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Renormalization of light-front Hamiltonian field theory

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We propose a regularization of light-front field theory which does not make a distinction between transverse and longitudinal coordinates. It consists of a subtraction of low order terms in the Taylor expansion of an amplitude with respect to external momenta, similar to the regularization in the BPHZ scheme, which has some advantages that do not exist in covariant field theory. Instantaneous parts and other longitudinal singularities are automatically removed irrespective of their explicit form. We argue therefore that these parts are meaningless. The local counterterms are equal to those of covariant field theory. We demonstrate the method in a number of examples, recovering the covariant results.

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I. INTRODUCTION

Because of the apparent breaking of manifest covariance in light-cone time-ordered perturbation theory people tend to treat the singularities and divergences in the longitudinal direction separately from the divergences in the perpendicular direction [1]. This separation makes it hard to recover the covariant result in lowest order perturbation theory [2]. Especially the longitudinal singularities are hard to interpret and lead to the belief that (confining) potentials occur owing to renormalization of light-front field theory [3].

We introduce a regularization scheme which maintains the symmetries of the theory such that the total amplitude is covariant order by order. We remove in our scheme the lowest orders in a Taylor expansion of the amplitude, which can be reabsorbed in counterterms to the original Lagrangian. The regularization also removes the longitudinal singularities without extra effort. This strengthens our belief that the instantaneous terms in low order diagrams are artifacts of the light-front formulation of the perturbation theory and hence meaningless. They might be considered to be local structures since they occur at finite order in the Taylor expansion, but they do not have the proper dimension to act as such.

The regularization scheme of subtracting low order terms in the Taylor expansion [4] might sound old-fashioned, but it has advantages over most other regularization schemes. First, it preserves covariance, and second, it is fully local. Third, it is a linear operation and can be applied to separate parts independently, and fourth, it gives finite integrands and allows for numerical implementation. Some of the problems that occur in manifestly covariant field theory, by means of Feynman diagrams, are not present in light-cone time-ordered perturbation theory with this regularization, since the external energy flow through the diagram is fixed from the start. The freedom to choose the flow of external momenta is the origin of most problems in the regularization of Feynman diagrams. The problem of the regularization procedure we propose is the separation of different orders

of divergence that occur in amplitudes with a nontrivial tensor structure. The separation in each energy denominator of internal and external coordinates, which is one of the advantages of light-front dynamics, turns out to solve this problem easily.

The dimensional analysis of each term in the light-cone time-ordered perturbative expansion tells us that they separately have the proper dimensions and that the same subtractions must be made as in the covariant case apart from the subtractions that, due to their tensorial structure, correspond to “negative powers” in the Taylor expansion. These terms are low order remnants of the instantaneous interactions, which are the implementation of constraints in Hamiltonian perturbation theory. We consider these negative terms meaningless and they automatically vanish if the proper divergences are subtracted from the diagram.

II. DIMENSIONAL ANALYSIS

In this section we restate some of the results already obtained in [5], and apply the methods developed there to specific examples in light-cone time-ordered perturbation theory. The starting point for dimensional analysis is the construction of an operator that has the “building blocks” of the perturbative expansion as its eigenfunctions with the “dimensions” of those building blocks as eigenvalues. For light-front field theory this dimensional operator is

$$\bar{\nabla} + D; \quad (1)$$

$\bar{\nabla}$ contains the internal variables k_i and D the external variables p_j :

$$\bar{\nabla} = \sum_i \left(k_i^+ \frac{\partial}{\partial k_i^+} + \vec{k}_{i\perp} \cdot \frac{\partial}{\partial \vec{k}_{i\perp}} \right), \quad (2)$$

$$D = \sum_i m_i \frac{\partial}{\partial m_i} + \sum_j p_j^\mu \frac{\partial}{\partial p_j^\mu}. \quad (3)$$

The dimensional operator acting on a light-front time-ordered diagram which contains l energy denominators, s fermion propagators, and $n - s$ boson propagators, returns the dimension of the diagram (schematically):

$$(\bar{\nabla} + D) \frac{(\not{q}_1 + m_1) \cdots (\not{q}_s + m_s)}{q_1^+ \cdots q_n^+ [p_1^- - H_1] \cdots [p_l^- - H_l]} \\ = (s - l - n) \frac{(\not{q}_1 + m_1) \cdots (\not{q}_s + m_s)}{q_1^+ \cdots q_n^+ [p_1^- - H_1] \cdots [p_l^- - H_l]}. \quad (4)$$

The momenta q_i are combinations of the internal (k) and the external momenta (p). The s spinor projections $(\not{q}_i + m_i)$ have an on-shell minus component: $q_i^- = (q_{i\perp}^2 + m_i^2)/(2q_i^+)$. The l energy denominators contain the difference between the incoming p^- -momentum and the sum of all the on-shell energies of the particles present in an intermediate state j , which are given by

$$H_j = \sum_{r \in j} \frac{q_{r\perp}^2 + m_r^2}{2q_r^+}. \quad (5)$$

With the help of partial integration [6,5], we can relate integrands with higher order divergences to integrands with lower order divergences, which forms the basis of dimensional regularization.

There are two major differences between the covariant approach and “old-fashioned” perturbation theory [7]. First, in light-cone time-ordered diagrams the longitudinal integration extends over a finite domain only, therefore, partial integration will lead to additional surface terms which renders a naive application of partial integration useless. Second, light-cone time-ordered diagrams contain singularities which cannot be associated with objects with the proper dimension as obtained in naive power counting.

Although partial integration leads to surface terms, each time-ordered diagram that corresponds to a single Feynman diagram appears to have the same dimension as the covariant amplitude. The same equation that provides us with the dimension of the full covariant amplitude, also gives the dimension of the time-ordered diagrams [5]:

$$D\mathcal{M}(p) = \delta\mathcal{M}(p). \quad (6)$$

The surface terms from the partial integration are canceled by the surface terms obtained by commuting the D operation with the integral:

$$\int_{p_1^+}^{p_2^+} dk^+ d^2k_\perp DI(k^+, \vec{k}_\perp) \\ = D \int_{p_1^+}^{p_2^+} dk^+ d^2k_\perp I - \int d^2k_\perp p_2^+ I(p_2^+, \vec{k}_\perp) \\ + \int d^2k_\perp p_1^+ I(p_1^+, \vec{k}_\perp). \quad (7)$$

In many cases the surface terms are infinite so this is a formal relation. The surface terms spoil many of the algebraic relations that exist in the standard cases [5]. Therefore we have invented a method which does not generate these surface terms and removes the end-point singularities.

A. Transverse dimensional analysis

Often, power counting in the longitudinal and in the perpendicular directions are separated. Although we wish to deal in our regularization scheme with all directions in the same way, the separation of dimensions gives some insight in the spin dependence of the degree of divergence. We apply the ideas of ordinary dimensional analysis to a general time-ordered diagram, Eq. (4). To do so, we introduce two dummy variables ξ and η , which are set to unity afterwards, and are used to determine the dimension of the spin structure:

$$\gamma^+ \rightarrow \xi \gamma^+, \quad (8)$$

$$\gamma^- \rightarrow \eta \gamma^-. \quad (9)$$

Also the transverse parts of gauge propagators must be multiplied with the same appropriate factors depending on the number of free $-$ and $+$ (upper) indices. We can define a transverse dimensional operator

$$\bar{\nabla}_T + D_T, \quad (10)$$

with the definitions

$$\bar{\nabla}_T = \sum_i \vec{k}_{i\perp} \cdot \frac{\partial}{\partial \vec{k}_{i\perp}}, \quad (11)$$

$$D_T = \sum_i m_i \frac{\partial}{\partial m_i} + 2 \sum_j p_j^- \frac{\partial}{\partial p_j^-} + \sum_j \vec{p}_{j\perp} \cdot \frac{\partial}{\partial \vec{p}_{j\perp}} \\ - \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta}. \quad (12)$$

An arbitrary time-ordered diagram will contain different eigenfunctions of the transverse dimensional operator. Note that the dimension in the minus direction is twice the dimension in the perpendicular direction. Applying this operator to the time-ordered diagram above, Eq. (4), will give us the transverse dimension

$$\Gamma_T = s - 2l - n^+ + n^-, \quad (13)$$

where n^+ (n^-) is the number of free $+$ ($-$) indices. This relation allows us to use dimensional regularization [6,2] for the perpendicular dimensions, since we can use partial integration and obtain the integrand identity

$$D_T I = (\Gamma_T + 2L) I. \quad (14)$$

Here, L is the number of loops. However, if there are longitudinal singularities present in the integrand, these would interfere with the perpendicular dimensional regularization. The equation (14) has a straightforward relation with the covariant dimension, Eq. (6): $(\Gamma_T + 2L) = \delta + n^- - n^+$. It is easy to check that if only covariant objects $\not{p}, p^2, p^\mu p^\nu$, etc. occur in the amplitude the two relations are equivalent.

The transverse dimension is our guideline for subtraction. Our regularization scheme is covariant, so if we subtract terms which remove the transverse divergence they automatically remove the associated longitudinal singularity. The longitudinal singularities are more complicated as guidelines since they are more difficult to in-

interpret and a diagram can have a transverse logarithmic divergence without longitudinal singularities.

From Eq. (13) we see that we need to separate different divergences if we treat an amplitude with nontrivial tensor structure. We will demonstrate this separation in several examples.

B. Longitudinal dimensional analysis

Using the ordinary dimensional analysis and the transverse dimensional analysis we can derive the appropriate longitudinal operator

$$\bar{\nabla}_L + D_L, \quad (15)$$

with the definitions

$$\bar{\nabla}_L = \sum_i k_i^+ \cdot \frac{\partial}{\partial k_i^+}, \quad (16)$$

$$D_L = - \sum_j p_j^- \frac{\partial}{\partial p_j^-} + \sum_j p_j^+ \frac{\partial}{\partial p_j^+} + \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta}. \quad (17)$$

The dimension it returns compensates for the inclusion of spin and the exclusion of the longitudinal direction in the transverse dimensional analysis:

$$(\bar{\nabla}_L + D_L)I(p, k) = (l - n + n^+ - n^-)I(p, k). \quad (18)$$

Note that the number $n - l = L$, the number of loops in the diagram.

III. MINUS REGULARIZATION

After the discussion of dimensional properties of light-cone time-ordered diagrams we propose a straightforward regularization for them, which we call “minus regularization.” We differentiate the diagram with respect to the external light-front energy p^- until the integral exists and we can integrate over the internal variables $k^+, \vec{k}_\perp$. Afterwards we integrate the integral again over the external light-front energy as often as we have differentiated the integrand (symbolically):

$$\int_0^{p^+} dk^+ \int d^2 k_\perp I(p, k) \equiv \left(\int_{\frac{p_\perp^2}{2p^+}}^{p^-} dp'^- \right)^n \int_0^{p^+} dk^+ \int d^2 k_\perp \left(\frac{\partial}{\partial p^-} \right)^n I(k, p) = \mathcal{F}(p) - P(p). \quad (19)$$

$P(p)$ is a polynomial in p^- , which equals the n th order Taylor expansion of the amplitude $\mathcal{F}$. The lower limit of the integration is the renormalization point, for which we have chosen $p^2 = 0$. One should take care that the renormalization point is below any production thresholds, otherwise the subtracted part $P(p)$ is not real and unitarity is violated. This procedure removes from the amplitude the lowest order terms in the Taylor expansion of the amplitude around the renormalization point:

$$\begin{aligned} \mathcal{F}(p) &= a_0 + a_1 \left(p^- - \frac{p_\perp^2}{2p^+} \right) + a_2 \left(p^- - \frac{p_\perp^2}{2p^+} \right)^2 + \dots + O \left(\left(p^- - \frac{p_\perp^2}{2p^+} \right)^n \right) \\ &= P(p) + O \left(\left(p^- - \frac{p_\perp^2}{2p^+} \right)^n \right). \end{aligned} \quad (20)$$

Thus the first $n - 1$ derivatives of the amplitude vanish if they are subjected to our regularization, Eq. (19). Different time-ordered diagrams can be subjected to this procedure separately and added together afterwards since the differentiations and integrations are linear operations and can be applied to each term separately, if the renormalization point is chosen the same for the separate pieces. One should only take care that the instantaneous diagrams are added to the diagrams with ordinary singularities, in order to avoid unnecessary differentiations.

In the case of scalar amplitudes this is the full story. The number of differentiations needed equals the order of divergence of the amplitude. In the case of spinors or vector bosons the amplitude has an internal structure and mixes different orders in p^- , thus different numbers of differentiations are needed to remove the divergence of different parts of the integrand. If we consider the truncated electron self-energy we know the general structure from covariance [8],

$$\Sigma(p) = \not{p} f(p^2) + m g(p^2), \quad (21)$$

but we see that this structure mixes different orders in p^- :

$$\Sigma(p) = \frac{\gamma^+}{2p^+} p^2 f(p^2) + \left(\frac{\gamma^+ p_\perp^2}{2p^+} + \gamma^- p^+ - \vec{\gamma}_\perp \cdot \vec{p}_\perp \right) f(p^2) + m g(p^2). \quad (22)$$

Renormalization in this case means $f(p^2) \rightarrow f(p^2) - f(0)$ and $g(p^2) \rightarrow g(p^2) - g(0)$. Part of the integrand should be differentiated twice and the rest of the integrand should be differentiated only once to remove the divergence. For the amplitude this separation is straightforward, but for the integrand it is more elaborate. In general the longitudinal momenta k^+ and p^+ are our guide. We will show this for different examples in separate sections below.

The truncated vacuum polarization contains even more orders in p^- [8]:

$$\Pi^{\mu\nu}(p) = (g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi(p^2). \quad (23)$$

In this case renormalization means $\Pi(p^2) \rightarrow \Pi(p^2) - \Pi(0)$. If we look at the tensor $p^\mu p^\nu$ we see three different orders in p^- :

$$\begin{pmatrix} p^- p^- & p^- p^+ & p^- p^1 & p^- p^2 \\ p^+ p^- & p^+ p^+ & p^+ p^1 & p^+ p^2 \\ p^1 p^- & p^1 p^+ & p^1 p^1 & p^1 p^2 \\ p^2 p^- & p^2 p^+ & p^2 p^1 & p^2 p^2 \end{pmatrix} = p^4 \begin{pmatrix} \frac{1}{(2p^+)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + p^2 \begin{pmatrix} \frac{2p_\perp^2}{(2p^+)^2} & \frac{p^+}{2p^+} & \frac{p^1}{2p^+} & \frac{p^2}{2p^+} \\ \frac{p^+}{2p^+} & 0 & 0 & 0 \\ \frac{p^1}{2p^+} & 0 & 0 & 0 \\ \frac{p^2}{2p^+} & 0 & 0 & 0 \end{pmatrix} \\ + \begin{pmatrix} \frac{p_\perp^4}{(2p^+)^2} & \frac{p^+ p_\perp^2}{2p^+} & \frac{p^1 p_\perp^2}{2p^+} & \frac{p^2 p_\perp^2}{2p^+} \\ \frac{p^+ p_\perp^2}{2p^+} & p^+ p^+ & p^+ p^1 & p^+ p^2 \\ \frac{p^1 p_\perp^2}{2p^+} & p^1 p^+ & p^1 p^1 & p^1 p^2 \\ \frac{p^2 p_\perp^2}{2p^+} & p^2 p^+ & p^2 p^1 & p^2 p^2 \end{pmatrix}. \quad (24)$$

For the amplitude this separation is again straightforward, but in terms of integrands it is more complicated. First we introduce internal variables as is done in front-form dynamics [9]. These coordinates have the advantage that the integrand changes in a trivial way under a Lorentz transformation of the external momenta. We can therefore separate the different structures and determine their appropriate degree of divergence. These terms are differentiated according to their divergence. For example, in Eq. (24) we have three orders. To subtract “ $\Pi(0)$ ” from the vacuum polarization the first term should be differentiated three times, the second term two times, and the last one only once.

Both in the electron self-energy and in the vacuum polarization there are instantaneous terms in a light-front perturbative expansion, but these do not contribute to the amplitude; they are removed by the regularization procedure. They are independent of p^- (or in some cases linear functions of p^-) and therefore drop under differentiations. The only assumption we made to introduce our regularization procedure was covariance of the amplitude. The fact that different diagrams together make up the complete covariant amplitude does not bother us since differentiation and integration are linear operations. The instantaneous parts and other longitudinal singularities are considered meaningless in this procedure; they correspond to the “negative order” Taylor expansion. For the fermion self-energy, for example, this term is

$$\frac{\gamma^+(p^- - \frac{p_\perp^2}{2p^+})}{p^2} \int dk^+ d^2 k_\perp \frac{p^+}{2(p^+ - k^+)k^+}. \quad (25)$$

There are no strict methods to determine the integration limits in this case. The precise form of the integrand depends on the implementation of the instantaneous terms [11]. These instantaneous terms drop automatically in our regularization procedure. We do not associate counterterms with these parts. It is a strange artifact of light-front field theory that these negative orders exist; it is the result of the linearity in p^2 of p^- . These terms can also be removed by removing the negative orders of the Tay-

lor expansion [10] in k^+ around the singularities in the time-ordered diagrams, which frees one of the combinatorial job to combine the time-ordered singularities with the instantaneous parts as done in the “blink” procedure [11].

A. Scalar loop

We consider the light-front time-ordered diagram for the scalar loop with two propagators, Fig. 1(a):

$$\mathcal{F}(p) = \int_0^{p^+} \frac{dk^+ d^2 k_\perp}{4k^+(p^+ - k^+)} \times \frac{1}{p^- - \frac{k_\perp^2 + m^2}{2k^+} - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}}. \quad (26)$$

The integration limits are those of the longitudinal integration in k^+ ; the perpendicular integration $k_\perp = (k^1, k^2)$ extends over the whole space. Throughout this paper we do not include the coupling constants and the additional factors ($2\pi i$, symmetry factors, -1 for fermion loops) to keep the formulas as simple as possible. After differentiating the integrand once with respect to p^- we can integrate over the internal variables $(k^+, \vec{k}_\perp)$, with the result

$$\frac{\partial \mathcal{F}}{\partial p^-} = \frac{\pi p^+}{p^2} - \frac{4\pi m^2 p^+}{p^4} \sqrt{\frac{p^2}{p^2 - 4m^2}} \times \text{arctanh} \sqrt{\frac{p^2}{p^2 - 4m^2}}. \quad (27)$$

FIG. 1. (a) The scalar loop; (b) the fermion self-energy; (c) the vacuum polarization.

The finite part of the amplitude is obtained by integrating over p^- :

$$\begin{aligned}\mathcal{F}_r(p) &= \mathcal{F}(p^2) - \mathcal{F}(0) \\ &= \int_{\frac{p_\perp^2}{2p^+}}^{p^-} dp'^- \frac{\partial \mathcal{F}(p')}{\partial p'^-} \\ &= -\pi + \pi \sqrt{\frac{p^2 - 4m^2}{p^2}} \operatorname{arctanh} \sqrt{\frac{p^2}{p^2 - 4m^2}}. \quad (28)\end{aligned}$$

The expressions simplify if we introduce the variables $x = k^+/p^+$, $y = x - \frac{1}{2}$, $\xi = \sqrt{\frac{p^2}{p^2 - 4m^2}}$, and $d\xi = -[4m^2 p^+ / \sqrt{p^2(p^2 - 4m^2)^3}] dp^-$. The proper imaginary part is obtained if we note that the mass has a small negative imaginary part and we are not on the principal branch for the $\operatorname{arctanh}$ for $p^2 > 4m^2$.

B. Fermion self-energy

The time-ordered diagram for the truncated fermion self-energy in the Feynman gauge is given by [Fig. 1(b)]

$$\Sigma(p) = \int_0^{p^+} \frac{dk^+ d^2 k_\perp}{4k^+(p^+ - k^+)} \frac{\gamma_\mu (\not{p} - \not{k} + m) \gamma^\mu}{p^- - \frac{k_\perp^2}{2k^+} - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}}. \quad (29)$$

The instantaneous term is included in the integrand which makes the spinor projection

$$\begin{aligned}\Sigma_r(p) &= \Sigma(p) - \Sigma(0) = -4\pi \not{p} \left\{ -\frac{1}{16} - \frac{m^2}{8p^2} + \left(\frac{1}{8} - \frac{m^4}{8p^4} \right) \ln \left[\frac{m^2 - p^2}{m^2} \right] \right\} \\ &\quad + 8\pi m \left\{ -\frac{1}{4} + \left(\frac{1}{4} - \frac{m^2}{4p^2} \right) \ln \left[\frac{m^2 - p^2}{m^2} \right] \right\}. \quad (34)\end{aligned}$$

C. Vacuum polarization

The light-cone time-ordered diagram for the vacuum polarization in QED is given by [Fig. 1(c)]

$$\begin{aligned}\Pi^{\mu\nu}(p) &= \int_0^{p^+} \frac{dk^+ d^2 k_\perp}{4k^+(p^+ - k^+)} \\ &\quad \times \frac{\operatorname{Tr}[\gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_2 + m)]}{p^- - \frac{k_\perp^2 + m^2}{2k^+} - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)}} , \quad (35)\end{aligned}$$

without instantaneous parts. Before we proceed, we introduce again internal and external variables which facilitate the separation of the different orders, counting powers of $q_\perp$:

$$k^+ = xp^+, \quad (36)$$

$$\vec{k}_\perp = x\vec{p}_\perp + \vec{q}_\perp, \quad (37)$$

$$\begin{aligned}\not{p} - \not{k} + m &= \gamma^+ \left(p^- - \frac{k_\perp^2}{2k^+} \right) + \gamma^- (p^+ - k^+) \\ &\quad - \vec{\gamma}_\perp \cdot (\vec{p}_\perp - \vec{k}_\perp) + m; \quad (30)\end{aligned}$$

without the instantaneous part the final result would be the same. Note also that there is a longitudinal singularity for $k^+ \rightarrow 0$. As we have seen, the self-energy mixes different orders of p^- . We separate the different parts of the integrand by going to internal and external light-front variables:

$$k^+ = xp^+, \quad (31)$$

$$\vec{k}_\perp = x\vec{p}_\perp + \vec{q}_\perp. \quad (32)$$

This separates the different orders of divergence:

$$\begin{aligned}\not{p} - \not{k} + m &= \gamma^+ \left(p^- - \frac{k_\perp^2}{2k^+} - (1-x) \frac{p_\perp^2}{2p^+} \right) \\ &\quad + \vec{\gamma}_\perp \cdot \vec{q}_\perp + [(1-x)\not{p} + m]. \quad (33)\end{aligned}$$

The last term contains the fraction $(1-x)$ of external momenta flowing through the fermion line. The fraction x is measured in terms of longitudinal momentum p^+ . The barred quantity $\bar{p}$ has an “on-shell” energy: $\bar{p}^- = \frac{p_\perp^2}{2p^+}$ and vanishes at the renormalization point $p^2 = 0$. The first term is differentiated twice with respect to p^- , the last term is differentiated only once. The middle term can be made finite with one or two differentiations, but being odd, it does not contribute. All integrals are then finite and we can integrate over the internal variables $x, \vec{q}_\perp$. Afterwards we integrate the first term twice over p^- and the last term once and obtain the finite part of the covariant amplitude:

$$p_1^\mu = x\bar{p}^\mu + q_\perp^\mu + g^{\mu+} \left(\frac{p_\perp \cdot q_\perp}{p^+} + \frac{q_\perp^2 + m^2}{2p^+ x} \right), \quad (38)$$

$$p_2^\nu = (x-1)\bar{p}^\nu + q_\perp^\nu + g^{\nu+} \left(\frac{p_\perp \cdot q_\perp}{p^+} + \frac{q_\perp^2 + m^2}{2p^+(x-1)} \right). \quad (39)$$

The barred p momentum has an “on-shell” minus component: $\bar{p}^- = \frac{p_\perp^2}{2p^+}$. It is part of the lowest order in the Taylor expansion around $p^2 = 0$; see Eq. (24). To remove all the divergences, the integrand should be differentiated three times. We will follow a different procedure. Because the separation of the different degrees of divergence makes the calculation lengthy and complicated, we will use a method which boils down to the same subtraction, but allows us to see which terms are removed where and when, while differentiations act as an almost black box. Upon differentiation terms are removed without a clear view on the algebraic structure of these terms. Since

the only p^- dependence is in the denominator we can see which terms would drop under differentiation. These are the constant, linear, and quadratic terms in an analytical expansion (around $p^2 = 0$) of the denominator:

$$I = \frac{1}{2p^+[x(1-x)p^2 - m^2 - q_\perp^2]} = -\frac{1}{2p^+(m^2 + q_\perp^2)} \sum_{j=0}^{\infty} \left(\frac{x(1-x)p^2}{m^2 + q_\perp^2} \right)^j. \quad (40)$$

We see that the removal of the constant term is equivalent to multiplying the integrand by a factor $x(1-x)p^2/(m^2 + q_\perp^2)$:

$$\mathcal{F}_r(p^2) \equiv \mathcal{F}(p^2) - \mathcal{F}(0) = \int_0^1 dx d^2q \left(\frac{x(1-x)p^2}{m^2 + q_\perp^2} \right) I. \quad (41)$$

Apparently, the procedure of differentiation followed by integration with respect to the variable p^- is equivalent to the calculation of certain moments of the variable $x(1-x)p^2/(m^2 + q_\perp^2)$. Note also that in this (covariant) calculation the singularities in the longitudinal direction are lowered at the same time as the divergences in the perpendicular direction. This is an even more easy way to calculate the scalar loop (Sec. III A). At each differentiation or calculation of a higher moment, the divergence in the perpendicular directions is lowered by two. The symmetries are maintained under differentiation and subtraction: rotational symmetry in the $\vec{q}_\perp$ plane, and the symmetry with respect to the interchange of x and $1-x$. The calculation of a higher moment is equivalent to subtracting certain low-order terms in a Taylor expansion. (See also the section on counterterms, Sec. V.) For the minus components of the four-vectors, Eqs. (38), (39), we can do this immediately:

$$p_1^\mu = x\bar{p}^\mu + q_\perp^\mu + g^{\mu+} \left(\frac{p_\perp \cdot q_\perp}{p^+} - \frac{(x-1)p^2}{2p^+} \right), \quad (42)$$

$$p_2^\nu = (x-1)\bar{p}^\nu + q_\perp^\nu + g^{\nu+} \left(\frac{p_\perp \cdot q_\perp}{p^+} - \frac{xp^2}{2p^+} \right). \quad (43)$$

We separate the different terms by performing the trace

$$4T = \text{Tr}[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_2 + m)] = 4[g^{\mu\nu}(m^2 - p_1 \cdot p_2) + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu]. \quad (44)$$

Owing to the rotational symmetry of the denominator, terms in the integrand with $(q_x)^2$ will lead to the same answer as those with $\frac{1}{2}q_\perp^2$. If we use this symmetry the trace yields

$$T = 2x(x-1)\bar{p}^\mu \bar{p}^\nu + g^{\mu\nu} \left(\frac{q_\perp^2 + m^2}{2x(1-x)} - q_\perp^2 \right) \quad (45)$$

$$+ \frac{g^{+\nu} \bar{p}^\mu q_\perp^2}{p^+} + \frac{g^{+\mu} \bar{p}^\nu q_\perp^2}{p^+} \quad (46)$$

$$+ [2x(1-x) - 1] \left(\frac{g^{+\nu} \bar{p}^\mu p^2}{2p^+} + \frac{g^{+\mu} \bar{p}^\nu p^2}{2p^+} \right) \quad (47)$$

$$+ 2x(x-1)g^{+\mu} g^{+\nu} \frac{p^4}{(2p^+)^2}. \quad (48)$$

For the terms without $q_\perp^2$ dependence we calculate the first moment; for those with $q_\perp^2$ dependence we calculate the second moment. The m^2 terms are treated in the same manner as the $q_\perp^2$ terms since in a Taylor expansion in p^2 the mass acts as an internal variable. That the mass is treated as an internal variable is the biggest difference between our regularization and dimensional regularization [6]. The integrals give us the finite part of the vacuum polarization:

$$\Pi_r^{\mu\nu}(p) = (g^{\mu\nu}p^2 - p^\mu p^\nu)[\Pi(p^2) - \Pi(0)] = (g^{\mu\nu}p^2 - p^\mu p^\nu) \left(\frac{\pi}{18} + \frac{2m^2 + p^2}{3p^2} \mathcal{F}_r \right). \quad (49)$$

IV. WARD-TAKAHASHI AND VERTEX CORRECTIONS

In the self-energy diagrams there is only one external momentum and the fraction of the longitudinal loop momentum plays the role of a Feynman parameter. In the case of one-loop three- and four-point functions there is still only one longitudinal momentum, but in a Feynman parametrization there would occur more Feynman parameters. In our regularization scheme, where we differentiate with respect to the external momenta, we acquired additional freedom, since we can choose with respect to which external momenta we differentiate. If a diagram is only logarithmically divergent, one differentiation is enough and we would have fixed the value of the amplitude for one external momentum while the others run freely. This choice is a freedom we have.

The remnant of current conservation, the Ward-Takahashi identities [12,13], fixes this freedom for gauge theories. It tells us how to do our differentiation and integration in order to keep the result gauge invariant. In the covariant case the Ward-Takahashi identities can be formulated as identities among integrands [8]. If we start with a self-energy diagram of a specific order and momentum p , the vertex function of the same order can be obtained by attaching a photon line, with momentum q , to the self-energy diagram at all possible fermion lines in the (truncated) diagram such that the outgoing momentum is $p + q$. If we now apply the identity

$$\frac{\not{p} + m}{p^2 - m^2} - \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} = \frac{\not{p} + m}{p^2 - m^2} \not{q} \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2}, \quad (50)$$

to all the diagrams at the places where the photon line is attached, we see that we end up with the difference of the (truncated) self-energy diagrams with momenta p and $p + q$. Sometimes we must shift the internal momenta to make this apparent; closed fermion loops in the diagram do not contribute. If we have chosen a proper regularization, this relation among the integrands should lead to the same relation among the integrals:

$$q^\mu \Gamma_\mu^{(n)}(p, p+q) = \Sigma^{(n)}(p) - \Sigma^{(n)}(p+q). \quad (51)$$

Similar relations can be established among the integrands of time-ordered diagrams. If our regularization procedure

does not destroy this relation we automatically satisfy the Ward-Takahashi identities. Therefore we should differentiate the integrands only with respect to p^- , not q^- . This is nothing special since Ward derived the $q \rightarrow 0$ limit of the Ward-Takahashi identity in a similar way [12]. There are no problems with the shift of coordinates here, since the minus momenta are fixed in a time-ordered diagram.

The one loop vertex correction is related to the fermion self-energy diagrams in the Feynman gauge, see Fig. 2. One should take proper care to include the instantaneous parts of all the diagrams ($0 < k^+ < p^+$):

FIG. 2. The Ward-Takahashi identities can be satisfied for each of the time-ordered diagrams, in their appropriate physical sectors, but only after inclusion of the instantaneous parts.

$$\begin{aligned}
 & \frac{\gamma^\mu \gamma^+ \gamma_\mu}{2(p^+ - k^+)} + \frac{\gamma^\mu \left(\gamma^+ \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} - \vec{k}) + m \right) \gamma_\mu}{4(p^+ - k^+)k^+ \left(p^- - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \\
 & - \frac{\gamma^\mu \gamma^+ \gamma_\mu}{2(p^+ + q^+ - k^+)} - \frac{\gamma^\mu \left(\gamma^+ \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} + \vec{q} - \vec{k}) + m \right) \gamma_\mu}{4(p^+ + q^+ - k^+)k^+ \left(p^- + q^- - \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \\
 & = \gamma^\mu \left(\frac{\gamma^+}{2(p^+ - k^+)} + \frac{\left(\gamma^+ \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} - \vec{k}) + m \right)}{4(p^+ - k^+) \left(p^- - \frac{(p_\perp - k_\perp)^2 + m^2}{2(p^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \right) \\
 & \times \not{q} \left(\frac{\gamma^+}{2(p^+ + q^+ - k^+)} + \frac{\left(\gamma^+ \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} + \vec{q} - \vec{k}) + m \right)}{4(p^+ + q^+ - k^+)k^+ \left(p^- + q^- - \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \right) \gamma_\mu. \quad (52)
 \end{aligned}$$

If $p^+ + q^+ > k^+ > p^+$, only the integrand of the self-energy diagram with the momentum $p + q$ contributes to the integrand identity:

$$\begin{aligned}
 & \frac{\gamma^\mu \left(\gamma^+ \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} + \vec{q} - \vec{k}) + m \right) \gamma_\mu}{4(p^+ + q^+ - k^+)k^+ \left(p^- + q^- - \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \\
 & = \gamma^\mu \left(\frac{-\gamma^+}{2(p^+ - k^+)} + \frac{\left(-\gamma^+ \frac{(k_\perp - p_\perp)^2 + m^2}{2(k^+ - p^+)} + \vec{\gamma} \cdot (\vec{p} - \vec{k}) + m \right)}{2(p^+ - k^+) \left(q^- - \frac{(k_\perp - p_\perp)^2 + m^2}{2(k^+ - p^+)} - \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} \right)} \right) \\
 & \times \not{q} \left(\frac{\left(\gamma^+ \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} + \vec{\gamma} \cdot (\vec{p} + \vec{q} - \vec{k}) + m \right)}{4(p^+ + q^+ - k^+)k^+ \left(p^- + q^- - \frac{(p_\perp + q_\perp - k_\perp)^2 + m^2}{2(p^+ + q^+ - k^+)} - \frac{k_\perp^2}{2k^+} \right)} \right) \gamma_\mu. \quad (53)
 \end{aligned}$$

In order to see how the regularization of the electron self-energy fixes the regularization of the vertex function we can best look at the truncated amplitudes, see Eq. (22):

$$q_\mu \Gamma_r^\mu(p, p+q) = \Sigma_r(p) - \Sigma_r(p+q). \quad (54)$$

We want to separate in the integrand of the vertex function the parts which give rise to different electron self-energies. If we separate the differentiation with respect to p^- in two pieces,

$$\frac{\partial}{\partial p^-} = \left(\frac{\partial}{\partial p^-} - \frac{\partial}{\partial q^-} \right) + \frac{\partial}{\partial q^-}, \quad (55)$$

the separate terms in the right-hand side of Eq. (54) vanish under the different terms in the differentiation Eq. (55). The lower integration limit of each term is now determined by the self-energy renormalization $\Sigma_r(0) = 0$. Also note that we do not have the same difficulties with the flow of external momenta through the diagram as in the covariant case, for in time-ordered diagrams the external energy variable is fixed in each diagram [14,15].

The differentiation and integration with respect to the two variables Eq. (55) lead precisely to the counterterms already obtained in the fermion self-energy diagram. Both act only on a single energy denominator of the two denominators in the first order correction in the vertex function. The subtracted part for each is the value of the integrand at $p^2 = 0$ and $(p+q)^2 = 0$, respectively. Hence the Ward-Takahashi identity is satisfied. Note the special role played by the plus component of the vertex correction Γ^+ , since it does not commute with the differentiations:

$$\left(\frac{\partial}{\partial p^-} - \frac{\partial}{\partial q^-} \right) q_\mu \Gamma^\mu = q_\mu \left[\left(\frac{\partial}{\partial p^-} - \frac{\partial}{\partial q^-} \right) \Gamma^\mu - \frac{g^{-\mu}}{q^-} \Gamma^+ \right], \quad (56)$$

$$\frac{\partial}{\partial q^-} q_\mu \Gamma^\mu = q_\mu \left[\frac{\partial}{\partial q^-} \Gamma^\mu + \frac{g^{-\mu}}{q^-} \Gamma^+ \right]. \quad (57)$$

The first term must be integrated over incoming momentum, p , from the point where $p^2 = 0$, the second over the outgoing momentum $p+q$. To remove the divergences of the vertex correction, in a gauge invariant way, we should not only use different renormalization points for the separate denominators, but we should also subtract and add a factor Γ^+/q^- .

V. COUNTERTERMS

In the section on the vacuum polarization we introduced a moment formula to subtract the lower order contributions of the integrand. This allows us to express the counterterms in terms of integrals. After regularizing the diagrams the infinite parts which are subtracted from the diagrams are added to the original La-

grangian to cancel against the counterterms [16,17]. The C 's are infinite constants which arose in the diagrams we considered in earlier sections; respectively ϕ^3 self-energy ($C_1\phi^2$), fermion self-energy ($C_2\bar{\psi}m\psi$, $C_3\bar{\psi}\not{p}\psi$), and vacuum polarization ($C_4F_{\mu\nu}F^{\mu\nu}$). These infinite numbers can be written in our regularization scheme as integrals over internal coordinates:

$$C_1 = - \int_0^1 dx d^2 q_\perp \frac{1}{2(m^2 + q_\perp^2)}, \quad (58)$$

$$C_2 = -2 \int_0^1 dx d^2 q_\perp \frac{1}{(xm^2 + q_\perp^2)}, \quad (59)$$

$$C_3 = \int_0^1 dx d^2 q_\perp \frac{1-x}{(xm^2 + q_\perp^2)}, \quad (60)$$

$$C_4 = - \int_0^1 dx d^2 q_\perp \frac{2x(1-x)}{(m^2 + q_\perp^2)}. \quad (61)$$

Note also the (integrable) infrared singularity in the electron self-energy for $(x \rightarrow 0, q_\perp \rightarrow 0)$, which would cause complications for on-shell renormalization [18]. Besides these terms we have counterterms which are specifically associated with the singularities of the instantaneous contributions. We saw that these contributions dropped automatically in our regularization procedure. These terms need no special treatment in our scheme. We maintain here that these contributions are meaningless and do not contain any physics.

VI. CONCLUSIONS

We have dealt with the divergences and singularities in light-front Hamiltonian field theory that are the result of local divergences. For these divergences we have defined a renormalization of light-cone time-ordered perturbation theory. We have recovered the covariant amplitude with light-cone time-ordered diagrams for the scalar two-propagator loop, the electron self-energy, and the vacuum polarization. Our method defines the regularization of the vertex correction if we wish to satisfy the Ward-Takahashi identities. Ambiguities as the result of the longitudinal singularities vanish.

Only after one has a proper method to remove all the local divergences can one deal with infrared divergences, long-range effects, and the strong-coupling regime. Often these two separate parts are intertwined in the discussions on the renormalization of light-front field theory, as the result of a lack of proper interpretation and method to deal with the local divergences. We hope that our way to deal with the local part of the theory will shed some light on the true nonperturbative content of light-front field theory. We have avoided the use of renormalization group methods because there is no sensible way to approach a point or an infinite constant, which are precisely the objects we have dealt with. Although some of the local structures look like potentials (infrared divergences) we have seen that these can be accounted for in the proper way as local terms or discarded as artifacts.

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