)$$

$$\begin{cases} \dot{x}_1 = (\frac{2x_2}{\frac{1}{2} + x_2} - 1)x_1 \\ \dot{x}_2 = x_3 - \frac{x_1x_2}{(\frac{1}{2} + x_2)} \\ \dot{x}_3 = \frac{1}{2}x_1 - x_3 \end{cases} \quad [(x_1^*, x_2^*, x_3^*) = (2x_3^*, \frac{1}{2}, x_3^*) = (1, \frac{1}{2}, \frac{1}{2})] \quad (28b)$$

$$\begin{cases} \dot{x}_1 = \left(\frac{2x_2}{\frac{1}{2} + x_2} - 1 - x_4 \right) x_1 \\ \dot{x}_2 = x_3 - \frac{x_1 x_2}{\frac{1}{2} + x_2} \\ \dot{x}_3 = \frac{1}{2} x_1 - x_3 + \frac{1}{4} x_1 x_4 + \frac{1}{2} x_4 \\ \dot{x}_4 = \frac{1}{2} x_4 x_1 - x_4 \end{cases} \quad \left[(x_1^*, x_2^*, x_3^*, x_4^*) = \left(2, x_2^*, \frac{2x_2^*}{\frac{1}{2} + x_2^*}, \frac{x_2^* - \frac{1}{2}}{\frac{1}{2} + x_2^*} \right) = (2, \frac{1}{2}, 1, 0) \right] \quad (28c)$$

The non-linear differential equations in (27a) - (28c) are linearized by means of Taylor's approximation method and the matrices A of the model (2) become:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} & A_2 &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} & A_3 &= \begin{bmatrix} 0 & 4 & 0 & -2 \\ -\frac{1}{2} & -2 & 1 & 0 \\ \frac{1}{2} & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ A_4 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & A_5 &= \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 \end{bmatrix} & A_6 &= \begin{bmatrix} 0 & 2 & 0 & -2 \\ -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The eigenvalues of the above six matrices, indicating stability of the corresponding linear differential equations, are given in the following table.

Matrix	Eigenvalues				Stability
A ₁	$\lambda_1=0$	$\lambda_2=-1$			Stable
A ₂	$\lambda_1=0.47$	$\lambda_2=-1.23+0.79i$	$\lambda_3=-1.23-0.79i$		Unstable
A ₃	$\lambda_1=0$	$\lambda_2=0.52$	$\lambda_3=-1.76+0.86i$	$\lambda_4=-1.76-0.86i$	Unstable
A ₄	$\lambda_1=0$	$\lambda_2=0$			Unstable
A ₅	$\lambda_1=0$	$\lambda_2=-0.50$	$\lambda_3=-1$		Stable
A ₆	$\lambda_1=0$	$\lambda_2=0$	$\lambda_3=-1+i$	$\lambda_4=-1-i$	Stable

Table 1. Eigenvalues of Matrices A₁, A₂, ..., A₆

The above matrices are part of two series of models, viz. on the one hand A₁, A₂, A₃ and on the other hand A₄, A₅ and A₆. Models A₂ and A₃ are both unstable because the real parts of one of the eigenvalues is positive while models A₅ and A₆ are stable with non-positive real parts of the eigenvalues. This can easily be seen when the eigenvalues are substituted into formula (19). The sensitivity of the elements a_{ij} of the (stable) matrices A₁, A₅ and A₆ is determined by formula (25). The results are given in Table 2. It has to be noted that the eigenvectors are determined up to their length so that the sensitivity of the elements a_{ij} have to be compared in a relative way.

Matrix A ₁	a ₁₁	a ₂₁	a ₁₂	a ₂₂
λ_1	1	2	0	0
λ_2	0	-2	0	1

Matrix A ₅	a ₁₁	a ₂₁	a ₃₁	a ₁₂	a ₂₂	a ₃₂	a ₁₃	a ₂₃	a ₃₃
λ_1	2	0	0	-1	0	0	1	0	0
λ_2	0	0	0	0	0	0	0	0	0
λ_3	0	0	0	1	0	-2	-1	0	2

Matrix A ₆	a ₁₁	a ₂₁	a ₃₁	a ₄₁	a ₁₂	a ₂₂	a ₃₂	a ₄₂	a ₁₃	a ₂₃	a ₃₃	a ₄₃	a ₁₄	a ₂₄	a ₃₄	a ₄₄
$\lambda_1=\lambda_2$	-4	-2	-2	0	2	1	1	0	0	0	0	0	2	1	1	0
$\text{re}(\lambda_3)=\text{re}(\lambda_4)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2. Sensitivity of Elements a_{ij} from the Stable Matrices A₁, A₅ and A₆.

5. Qualitative Stability

5.1. Introduction.

The stability conditions with linear differential and linear difference equations measured in a cardinal sense were discussed in the previous section. A matrix A with elements a_{ij} is called asymptotically stable, if the real part of each of its eigenvalues is negative. The purpose of the present section is to analyse stability conditions if only qualitative information about the interaction element a_{ij} is available, viz. knowledge about the signs $(-, +, 0)$. This means that we will analyse whether or not a system will tend toward an equilibrium point after a perturbation. Qualitative stability or sign stability can be interpreted easily in the following way (see also Jeffries, 1974; Quirk, 1981; Roberts, 1978; Wilson, 1981). Suppose the magnitudes (but not the sign) of the non-zero cell-elements of the matrix A are changed at random. If all of these matrices A are asymptotically stable, matrix A which only contains qualitative information, will be called sign stable or qualitative stable.

Necessary conditions for qualitative stability (in an asymptotic way) of a linear system of differential equations like (2), are formulated by Quirk and Ruppert (1965). These conditions, which are related to the so-called Routh-Hurwitz theorem, become:

- (1) the diagonal elements of A be all non-positive, i.e. $a_{ii} \leq 0$ for all i .
 - (2) $a_{ii} < 0$ for at least one i ($a_{ii} < 0$ occurs when this variable or species is self-regulating).
 - (3) $a_{ij} a_{ji} < 0$ for all $i \neq j$.
 - (4) if $i_1 \neq i_2 \neq \dots \neq i_m$ ($m \geq 3$), $a_{i_1 i_2} \neq 0$, $a_{i_2 i_3} \neq 0$, $a_{i_{m-1} i_m} \neq 0$ then $a_{i_m i_1} = 0$, or: $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_m i_1} = 0$
- for any sequence of three or more distinct indices $i_1, \dots, i_m$. This means there are no cycles of length m (for $m \geq 3$) in the graph which corresponds to the matrix A .
- (5) The matrix A is non-singular.

The above mentioned conditions are only necessary, but by no means sufficient. For instance, Jeffries et al (1977) have developed an example which fulfills all above mentioned conditions but is not asymptotically stable. Consider the following matrix A:

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{sign}(A) = \begin{bmatrix} 0 & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{bmatrix}$$

The eigenvalues of this matrix are (approximately) equal to -0.36 , i , $-i$, $-0.32 + 0.16i$ and $-0.32 - 0.16i$. Although these five conditions hold for the matrix A, two out of five eigenvalues do not contain negative real parts. Unsufficiency of the five conditions can also be seen when the following trajectory is considered (see also Jeffries et al, 1977).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \\ 0 \\ \sin t \\ -\cos t \end{bmatrix}$$

The changes of $x_1, \dots, x_5$ due to small changes in time do not stabilize but they fluctuate over time by means of the noted sine and cosine functions.

The above mentioned five necessary conditions have been reformulated so that necessary as well as sufficient conditions for qualitative stability are obtained (see also Jeffries, 1974 and Jeffries et al, 1977). The use of signed directed graphs (or, shortly signed digraphs) is a meaningful tool to visualize qualitative asymptotic stability. A digraph consists of n points (called vertices) connected to each other by at most n^2 directed lines (edges) representing interaction effects between the variables. The qualitative nature of the effects is denoted by a "+", a "-", or a zero-sign. The qualitative matrix A can be represented by sign digraphs in the following way:

Figure 2. Graph Representation of a Qualitative Matrix.

The above mentioned figure for analysing stability in a qualitative way is applied especially with growth patterns in ecological/economic systems with only limited information. An illustration in ecological terms is given in the following section.

5.2. Illustration for Ecological Modeling.

The graph representation in figure 2 consists of four 2-cycles, while each 2-cycle consists of one positive effect and one negative effect; such 2-cycles can be interpreted as a predator and a prey, also called a predation link (when applied to biology). Variable 3 is called a self-regulating species because

$a_{33} < 0$. A predation community is also defined for necessary conditions of qualitative stability. A predation community is defined by the maximum set of species which are related to each other by predation links. A species which is not related to other species by a predation link is also called a predation community. From this follows that a digraph with no predation links consists of n predation communities. The signed digraph from figure 2 consists of only one predation community with species (1,2,3,4,5). Condition (2) appears to be a necessary condition for qualitative stability. If in addition to $a_{ii} < 0$ for at least one i the following condition also holds, one finds the sufficient conditions:

- (2)' each predation community in a signed digraph with elements (a_{ij}) fails the color test.

A predation community is in agreement with the color test when coloring black and white and:

- a) each self-regulating point is colored black.
- b) there is at least one white point.
- c) each white point is connected by a predation link to at least one other white point.
- d) when a black point is connected to a white point by a predation link, it also is connected by a predation link to at least one other white point.

Sufficient conditions for asymptotic stability are given by (1), (2)', (3), (4) and (5). The signed digraph of figure 2 is in agreement with the color test when species 3 is colored black and the other species colored white, and the signed digraph is not necessarily asymptotically stable. Asymptotic stability of this digraph holds when either species 1 or species 5 (or both) are self-regulating.

When species 1 is considered to be self-regulating, the matrix A of interaction effects with its (approximated) eigenvalues becomes:

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \lambda_1 &= -0.78 \\ \lambda_2 &= -0.51 + 1.51i \\ \lambda_3 &= -0.51 - 1.51i \\ \lambda_4 &= -0.10 + 1.09i \\ \lambda_5 &= -0.10 - 1.09i \end{aligned}$$

The color test does not pass when all species are self-regulating points (i.e. $a_{ii} < 0$ for all i) and the qualitative matrix A is sign stable when condition (3) and (4) also hold (the matrix A is non-singular since $a_{ii} < 0$ for all i).

The conditions (1), (2)', (3), (4) and (5) are related to the matrices A_1 , A_2 , ..., A_6 from the previous section and if some matrix passes all these conditions (represented by a "+" sign in the following table), it is qualitative stable.

Condition						
Matrix	(1)	(2)'	(3)	(4)	(5)	Qualitative Stable
A_1	+	+	+	+	-	No
A_2	+	+	+	-	+	No
A_3	+	+	+	-	-	No
A_4	+	+	+	+	-	No
A_5	+	+	+	+	-	No
A_6	+	+	+	-	-	No

Table 3. Qualitative Stability of Matrices $A_1, \dots, A_6$.

The five necessary as well as sufficient conditions for qualitative stability are formulated in this section in ecological terms.

A self-regulating point as well as a predation link can be reformulated in general terms. A self-regulating point has a negative impact of the growth-rate into itself (i.e. a negative diagonal cell-entry). A predation link consists of two points which are mutually related to each other with reversed effects (i.e. a positive and a negative cell-entry). When reformulating the conditions in general terms, the color test becomes a useful tool in the analysis of qualitative stability in dynamic systems.

6. Conclusion

This paper deals with qualitative analysis of model structures concerning the sign-solvability conditions and the stability properties of matrices characterized by the sign pattern $(+,-,0)$ of their entries. Necessary as well as sufficient conditions for full sign-solvability are given which make use of a graph-theoretical representation of the qualitative interaction-effects.

In addition to it we discussed stability in large-scale systems that are highly interacted with qualitative information of the sign patterns. Necessary as well as sufficient conditions for sign stability are formulated.

The qualitative aspects of stability and sign-solvability are rather promising when only limited information is available, especially in economics, biology, ecology and chemistry.

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