VU Research Portal

An alternative proof of uniqueness in non-cooperative bargaining

Houba, H.

1993

document version

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

citation for published version (APA)

Houba, H. (1993). *An alternative proof of uniqueness in non-cooperative bargaining*. (Serie Research Memoranda; No. 1993-15). Faculty of Economics and Business Administration, Vrije Universiteit Amsterdam.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal ?

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

E-mail address:

vuresearchportal.ub@vu.nl

Serie Research Memoranda

An Alternative Proof of Uniqueness in Non-Cooperative Bargaining

Harold Houba

Research-Memorandum 1993-15
March 1993

An Alternative Proof of Uniqueness in Non-Cooperative Bargaining*

Harold Houba
Department of Econometrics,
Free University,
De Boelelaan 1105,
1081 HV Amsterdam,
The Netherlands.

Abstract

In this paper the standard alternating offer model is analysed. An alternative proof is suggested to show that the system of equations that characterises the stationary SPE proposals admits a unique solution. Furthermore, this proof is more general than the existing proofs.

* The author thanks Hans Haller, Gerard van der Laan and Aart de Zeeuw.

JEL-code: C78

1 Introduction

The bargaining model in this note is the alternating offer model (Rubinstein 1982) in which players bargain over a set of feasible payoff vectors. Furthermore, it is assumed that the players face an exogenously given probability of a terminal break down after each bargaining round. If the bargaining breaks down while the players have not agreed in the past, then the players get their exogenous disagreement payoffs.

The bargaining problem in this particular setting has been analysed in van Damme (1991, chapter 7) and Okada (1991).¹ In Okada (1991) it is assumed that the Pareto frontier is concave and piecewise linear and the proof is quite long. In van Damme (1991) it is assumed that the disagreement payoffs are equal to 0 and that the Pareto frontier is differentiable, concave and strictly decreasing. In both papers the standard approach to prove uniqueness of the subgame perfect equilibrium (SPE) is followed. This is to establish existence of a stationary SPE first, then prove that the system of equations that characterises stationary SPE's admits at most one solution and, finally, apply the method in Shaked and Sutton (1984) to prove that no non-stationary equilibria exist [see, e.g., van Damme (1991) and Osborne and Rubinstein (1990)].

The aim of this paper is to suggest an alternative proof to show that the system of equations that characterises the stationary SPE proposals admits at most one solution. Furthermore, this proof requires weaker conditions than the conditions imposed in van Damme (1991) and Okada (1991), namely that the Pareto frontier is continuous, concave and strictly decreasing.

¹In Okada (1991) the players bargaining over a stream of payoffs and the disagreement payoffs are endogenously determined by the players. However, it should be acknowledged that Okada's arguments are also valid in the bargaining model analysed in this note.

2 Results

The bargaining problem is defined as a couple (S, d) , where $S \in \mathbb{R}^2$ denotes the non-empty set of feasible payoffs and $d \in S$ the disagreement payoff vector. It is assumed that the Pareto efficient frontier of $\{x \in S | x \geq d\}$ ($x \in S$ is Pareto efficient if $\nexists y \in S$ s.t. $y_i > x_i, i = 1, 2$) can be described by a strictly decreasing, concave and continuous function. If the set S is closed, convex and strongly comprehensive (strongly comprehensive means that if $x \in S$ and $y \leq x$, then $y \in S$) and $\{x \in S | x \geq d\}$ is compact, then such a function exists. This function is denoted by $f_i(z_j)$, $i, j = 1, 2, i \neq j$, and specifies player i 's maximum payoff in S given the payoff z_j for player j . Furthermore, f_2 is the inverse function of f_1 and visa versa. It is also assumed that $d \in S$ is not Pareto efficient, that is $d_1 < f_1(d_2)$, to rule out trivial bargaining problems. Finally, $\delta \in [0, 1)$ is defined as the exogenous probability that the bargaining process continues for one more bargaining round after the proposal in the current bargaining round is rejected and $(1 - \delta)$ is the probability that the bargaining breaks down after the proposal in the current bargaining round is declined by the responding player.

The bargaining procedure is the same as in the standard alternating offer model (Rubinstein 1982). In every odd bargaining round player 1 proposes a vector of payoffs $x \in S$ to player 2. Player 2 either accepts or rejects this proposal. If player 2 accepts then the game ends. However, if player 2 rejects, then this player proposes a vector of payoffs $y \in S$ in the next (even) bargaining round (provided there is one). Player 1 can either accept or reject player 2's proposal. Again, if player 1 accepts then the game ends, but if player 1 rejects then this player has the initiative to propose in the next (odd) bargaining round (provided there is one). As long as the two players do not agree the bargaining process continues, unless the bargaining breaks down and $d \in S$ is the terminal payoff vector to the players.

As mentioned in the introduction, the standard approach to prove uniqueness of the subgame perfect equilibrium is followed. For every stationary SPE it holds that a responding player is indifferent between accepting or rejecting the current proposal

and the proposing player obtains the remainder [see, e.g., van Damme (1991) and Osborne and Rubinstein (1990)]. This leads to the following system of equations that completely characterises the stationary SPE proposal x of player 1 and the stationary SPE proposal y of player 2, namely

$$x_1 = f_1((1 - \delta)d_2 + \delta y_2), \quad x_2 = (1 - \delta)d_2 + \delta y_2$$

and

$$y_1 = (1 - \delta)d_1 + \delta x_1, \quad y_2 = f_2((1 - \delta)d_1 + \delta x_1).$$

Each pair (x, y) corresponds to one stationary SPE and visa versa.

In order to prove that the couple (x, y) is unique the function $p : S \rightarrow S$ is defined as

$$p_2(y) = (1 - \delta)d_2 + \delta y_2, \quad p_1(y) = f_1(p_2(y)),$$

and the function $q : S \rightarrow S$ is defined as

$$q_1(x) = (1 - \delta)d_1 + \delta x_1, \quad q_2(x) = f_2(q_1(x)).$$

Brouwer's fixed point theorem can be applied to prove that $p \times q : S \times S \rightarrow S \times S$ has a fixed point $(x, y) = (p(y), q(x))$. This establishes the existence of at least one stationary SPE.

It is easy to show that each pair (x, y) possesses the properties known from the standard alternating offer model (Rubinstein 1982). By construction of the functions p and q it follows that the proposals x and y are Pareto efficient. The concavity of f_1 and f_2 can be used to show that x and y are individually rational with respect to d , e.g.

$$x_1 = f_1((1 - \delta)d_2 + \delta y_2) \geq (1 - \delta)f_1(d_2) + \delta f_1(y_2) > (1 - \delta)d_1 + \delta[(1 - \delta)d_1 + \delta x_1]$$

and, therefore, $x_1 > d_1$ and also $y_1 = q_1(x) = (1 - \delta)d_1 + \delta x_1 \in (d_1, x_1)$. Moreover, the latter result implies that x lies to the right of y on the Pareto frontier, a property that is better known as the first mover advantage.

Both in van Damme (1991) and Okada (1991) it is first proved that the couple (x, y) is unique and then it is shown that the corresponding x and y have the same

Nash product. This latter result is used to prove that x and y converge to the Nash bargaining solution with threat vector d as δ goes to 1. The proofs in these publications make use of different mathematical techniques to establish the uniqueness of (x, y) . However, lemma 2.1 states that, even before uniqueness is established, for every fixed point (x, y) of $p \times q$ it holds that x and y have the same Nash product. This insight is then used to construct an alternative proof, which requires less stringent conditions than the conditions in van Damme (1991) and Okada (1991). This alternative proof exploits the geometry of the bargaining problem and is the main result presented in this note.

Lemma 2.1 *If (x, y) is a fixed point of $p \times q$, then x and y have the same Nash product with threat vector d .*

proof.

The points x and y have the same Nash product with threat vector d , because $x_2 = p_2(y)$ and $y_1 = q_1(x)$ imply that

$$(x_1 - d_1)(x_2 - d_2) = \delta(x_1 - d_1)(y_2 - d_2)$$

and

$$(y_1 - d_1)(y_2 - d_2) = \delta(x_1 - d_1)(y_2 - d_2).$$

This completes the proof. □

Proposition 2.1 *The function $p \times q$ has a unique fixed point (x, y) .*

Proof.

Convexity of the set S implies that each curve of Nash products (this is a curve of points in $\mathbb{R}^2$ that all have the same Nash product and the same threat vector) intersects the Pareto frontier of S at most twice. Suppose (x, y) and $(\hat{x}, \hat{y})$ are both fixed points of $p \times q$, then x (y) and $\hat{x}$ ($\hat{y}$) must lie on distinct curves of Nash products (if not, either the curve would intersect the Pareto frontier four times, which is a contradiction, or $x = \hat{x}$ and $y = \hat{y}$). Therefore, their Nash products differ. Without

loss of generality assume that the Nash product of x (y) is strictly larger than $\hat{x}$ ($\hat{y}$). This implies that $\hat{x}$ lies to the right of x and $\hat{y}$ lies to the left of y (all four points lie on the Pareto frontier of S). Thus, $\hat{x}_1 > x_1$ and $\hat{y}_2 > y_2$. However, $\hat{x}_1 > x_1$ implies that $\hat{y}_1 = q_1(\hat{x}) > q_1(x) = y_1$ and, hence, $\hat{y}_2 = f_2(\hat{y}_1) < f_2(y_1) = y_2$, which contradicts $\hat{y}_2 > y_2$ above. Hence, (x, y) is unique. $\square$

The final step in proving that the unique stationary SPE is the only SPE is based upon the method in Shaked and Sutton (1984). Define $x^M \in S$ as the best SPE payoff vector for player 1 whenever this player has the initiative to propose and $y^m \in S$ as the worst SPE payoff vector for player 2 whenever this player has the initiative to propose. Using the arguments in van Damme (1991) it follows that x^M and y^m satisfy the same relations as x and y , namely

$$x_1^M = f_1((1 - \delta)d_2 + \delta y_2^m), \quad x_2^M = (1 - \delta)d_2 + \delta y_2^m$$

and

$$y_1^m = (1 - \delta)d_1 + \delta x_1^M, \quad y_2^m = f_2((1 - \delta)d_1 + \delta x_1^M).$$

Therefore, (x^M, y^m) is a fixed point of $p \times q$. Hence, $(x^M, y^m) = (x, y)$, where (x, y) denotes the unique fixed point of proposition 2.1. Similar definitions and arguments imply $(x^m, y^M) = (x, y)$. Hence, the stationary SPE is the unique SPE.

Finally, it is also easy to prove that x and y converge to the Nash bargaining solution with threat vector $d \in S$ as δ goes to 1. Firstly, x and y converge to the same limit vector, because

$$0 \leq x_1 - y_1 = (1 - \delta)[f_1((1 - \delta)d_2 + \delta y_2) - d_1] \leq (1 - \delta)[f_1(d_2) - d_1] \rightarrow 0,$$

as δ goes to 1. This latter argument is also simpler than the arguments in van Damme (1991) and Okada (1991). Finally, following van Damme (1991), for every $\delta \in [0, 1)$ it holds that the points x and y are two distinct points that lie on the same Nash product curve and that have a common limit vector, which must be the unique Pareto efficient point where the Nash product curve $(z_1 - d_1)(z_2 - d_2)$ is tangent to the Pareto frontier, i.e. the limit vector is the Nash bargaining solution with threat point d .

3 References

- van Damme, E., 1991, Stability and perfection of Nash equilibria, second edition (Springer-Verlag, New York/Berlin).
- Okada, A., 1991, A non-cooperative approach to the Nash bargaining problem, in: R. Selten, ed., Game theoretic equilibrium models III: strategic bargaining (Springer-Verlag, New York/Berlin), 7-33.
- Osborne, M., and A. Rubinstein, 1990, Bargaining and markets, (Academic Press, San Diego).
- Rubinstein, A., 1982, Perfect equilibrium in a bargaining model, *Econometrica* 50, 97-109.
- Shaked, A., and J. Sutton, 1984, Involuntary unemployment as a perfect equilibrium in a bargaining model, *Econometrica* 52, 1351-1364.