VU Research Portal

Local balance in queueing networks with positive and negative customers

Boucherie, R.J.; van Dijk, N.M.

1992

document version

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

citation for published version (APA)

Boucherie, R. J., & van Dijk, N. M. (1992). *Local balance in queueing networks with positive and negative customers*. (Serie Research Memoranda; No. 1992-1). Faculty of Economics and Business Administration, Vrije Universiteit Amsterdam.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal ?

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

E-mail address:

vuresearchportal.ub@vu.nl

1992-1

ET

05348

Faculteit der Economische Wetenschappen en Econometrie

Bibliotheek Economie
Tijdschriften/reeksen
Kaer 32-02

Serie Research Memoranda

Local balance in queueing networks with
positive and negative customers

R.J. Boucherie
N.M. van Dijk

Research Memorandum 1992-1
Januari 1992

Local balance in queueing networks with positive and negative customers

Richard J. Boucherie* Nico M. van Dijk†

Abstract

This paper investigates to what extent a recently developed new product form result for queueing networks with positive and negative customers fits into the class of product form queueing networks that satisfy a notion of partial or local balance. As such this paper investigates whether this new product form is still a consequence of an appropriate notion of local balance. To this end a new and non-standard type of local balance is introduced as an extension of standard local balance. This new type of local balance appears more restrictive and is no longer directly sufficient for global balance. Nevertheless, based on this new type of local balance, some extensions such as blocking phenomena for queueing networks with positive and negative customers can be concluded.

Keywords: Local balance, Negative customers, Product form, Queueing network, Traffic equations.

1 Introduction

Recently, in [6], a queueing network with positive and negative customers was introduced, motivated by neural networks. By analogy with neural networks a station represents a neuron. Positive customers routing from one station to another represent excitation signals, adding one unit to the neuron potential when it arrives, whereas negative customers routing from one station to another represent inhibition signals, reducing the potential of the neuron to which it arrives by one unit [4]. In queueing networks with positive and negative customers the positive customers behave similar to normal customers in a Jackson network [1], [9]. Negative customers, however, behave quite differently. When a negative customer enters a station the queue length is reduced by one. The queue at a station consists of positive customers only. Upon leaving a station a customer either remains a positive customer or, with fixed probability, becomes a negative customer. The routing of customers depends on their sign,

*Department of Econometrics, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands

†Department of Econometrics, University of Amsterdam, Jodenbreestraat, 1011 NH Amsterdam, The Netherlands

and is assumed to be state-independent as in Jackson networks. In addition to customers routing among the stations, customers enter the queueing network according to a Poisson process. In [6] the queue length at the stations of the queueing network is allowed to be non-negative only. This represents the additional assumption made in [4] that the neuron potential is not allowed to become negative. If a negative customer arrives at a station with queue length 0 it is lost. Another application of queueing networks with positive and negative customers is to multiple resource systems with requests for service (positive customers) and decisions to cancel a request (negative customers) [6]. In this application it is obvious that the number of requests can be non-negative only. Decisions to cancel a request are ignored if no requests are present. When negative customers are involved, however, it seems logical to allow a negative number of customers at a station too (backlog). A negative number of customers represents that a number of negative customers has entered the station when no positive customers were present or that a number of service completions has occurred when no customers were present. Due to these completions a shortage of customers is generated at the station, i.e. the number of customers has become negative.

Let us first give an example to illustrate the behaviour of a queueing network with positive and negative customers. This example will be used throughout the paper to illustrate our results and is discussed extensively in Section 5. The blocking results that will be obtained for this example are characteristic for the results for general networks and the limitations already illustrate that there is little hope for much extensions of the results presented in this paper. Note that the results of this paper are not limited to the case of a two station queueing network.

Example 1.1 (Two station queueing network) Consider the two station queueing network depicted in Figure 1. Both stations only have external arrivals of positive customers. Customers completing service at a station route to the other station as negative customers. If the service rate at the stations is "fast enough" then this queueing network is stable and possesses a product form equilibrium distribution (cf. Section 5). Note that there are no customers leaving the queueing network. This shows that queueing networks with negative customers are a very special type of queueing network. $\square$

In [6] it is shown that a queueing network with negative and positive customers and fixed routing probabilities as described above, possesses a product form equilibrium distribution if a solution exists to a set of non-linear customer flow equations. The customer flow equations presented in [6] do not seem to express a notion of local balance. As a consequence, the queueing network with positive and negative customers introduced in [6] seems to be a counter-example to the generally believed statement for queueing networks that a form of local balance is directly related to a product form equilibrium distribution. Here local (or partial) balance is to be read as a more detailed way by which global balance equations of rates into and out of one and the same state are satisfied in a decomposed manner of balanced subsets of these equations.

Figure 1. Two station queueing network

This paper aims to investigate the relation between the product form equilibrium distribution obtained in [6] and an appropriate notion of local balance. Indeed, this paper shows that the product form result obtained in [6] is not a consequence of any notion of standard local balance. However, a new and non-standard type of local balance which is specially dedicated to the specific model is introduced. The product form obtained in [6] is shown to satisfy this new and non-standard type of local balance. The major difference between standard local balance and local balance for queueing networks with positive and negative customers is that in contrast to local balance appearing in the literature, local balance for queueing networks with positive and negative customers does not compare the probability flow into and out of one and the same state (cf. Remark 2.5). As such it is shown that queueing networks with positive and negative customers do not fit into any class of queueing networks which satisfy a standard form of local balance.

From the literature on product form queueing networks we obtain that a product form equilibrium distribution $\pi(\bar{n}) = \prod_i q_i^{n_i}$ satisfies local balance if the coefficients $\{q_i\}$ are a solution of the traffic equations. The traffic equations state that the average input rate of a station equals the average output rate of that station. As is shown in Remark 3.3 the traffic equations introduced here do state this equivalence. These traffic equations are a set of non-linear equations. Note that for processes with product form equilibrium distribution this is not uncommon in the literature (e.g. polymerization processes [2], [10], [12], batch routing queueing networks [3], [7]). Therefore, in this respect local balance for queueing networks with positive and negative customers does fit into local balance as appearing in the literature.

We thus incorporate queueing networks with positive and negative customers introduced in [6] into an extended class of queueing networks which satisfy local

balance and extend local balance to processes for which the rate out of a state is not balanced by the rate into that state as a direct consequence of local balance. Local balance gives insight into the behaviour of the Markov process describing the queueing network. Most particularly, local balance allows us to introduce boundaries to the state space, i.e. to consider blocking phenomena.

In contrast to a standard queueing network (Jackson network), where a unique solution to the traffic equations is guaranteed if the process is irreducible, for a queueing network with positive and negative customers a unique solution to the traffic equations is not a direct consequence of irreducibility. For the two station queueing network discussed in Example 1.1 it is shown that a unique positive solution exists to the traffic equations. For queueing networks consisting of more than two stations some classes of queueing networks for which a unique solution to the traffic equations exists are discussed. Finally it is argued that the traffic equations are characteristic equations to the queueing network as they determine not only the invariant measure, but also the shape of the state space and the blocking conditions used at the boundaries of the state space (cf. Section 4.5).

Summarizing, in this paper, for the queueing network with positive and negative customers introduced in [6] it is shown that the product form equilibrium distribution obtained in [6] is not a consequence of standard local balance, but is a consequence of a very special type of local balance. To this end we

- introduce a new and non-standard type of local balance;
- introduce the traffic equations for a queueing network with positive and negative customers;
- show that Gelenbe's result [6] fits into the extended class of product form queueing networks satisfying local balance.

The results obtained in [6] are generalized to

- a necessary and sufficient condition for the existence of a product form equilibrium distribution;
- blocking phenomena due to capacity constraints at the stations;
- also include a negative number of customers.

The paper is organized as follows. Section 2 presents the model. We have chosen to use a theoretical setup and give an interpretation of the model as queueing network with positive and negative customers afterwards as this is mathematically more elegant. In Section 3 blocking of transitions due to local balance is considered. In particular, lower and upper bounds are discussed extensively. Section 4 considers the traffic equations. Conditions guaranteeing a unique positive solution to the traffic equations are given. Furthermore, Section 4.5 argues that the traffic equations are characteristic equations to the queueing network. Section 5 discusses the two station queueing network introduced in Example 1.1 and finally Section 6 concludes the paper with some general remarks on generalizations of the model discussed in this paper.

2 Model and local balance

Consider a stable, regular continuous-time Markov chain, X , at state space $S \subset Z^N = \{\dots, -2, -1, 0, 1, 2, \dots\}^N$. A state $\bar{n} \in S$ is a vector with components n_i , $i = 1, \dots, N$. The transition rates, $Q = (q(\bar{n}, \bar{n}'), \bar{n}, \bar{n}' \in Z^N)$, are given by

$$q(\bar{n}, \bar{n}') = \begin{cases} \Lambda(i), & \bar{n}' = \bar{n} + e_i, i = 1, \dots, N, \\ r(i)d(i) + \lambda(i), & \bar{n}' = \bar{n} - e_i, i = 1, \dots, N, \\ r(i)p^+(i, j), & \bar{n}' = \bar{n} - e_i + e_j, i, j = 1, \dots, N, \\ r(i)p^-(i, j) + r(j)p^-(j, i), & \bar{n}' = \bar{n} - e_i - e_j, i, j = 1, \dots, N, \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

where $\Lambda, r, d : \{1, \dots, N\} \rightarrow [0, \infty)$, $p^+, p^- : \{1, \dots, N\}^2 \rightarrow [0, \infty)$ and $d(i)$, $i = 1, \dots, N$, $p^+(i, j)$, $p^-(i, j)$, $i, j = 1, \dots, N$, are such that for all i , $i = 1, \dots, N$,

$$d(i) + \sum_{j=1}^N \{p^+(i, j) + p^-(i, j)\} = 1, \quad (2.2)$$

and $p^+(i, i) = p^-(i, i) = 0$, $i = 1, \dots, N$.

Interpretation 2.1 (Queueing network) The Markov process with transition rates (2.1) can be interpreted as representing a queueing network with positive and negative customers. To this end consider an open queueing network consisting of N stations. The state of a queue represents the number of customers at the queue, i.e. n_i is the number of customers at queue i , $i = 1, \dots, N$. Customers at queue i are served at exponential rate $r(i)$ independent of the state of the other queues. Upon departure from a queue, say queue i , for example due to external influence, a customer receives a positive sign with probability $p^+(i)$ and a negative sign with probability $p^-(i)$. Subsequently, a positive customer is routed to queue j with probability $\tilde{p}^+(i, j)$ and leaves the network with probability $d^+(i)$, and a negative customer is routed to queue j with probability $\tilde{p}^-(i, j)$ and leaves the network with probability $d^-(i)$. When a positive customer enters a queue the number of customers at that queue increases by one and when a negative customer enters a queue the number of customers at that queue decreases by one. External arrivals to the queueing network can either be positive customers which arrive at queue i according to a Poisson process with rate $\Lambda(i)$ or negative customers arriving at queue i according to a Poisson process with rate $\lambda(i)$. As negative customers are involved it is natural to allow the number of customers at a queue to be negative. A negative number of customers represents the absence of a number of customers due to the arrival of negative customers or the departure of customers. The transition rates of the Markov process representing this queueing network are given in (2.1), where $p^+(i, j) = p^+(i)\tilde{p}^+(i, j)$, $p^-(i, j) = p^-(i)\tilde{p}^-(i, j)$, $d(i) = p^+(i)d^+(i) + p^-(i)d^-(i)$, and $r(i)$ is the rate at which customers are expelled from station i . $\square$

The following theorem gives a sufficient condition for X to possess an invariant measure, i.e. a set of non-negative numbers, $m = (m(\bar{n}), \bar{n} \in Z^N)$, that satisfies the global balance equations

$$\sum_{\bar{n}' \neq \bar{n}} \{m(\bar{n})q(\bar{n}, \bar{n}') - m(\bar{n}')q(\bar{n}', \bar{n})\} = 0, \quad \bar{n} \in Z^N. \quad (2.3)$$

Theorem 2.2 If $\{q_i\}_{i=1}^N$ is a positive solution of

$$\gamma_i r(i) + \sum_{j=1}^N \gamma_i \gamma_j r(j) p^-(j, i) + \gamma_i \lambda(i) = \sum_{j=1}^N \gamma_j r(j) p^+(j, i) + \Lambda(i), \quad i = 1, \dots, N, \quad (2.4)$$

then X possesses an invariant measure, m , at Z^N given by

$$m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}, \quad \bar{n} \in Z^N. \quad (2.5)$$

Proof Insertion of m in the global balance equations (2.3) gives for $\bar{n} \in Z^N$

$$\begin{aligned} & \sum_{\bar{n}' \neq \bar{n}} \{m(\bar{n})q(\bar{n}, \bar{n}') - m(\bar{n}')q(\bar{n}', \bar{n})\} \frac{1}{m(\bar{n})} \\ &= \sum_{i=1}^N \left\{ \Lambda(i) + r(i)d(i) + \lambda(i) + \sum_{j=1}^N \{r(i)p^+(i, j) + r(i)p^-(i, j)\} \right\} \\ & \quad - \sum_{i=1}^N \left\{ \frac{1}{q_i} \Lambda(i) + q_i r(i)d(i) + q_i \lambda(i) + \sum_{j=1}^N \left\{ \frac{q_j}{q_i} r(i)p^+(i, j) + q_i q_j r(i)p^-(i, j) \right\} \right\} \\ &\stackrel{(*)}{=} \sum_{i=1}^N \left\{ \Lambda(i) - \sum_{j=1}^N q_i q_j r(i)p^-(i, j) - q_i r(i)d(i) - q_i \lambda(i) \right\} \\ & \quad + \sum_{i=1}^N \left\{ r(i) + \lambda(i) - \sum_{j=1}^N \frac{q_j}{q_i} r(i)p^+(i, j) - \frac{1}{q_i} \Lambda(i) \right\} \\ &\stackrel{(**)}{=} \sum_{i=1}^N \left\{ \Lambda(i) + \sum_{j=1}^N q_j r(j)p^+(j, i) - q_i r(i) - \sum_{j=1}^N q_i q_j r(i)p^-(i, j) - q_i \lambda(i) \right\} \\ & \quad + \sum_{i=1}^N \frac{1}{q_i} \left\{ q_i r(i) + \sum_{j=1}^N q_i q_j r(i)p^-(j, i) + q_i \lambda(i) - \sum_{j=1}^N q_j r(j)p^+(j, i) - \Lambda(i) \right\} \\ &\stackrel{(***)}{=} 0, \end{aligned}$$

where in $(*)$ terms are rearranged such that the global balance equations resemble (2.4) and $(**)$ is obtained from (2.2). $\square$

Remark 2.3 (Normalization) The invariant measure, m , given in (2.5) cannot be normalized at Z^N . Thus, although X possesses a product form invariant measure at Z^N , X does not possess a product form equilibrium distribution at Z^N . If X is truncated to a subset of Z^N which is bounded from above or bounded from below, m can be normalized. Based on local balance, X can be truncated. Local balance is discussed below and the truncation of X is discussed in Sections 3.1 and 3.2. $\square$

Remark 2.4 (Traffic equations (2.4)) Equation (2.4) is the generalization of the traffic equations to networks that allow negative customers too. This can be seen by setting $p^-(i, j) = 0$, $i, j = 1, \dots, N$, $\lambda(i) = 0$, $i = 1, \dots, N$, thus removing the influence of negative customers from (2.4). Then, for $c_i = \gamma_i r(i)$, $i = 1, \dots, N$, (2.4) reduces to the standard traffic equations

$$c_i = \Lambda(i) + \sum_{j=1}^N c_j p^+(j, i), \quad i = 1, \dots, N. \quad (2.6)$$

Therefore, henceforth (2.4) will be called traffic equations. $\square$

Remark 2.5 (Local balance; traffic equations) As can be seen from the proof of Theorem 2.2 above, the assumption that $\{q_i\}_{i=1}^N$ satisfies the traffic equations (2.4) can be relaxed to $\{q_i\}_{i=1}^N$ is such that (***) is satisfied. However, similar to the assumptions in a standard queueing network, where one assumes that a solution exists to the standard traffic equations (2.6), the assumption that $\{q_i\}_{i=1}^N$ satisfies the traffic equations (2.4) implies that the invariant measure m given in (2.5) satisfies a type of local balance. To this end, define q_i^* , the transition rates for negative customers leaving station i to route to a station of the queueing network: $q_i^*(\bar{n}, \bar{n} - e_i - e_j) = r(i)p^-(i, j)$. Then $q(\bar{n}, \bar{n} - e_i - e_j) = q_i^*(\bar{n}, \bar{n} - e_i - e_j) + q_j^*(\bar{n}, \bar{n} - e_i - e_j)$. With this definition, from the traffic equations we obtain that m satisfies for all $\bar{n} \in Z^N$, $i = 1, \dots, N$

$$\begin{aligned} & \sum_{j=0}^N m(\bar{n}) q(\bar{n}, \bar{n} - e_i + e_j) \\ & + \sum_{j=1}^N \left\{ m(\bar{n}) q_i^*(\bar{n}, \bar{n} - e_i - e_j) + m(\bar{n} + e_j) q_j^*(\bar{n} + e_j, \bar{n} - e_i) \right\} \\ & = \sum_{j=0}^N m(\bar{n} - e_i + e_j) q(\bar{n} - e_i + e_j, \bar{n}). \end{aligned} \quad (2.7)$$

This notion of *local balance for queueing networks with positive and negative customers* expresses that the outate of station i due to the arrival of negative customers to station i from the outside (first term in the left-hand side), due to the departure of customers from station i (first and second term in the left-hand side of (2.7)), and due to the arrival of negative customers from other stations (third term on the left-hand side) balances with the inrate of station i due to the arrival of positive customers. In the case of lower or upper bounds, the traffic equations are necessary and sufficient for the existence of a product form invariant measure as is shown in Sections 3.1 and 3.2. Therefore, the assumption that a solution exists to the traffic equations (2.4) is a very natural assumption. $\square$

Remark 2.6 (Local balance; relation to global balance) In Remark 2.5 above, the notion of local balance for queueing networks with positive and negative customers is introduced. Local balance as given in (2.7) expresses that the outate of station i

due to the departure of a customer either due to a service completion or to the arrival of a negative customer balances with the inrate of station i due to the arrival of a positive customer. However, note that *the third term in the left-hand side does not correspond to the outrate of customers from station i in state $\bar{n}$ but in state $\bar{n} + e_j$* . Therefore, local balance as expressed by (2.7) is not of the standard type where global balance decomposes into local balance for each state separately. As a consequence, if a measure satisfies local balance it remains to be shown that this measure satisfies global balance. This cannot be concluded by summation of the local balance equations over the stations as is the case for a standard queueing network. Also, the result of Theorem 2.2 is not a consequence of standard local balance, but is obtained by applying (2.4) for several states, once for state $\bar{n}$ and once for state $\bar{n} - e_i$, $i = 1, \dots, N$. Therefore, *the result of Theorem 2.2 is not a consequence of standard local balance*. In fact, the notion of local balance introduced in (2.7) does not correspond to notions of local balance as reported in the literature (also see Remark 3.3, where the interpretation of (2.4) is formalized and Remark 2.7, where local balance for queueing networks with negative customers is contrasted to local balance as appearing in the literature). $\square$

Remark 2.7 (Local balance; literature) This remark shows that the result of Theorem 2.2 cannot be concluded from local balance as described in the literature. To this end, local balance as expressed by (2.7) is compared to local balance as appearing in the literature.

- **Standard Jackson network:** Transitions in which customers leave at two stations simultaneously, i.e. transitions of the type $\bar{n} \rightarrow \bar{n} - e_i - e_j$, are not allowed in a single changes queueing network. Therefore (2.7) cannot be concluded from local balance as appearing in the literature on single changes queueing networks.
- **Batch routing network:** In batch routing queueing networks, for the network to possess a product form equilibrium distribution, it is assumed that a batch $\bar{g} = (g_1, \dots, g_N)$ that can leave is allowed to enter also (cf. [3]). This can be seen when considering the batch traffic equations. For a batch routing queueing network with state independent routing, for the process to possess a product form equilibrium distribution it is necessary and sufficient that for all $\bar{g} \in N_0^N$ a non-negative solution $\{q_i\}_{i=1}^N$ exists to

$$\sum_{\bar{g}' \neq \bar{g}} \left\{ \prod_{k=1}^N \gamma_k^{g_k} p(\bar{g}, \bar{g}') - \prod_{k=1}^N \gamma_k^{g'_k} p(\bar{g}', \bar{g}) \right\} = 0. \quad (2.8)$$

As a transition $\bar{n} \rightarrow \bar{n} - e_i - e_j$ in which a customer leaves at both queue i and queue j is allowed, it must be the case that $p(e_i + e_j, \bar{0}) > 0$. However, a transition in which a customer arrives at both queue i and queue j is not allowed in a queueing network with positive and negative customers, thus for all $\bar{g}'$ it must be the case that $p(\bar{g}', e_i + e_j) = 0$. Therefore, for a queueing network with positive and negative customers (2.8) cannot have a positive solution $\{q_i\}_{i=1}^N$ as is required in Theorem 2.2.

- Multiple types of customers: In single changes queueing networks with multiple types of customers a transition $\bar{n} \rightarrow \bar{n} - e_i - e_j$ is not allowed. In batch routing queueing networks with multiple types of customers, similar to the arguments given above, (2.8) generalized to also include multiple customer types cannot posses a positive solution $\{q_i\}_{i=1}^N$.

The main difference between local balance appearing in the literature and local balance for queueing networks with positive and negative customers is the following. Standard local balance states:

$$\begin{aligned} & \text{outrate due to the departure of } \bar{g} \\ & = \\ & \text{inrate due to the arrival of } \bar{g}. \end{aligned}$$

In contrast, local balance for queueing networks with positive and negative customers states:

$$\begin{aligned} & \text{outrate due to the departure of } e_i \text{ and } e_i + e_j, j = 1, \dots, N \\ & = \\ & \text{inrate due to the arrival of } e_i. \end{aligned}$$

Comparison of these statements shows that (2.4) and (2.7) cannot be obtained from standard local balance as appearing in the literature unless $p^-(i, j) = 0$ for all i, j . $\square$

Example 2.8 (Two station queueing network) This example illustrates local balance for queueing networks with positive and negative customers as expressed by (2.4) and (2.7) for the special case of a two station queueing network. In Figure 2a for station 1 (i.e. $i = 1$), in Figure 2b for $i = 2$ and in Figure 2c for the outside both (2.4) and (2.6) are depicted. Note that in a standard queueing network local balance for the outside is obtained by comparing the outrate of all stations to the inrate of all stations, e.g. by summing (2.6) over $i = 1, 2$. This construction is used to obtain the case $i = 0$ for a queueing network with negative and positive customers also. This is justified since for $p^-(1, 2) = p^-(2, 1) = \lambda(1) = \lambda(2) = 0$ the queueing network with negative customers reduces to a standard queueing network. Global balance at state $\bar{n}$ for a standard queueing network is obtained by adding local balance for $i = 0, 1, 2$ as depicted in Figure 3a. For a queueing network with positive and negative customers adding local balance for $i = 0, 1, 2$ does not give local balance. As is shown in Figure 3b the transitions $\bar{n} + e_1 \rightarrow \bar{n} - e_2$ and $\bar{n} + e_2 \rightarrow \bar{n} - e_1$ destroy this decomposition of global balance into local balance. $\square$

Figure 2a. $i = 1$

Figure 2b. $i = 2$

Figure 2c. $i = 0$

positive and negative customers

positive customers only

sum of local balance at $\bar{n}$ = global balance at $\bar{n}$

Figure 3a. standard queueing network

sum of local balance at $\bar{n}$ $\neq$ global balance at $\bar{n}$

Figure 3b. queueing network with positive and negative customers

3 State space truncation based on local balance

In Section 2 an invariant measure for X with transition rates (2.1) is derived. As a direct consequence of the unrestricted state space, Z^N , chosen in Section 2 this invariant measure cannot be normalized. Also, in applications it is more natural to constrain the number of customers at the stations, say $-\infty < N_i \leq n_i \leq M_i < \infty$ for some $N_i, M_i \in Z, i = 1, \dots, N$. For example if $N_i = 0, i = 1, \dots, N$, the number of customers at the stations is allowed to be non-negative only which yields the model discussed in [6]. As can be seen from Figure 3b for X to be truncated to a state space with lower and upper bounds the transition rates at the boundaries must be modified since transitions leaving the state space are not allowed. As is shown in Theorem 2.2, m as given in (2.5) satisfies the global balance equations at the interior of the state space. For m to satisfy the global balance equations at the boundaries the modification of the transition rates at the boundaries must be chosen very carefully. As the proof of Theorem 2.2 is based on local balance it is sufficient to consider local balance when modifying the transition rates. In Section 3.1 we first discuss lower bounds as these are easy to deal with in standard queueing networks and as is shown below are relatively easy to deal with in the present setting. In Section 3.2 we discuss upper bounds.

3.1 Lower bounds

This section considers X truncated to $S \subset Z^N$, where

$$S = \{\bar{n} : -\infty < N_i \leq n_i, i = 1, \dots, N\}.$$

The transition rates at the boundary of S are modified such that local balance at the boundary is preserved. To this end consider Example 2.8. As can be seen from Figures 2a, 2b for $n_1 \geq N_1, n_2 = N_2$ local balance for $i = 2$ does not cross this boundary. In contrast, for $i = 1$ the transition $\bar{n} \rightarrow \bar{n} - e_1 - e_2$ leaves S . For local balance to be preserved this transition must be modified. A straightforward modification is to simply replace the transition $\bar{n} \rightarrow \bar{n} - e_1 - e_2$ by $\bar{n} \rightarrow \bar{n} - e_1$ with the same rate. This is depicted in Figure 4. The physical interpretation of this modification is that a negative customer leaving station 1 leaves the queueing network since it cannot enter station 2 or that a negative customer entering station 2 when $n_2 = N_2$ is lost. For the general process this modification gives that the transition rates $Q = (q(\bar{n}, \bar{n}'), \bar{n}, \bar{n}' \in S)$ are

$$q(\bar{n}, \bar{n} - e_i) = r(i)d(i) + \lambda(i) + \sum_{j=1}^N r(i)p^-(i, j)1(n_j = N_j), \quad i = 1, \dots, N, \quad (3.1)$$

and $q(\bar{n}, \bar{n}'), \bar{n}, \bar{n}' \in S$, as given in (2.1) otherwise.

The following theorem shows that under this modification the traffic equations are necessary and sufficient for X to have a product form equilibrium distribution at S .

Figure 4. Modification at lower bound

Theorem 3.1 *X possesses a unique, positive, product form equilibrium distribution, π , at $S = \{\bar{n} : n_i \geq N_i, i = 1, \dots, N, N_i \in \mathbb{Z}, i = 1, \dots, N\}$ given by*

$$\pi(\bar{n}) = \prod_{k=1}^N \frac{1 - q_k}{q_k^{N_k}} q_k^{n_k}, \quad \bar{n} \in S, \quad (3.2)$$

if and only if $\{q_i\}_{i=1}^N$ is the unique positive solution of (2.4) such that $q_i < 1, i = 1, \dots, N$.

Proof Assume that $\{q_i\}_{i=1}^N$ is a positive solution of (2.4). For $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ we obtain

$$\begin{aligned} & \sum_{\bar{n}' \neq \bar{n}} \{m(\bar{n})q(\bar{n}, \bar{n}') - m(\bar{n}')q(\bar{n}', \bar{n})\} \\ &= \sum_{i=1}^N m(\bar{n}) \left\{ q(\bar{n}, \bar{n} + e_i) + q(\bar{n}, \bar{n} - e_i) \right. \\ & \quad \left. + \sum_{j=1}^N \{q(\bar{n}, \bar{n} - e_i + e_j) + q(\bar{n}, \bar{n} - e_i - e_j)\} \right\} \\ & \quad - \sum_{i=1}^N \left\{ m(\bar{n} - e_i)q(\bar{n} - e_i, \bar{n}) + m(\bar{n} + e_i)q(\bar{n} + e_i, \bar{n}) \right. \\ & \quad \left. + \sum_{j=1}^N \{m(\bar{n} - e_i + e_j)q(\bar{n} - e_i + e_j, \bar{n}) \right. \\ & \quad \left. + m(\bar{n} + e_i + e_j)q(\bar{n} + e_i + e_j, \bar{n})\} \right\} \end{aligned}$$

$$\begin{aligned}
&= m(\bar{n}) \sum_{i=1}^N \left\{ \Lambda(i) + \lambda(i) \mathbf{1}(n_i > N_i) + r(i) d(i) \mathbf{1}(n_i > N_i) \right. \\
&\quad + \sum_{j=1}^N r(i) p^-(i, j) \mathbf{1}(n_j = N_j, n_i > N_i) \\
&\quad + \sum_{j=1}^N r(i) p^+(i, j) \mathbf{1}(n_i > N_i) + \sum_{j=1}^N r(i) p^-(i, j) \mathbf{1}(n_j > N_j, n_i > N_i) \\
&\quad - \frac{1}{q_i} \Lambda(i) \mathbf{1}(n_i > N_i) - q_i \lambda(i) - q_i r(i) d(i) \\
&\quad - \sum_{j=1}^N q_i r(i) p^-(i, j) \mathbf{1}(n_j = N_j) \\
&\quad \left. - \sum_{j=1}^N \frac{q_j}{q_i} r(j) p^+(j, i) \mathbf{1}(n_i > N_i) - \sum_{j=1}^N q_i q_j r(i) p^-(i, j) \right\} \\
&\stackrel{(*)}{=} m(\bar{n}) \sum_{i=1}^N \left\{ \Lambda(i) - q_i r(i) d(i) - \sum_{j=1}^N q_i r(i) p^-(i, j) \mathbf{1}(n_j = N_j) \right. \\
&\quad \left. - \sum_{j=1}^N q_i q_j r(i) p^-(i, j) - q_i \lambda(i) \right\} \\
&\quad + m(\bar{n}) \sum_{i=1}^N \left\{ r(i) + \lambda(i) - \frac{1}{q_i} \Lambda(i) - \sum_{j=1}^N \frac{q_j}{q_i} r(j) p^+(j, i) \right\} \mathbf{1}(n_i > N_i) \\
&\stackrel{(**)}{=} \sum_{i=1}^N m(\bar{n}) \left\{ \Lambda(i) + \sum_{j=1}^N q_j r(j) p^+(j, i) \right. \\
&\quad \left. - q_i r(i) - \sum_{j=1}^N q_i q_j r(j) p^-(j, i) - q_i \lambda(i) \right\} \\
&\quad + \sum_{i=1}^N m(\bar{n} - e_i) \left\{ q_i r(i) + \sum_{j=1}^N q_i q_j r(j) p^-(j, i) + q_i \lambda(i) \right. \\
&\quad \left. - \sum_{j=1}^N q_j r(j) p^+(j, i) - \Lambda(i) \right\} \mathbf{1}(n_i > N_i) \\
&= 0,
\end{aligned}$$

where (*) is obtained by rearranging terms using (2.2) and (**) by adding and subtracting $\sum_{i,j=1}^N q_j r(j) p^+(j, i)$ and $\sum_{i,j=1}^N q_j r(j) p^-(j, i) \mathbf{1}(n_i > N_i)$. Now assume that $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ is a positive invariant measure at S . Then, as $m(\bar{n}) > 0$ for all $\bar{n} \in S$, for $\bar{n} \in S$ we obtain for $i = 1, \dots, N$

$$\frac{m(\bar{n} + e_i)}{m(\bar{n})} = q_i.$$

Thus, $q_i > 0$, $i = 1, \dots, N$. Inserting $m(\bar{n})$ into the global balance equations gives, for $\bar{n} = (N_1, \dots, N_N)$, the left lower corner of the state space

$$\begin{aligned} m(\bar{n}) \sum_{i=1}^N \Lambda(i) &= \sum_{i=1}^N m(\bar{n} + e_i) \left\{ r(i)d(i) + \sum_{j=1}^N r(i)p^-(i, j) + \lambda(i) \right\} \\ &\quad + \sum_{i,j=1}^N m(\bar{n} + e_i + e_j) r(i)p^-(i, j), \end{aligned}$$

which implies that

$$\sum_{i=1}^N \Lambda(i) + \sum_{i,j=1}^N q_i r(i)p^+(i, j) = \sum_{i=1}^N q_i \{r(i) + \lambda(i)\} + \sum_{i,j=1}^N q_i q_j r(i)p^-(i, j). \quad (3.3)$$

For $\bar{n} = (N_1, \dots, N_N) + e_k$, $k = 1, \dots, N$, we obtain

$$\begin{aligned} m(\bar{n}) &\left\{ \sum_{i=1}^N \Lambda(i) + \lambda(k) + r(k) \right\} \\ &= \sum_{i=1}^N m(\bar{n} + e_i) \left\{ r(i)d(i) + \sum_{j \neq k}^N r(i)p^-(i, j) + \lambda(i) \right\} \\ &\quad + m(\bar{n} - e_k) \Lambda(k) + \sum_{i,j=1}^N m(\bar{n} + e_i + e_j) r(i)p^-(i, j) \\ &\quad + \sum_{j=1}^N m(\bar{n} - e_k + e_j) r(j)p^+(j, k), \quad k = 1, \dots, N. \end{aligned}$$

From (3.3) and rearranging terms

$$\begin{aligned} \lambda(k) + r(k) + \sum_{i=1}^N q_i r(i) \\ &= \sum_{i,j=1}^N q_i r(i)p^+(i, j) + \sum_{i=1}^N q_i \left\{ r(i)d(i) + \sum_{j \neq k}^N r(i)p^-(i, j) \right\} \\ &\quad + \frac{1}{q_k} \Lambda(k) + \sum_{j=1}^N \frac{q_j}{q_k} r(j)p^+(j, k), \quad k = 1, \dots, N. \end{aligned}$$

Using (2.2) this gives that $\{q_i\}_{i=1}^N$ satisfies (2.4). Thus $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ is a positive invariant measure if and only if $\{q_i\}_{i=1}^N$ is a positive solution of (2.4).

If $\{q_i\}_{i=1}^N$ is the unique positive solution of (2.4) such that $q_i < 1$, $i = 1, \dots, N$, then, as is shown above, $\pi(\bar{n})$ as given in (3.2) is an invariant measure. As $0 < q_i < 1$, $i = 1, \dots, N$, π is a positive product form equilibrium distribution. Now assume $\tilde{\pi}$ is another positive product form equilibrium distribution, i.e. for a set of constants $\{\tilde{q}_i\}_{i=1}^N$ $\tilde{\pi}$ can be written $\tilde{\pi}(\bar{n}) = B \prod_{k=1}^N \tilde{q}_k^{n_k}$, $\bar{n} \in S$ and there exists an $\bar{n} \in S$ such that $\tilde{\pi}(\bar{n}) \neq \pi(\bar{n})$ and $\tilde{\pi}$ satisfies the global balance equations at S . Then, as $\tilde{\pi}$ is a positive probability distribution, it must be the case that $0 < \tilde{q}_i < 1$, $i = 1, \dots, N$.

Furthermore, as is shown above, $\{\tilde{q}_i\}_{i=1}^N$ satisfies (2.4). As $\tilde{\pi}(\bar{n}) \neq \pi(\bar{n})$ for some $\bar{n} \in S$, it must be the case that $\tilde{q}_i \neq q_i$ for some i which is in contradiction to the assumption on the uniqueness of the solution to (2.4).

To prove the reversed statement, note that as π given in (3.2) is a probability distribution, $0 < q_i < 1$, $i = 1, \dots, N$. As is shown above, $\{q_i\}_{i=1}^N$ satisfies (2.4). Now assume $\{\tilde{q}_i\}_{i=1}^N$ such that $0 < \tilde{q}_i < 1$, $i = 1, \dots, N$, also satisfies (2.4) and $\tilde{q}_i \neq q_i$ for some i . Then $\tilde{\pi}$ given by $\tilde{\pi}(\bar{n}) = B \prod_{k=1}^N \tilde{q}_k^{n_k}$, $\bar{n} \in S$, is an equilibrium distribution. As X possesses a unique product form equilibrium distribution, for $\bar{n} \in S$ we obtain for $i = 1, \dots, N$

$$q_i = \frac{\pi(\bar{n} + e_i)}{\pi(\bar{n})} = \frac{\tilde{\pi}(\bar{n} + e_i)}{\tilde{\pi}(\bar{n})} = \tilde{q}_i.$$

Thus $\{q_i\}_{i=1}^N$ is the unique solution to (2.4) such that $0 < q_i < 1$, $i = 1, \dots, N$. $\square$

Remark 3.2 (Product form; uniqueness of π) The statement of Theorem 3.1 is that X possesses a unique positive product form equilibrium distribution, π , as given in (3.2). This does not imply that π is the unique equilibrium distribution. For π to be the unique equilibrium distribution we have to assume that X is irreducible. Sufficient conditions for X to be irreducible are, amongst others, that

1. $\Lambda(i) > 0$, $i = 1, \dots, N$, $\lambda(i) > 0$, $i = 1, \dots, N$,
2. $\Lambda(i) > 0$, $i = 1, \dots, N$, $r(i)p^-(i, j) > 0$, $i, j = 1, \dots, N$,
3. $\Lambda(i) > 0$, $i \in \mathcal{I} \subset \{1, \dots, N\}$, $r(i)p^+(i, j) > 0$, $i \in \mathcal{I}$, $j \in \{1, \dots, N\} \setminus \mathcal{I}$, $\lambda(i) > 0$, $i = 1, \dots, N$,
4. there exists a unique, positive solution $\{c_i\}_{i=1}^N$ to

$$\gamma_i \left\{ d(i) + \sum_{j=1}^N p^+(i, j) \right\} = \Lambda(i) + \sum_{j=1}^N \gamma_j p^+(j, i), \quad i = 1, \dots, N.$$

This condition states that if the network with positive customers only is irreducible, then the network with positive and negative customers is also irreducible. This insight is due to local balance.

If X is irreducible and a solution $\{q_i\}_{i=1}^N$ such that $0 \leq q_i < 1$, $i = 1, \dots, N$, exists to the traffic equations then π as given in (3.2) is the unique equilibrium distribution to the process. $\square$

Remark 3.3 (Interpretation of (2.4)) Under the product form equilibrium distribution, π given in (3.2), the traffic equations (2.4) state that the expected probability flow out of queue i equals the expected probability flow into queue i , $i = 1, \dots, N$. This can be seen by computing these expectations. The average output of queue i

due to the departure of positive customers from queue i routing to queue j is given by

$$\begin{aligned} Eq_{ij} &= \sum_{\bar{n} \in S} \pi(\bar{n}) q(\bar{n}, \bar{n} - e_i + e_j) \\ &= \prod_{k=1}^N \sum_{n_k \geq N_k} \frac{1 - q_k}{q_k} q_k^{n_k} r(i) p^+(i, j) 1(n_i > N_i) \\ &= q_i r(i) p^+(i, j). \end{aligned}$$

The average output of queue i due to the arrival of negative customers from queue j is given by

$$\begin{aligned} Eq_{ij}^- &= \prod_{k=1}^N \sum_{n_k \geq N_k} \frac{1 - q_k}{q_k} q_k^{n_k} r(j) p^-(j, i) 1(n_j > N_j, n_i > N_i) \\ &= q_i q_j r(j) p^-(j, i). \end{aligned}$$

Similarly the other terms can be computed. This gives:

$q_i r(i)$	=	average output of queue i due to the departure of negative and positive customers
$\sum_{j=1}^N q_i q_j r(j) p^-(j, i)$	=	average output of queue i due to the arrival of negative customer from other queues
$q_i \lambda(i)$	=	average output of queue i due to the arrival of negative customers from outside the network
$\sum_{j=1}^N q_j r(j) p^+(j, i)$	=	average arrival rate at queue i due to the arrival of positive customers from other queues
$\Lambda(i)$	=	average arrival rate at queue i due to the arrival of positive customers from outside the network

The left-hand side of (2.4) represents the average rate of departures of customers from queue i and the right-hand side represents the average rate of arrivals at queue i , $i = 1, \dots, N$. Thus, the traffic equations are natural equations for X at S . Moreover, this gives a second motivation for calling (2.4) the traffic equations. $\square$

In [6] the queueing network with positive and negative customers discussed above is introduced for the case $N_i = 0$, $i = 1, \dots, N$. A sufficient condition for X to possess a product form equilibrium distribution is given only. The following lemma relates the sufficient condition given in [6], (3.4a)–(3.4c), to the traffic equations. In Remark 3.5 Gelenbe's result is related to the result of the present paper.

Lemma 3.4 $\{q_i\}_{i=1}^N$ is a solution to

$$\lambda^+(i) = \sum_{j=1}^N \gamma_j r(j) p^+(j, i) + \Lambda(i), \quad i = 1, \dots, N, \quad (3.4a)$$

$$\lambda^-(i) = \sum_{j=1}^N \gamma_j r(j) p^-(j, i) + \lambda(i), \quad i = 1, \dots, N, \quad (3.4b)$$

$$\gamma_i = \frac{\lambda^+(i)}{r(i) + \lambda^-(i)}, \quad i = 1, \dots, N, \quad (3.4c)$$

if and only if $\{q_i\}_{i=1}^N$ is a solution to

$$\gamma_i r(i) + \sum_{j=1}^N \gamma_j r(j) p^-(j, i) + \gamma_i \lambda(i) = \sum_{j=1}^N \gamma_j r(j) p^+(j, i) + \Lambda(i), \quad i = 1, \dots, N. \quad (3.5)$$

Proof If $\{q_i\}_{i=1}^N$ is a positive solution of (3.4a) – (3.4c). Substitution of (3.4c) into (3.4a) gives for $i = 1, \dots, N$

$$\gamma_i (r(i) + \lambda^-(i)) = \sum_{j=1}^N \gamma_j r(j) p^+(j, i) + \Lambda(i). \quad (3.6)$$

Substitution of (3.4b) into (3.6) gives $\{q_i\}_{i=1}^N$ satisfies (3.5).

Now assume $\{q_i\}_{i=1}^N$ satisfies (3.5). Define $\lambda^+(i)$, $\lambda^-(i)$, $i = 1, \dots, N$ as

$$\lambda^+(i) = \sum_{j=1}^N q_j r(j) p^+(j, i) + \Lambda(i), \quad \lambda^-(i) = \sum_{j=1}^N q_j r(j) p^-(j, i) + \lambda(i).$$

Substitution of $\lambda^+(i)$, $\lambda^-(i)$ into (3.5) gives for $i = 1, \dots, N$

$$q_i r(i) + q_i \lambda^-(i) = \lambda^+(i).$$

Thus $\{q_i\}_{i=1}^N$ satisfies (3.4a)–(3.4c). □

Remark 3.5 (Gelenbe's result) In [6] it is proven by substitution into the global balance equations, that if $\{q_i\}_{i=1}^N$ is a positive solution to (3.4a)–(3.4c) then X possesses a product form equilibrium distribution, π , given in (3.2). From Lemma 3.4 and Theorem 3.1 we obtain that we have reproduced this result. The result obtained in Theorem 3.1 is more general than Gelenbe's result as Theorem 3.1 gives a necessary and sufficient condition for π to be of product form. Furthermore, the result of Theorem 3.1 is obtained by considering *local balance*. The insight obtained through local balance is an essential contribution to the theory on queueing networks with positive and negative customers. Local balance allows us to introduce boundaries as is shown in the present section, Section 3.2 and Section 5 and incorporates blocking phenomena into the theory.

The coefficients $\lambda^+(i)$, $\lambda^-(i)$ defined in (3.4a), (3.4b) are related to the average arrival rate of customers at queue i , $i = 1, \dots, N$. At first glance one suspects λ^+ , λ^- to represent the average arrival rate of positive and negative customers respectively [6]. Be aware, however, that λ^+ , λ^- are indeed related to these arrival rates, but as follows (cf. Remark 3.3):

$$\begin{aligned} \lambda^+(i) &= \text{average arrival rate of positive customers at queue } i; \\ q_i \lambda^-(i) &= \text{average arrival rate of negative customers at queue } i. \end{aligned}$$

As the average output rate of customers from queue i is the average output rate due to the arrival of negative customers plus the average output rate due to service, i.e. $q_i \lambda^-(i) + q_i r(i)$, (3.4c) expresses that the average input rate equals the average output rate. □

Figure 5. Modification at upper bound

3.2 Upper bounds

In Section 3.1, based on local balance, the transition rates are modified such that X possesses a product form equilibrium distribution at $S = \{\bar{n} : n_i \geq N_i, i = 1, \dots, N\}$. This section addresses the complementary problem: upper limit blocking at the queues. Based on local balance, the transition rates are modified such that X possesses a product form equilibrium distribution at

$$S = \{\bar{n} \in Z^N : n_i \leq M_i < \infty, i = 1, \dots, N\},$$

i.e. the number of customers at queue i is constrained not to exceed M_i .

For standard queueing networks it is known that in the case of upper limit blocking special blocking protocols, such as the stop protocol, must be applied to preserve a product form equilibrium distribution (cf. [3], [8], [11]). For a queueing network with positive and negative customers these protocols do not preserve a product form equilibrium distribution. To show this reconsider Example 2.8. For a standard queueing network the stop protocol stops all transitions in which a customer leaves station 2 if station 1 is saturated, i.e. if $n_1 = M_1$. In Figure 3a this comes down to blocking the transitions $\bar{n} \leftrightarrow \bar{n} \pm e_2$. As can immediately be seen from Figure 3a, at this boundary global balance reduces to local balance for $i = 1$. For a queueing network with positive and negative customers the stop protocol does not preserve product form. This can be seen from Figure 2b. Blocking of transitions in which a customer leaves station 2 if $n_1 = M_1$ also stops the transition $\bar{n} \rightarrow \bar{n} - e_1 - e_2$. As a consequence local balance for $i = 1$ at $\bar{n} - e_2$ is destroyed. Thus the stop protocol does not preserve product form. A similar argument can be used for other blocking protocols for standard queueing networks.

From local balance we obtain that the following modification of the transition rates $Q = (q(\bar{n}, \bar{n}'), \bar{n}, \bar{n}' \in S)$ preserves product form:

$$\begin{aligned} q(\bar{n}, \bar{n} + e_i) &= \Lambda(i) + \sum_{k=1}^N q_k r(k) p^+(k, i) \mathbf{1}(n_k = M_k), \quad i = 1, \dots, N, \\ q(\bar{n}, \bar{n} - e_i) &= r(i) d(i) + \lambda(i) \\ &+ \sum_{k=1}^N q_k r(k) p^-(k, i) \mathbf{1}(n_k = M_k) + \sum_{k=1}^N r(i) p^+(i, k) \mathbf{1}(n_k = M_k), \quad i = 1, \dots, N, \end{aligned} \quad (3.7)$$

where $\{q_i\}_{i=1}^N$ are arbitrary numbers such that $q_i > 1$, $i = 1, \dots, N$ and $q(\bar{n}, \bar{n}')$ as given in (2.1) otherwise. For the two station queueing network described in Example 2.8 this modification is illustrated in Figure 5. The additional terms have the following interpretation. If $n_k = M_k$, then customers leaving queue i to route to queue k are lost, which explains $\sum_{k=1}^N r(i)p^+(i, k)1(n_k = M_k)$. Furthermore, if $n_k = M_k$ at queue i additional arrivals from the outside are generated at rate $q_k r(k)p^+(k, i)$ for positive customers and $q_k r(k)p^-(k, i)$ for negative customers.

The following theorem shows that X thus modified possesses a product form equilibrium distribution at S . At first glance, both Theorem 3.6 and its proof are very similar to Theorem 3.1 and its proof. However, proving global balance in the proof of these theorems differs substantially. (For example see equation (**) below.)

Theorem 3.6 *X possesses a unique, positive, product form equilibrium distribution, π , at $S = \{\bar{n} : n_i \leq M_i, i = 1, \dots, N, M_i \in \mathbb{Z}, i = 1, \dots, N\}$ given by*

$$\pi(\bar{n}) = \prod_{k=1}^N \frac{q_k - 1}{q_k^{M_k+1}} q_k^{n_k}, \quad \bar{n} \in S, \quad (3.8)$$

if and only if $\{q_i\}_{i=1}^N$ is the unique positive solution of (2.4) such that $q_i > 1$, $i = 1, \dots, N$.

Remark 3.7 (Coefficients $\{q_i\}_{i=1}^N$) Note that the coefficients $\{q_i\}_{i=1}^N$ appearing in the modification (3.7) are appearing in the traffic equations too. Thus, in the theorem above, it is implicitly assumed that the coefficients used in the modifications also satisfy the traffic equations. This problem will be considered in Section 4.5. $\square$

Proof Assume that $\{q_i\}_{i=1}^N$ is a positive solution to the traffic equations (2.4). Insertion of $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ into the global balance equations at S gives

$$\begin{aligned} & \sum_{\bar{n}' \neq \bar{n}} \{m(\bar{n})q(\bar{n}, \bar{n}') - m(\bar{n}')q(\bar{n}', \bar{n})\} \\ &= m(\bar{n}) \sum_{i=1}^N \left\{ \Lambda(i)1(n_i < M_i) + \sum_{k=1}^N q_k r(k)p^+(k, i)1(n_k = M_k, n_i < M_i) \right. \\ & \quad + r(i)d(i) + \lambda(i) \\ & \quad + \sum_{k=1}^N q_k r(k)p^-(k, i)1(n_k = M_k) + \sum_{k=1}^N r(i)p^+(i, k)1(n_k = M_k) \\ & \quad + \sum_{j=1}^N r(i)p^+(i, j)1(n_j < M_j) + \sum_{j=1}^N r(i)p^-(i, j) \\ & \quad - \frac{1}{q_i} \Lambda(i) - \sum_{k=1}^N \frac{q_k}{q_i} r(k)p^+(k, i)1(n_k = M_k) \\ & \quad - q_i r(i)d(i)1(n_i < M_i) - q_i \lambda(i)1(n_i < M_i) \\ & \quad \left. - \sum_{k=1}^N q_i q_k r(k)p^-(k, i)1(n_k = M_k, n_i < M_i) \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^N q_i r(i) p^+(i, k) \mathbf{1}(n_k = M_k, n_i < M_i) \\
& - \sum_{j=1}^N \frac{q_i}{q_j} r(i) p^+(i, j) \mathbf{1}(n_i < M_i) \\
& - \sum_{j=1}^N q_i q_j r(i) p^-(i, j) \mathbf{1}(n_i < M_i, n_j < M_j) \Big\} \\
\stackrel{(*)}{=} & m(\bar{n}) \sum_{i=1}^N \Big\{ \left(\Lambda(i) - q_i r(i) d(i) - q_i \lambda(i) - \sum_{k=1}^N q_i q_k r(k) p^-(k, i) \right) \mathbf{1}(n_i < M_i) \\
& + r(i) + \lambda(i) - \frac{1}{q_i} \Lambda(i) - \sum_{k=1}^N \frac{q_k}{q_i} r(k) p^+(k, i) \\
& + \sum_{k=1}^N q_k r(k) p^-(k, i) \mathbf{1}(n_k = M_k) \\
& + \sum_{k=1}^N q_k r(k) p^+(k, i) \mathbf{1}(n_k = M_k, n_i < M_i) \\
& - \sum_{k=1}^N q_i r(i) p^+(i, k) \mathbf{1}(n_k = M_k, n_i < M_i) \Big\} \\
\stackrel{(**)}{=} & m(\bar{n}) \sum_{i=1}^N \Big\{ \Lambda(i) + \sum_{k=1}^N q_k r(k) p^+(k, i) \\
& - q_i r(i) - \sum_{k=1}^N q_i q_k r(k) p^-(k, i) - q_i \lambda(i) \Big\} \mathbf{1}(n_i < M_i) \\
& + \sum_{i=1}^N m(\bar{n} - e_i) \Big\{ q_i r(i) + \sum_{k=1}^N q_i q_k r(k) p^-(k, i) + q_i \lambda(i) \\
& - \Lambda(i) - \sum_{k=1}^N q_k r(k) p^+(k, i) \Big\} \\
= & 0,
\end{aligned}$$

where (*) is obtained by rearranging terms, and (**) is obtained from (2.2) and the following identity

$$\begin{aligned}
& \sum_{i,j=1}^N q_i r(i) p^+(i, j) \mathbf{1}(n_i < M_i) + \sum_{i,j=1}^N q_i r(i) p^+(i, j) \mathbf{1}(n_j < M_j, n_i = M_i) \\
& = \sum_{i,j=1}^N q_i r(i) p^+(i, j) \mathbf{1}(n_j < M_j) + \sum_{i,j=1}^N q_i r(i) p^+(i, j) \mathbf{1}(n_i < M_i, n_j = M_j).
\end{aligned}$$

Now assume that $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ is a positive invariant measure at S . Then, as $m(\bar{n}) > 0$ for all $\bar{n} \in S$ we obtain that $q_i > 0$, $i = 1, \dots, N$. Insertion of $m(\bar{n})$ into the global balance equations gives for $\bar{n} = (M_1, \dots, M_N)$ the right upper corner of the state space

$$\sum_{i=1}^N \left\{ r(i) + \sum_{k=1}^N q_k r(k) p^-(k, i) + \lambda(i) \right\} = \sum_{i=1}^N \frac{1}{q_i} \left\{ \Lambda(i) + \sum_{k=1}^N q_k r(k) p^+(k, i) \right\}. \quad (3.9)$$

For $\bar{n} = (M_1, \dots, M_N) - e_s$, we obtain

$$\begin{aligned} & \sum_{i=1}^N \left\{ r(i) d(i) + \lambda(i) + \sum_{k \neq s} q_k r(k) p^-(k, i) + \sum_{k \neq s} r(i) p^+(i, k) \right. \\ & \quad \left. + \sum_j r(i) p^-(i, j) + \Lambda(s) + \sum_{k \neq s} q_k r(k) p^+(k, s) + r(i) p^+(i, s) \right\} \\ &= \sum_{i=1}^N \left\{ \frac{1}{q_i} \Lambda(i) + \sum_{k \neq s, i} \frac{q_k}{q_i} r(k) p^+(k, i) + q_s r(s) d(s) + q_s \lambda(s) \right. \\ & \quad \left. + q_s \sum_{k \neq s} q_k r(k) p^-(k, s) + q_s \sum_{k \neq s} r(s) p^+(s, k) + \frac{q_s}{q_i} r(s) p^+(s, i) \right\}. \end{aligned}$$

Rearranging terms and insertion of (3.9) gives that $\{q_i\}_{i=1}^N$ satisfies the traffic equations. The remainder of the proof is similar to the proof of Theorem 3.1. $\square$

4 Characteristic equations

In the previous sections it is assumed that a unique positive solution exists to the traffic equations (2.4) such that either $q_i < 1$, $i = 1, \dots, N$, (Section 3.1) or $q_i > 1$, $i = 1, \dots, N$, (Section 3.2). This section addresses the problem of existence and uniqueness of a positive solution to the traffic equations. As is shown below in Section 4.1 for the queueing network introduced in Example 2.8, this problem is not easy to solve. For the general queueing network consisting of more than two stations, for some very special cases existence and uniqueness of a positive solution to the traffic equations can be proven. In these cases, discussed in Sections 4.2 and 4.3, it can easily be shown that it may be the case that $q_i < 1$, $q_j > 1$, and $q_k = 1$ for $i \neq j \neq k$. In Section 4.5 the traffic equations are discussed. It is argued that the traffic equations are characteristic equations for the process.

4.1 Uniqueness of solution

This section reconsiders the two station queueing network discussed in Examples 1.1 and 2.8. For this queueing network Theorem 4.1 shows the traffic equations possess a unique positive solution. The first condition, irreducibility, is sufficient for the existence of a positive solution of the traffic equations in a standard queueing network. The second condition stating that the queueing network is not pathological is discussed in more detail in Section 4.4 and states that positive customers can enter each queue.

Theorem 4.1 (Uniqueness of solution) *If the Markov process, X , describing a two station queueing network is irreducible and not pathological, which means that*

$$\Lambda(1) > 0 \text{ or } r(2)p^+(2,1) > 0, \quad (4.1a)$$

and

$$\Lambda(2) > 0 \text{ or } r(1)p^+(1,2) > 0, \quad (4.1b)$$

and

$$\Lambda(1) > 0 \text{ or } \Lambda(2) > 0, \quad (4.1c)$$

then the traffic equations possess a unique positive solution q_1, q_2 .

Proof The traffic equations (2.4) for this special case read:

$$\gamma_1 r(1) + \gamma_1 \gamma_2 r(2) p^-(2,1) + \gamma_1 \lambda(1) = \gamma_2 r(2) p^+(2,1) + \Lambda(1), \quad (4.2a)$$

$$\gamma_2 r(2) + \gamma_2 \gamma_1 r(1) p^-(1,2) + \gamma_2 \lambda(2) = \gamma_1 r(1) p^+(1,2) + \Lambda(2). \quad (4.2b)$$

First, consider the case where all parameters $r(1), r(2)$, etc. are positive. Then (4.2a) can be written

$$\gamma_1 = \frac{\gamma_2 r(2) p^+(2,1) + \Lambda(1)}{r(1) + \gamma_2 r(2) p^-(2,1) + \lambda(1)}. \quad (4.2c)$$

Substitution of (4.2c) into (4.2b) and rearranging terms gives that q_2 must satisfy

$$\begin{aligned} & \gamma_2^2 \{ r(2)^2 p^-(2,1) + r(1) r(2) p^+(2,1) p^-(1,2) + \lambda(2) r(2) p^-(2,1) \} \\ & + \gamma_2 \{ r(1) r(2) + \lambda(1) \lambda(2) + \Lambda(1) r(1) p^-(1,2) + r(1) \lambda(2) + \lambda(1) r(2) \\ & \quad - r(1) r(2) p^+(1,2) p^+(2,1) - \Lambda(2) r(2) p^-(2,1) \} \\ & + \{ -\Lambda(1) r(1) p^+(1,2) - r(1) \Lambda(2) - \lambda(1) \Lambda(2) \} = 0 \end{aligned} \quad (4.3)$$

As all parameters are positive the coefficient of γ_2^2 is positive and the coefficient of γ_2^0 is negative. This implies that (4.3) possesses two distinct solutions, one positive and one negative solution. Thus (4.2a), (4.2b) possess a unique positive solution q_2^* . By symmetry, (4.2a), (4.2b) possess a unique positive solution q_1^* also. Insertion of q_2^* into (4.2c) gives that q_1^* corresponds to q_2^* which implies that the traffic equations possess a unique positive solution $q_1 = q_1^*, q_2 = q_2^*$.

Second, consider the case $r(2) = 0$, i.e. no service at queue 2. From the irreducibility assumption we obtain $\lambda(2) > 0$ or $r(1)p^-(1,2) > 0$ and $r(1) > 0$ or $\lambda(1) > 0$ as both at queue 1 and queue 2 in some state $\bar{n}$ the number of customers must be able to decrease. Then, from (4.2a) and (4.1a) we obtain $q_1 = \frac{\Lambda(1)}{r(1)+\lambda(1)} > 0$. Insertion of q_1 into (4.2b) gives that $q_2 > 0$ since $q_1 r(1) p^-(1,2) + \lambda(2) > 0$ by irreducibility and

$q_1 r(1) p^+(1, 2) + \Lambda(2) > 0$ from (4.1b). Thus the traffic equations possess a unique positive solution q_1, q_2 . Similarly, if $r(1) = 0$ the traffic equations possess a unique positive solution q_1, q_2 also.

Now assume that $r(1) > 0$ and $r(2) > 0$. Then (4.2c) is well-defined and q_2 must satisfy (4.3). If the coefficients of γ_2^2, γ_2^1 and γ_2^0 are non-zero then (4.3) possesses a unique positive solution q_2 . Insertion of $\gamma_2 > 0$ into (4.2c) gives that $\gamma_1 > 0$ since the network is assumed to be non-pathological. Insertion of $\gamma_2 < 0$ into

$$\gamma_2 = \frac{\gamma_1 r(1) p^+(1, 2) + \Lambda(2)}{r(2) + \gamma_1 r(1) p^-(1, 2) + \lambda(2)},$$

which is obtained from (4.2b) and is well-defined since $r(2) > 0$, shows that $\gamma_1 < 0$. As the traffic equations possess at most two distinct solutions q_1 we obtain that the traffic equations possess a unique positive solution q_1, q_2 if the coefficients in (4.3) are non-zero. If the coefficients of γ_2^2 or γ_2^0 vanish, however, this must be shown separately.

The coefficient of γ_2^2 vanishes if both $p^-(2, 1) = 0$ and $p^+(2, 1) p^-(1, 2) = 0$. If $p^-(2, 1) = 0$ and $p^-(1, 2) = 0$ then the coefficient of γ_2^1 is non-negative. As the coefficient of γ_2^0 is non-positive, we obtain that $q_2 \geq 0$ and from (4.2c) that $q_1 \geq 0$. Insertion of $q_1 = 0$ into (4.2a) implies that $q_2 r(2) p^+(2, 1) + \Lambda(1) = 0$. As $q_2 \geq 0$ this implies that $\Lambda(1) = 0$. Then, since the network is assumed to be non-pathological, $r(2) p^+(2, 1) > 0$ and thus $q_2 = 0$ which implies that $\Lambda(2) = 0$ which contradicts (4.1c). Similarly, if $p^-(2, 1) = 0$ and $p^+(2, 1) = 0$ the coefficient of γ_2^1 is non-negative and (4.2a), (4.2b) possess a unique positive solution q_1, q_2 . From (4.1c) we obtain that the coefficient of γ_2^0 vanishes only if $r(1) = \lambda(1) = 0$. $\square$

Note that all combinations $(q_1 > 1, q_2 > 1)$, $(q_1 > 1, q_2 < 1)$, etc. are possible. This can be seen in the following example. Assume that $\lambda(1) = \lambda(2) = p^+(1, 2) = p^+(2, 1) = d(1) = d(2) = 0$ and $r(1) = r(2) = 1$, then (4.3) reads

$$q_1(1 + q_2) = \Lambda(1), \quad q_2(1 + q_1) = \Lambda(2).$$

The following table gives some possible combinations.

$\Lambda(1)$	$\Lambda(2)$	q_1	q_2
$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{2}$	1	1	$\frac{1}{2}$
3	$\frac{3}{2}$	2	$\frac{1}{2}$

As is shown in the simple case of a two station queueing network it is very difficult to prove that a unique positive solution to the traffic equations exists. Moreover, in this special case we have to assume that the queueing network is non-pathological. Thus,

in general, it does not seem to be possible to prove the existence and uniqueness of a positive solution to the traffic equations without further assumptions. In some special cases this problem is solved (cf. [6]). These cases are reviewed below in Section 4.2, where feedforward networks are discussed and Section 4.3, which considers balanced networks. As is shown in Section 4.4, it is necessary for the existence of a positive solution to the traffic equations that the queueing network is not pathological.

4.2 Feedforward networks

As is argued above, general conditions for the existence and uniqueness of a positive solution $\{q_i\}_{i=1}^N$ are hard to give. For an important class of networks, so-called feedforward networks, such a solution exists (cf. [6]).

A network is said to be feedforward if the queues can be numbered such that $p^+(i, j) = 0$ if $j < i$ and $p^-(i, j) = 0$ if $j < i$, i.e. if customers can go from queue i to queues with a larger number only. For such a network $\{q_i\}_{i=1}^N$ can be calculated recursively (cf. [6]). From the traffic equations we first obtain

$$q_1 = \frac{\Lambda(1)}{r(1) + \lambda(1)}.$$

If $q_i, i = 1, \dots, s-1$, are known then q_s can be calculated as follows

$$q_s = \frac{\sum_{j < s} q_j r(j) p^+(j, s) + \Lambda(s)}{r(s) + \sum_{j < s} q_j r(j) p^-(j, s) + \lambda(s)}.$$

As all terms $r(1), r(2)$, etc. are non-negative, and the calculating procedure is unique, this gives a unique positive solution to the traffic equations.

4.3 Balanced networks

A very special class of networks considered in [6] is the class of balanced networks. A network with negative and positive customers is balanced if a constant solution $q_i = q, i = 1, \dots, N$, exists to the traffic equations. If such a solution exists, then a positive constant solution exists and this positive constant solution is the unique positive constant solution to the traffic equations (cf. [6]). Be aware, however, that this does not imply that it is the unique positive solution to the traffic equations.

Assume that the network is balanced. Insertion of $q_i = q, i = 1, \dots, N$, into the traffic equations gives (cf. [6])

$$q = \frac{\sum_{j=1}^N r(j) p^+(j, i) - \lambda(i) - r(i) + \left[\left(\sum_{j=1}^N r(j) p^+(j, i) - \lambda(i) - r(i) \right)^2 + 4 \sum_{j=1}^N r(j) p^-(j, i) \Lambda(i) \right]^{\frac{1}{2}}}{2 \sum_{j=1}^N r(j) p^-(j, i)}.$$

Note that the right-hand side is independent of i as q is independent of i . Further note that q can take arbitrary positive values as can be seen by considering the right-hand side. If

$$2 \sum_{j=1}^N r(j)p^-(j,i) < \sum_{j=1}^N r(j)p^+(j,i) - \lambda(i) - r(i),$$

then $q > 1$, otherwise if

$$r(i) + \sum_{j=1}^N r(j)p^-(j,i) + \lambda(i) > \Lambda(i) + \sum_{j=1}^N r(j)p^+(j,i) \quad (4.4)$$

then $q < 1$, if (4.4) holds with equality then $q = 1$ and if $>$ is replaced by $<$ then $q > 1$. Note that (4.4) expresses that the departure rate at queue i is larger than the arrival rate at queue i . In all cases, X possesses an invariant measure m given by

$$m(\bar{n}) = q^{\sum_{i=1}^N n_i}.$$

4.4 Pathological network

The assumption that the queueing network is not pathological made in Section 4.1 is a very natural assumption. For, assume that the network is pathological, then positive customers cannot enter the queues, i.e.

$$\Lambda(i) = 0, \text{ for all } i, \text{ and } r(j)p^+(j,i) = 0, \text{ for all } i,j.$$

If this is the case then the number of customers at the queues cannot increase, thus in case of a lower bound we have $\pi((N_1, \dots, N_N)) = 1$ and in case of no lower bounds the process is transient, i.e. X does not possess an equilibrium distribution.

Further note that the assumption that the network is non-pathological is necessary for the existence of a positive solution. To this end, assume that the network is pathological. Then the traffic equations read

$$q_i \left(r(i) + \sum_{j=1}^N q_j r(j)p^-(j,i) + \lambda(i) \right) = 0, \quad i = 1, \dots, N,$$

with solution

$$q_i = 0 \text{ or } r(i) + \sum_{j=1}^N q_j r(j)p^-(j,i) + \lambda(i) = 0, \quad i = 1, \dots, N.$$

Note that $r(i) = \lambda(i) = 0$ for all i implies that there are no customers routing in the network, therefore it must be the case that $r(i) + \lambda(i) > 0$ for some i , say i_0 . Then for i_0 we obtain

$$\sum_{j=1}^N q_j r(j)p^-(j,i_0) = -r(i_0) - \lambda(i_0) < 0.$$

If there exists a non-zero solution, it must be the case that $q_k < 0$ for some k .

4.5 Characteristic result

As is illustrated above, for the Markov process to possess a product form equilibrium distribution, the traffic equations completely characterize the process. In Sections 4.1–4.4 some insight is given in the behaviour of the traffic equations and more specifically in the existence of a unique positive solution to the traffic equations. As is observed in these sections, some special conditions, additional to the conditions imposed in a standard queueing network are necessary for the existence of a unique positive solution to the traffic equations. In this section we argue that the traffic equations are characteristic equations for the Markov process describing the queueing network in the sense that they determine not only the equilibrium distribution but also determine the shape of the state space and the blocking conditions at the boundaries of the state space. To this end first note that the Markov process possesses a product form invariant measure only if the traffic equations possess a non-negative solution, $\{q_i\}_{i=1}^N$, as the invariant measure must satisfy the global balance equations at the interior of the state space. Second, the value of q_i , $i = 1, \dots, N$, characterizes the state space at which the invariant measure can be normalized. To this end, note that for $m(\bar{n}) = \prod_{k=1}^N q_k^{n_k}$ to be normalizable, if $q_i < 1$ the state space must be such that $-\infty < N_i \leq n_i$, if $q_i > 1$ the state space must be such that $n_i \leq M_i < \infty$ and if $q_i = 1$ it must be the case that $-\infty < N_i \leq n_i \leq M_i < \infty$. Third, at the boundaries, for the Markov process to possess a product form equilibrium distribution, the transition rates must be modified as given in (3.1) and (3.7). These modifications involve $\{q_i\}_{i=1}^N$, the solution to the traffic equations. Therefore, the traffic equations are characteristic equations for the Markov process as they determine the state space and the equilibrium distribution. This is formalized in Theorem 4.2 below. The proof is omitted since it is a combination of the proofs of Theorems 3.1, and 3.6. Note that the corner points $\bar{n} = (n_1, \dots, n_N)$ such that $n_i = N_i$ for some i and $n_j = M_j$ for some $j \neq i$ must be checked explicitly since these points are not considered in Theorems 3.1 and 3.6.

Theorem 4.2 (Characteristic result) *Assume a non-negative solution $\{q_i\}_{i=1}^N$ exists to*

$$\gamma_i r(i) + \sum_{j=1}^N \gamma_i \gamma_j r(j) p^-(j, i) + \gamma_i \lambda(i) = \sum_{j=1}^N \gamma_j r(j) p^+(j, i) + \Lambda(i), \quad i = 1, \dots, N, \quad (4.5)$$

such that with $I_<$, $I_=$, $I_>$ a partition of $\{1, \dots, N\}$

$$\begin{aligned} q_i &< 1 & \text{if } i \in I_<, \\ q_i &= 1 & \text{if } i \in I_=, \\ q_i &> 1 & \text{if } i \in I_>. \end{aligned}$$

For $N_i, M_i \in \mathbb{Z}$, $i = 1, \dots, N$, such that

$$\begin{aligned} N_i &> -\infty & \text{for } i \in I_< \cup I_=, \\ M_i &< \infty & \text{for } i \in I_> \cup I_=, \end{aligned}$$

the Markov process, X , with transition rates $Q = (q(\bar{n}, \bar{n}'), \bar{n}, \bar{n}' \in \mathbb{Z}^N)$ defined as

$$q(\bar{n}, \bar{n}') = \begin{cases} \Lambda(i) + \sum_{k=1}^N q_k r(k) p^+(k, i) \mathbf{1}(n_k = M_k), & \bar{n}' = \bar{n} + e_i, n_i < M_i, \\ r(i) d(i) + \lambda(i) \\ \quad + \sum_{j=1}^N r(i) p^-(i, j) \mathbf{1}(n_j = N_j) \\ \quad + \sum_{k=1}^N q_k r(k) p^-(k, i) \mathbf{1}(n_k = M_k) \\ \quad + \sum_{k=1}^N r(i) p^+(i, k) \mathbf{1}(n_k = M_k), & \bar{n}' = \bar{n} - e_i, n_i > N_i, \\ r(i) p^+(i, j), & \bar{n}' = \bar{n} - e_i + e_j, n_i > N_i, n_j < M_j, \\ r(i) p^-(i, j) + r(j) p^-(j, i), & \bar{n}' = \bar{n} - e_i - e_j, n_i > N_i, n_j > N_j, \\ 0, & \text{otherwise,} \end{cases} \quad (4.6)$$

possesses an equilibrium distribution, π , at state space

$$S = \{\bar{n} \in Z^N : N_i \leq n_i \leq M_i, i = 1, \dots, N\}$$

given by

$$\pi(\bar{n}) = \prod_{i \in I_-} \frac{1 - q_i}{q_i^{N_i}} \prod_{i \in I_0} \frac{1 - q_i}{q_i^{N_i} - q_i^{M_i+1}} \prod_{i \in I_+} \frac{1 - q_i}{-q_i^{M_i+1}} \prod_{k=1}^N q_k^{n_k}, \quad \bar{n} \in S.$$

5 Two station queueing network

Reconsider the two station queueing network discussed in Example 1.1 and depicted in Figure 1. At both stations external arrivals of positive customers at rate Λ_i at station i , $i = 1, 2$, occur only. At the stations service is provided in FCFS order by a single server at exponential rate r_i at station i , $i = 1, 2$. Customers completing service at station 1 route to station 2 as negative customers and customers completing service at station 2 route to station 1 as negative customers. The parameters for this queueing network are

$$\begin{aligned} \Lambda(1) &= \Lambda_1 > 0, \Lambda(2) = \Lambda_2 > 0, \lambda(1) = 0, \lambda(2) = 0, \\ r(1) &= r_1 > 0, r(2) = r_2 > 0, \\ p^-(1, 2) &= 1, p^-(2, 1) = 1, p^+(1, 2) = 0, p^+(2, 1) = 0, d(1) = 0, d(2) = 0, \end{aligned}$$

and the traffic equations read

$$\gamma_1 r(1) + \gamma_1 \gamma_2 r(2) = \Lambda(1), \quad (5.1a)$$

$$\gamma_2 r(2) + \gamma_1 \gamma_2 r(1) = \Lambda(2). \quad (5.1b)$$

The traffic equations represent generalized station balance for stations 1 and 2. To this end note that (5.1a) states that the rate into station 1 due to external arrivals, $\Lambda(1)$, equals the rate out of station 1 due to service completion at station 1, $\gamma_1 r(1)$, and due to arrivals from station 2, $\gamma_1 \gamma_2 r(2)$. In Figure 6 the traffic equations are expressed graphically. For $i = 1$ and $i = 2$ local balance at state $\bar{n}$ and local balance

Figure 6. Local balance for queueing network of Figure 1

at state $\bar{n} - e_i$ is depicted. From Remark 2.6 we obtain that global balance at state $\bar{n}$ is obtained by adding these 4 partial balance relations. Combining local balance at state $\bar{n}$ for $i = 1$ and local balance at state $\bar{n} - e_2$ for $i = 2$ gives “half of” global balance. This is depicted in the bottom line of Figure 5. Thus we have obtained an additional type of local balance for this specific model. This type of local balance is used below to further reduce the state space, i.e., to introduce additional boundaries.

Since all parameters Λ_1 , Λ_2 , q_1 , q_2 are positive the process is irreducible and from Theorem 3.6 we obtain that the traffic equations possess a unique positive solution q_1 , q_2 . As a consequence the queueing network possesses a unique invariant measure m at state space $S = Z^N$ given by

$$m(\bar{n}) = q_1^{n_1} q_2^{n_2}. \quad (5.2)$$

From (4.3) we obtain that $q_2 < 1$ iff

Figure 7. State space

$$r(2)^2 + r(1)r(2) + \Lambda(1)r(1) - \Lambda(2)r(2) - \Lambda(2)r(1) > 0, \quad (5.3a)$$

and by symmetry that $q_1 < 1$ iff

$$r(1)^2 + r(1)r(2) + \Lambda(2)r(2) - \Lambda(1)r(1) - \Lambda(1)r(2) > 0, \quad (5.3b)$$

If the coefficients $\Lambda(1)$, $\Lambda(2)$, $r(1)$, $r(2)$ satisfy (5.3a), (5.3b) then from Theorem 3.1 we obtain that for $N_1, N_2 \in \mathbb{Z}$ the process possesses a unique equilibrium distribution at $S = \{\bar{n} : n_1 \geq N_1, n_2 \leq N_2\}$. For $N_1 = N_2 = 0$

$$\pi(\bar{n}) = (1 - q_1)q_1^{n_1}(1 - q_2)q_2^{n_2}, \quad n_1, n_2 \geq 0.$$

If (5.3a), (5.3b) are satisfied with $>$ replaced by $<$ then $q_1 > 1$ and $q_2 > 1$ and Theorem 4.1 can be applied to produce a unique equilibrium distribution at a state space that is bounded from above.

In addition to these types of blocking, observation of local balance depicted in Figure 6 shows that, without additional stringent assumptions on the transition rates, the state space can be reduced to a band as depicted in Figure 7. To this end note that combination of local balance for $i = 1$ with reference state $\bar{n}$ and $i = 2$ with reference state $\bar{n} - e_2$ shows that for transitions on the right-hand side of the diagonal the rate into $\bar{n}$ balances with the rate out of $\bar{n}$. Similarly for the left-hand side of the diagonal. At boundary a service at station 2 is stopped and at boundary b service at station 1 is stopped to preserve a product form equilibrium distribution.

6 Conclusion

In this paper we have shown that a recently developed product form result for queueing networks with positive and negative customers is not a consequence of standard

local balance equating in each state the “rate in” and the “rate out” at each station separately. To this end we have derived the traffic equations for queueing networks with positive and negative customers. Based on these traffic equations we have derived a new and non-standard type of local balance. In contrast to standard local balance appearing in the literature, local balance for queueing networks with positive and negative customers equates the “rate in” and the “rate out” at different states. For $q(\bar{n}, \bar{n} - e_i + e_j)$ the transition rate due to a positive customer routing from station i to station j , $i, j = 0, \dots, N$, and $q_i^*(\bar{n}, \bar{n} - e_i - e_j)$ the transition rate for a negative customer leaving station i to route to station j , $i, j = 1, \dots, N$, the equilibrium distribution π satisfies *local balance for queueing networks with positive and negative customers* if

$$\begin{aligned} \sum_{j=0}^N \pi(\bar{n}) q(\bar{n}, \bar{n} - e_i + e_j) \\ + \sum_{j=1}^N \left\{ \pi(\bar{n}) q_i^*(\bar{n}, \bar{n} - e_i - e_j) + \pi(\bar{n} + e_j) q_j^*(\bar{n} + e_j, \bar{n} - e_i) \right\} \\ = \sum_{j=0}^N \pi(\bar{n} - e_i + e_j) q(\bar{n} - e_i + e_j, \bar{n}). \end{aligned} \quad (6.1)$$

The third term in the left-hand side is neither related to the rate into $\bar{n}$ nor to the rate out of $\bar{n}$ thus “destroying” standard local balance. The behaviour of the queueing network with negative and positive customers under local balance (6.1) is similar to the behaviour of standard queueing networks under standard local balance. For example, blocking phenomena can be obtained from (6.1).

Local balance for queueing networks with positive and negative customers relates the probability flow out of a state to the probability flow into that state *and* its neighbours. Furthermore, the service rates and routing probabilities are intertwined in the traffic equations. Therefore, in contrast with standard queueing networks (Jackson networks), the result does not seem to give much room for generalizations, such as to include state-dependent service rates. Only for single-server stations in isolation (not part of a queueing network) generalizations are possible [5].

References

- [1] Baskett, F., Chandy, K. M., Muntz, R. R. and Palacios, F. G. (1975) Open, closed and mixed networks of queues with different classes of customers. *J. A. C. M.* Vol. 22 248-260.
- [2] Boucherie, R. J. and Van Dijk, N. M. (1990) Spatial birth-death processes with multiple changes and applications to batch service networks and clustering processes. *Adv. Appl. Prob* 22, 433-455.
- [3] Boucherie, R. J. and Van Dijk, N. M. (1991) Product forms for queueing networks with state dependent multiple job transitions. *Adv. Appl. Prob* 23, 152-187.

- [4] Gelenbe, E. (1989) Random neural networks with negative and positive signals and product form solution. *Neural Computation* **1**, 502-510.
- [5] Gelenbe, E., Glynn, P. and Sigman, K. (1991) Queues with negative arrivals. *J. Appl. Prob.* **28**, 245-250.
- [6] Gelenbe, E. (1991) Product form networks with negative and positive customers. *Research report*
- [7] Henderson, W., Pearce, C. E. M., Taylor, P. G. and Van Dijk, N. M. (1988) Closed queueing networks with batch services. *QUESTA* **6**, 59-70.
- [8] Hordijk, A. and Van Dijk, N. M. (1983) Networks of queues Part I: Job-local-balance and the adjoint process. Part II: General routing and service characteristics. *Lecture notes in Control and Informational Sciences*. Springer-Verlag. Vol. **60** 158-205.
- [9] Jackson, J. R. (1963) Jobshop-like queueing systems. *Management Science* **10** 131-142.
- [10] Kelly, F. P. (1979) *Reversibility and stochastic networks*. Wiley.
- [11] Serfozo, R. F. (1989) Markovian network processes: congestion dependent routing and processing. *QUESTA* **5**, 5-36.
- [12] Whittle, P. (1986) *Systems in stochastic equilibrium*. Wiley.