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Probabilistic Epistemic Updates on Algebras

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The present article contributes to the development of the mathematical theory of epistemic updates using the tools of duality theory. Here, we focus on Probabilistic Dynamic Epistemic Logic (PDEL). We dually characterize the product update construction of PDEL-models as a certain construction transforming the complex algebras associated with the given model into the complex algebra associated with the updated model. Thanks to this construction, an interpretation of the language of PDEL can be defined on algebraic models based on Heyting algebras. This justifies our proposal for the axiomatization of the intuitionistic counterpart of PDEL.

CCS Concepts: • **Theory of computation** → **Modal and temporal logics; Constructive mathematics;**

Additional Key Words and Phrases: Intuitionistic probabilistic dynamic epistemic logic, duality, intuitionistic modal logic, algebraic models, pointfree semantics

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1 INTRODUCTION

This article pertains to a line of research aimed at exploring agency and information flow in situations in which truth is socially constructed. Such situations are ubiquitous in the real world. A

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prime example is the validity of contracts. Establishing that an agreement constitutes a valid contract appeals to notions such as legal competency and bona fide offers, which are inherently socially constructed. The ultimate way in which the validity of a contract can be ascertained is for it to be tested in a court of law. In this last instance, the validity of a contract is thus procedural and may also admit of situations in which it is indeterminate, such as when the court declares itself incompetent. These are features at odds with standard classical logic. Accommodating these features within classical logic requires additional encoding mechanisms. The alternative is working with logics that are specifically designed to accommodate these characteristics of socially constructed truth.

Examples of situations where truth is socially constructed are certainly not confined to contract law but are easy to find in many other contexts. These include establishing public opinion in a binding way, such as referendums, establishing whether a certain item of clothing is fashionable, and determining the value of products in a market.

There is a large literature on logics that very adequately capture agency and information flow (see Reference [39] and references therein) but assume a notion of truth that is classical. There is therefore a need for a uniform methodology for transferring these logics onto nonclassical bases. In References [34, 36], a uniform methodology is introduced for defining the nonclassical counterparts of dynamic epistemic logics. This methodology, further pursued in References [5, 6, 38], is grounded on semantics and is based on the dual characterizations of the transformations of models that interpret epistemic actions.

The present article expands on Reference [16] and applies the methodology of References [34, 36] to obtain nonclassical counterparts of probabilistic dynamic epistemic logic (PDEL) [33, 40]. Intuitionistic logic is the propositional base of the version of PDEL introduced in the present article. The intuitionistic base lends itself to a finer-grained analysis of dynamic updates than the classical base. For instance, the fact that the box-type and the diamond-type modalities are no longer interdefinable makes several axioms non-equivalent that are logically equivalent in the classical setting. Moreover, the intuitionistic relational semantics allows for propositions to be neither true nor false at some states, a feature that, as discussed above, accommodates more naturally than classical models the characteristic features of the truth of statements regarding social constructs such as the validity of contracts or the bankruptcy of a firm (the latter being the focal notion of the case study discussed in Section 7). Perhaps even more importantly, the intuitionistic environment helps to highlight the fact that in situations such as those described in the case study, decisions are taken based on beliefs and expectations rather than on facts-of-the-matter, the truth of which might be *undefined* when the decision-making takes place.

Moving to the intuitionistic base also requires the use of intuitionistic probability theory (cf. References [3, 23]) as the background framework for probabilistic reasoning. From the point of view of applications, this generalization is needed to account for situations in which the probability of a certain proposition p is interpreted as an agent's propensity to bet on p given some evidence for or against p . If there is little or no evidence for or against p , it should be reasonable to attribute low probability values to both p and $\neg p$, which is forbidden by classical probability theory (cf. Reference [44]).

Finally, these mathematical developments appear in tandem with interesting analyses on the philosophical side of formal logic (e.g., Reference [4]), exploring epistemic logic in an evidentialist key, which is congenial to the kind of social situations targeted by our research program.

Our methodology is based on the dual characterization of the product update construction for standard PDEL-models as a certain construction transforming the complex algebras associated with a given model into the complex algebra associated with the updated model. This dual characterization naturally generalizes to much wider classes of algebras, which include arbitrary classical S5 algebras and certain monadic Heyting algebras. As an application of this dual characterization,

we introduce the axiomatization of the intuitionistic analogue of PDEL semantically arising from this construction and prove its soundness and completeness with respect to the class of so-called *algebraic probabilistic epistemic models* (see Definition 5.3).

Structure of the article. In Section 2, we recall the definition of classical PDEL and its relational semantics. We give an alternative presentation of the product update construction that consists of two steps, as done in Reference [34]. The two-step construction highlights the elements that will be key in the dualization. In Section 3, we expand on the methodology making use of Stone duality. Section 4 is the main section, in which the construction of the PDEL-updates on epistemic Heyting algebras is introduced. In Section 5, we define axiomatically the intuitionistic version of PDEL (IPDEL) and its interpretation on algebraic probabilistic epistemic models and discuss the proof of its soundness. In Section 6, we introduce the relational semantics of IPDEL. In Section 7, we discuss the case study of decision-making under uncertainty in a situation to which socially constructed truth is key. In Section 8, we collect conclusions and further directions. Appendix A collects some proofs of Section 4. Appendix B contains the proof of soundness of IPDEL with respect to algebraic probabilistic epistemic models. Appendix C contains the proof of completeness of IPDEL with respect to algebraic probabilistic epistemic models.

2 PDEL LANGUAGE AND UPDATES

The present section is mostly devoted to preliminaries on classical PDEL, but contains also an original contribution in the form of an alternative, two-step account of the product update construction on PDEL-models. This account is similar to the treatment of epistemic updates in References [34, 36], and, as explained in Section 3, lays the groundwork to the dualization procedure that motivates the construction introduced in Section 4. The specific PDEL framework we report on shares common features with those of References [2, 8], and [40].

In Section 2.1, we recall basic facts about probability theory and collect preliminaries on classical PDEL and its associated models and event structures. In Section 2.2, we give an alternative presentation of the construction of the epistemic update of a PES-model induced by a probabilistic event structure. This alternative presentation is closer to the update construction we introduce in the subsequent sections, which is the core technical contribution of this article. In Sections 2.3 and 2.4 s respectively, we collect preliminaries on the interpretation of PDEL-formulas on PES-models and on the axiomatization of classical PDEL.

2.1 Language and Models of PDEL

Let us start by recalling the standard definitions of probability distribution and probability measure. For any finite set X , a *probability distribution* P over X is a map

$$P : X \rightarrow [0, 1]$$

such that

$$\sum_{x \in X} P(x) = 1.$$

A *probability measure* on $\mathcal{P}X$ is a map

$$\mu : \mathcal{P}X \rightarrow [0, 1]$$

satisfying the following properties:

- (1) $\mu(\emptyset) = 0$,
- (2) $\mu(X) = 1$,
- (3) for any $A, B \subseteq X$, we have $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

The probability measure $\mu : \mathcal{P}X \rightarrow [0, 1]$ associated with a probability distribution P over X is defined as follows: for any $S \subseteq X$,

$$\mu(S) := \sum_{x \in S} P(x).$$

In the remainder of the article, we fix a countable set AtProp of proposition letters p, q , and a non-empty finite set Ag of agents i . We let $\alpha_1, \dots, \alpha_n, \beta$ denote rational numbers.

Definition 2.1 (PDEL Syntax). The set \mathcal{L} of PDEL-formulas φ and the class of probabilistic event structures \mathcal{E} over \mathcal{L} (see Definition 2.4) are built by simultaneous recursion as follows:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \diamond_i \varphi \mid \square_i \varphi \mid \langle \mathcal{E}, e \rangle \varphi \mid [\mathcal{E}, e] \varphi \mid \left(\sum_{k=1}^n \alpha_k \mu_i(\varphi) \right) \geq \beta,$$

where $p \in \text{AtProp}$, $i \in \text{Ag}$, $\alpha_1, \dots, \alpha_n, \beta$ are rational numbers,¹ and the event structures \mathcal{E} are as in Definition 2.4.

The connectives \top , \neg , and \leftrightarrow are defined by the usual abbreviations.

In the literature in probabilistic logic, it is common to use, e.g., $\alpha P_i(\varphi) \geq \beta$ as the syntax of formulas describing the probabilities of events. However, in the models defined below, the maps P_i are probability *distributions*, which take states of the model rather than extensions of formulas in input. To avoid any ambiguities, in the definition above, we use the syntactic μ_i s to stress the fact that their intended interpretation is as probability *measures* rather than probability distributions.

Definition 2.2 (PES-Model). A probabilistic epistemic state model (PES-model) is a structure

$$\mathbb{M} = \left\langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \right\rangle$$

such that

- (1) S is a finite set,
- (2) each binary relation \sim_i is an equivalence relation on S ,
- (3) each map $P_i : S \rightarrow]0, 1]$ restricts to a strictly positive probability distribution over each \sim_i -equivalence class, (i.e., $\sum \{P_i(s') : s' \sim_i s\} = 1$), and
- (4) the map $\llbracket \cdot \rrbracket : \text{AtProp} \rightarrow \mathcal{P}S$ is a valuation.

The interpretation of PDEL-formulas in PES-models is reported in Section 2.3 and is based on the epistemic update construction discussed in Section 2.2. As usual, the map $\llbracket \cdot \rrbracket$ will be identified with its unique extension to \mathcal{L} , so we will be able to write $\llbracket \varphi \rrbracket$ for every $\varphi \in \mathcal{L}$.

Remark 1. In the definition above, the assumption that the probability of each state is strictly positive is needed for the update construction in Definition 2.7 to be well defined, since it involves taking a quotient. This is also the convention followed in Reference [42], where subjective probabilities are identified with “lotteries” assigned to each agent, and are hence taken to be strictly positive. For the same reason, an analogous assumption is made of the event structures of Definition 2.4 below, an assumption that is also made of the event structures in References [1, 42].

A key difference between PDEL and dynamic epistemic logic (DEL) concerns the nature of the actions or events: while in DEL the actions are epistemic, and as such their main feature is that

¹Using rational numbers rather than, e.g., real numbers guarantees that the language remains denumerable.

they do not change the truth values of atomic propositions, PDEL deals with events that might change the truth values of atomic propositions. The following auxiliary definition feeds into the main Definition 2.4 and provides a formal account of this feature of events:

Definition 2.3 (Substitution Function). A substitution function

$$\sigma : \text{AtProp} \rightarrow \mathcal{L}$$

is a function that maps all but a finite² number of proposition letters to themselves.

We let the set

$$\text{supp}(\sigma) := \{p \in \text{AtProp} \mid \sigma(p) \neq p\}$$

be called the *support* of σ .

Let $\text{Sub}_{\mathcal{L}}$ denote the set of all substitution functions and ϵ the identity substitution.

Definition 2.4 (Probabilistic Event Structure Over a Language). A probabilistic event structure over \mathcal{L} is a tuple

$$\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \mathbf{sub}),$$

such that

- (1) E is a non-empty finite set,
- (2) each \sim_i is an equivalence relation on E ,
- (3) each $P_i : E \rightarrow]0, 1]$ restricts to a strictly positive probability distribution over each \sim_i -equivalence class, i.e.,

$$\sum \{P_i(e') \mid e' \sim_i e\} = 1,$$

- (4) Φ is a finite set of pairwise inconsistent \mathcal{L} -formulas, and
- (5) pre assigns a probability distribution $\text{pre}(\bullet \mid \phi)$ over E for every $\phi \in \Phi$.
- (6) $\mathbf{sub} : E \rightarrow \text{Sub}_{\mathcal{L}}$ assigns a substitution function to each event in E .

Informally, elements of E encode possible events, the relations \sim_i encode as usual the epistemic uncertainty of the agent i , who assigns probability $P_i(e)$ to e being the actually occurring event, formulas in Φ are intended as the preconditions of the event, and $\text{pre}(e \mid \phi)$ expresses the prior probability that the event $e \in E$ might occur in a(ny) state satisfying precondition ϕ . In addition, the substitution map $\mathbf{sub}(e)$ assigned to each event $e \in E$ describes how the event e changes the atomic facts of the world as represented by the proposition letters. In what follows, we will refer to the structures \mathcal{E} defined above as *event structures over \mathcal{L}* .

NOTATION 1. For any probabilistic epistemic state model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, [\![\cdot]\!] \rangle$, any probabilistic event structure \mathcal{E} , any $s \in S$ and $e \in E$, we let $\text{pre}(e \mid s)$ denote the value $\text{pre}(e \mid \phi)$, for the unique $\phi \in \Phi$ such that $\mathbb{M}, s \Vdash \phi$ (recall that the formulas in Φ are pairwise inconsistent). If no such ϕ exists, then we let $\text{pre}(e \mid s) = 0$.

2.2 Epistemic Updates

In this subsection, we introduce an alternative and equivalent presentation of the update construction on PES-models. This presentation is a variant of those introduced in References [34, 36] for models of public announcement logic and dynamic epistemic logic and consists of a two-step process—namely, a co-product-type construction followed by a subobject-type construction. We can then dualize the two steps separately, and thus obtain the construction of (probabilistic) epistemic updates on algebras as the composition of the two dualized constructions. The two steps

²This assumption guarantees that events affect only a finite number of facts.

are given in Definition 2.5 and Definition 2.7, and Lemma 2.8 proves that the updated model of a PES-model is also a PES-model.

Definition 2.5 (Intermediate Structure). For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$ and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} , let the *intermediate structure* of \mathbb{M} and \mathcal{E} be the tuple

$$\coprod_{\mathcal{E}} \mathbb{M} := \left\langle \coprod_{|E|} S, (\sim_i^{\coprod})_{i \in \text{Ag}}, (P_i^{\coprod})_{i \in \text{Ag}}, \llbracket \cdot \rrbracket_{\coprod} \right\rangle,$$

where

- (1) $\coprod_{|E|} S \cong S \times E$ is the $|E|$ -fold coproduct of S ,
- (2) each binary relation \sim_i^{\coprod} on $\coprod_{|E|} S$ is defined as follows:

$$(s, e) \sim_i^{\coprod} (s', e') \quad \text{iff} \quad s \sim_i s' \text{ and } e \sim_i e',$$

- (3) each map $P_i^{\coprod} : \coprod_{|E|} S \rightarrow [0, 1]$ is defined by

$$(s, e) \mapsto P_i(s) \cdot P_i(e) \cdot \text{pre}(e \mid s),$$

- (4) and the valuation $\llbracket \cdot \rrbracket_{\coprod} : \text{AtProp} \rightarrow \mathcal{P}S$ is defined by

$$\llbracket p \rrbracket_{\coprod} := \{(s, e) \mid s \in \llbracket p \rrbracket_{\mathbb{M}}\} = \llbracket p \rrbracket_{\mathbb{M}} \times E$$

for every $p \in \text{AtProp}$.

Notice that, because $\text{pre}(e \mid s)$ can be zero, by construction P_i^{\coprod} is not strictly positive. Moreover, in general, P_i^{\coprod} does not induce probability distributions over the \sim_i^{\coprod} -equivalence classes. Hence, $\coprod_{\mathcal{E}} \mathbb{M}$ is not a PES-model. However, the second step of the construction will yield a PES-model.

Finally, to define the updated model, observe that the map $\text{pre} : E \times \Phi \rightarrow [0, 1]$ in \mathcal{E} induces the map $\text{pre} : E \rightarrow \mathcal{L}$ defined below.

Definition 2.6. For any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} , let the map pre be defined as follows:

$$\begin{aligned} \text{pre} : E &\rightarrow \mathcal{L} \\ e &\mapsto \bigvee \{\phi \in \Phi \mid \text{pre}(e \mid \phi) \neq 0\}. \end{aligned}$$

Definition 2.7 (Updated Model). For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$ and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} , let the *epistemic update* $\mathbb{M}^{\mathcal{E}}$ of the model \mathbb{M} by the probabilistic event structure \mathcal{E} be as follows:

$$\mathbb{M}^{\mathcal{E}} := \left\langle S^{\mathcal{E}}, (\sim_i^{\mathcal{E}})_{i \in \text{Ag}}, (P_i^{\mathcal{E}})_{i \in \text{Ag}}, \llbracket \cdot \rrbracket_{\mathbb{M}^{\mathcal{E}}} \right\rangle$$

with

- (1) $S^{\mathcal{E}} := \{(s, e) \in \coprod_{|E|} S \mid \mathbb{M}, s \Vdash \text{pre}(e)\}$;
- (2) $\sim_i^{\mathcal{E}} = \sim_i^{\coprod} \cap (S^{\mathcal{E}} \times S^{\mathcal{E}})$ for any $i \in \text{Ag}$;
- (3) each map $P_i^{\mathcal{E}} : S^{\mathcal{E}} \rightarrow [0, 1]$ is defined by the assignment

$$(s, e) \mapsto \frac{P_i^{\coprod}(s, e)}{\sum \{P_i^{\coprod}(s', e') \mid (s, e) \sim_i (s', e')\}};$$

(4) the map $\llbracket \cdot \rrbracket_{\mathbb{M}^\mathcal{E}} : \text{AtProp} \rightarrow \mathcal{P}(S^\mathcal{E})$ is defined as follows:

$$\llbracket p \rrbracket_{\mathbb{M}^\mathcal{E}} := \llbracket \text{sub}(p) \rrbracket_{\mathbb{I}} \cap S^\mathcal{E},$$

where the map $\text{sub}(p) : E \rightarrow \mathcal{L}$ is given by:

$$\text{sub}(p)(e) := \begin{cases} \mathbf{sub}(e)(p) & \text{if } p \in \text{supp}(\mathbf{sub}(e)) \\ p & \text{otherwise.} \end{cases}$$

Item (4) in the definition above captures in a formal way the change in the states of affairs brought about by the event taking place (see also discussion before Definition 2.3).

LEMMA 2.8. *For any PES-model \mathbb{M} and any probabilistic event structure \mathcal{E} over \mathcal{L} , the epistemic update $\mathbb{M}^\mathcal{E}$ of the model \mathbb{M} by the probabilistic event structure \mathcal{E} is a PES-model.*

PROOF. To prove that $\mathbb{M}^\mathcal{E}$ is a PES-model (Definition 2.2), we need to show that it satisfies the following properties:

- (1) the set $S^\mathcal{E}$ is finite,
- (2) each relation $\sim_i^\mathcal{E}$ is an equivalence relation on S ,
- (3) each map $P_i^\mathcal{E} : S^\mathcal{E} \rightarrow]0, 1]$ assigns a strictly positive probability distribution over each $\sim_i^\mathcal{E}$ -equivalence class,
- (4) the map $\llbracket \cdot \rrbracket : \text{AtProp} \rightarrow \mathcal{P}S^\mathcal{E}$ is well defined.

Item (1) immediately follows by construction, since the product of finite sets is finite. Likewise, item (2) follows from the intersection of equivalence relations being an equivalence relation. As to item (3), by construction, $P_i^\mathcal{E}$ is a probability distribution over $\sim_i^\mathcal{E}$ -equivalence classes. To show that $P_i^\mathcal{E}$ is strictly positive, it is enough to show that $P_i^\mathbb{I}(s, e)$ is strictly positive for every $(s, e) \in S^\mathcal{E}$. By Definition 2.5, $P_i^\mathbb{I}(s, e) := P_i(s) \cdot P_i(e) \cdot \text{pre}(e \mid s)$. Hence, the statement follows from the fact that both $P_i(s)$ and $P_i(e)$ are strictly positive (cf. items (3) of Definitions 2.2 and 2.4), and the fact that if $(s, e) \in S^\mathcal{E}$, then by definition $\mathbb{M}, s \Vdash \text{pre}(e) = \bigvee \{\phi \in \Phi \mid \text{pre}(e \mid \phi) \neq 0\}$, i.e., $\mathbb{M}, s \Vdash \phi$ for some (unique) $\phi \in \Phi$ such that $\text{pre}(e \mid \phi) \neq 0$ and hence, by Notation 1, $\text{pre}(e \mid s) = \text{pre}(e \mid \phi) > 0$. By construction, the map in item (4) is well defined, which completes the proof. \square

2.3 Interpretation of PDEL-formulas on PES-models

Definition 2.9 (Probability Measure). Every PES-model

$$\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle,$$

induces the probability measure $\mu_i^\mathbb{M} : S \times \mathcal{L} \rightarrow [0, 1]$ defined as follows: for any $\phi \in \mathcal{L}$,

$$\mu_i^\mathbb{M}(s, \phi) := \sum_{\substack{s \sim_i s' \\ s' \in \llbracket \phi \rrbracket}} P_i(s').$$

Notice that μ_i defines a probability measure on each \sim_i -equivalence class.

Definition 2.10 (Semantics of PDEL). For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$, the interpretation of \mathcal{L} -formulas at states in \mathbb{M} is defined by the following recursion:

$$\begin{aligned}
\mathbb{M}, s \Vdash \perp & \text{ iff } \textit{never} \\
\mathbb{M}, s \Vdash p & \text{ iff } s \in \llbracket p \rrbracket \\
\mathbb{M}, s \Vdash \phi \wedge \psi & \text{ iff } \mathbb{M}, s \Vdash \phi \text{ and } \mathbb{M}, s \Vdash \psi \\
\mathbb{M}, s \Vdash \phi \vee \psi & \text{ iff } \mathbb{M}, s \Vdash \phi \text{ or } \mathbb{M}, s \Vdash \psi \\
\mathbb{M}, s \Vdash \phi \rightarrow \psi & \text{ iff } \mathbb{M}, s \Vdash \phi \text{ implies } \mathbb{M}, s \Vdash \psi \\
\mathbb{M}, s \Vdash \diamond_i \phi & \text{ iff there exists } s' \sim_i s \text{ such that } \mathbb{M}, s' \Vdash \phi \\
\mathbb{M}, s \Vdash \square_i \phi & \text{ iff } \mathbb{M}, s' \Vdash \phi \text{ for all } s' \sim_i s \\
\mathbb{M}, s \Vdash \langle \mathcal{E}, e \rangle \phi & \text{ iff } \mathbb{M}, s \Vdash \textit{pre}(e) \text{ and } \mathbb{M}^{\mathcal{E}}, (s, e) \Vdash \phi \\
\mathbb{M}, s \Vdash [\mathcal{E}, e] \phi & \text{ iff } \mathbb{M}, s \Vdash \textit{pre}(e) \text{ implies } \mathbb{M}^{\mathcal{E}}, (s, e) \Vdash \phi \\
\mathbb{M}, s \Vdash \left(\sum_{k=1}^n \alpha_k \mu_i(\varphi) \right) \geq \beta & \text{ iff } \sum_{k=1}^n \alpha_k \mu_i^{\mathbb{M}}(s, \varphi) \geq \beta,
\end{aligned}$$

where in the last clause α_k and β are rational numbers (cf. Footnote 1) and the probability measures $\mu_i^{\mathbb{M}}$ are defined as in Definition 2.9.

2.4 Axiomatization

PDEL is a logical framework bringing together epistemics, dynamics, and probabilities. Hence, its axiomatization describes the behavior of each of these components as well as their interactions. The full axiomatization of PDEL is given in Table 1 and includes the axioms of classical multimodal logic S5, understood as the basic epistemic logic, axioms capturing the theory of linear inequalities with rational coefficients (cf. Reference [21, Theorem 4.3], so, for instance, axioms n2 and n3 capture the commutativity and monotonicity of $+$, and distributivity of \cdot), axioms capturing basic classical probability theory (cf. References [1, 20, 21, 40, 42]), and axioms encoding the interaction between the dynamic modalities and the other logical connectives [1, 40], as well as the following inference rules: modus ponens, uniform substitution (see Reference [43]), necessitation for the static and dynamic modalities, and a substitution rule for the probabilistic operators μ_i (cf. References [1, 20, 40, 42]).

LEMMA 2.11 (SOUNDNESS AND COMPLETENESS). *PDEL is sound and complete w.r.t. the axiomatization given in Table 1.*

PROOF. The statement follows from the general proof in Appendix C and Stone-type duality. \square

3 METHODOLOGY

Let us expand on the methodology of the article. In the previous section, we gave a two-step account of the *product update* construction that, for any PES-model \mathbb{M} and any event model \mathcal{E} over \mathcal{L} , yields the updated model $\mathbb{M}^{\mathcal{E}}$ as a certain *submodel* of a certain *intermediate model* $\coprod_{\mathcal{E}} \mathbb{M}$. This account is analogous to those given in References [34, 36] of the product updates of models of PAL and of Baltag-Moss-Solecki's dynamic epistemic logic EAK. In each instance, the original product update construction can be illustrated by the following diagram (which uses the notation introduced in the instance treated in the previous section):

$$\mathbb{M} \leftrightarrow \coprod_{\mathcal{E}} \mathbb{M} \leftrightarrow \mathbb{M}^{\mathcal{E}}.$$

Table 1. AXIOMS OF PDEL

Axioms of classical modal logic S5	
Tautologies of classical propositional logic	
k.	$\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$
dual.	$\Box_i\varphi \leftrightarrow \neg\Diamond_i\neg\varphi$
t.	$\Box_i\varphi \rightarrow \varphi$
iv.	$\Box_i\varphi \rightarrow \Box_i\Box_i\varphi$
v.	$\neg\Box_i\varphi \rightarrow \Box_i\neg\Box_i\varphi$
Axioms capturing the theory of linear inequalities with rational coefficients	
n0.	$t \geq t$
n1.	$(t \geq \beta) \leftrightarrow (t + 0 \cdot \mu_i(\varphi) \geq \beta)$
n2.	$(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \rightarrow (\sum_{k=1}^n \alpha_{\rho(k)} \cdot \mu_i(\varphi_{\rho(k)}) \geq \beta)$ for any permutation ρ over $\{1, \dots, n\}$
n3.	$((\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \wedge (\sum_{k=1}^n \alpha'_k \cdot \mu_i(\varphi_k) \geq \beta')) \rightarrow (\sum_{k=1}^n (\alpha_k + \alpha'_k) \cdot \mu_i(\varphi_k) \geq (\beta + \beta'))$
n4.	$((t \geq \beta) \wedge (d \geq 0)) \rightarrow (d \cdot t \geq d \cdot \beta)$
n5.	$(t \geq \beta) \vee (\beta \geq t)$
n6.	$(t \geq \beta) \rightarrow (t > \gamma)$ for all $\gamma < \beta$
Axioms capturing basic classical probability theory	
p1.	$\mu_i(\perp) = 0$
p2.	$\mu_i(\top) = 1$
p3.	$\mu_i(\varphi \wedge \psi) + \mu_i(\varphi \wedge \neg\psi) = \mu_i(\varphi)$
p4.	$\Box_i\varphi \leftrightarrow (\mu_i(\varphi) = 1)$
p5.	$(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \rightarrow \Box_i(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta)$
Reduction Axioms	
i1.	$[\mathcal{E}, e]p \leftrightarrow (\text{pre}(e) \rightarrow \text{sub}(e)(p))$
i2.	$[\mathcal{E}, e]\neg\varphi \leftrightarrow (\text{pre}(e) \rightarrow \neg[\mathcal{E}, e]\varphi)$
i4.	$[\mathcal{E}, e](\varphi \wedge \psi) \leftrightarrow ([\mathcal{E}, e]\varphi \wedge [\mathcal{E}, e]\psi)$
i5.	$[\mathcal{E}, e]\Box_i A \leftrightarrow (\text{pre}(e) \rightarrow \bigwedge \{\Box_i[\mathcal{E}, f]A \mid e \sim_i f\})$
i6.	$[\mathcal{E}, e](\sum_{k=1}^n \alpha_k \cdot \mu_i(\psi_k) \geq \beta) \leftrightarrow (\text{pre}(e) \rightarrow C \geq D)$ with $C = \sum_{\phi \in \Phi} \sum_{e \sim_i f} \sum_{k=1}^n \alpha_k \cdot \text{pre}(f \mid \phi) \cdot \mu_i(\phi \wedge [\mathcal{E}, f]\psi_k)$ and $D = \sum_{\phi \in \Phi} \sum_{e \sim_i f} \beta \cdot \text{pre}(f \mid \phi) \cdot \mu_i(\phi)$
Inference Rules	
MP	if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$
Nec _i	if $\vdash A$, then $\vdash \Box_i A$
Nec _α	if $\vdash A$, then $\vdash [\mathcal{E}, e]A$
Sub _μ	if $\vdash A \rightarrow B$, then $\vdash \mu_i(A) \leq \mu_i(B)$
SubEq	if $\vdash A \leftrightarrow B$, then $\vdash \phi \leftrightarrow \phi[A/B]$

As is well known (see, e.g., Reference [19]) in duality theory, coproducts can be dually characterized as products, and subobjects as quotients. In light of this fact, the construction of *product update*, regarded as a “subobject after coproduct” concatenation, can be dually characterized on the algebras dual to the relational structures of PES-models by means of a “quotient after product” concatenation, as illustrated in the following diagram:

$$\mathbb{A} \leftarrow \prod_{\mathcal{E}} \mathbb{A} \rightarrow \mathbb{A}^{\mathcal{E}},$$

resulting in the following two-step process: First, the coproduct $\coprod_{\mathcal{E}} M$ is dually characterized as a certain *product* $\prod_{\mathcal{E}} \mathbb{A}$, indexed as well by the states of \mathcal{E} , and such that \mathbb{A} is the algebraic dual of \mathbb{M} ; second, an appropriate *quotient* of $\prod_{\mathcal{E}} \mathbb{A}$ is then taken, which dually characterizes the submodel

step. On which algebras are we going to apply the “quotient after product” construction? The prime candidates are the algebras associated with the PES-models via standard Stone-type duality:

Definition 3.1 (Complex Algebra). For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, [\![\cdot]\!] \rangle$, its *complex algebra* is the tuple

$$\mathbb{M}^+ := (\mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}}, (P_i^+)_{i \in \text{Ag}}),$$

where for each $i \in \text{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned} \diamond_i X &= \{s \in S \mid \exists x (s \sim_i x \text{ and } x \in X)\}, \\ \square_i X &= \{s \in S \mid \forall x (s \sim_i x \Rightarrow x \in X)\}, \\ \text{dom}(P_i^+) &= \{X \in \mathcal{P}S \mid \exists y \forall x (x \in X \Rightarrow x \sim_i y)\}, \\ P_i^+ X &= \sum_{x \in X} P_i(x). \end{aligned}$$

Notice that the domain of P_i^+ consists of all the subsets of the equivalence classes of \sim_i .

In this setting, the “quotient after product” construction behaves exactly in the desired way, in the sense that one can check *a posteriori* that the following holds:

PROPOSITION 3.2. *For every PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} , the algebraic structures $(\mathbb{M}^+)^{\mathcal{E}}$ and $(\mathbb{M}^{\mathcal{E}})^+$ can be identified.*

PROOF. This results follows from: (1) Fact 12 in Reference [34] that states that for any (non probabilistic) Kripke model \mathbb{N} , the structures $(\mathbb{N}^+)^{\mathcal{E}}$ and $(\mathbb{N}^{\mathcal{E}})^+$ can be identified, and (2) Lemma 4.32 on page 23 that states that the probability measures on the complex algebras $(\mathbb{M}^+)^{\mathcal{E}}$ and $(\mathbb{M}^{\mathcal{E}})^+$ are the same. \square

Moreover, the “quotient after product” construction holds in much greater generality than the class of complex algebras of PES-models, which is exactly its added value over the update on relational structures. In the following section, we are going to define it in detail in the setting of epistemic Heyting algebras.

4 PROBABILISTIC DYNAMIC EPISTEMIC UPDATES ON FINITE HEYTING ALGEBRAS

The present section aims at introducing the algebraic counterpart of the event update construction of Section 2. For the sake of enforcing a neat separation between syntax and semantics, throughout the present section, we disregard the logical language \mathcal{L} , and work on *algebraic probabilistic epistemic structures* (APE-structures, see Definition 4.7) rather than on APE-models (i.e., APE-structures endowed with valuations). Consequently, to be able to define the update construction, we will base our treatment on a modified definition of event structure over an *algebra*, rather than over \mathcal{L} . In Section 4.1, we introduce epistemic Heyting algebras. In Section 4.2, we recall the definition of intuitionistic probability from Reference [44] and endow epistemic Heyting algebras with measures to define algebraic probabilistic epistemic structures. In Section 4.3, we define probabilistic event structures over epistemic algebras as the intuitionistic algebraic counterparts of classical probabilistic event structures. In Section 4.4, we introduce the construction of intermediate pre-probabilistic event structure as the first step of the algebraic event update construction. Finally, in Section 4.5, we introduce the pseudo-quotient update construction and define the event update on algebraic probabilistic epistemic structures.

4.1 Epistemic Heyting Algebras

We start this subsection by recalling the definition of monadic Heyting algebras, i.e., the algebraic semantics of the logic MIPC, the intuitionistic analogue of the classical modal logic S5 (cf. References [11, 12, 34]). Then, we introduce the notion of i -minimal element of a monadic Heyting algebra. Finally, we define epistemic Heyting algebras as a subclass of monadic Heyting algebras with enough i -minimal elements to describe certain subalgebras of interest for the developments of the next sections.

Definition 4.1 (Monadic Heyting Algebra (cf. Reference [11])). A monadic Heyting algebra is a tuple

$$\mathbb{A} := (\mathbb{L}, (\Diamond_i)_{i \in \text{Ag}}, (\Box_i)_{i \in \text{Ag}})$$

such that \mathbb{L} is a Heyting algebra, and each \Diamond_i and \Box_i is a monotone unary operation on \mathbb{L} such that for all $a, b \in \mathbb{L}$,

$$\begin{array}{ll} a \leq \Diamond_i a & \text{(M1)} \\ \Box_i a \leq a & \text{(M2)} \\ \Diamond_i(a \vee b) \leq \Diamond_i a \vee \Diamond_i b & \text{(M3)} \\ \Box_i(a \rightarrow b) \leq \Box_i a \rightarrow \Box_i b & \text{(M4)} \\ \Diamond_i a \leq \Box_i \Diamond_i a & \text{(M5)} \\ \Diamond_i \Box_i a \leq \Box_i a & \text{(M6)} \\ \Box_i(a \rightarrow b) \leq \Diamond_i a \rightarrow \Diamond_i b & \text{(M7)} \\ \Diamond_i \perp \leq \perp & \text{(M8)} \\ \top \leq \Box_i \top & \text{(M9)} \end{array}$$

Remark 2. The algebraic and duality theoretic treatment of monadic Heyting algebras has been developed in References [11] and [12]. In particular, as mentioned in Reference [11, Lemma 2], in the presence of (M9), axiom (M4) is equivalent to $\Box_i a \wedge \Box_i b \leq \Box_i(a \wedge b)$, so all modalities are normal, and $\Diamond_i \Diamond_i a \leq \Diamond_i a$ and $\Box_i a \leq \Box_i \Box_i a$ are derivable from the axioms. These conditions correspond also in the best-known intuitionistic settings to the transitivity of the associated accessibility relations (cf. Reference [17]). This implies in particular that \Diamond_i is a closure operator for each $i \in \text{Ag}$.

The next definition intends to capture algebraically the notion of equivalence cell in the epistemic space of agents. Notice that for any equivalence relation R on a set X and any $x \in X$, the equivalence cell $R[x] = R^{-1}[x] = \langle R \rangle \{x\}$ is a minimal nonempty fixed point of $\langle R \rangle$.³ This justifies the following definition:

Definition 4.2 (i-Minimal Elements). Let \mathbb{A} be a monadic Heyting algebra. An element $a \in \mathbb{A}$ is i -minimal if

- (1) $a \neq \perp$,
- (2) $\Diamond_i a = a$, and
- (3) if $b \in \mathbb{A}$, $b < a$ and $\Diamond_i b = b$, then $b = \perp$.

Let $\text{Min}_i(\mathbb{A})$ denote the set of the i -minimal elements of \mathbb{A} .

Remark 3. Notice that, for any $b \in \mathbb{A} \setminus \{\perp\}$, there exists at most one $a \in \text{Min}_i(\mathbb{A})$ such that $b \leq a$. Indeed, every such a must coincide with $\Diamond_i b$.

³Recall that, for any binary relation $R \subseteq X \times X$, we define the maps R , R^{-1} and $\langle R \rangle$ as follows:

$$\begin{array}{ll} R : X \rightarrow \mathcal{P}X & R^{-1} : X \rightarrow \mathcal{P}X \\ x \mapsto \{x' \in X \mid (x, x') \in R\} & x \mapsto \{x' \in X \mid (x', x) \in R\} \end{array}$$

$$\begin{array}{l} \langle R \rangle : \mathcal{P}X \rightarrow \mathcal{P}X \\ S \mapsto \{x' \in X \mid \exists x \in S, (x', x) \in R\}. \end{array}$$

Definition 4.3 (Epistemic Heyting Algebra). An *epistemic Heyting algebra* is a finite monadic Heyting algebra

$$\mathbb{A} := (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$$

such that for every $i \in \text{Ag}$ and every $a \in \mathbb{A}$ the following holds:

$$\diamond_i a \vee \neg \diamond_i a = \top. \quad (\text{E})$$

Remark 4. As we will see in Lemma 4.4, condition (E) above implies that, for every agent i , the range of \diamond_i can be endowed with a Boolean algebra structure as a subalgebra of \mathbb{A} . Condition (E) captures algebraically the requirement that i -minimal elements, representing cells in the partition cover the whole space, in the sense that every element of \mathbb{A} is below the disjunction of i -minimal elements. More discussion about this condition can be found in the conclusions.

In the remainder of the present section, \mathbb{A} will denote an epistemic Heyting algebra.

LEMMA 4.4. *If \mathbb{A} is an Epistemic Heyting algebra, then, for every agent i ,*

$$\diamond_i \mathbb{A} := \{\diamond_i a \in \mathbb{A} \mid a \in \mathbb{A}\}$$

is a Boolean sub-algebra of \mathbb{A} . Furthermore, if

$$\square_i \mathbb{A} := \{\square_i a \in \mathbb{A} \mid a \in \mathbb{A}\},$$

then $\diamond_i \mathbb{A} = \square_i \mathbb{A}$.

PROOF. That $\diamond_i \mathbb{A}$ is a subalgebra of \mathbb{A} follows from the fact that the equalities

$$\diamond_i(\diamond_i a \wedge \diamond_i b) = \diamond_i a \wedge \diamond_i b \quad \text{and} \quad \diamond_i(\diamond_i a \rightarrow \diamond_i b) = \diamond_i a \rightarrow \diamond_i b$$

hold in every monadic Heyting algebra (see for example Reference [11, Definition 1, Lemma 2]⁴). Using the second identity and $\diamond_i \perp = \perp$, one can readily show that $\neg \diamond_i a = \diamond_i(\neg \diamond_i a) \in \diamond_i \mathbb{A}$ for every $a \in \mathbb{A}$. Then $\diamond_i a \wedge \neg \diamond_i a \leq \perp$ follows by Heyting residuation, and by condition (E), $\diamond_i a \vee \neg \diamond_i a = \top$, which completes the proof that $\diamond_i \mathbb{A}$ is a Boolean algebra.

Finally, let us show that $\diamond_i \mathbb{A} = \square_i \mathbb{A}$. Axiom (M1) implies that $\square_i a \leq \diamond_i \square_i a$ for every $a \in \mathbb{A}$. Combining this with axiom (M6), we obtain that $\square_i a = \diamond_i \square_i a \in \diamond_i \mathbb{A}$ for every $a \in \mathbb{A}$, which shows that $\square_i \mathbb{A} \subseteq \diamond_i \mathbb{A}$. The converse inclusion is shown similarly using axioms (M2) and (M5). \square

Remark 5. Given the fact that Epistemic Heyting algebras are finite and since $\diamond_i \mathbb{A}$ is a Boolean algebra, it is not hard to see that i -minimal elements are the atoms of $\diamond_i \mathbb{A}$, and hence $\bigvee \text{Min}_i(\mathbb{A}) = \top$.

NOTATION 2. *For any poset (partially ordered set) $\mathbb{P} = (P, \leq)$, we let*

$$\downarrow_{\mathbb{P}} : \mathcal{P}\mathbb{P} \rightarrow \mathcal{P}\mathbb{P}$$

$$X \mapsto X \downarrow_{\mathbb{P}} := \{x' \in \mathbb{P} \mid x' \leq x \text{ for some } x \in X\}.$$

For the sake of readability, we drop the subscript and let $X \downarrow$ denote the downset generated by X . In addition, if $X = \{x\}$, we let $x \downarrow$ denote the downset generated by $\{x\}$.

4.2 Algebraic Probabilistic Epistemic Structures

In this Section, we introduce i -premeasures and i -measures and define algebraic pre-probabilistic and probabilistic epistemic structures that will serve as the underlying structures of intuitionistic probabilistic epistemic logic.

⁴As to the first equality, by Reference [11, Definition 1], $\diamond_i(\diamond_i a \wedge b) = \diamond_i a \wedge \diamond_i b$ holds, and instantiating it with $b := \diamond_i b$, one has $\diamond_i(\diamond_i a \wedge \diamond_i b) = \diamond_i a \wedge \diamond_i \diamond_i b$; then the required identity follows by observing that $\diamond_i \diamond_i b = \diamond_i b$.

The following definition is an adaptation of a proposal of Weatherson's (see Reference [44, page 2]) in which the notion of probability is generalized and made parametric in a given consequence relation. Even though there is no consensus on what an intuitionistic probability function should be, Weatherson's proposal captures necessary conditions for such a function and establishes a systematic link between logic and probability. We refer to his paper for an explanation and motivation of his proposal. The definition below has also been adopted by References [3, 23].

Definition 4.5 (Intuitionistic Probability Measures). Let \mathbb{H} be a Heyting algebra. A function $\Pr : \mathbb{H} \rightarrow [0, 1]$ is an *intuitionistic probability measure* if the following conditions are satisfied: for all $a, b \in \mathbb{H}$,

- (1) $\Pr(\perp) = 0$,
- (2) $\Pr(\top) = 1$,
- (3) if $a \leq_{\mathbb{H}} b$, then $\Pr(a) \leq \Pr(b)$,
- (4) $\Pr(a) + \Pr(b) = \Pr(a \vee b) + \Pr(a \wedge b)$.

Notice that, for intuitionistic probability measures, it does no longer hold that $\Pr(p \vee \neg p) = 1$, which, as observed in the introduction, makes it possible to assign low subjective probability to an event and to its negation, when, e.g., one does not have evidence in favor of either.

In classical PDEL, the probability functions range over equivalence classes instead of the whole model. We capture this fact on the side of the algebras by defining probability functions that are probability measures on the quotient algebras generated by *i*-minimal elements.

Definition 4.6 (i-Premeasure & i-Measure). A partial function $\mu : \mathbb{A} \rightarrow \mathbb{R}^+$ is an *i-premeasure* on \mathbb{A} if it satisfies the following properties:

- (1) $\text{dom}(\mu) = \text{Min}_i(\mathbb{A})\downarrow$;
- (2) μ is order-preserving;
- (3) for every $a \in \text{Min}_i(\mathbb{A})$ and all $b, c \in a\downarrow$, we have $\mu(b \vee c) = \mu(b) + \mu(c) - \mu(b \wedge c)$;
- (4) $\mu(\perp) = 0$ if $\perp \in \text{dom}(\mu)$.

An *i-premeasure* on \mathbb{A} is an *i-measure*, if it satisfies the following properties:

- (5) $\mu(a) = 1$ for every $a \in \text{Min}_i(\mathbb{A})$.
- (6) for every $a \in \text{Min}_i(\mathbb{A})$ and all $b, c \in a\downarrow$ such that $b < c$, it holds that $\mu(b) < \mu(c)$;

Condition (1) ensures that the probability measures are defined on the quotient algebras generated by *i*-minimal elements. Conditions (2) to (5) are imported from Weatherson's definition of intuitionistic probabilistic functions. Condition (6) is classically but not intuitionistically equivalent to the classical condition that probability distributions over the states of each equivalence class are strictly positive (see Definition 2.2 and Remark 1). In classical logic, this is captured by the condition that all formulas that are not logically equivalent to \perp have non-zero probability, because all singleton subsets are elements of the complex algebra of a Kripke frame, and hence they are potential extensions of formulas. However, in the intuitionistic setting this is not enough: since valuations are persistent, singletons are not in general elements of the complex algebra of an intuitionistic Kripke frames, so conditions on intuitionistic formulas capture properties of principal up-sets (or down-sets) of states in intuitionistic Kripke frames, but not in general of states. This is part of the reason why we need to resort to the intuitionistically stronger requirement (6). However, and more importantly, in the intuitionistic setting it would not be reasonable to talk about the subjective probability distribution of a single state without relating it to all its *successor* states, and this is what condition (6) also captures algebraically: it guarantees *both* that the subjective probability of any join-irreducible (hence, non \perp) element is strictly positive, *and* also that the probability distribution of successor states is added to the overall probability measure of extensions of formulas in intuitionistic Kripke frames. Finally, notice that when $\text{Min}_i(\mathbb{A})\downarrow = \emptyset$,

there exists a unique i -(pre)measure—namely, the empty function. Throughout this section, all the results regarding i -minimal elements and i -(pre)measure hold vacuously when $\text{Min}_i(\mathbb{A})\downarrow = \emptyset$.

Definition 4.7 (ApPE-Structure & APE-Structure). An algebraic pre-probabilistic epistemic structure (ApPE-structure) is a tuple

$$\mathcal{F} := (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$$

such that

- (1) \mathbb{A} is an epistemic Heyting algebra (see Definition 4.3), and
- (2) each μ_i is an i -premeasure on \mathbb{A} .

An ApPE-structure \mathcal{F} is an algebraic probabilistic epistemic structure (APE-structure) if each μ_i is an i -measure on \mathbb{A} .

We refer to \mathbb{A} as the *support* of \mathcal{F} and we denote it $\text{support}(\mathcal{F})$.

4.2.1 The Algebraic Epistemic Structure Associated with a Classical Model.

LEMMA 4.8. For any PES-model \mathbb{M} , the i -minimal elements of its complex algebra \mathbb{M}^+ are exactly the equivalence classes of \sim_i .

PROOF. See Appendix A, page 44. □

PROPOSITION 4.9. For any PES-model \mathbb{M} , its complex algebra \mathbb{M}^+ (see Definition 3.1) is an APE-structure (see Definition 4.7).

PROOF. See Appendix A, page 44. □

4.3 Probabilistic Event Structures over Epistemic Heyting Algebras

In this section, we introduce intuitionistic event structures, which are needed to correctly generalize probabilistic epistemic updates to an intuitionistic metatheory.

We will find it useful to introduce the following auxiliary definitions. Recall that a *multiset* is a generalization of the concept of a set that allows multiple instances of the same element. Hence, $\{a, a, b\}$ and $\{a, b\}$ are the same set, but different multisets. However, order does not matter, so $\{a, a, b\}$ and $\{a, b, a\}$ are the same multiset. Let Φ be a multiset on the set X and $a, b \in \Phi$. We say that a and b *arise from the same element* if a and b are copies of the same element from X . We denote it $a =_X b$.

Definition 4.10 (Ordered Multiset on a Lattice). Let $\mathbb{L} = (L, \leq)$ be a finite lattice. An *ordered multiset* $\Phi = (\Phi, <)$ on \mathbb{L} is a multiset Φ of elements of L equipped with a strict order $<$ such that, for all pairwise distinct elements $x, y, z \in \Phi$,

- (1) if $x < y$, then $x \leq_{\mathbb{L}} y$;
- (2) if $x \neq \perp$ and $x \leq_{\mathbb{L}} y$, then $x < y$ or $y < x$;
- (3) if $x < y$ and $x < z$, then $y < z$ or $z < y$.

In the present article, we use the membership symbol \in in the context of multisets on \mathbb{L} always referring to the copies of a given element of \mathbb{L} . For instance, the variable y in the symbol $y \in \Phi$ refers to one specific copy of some element of \mathbb{L} .

Remark 6. In Section 5, we will be working with event structures over logical languages rather than with event structures over algebras (see Definition 4.11). Event structures over languages (see Definition 5.2) are tuples where Φ is a set of formulas, each pair of which is made either of incompatible formulas or of formulas one of which implies the other (see also Remark 8). However, some of these formulas might be identified with each other under some valuations. To define updates

on algebras independently from logic, in Definition 4.11 the ordered multisets above will play the same role played by the sets Φ in event structures over languages. Specifically, the multiset structure serves to keep track of the fact that some elements of the lattice might be the interpretation of more than one formula in the set Φ , and the order on the multiset Φ helps to keep track of the logical structure of the set Φ . Finally, condition (3) makes sure that the order structure of Φ is an upward forest, and conditions (1) and (2) together guarantee that, with the exception of formulas that are mapped to \perp , the logical structure of the set Φ is preserved and reflected by the order $<$.

Now let us introduce probabilistic event structures in the intuitionistic setting:

Definition 4.11 (Probabilistic Event Structure Over an Epistemic Heyting Algebra). For any epistemic Heyting algebra \mathbb{A} (see Definition 4.3), a *probabilistic event structure over \mathbb{A}* is a tuple

$$\mathbb{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \overline{\text{pre}})$$

such that

- (1) E is a non-empty finite set;
- (2) each \sim_i is an equivalence relation on E ;
- (3) each $P_i : E \rightarrow]0, 1]$ assigns a probability distribution over each \sim_i -equivalence class, i.e.,

$$\sum \{P_i(e') \mid e' \sim_i e\} = 1;$$

- (4) $\Phi = (\Phi, <)$ is a finite ordered multiset on \mathbb{A} such that, for all $a, b \in \Phi$ that arise from distinct elements in \mathbb{A} , either

$$a \wedge_{\mathbb{A}} b = \perp \quad \text{or} \quad a <_{\mathbb{A}} b \quad \text{or} \quad b <_{\mathbb{A}} a;$$

- (5) the map $\overline{\text{pre}} : E \times \Phi \rightarrow [0, 1]$ assigns a probability distribution $\overline{\text{pre}}(\bullet|a)$ over E for every $a \in \Phi$;
- (6) for all $a \in \Phi$ and $e \in E$, if $\overline{\text{pre}}(e|a) = 0$, then $\overline{\text{pre}}(e|b) = 0$ for all $b \in \Phi$ such that $a < b$.

The definition above is a proper generalization of the analogous definition given in the classical setting (Definition 2.4). The main generalization concerns the fact that the elements in Φ (which are the potential interpretants of formulas) are no longer required to be mutually inconsistent but may also be “logically dependent.” In this latter case, the precondition function is required to satisfy an additional compatibility condition that is similar to the one adopted in Reference [3]. For the sake of readability, in what follows, we will simply refer to probabilistic event structures over epistemic Heyting algebras as *event structures*.

Remark 7 (The substitution map). Clearly, a purely algebraic counterpart of the substitution map that was part of the definition of *probabilistic event structures over a language* (see Definition 2.4) cannot be given.

Remark 8 (The order $\leq_{\mathbb{A}}$ on the set Φ). The classical and the intuitionistic setting are distinguished by the fact that states are pairwise incomparable in the classical setting and (non-trivially) ordered in the intuitionistic setting. Thus, in probabilistic event structures over a language (see Definition 2.4), it is enough to require the set Φ to contain mutually inconsistent formulas to tell apart states of the Kripke model. However, due to the order between states of intuitionistic Kripke frames, mutually incompatible formulas are not enough to separate distinct but comparable states. To overcome this hurdle, we require Φ to satisfy the following condition: for all $a_k, a_j \in \Phi$,

$$a_j \wedge a_k = \perp \quad \text{or} \quad a_j < a_k \quad \text{or} \quad a_k < a_j.$$

This condition makes it possible to compute the probabilities of a given non-maximal state, even if there is no proposition uniquely identifying this state (cf. Definition 4.15).

4.4 The Intermediate (Pre-)probabilistic Epistemic Structure

In the present subsection, we define the intermediate ApPE-structure $\prod_{\mathbb{E}} \mathcal{F}$ associated with any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support of \mathcal{F} (see Definition 4.7 for the definition of support):

$$\prod_{\mathbb{E}} \mathcal{F} := \left(\prod_{\mathbb{E}} \mathbb{A}, (\mu'_i)_{i \in \text{Ag}} \right). \quad (4.1)$$

Structure of the Subsection. First, we define the intermediate algebra $\prod_{|E|} \mathbb{A}$ that will become the support of the intermediate ApPE-structure $\prod_{\mathbb{E}} \mathcal{F}$ (see Definition 4.12 and Proposition 4.13) and we identify its i -minimal elements (see Proposition 4.14). Then, we introduce the i -premeasures on the intermediate algebra (see Definition 4.17 and Proposition 4.18). Finally, we show that the definition ApPE-structure is coherent with the relational semantics in the classical case (see Proposition 4.21).

4.4.1 The Intermediate Algebra and Its i -minimal Elements.

Definition 4.12 (Intermediate Algebra). For every epistemic Heyting algebra $\mathbb{A} = (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$ and every event structure $\mathbb{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \overline{\text{pre}})$ over \mathbb{A} , let *the intermediate algebra* be

$$\prod_{\mathbb{E}} \mathbb{A} := \left(\prod_{|E|} \mathbb{L}, \{\diamond'_i, \square'_i \mid i \in \text{Ag}\} \right),$$

where

- (1) $\prod_{|E|} \mathbb{L}$ is the $|E|$ -fold power of \mathbb{L} , the elements of which can be seen either as $|E|$ -tuples of elements in \mathbb{A} or as maps $f : E \rightarrow \mathbb{A}$;
- (2) for any $f : E \rightarrow \mathbb{A}$, let us define $\diamond'_i(f)$ as follows:

$$\begin{aligned} \diamond'_i(f) &: E \rightarrow \mathbb{A} \\ e &\mapsto \bigvee \{ \diamond_i f(e') \mid e' \sim_i e \}; \end{aligned}$$

- (3) for any $f : E \rightarrow \mathbb{A}$, let us define $\square'_i(f)$ as follows:

$$\begin{aligned} \square'_i(f) &: E \rightarrow \mathbb{A} \\ e &\mapsto \bigwedge \{ \square_i f(e') \mid e' \sim_i e \}. \end{aligned}$$

Below, the algebra $\prod_{\mathbb{E}} \mathbb{A}$ will be sometimes abbreviated as \mathbb{A}' .

We refer to Reference [34, Section 3.1] for an extensive justification of the definition of the operations \diamond'_i and \square'_i .

PROPOSITION 4.13. *For every epistemic Heyting algebra \mathbb{A} and every event structure \mathbb{E} over \mathbb{A} , the algebra \mathbb{A}' is an epistemic Heyting algebra.*

PROOF. To prove that \mathbb{A}' is an epistemic Heyting algebra (Definition 4.3), we need to show that \mathbb{A}' is a monadic Heyting algebra such that for every $i \in \text{Ag}$ and every $f \in \mathbb{A}'$, we have: $\diamond_i f \vee \neg \diamond_i f = \top$.

The proof that \mathbb{A}' is a monadic Heyting algebra can be found in Reference [34, Proposition 3.1]. Let $i \in \text{Ag}$, $f \in \mathbb{A}'$, and $e \in E$. We have:

$$\begin{aligned} (\diamond'_i f \vee \neg \diamond'_i f)(e) &= (\diamond'_i f)(e) \vee \neg(\diamond'_i f)(e) \\ &= \bigvee \{ \diamond_i(f(e')) \mid e' \sim e \} \vee \neg \bigvee \{ \diamond_i(f(e')) \mid e' \sim e \} && \text{(by definition of } \diamond'_i) \\ &= \diamond_i \bigvee \{ f(e') \mid e' \sim e \} \vee \neg \diamond_i \bigvee \{ f(e') \mid e' \sim e \} && \text{(by the normality of } \diamond_i) \\ &= \top. && \text{(since } \diamond_i a \vee \neg \diamond_i a = \top) \end{aligned}$$

Hence, $(\diamond'_i f \vee \neg \diamond'_i f)(e) = \top$ for all $e \in E$, which by definition yields that $\diamond'_i f \vee \neg \diamond'_i f = \top$. \square

PROPOSITION 4.14. For every epistemic Heyting algebra \mathbb{A} and every agent $i \in \text{Ag}$,

$$\text{Min}_i(\mathbb{A}') = \{f_{e,a} \mid e \in E \text{ and } a \in \text{Min}_i(\mathbb{A})\},$$

where for any $e \in E$ and $a \in \text{Min}_i(\mathbb{A})$, the map $f_{e,a}$ is defined as follows:

$$\begin{aligned} f_{e,a} : E &\rightarrow \mathbb{A} \\ e' &\mapsto \begin{cases} a & \text{if } e' \sim_i e \\ \perp & \text{otherwise.} \end{cases} \end{aligned}$$

PROOF. See Appendix A, page 44. \square

4.4.2 The i -premeasures on the Intermediate Algebra. Before providing i -premeasures for the product epistemic algebra (Definition 4.17 and Proposition 4.18), we present an auxiliary definition:

Definition 4.15. Let $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ be an APE-structure and let $\mathbb{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \overline{\text{pre}})$ be an event structure over \mathbb{A} . For all $a \in \Phi$ and $i \in \text{Ag}$, we define the partial function $\mu_i^a : \mathbb{A} \rightarrow \mathbb{R}^+$ by

$$\mu_i^a(x) := \mu_i(x \wedge a) - \sum_{b \in \text{mb}(a)} \mu_i(x \wedge b), \quad (4.2)$$

where $\text{mb}(a)$ denotes the multiset of the \prec -maximal elements of Φ \prec -below a .

We make the following observations regarding μ_i^a :

PROPOSITION 4.16. For every APE-structure $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ and every event structure \mathbb{E} over \mathbb{A} , μ_i^a is an i -premeasure over \mathbb{A} . Furthermore, if $a \leq y$, then $\mu_i^a(x) = \mu_i^a(x \wedge y)$.

PROOF. See Appendix A, page 44. \square

Remark 9. Notice that if $a \leq y$, then for every $b \in \text{mb}(a)$, we have $b \leq y$, thus $\mu_i(x \wedge y \wedge a) = \mu_i(x \wedge a)$ and $\mu_i(x \wedge y \wedge b) = \mu_i(x \wedge b)$, which implies that $\mu_i^a(x) = \mu_i^a(x \wedge y)$.

Definition 4.17 (Intermediate Structure). For any APE-structure $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ and any event structure $\mathbb{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \overline{\text{pre}})$ over \mathbb{A} , let the intermediate structure be

$$\prod_{\mathbb{E}} \mathcal{F} := \left(\prod_{\mathbb{E}} \mathbb{A}, (\mu'_i)_{i \in \text{Ag}} \right),$$

where

- (1) $\prod_{\mathbb{E}} \mathbb{A} = \mathbb{A}'$ is defined as in Definition 4.12;
- (2) each μ'_i is defined as follows:

$$\begin{aligned} \mu'_i : \text{Min}_i(\mathbb{A}') \downarrow &\rightarrow \mathbb{R}^+ \\ f &\mapsto \sum_{e \in E} \sum_{a \in \Phi} P_i(e) \cdot \mu_i^a(f(e)) \cdot \overline{\text{pre}}(e \mid a). \end{aligned} \quad (4.3)$$

PROPOSITION 4.18. *For every APE-structure \mathcal{F} and every event structure \mathbb{E} over the support of \mathcal{F} , the intermediate structure $\prod_{\mathbb{E}} \mathcal{F}$ is an ApPE-structure (see Definition 4.7). Furthermore, if $\bigvee_{a \in \Phi} a \leq y$, then $\mu'_i(x) = \mu'_i(x \wedge y)$.*

PROOF. Proposition 4.13 states that $\prod_{\mathbb{E}} \mathbb{A}$ is an epistemic Heyting algebra. To prove that $\prod_{\mathbb{E}} \mathcal{F}$ is an ApPE-structure, it remains to show that for every $i \in \text{Ag}$, the map μ'_i is an i -premeasure (see items (1–4) of Definition 4.6). Fix $i \in \text{Ag}$. The map μ'_i is clearly well defined. Since the maps $\{\mu_i^a\}_{a \in \Phi}$ are i -premeasures, the items (1), (2), and (4) are trivially true.

Proof of item 3. By Proposition 4.14, i -minimal elements of \mathbb{A}' are of the form $f_{e,b} : E \rightarrow \mathbb{A}$ for some $e \in E$ and some i -minimal element $b \in \text{Min}_i(\mathbb{A})$. Fix one such element $f_{e,b} \in \text{Min}_i(\mathbb{A}')$, and let $g, h : E \rightarrow \mathbb{A}$ such that $g, h \leq f_{e,b}$. By definition, $f \leq f_{e,b}$ can be rewritten as $f(e') \leq f_{e,b}(e')$ for any $e' \in E$. Since $f_{e,b}(e') = \perp$ for any $e' \approx_i e$, we can deduce that $g(e') = h(e') = \perp$ for any $e' \approx_i e$. Similarly, we can deduce that $g(e') \leq b$ and $h(e') \leq b$ for any $e' \sim_i e$. Hence,

$$\begin{aligned} \mu'_i(g \vee h) &= \sum_{e' \in E} \sum_{a \in \Phi} P_i(e') \cdot \mu_i^a(g(e') \vee h(e')) \cdot \overline{\text{pre}}(e' \mid a) && \text{(by definition)} \\ &= \sum_{e' \in E} \sum_{a \in \Phi} P_i(e') \cdot (\mu_i^a(g(e')) + \mu_i^a(h(e')) - \mu_i^a(g(e') \wedge h(e'))) \cdot \overline{\text{pre}}(e' \mid a) \\ &\quad (\mu_i^a \text{ is an } i\text{-premeasure, } b \in \text{Min}_i(\mathbb{A}), \text{ and } g(e') \leq b \text{ and } h(e') \leq b \text{ for any } e' \in E) \\ &= \mu'_i(g) + \mu'_i(h) - \mu'_i(g \wedge h). && \text{(by definition)} \end{aligned}$$

Finally, the fact that if $(\bigvee_{a \in \Phi} a) \leq y$, then $\mu'_i(x) = \mu'_i(x \wedge y)$ follows from Proposition 4.16. \square

4.4.3 *The Intermediate Algebra for the Classical Case.* Here, we show that the construction described above, applied to the complex algebras of classical models, dualizes the construction of the intermediate model of Section 2.2. This is the first step towards the result stated in Proposition 3.2.

Definition 4.19. For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$ (see Definition 2.2) and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre})$ over \mathcal{L} (see Definition 2.4), let the probabilistic event structure over \mathbb{M}^+ (see Definitions 3.1 and 4.11) be

$$\mathbb{E}_{\mathcal{E}} := (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi_{\mathbb{M}}, \overline{\text{pre}}_{\mathbb{M}}),$$

where

- $\Phi_{\mathbb{M}} = (\Phi_{\mathbb{M}}, <_{\mathbb{M}})$ is the ordered multiset such that $\Phi_{\mathbb{M}} := \{\llbracket \phi \rrbracket_{\mathbb{M}} \mid \phi \in \Phi\}$ and the strict order $<_{\mathbb{M}}$ is the empty relation;
- the map $\overline{\text{pre}}_{\mathbb{M}} : \Phi_{\mathbb{M}} \rightarrow (E \rightarrow [0, 1])$ assigns a probability distribution $\overline{\text{pre}}_{\mathbb{M}}(\bullet \mid \llbracket \phi \rrbracket) : E \rightarrow [0, 1]$ over E for every $\phi \in \Phi$ such that:

$$\begin{aligned} \overline{\text{pre}}_{\mathbb{M}}(\bullet \mid \llbracket \phi \rrbracket) : E &\rightarrow [0, 1] && (4.4) \\ e &\mapsto \text{pre}(e \mid \phi). \end{aligned}$$

PROPOSITION 4.20. *For any PES-model \mathbb{M} (see Definition 2.2) and any event structure \mathcal{E} over \mathcal{L} (see Definition 2.4), the tuple $\mathbb{E}_{\mathcal{E}}$ is an event structure over the epistemic Heyting algebra underlying \mathbb{M}^+ .*

PROOF. We need to verify that the tuple $\mathbb{E}_{\mathcal{E}}$ satisfies Definition 4.11. Items (1)–(3) are trivially satisfied. Hence, we only need to prove that

- (4) $\Phi_{\mathbb{M}} = (\Phi_{\mathbb{M}}, <_{\mathbb{M}})$ is a finite ordered multiset on \mathbb{M}^+ such that, for all $a, b \in \Phi_{\mathbb{M}}$ that arise from distinct elements in \mathbb{M}^+ , either

$$a \wedge_{\mathbb{M}^+} b = \perp \quad \text{or} \quad a <_{\mathbb{M}^+} b \quad \text{or} \quad b <_{\mathbb{M}^+} a;$$

- (5) the map $\overline{\text{pre}}_{\mathbb{M}} : E \times \Phi_{\mathbb{M}} \rightarrow [0, 1]$ assigns a probability distribution $\overline{\text{pre}}_{\mathbb{M}}(\bullet|a)$ over E for every $a \in \Phi_{\mathbb{M}}$;
- (6) for all $a \in \Phi$ and $e \in E$, if $\overline{\text{pre}}_{\mathbb{M}}(e|a) = 0$, then $\overline{\text{pre}}_{\mathbb{M}}(e|b) = 0$ for all $b \in \Phi$ such that $a < b$.

Proof of 4. First, we need to prove that $\Phi_{\mathbb{M}}$ is an ordered multiset (Definition 4.10). $\Phi_{\mathbb{M}}$ is clearly a multiset, hence, we only need to prove that the empty relation $<_{\mathbb{M}}$ satisfies the following conditions: for all pairwise distinct elements $x, y, z \in \Phi_{\mathbb{M}}$,

- (i) if $x <_{\mathbb{M}} y$, then $x \leq_{\mathbb{M}^+} y$;
- (ii) if $x \neq \perp_{\mathbb{M}^+}$ and $x \leq_{\mathbb{M}^+} y$, then $x <_{\mathbb{M}} y$ or $y <_{\mathbb{M}} x$;
- (iii) if $x <_{\mathbb{M}} y$ and $x <_{\mathbb{M}} z$, then $y <_{\mathbb{M}} z$ or $z <_{\mathbb{M}} y$.

Conditions (i) and (iii) are trivially satisfied. Notice that, since \mathcal{E} is a (classical) probabilistic event structure, Φ is a finite set of pairwise inconsistent \mathcal{L} -formulas. Assume that $[[\phi]], [[\psi]] \in \Phi_{\mathbb{M}}$ are pairwise distinct (i.e., $\phi \neq \psi$ in the language \mathcal{L}) and such that $[[\phi]] \leq_{\mathbb{M}^+} [[\psi]]$. One can easily verify that $\phi \wedge \psi = \perp$ implies that $[[\phi]] = \perp_{\mathbb{M}^+}$. Hence, $<_{\mathbb{M}}$ satisfies condition (ii). This finishes the proof that the ordered multiset $\Phi_{\mathbb{M}}$ is well defined.

Let $[[\phi]], [[\psi]] \in \Phi_{\mathbb{M}}$ arise from distinct elements in \mathbb{M}^+ . By definition, $\phi \wedge \psi = \perp$. Hence, $a \wedge_{\mathbb{M}^+} b = \perp$, which proves item 4.

Proof of 5. Since \mathcal{E} is a (classical) probabilistic event structure, pre assigns a probability distribution $\text{pre}(\bullet|\phi)$ over E for every $\phi \in \Phi$. Hence, the map $\overline{\text{pre}}_{\mathbb{M}}$ is well defined.

Proof of 6. Since $<_{\mathbb{M}}$ is the empty relation, this condition is trivially true. \square

Remark 10. Notice that, in the classical case, $\text{mb}(a) = \emptyset$ for all $a \in \Phi$. Indeed, $\text{mb}(a)$ denotes the multiset of the $<$ -maximal elements of Φ $<$ -below a . But, since in the classical case $<_{\mathbb{M}}$ is the empty relation, there is no element below a in Φ .

PROPOSITION 4.21. *For every PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} ,*

$$\left(\coprod_{\mathcal{E}} \mathbb{M} \right)^+ \cong \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{M}^+.$$

PROOF. See Appendix A, page 44. \square

COROLLARY 4.22. *For every PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} , the complex algebra $(\coprod_{\mathcal{E}} \mathbb{M})^+$ of the intermediate structure $\coprod_{\mathcal{E}} \mathbb{M}$ is an ApPE-structure.*

4.5 The Pseudo-quotient and the Updated APE-structure

In the present subsection, we define the APE-structure $\mathcal{F}^{\mathbb{E}}$, resulting from the update of the APE-structure \mathcal{F} with the event structure \mathbb{E} over the support of \mathcal{F} , by taking a suitable pseudo-quotient of the intermediate APE-structure $\coprod_{\mathbb{E}} \mathcal{F}$. Some of the results that are relevant for the ensuing treatment (such as the characterization of the i -minimal elements in the pseudo-quotient) are independent of the fact that we will be working with the intermediate algebra. Therefore, in what follows, we will discuss them in the more general setting of arbitrary epistemic Heyting algebras \mathbb{A} .

Structure of the Subsection. First, we define the pseudo-quotient algebra (Definition 4.23) and prove that it is an epistemic Heyting algebra (Proposition 4.24). Then, we characterize the i -minimal elements of the pseudo-quotient algebra (Proposition 4.27). Finally, we define the APE-structure $\mathcal{F}^{\mathbb{E}}$, resulting from the update of the APE-structure \mathcal{F} with the event structure \mathbb{E} (Definition 4.28 and Proposition 4.30) and show that this definition is compatible with the update on PES-models (Lemma 4.32).

Pseudo-quotient Algebra.

Definition 4.23 (Pseudo-Quotient Algebra). (cf. Reference [36, Sections 3.2, 3.3]) For any epistemic Heyting algebra $\mathbb{A} := (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$, and any $a \in \mathbb{A}$, let the *pseudo-quotient algebra* be

$$\mathbb{A}^a := (\mathbb{L}/\cong_a, (\diamond_i^a)_{i \in \text{Ag}}, (\square_i^a)_{i \in \text{Ag}}),$$

where

- \cong_a is defined as follows: for all $b, c \in \mathbb{L}$,

$$b \cong_a c \quad \text{iff} \quad b \wedge a = c \wedge a,$$

- for every $i \in \text{Ag}$, the operations \diamond_i^a and \square_i^a are defined as follows:

$$\begin{array}{ccc} \diamond_i^a : \mathbb{A}^a \rightarrow \mathbb{A}^a & \text{and} & \square_i^a : \mathbb{A}^a \rightarrow \mathbb{A}^a \\ b \mapsto [\diamond_i(b \wedge a)] & & b \mapsto [\square_i(a \rightarrow b)], \end{array}$$

where $[c]$ denotes the \cong_a -equivalence class of $c \in \mathbb{A}$.

PROPOSITION 4.24. (cf. Reference [36, Fact 12]) *For any epistemic Heyting algebra \mathbb{A} , the pseudo-quotient algebra \mathbb{A}^a (see Definition 4.23) is an epistemic Heyting algebra.*

PROOF. The proof that \mathbb{A}^a is a monadic Heyting algebra can be found in Reference [36, Fact 12]. To show that \mathbb{A}^a is an epistemic Heyting algebra (see Definition 4.3), it remains to prove that $\diamond_i^a[b] \vee \neg \diamond_i^a[b] = [\top]$ for all $i \in \text{Ag}$ and $b \in \mathbb{A}^a$. We have that $\diamond_i^a[b] = [\diamond_i(b \wedge a)]$ and that $\neg \diamond_i^a[b] = \neg[\diamond_i(b \wedge a)] = [\neg \diamond_i(b \wedge a)]$. Hence,

$$\diamond_i^a[b] \vee \neg \diamond_i^a[b] = [\diamond_i(b \wedge a) \vee \neg \diamond_i(b \wedge a)] = [\top],$$

since \mathbb{A} is an epistemic Heyting algebra. □

The i -minimal Elements of the Pseudo-quotient Algebra.

LEMMA 4.25. *For any epistemic Heyting algebra \mathbb{A} and any $a \in \mathbb{A}$, if $b \in \text{Min}_i(\mathbb{A})$ and $b \wedge a \neq \perp$, then $[b] \in \text{Min}_i(\mathbb{A}^a)$.*

PROOF. Fix some $b \in \text{Min}_i(\mathbb{A})$ such that $b \wedge a \neq \perp$. We need to prove that $[b] \in \mathbb{A}^a$ satisfies items (1), (2), and (3) of Definition 4.2.

Proof of item 1. By assumption, $[b] \neq \perp$, hence $[b]$ satisfies item (1).

Proof of item 2. To show that $\diamond_i^a[b] = [b]$, it is enough to show that $\diamond_i(b \wedge a) \wedge a = b \wedge a$. Clearly, $b \wedge a \leq b$ implies that $\diamond_i(b \wedge a) \wedge a \leq \diamond_i b \wedge a = b \wedge a$, making use that $\diamond_i b = b$. Conversely, recalling that \diamond_i is reflexive (Definition 4.1, axiom (M1)), we have $b \wedge a = (b \wedge a) \wedge a \leq \diamond_i(b \wedge a) \wedge a$. Hence, $\diamond_i^a[b] = [b]$.

Proof of item 3. We need to prove that $[b]$ is a minimal fixed point of \diamond_i^a . Let $[\perp] \neq [c] \leq [b]$ such that $\diamond_i^a[c] = [c]$, and let us show that $[c] = [b]$. It is enough to show that $c \wedge a = b \wedge a$. The assumption that $[c] \leq [b]$ implies that $c \wedge a \leq b \wedge a \leq b$. Hence, $\diamond_i(c \wedge a) \leq \diamond_i b = b$. Notice that the assumption that \diamond_i is transitive (Definition 4.1, axiom (M6)) implies that $\diamond_i \diamond_i(c \wedge a) = \diamond_i(c \wedge a)$; that is, $\diamond_i(c \wedge a)$ is a fixed point of \diamond_i . Moreover, $\perp \neq c \wedge a \leq \diamond_i(c \wedge a)$ implies that $\diamond_i(c \wedge a) \neq \perp$. Hence, by the i -minimality of b in \mathbb{A} , we conclude that $\diamond_i(c \wedge a) = b$, and hence $\diamond_i(c \wedge a) \wedge a = b \wedge a$. Moreover, the assumption that $\diamond_i^a[c] = [c]$ implies that $\diamond_i(c \wedge a) \wedge a = c \wedge a$. Thus, the following chain of identities holds $c \wedge a = \diamond_i(c \wedge a) \wedge a = b \wedge a$, as required. □

LEMMA 4.26. *For any epistemic Heyting algebra \mathbb{A} and any $a \in \mathbb{A}$, if $[b] \in \text{Min}_i(\mathbb{A}^a)$, then $\diamond_i(b \wedge a)$ is the unique i -minimal element of \mathbb{A} which belongs to $[b]$.*

PROOF. See Appendix A, page 44. □

Combining the two lemmas above, we obtain the following result:

PROPOSITION 4.27. *The following are equivalent for any \mathbb{A} and any $a \in \mathbb{A}$:*

- (1) $[b] \in \text{Min}_i(\mathbb{A}^a)$;
- (2) $[b] = [b']$ for a unique $b' \in \text{Min}_i(\mathbb{A})$ such that $b' \wedge a \neq \perp$.

NOTATION 3. *In what follows, whenever $[b] \in \text{Min}_i(\mathbb{A}^a)$, we will assume w.l.o.g. that $b \in \text{Min}_i(\mathbb{A})$ is the “canonical” (in the sense of Proposition 4.27) representant of $[b]$.*

The Updated APE-structure. For any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support \mathbb{A} of \mathcal{F} , the map $\overline{\text{pre}}$ in \mathbb{E} induces the map $\overline{\text{pre}}$ defined as follows:

$$\begin{aligned} \overline{\text{pre}} : E &\rightarrow \mathbb{A} \\ e &\mapsto \bigvee_{\substack{a \in \Phi \\ \overline{\text{pre}}(e|a) \neq 0}} a. \end{aligned} \quad (4.5)$$

It immediately follows from Propositions 4.14 and 4.27 that the i -minimal elements of $\mathbb{A}^{\mathbb{E}}$ are exactly the elements $[f_{e,b}]$ for $e \in E$ and $b \in \text{Min}_i(\mathbb{A})$ such that $b \wedge \overline{\text{pre}}(e') \neq \perp$ for some $e' \sim_i e$.

Definition 4.28 (Updated APE-Structure). For any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support of \mathcal{F} , the updated APE-structure is the tuple

$$\mathcal{F}^{\mathbb{E}} := (\mathbb{A}^{\mathbb{E}}, (\mu'_i)^{\mathbb{E}})_{i \in \text{Ag}}$$

such that:

- (1) $\mathbb{A}^{\mathbb{E}}$ is obtained by instantiating Definition 4.23 to $\prod_{\mathbb{E}} \mathbb{A}$ and $\overline{\text{pre}} \in \prod_{\mathbb{E}} \mathbb{A}$, i.e.,

$$\mathbb{A}^{\mathbb{E}} := \left(\prod_{\mathbb{E}} \mathbb{A} \right)^{\overline{\text{pre}}};$$

- (2) The maps $\mu'_i{}^{\mathbb{E}}$ are defined as follows:

$$\begin{aligned} \mu'_i{}^{\mathbb{E}} : \text{Min}_i(\mathbb{A}^{\mathbb{E}}) \downarrow &\rightarrow [0, 1] \\ [g] &\mapsto \begin{cases} 0 & \text{if } [g] = \perp, \\ \frac{\mu'_i(g)}{\mu'_i(f)} & \text{otherwise,} \end{cases} \end{aligned}$$

where $[f]$ is the only element in $\text{Min}_i(\mathbb{A}^{\mathbb{E}})$ such that $[g] \leq [f]$.⁵

LEMMA 4.29. *For any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support of \mathcal{F} , the maps $(\mu'_i)^{\mathbb{E}})_{i \in \text{Ag}}$ of the updated APE-structure $\mathcal{F}^{\mathbb{E}} := (\mathbb{A}^{\mathbb{E}}, (\mu'_i)^{\mathbb{E}})_{i \in \text{Ag}}$ are well defined.*

PROOF. Let us first prove the following claim: □

Claim 1. For each $[h] \in \text{Min}_i(\mathbb{A}^{\mathbb{E}}) \downarrow$ such that $[h] \neq \perp$, we have $\mu'_i([h]) \neq 0$.

PROOF OF CLAIM. Let $e \in E$ be such that $(h \wedge \overline{\text{pre}})(e) \neq \perp$. Notice that

$$(h \wedge \overline{\text{pre}})(e) \neq \perp \quad \text{iff} \quad h(e) \wedge \bigvee_{\substack{a \in \Phi \\ \overline{\text{pre}}(e|a) \neq 0}} a \neq \perp.$$

This implies that there is $a \in \Phi$ such that

$$\overline{\text{pre}}(e | a) > 0 \quad \text{and} \quad h(e) \wedge a \neq \perp.$$

⁵See Definition 4.17 for the definition of the maps $(\mu'_i)_{i \in \text{Ag}}$.

Since μ_i is an i -measure (see Definition 4.6), we have $\mu_i((h \wedge a)(e)) > 0$. Then, the following set is non-empty

$$\{a \in \Phi \mid \mu_i((h \wedge a)(e)) > 0 \text{ and } \overline{\text{pre}}(e|a) > 0\}.$$

Since Φ is finite, it is well founded with respect to the order of the multiset $<$, hence, it contains at least one minimal element. Let a_0 be such a minimal element. From item (6) of Definition 4.11, we deduce that, for every $b \in \Phi$ such that $b < a_0$, it is the case that $\overline{\text{pre}}(e|b) > 0$. The minimality of a_0 implies that, for every $b \in \Phi$ such that $b < a_0$, we have $\mu_i((h \wedge b)(e)) = 0$. This implies that, for all $b \in \text{mb}(a)$, we have $\mu_i((h \wedge b)(e)) = 0$. Hence,

$$\begin{aligned} \mu_i^{a_0}(h(e)) &= \mu_i(g(e) \wedge a_0) - \sum_{b \in \text{mb}(a_0)} \mu_i(h(e) \wedge b) && \text{(see Definition 4.15)} \\ &= \mu_i(h(e) \wedge a_0). \end{aligned}$$

Therefore, $\mu_i^{a_0}(h(e)) > 0$ and $P_i(e) \cdot \mu_i^{a_0}(h(e)) \cdot \overline{\text{pre}}(e|a_0) > 0$. This guarantees that $\mu'_i([h]) \neq 0$. This finishes the proof of the claim. \blacksquare

Now, let us prove that the map $\mu_i^{\mathbb{E}}$ is well defined. Recall that, if $[g] \neq \perp$, then $[f]$ is unique (see Remark 3). From the claim above, it follows that the division $\frac{\mu'_i(g)}{\mu'_i(f)}$ is defined. Finally, let us verify that $\mu_i^{\mathbb{E}}$ assigns exactly one value to every $[g] \in \text{Min}_i(\mathbb{A}^{\mathbb{E}})\downarrow$. Let $g_1, g_2 \in [g]$. Then, we have $\mu'_i(g_1) = \mu'_i(g_1 \wedge \overline{\text{pre}}e) = \mu'_i(g_2 \wedge \overline{\text{pre}}e) = \mu'_i(g_2)$ (see Proposition 4.18). Since μ'_i is order-preserving, strictly positive for $[f] \neq \perp$ and $\mu_i^{\mathbb{E}}([g]) = \frac{\mu'_i(g)}{\mu'_i(f)}$ with $0 < [g] \leq [f]$, we have that the division $\frac{\mu'_i(g)}{\mu'_i(f)}$ is defined and $\frac{\mu'_i(g)}{\mu'_i(f)} \leq 1$. Hence, $\mu_i^{\mathbb{E}}$ is well defined for any $i \in \text{Ag}$.

PROPOSITION 4.30. *For any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support of \mathcal{F} , the tuple $\mathcal{F}^{\mathbb{E}} = (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i \in \text{Ag}})$ is an APE-structure.*

PROOF. See Appendix A, page 44. \square

The Updated Algebra for the Classical Case. In this section, we conclude the proof of Proposition 3.2 by showing that the pseudo-quotient construction described above, applied to the complex algebras of the intermediate classical models, dualizes the submodel construction in Section 2.2.

The definition of the complex algebra of a PES-model (Definition 3.1) can be equivalently reformulated as follows:

Definition 4.31 (Complex Algebra). For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, [\cdot] \rangle$, its complex algebra is the tuple

$$\mathbb{M}^+ := (\mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}}, (P_i^+)_{i \in \text{Ag}}),$$

where

- (1) for each $i \in \text{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned} \diamond_i X &= \{s \in S \mid \exists x (s \sim_i x \text{ and } x \in X)\}, \\ \square_i X &= \{s \in S \mid \forall x (s \sim_i x \Rightarrow x \in X)\}, \end{aligned}$$
- (2) $\mathbb{A} := \langle \mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}} \rangle$ is an epistemic Heyting algebra,
- (3) for each $i \in \text{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned} P_i^+ : \text{Min}_i(\mathbb{A})\downarrow &\rightarrow \mathbb{A} \\ X &\mapsto \sum_{x \in X} P_i(x). \end{aligned}$$

Notice that the domain of P_i^+ consists of all the subsets of the equivalence classes of \sim_i .

LEMMA 4.32. For any PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} ,

$$(P_i^+)^{\mathbb{E}_{\mathcal{E}}} \cong (P_i^{\mathcal{E}})^+.$$

PROOF. See Appendix A, page 44. □

5 INTUITIONISTIC PDEL

In this section, we introduce the Intuitionistic Probabilistic Dynamic Epistemic Logic (IPDEL). We define its syntax in Section 5.1 and its algebraic semantics (Definition 5.6) in Section 5.2. Then, in Section 5.3, we introduce the axiomatization of IPDEL (Table 2) and state its soundness and completeness. For the proofs, see Appendices B and C.

5.1 The Language of IPDEL

Definition 5.1 (IPDEL Language). The set \mathcal{L} of IPDEL-formulas φ and the class of intuitionistic probabilistic event structures \mathbb{E} over \mathcal{L} are built by simultaneous recursion as follows:

$$\varphi ::= p \mid \perp \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \diamond_i \varphi \mid \square_i \varphi \mid \langle \mathcal{E}, e \rangle \varphi \mid [\mathcal{E}, e] \varphi \mid \left(\sum_{k=1}^n \alpha_k \mu_i(\varphi) \right) \geq \beta,$$

where $i \in \text{Ag}$, and, following Reference [20], we let α_k, β be rational numbers, and the event structures \mathcal{E} are as in Definition 5.2.

The connectives \top , \neg , and \leftrightarrow are defined by the usual abbreviations.

Definition 5.2 (Intuitionistic Probabilistic Event Structure). An intuitionistic probabilistic event structure over \mathcal{L} is a tuple

$$\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub}),$$

such that

- E is a non-empty finite set;
- each \sim_i is an equivalence relation on E ;
- each $P_i : E \rightarrow]0, 1]$ assigns a probability distribution over each \sim_i -equivalence class, i.e.,

$$\sum \{P_i(e') \mid e' \sim_i e\} = 1;$$

- Φ is a finite set of formulas in \mathcal{L} such that, for all $\phi_k, \phi_j \in \Phi$, one and only one of the following conditions is true:
 - $\vdash (\phi_j \wedge \phi_k) \rightarrow \perp$,
 - $\vdash \phi_k \rightarrow \phi_j$,
 - $\vdash \phi_j \rightarrow \phi_k$;
- the map $\text{pre} : E \times \Phi \rightarrow [0, 1]$ assigns a probability distribution $\text{pre}(\bullet \mid \phi)$ over E for every $\phi \in \Phi$;
- the map $\text{sub} : E \rightarrow \text{Sub}_{\mathcal{L}}$ assigns a substitution function (see Definition 2.3) to each event in E ;
- for all $\phi_j \in \Phi$ and $e \in E$, if $\text{pre}(e \mid \phi_j) = 0$, then $\text{pre}(e \mid \phi_k) = 0$ for all $\phi_k \in \Phi$ such that $\vdash \phi_j \rightarrow \phi_k$.

Remark 11. The conditions on Φ match the conditions of Φ given in Definition 4.11 (cf. Definition 5.4). The requirement in Definition 4.11 that Φ is a multiset stems from the fact that the interpretation of distinct formulas ϕ_k, ϕ_j such that $\phi_k \rightarrow \phi_j$ might coincide in a model.

Remark 12. The conditions on the preconditions are given using \vdash . One should refer to Section 5.3 and Table 2 for the axiomatization of IPDEL.

5.2 Algebraic Semantics

In what follows, we define the models, the event structures on the language, the event structures on the model, the updated models, and the semantics. Notice that the definition of the event structure on the model relies on the definition of the event structure on the language, and that the definitions of the event structure on the model, the updated models, and the semantics are given by simultaneous induction.

Definition 5.3 (APE-Models). Algebraic probabilistic epistemic models (APE-models) are tuples $\mathcal{M} = \langle \mathcal{F}, \nu \rangle$ such that $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ is an APE-structure and $\nu : \text{AtProp} \rightarrow \mathbb{A}$.

The update construction of Section 4 extends from APE-structures to APE-models. Indeed, for any APE-model $\mathcal{M} = \langle \mathbb{A}, (\mu_i)_{i \in \text{Ag}}, \nu \rangle$ and any event structure \mathcal{E} (see Definition 5.2), the event structure \mathcal{E} induces an event structure over the algebra \mathbb{A} (see Definition 4.11) as follows:

Definition 5.4. For any APE-model $\mathcal{M} = \langle \mathbb{A}, (\mu_i)_{i \in \text{Ag}}, \nu \rangle$ and any event structure

$$\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub}),$$

over \mathcal{L} , the following tuple is an event structure over \mathbb{A} :

$$\mathbb{E}_{\mathcal{E}} := (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi_{\mathcal{M}}, \overline{\text{pre}}_{\mathcal{M}}),$$

where

- $\Phi := (\Phi_{\mathcal{M}}, <_{\mathcal{M}})$ with $\Phi_{\mathcal{M}} := \{[\![\phi]\!]_{\mathcal{M}} \mid \phi \in \Phi\}$ and $<_{\mathcal{M}} := \{([\![\phi_j]\!], [\![\phi_k]\!]) \mid \vdash \phi_j \rightarrow \phi_k\}$, and
- $\overline{\text{pre}}_{\mathcal{M}}$ assigns a probability distribution $\overline{\text{pre}}_{\mathcal{M}}(\bullet \mid a)$ over E for every $a \in \Phi_{\mathcal{M}}$.

It is straightforward to verify that $\mathbb{E}_{\mathcal{E}}$ defined above is an event structure.

Definition 5.5 (Updated Model). The update of the APE-model $\mathcal{M} = \langle \mathcal{F}, \nu \rangle$ by the intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ is given by the APE-model

$$\mathcal{M}^{\mathcal{E}} := \langle \mathcal{F}^{\mathcal{E}}, \nu^{\mathcal{E}} \rangle,$$

where

- $\mathcal{F}^{\mathcal{E}} := \mathcal{F}^{\mathbb{E}_{\mathcal{E}}}$ as in Definition 4.28,
- and the map $\nu^{\mathcal{E}}$ is defined as follows:

$$\nu^{\mathcal{E}} : \text{AtProp} \rightarrow \mathbb{A}^{\mathbb{E}_{\mathcal{E}}}$$

$$p \mapsto \begin{cases} [\![v^{\Pi}(\text{sub}(e)(p))]\!] & \text{if } p \in \text{dom}(\text{sub}(e)) \\ [\![v^{\Pi}(p)]\!] & \text{otherwise} \end{cases}$$

where

$$v^{\Pi}(p) : E \rightarrow \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A} \quad \text{and} \quad v^{\Pi}(\text{sub}(e)(p)) : E \rightarrow \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A}$$

$$e \mapsto v(p) \quad \quad \quad e \mapsto v(\text{sub}(e)(p)).$$

NOTATION 4. We define the e th projection π_e for every $e \in E$, the quotient map π and the map ι as follows:

$$\begin{aligned} \pi_e : \prod_{\mathbb{E}_e} \mathbb{A} &\rightarrow \mathbb{A} & \text{and} & & \pi : \prod_{\mathbb{E}_E} \mathbb{A} &\rightarrow \mathbb{A}^{\mathbb{E}_E} & \text{and} & & \iota : \mathbb{A}^{\mathbb{E}_E} &\rightarrow \prod_{\mathbb{E}_E} \mathbb{A} \\ g &\mapsto g(e) & & & g &\mapsto [g] & & & [g] &\mapsto g \wedge \overline{pre}. \end{aligned}$$

As explained in Reference [36, Section 3.2], the map ι is well defined.

Definition 5.6 (Semantics). The interpretation of \mathcal{L} -formulas on any APE-model \mathcal{M} is defined recursively as follows:

$$\begin{aligned} \llbracket \perp \rrbracket_{\mathcal{M}} &= \perp^{\mathbb{A}} & \llbracket \top \rrbracket_{\mathcal{M}} &= \top^{\mathbb{A}} \\ \llbracket p \rrbracket_{\mathcal{M}} &= v(p) & \llbracket \varphi \rightarrow \psi \rrbracket_{\mathcal{M}} &= \llbracket \varphi \rrbracket_{\mathcal{M}} \rightarrow^{\mathbb{A}} \llbracket \psi \rrbracket_{\mathcal{M}} \\ \llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}} &= \llbracket \varphi \rrbracket_{\mathcal{M}} \wedge^{\mathbb{A}} \llbracket \psi \rrbracket_{\mathcal{M}} & \llbracket \varphi \vee \psi \rrbracket_{\mathcal{M}} &= \llbracket \varphi \rrbracket_{\mathcal{M}} \vee^{\mathbb{A}} \llbracket \psi \rrbracket_{\mathcal{M}} \\ \llbracket \diamond_i \varphi \rrbracket_{\mathcal{M}} &= \diamond_i \llbracket \varphi \rrbracket_{\mathcal{M}} & \llbracket \square_i \varphi \rrbracket_{\mathcal{M}} &= \square_i \llbracket \varphi \rrbracket_{\mathcal{M}} \\ \llbracket \langle \mathcal{E}, e \rangle \varphi \rrbracket_{\mathcal{M}} &= \llbracket pre(e) \rrbracket_{\mathcal{M}} \wedge^{\mathbb{A}} \pi_e \circ \iota(\llbracket \varphi \rrbracket_{\mathcal{M}^{\mathbb{E}_E}}) & \llbracket [\mathcal{E}, e] \varphi \rrbracket_{\mathcal{M}} &= \llbracket pre(e) \rrbracket_{\mathcal{M}} \rightarrow^{\mathbb{A}} \pi_e \circ \iota(\llbracket \varphi \rrbracket_{\mathcal{M}^{\mathbb{E}_E}}) \\ \llbracket \left(\sum_{k=1}^n \alpha_k \mu_i(\varphi_k) \right) \geq \beta \rrbracket_{\mathcal{M}} &= \bigvee \left\{ a \in \mathbb{A} \mid a \in \text{Min}_i(\mathbb{A}) \text{ and } \left(\sum_{k=1}^n \alpha_k \mu_i(\llbracket \varphi_k \rrbracket_{\mathcal{M}} \wedge a) \right) \geq \beta \right\}. \end{aligned}$$

5.3 Axiomatization

IPDEL is intended as the intuitionistic counterpart of classical PDEL. The full axiomatization of IPDEL is given in Table 2 (see page 26). This axiomatization differs from the one of classical PDEL (cf. Table 1) in that the axioms for S5 are replaced by the axioms of intuitionistic modal logic MIPC and axiom E (see Definition 4.3), and the axioms capturing classical probability theory are replaced by axioms capturing intuitionistic probability theory. In particular, axioms p3 and p4 in Table 1 are different from the axioms P3 and P4 in Table 2. It is not hard to see that axiom p3 implies P3 and $\mu_i(\varphi) + \mu_i(\neg\varphi) = 1$ in the presence of p1 and p2. Axiom P3 is strictly weaker than p3, since the aforementioned equality is generally false for intuitionistic probabilities. Axioms p4 and P4 are classically equivalent. In intuitionistic logic, P4 is strictly stronger than p4. Indeed, as Lemma 5.8 shows, p4 is not strong enough to express the strict monotonicity of i -measures. Finally, notice that axioms M8 and M9 from Definition 4.1 are not in Table 2. Indeed, they follow from the remaining axioms and the necessitation rules (see Lemma 5.7 and also compare with Reference [11]). Compared to classical PDEL, the greater number of reduction axioms reflects the fact that, in the intuitionistic setting, connectives are not interdefinable. However, of course the intuitionistic reduction axioms are classically derivable and serve the same purpose as the classical reduction axioms, namely to reduce the completeness of IPDEL to the completeness of its “static” fragment.

LEMMA 5.7. Axioms M8 and M9 from Definition 4.1 are derivable from rules and axioms in Table 2.

PROOF. Axiom M9 (i.e., $\top \leq \square_i \top$) is a direct consequence of the necessitation rule. Axiom M8 (i.e., $\diamond_i \perp \leq \perp$) can be derived as follows: by instantiating axiom M6 with \perp , one gets $\diamond_i \square_i \perp \rightarrow \square_i \perp$; by instantiating axiom M2 with \perp , one gets $\square_i \perp \rightarrow \perp$; since, in addition, $\perp \rightarrow \square_i \perp$ (axiom H9), one gets that $\square_i \perp \leftrightarrow \perp$; by substitution of logical equivalence (rule SubEq) in $\diamond_i \square_i \perp \rightarrow \square_i \perp$, one gets $\diamond_i \perp \rightarrow \perp$, as required. \square

Table 2. AXIOMS OF IPDEL

Axioms of IPL	
H1. $A \rightarrow (B \rightarrow A)$	H2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
H3. $A \rightarrow (B \rightarrow A \wedge B)$	H4. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
H5. $A \wedge B \rightarrow A$	H6. $A \wedge B \rightarrow B$
H7. $A \rightarrow A \vee B$	H8. $B \rightarrow A \vee B$
H9. $\perp \rightarrow A$	
Axioms for static modalities	
M1. $p \rightarrow \Diamond_i p$	M2. $\Box_i p \rightarrow p$
M3. $\Diamond_i(p \vee q) \rightarrow \Diamond_i p \vee \Diamond_i q$	M4. $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$
M5. $\Diamond_i p \rightarrow \Box_i \Diamond_i p$	M6. $\Diamond_i \Box_i p \rightarrow \Box_i p$
M7. $\Box_i(p \rightarrow q) \rightarrow (\Diamond_i p \rightarrow \Diamond_i q)$	E. $\Diamond_i p \vee \neg \Diamond_i p$
Axioms for inequalities	
N0. $t \geq t$	N1. $(t \geq \beta) \leftrightarrow (t + 0 \cdot \mu_i(\varphi) \geq \beta)$
N2. $(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \rightarrow (\sum_{k=1}^n \alpha_{\rho(k)} \cdot \mu_i(\varphi_{\rho(k)}) \geq \beta)$ for any permutation ρ over $\{1, \dots, n\}$	
N3. $(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \wedge (\sum_{k=1}^n \alpha'_k \cdot \mu_i(\varphi_k) \geq \beta') \rightarrow (\sum_{k=1}^n (\alpha_k + \alpha'_k) \cdot \mu_i(\varphi_k) \geq (\beta + \beta'))$	
N4. $((t \geq \beta) \wedge (d \geq 0)) \rightarrow (d \cdot t \geq d \cdot \beta)$	N5. $(t \geq \beta) \vee (\beta \geq t)$
N6. $(t \geq \beta) \rightarrow (t > \gamma)$ for all $\gamma < \beta$	
Axioms for Intuitionistic Probabilities	
P1. $\mu_i(\perp) = 0$	P2. $\mu_i(\top) = 1$
P3. $\mu_i(\varphi) + \mu_i(\psi) = \mu_i(\varphi \vee \psi) + \mu_i(\varphi \wedge \psi)$	P4. $((\Box_i(\varphi \rightarrow \psi)) \wedge (\mu_i(\varphi) = \mu_i(\psi))) \leftrightarrow \Box_i(\psi \leftrightarrow \varphi)$
P5. $(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \rightarrow \Box_i(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta)$	
Reduction Axioms	
I1. $[\mathcal{E}, e]p \leftrightarrow \text{pre}(e) \rightarrow \text{sub}(e, p)$	I2. $\langle \mathcal{E}, e \rangle p \leftrightarrow \text{pre}(e) \wedge \text{sub}(e, p)$
I3. $[\mathcal{E}, e] \top \leftrightarrow \top$	I4. $\langle \mathcal{E}, e \rangle \top \leftrightarrow \text{pre}(e)$
I5. $[\mathcal{E}, e] \perp \leftrightarrow \neg \text{pre}(e)$	I6. $\langle \mathcal{E}, e \rangle \perp \leftrightarrow \perp$
I7. $[\mathcal{E}, e](\psi_1 \wedge \psi_2) \leftrightarrow [\mathcal{E}, e]\psi_1 \wedge [\mathcal{E}, e]\psi_2$	I8. $\langle \mathcal{E}, e \rangle(\psi_1 \wedge \psi_2) \leftrightarrow \langle \mathcal{E}, e \rangle\psi_1 \wedge \langle \mathcal{E}, e \rangle\psi_2$
I9. $[\mathcal{E}, e](\psi_1 \vee \psi_2) \leftrightarrow \text{pre}(e) \rightarrow \langle \mathcal{E}, e \rangle\psi_1 \vee \langle \mathcal{E}, e \rangle\psi_2$	I10. $\langle \mathcal{E}, e \rangle(\psi_1 \vee \psi_2) \leftrightarrow \langle \mathcal{E}, e \rangle\psi_1 \vee \langle \mathcal{E}, e \rangle\psi_2$
I11. $[\mathcal{E}, e](\psi_1 \rightarrow \psi_2) \leftrightarrow \langle \mathcal{E}, e \rangle\psi_1 \rightarrow \langle \mathcal{E}, e \rangle\psi_2$	I12. $\langle \mathcal{E}, e \rangle(\psi_1 \rightarrow \psi_2) \leftrightarrow \text{pre}(e) \wedge (\langle \mathcal{E}, e \rangle\psi_1 \rightarrow \langle \mathcal{E}, e \rangle\psi_2)$
I13. $[\mathcal{E}, e] \Diamond_i \psi \leftrightarrow \text{pre}(e) \rightarrow \bigvee_{e' \sim_i e} \Diamond_i(\langle \mathcal{E}, e' \rangle \psi)$	I14. $\langle \mathcal{E}, e \rangle \Diamond_i \psi \leftrightarrow \text{pre}(e) \wedge \bigvee_{e' \sim_i e} \Diamond_i(\langle \mathcal{E}, e' \rangle \psi)$
I15. $[\mathcal{E}, e] \Box_i \psi \leftrightarrow \text{pre}(e) \rightarrow \bigwedge_{e' \sim_i e} \Box_i([\mathcal{E}, e'] \psi)$	I16. $\langle \mathcal{E}, e \rangle \Box_i \psi \leftrightarrow \text{pre}(e) \wedge \bigwedge_{e' \sim_i e} \Box_i([\mathcal{E}, e'] \psi)$
I17. $[\mathcal{E}, e](\alpha \mu_i(\psi) \geq \beta) \leftrightarrow \text{pre}(e) \rightarrow (C + D \geq 0)$	I18. $\langle \mathcal{E}, e \rangle(\alpha \mu_i(\psi) \geq \beta) \leftrightarrow \text{pre}(e) \wedge (C' + D \geq 0)$
<p>where</p> $C := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi([\mathcal{E}, e'] \psi) \quad C' := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\langle \mathcal{E}, e' \rangle \psi)$ $D := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\top),$ <p>with</p> $\mu_i^\phi(\psi) := \mu_i(\psi \wedge \phi) - \sum_{\sigma \in \text{mb}(\phi)} \mu_i(\psi \wedge \sigma) \quad \text{and} \quad \text{mb}(\phi) := \max_{\rightarrow} \Phi \cap \downarrow \phi.$	
Inference Rules	
MP. if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$	
Nec _i if $\vdash A$, then $\vdash \Box_i A$	Nec _α if $\vdash A$, then $\vdash [\mathcal{E}, e]A$
Sub _μ if $\vdash A \rightarrow B$, then $\vdash \mu_i(A) \leq \mu_i(B)$	SubEq if $\vdash A \leftrightarrow B$, then $\vdash \phi \rightarrow \phi[A/B]$

LEMMA 5.8. *Axiom P4 in Table 2 implies axiom p4 in Table 1. In classical logic, the two formulas are equivalent in the context of the rest of the axioms. Finally, there exists an ApPE-structure that validates axiom p4 but doesn't validate axiom P4.*

PROOF. Recall that

$$\begin{aligned} \text{(P4)} \quad & ((\Box_i(\phi \rightarrow \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi))) \leftrightarrow \Box_i(\psi \leftrightarrow \phi), \\ \text{(p4)} \quad & \Box_i\phi \leftrightarrow (\mu_i(\phi) = 1). \end{aligned}$$

That P4 implies p4 follows immediately by replacing ψ with \top . Now, let us prove that p4 implies P4 in classical logic. We first show that p4 implies $\Box_i(\psi \leftrightarrow \phi) \rightarrow ((\Box_i(\phi \rightarrow \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi)))$ as follows:

$$\begin{aligned} \Box_i(\psi \leftrightarrow \phi) &\Leftrightarrow \mu_i(\psi \leftrightarrow \phi) = 1 && \text{(Axiom p4)} \\ &\Leftrightarrow \mu_i((\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi)) = 1 && \text{(classical logic equivalence)} \end{aligned}$$

Notice that

$$(\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi) \rightarrow (\neg\psi \vee \phi) \quad (5.1)$$

$$(\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi) \rightarrow (\neg\phi \vee \psi) \quad (5.2)$$

Hence, using the rule $\text{Sub}_\mu : \text{if } \vdash A \rightarrow B, \text{ then } \vdash \mu_i(A) \leq \mu_i(B)$, the equality $\mu_i((\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi)) = 1$, and the Equations (5.1) and (5.2), one can prove that

$$(\mu_i(\neg\psi \vee \phi) = 1) \wedge (\mu_i(\neg\phi \vee \psi) = 1).$$

Using p4, we can derive that $\Box_i(\phi \rightarrow \phi)$. It remains to derive that $\mu_i(\psi) = \mu_i(\phi)$ as follows:

$$\begin{aligned} & (\mu_i(\neg\psi \vee \phi) = 1) \wedge (\mu_i(\neg\phi \vee \psi) = 1) \\ \Rightarrow & (\mu_i(\neg(\neg\psi \vee \phi)) = 0) \wedge (\mu_i(\neg\phi \vee \psi) = 1) && (\mu_i(\phi) = 1 - \mu_i(\neg\phi) \text{ in PDEL, see Table 1}) \\ \Rightarrow & (\mu_i(\psi \wedge \neg\phi) = 0) \wedge (\mu_i(\neg\phi \vee \psi) = 1) && \text{(De Morgan laws)} \\ \Rightarrow & (\mu_i(\psi \wedge \neg\phi) = 0) \wedge (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\psi \wedge \neg\phi) = 1) \\ & (\mu_i(\phi) + \mu_i(\psi) = \mu_i(\phi \vee \psi) + \mu_i(\phi \wedge \psi) \text{ in PDEL, see Table 1}) \\ \Rightarrow & \mu_i(\neg\phi) + \mu_i(\psi) = 1 \\ \Rightarrow & \mu_i(\neg\phi) + \mu_i(\psi) = \mu_i(\phi) + \mu_i(\neg\phi) && (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL, by axioms p2 and p3}) \\ \Rightarrow & \mu_i(\psi) = \mu_i(\phi). \end{aligned}$$

Now, we show that p4 implies $((\Box_i(\phi \rightarrow \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi))) \rightarrow \Box_i(\psi \leftrightarrow \phi)$ as follows:

$$\begin{aligned} & \Box_i(\phi \rightarrow \psi) \wedge (\mu_i(\phi) = \mu_i(\psi)) \\ \Rightarrow & (\mu_i(\neg\phi \vee \psi) = 1) \wedge (\mu_i(\phi) = \mu_i(\psi)) && \text{(Axiom p4)} \\ \Rightarrow & (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \wedge (\mu_i(\phi) = \mu_i(\psi)) \\ & (\mu_i(\phi) + \mu_i(\psi) = \mu_i(\phi \vee \psi) + \mu_i(\phi \wedge \psi) \text{ in PDEL}) \\ \Rightarrow & (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \wedge (\mu_i(\neg\phi) = \mu_i(\neg\psi)) && (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\ \Rightarrow & (\mu_i(\neg\psi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \\ \Rightarrow & (1 - \mu_i(\neg\phi \wedge \psi) = 1) && (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\ \Rightarrow & (\mu_i(\neg\phi \wedge \psi) = 0) && (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\ \Rightarrow & (\mu_i(\phi \vee \neg\psi) = 1) && (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\ \Rightarrow & (\mu_i(\psi \rightarrow \phi) = 1) \\ \Rightarrow & \Box_i(\psi \rightarrow \phi) && \text{(Axiom p4)} \end{aligned}$$

Fig. 1. Heyting algebra \mathbb{H} .

This concludes the proof that in classical logic $p4$ and $P4$ are equivalent. Finally, consider the Heyting algebra \mathbb{H} in Figure 1 with

$$\diamond x := \begin{cases} \top & \text{if } x \neq \perp, \\ \perp & \text{if } x = \perp \end{cases} \quad \Box x := \begin{cases} \perp & \text{if } x \neq \top, \\ \top & \text{if } x = \top \end{cases}$$

and $\mu(\perp) = 0$, $\mu(a) = 0.5$, $\mu(b) = 0.5$ and $\mu(\top) = 1$.

It is easy to see that the Heyting algebra in Figure 1 satisfies all axioms of IPDEL except for $P4$ and it satisfies $p4$. It falsifies $P4$, because $(\Box(a \rightarrow b)) \wedge (\mu(a) = \mu(b)) = \top$, while $\Box(a \leftrightarrow b) = \perp$. \square

THEOREM 5.9 (SOUNDNESS). *The axiomatization for IPDEL given in Table 2 is sound w.r.t. APE-models.*

THEOREM 5.10 (COMPLETENESS). *The axiomatization for IPDEL given in Table 2 is weakly complete w.r.t. APE-models.*

The proof of soundness is given in Appendix B, and the proof of completeness is given in Appendix C.

6 RELATIONAL SEMANTICS

In this section, we introduce the finite relational semantics of IPDEL as the dual structures of epistemic Heyting algebras within the duality between monadic Heyting algebras and MIPC-frames (cf. References [12, 34]). Specifically, we specialize this duality⁶ by identifying the condition corresponding to axiom E. Moreover, we present a dual correspondence between the probability distributions on intuitionistic Kripke frames and measures on epistemic Heyting algebras. This correspondence appears in Reference [23] in the context of finite GBL-algebras. Furthermore, we generalize the model-theoretic constructions presented in Section 2.2 for the Boolean setting and show that they dually correspond to the constructions presented in Section 4. Finally, notice that these results readily imply the completeness and the finite model property of IPDEL with respect to this class of relational structures via the algebraic completeness presented in Appendix C.

Structure of This Section. In Section 6.1, we introduce the *epistemic intuitionistic Kripke frames* as the class of relational structures dually corresponding to epistemic Heyting algebras. In Section 6.2, we introduce the probability distributions associated with any agent i and prove that each dually corresponds to an i -measure. In Section 6.3, we introduce the construction of intermediate epistemic intuitionistic Kripke frames and prove that it dually corresponds to the construction of intermediate epistemic Heyting algebras presented in Section 4.4. In Section 6.4, we define the dual construction to the pseudo-quotient defined in 4.5. Finally, in Section 6.5, we use this construction to define the interpretation of IPDEL-formulas on IPDEL-models.

⁶Because we consider only finite algebras and finite relational structures, we can dispense with the topology.

6.1 Epistemic Heyting Algebras and Epistemic Intuitionistic Kripke Frames

We first recall the definition on the objects of the duality between finite monadic Heyting algebras and MIPC-frames.⁷ We then identify the MIPC-frames corresponding to epistemic intuitionistic Kripke frames and show that their dual algebras exactly correspond to epistemic Heyting algebras.

Definition 6.1 (Finite MIPC-Frames). A finite MIPC-frame is a tuple

$$\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$$

such that (S, \leq) is a finite poset and each R_i is an equivalence relation on S such that

$$(R_i \circ \geq) \subseteq (\geq \circ R_i) \quad R_i = (\geq \circ R_i) \cap (R_i \circ \leq).$$

NOTATION 5. For any poset (S, \leq) and any set $X \subseteq S$, we define the down-set and the up-set generated by X as

$$X \downarrow = \{w \in S \mid \exists v \in X, w \leq v\} \quad \text{and} \quad X \uparrow = \{w \in S \mid \exists v \in X, w \geq v\},$$

respectively. We let $\mathcal{P}^\downarrow(S) = \{X \downarrow \mid X \subseteq S\}$ be the set of all downsets of S .

Definition 6.2 (Complex Algebra of a Finite MIPC-Frame). For any finite MIPC-frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, let its complex algebra be:

$$\mathbb{F}^+ = (\mathcal{P}^\downarrow(S), \wedge, \vee, \rightarrow, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}}, \perp)$$

where

$$X \wedge Y := X \cap Y, \tag{6.1}$$

$$X \vee Y := X \cup Y, \tag{6.2}$$

$$X \rightarrow Y := S \setminus ((X \cap (S \setminus Y)) \uparrow), \tag{6.3}$$

$$\diamond_i X := R_i^{-1}[X], \tag{6.4}$$

$$\square_i X := S \setminus (\geq \circ R_i)^{-1}[S \setminus X], \tag{6.5}$$

$$\perp := \emptyset. \tag{6.6}$$

We also use the standard notation

$$\top := S, \tag{6.7}$$

$$\neg X := X \rightarrow \perp = S \setminus X \uparrow. \tag{6.8}$$

Definition 6.3 (MIPC Frame Associated to a Finite Monadic Heyting Algebra). For any finite monadic Heyting algebra⁸ $\mathbb{A} = (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$, let its associated frame be:

$$\mathbb{A}_+ = \langle \mathcal{J}(\mathbb{A}), \leq, (R_i)_{i \in \text{Ag}} \rangle,$$

where

- $\mathcal{J}(\mathbb{A})$ is the set of join-irreducible elements of \mathbb{A} ;
- $\leq \subseteq \mathcal{J}(\mathbb{A}) \times \mathcal{J}(\mathbb{A})$ is the order inherited from \mathbb{A} , i.e., $j \leq j'$ iff $j \leq_{\mathbb{A}} j'$ for all $j, j' \in \mathcal{J}(\mathbb{A})$;
- $R_i \subseteq \mathcal{J}(\mathbb{A}) \times \mathcal{J}(\mathbb{A})$ is defined as follows: $j R_i j'$ if and only if $\diamond_i j = \diamond_i j'$ for all $j, j' \in \mathcal{J}(\mathbb{A})$ and every $i \in \text{Ag}$.

The following lemma is stated in Reference [34, Fact 4.5, Proposition 4.6] and Reference [12]:

LEMMA 6.4. *If \mathbb{F} is a finite MIPC-frame, then \mathbb{F}^+ is a finite monadic Heyting algebra. If \mathbb{A} is a finite monadic Heyting algebra, then \mathbb{A}_+ is a finite MIPC-frame. Furthermore $(\mathbb{F}^+)_+ \cong \mathbb{F}$ and $(\mathbb{A}_+)^+ \cong \mathbb{A}$.*

⁷A complete exposition can be found in Reference [12].

⁸see Definition 4.1, page 11.

NOTATION 6. Let $\eta : \mathbb{A} \rightarrow (\mathbb{A}_+)^+$ and $\epsilon : \mathbb{F} \rightarrow (\mathbb{F}^+)_+$ denote the natural isomorphisms inherited from the object dualities $(\mathbb{A}_+)^+ \cong \mathbb{A}$ and $(\mathbb{F}^+)_+ \cong \mathbb{F}$. (see Reference [19] for more details on η and ϵ .)

DEFINITION 6.5 (Epistemic Intuitionistic Kripke Frame). An epistemic intuitionistic Kripke frame is a finite MIPC-frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$ such that, for every $i \in \text{Ag}$, the equivalence relation R_i is upwards and downwards closed w.r.t. the order relation \leq .

The following lemma characterizes the dual spaces of epistemic Heyting algebras⁹:

LEMMA 6.6. If \mathbb{A} is an epistemic Heyting algebra, then \mathbb{A}_+ is an epistemic intuitionistic Kripke frame. If \mathbb{F} is an epistemic intuitionistic Kripke frame, then \mathbb{F}^+ is an epistemic Heyting algebra.

PROOF. Since, by definition, all epistemic Heyting algebras are finite monadic Heyting algebras, it follows from Lemma 6.4 that their dual spaces are finite MIPC-frames.

Let $\mathbb{A} = (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$ and $\mathbb{A}_+ = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$. By Lemma 6.4, it is enough to show that the equivalence relations R_i are upwards and downwards closed. Since R_i is symmetric, it is enough to show that R_i is upwards closed.

Assume, for contradiction, that the equivalence relation R_i is not upwards closed for some $i \in \text{Ag}$. Hence, there is at least one equivalence class defined by the relation R_i that is not upwards closed. Since the empty set is upwards and downwards closed, this equivalence class is non-empty. Let $w \in S$ be an element of that class, let $v \in S$ be such that $v \geq w$ and $v \notin R_i[w]$, and let a be the element of the dual algebra corresponding to the downset generated by w . Then, $\diamond_i a = R_i^{-1}[w\downarrow]$.

First, let us show that $v \notin R_i^{-1}[w\downarrow]$. Heading towards a contradiction, let us assume that $v \in R_i^{-1}[w\downarrow]$. This means that there exists $z \in S$ such that $z \leq w$ and $(v, z) \in R_i$; therefore, $(v, w) \in (R_i \circ \leq)$. Furthermore, we have that $(v, w) \in (\geq \circ R_i)$, because $(w, w) \in R_i$ and $v \geq w$. Since $R_i = (\geq \circ R_i) \cap (R_i \circ \leq)$, we deduce that $(v, w) \in R_i$, which is a contradiction. This proves that $v \notin R_i^{-1}[w\downarrow]$.

From (6.8), we have that $\neg \diamond_i a = S \setminus ((R_i^{-1}[w\downarrow])\uparrow)$. By assumption, $w \leq v$, hence, $v \in (R_i^{-1}[w\downarrow])\uparrow$ and $v \notin \neg \diamond_i a$. Hence, $v \notin \diamond_i a \vee \neg \diamond_i a$, and therefore axiom E does not hold, contradicting the assumption that \mathbb{A} is an epistemic Heyting algebra. Hence, R_i is upwards closed.

As to the second part of the statement, let $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$ and $\mathbb{F}^+ = (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$. By Lemma 6.4, it remains to prove that \mathbb{F}^+ satisfies axiom E (i.e., $\diamond_i a \vee \neg \diamond_i a = \top$) for every $i \in \text{Ag}$. Since R_i is upwards closed for every $i \in \text{Ag}$, it follows that $(R_i^{-1}[X\downarrow])\uparrow = R_i^{-1}[X\downarrow]$. Therefore, $R_i^{-1}[X\downarrow] \cup (S \setminus ((R_i^{-1}[X\downarrow])\uparrow)) = S$, i.e., axiom E holds in \mathbb{F}^+ , as required. \square

DEFINITION 6.7 (Epistemic Intuitionistic Kripke Model). An epistemic intuitionistic Kripke model is a tuple $\mathbb{M} = \langle \mathbb{F}, V \rangle$ such that \mathbb{F} is an epistemic intuitionistic Kripke frame and $V : \text{AtProp} \rightarrow \mathcal{P}^\downarrow(S)$.

COROLLARY 6.8. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, the i -minimal elements of \mathbb{F}^+ are exactly the equivalence cells of R_i .

PROOF. Recall (cf. Definition 4.2) that an element $a \in \mathbb{F}^+$ is i -minimal if

- (1) $a \neq \perp$,
- (2) $\diamond_i a = a$, and
- (3) if $b \in \mathbb{F}^+$, $b < a$, and $\diamond_i b = b$, then $b = \perp$.

Let $X \subseteq S$ be an R_i -equivalence cell of \mathbb{F} . Hence, X is a non-empty set, which proves item (1). Moreover, by definition of \diamond (see (6.4)), we have $\diamond_i X := R_i^{-1}[X] = X$, which proves item (2). Finally, if $\emptyset \neq Y \subseteq X$ then $\diamond_i Y = R_i^{-1}[Y] = X$, which proves item (3).

⁹see Definition 4.3, page 12.

Let $a \in \mathbb{F}^+ = \mathcal{P}^\downarrow(S)$ be an i -minimal element. To prove that a is an equivalence cell of R_i , we need to show that $a = R_i^{-1}[w]$ for some $w \in S$. By item (1), $a \neq \emptyset$; hence, there exists $w \in a$. Recall that $\diamond_i X := R_i^{-1}[X]$ (see (6.4)). By item (2), $a = \diamond_i a = R_i^{-1}[a]$; hence, a is the union of equivalence cells. By item (3), the only equivalence cell or union of equivalence cells smaller than a is the empty set; hence, a contains exactly one equivalence cell. \square

COROLLARY 6.9. *For every epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, and every join-prime element j of \mathbb{F}^+ , there exists some i -minimal element a such that $j \leq a$.*

PROOF. If j is a join-prime element of \mathbb{F}^+ , then $j = w \downarrow$ for some $w \in S$. Let $a = R_i^{-1}[w]$, which is an i -minimal element by Corollary 6.8. Since the equivalence relation R_i is upwards and downwards closed for every $i \in \text{Ag}$, we have $w \downarrow \subseteq R_i^{-1}[w]$, as required. \square

6.2 Epistemic Intuitionistic Kripke Frames and Probabilities

In this section, we define i -probability distributions. Applying ideas of Reference [23] to the setting of epistemic Heyting algebras, we define a correspondence between maps from epistemic intuitionistic Kripke frames to non-negative reals and premeasures on epistemic Heyting algebras (see Definition 4.6).

Definition 6.10 (i -Probability Distribution). Let $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$ be an epistemic intuitionistic Kripke frame. An i -probability distribution over S is a map $P_i : S \rightarrow]0, 1]$ such that $\sum_{w \in X} P_i(w) = 1$ for each equivalence cell X of R_i .

LEMMA 6.11. *For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, any map $f : S \rightarrow \mathbb{R}^+$ defines the i -premeasure f^+ on \mathbb{F}^+ as follows:*

$$\begin{aligned} f^+ : \text{Min}_i(\mathbb{A}) \downarrow &\rightarrow \mathbb{R}^+ \\ a &\mapsto \sum_{x \in a} P_i(x). \end{aligned} \tag{6.9}$$

Moreover, if f is an i -probability distribution, then the map f^+ is an i -measure (see Definition 4.6) on \mathbb{F}^+ .

PROOF. This result directly follows from the definition of f^+ and Corollary 6.8. \square

Definition 6.12. For any finite monadic Heyting algebra $\mathbb{A} = (\mathbb{L}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$ and any i -premeasure μ_i on \mathbb{A} , let

$$\begin{aligned} (\mu_i)_+ : \mathcal{J}(\mathbb{A}) &\rightarrow \mathbb{R}^+ \\ b &\mapsto \mu_i(b) - \mu_i \left(\bigvee_{c < b} c \right). \end{aligned} \tag{6.10}$$

It follows from the monotonicity of μ_i that $(\mu_i)_+$ is well defined.

LEMMA 6.13. *Let \mathbb{A} be an epistemic Heyting algebra equipped with an i -premeasure μ_i . Let the map $\eta : \mathbb{A} \rightarrow (\mathbb{A}_+)^+$ be the natural isomorphism (see Notation 6). Then, $((\mu_i)_+)^+(\eta(a)) = \mu_i(a)$ for every $a \in \mathbb{A}$.*

PROOF. Notice that, by definition,

$$\begin{aligned} ((\mu_i)_+)^+ \circ \eta : \text{Min}_i(\mathbb{A}) \downarrow &\rightarrow \mathbb{R}^+ \\ b &\mapsto \sum_{x \in b} (\mu_i)_+(x) = \sum_{x \in b} \left(\mu_i(x) - \mu_i \left(\bigvee_{c < x} c \right) \right). \end{aligned}$$

Since \mathbb{A} is a finite poset, we can define the *height* of its elements as follows: for every $a \in \mathbb{A}$,

$$h(a) := \begin{cases} 0 & \text{if } a = \perp \\ \max\{h(b) \mid b < a\} + 1 & \text{otherwise.} \end{cases}$$

Notice that the only element of height 0 is \perp . The proof will proceed by induction on the height of the elements of \mathbb{A} below the i -minimal elements.

As to the base case, it is immediate to see that $((\mu_i)_+)^+(\eta(\perp)) = ((\mu_i)_+)^+(\emptyset) = \mu_i(\perp) = 0$.

As to the induction step, assume that $\mu_i(a) = ((\mu_i)_+)^+(\eta(a))$ for all $a \in \text{Min}_i(\mathbb{A})$ such that $h(a) \leq n$. Now, let b be such that $h(b) = n + 1$. If b is a join prime element of \mathbb{A} , then $\eta(b) = b \downarrow$ and by definition $(\bigvee_{c < b} c) < b$. This implies that $h(\bigvee_{c < b} c) < h(b)$. Hence, by induction hypothesis,

$$\mu_i\left(\bigvee_{c < b} c\right) = ((\mu_i)_+)^+\left(\eta\left(\bigvee_{c < b} c\right)\right) = ((\mu_i)_+)^+(b \downarrow \setminus \{b\}).$$

Therefore,

$$\begin{aligned} ((\mu_i)_+)^+(b \downarrow) &= \sum_{x \in b \downarrow} ((\mu_i)_+)(x) \\ &= ((\mu_i)_+)(b) + \sum_{x \in b \downarrow \setminus \{b\}} ((\mu_i)_+)(x) \\ &= ((\mu_i)_+)(b) + ((\mu_i)_+)^+(b \downarrow \setminus \{b\}) \\ &= \mu_i(b) - \mu_i\left(\bigvee_{c < b} c\right) + ((\mu_i)_+)^+\left(\eta\left(\bigvee_{c < b} c\right)\right) \\ &= \mu_i(b). \end{aligned} \quad (\text{by induction hypothesis})$$

If b is not a join prime element, then it can be written as the union of elements strictly below it. Since both μ_i and $((\mu_i)_+)^+$ satisfy condition (3) of Definition 4.6 and have the same values on elements of height strictly smaller than $n + 1$, it follows that $\mu_i(b) = ((\mu_i)_+)^+(\eta(b))$. \square

COROLLARY 6.14. *Let \mathbb{A} be an epistemic Heyting algebra equipped with an i -measure $\mu_i : \text{Min}_i(\mathbb{A}) \downarrow \rightarrow \mathbb{R}^+$. Then the map*

$$(\mu_i)_+ : \mathcal{J}(\mathbb{A}) \rightarrow]0, 1] \quad (6.11)$$

$$a \mapsto \mu_i(a) - \mu_i\left(\bigvee_{b < a} b\right)$$

is an i -probability distribution over \mathbb{A}_+ .

PROOF. The map $(\mu_i)_+$ is well defined. Indeed, $(\mu_i)_+(b)$ is strictly positive for any $b \in \mathcal{J}(\mathbb{A})$, because μ_i is strictly monotone (see Definition 4.6, item 6) and $(\mu_i)_+(b) \leq 1$, because there exists an i -minimal element a such that $b \leq a$ (see Corollary 6.9) and because $\mu_i(a) = 1$ (see Definition 4.6, item 5). Lemma 6.13 implies that $1 = \mu_i(a) = ((\mu_i)_+)^+(a) = \sum_{x \in X} (\mu_i)_+(x)$ for every i -minimal element a , which shows that $(\mu_i)_+$ is an i -probability distribution over \mathbb{A}_+ , as required. \square

LEMMA 6.15. *Let \mathbb{F} be an epistemic intuitionistic Kripke frame equipped with a probability distribution P_i . Let the map $\epsilon : \mathbb{F} \rightarrow (\mathbb{F}^+)_+$ be the natural isomorphism (see Notation 6). Then, $((P_i)^+)_+(\epsilon(w)) = P_i(w)$ for every $w \in \mathbb{F}$.*

PROOF. For every join prime element $w\downarrow$ of \mathbb{F}^+ , we have that $v \in w\downarrow$ if and only if $v \leq w$. Thus, we obtain:

$$((P_i^+)^+(\epsilon(w))) = (P_i)^+(w\downarrow) - (P_i)^+\left(\bigvee_{b < w\downarrow} b\right) = \sum_{v \leq w} P_i(v) - \sum_{v < w} P_i(v) = P_i(w).$$

□

6.3 Dualizing the Product Updates of APE Structures

In this section, we introduce the generalization of the construction of the intermediate structure presented in Section 2.2 and show that it dualizes the intermediate construction on algebras presented in Section 4.4.

Definition 6.16 (Intermediate Intuitionistic Structure). For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} (see Definition 5.2), let the intermediate intuitionistic structure of \mathbb{M} and \mathcal{E} be the tuple:

$$\coprod_{\mathcal{E}} \mathbb{M} := \left\langle \coprod_{|E|} S, \leq^{\coprod}, (R_i^{\coprod})_{i \in \text{Ag}}, \llbracket \cdot \rrbracket_{\coprod} \right\rangle,$$

where

- $\coprod_{|E|} S \cong S \times E$ is the $|E|$ -fold coproduct of S ,
- the order relation \leq^{\coprod} on $\coprod_{|E|} S$ is defined as follows:

$$(s, e) \leq_i^{\coprod} (s', e') \quad \text{iff} \quad s \leq_i s' \text{ and } e = e',$$

- each binary relation R_i^{\coprod} on $\coprod_{|E|} S$ is defined as follows:

$$(s, e) R_i^{\coprod} (s', e') \quad \text{iff} \quad s R_i s' \text{ and } e \sim_i e',$$

- and the valuation $\llbracket \cdot \rrbracket_{\coprod} : \text{AtProp} \rightarrow \mathcal{P}S$ is defined by

$$\llbracket p \rrbracket_{\coprod} := \{(s, e) \mid s \in \llbracket p \rrbracket_{\mathbb{M}}\} = \llbracket p \rrbracket_{\mathbb{M}} \times E$$

for every $p \in \text{AtProp}$.

For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, \llbracket \cdot \rrbracket \rangle$, let

$$\coprod_{\mathcal{E}} \mathbb{F} := \left\langle \coprod_{|E|} S, \leq^{\coprod}, (R_i^{\coprod})_{i \in \text{Ag}} \right\rangle.$$

LEMMA 6.17. *Let $\mathbb{M} = \langle \mathbb{F}, \llbracket \cdot \rrbracket \rangle$ be an epistemic intuitionistic Kripke model. Then $(\coprod_{\mathcal{E}} \mathbb{F}, \llbracket \cdot \rrbracket_{\coprod})$ is also an epistemic intuitionistic Kripke model. Moreover, $(\coprod_{\mathcal{E}} \mathbb{F})^+ = \prod_{\mathcal{E}} (\mathbb{F}^+)$.*

PROOF. Given Reference [34, Fact 4.8], Lemmas 6.4 and 6.6, it remains to show that each R_i^{\coprod} is upwards closed. This follows from each R_i being upwards closed and the definition of \leq^{\coprod} . □

Definition 6.18. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, \llbracket \cdot \rrbracket \rangle$, any i -probability distribution P_i on \mathbb{F} (see Definition 6.10), and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} , let us define the function $P_i^{\coprod} : S \times E \rightarrow \mathbb{R}^+$ by recursion on the order \leq^{\coprod} as follows:

$$P_i^{\coprod}(w, e) = \left(\sum_{\varphi \in \Phi} P_i(e) \cdot P_i^\varphi(w) \cdot \text{pre}(e \mid \varphi) \right) - \sum_{v < w} P_i^{\coprod}(v, e), \quad (6.12)$$

where

$$P_i^\varphi(w) = \sum_{v \leq w} \{P_i(v) \mid \mathbb{M}, v \models \varphi \text{ and } \mathbb{M}, v \not\models \psi \text{ for all } \psi \in \text{mb}(\llbracket \varphi \rrbracket)\}. \quad (6.13)$$

Recall that $\text{mb}(a)$ denotes the multiset of the $<$ -maximal elements of Φ $<$ -below a (see Definition 4.15).

LEMMA 6.19. *For every \mathbb{M} , P_i and \mathcal{E} as in Definition 6.18 and for every $w \in S$,*

$$P_i^\varphi(w) = ((P_i)^+)^{\llbracket \varphi \rrbracket}(w \downarrow). \quad (6.14)$$

PROOF.

$$\begin{aligned} P_i^\varphi(w) &= \sum_{v \leq w} \{P_i(v) \mid \mathbb{M}, v \models \varphi \text{ and } \mathbb{M}, v \not\models \psi \text{ for all } \psi \in \text{mb}(\llbracket \varphi \rrbracket)\} \\ &= \sum_{v \leq w} \{P_i(v) \mid M, v \models \varphi\} - \sum_{v \leq w} \left\{ P_i(v) \mid M, v \models \bigvee_{\llbracket \psi \rrbracket \in \text{mb}(\llbracket \varphi \rrbracket)} \psi \right\} \\ &= (P_i)^+(w \downarrow \wedge \llbracket \varphi \rrbracket) - (P_i)^+\left(w \downarrow \wedge \bigvee_{\llbracket \psi \rrbracket \in \text{mb}(\llbracket \varphi \rrbracket)} \llbracket \psi \rrbracket\right) \quad (\text{see Lemma 6.11 and Equation (6.9)}) \\ &= ((P_i)^+)^{\llbracket \varphi \rrbracket}(w \downarrow). \quad (\text{see Definition 4.15 and Equation (4.2)}) \end{aligned}$$

□

LEMMA 6.20. *For every \mathbb{M} , P_i , and \mathcal{E} as in Definition 6.18,*

$$(P_i^{\text{II}})^+ = ((P_i)^+)'.$$

PROOF. Recall that

$$\begin{aligned} ((P_i)^+)' : \text{Min}_i \left(\prod_{\mathbb{E}} \mathbb{A} \right) \downarrow &\rightarrow \mathbb{R}^+ && (\text{see Definition 4.17}) \\ f &\mapsto \sum_{e \in E} \sum_{a \in \Phi} P_i(e) \cdot \mu_i^a(f(e)) \cdot \overline{\text{pre}}(e \mid a). \end{aligned}$$

By Lemmas 6.13 and 6.15, it is enough to show that $P_i^{\text{II}} = (((P_i)^+)'_+)$. We show this by induction on the well-founded order \leq^{II} .

The induction hypothesis, for an element $f \in \text{Min}_i(\prod_{\mathbb{E}} \mathbb{A}) \downarrow$ is

$$((P_i)^+)'(f) = (P_i^{\text{II}})^+(f) = \sum_{v < w} P_i^{\text{II}}(v, e). \quad (\text{IH}_f)$$

The case where $f = \perp$ is trivially true. Notice that the element $(w, e) \downarrow$ corresponds to the map $g_{(w, e)} : E \rightarrow S$ such that $g_{(w, e)}(e) = w \downarrow$ and $g_{(w, e)}(e') = \emptyset$ for every $e' \neq e$. Hence, we have:

$$\begin{aligned} ((P_i)^+)'(g_{(w, e)}) &= \sum_{\varphi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \varphi \rrbracket}(w \downarrow) \cdot \overline{\text{pre}}(e \mid \llbracket \varphi \rrbracket) \\ &= \sum_{\varphi \in \Phi} P_i(e) \cdot P_i^\varphi(w) \cdot \text{pre}(e \mid \varphi). \quad (\text{Lemma 6.19 and (4.4)}) \end{aligned}$$

Notice that

$$(((P_i)^+)'_+)((w, e)) = ((P_i)^+)'(g_{(w, e)}) - ((P_i)^+)'(f) \quad (\text{see Definition 6.12})$$

with $f(e) = w \downarrow \setminus \{w\}$ and $f(e') = \emptyset$ for $e' \neq e$.

Notice that $f < g$. Hence, by the induction hypothesis on f , we have

$$((P_i)^+)'(f) = (P_i^{\text{II}})^+(f) = \sum_{v < w} P_i^{\text{II}}(v, e).$$

Hence, we get

$$\begin{aligned} (((P_i)^+)'_+)((w, e)) &= ((P_i)^+)'(g_{(w, e)}) - \sum_{v < w} P_i^{\text{II}}(v, e) \\ &= \sum_{\varphi \in \Phi} P_i(e) \cdot P_i^\varphi(w) \cdot \text{pre}(e \mid \varphi) - \sum_{v < w} P_i^{\text{II}}(v, e) \\ &= P_i^{\text{II}}((w, e)). \end{aligned} \quad (\text{see Definition 6.18})$$

□

6.4 Dualizing the Updated APE Structures

In the present section, we introduce the generalization of the construction of the update model presented in Section 2.2 and show that it dualizes the construction of the updated APE structure presented in Section 4.5.

Definition 6.21. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = \langle E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub} \rangle$ over \mathcal{L} , let

$$\begin{aligned} \text{pre} : E &\rightarrow \mathcal{L} \\ e &\mapsto \bigvee \{ \phi \in \Phi \mid \text{pre}(e \mid \phi) \neq 0 \}. \end{aligned}$$

Definition 6.22 (Updated Intuitionistic Structure). For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \text{Ag}}, [\![\cdot]\!] \rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = \langle E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub} \rangle$ over \mathcal{L} (see Definition 6.10), let the updated intuitionistic structure of \mathbb{M} and \mathcal{E} be the tuple:

$$\mathbb{M}^{\mathcal{E}} := \langle S^{\mathcal{E}}, \leq^{\mathcal{E}}, (R_i^{\mathcal{E}})_{i \in \text{Ag}}, [\![\cdot]\!]^{\mathcal{E}} \rangle,$$

where

- $S^{\mathcal{E}} = \{(w, e) \in \coprod_{|E|} S \mid \mathbb{M}, w \models \text{pre}(e)\}$,
- $\leq^{\mathcal{E}} = \leq^{\text{II}} \cap (S^{\mathcal{E}} \times S^{\mathcal{E}})$,
- $R_i^{\mathcal{E}} = R_i^{\text{II}} \cap (S^{\mathcal{E}} \times S^{\mathcal{E}})$ for each $i \in \text{Ag}$,
- $[\![\cdot]\!]^{\mathcal{E}} : \text{AtProp} \rightarrow \mathcal{PS}$ is defined by

$$[\![p]\!]^{\mathcal{E}} := \{(w, e) \in S^{\mathcal{E}} \mid \mathbb{M}, w \models \text{sub}(e)(p)\}$$

for every $p \in \text{AtProp}$.

For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$, let

$$\mathbb{F}^{\mathcal{E}} := \langle S^{\mathcal{E}}, \leq^{\mathcal{E}}, (R_i^{\mathcal{E}})_{i \in \text{Ag}} \rangle.$$

LEMMA 6.23. *If $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$ is an epistemic intuitionistic Kripke model, then so is $\mathbb{M}_{\mathcal{E}}$. Moreover, $(F^{\mathcal{E}})^+ = (F^+)^{\mathbb{E}^{\mathcal{E}}}$.*

PROOF. It follows from Reference [34, Definition 4.7, Fact 4.8] and Lemma 6.17. \square

Definition 6.24. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \text{Ag}} \rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, \llbracket \cdot \rrbracket \rangle$, any i -probability distribution P_i on \mathbb{F} (see Definition 6.10), and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \text{pre}, \text{sub})$ over \mathcal{L} , the updated i -probability distribution $P_i^\mathcal{E} : S^\mathcal{E} \rightarrow]0, 1]$ is defined as follows:

$$P_i^\mathcal{E}(w, e) := \frac{P_i^{\text{II}}(w, e)}{\sum \{P_i^{\text{II}}(w', e') \mid (w', e')R_i^\mathcal{E}(w, e)\}}, \quad (6.15)$$

where P_i^{II} is as for Definition 6.18.

LEMMA 6.25. For every \mathbb{M} , P_i , and \mathcal{E} as in Definition 6.24,

$$(P_i^\mathcal{E})^+ = ((P_i)^+)^{\mathbb{E}_\mathcal{E}}.$$

PROOF. By Corollary 6.8 and Lemma 6.17, the i -minimal elements of $(\mathbb{M}^\mathcal{E})^+$ are the equivalence cells of R_i . Now, let $g \in (\mathbb{M}^\mathcal{E})^+$, f the i -minimal element above g , and $(w, e) \in g$. By Lemma 6.20, $\sum \{P_i^{\text{II}}(w', e') \mid (w', e')R_i^\mathcal{E}(w, e)\} = (P_i^{\text{II}})^+(f)$ and $\sum_{(w', e') \in g} P_i^{\text{II}}(w', e') = (P_i^{\text{II}})^+(g)$. Therefore:

$$\begin{aligned} ((P_i)^+)^{\mathbb{E}_\mathcal{E}}(g) &= \frac{(P_i^{\text{II}})^+(g)}{(P_i^{\text{II}})^+(f)} \\ &= \frac{\sum_{(w', e') \in g} P_i^{\text{II}}(w', e')}{\sum \{P_i^{\text{II}}(w', e') \mid (w', e')R_i^\mathcal{E}(w, e)\}} \\ &= \sum_{(w', e') \in g} \frac{P_i^{\text{II}}(w', e')}{\sum \{P_i^{\text{II}}(w', e') \mid (w', e')R_i^\mathcal{E}(w, e)\}} \\ &= \sum_{(w', e') \in g} P_i^\mathcal{E}(w, e) \\ &= (P_i^\mathcal{E})^+(g). \quad \square \end{aligned}$$

6.5 Relational Semantics for IPDEL

Definition 6.26. An IPDEL-model is a structure $\mathbb{N} = \langle \mathbb{M}, (P_i)_{i \in \text{Ag}} \rangle$ such that $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$ is an epistemic intuitionistic Kripke model, and P_i is a probability distribution over S for every $i \in \text{Ag}$. For every IPDEL-model \mathbb{N} and every event structure \mathcal{E} , we let $\mathbb{N}^\mathcal{E} = \langle \mathbb{M}^\mathcal{E}, (P_i^\mathcal{E})_{i \in \text{Ag}} \rangle$ (cf. Definitions 6.22 and 6.24).

It can be verified straightforwardly that for every IPDEL-model \mathbb{N} and every event structure \mathcal{E} , the structure $\mathbb{N}^\mathcal{E}$ is an IPDEL-model.

Definition 6.27 (Semantics of IPDEL). For every IPDEL-model $\mathbb{N} = \langle \mathbb{M}, (P_i)_{i \in \text{Ag}} \rangle$ where $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \text{Ag}}, \llbracket \cdot \rrbracket \rangle$, the IPDEL-formulas are interpreted on \mathbb{N} as follows:

$$\begin{aligned}
 \mathbb{N}, s \models \perp & \text{ iff } \textit{never} \\
 \mathbb{N}, s \models p & \text{ iff } s \in \llbracket p \rrbracket \\
 \mathbb{N}, s \models \phi \wedge \psi & \text{ iff } \mathbb{N}, s \models \phi \text{ and } \mathbb{N}, s \models \psi \\
 \mathbb{N}, s \models \phi \vee \psi & \text{ iff } \mathbb{N}, s \models \phi \text{ or } \mathbb{N}, s \models \psi \\
 \mathbb{N}, s \models \phi \rightarrow \psi & \text{ iff } \mathbb{N}, s' \models \phi \text{ implies } \mathbb{N}, s' \models \psi \text{ for every } s' \leq s \\
 \mathbb{N}, s \models \Diamond_i \phi & \text{ iff there exists } s' R_i s \text{ such that } \mathbb{N}, s' \models \phi \\
 \mathbb{N}, s \models \Box_i \phi & \text{ iff } \mathbb{N}, s' \models \phi \text{ for all } s' (\geq \circ R_i) s \\
 \mathbb{N}, s \models \langle \mathcal{E}, e \rangle \phi & \text{ iff } \mathbb{N}, s \models \textit{pre}(e) \text{ and } \mathbb{N}^{\mathcal{E}}, (s, e) \models \phi \\
 \mathbb{N}, s \models [\mathcal{E}, e] \phi & \text{ iff } \mathbb{N}, s \models \textit{pre}(e) \text{ implies } \mathbb{N}^{\mathcal{E}}, (s, e) \models \phi \\
 \mathbb{N}, s \models \left(\sum_{k=1}^n \alpha_k \mu_i(\varphi) \right) \geq \beta & \text{ iff } \sum_{k=1}^n \alpha_k (P_i)^+ (\llbracket \varphi \rrbracket \cap R_i[s]) \geq \beta.
 \end{aligned}$$

Recalling that in epistemic intuitionistic Kripke frames, and hence on IPDEL-models, the relations R_i are both upwards and downwards closed, this implies that the seventh clause in the definition above can be simplified as follows:

$$\mathbb{N}, s \models \Box_i \phi \text{ iff } \mathbb{N}, s' \models \phi \text{ for all } s' R_i s.$$

7 CASE STUDY: DECISION-MAKING UNDER UNCERTAINTY

In the present section, we illustrate the relational semantic update process described in Section 6 by means of a case study that involves the assessment of the likelihood of a socially constructed event (a bankruptcy), taking place at some point in the future.

The focal feature of the case study is that this assessment depends to a greater extent on the actions, beliefs, and expectations of the agents than on factual information.

In what follows, we first present the case study informally, and then we introduce a simplified formalization of the problem using probabilistic epistemic intuitionistic Kripke models and probabilistic intuitionistic epistemic event structures.

7.1 Informal Presentation

Around 1950, there was a small businessman w in Amsterdam whose main business was to sell the products of foreign textile manufacturers to Dutch clothing firms. Like most small businessmen in Amsterdam at the time, he banked with the Amsterdamsche Bank (which later became the present ABN AMRO).

One day, w received an invitation to lunch with one of the directors of that bank. This invitation puzzled him a great deal, because he did not know this director personally, and a small businessman like him usually only dealt with bank employees at much lower levels. However, he accepted the invitation and showed up for the lunch at the top floor of the bank's headquarters in the city center.

During the copious lunch, the bank director talked about all kinds of general subjects and asked w 's opinion about the economic climate in Amsterdam. Rather than being flattered, w found it hard to imagine he was invited to provide opinions about matters the bank knew better than he. When the dessert was served, the banker mentioned aside some other matter the name of a certain Amsterdam firm f , which was an important client of w . This firm, the bank director said, was doing very well under the present solid leadership.

The small businessman realized that this must have been the point of the whole lunch. And if this large bank went to so much effort to increase the confidence of one small businessman in this firm, it must have been very important to the bank that w believed that f was doing well.

The small businessman said he wanted to wash his hands, although coffee still needed to be served, but instead of walking to the bathroom he ran down the stairs and onto the street to find a telephone booth and call the office to stop all deliveries to f and also claim back any supplies that had already been delivered.

Two weeks later, f went bankrupt and it turned out that the bank not only was its major creditor but also had preferential right to sell off any stocks in the possession of f to pay back the debt to the bank before other creditors would be satisfied.

7.2 Analysis of the Situation

Let B_f be the following proposition:

“Firm f will bankrupt within a month.”

Notice that, while being two-valued, intuitionistic logic allows for B_f to be either true, false, or undecided in a model, and the availability of the third option seems to adequately reflect real-life situations. Indeed, there is a strict judicial procedure that establishes the truth of B_f , and when this procedure is not (yet) in place, it seems reasonable to not assign it a truth value.

Accordingly, the sum of the probability attributed to B_f by w and the probability attributed to $\neg B_f$ by w does not need to be 1.

For simplicity, we regard everything that happened from the invitation to the banker’s utterance about firm f as one single event. We also propose that the uncertainty of w concerns how to interpret this event, and very much simplifying this story, the two mutually inconsistent interpretations of this event are

e_1 : *“The banker is trying to manipulate my opinions.”*

e_2 : *“The banker only wants to exchange information.”*

The uncertainty of w about how to interpret the event is encoded in the shape of the event structure, which consists of two states, corresponding to e_1 and e_2 above, respectively, to each of which w assigns his (subjective) probability.

For the sake of illustrating how the substitution map works and to simplify the subsequent treatment, we also include the following atomic proposition M in our language, the intended meaning of which is:

“The banker is manipulative.”

7.3 Formalization: Initial Model and Event Structure

Let the set of atomic propositions be $\text{AtProp} := \{B_f, M\}$ as discussed above.

Initial Model. In the formalization discussed below, we only consider the viewpoint of agent w ; hence, in the model and the event structure, we specify only the subjective probabilities of agent w . The initial model is

$$\mathbb{M} := \langle S, \leq, \sim_w, P_w, \llbracket \cdot \rrbracket \rangle$$

with:

- $S := \{s_0, s_1, s_2\}$,
- $\leq := \{(s, s) \mid s \in S\} \cup \{(s_1, s_0), (s_2, s_0)\}$,
- $\sim_w := S \times S$,
- $P_w : S \rightarrow]0, 1]$ with

$$P_w(s_0) := 0.1, \quad P_w(s_1) := 0.1, \quad P_w(s_2) := 0.8,$$

- $\llbracket \cdot \rrbracket : \text{AtProp} \rightarrow \mathcal{P}S$ is such that $\llbracket B_f \rrbracket := \{s_1\}$ and $\llbracket M \rrbracket := \perp$.

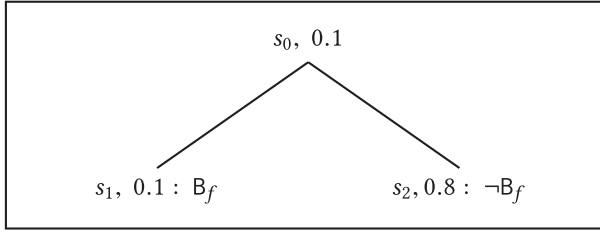

 Fig. 2. Initial model \mathbb{M} .

 Fig. 3. Event structure \mathbb{E} .

This model represents a situation in which w has no additional information about the financial health of firm f . Hence, we assume that the probability assigned by w to each state of the model reflects the average risk of bankruptcy of firms in that industry during that period. For w to be willing to do business with f , it is not just enough that f does not have a higher probability of bankruptcy than the average firm, but also the probability of being in an uncertain state should be low. The model \mathbb{M} is drawn in Figure 2.

Event Structure. We consider the following pointed event structure:

$$(\mathcal{E}, e_1) := (E, \sim_w, P_w, \Phi, \text{pre}, \text{sub}),$$

where

- $E := \{e_1, e_2\}$,
- $\sim_w := E \times E$,
- $P_w(e_1) = 0.95$ and $P_w(e_2) = 0.05$,
- $\Phi = \{\top, B_f, \neg B_f\}$,
- $\text{pre} : E \times \Phi \rightarrow [0, 1]$ is given in Figure 4.
- the definition of the map $\text{sub} : E \times \{\mathbb{M}\} \rightarrow \mathcal{L}$ is given in Figure 5,

where e_1 and e_2 correspond to the two interpretations of the event discussed in the previous section (that is, the probability w assigns to e_1 is 19.8 times the probability w assigns to e_2). The event structure \mathbb{E} is partially represented in Figure 3.

By stipulating that $P_w(e_1) = 0.95$ and $P_w(e_2) = 0.05$, we indicate that w believes that it is far more likely that the banker is trying to manipulate his opinion on f .

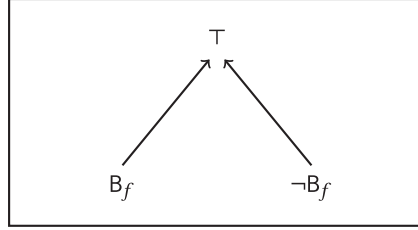
The map pre provides the objective probability $\text{pre}(e \mid \phi)$ of each event $e \in E$ happening when one assumes that the formula $\phi \in \Phi$ holds. Each line of Figure 4 gives the probability distribution $\text{pre}(\bullet \mid \phi) : E \times [0, 1]$ for each $\phi \in \Phi$. The values in Figure 4 are based on the following assumptions:

- If we consider the row where $\phi = \top$, which corresponds to the state in which the bankruptcy of f is undetermined, it is reasonable to assume that the probability of e_1 , namely the banker trying to manipulate w 's opinion on f , is significantly higher than that of e_2 .
- If we consider the row where $\phi = B_f$, which corresponds to the state in which f is going to be bankrupt within a month, it is reasonable to regard e_1 as almost certain.

	e_1	e_2
\top	0.8	0.2
B_f	0.99	0.01
$\neg B_f$	0.05	0.95

Fig. 4. The map pre .

	e_1	e_2
M	\top	\perp

Fig. 5. The map sub .Fig. 6. The partial order given by (Φ, \rightarrow) .

- If we consider the row where $\phi = \neg B_f$, which corresponds to the state in which f is financially healthy, then it is reasonable to assign a very low probability to the event in which the banker wants to manipulate w 's opinion about f , since the banker has nothing to gain from it.

Remark 13. The poset Φ ordered by logical implication is a tree and is drawn in Figure 6.

7.4 Updated Model

In this section, we show how the initial model described in the section above is updated with the event structure. The updated model

$$\mathbb{M}^{(\mathcal{E}, e_1)} := \langle S', \leq', \sim'_w, P'_w, \llbracket \cdot \rrbracket' \rangle$$

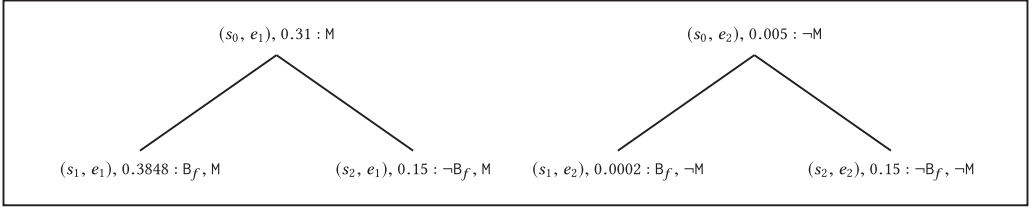
is defined as follows:

- $S' := S \times E$,
- $(s, e) \leq' (s', e')$ iff $s \leq s'$ and $e = e'$ for all $(s, e), (s', e') \in S'$,
- $(s, e) \sim'_w (s', e')$ iff $s \sim_w s'$ and $e \sim_w e'$ for all $(s, e), (s', e') \in S'$,
- the map P'_w is shown in Figure 7, where the actual values are rounded off,
- the map $\llbracket \cdot \rrbracket' : \text{AtProp} \rightarrow \mathcal{P}S'$ is defined as follows:

$$\begin{aligned} \llbracket B_f \rrbracket' &:= \llbracket B_f \rrbracket \times E; \\ \llbracket M \rrbracket' &:= (\llbracket \text{sub}(e_1, M) \rrbracket \times \{e_1\}) \cup (\llbracket \text{sub}(e_2, M) \rrbracket \times \{e_2\}) \\ &= \{(s_0, e_1), (s_1, e_1), (s_2, e_1)\}. \end{aligned}$$

The updated model $\mathbb{M}^{(\mathcal{E}, e_1)}$ is drawn in Figure 7.

As expected, the fact that w assigns a much greater probability to e_1 than e_2 implies that the probabilistic weight of the model above is concentrated among the three leftmost states. Of these three states, the weight is shared almost equally between the two in which B_f is either true or undecided, which reverses the subjective probability assigned in the initial model. This reversal captures w 's decision to abruptly stop all deliveries to f .


 Fig. 7. Updated model $\mathbb{M}^{\mathcal{E}}$.

7.5 Syntactic Inference of a Formal Condition about the Case Study

In the present section, we use the Hilbert style presentation of IPDEL to derive the logical equivalence (7.1). The right-hand side is a dynamic formula expressing that after the event, w reverts his opinion and assigns greater probability to the bankruptcy of f (and to the banker's manipulative intentions). The left-hand side of this equivalence describes the threshold of reasonable optimism that enables w to revert his opinion after the event. Namely, for the event (\mathcal{E}, e_1) to revert w' opinion, the ratio between the probability that w assigns to $\neg B_f$ and to B_f needs to be smaller than the value (19.8) of the ratio between the probability w assigns to e_1 and to e_2 .

PROPOSITION 7.1. *The formula*

$$(19.8\mu_w(B_f) > \mu_w(\neg B_f)) \leftrightarrow [\mathcal{E}, e_1](\mu_w(M \wedge B_f) > \mu_w(\neg M \wedge \neg B_f)), \quad (7.1)$$

where $\alpha\mu_i(\varphi) > \beta\mu_i(\psi)$ is shorthand for $(\beta\mu_i(\psi) \geq \alpha\mu_i(\varphi)) \rightarrow \perp$, is derivable in IPDEL.

PROOF. To show the equivalence (7.1), we will use the IPDEL axioms to equivalently rewrite its right-hand side into its left-hand side.

$$\begin{aligned} & [\mathcal{E}, e_1](\mu_w(M \wedge B_f) > \mu_w(\neg M \wedge \neg B_f)) \\ \text{iff} & \quad [\mathcal{E}, e_1] \left((\mu_w(\neg M \wedge \neg B_f) \geq \mu_w(M \wedge B_f)) \rightarrow \perp \right) && \text{(notation for } > \text{)} \\ \text{iff} & \quad \langle \mathcal{E}, e_1 \rangle (\mu_w(\neg M \wedge \neg B_f) \geq \mu_w(M \wedge B_f)) \rightarrow \langle \mathcal{E}, e_1 \rangle \perp && \text{(I11 in Table 2)} \\ \text{iff} & \quad \langle \mathcal{E}, e_1 \rangle (\mu_w(\neg M \wedge \neg B_f) \geq \mu_w(M \wedge B_f)) \rightarrow \perp. && \text{(I6 in Table 2)} \end{aligned}$$

In what follows, we focus on equivalently rewriting the antecedent of the implication above.

$$\begin{aligned} & \langle \mathcal{E}, e_1 \rangle (\mu_w(\neg M \wedge \neg B_f) \geq \mu_w(M \wedge B_f)) \\ \text{iff} & \quad \langle \mathcal{E}, e_1 \rangle (\mu_w(\neg M \wedge \neg B_f) - \mu_w(M \wedge B_f) \geq 0) \\ \text{iff} & \quad \sum_{\substack{e' \in \mathcal{E} \\ \phi \in \Phi}} P_w(e') \cdot \text{pre}(e' \mid \phi) \cdot \mu_w^\phi(\langle \mathcal{E}, e' \rangle (\neg M \wedge \neg B_f)) \\ & \quad - \sum_{\substack{e' \in \mathcal{E} \\ \phi \in \Phi}} P_w(e') \cdot \text{pre}(e' \mid \phi) \cdot \mu_w^\phi(\langle \mathcal{E}, e' \rangle (M \wedge B_f)) \geq 0 && \text{(I18 in Table 2)} \end{aligned}$$

$$\begin{aligned}
& \text{iff } \sum_{\substack{e' \in E \\ \phi \in \Phi}} P_w(e') \cdot \text{pre}(e' \mid \phi) \cdot \mu_w^\phi(\langle \mathcal{E}, e' \rangle (\neg M \wedge \neg B_f)) \\
& \quad \geq \sum_{\substack{e' \in E \\ \phi \in \Phi}} P_w(e') \cdot \text{pre}(e' \mid \phi) \cdot \mu_w^\phi(\langle \mathcal{E}, e' \rangle (M \wedge B_f)) \\
& \text{iff } P_w(e_2) \cdot \text{pre}(e_2 \mid \neg B_f) \cdot \mu_w(\neg B_f) \geq P_w(e_1) \cdot \text{pre}(e_1 \mid B_f) \cdot \mu_w(B_f) && \text{(by Lemma 7.2)} \\
& \text{iff } 0.05 \cdot 0.95 \cdot \mu_w(\neg B_f) \geq 0.95 \cdot 0.99 \cdot \mu_w(B_f) && \text{(Definition of } (\mathcal{E}, e_1)) \\
& \text{iff } 0.05 \cdot \mu_w(\neg B_f) \geq 0.99 \cdot \mu_w(B_f) && \text{(by Lemma C.4)} \\
& \text{iff } \mu_w(\neg B_f) \geq 19.8 \mu_w(B_f). && \text{(by Lemma C.4)}
\end{aligned}$$

Hence,

$$\begin{aligned}
& \langle \mathcal{E}, e_1 \rangle (\mu_w(\neg M \wedge \neg B_f) \geq \mu_w(M \wedge B_f)) \rightarrow \perp \\
& \text{iff } (\mu_w(\neg B_f) \geq 19.8 \mu_w(B_f)) \rightarrow \perp \\
& \text{iff } 19.8 \mu_w(B_f) > \mu_w(\neg B_f),
\end{aligned}$$

as required. \square

LEMMA 7.2. *The following propositions are provable in IPDEL.*

- (1) $\langle \mathcal{E}, e_1 \rangle (M \wedge B_f) \leftrightarrow B_f$ and $\langle \mathcal{E}, e_1 \rangle (\neg M \wedge \neg B_f) \leftrightarrow \perp$;
- (2) $\langle \mathcal{E}, e_2 \rangle (\neg M \wedge \neg B_f) \leftrightarrow \neg B_f$ and $\langle \mathcal{E}, e_2 \rangle (M \wedge B_f) \leftrightarrow \perp$;
- (3) $\mu_w^\top(B_f) = 0$ and $\mu_w^\top(\neg B_f) = 0$;
- (4) $\mu_w^{B_f}(\neg B_f) = 0$ and $\mu_w^{\neg B_f}(B_f) = 0$;
- (5) $\mu_w^{B_f}(B_f) = \mu_w(B_f)$ and $\mu_w^{\neg B_f}(\neg B_f) = \mu_w(\neg B_f)$.

PROOF. Proof of item (1).

$$\begin{aligned}
& \langle \mathcal{E}, e_1 \rangle (M \wedge B_f) \\
& \text{iff } \langle \mathcal{E}, e_1 \rangle M \wedge \langle \mathcal{E}, e_1 \rangle B_f && \text{(I8 in Table 2)} \\
& \text{iff } \text{pre}(e_1) \wedge \mathbf{sub}(e_1, M) \wedge \text{pre}(e_1) \wedge \mathbf{sub}(e_1, B_f) && \text{(I2 in Table 2)} \\
& \text{iff } \mathbf{sub}(e_1, M) \wedge \mathbf{sub}(e_1, B_f) && (\text{pre}(e_1) \text{ is } \top \vee B_f \vee \neg B_f) \\
& \text{iff } \top \wedge B_f && \text{(Definition of sub)} \\
& \text{iff } B_f
\end{aligned}$$

and

$$\begin{aligned}
& \langle \mathcal{E}, e_1 \rangle (\neg M \wedge \neg B_f) \\
& \text{iff } \langle \mathcal{E}, e_1 \rangle \neg M \wedge \langle \mathcal{E}, e_1 \rangle \neg B_f && \text{(I8 in Table 2)} \\
& \text{iff } (\text{pre}(e_1) \wedge \neg \langle \mathcal{E}, e_1 \rangle M) \wedge (\text{pre}(e_1) \wedge \neg \langle \mathcal{E}, e_1 \rangle B_f) && \text{(I12 and I6 in Table 2)} \\
& \text{iff } \text{pre}(e_1) \wedge \neg(\text{pre}(e_1) \wedge \mathbf{sub}(e_1, M)) \wedge \text{pre}(e_1) \wedge \neg(\text{pre}(e_1) \wedge \mathbf{sub}(e_1, B_f)) && \text{(I2 in Table 2)} \\
& \text{iff } \neg \mathbf{sub}(e_1, M) \wedge \neg \mathbf{sub}(e_1, B_f) && (\text{pre}(e_1) \text{ is } \top \vee B_f \vee \neg B_f) \\
& \text{iff } \neg \top \wedge \neg B_f && \text{(Definition of sub)} \\
& \text{iff } \perp. && (\neg \top \leftrightarrow \perp)
\end{aligned}$$

Proof of item (2). The proof is similar to that of item (1).

Proof of item (3). Notice that $\mu_w^\top(B_f)$ is shorthand for $\mu_w(\top \wedge B_f) - (\mu_w(B_f \wedge B_f) + \mu_w(\neg B_f \wedge B_f))$ (cf. Definition 4.15). Therefore:

$$\begin{aligned} & \mu_w^\top(B_f) = 0 \\ \text{iff } & \mu_w(\top \wedge B_f) - (\mu_w(B_f \wedge B_f) + \mu_w(\neg B_f \wedge B_f)) = 0 \\ \text{iff } & \mu_w(\top \wedge B_f) - \mu_w(B_f \wedge B_f) = 0 && \text{(P1 in Table 2 and Lemma C.4)} \\ \text{iff } & \mu_w(B_f) - \mu_w(B_f) = 0. \end{aligned}$$

The last equality follows by N0 in Table 2. The proof of the second inequality is similar.

Proof of item (4). Notice that $\mu_w^{B_f}(\neg B_f)$ is shorthand for $\mu_w(B_f \wedge \neg B_f)$ and $\mu_w^{\neg B_f}(B_f)$ is shorthand for $\mu_w(\neg B_f \wedge B_f)$. Hence, the equality follows from Axiom P1 in Table 2.

Proof of item (5). Notice that $\mu_w^{B_f}(B_f)$ is shorthand for $\mu_w(B_f \wedge B_f)$ and $\mu_w^{\neg B_f}(\neg B_f)$ is shorthand for $\mu_w(\neg B_f \wedge \neg B_f)$. Hence, the required equality is straightforwardly true. \square

Since, as discussed in Section 6, IPDEL is sound and complete with respect to the class of relational models, Proposition 7.1 implies that every IPDEL model \mathcal{M} that supports the left-hand side of the equivalence (7.1) will be updated by the event (E, e_1) to a model that satisfies $\mu_w(\mathcal{M} \wedge B_f) > \mu_w(\neg \mathcal{M} \wedge \neg B_f)$. Hence, in each such model, agent w will update his subjective probabilities concerning B_f analogously to the model in the example above (see Section 7.4 and Figure 7).

8 CONCLUSION

Present Contributions. In this article, we have introduced the logic IPDEL, the intuitionistic counterpart of classical PDEL, as an instance of a general methodology based on the mathematical construction of updates on algebras, which makes it possible to define non-classical counterparts of DEL-type logics on different propositional bases. This methodology makes it possible to also obtain the update construction on relational and topological models via appropriate (extended) dualities, and hence define relational semantics for the defined logics. In this way, we have shown that IPDEL—which is sound by construction with respect to the class of algebraic probabilistic epistemic models (cf. Definition 5.3)—is also complete with respect to APE-models, and hence also with respect to their dual relational structures. Since these structures are finite by definition, this result immediately implies that IPDEL has the finite model property. The logic IPDEL is intended as a tool to analyze decision-making under uncertainty in situations in which truth is socially constructed, and hence decisions are taken in contexts in which the truth value of certain states of affair might be undetermined. To show IPDEL at work, we partially formalize one such situation.

Generalizing APE-structures. APE-structures are based on epistemic Heyting algebras (cf. Definition 4.3), the definition of which requires the image of each diamond operator to have a Boolean algebra structure. Thus, epistemic Heyting algebras are a proper subclass of monadic Heyting algebras. This additional condition guarantees that the i -minimal elements induce a partition on the dual structure of each epistemic Heyting algebra, and hence that axioms such as $\mu(\top) = 1$ or $(\mu(\varphi) \geq \alpha) \vee (\mu(\varphi) < \alpha)$ are valid. One natural question that presents itself is whether this condition can be dropped, and hence APE-structures can be based on general monadic Heyting algebras. Addressing this question requires solving issues of technical and conceptual nature. On the technical side, the additional requirement plays a role in the completeness theorem, and specifically makes sure that—in the finite lattice that we extract from the Lindenbaum-Tarski algebra—a sublattice can be defined out of the image of each diamond (cf. Lemma C.3). This issue would partially be addressed by relaxing the condition that APE-structures be finite (see paragraph below). On the conceptual side, we would need to restructure the definition of probabilistic measure. The axiom

$(\mu(\varphi) \geq \alpha) \vee (\mu(\varphi) < \alpha)$ is tightly linked to the metatheory of the real numbers and in particular to the validity of trichotomy. Hence, in the context of a different metatheory in which trichotomy does not hold such as the constructive metatheory of real numbers, it seems reasonable that this axiom might be dropped. However, the condition $\mu(\top) = 1$ expresses the link between probability and the underlying logic. For this reason, this axiom should arguably be kept.

Finite to Infinite Models. Another natural question is whether we can drop the condition that APE-structures be finite. A first step would be to investigate the case of APE-structures based on perfect Heyting algebras, i.e., those Heyting algebras that are isomorphic to algebras of up-sets or down-sets of given posets. Does every probability measure on such a Heyting algebra correspond to a discrete probability distribution on the corresponding dual poset? More generally, possibly infinite APE-structures would dually correspond to relational Esakia spaces endowed with probability distributions. Are there purely algebraic conditions on probability measures guaranteeing that the corresponding probability distribution be discrete?

Proof Theory for Probabilistic Logics. As mentioned in the introduction, the present article pertains to a line of research aimed at studying the phenomenon of dynamic (probabilistic epistemic) updates in contexts at odds with classical truth. The language and semantics of the formal settings previously studied (i.e., those of the nonclassical versions of PAL and EAK) have served as a basis for a research program in structural proof theory aimed at developing a uniform methodology for endowing dynamic logics with so-called *analytic calculi* (see References [15, 30]). This research program has successfully addressed PAL and DEL [25–27, 29], and PDL [24], and has been further generalized into the proof-theoretic framework of *multi-type calculi* [25]. This methodology has been successfully deployed to introduce analytic calculi for logics particularly impervious to the standard treatment [13, 28, 31, 32] and is now ready to be applied to the issue of endowing PDEL and its non-classical versions with analytic calculi.

APPENDICES

A PROOFS OF SECTION 4

Proof of Lemma 4.8

Lemma 4.8. For any PES-model \mathbb{M} , the i -minimal elements of its complex algebra \mathbb{M}^+ are exactly the equivalence classes of \sim_i .

PROOF. Let $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, [\cdot] \rangle$ be a PES-model and $\mathbb{M}^+ = (\mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\Box_i)_{i \in \text{Ag}}, (P_i^+)_{i \in \text{Ag}})$ be its complex algebra. For any $i \in \text{Ag}$ and any $s \in S$, let $[s]_i$ be the \sim_i -equivalence cell of s . Fix $i \in \text{Ag}$.

First, let us prove that any \sim_i -equivalence cell corresponds to an i -minimal element of \mathbb{M}^+ . Since \sim_i is reflexive, $[s]_i \neq \emptyset$. Since \sim_i is symmetric and transitive, $[s]_i = \diamond_i\{s\} = \diamond_i\diamond_i\{s\} = \diamond_i[s]_i$. This shows that $[s]_i$ is a fixed-point of \diamond_i . It remains to show that $[s]_i$ is a minimal fixed-point \diamond_i . Let $X \subseteq S$ be an i -minimal element of \mathbb{M}^+ . By definition, we have that $X \subseteq [s]_i$, $X \neq \emptyset$ and $\diamond_i X = X$. The assumption that $\diamond_i X = X$ implies that $X = \bigcup_{x \in X} \diamond_i\{x\} = \bigcup_{x \in X} [x]_i$. The assumption that $X \subseteq [s]_i$ implies that all $x \in X$ must be \sim_i -equivalent to s , and hence to each other. Therefore, X cannot be the union of more than one equivalence cell. Moreover, the assumption that $X \neq \emptyset$ implies that there exists at least one equivalence cell in $\bigcup_{x \in X} [x]_i$. This concludes the proof that, for any $s \in S$, its \sim_i -equivalence cell $[s]_i$ corresponds to an i -minimal element of \mathbb{M}^+ , as required.

Now, let us prove that any i -minimal element of \mathbb{M}^+ corresponds to the \sim_i -equivalence cell of an element $s \in S$. Let X be an i -minimal element of \mathbb{M}^+ . The assumption that $X = \diamond_i X$ implies that $X = \bigcup_{x \in X} [x]_i$. The assumption that $X \neq \emptyset$ implies that there exists at least one equivalence cell $[s]_i$ in $\bigcup_{x \in X} [x]_i$. Since $[s]_i$ is an i -minimal element of \mathbb{M}^+ and $[s]_i \subseteq X$, we have $X = [s]_i$ by minimality of X . \square

Proof of Proposition 4.9

Proposition 4.9. For any PES-model \mathbb{M} , its complex algebra \mathbb{M}^+ (see Definition 3.1) is an APE-structure (see Definition 4.7).

PROOF. Let $\mathbb{M} = \langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, [\cdot] \rangle$ be a PES-model (see Definition 2.2) and let $\mathbb{M}^+ = \langle \mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}}, (P_i^+)_{i \in \text{Ag}} \rangle$ be its complex algebra. \mathbb{M}^+ is an APE-structure if its support is an epistemic Heyting algebra and if each P_i^+ is an i -measure over $\langle S, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}} \rangle$. Clearly, $(\mathcal{P}S, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}})$ is an epistemic Heyting algebra (see Definition 4.3), since \sim_i is an equivalence relation and $\mathcal{P}S$ is a Boolean algebra. To finish the proof, we need to show that each P_i^+ is an i -measure on $\text{support}(\mathbb{M}^+)$. Hence, for every $i \in \text{Ag}$, we need to prove the following properties:

- (a) $\text{dom}(P_i^+) = \text{Min}_i(\text{support}(\mathbb{M}^+))\downarrow$;
- (b) P_i^+ is order-preserving;
- (c) for every i -minimal element $X \in \mathcal{P}S$ and all $Y_1, Y_2 \in X\downarrow$, we have

$$P_i^+(Y_1 \cup Y_2) = P_i^+(Y_1) + P_i^+(Y_2) - P_i^+(Y_1 \cap Y_2);$$

- (d) $P_i^+(\emptyset) = 0$ if $\text{dom}(P_i^+) \neq \emptyset$;
- (e) for every i -minimal element $X \in \mathcal{P}S$, we have $P_i^+(X) = 1$.
- (f) for every i -minimal element $X \in \mathcal{P}S$ and all $Y_1, Y_2 \in X\downarrow$ such that $Y_1 \subset Y_2$, it holds that

$$P_i^+(b) < P_i^+(c).$$

Fix $i \in \text{Ag}$.

Proof of (a). By definition, $\text{dom}(P_i^+) = \{X \in \mathcal{P}S \mid \exists y \forall x (x \in X \Rightarrow x \sim_i y)\}$. Notice that

$$\{X \in \mathcal{P}S \mid \exists y \forall x (x \in X \Rightarrow x \sim_i y)\} = \{X \mid X \subseteq [s] \text{ and } s \in S\}.$$

By Lemma 4.8, we deduce that $\text{dom}(P_i^+) = \text{Min}_i(\text{support}(\mathbb{M}^+))\downarrow$.

Proof of (b). Since $P_i(s) \geq 0$ for all $s \in S$, the maps P_i^+ are monotone.

Proof of (c). By Lemma 4.8, if X is an i -minimal element of \mathbb{M}^+ , then $X = [s]$ for some $s \in S$. If $Y_1, Y_2 \in X\downarrow$, then $Y_1 \cup Y_2 \subseteq [s]$. Hence,

$$\begin{aligned} P_i^+(Y_1 \cup Y_2) &= \sum_{x \in Y_1 \cup Y_2} P_i(x) && \text{(Definition of } P_i^+) \\ &= \sum_{x \in Y_1} P_i(x) + \sum_{x \in Y_2} P_i(x) - \sum_{x \in Y_1 \cap Y_2} P_i(x) \\ &= P_i^+(Y_1) + P_i^+(Y_2) - P_i^+(Y_1 \cap Y_2). && \text{(Definition of } P_i^+) \end{aligned}$$

Proof of (d). By definition, $P_i^+(\emptyset) = 0$.

Proof of (e). Let $X \in \mathcal{P}S$ be an i -minimal element. By Lemma 4.8, there exists an $s \in S$ such that $[s] = X$. Hence, using the definition of P_i (see Definition 3.1), we have:

$$P_i^+(X) = \sum_{x \in [s]} P_i(x) = 1.$$

Proof of (f). Let $X \in \mathcal{P}S$ be i -minimal element and $Y_1, Y_2 \in X\downarrow$ such that $Y_1 \subset Y_2$. By definition, we have that

$$P_i^+(Y_2) = \sum_{x \in Y_2} P_i(x) = \sum_{x \in Y_1} P_i(x) + \sum_{x \in Y_2 \setminus Y_1} P_i(x) = P_i^+(Y_1) + \sum_{x \in Y_2 \setminus Y_1} P_i(x).$$

Since $Y_1 \subset Y_2$, there exists $s \in Y_2 \setminus Y_1$. Since $P_i : S \rightarrow]0, 1]$, we have $P_i(s) > 0$ for all $s \in Y_2 \setminus Y_1$. Hence, $\sum_{x \in Y_2 \setminus Y_1} P_i(x) > 0$ and $P_i^+(Y_1) < P_i^+(Y_2)$. \square

Proof of Proposition 4.14

Proposition 4.14. for every epistemic Heyting algebra \mathbb{A} and every agent $i \in \text{Ag}$,

$$\text{Min}_i(\mathbb{A}') = \{f_{e,a} \mid e \in E \text{ and } a \in \text{Min}_i(\mathbb{A})\},$$

where for any $e \in E$ and $a \in \text{Min}_i(\mathbb{A})$, the map $f_{e,a}$ is defined as follows:

$$f_{e,a} : E \rightarrow \mathbb{A}$$

$$e' \mapsto \begin{cases} a & \text{if } e' \sim_i e \\ \perp & \text{otherwise.} \end{cases}$$

PROOF. Recall that $f \in \mathbb{A}'$ is an i -minimal element (see Definition 4.2) if it satisfies the following conditions: (1) $f \neq \perp$, (2) $\diamond_i f = f$, and (3) if $g \in \mathbb{A}$, $g < f$ and $\diamond_i g = g$, then $g = \perp$.

Let us first prove that any map $f_{e,a}$ as above is an i -minimal element of \mathbb{A}' . By definition, $f_{e,a}(e) = a \neq \perp_{\mathbb{A}}$. Hence, $f_{e,a} \neq \perp_{\mathbb{A}'}$. As to showing that $\diamond'_i f_{e,a} = f_{e,a}$, fix $e' \in E$, and let us show that $(\diamond'_i f_{e,a})(e') = f_{e,a}(e')$. By definition,

$$\diamond'_i f_{e,a}(e') = \bigvee \{\diamond_i f_{e,a}(e'') \mid e'' \sim_i e'\}.$$

We proceed by cases: (a) If $e' \sim_i e$, then:

$$\begin{aligned} \diamond'_i f_{e,a}(e') &= \bigvee \{\diamond_i f_{e,a}(e'') \mid e'' \sim_i e'\} && \text{(by definition)} \\ &= \bigvee \{\diamond_i a \mid e'' \sim_i e'\} && (f_{e,a}(e'') = a, \text{ since } e \sim_i e' \text{ and } \sim_i \text{ symmetric and transitive}) \\ &= \diamond_i a && \text{(the join is nonempty since } \sim_i \text{ is reflexive)} \\ &= a && (a \text{ is } i\text{-minimal, hence is a fixed point of } \diamond_i) \\ &= f_{e,a}(e'). && \text{(definition of } f_{e,a} \text{ and } e' \sim_i e) \end{aligned}$$

(b) If $e' \not\sim_i e$, then:

$$\begin{aligned} \diamond'_i f_{e,a}(e') &= \bigvee \{\diamond_i f_{e,a}(e'') \mid e'' \sim_i e'\} && \text{(by definition)} \\ &= \bigvee \{\diamond_i \perp \mid e'' \sim_i e'\} && (e \not\sim_i e') \\ &= \diamond_i \perp \\ &= \perp && (\diamond_i \perp = \perp) \\ &= f_{e,a}(e'). \end{aligned}$$

Finally, we need to show that $f_{e,a}$ is a minimal non-bottom fixed-point of \diamond'_i . Notice preliminarily that if $g : E \rightarrow \mathbb{A}$ is a fixed point for \diamond'_i , then

$$g(e) = g(e') \text{ whenever } e \sim_i e'. \quad (\text{A.1})$$

Indeed,

$$g(e) = (\diamond'_i g)(e) = \bigvee \{\diamond_i g(e'') \mid e'' \sim_i e\} = \bigvee \{\diamond_i g(e'') \mid e'' \sim_i e'\} = (\diamond'_i g)(e') = g(e').$$

Given that \sim_i is reflexive, this implies in particular that, for every $e' \in E$,

$$(\diamond'_i g)(e') = \diamond_i g(e'). \quad (\text{A.2})$$

Let g be as above, assume that $\perp \neq g \leq f_{e,a}$, and let us show that $g = f_{e,a}$. Clearly, the assumption $g \leq f_{e,a}$ implies that $g(e') = \perp$ for every $e' \in E$ such that $e' \not\sim_i e$. Let $e' \in E$ such that $g(e') \neq \perp$. Together with the assumption that $g \leq f_{e,a}$, this implies that $f_{e,a}(e') \neq \perp$, hence $e' \sim_i e$

and $\perp \neq g(e') \leq a$. To prove that $g(e) = a$, by the i -minimality of a , it suffices to show that $g(e')$ is a fixed point of \diamond_i . Indeed, by (A.2):

$$\diamond_i g(e') = (\diamond'_i g)(e') = g(e'),$$

as required. Finally, the fact above and the preliminary observation (A.1) imply that $g(e') = a$ for every $e' \in E$ such that $e' \sim_i e$.

This finishes the proof that $f_{e,a}$ is i -minimal.

Conversely, let $g : E \rightarrow \mathbb{A}$ be i -minimal in \mathbb{A}' , and let us show that $g = f_{e,a}$ for some $e \in E$ and some i -minimal element $a \in \mathbb{A}$. The assumption that $g \neq \perp$ implies that $g(e) \neq \perp$ for some $e \in E$. Let $g(e) = a \in \mathbb{A}$. Then, the assumption that $g = \diamond'_i g$ and the observation (A.1) imply that $g(e') = a$ for every $e' \in E$ such that $e' \sim_i e$. Then, the proof is finished if we show that a is i -minimal in \mathbb{A} . Indeed, then, by construction, we would have $\perp \neq f_{e,a} \leq g$, hence the minimality of g would yield $f_{e,a} = g$.

By definition, we have that $a = g(e') \neq \perp$. By observation (A.2),

$$\diamond_i a = \diamond_i g(e) = (\diamond'_i g)(e) = g(e) = a,$$

which shows that a is a fixed point of \diamond_i . Finally, let $\perp \neq b \leq a$ such that $\diamond_i b = b$. Then, with an argument analogous to the one given above, the map $f_{e,b} : E \rightarrow \mathbb{A}$ would be proven to be a non-bottom fixed-point of \diamond'_i . Moreover, $f_{e,b} \leq g$, and hence the i -minimality of g would yield $f_{e,b} = g$, hence $a = b$. \square

Proof of Proposition 4.16

Proposition 4.16. For every APE-structure $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ and every event structure \mathbb{E} over \mathbb{A} , μ_i^a is an i -premeasure over \mathbb{A} . Furthermore, if $a \leq y$, then $\mu_i^a(x) = \mu_i^a(x \wedge y)$.

PROOF. For every $a \in \Phi$ and every $i \in \text{Ag}$, we want to prove that μ_i^a is an i -premeasure over \mathbb{A} ; hence, we need to prove that μ_i^a is a partial function $\mathbb{A} \rightarrow \mathbb{R}^+$ that satisfies items (1–4) of Definition 4.6. Fix $a \in \Phi$ and $i \in \text{Ag}$.

Proof of item (1). We want to prove that $\text{dom}(\mu) = \text{Min}_i(\mathbb{A})\downarrow$. The map μ_i is an i -premeasure, hence $\text{dom}(\mu_i) = \text{Min}_i(\mathbb{A})\downarrow$. Therefore, the map μ_i^a is defined on every $x \in \text{Min}_i(\mathbb{A})\downarrow$ and we can restrict its domain as follows: $\text{dom}(\mu_i^a) := \text{Min}_i(\mathbb{A})\downarrow$.

Proof that μ_i^a is well defined. We need to prove that $\mu_i^a(x) \geq 0$ for all $x \in \text{Min}_i(\mathbb{A})\downarrow$. Recall that Φ is a finite ordered multiset of elements of \mathbb{A} such that, for all distinct $b, c \in \Phi$, either $b \wedge c = \perp$ or $b < c$ or $c < b$ (see Definition 4.11 and Remark 8). Hence, for every $b, c \in \text{mb}(a)$, we have $b \wedge c = \perp$. Indeed, by item (2) of Definition 4.10 and what was mentioned above, if $b \wedge c \neq \perp$, then either $b < c$ or $c < b$. Hence, they cannot both be maximal.

Fix $x \in \text{Min}_i(\mathbb{A})\downarrow$. Let us prove by induction on the size of S that for any $S \subseteq \text{mb}(a)$,

$$\mu_i \left(\bigvee_{b \in S} x \wedge b \right) = \sum_{b \in S} \mu_i(x \wedge b). \quad (\text{A.3})$$

Base case: $|S| = 0$. Assume that $S = \emptyset$. Then, we trivially have that

$$\mu_i \left(\bigvee_{b \in S} x \wedge b \right) = \mu_i(\perp) = 0 = \sum_{b \in S} \mu_i(x \wedge b). \quad (\text{IH}_0)$$

Induction step: $\text{IH}_n \Rightarrow \text{IH}_{n+1}$. Assume that, for any set S' that contains exactly n elements, we have

$$\mu_i \left(\bigvee_{b' \in S'} x \wedge b' \right) = \sum_{b' \in S'} \mu_i(x \wedge b'). \quad (\text{IH}_n)$$

Let S contain exactly $n + 1$ elements, $S' \subset S$ contain exactly n elements, and $S = S' \cup \{c\}$. Let us prove IH_{n+1} :

$$\begin{aligned} & \mu_i \left(\bigvee_{b \in S} x \wedge b \right) \\ &= \mu_i \left((x \wedge c) \vee \bigvee_{b' \in S'} (x \wedge b') \right) && (S = S' \cup \{c\}) \\ &= \mu_i(x \wedge c) + \mu_i \left(\bigvee_{b' \in S'} x \wedge b' \right) - \mu_i \left((x \wedge c) \wedge \bigvee_{b' \in S'} (x \wedge b') \right) && (\mu_i \text{ is an } i\text{-premeasure}) \\ &= \mu_i(x \wedge c) + \mu_i \left(\bigvee_{b' \in S'} x \wedge b' \right) - \mu_i \left(\bigvee_{b' \in S'} x \wedge c \wedge x \wedge b' \right) && (\wedge \text{ distributes over } \vee) \\ &= \mu_i(x \wedge c) + \mu_i \left(\bigvee_{b' \in S'} x \wedge b' \right) - \mu_i(\perp) && (c \neq b' \text{ implies } c \wedge b' = \perp) \\ &= \mu_i(x \wedge c) + \sum_{b' \in S'} \mu_i(x \wedge b') && (\mu_i(\perp) = 0 \text{ and } (\text{IH}_n)) \\ &= \sum_{b \in S} \mu_i(x \wedge b). && (S = S' \cup \{c\}) \end{aligned}$$

By induction, for any $x \in \text{Min}_i(\mathbb{A})\downarrow$, we have $\mu_i(\bigvee_{b \in \text{mb}(a)} x \wedge b) = \sum_{b \in \text{mb}(a)} \mu_i(x \wedge b)$.

Since $\text{mb}(a)$ denotes the set of the $<$ -maximal elements of $(\Phi \cap \downarrow a) \setminus \{a\}$, we have that $\bigvee_{b \in \text{mb}(a)} x \wedge b \leq x \wedge a$. By monotonicity of μ_i , we get that

$$\sum_{b \in \text{mb}(a)} \mu_i(x \wedge b) = \mu_i \left(\bigvee_{b \in \text{mb}(a)} x \wedge b \right) \leq \mu_i(x \wedge a).$$

Hence, $\mu_i^a(x) \geq 0$ for any $x \in \text{Min}_i(\mathbb{A})\downarrow$, as required.

Proof of item (2). We want to show that μ_i^a is order-preserving. Using (A.3) and the fact that \wedge distributes over \vee , we get that: for any $x \in \text{Min}_i(\mathbb{A})\downarrow$,

$$\sum_{b \in \text{mb}(a)} \mu_i(x \wedge b) = \mu_i \left(\bigvee_{b \in \text{mb}(a)} x \wedge b \right) = \mu_i \left(x \wedge \bigvee_{b \in \text{mb}(a)} b \right). \quad (\text{A.4})$$

Fix $x, y \in \text{Min}_i(\mathbb{A})\downarrow$ such that $x \leq y$. Notice that $\bigvee_{b \in \text{mb}(a)} b \leq a$ and $x \wedge a \wedge y = x$. Furthermore, $x \wedge a \leq y \wedge a$ and $y \wedge (\bigvee_{b \in \text{mb}(a)} b) \leq y \wedge a$. Hence, $(x \wedge a) \vee (y \wedge (\bigvee_{b \in \text{mb}(a)} b)) \leq y \wedge a$. From

this, we can deduce that:

$$\begin{aligned}
 & (x \wedge a) \vee \left(y \wedge \left(\bigvee_{b \in \text{emb}(a)} b \right) \right) \leq y \wedge a \\
 \Rightarrow & \mu_i \left((x \wedge a) \vee \left(y \wedge \bigvee_{b \in \text{emb}(a)} b \right) \right) \leq \mu_i(y \wedge a) && (\mu_i \text{ is order-preserving}) \\
 \Leftrightarrow & \mu_i(x \wedge a) + \mu_i \left(y \wedge \bigvee_{b \in \text{emb}(a)} b \right) - \mu_i \left(x \wedge a \wedge y \wedge \bigvee_{b \in \text{emb}(a)} b \right) \leq \mu_i(y \wedge a) && (\mu_i \text{ is an } i\text{-premeasure}) \\
 \Leftrightarrow & \mu_i(x \wedge a) + \mu_i \left(y \wedge \bigvee_{b \in \text{emb}(a)} b \right) - \mu_i \left(x \wedge \bigvee_{b \in \text{emb}(a)} b \right) \leq \mu_i(y \wedge a) && (x \wedge a \wedge y = x) \\
 \Leftrightarrow & \mu_i(x \wedge a) - \mu_i \left(x \wedge \bigvee_{b \in \text{emb}(a)} b \right) \leq \mu_i(y \wedge a) - \mu_i \left(y \wedge \bigvee_{b \in \text{emb}(a)} b \right) \\
 \Leftrightarrow & \mu_i(x \wedge a) - \sum_{b \in \text{emb}(a)} \mu_i(x \wedge b) \leq \mu_i(y \wedge a) - \sum_{b \in \text{emb}(a)} \mu_i(y \wedge b) && (\text{by (A.4)}) \\
 \Leftrightarrow & \mu_i^a(x) \leq \mu_i^a(y).
 \end{aligned}$$

Proof of item (3). We need to show that $\mu_i^a(x \vee y) = \mu_i^a(x) + \mu_i^a(y) - \mu_i^a(x \wedge y)$ for all $x, y \in \text{Min}_i(\mathbb{A}) \downarrow$. We have:

$$\begin{aligned}
 \mu_i^a(x \vee y) &= \mu_i((x \vee y) \wedge a) - \sum_{b \in \text{emb}(a)} \mu_i((x \vee y) \wedge b) \\
 &= \mu_i((x \wedge a) \vee (y \wedge a)) - \sum_{b \in \text{emb}(a)} \mu_i((x \wedge b) \vee (y \wedge b)) && (\text{distributivity}) \\
 &= (\mu_i(x \wedge a) + \mu_i(y \wedge a) - \mu_i(x \wedge y \wedge a)) - \sum_{b \in \text{emb}(a)} (\mu_i(x \wedge b) + \mu_i(y \wedge b) - \mu_i(x \wedge y \wedge b)) \\
 & && (\mu_i \text{ is an } i\text{-measure}) \\
 &= \mu_i^a(x) + \mu_i^a(y) - \mu_i^a(x \wedge y).
 \end{aligned}$$

Proof of item (4). If $\text{Min}_i(\mathbb{A}) \downarrow \neq \emptyset$, it follows from $\mu_i(\perp) = 0$ (because μ_i is an i -premeasure) that $\mu_i^a(\perp) = 0$. \square

Proof of Proposition 4.21

Proposition 4.21. For every PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} ,

$$\left(\prod_{\mathcal{E}} \mathbb{M} \right)^+ \cong \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{M}^+.$$

PROOF. The proof that the supports of the two APE-structures (Definition 4.7) can be identified is essentially the same as that of Reference [34, Fact 23.3] and is omitted. Recall that the basic identification between $\mathcal{P}(\prod_{|E|} S)$ and $\prod_{|E|} \mathcal{P}(S)$ associates every subset $X \subseteq \prod_{|E|} S$ with the map

$$\begin{aligned}
 g : E &\rightarrow \mathcal{P}(S) \\
 e &\mapsto X_e := \{s \in S \mid (s, e) \in X\}.
 \end{aligned}$$

Let us prove that this identification induces an identification between the maps¹⁰

$$(P_i^+)' : \prod_{|E|} \mathcal{P}(S) \rightarrow [0, 1] \quad \text{and} \quad (P_i^{\llbracket \cdot \rrbracket})^+ : \mathcal{P}\left(\prod_{|E|} S\right) \rightarrow [0, 1].$$

In what follows, we fix a subset $X \subseteq \prod_{|E|} S$ in the domain of $P_i^{\llbracket \cdot \rrbracket}$ and let $g \in \prod_{|E|} \mathcal{P}(S)$ be defined as its counterpart as discussed above. Recall that for any $s \in S$ and $e \in E$, $\text{pre}(e \mid s)$ denotes the value $\text{pre}(e \mid \phi)$ for the unique $\phi \in \Phi$ such that $\mathbb{M}, s \Vdash \phi$ (see Notation 1). Then, we have:

$$\begin{aligned} (P_i^{\llbracket \cdot \rrbracket})^+(X) &= \sum_{(s,e) \in X} P_i^{\llbracket \cdot \rrbracket}((s, e)) && \text{(Definition 3.1 on } P_i^{\llbracket \cdot \rrbracket}) \\ &= \sum_{(s,e) \in X} P_i(s) \cdot P_i(e) \cdot \text{pre}(e \mid s) && \text{(Definition 2.5)} \\ &= \sum_{e \in E} \sum_{s \in X_e} P_i(s) \cdot P_i(e) \cdot \text{pre}(e \mid s) && (X_e := \{s \in S \mid (s, e) \in X\}) \\ &= \sum_{e \in E} P_i(e) \cdot \left(\sum_{s \in X_e} P_i(s) \cdot \text{pre}(e \mid s) \right) \\ &= \sum_{e \in E} P_i(e) \cdot \sum_{\phi \in \Phi} \left(\sum_{s \in X_e \cap \llbracket \phi \rrbracket} P_i(s) \cdot \text{pre}(e \mid s) \right) && (\Phi \text{ provides a partition of } \{s \in S \mid \text{pre}(e \mid s) \neq 0\}) \\ &= \sum_{e \in E} P_i(e) \cdot \left(\sum_{\phi \in \Phi} \left(\sum_{s \in X_e \cap \llbracket \phi \rrbracket} P_i(s) \right) \cdot \text{pre}(e \mid \phi) \right) && \text{(Notation 1)} \\ &= \sum_{e \in E} P_i(e) \cdot \left(\sum_{\phi \in \Phi} P_i^+(X_e \cap \llbracket \phi \rrbracket) \cdot \overline{\text{pre}}_{\mathbb{M}}(e \mid \llbracket \phi \rrbracket) \right) && \text{(Definition 3.1)} \\ &= \sum_{e \in E} P_i(e) \cdot \left(\sum_{\phi \in \Phi} (P_i^+)^{\llbracket \phi \rrbracket}(X_e) \cdot \overline{\text{pre}}_{\mathbb{M}}(e \mid \llbracket \phi \rrbracket) \right) && \text{(Remark 10 : } \text{mb}(\llbracket \phi \rrbracket) = \emptyset) \\ &= \sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(X_e) \cdot \overline{\text{pre}}_{\mathbb{M}}(e \mid \llbracket \phi \rrbracket) \\ &= \sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(g(e)) \cdot \overline{\text{pre}}_{\mathbb{M}}(e \mid \llbracket \phi \rrbracket) \\ &= (P_i^+)'(g) && \text{(Definition 4.17 on } \mathbb{M}^+) \end{aligned}$$

□

Proof of Lemma 4.26

Lemma 4.26. For any epistemic Heyting algebra \mathbb{A} and any $a \in \mathbb{A}$, if $[b] \in \text{Min}_i(\mathbb{A}^a)$, then $\diamond_i(b \wedge a)$ is the unique i -minimal element of \mathbb{A} that belongs to $[b]$.

PROOF. Let us first prove that $\diamond_i(b \wedge a) \in [b]$. By assumption, $[b] \in \text{Min}_i(\mathbb{A}^a)$, hence $[b] = \diamond_i^a[b] = b \wedge a = \diamond_i(b \wedge a) \wedge a$. This implies that $\diamond_i(b \wedge a) \in [b]$.

¹⁰Refer to Definitions 2.5 and 3.1 for the definitions of the intermediate structure $\prod_{|E|} \mathbb{M}$ and of the complex algebra associated to a model.

Now, we need to show that $\diamond_i(b \wedge a)$ is an i -minimal element of \mathbb{A} . Hence, we need to prove that $\diamond_i(b \wedge a)$ satisfies items (1), (2), and (3) of Definition 4.2.

Proof of item (1). By assumption, $[b] \in \text{Min}_i(\mathbb{A}^a)$, hence $[b] \neq \perp$ and $b \wedge a \neq \perp$. Since \diamond_i is reflexive (Definition 4.1, axiom (M1)), $\perp \neq b \wedge a \leq \diamond_i(b \wedge a)$, which shows that $\diamond_i(b \wedge a) \neq \perp$, as required.

Proof of item (2). Since \diamond_i is transitive (Definition 4.1, axiom (M6)), we have that $\diamond_i(b \wedge a) = \diamond_i \diamond_i(b \wedge a)$, as required.

Proof of item (3). Let $c \in \text{Min}_i(\mathbb{A})$ and $c \leq \diamond_i(b \wedge a)$. We need to prove that $c = \diamond_i(b \wedge a)$. To do so, we follow the following steps:

- (i) we prove that $[b] = [c]$,
- (ii) we show that $c \wedge a \neq \perp$,
- (iii) we prove that $\diamond_i(b \wedge a)$.

Step (i). From the assumptions that $c \leq \diamond_i(b \wedge a)$ and that $[b] = \diamond_i^a[b]$, we get that $c \wedge a \leq \diamond_i(b \wedge a) \wedge a = b \wedge a$, which proves that $[c] \leq [b]$.

Step (ii). Since $c \leq \diamond_i(b \wedge a)$, we have that $c \leq \diamond_i a$; that is, $c = c \wedge \diamond_i a$. This gives the following chain of equalities:

$$c = c \wedge \diamond_i a = \diamond_i c \wedge \diamond_i a = \diamond_i(\diamond_i c \wedge a).$$

The last equality is true in all monadic Heyting algebras (see, e.g., Reference [11, Definition 1]). Now, since $\diamond_i c = c$, we get that $c = \diamond_i(c \wedge a)$, which implies $\diamond_i(c \wedge a) \neq \perp$ and $c \wedge a \neq \perp$.

Step (iii). By Lemma 4.25, $[c] \in \text{Min}_i(\mathbb{A}^a)$. By the i -minimality of $[b]$, we get $[b] = [c]$; that is, $b \wedge a = c \wedge a$. Hence, $\diamond_i(b \wedge a) = \diamond_i(c \wedge a) \leq \diamond_i(c) = c$, which, together with the assumption that $c \leq \diamond_i(b \wedge a)$, proves that $\diamond_i(b \wedge a) = c$, as required. This finishes the proof that $\diamond_i(b \wedge a)$ is an i -minimal element of \mathbb{A} .

To show the uniqueness, let $c_1, c_2 \in [b]$ and assume that both c_1 and c_2 are i -minimal elements of \mathbb{A} . Then, $c_1 \wedge a = c_2 \wedge a$, and hence $\diamond_i(c_1 \wedge a) = \diamond_i(c_2 \wedge a)$. Reasoning as above, one can show that $\perp \neq \diamond_i(c_j \wedge a) \leq c_j$ and $\diamond_i(c_j \wedge a)$ is a fixed point of \diamond_i for $1 \leq j \leq 2$. Hence, the i -minimality of c_j implies that $\diamond_i(c_j \wedge a) = c_j$. Thus, the following chain of identities holds:

$$c_1 = \diamond_i(c_1 \wedge a) = \diamond_i(c_2 \wedge a) = c_2. \quad \square$$

Proof of Proposition 4.30

Proposition 4.30. For any APE-structure \mathcal{F} and any event structure \mathbb{E} over the support of \mathcal{F} , the tuple $\mathcal{F}^{\mathbb{E}} = (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i \in \text{Ag}})$ is an APE-structure.

PROOF. Let $\mathbb{E} = (E, (\sim_i)_{i \in \text{Ag}}, (P_i)_{i \in \text{Ag}}, \Phi, \overline{\text{pre}})$ be an event structure and $\mathcal{F} := (\mathbb{A}, (\mu_i)_{i \in \text{Ag}})$ be an APE-structure. To prove that $\mathcal{F}^{\mathbb{E}}$ is an APE-structure (see Definition 4.7), we need to prove that $\mathbb{A}^{\mathbb{E}}$ is an epistemic Heyting algebra (see Definition 4.3) and that each map $\mu_i^{\mathbb{E}}$ is an i -measure on $\mathbb{A}^{\mathbb{E}}$. By Proposition 4.24, $\mathbb{A}^{\mathbb{E}}$ is an epistemic Heyting algebra. Hence, it remains to prove that, for each $i \in \text{Ag}$, the map $\mu_i^{\mathbb{E}}$ is an i -measure (see Definition 4.6), i.e., we need to prove that:

- (1) $\text{dom}(\mu_i^{\mathbb{E}}) = \text{Min}_i(\mathbb{A}^{\mathbb{E}})\downarrow$;
- (2) $\mu_i^{\mathbb{E}}$ is order-preserving;
- (3) for every $a \in \text{Min}_i(\mathbb{A}^{\mathbb{E}})$ and all $b, c \in a\downarrow$, it holds that $\mu_i^{\mathbb{E}}(b \vee c) = \mu_i^{\mathbb{E}}(b) + \mu_i^{\mathbb{E}}(c) - \mu_i^{\mathbb{E}}(b \wedge c)$;
- (4) $\mu_i^{\mathbb{E}}(\perp) = 0$ if $\text{dom}(\mu_i^{\mathbb{E}}) \neq \emptyset$;

- (5) $\mu_i^{\mathbb{E}}(a) = 1$ for every $a \in \text{Min}_i(\mathbb{A}^{\mathbb{E}})$;
 (6) for every $a \in \text{Min}_i(\mathbb{A}^{\mathbb{E}})$ and all $b, c \in a \downarrow$ such that $b < c$, it holds that $\mu_i^{\mathbb{E}}(b) < \mu_i^{\mathbb{E}}(c)$.

Proof of (1). This condition is satisfied by definition.

The remaining items are trivially satisfied if the domain of $\mu_i^{\mathbb{E}}$ is empty. For the remaining of the proof, let us assume that the domain of $\mu_i^{\mathbb{E}}$ is non-empty.

Proof of item (2). The definition of μ'_i (see Definition 4.17), the Proposition 4.16, and the fact that, if $\overline{\text{pre}}(e | a) \neq 0$, then $a \leq \overline{\text{pre}}(e)$ (see Definition of $\overline{\text{pre}}$ (4.5)), imply that $\mu'_i(g) = \mu'_i(g \wedge \overline{\text{pre}})$. Assume that $[g_1] \leq [g_2] \leq [f_{e,a}]$. This means that $g_1 \wedge \overline{\text{pre}} \leq g_2 \wedge \overline{\text{pre}}$. Since μ'_i is an i -premeasure (Proposition 4.18), it is monotone. Hence, $\mu'_i(g_1) = \mu'_i(g_1 \wedge \overline{\text{pre}}) \leq \mu'_i(g_2 \wedge \overline{\text{pre}}) = \mu'_i(g_2)$. This implies that

$$\frac{\mu'_i(g_1)}{\mu'_i(f_{e,a})} \leq \frac{\mu'_i(g_2)}{\mu'_i(f_{e,a})};$$

that is, $\mu_i^{\mathbb{E}}([g_1]) \leq \mu_i^{\mathbb{E}}([g_2])$.

Proof of item (3). Let $[g_1]$ and $[g_2]$ in $\mathcal{F}^{\mathbb{E}}$ such that $[g_1] \leq [f_{e,a}]$ and $[g_2] \leq [f_{e,a}]$. We have:

$$\begin{aligned} & \mu_i^{\mathbb{E}}([g_1] \vee [g_2]) \\ &= \frac{\mu'_i((g_1 \wedge \overline{\text{pre}}) \vee (g_2 \wedge \overline{\text{pre}}))}{\mu'_i(f_{e,a})} \\ &= \frac{\mu'_i(g_1 \wedge \overline{\text{pre}}) + \mu'_i(g_2 \wedge \overline{\text{pre}}) - \mu'_i((g_1 \wedge g_2) \wedge \overline{\text{pre}})}{\mu'_i(f_{e,a})} \quad (\text{Proposition 4.18. } \mu'_i \text{ is an } i\text{-premeasure}) \\ &= \frac{\mu'_i(g_1 \wedge \overline{\text{pre}})}{\mu'_i(f_{e,a})} + \frac{\mu'_i(g_2 \wedge \overline{\text{pre}})}{\mu'_i(f_{e,a})} - \frac{\mu'_i((g_1 \wedge g_2) \wedge \overline{\text{pre}})}{\mu'_i(f_{e,a})} \\ &= \frac{\mu'_i(g_1)}{\mu'_i(f_{e,a})} + \frac{\mu'_i(g_2)}{\mu'_i(f_{e,a})} - \frac{\mu'_i(g_1 \wedge g_2)}{\mu'_i(f_{e,a})} \\ &= \mu_i^{\mathbb{E}}([g_1]) + \mu_i^{\mathbb{E}}([g_2]) - \mu_i^{\mathbb{E}}([g_1 \wedge g_2]). \end{aligned}$$

Proof of Items (4) and (5). Trivial.

Proof of item (6). Recall that, if $[g] \neq \perp$, then $\mu_i^{\mathbb{E}}([g]) > 0$ (see Claim in Lemma 4.29). Let $\perp \neq [g] < [h]$. The monotonicity of the μ_i^a guarantees that, for all $e \in E$ and $a \in \Phi$, we have

$$P_i(e) \cdot \mu_i^a(g(e)) \cdot \overline{\text{pre}}(e|a) \leq P_i(e) \cdot \mu_i^a(h(e)) \cdot \overline{\text{pre}}(e|a).$$

Furthermore, since $[g] < [h]$, there exists an $e \in E$ such that the set

$$\{ a \in \Phi \mid \overline{\text{pre}}(e|a) > 0 \text{ and } g(e) \wedge a < h(e) \wedge a \}$$

is non-empty. Since Φ is finite, the order $<$ is well founded and the aforementioned set contains at least one minimal element. Let a_0 be such a minimal element. From Definition 4.11, we have that $\overline{\text{pre}}(e|b) > 0$ for all $b \in \Phi$ with $b < a_0$. By the minimality of a_0 , we have that $g(e) \wedge b = h(e) \wedge b$ for all such $b < a_0$. Hence,

$$\sum_{b \in \text{mb}(a_0)} \mu_i(g(e) \wedge b) = \sum_{b \in \text{mb}(a_0)} \mu_i(h(e) \wedge b),$$

where $\text{mb}(a)$ denotes the multiset of the \prec -maximal elements of Φ \prec -below a (see Definition 4.15). Since \mathcal{F} is an APE-structure, μ_i is strictly monotone. Hence, $g(e) \wedge a_0 < h(e) \wedge a_0$ implies that

$$\begin{aligned} \mu_i^{a_0}(g(e)) &= \mu_i(g(e) \wedge a_0) - \sum_{b \in \text{mb}(a_0)} \mu_i(g(e) \wedge b) \\ &< \mu_i(h(e) \wedge a_0) - \sum_{b \in \text{mb}(a_0)} \mu_i(h(e) \wedge b) \\ &= \mu_i^{a_0}(h(e)). \end{aligned}$$

Hence, for some $e \in E$ and $a \in \Phi$, we have

$$P_i(e) \cdot \mu_i^a(g(e)) \cdot \overline{\text{pre}}(e|a) < P_i(e) \cdot \mu_i^a(h(e)) \cdot \overline{\text{pre}}(e|a).$$

The inequality above, the definition of μ'_i (see Definition 4.17), and the monotonicity of μ'_i (see Proposition 4.18) imply that $\mu'_i([g]) < \mu'_i([h])$, which in turn implies that $\mu_i^{\mathbb{E}}([g]) < \mu_i^{\mathbb{E}}([h])$. \square

Proof of Lemma 4.32

Lemma 4.32. For any PES-model \mathbb{M} and any event structure \mathcal{E} over \mathcal{L} ,

$$(P_i^+)^{\mathbb{E}_{\mathcal{E}}} \cong (P_i^{\mathcal{E}})^+.$$

PROOF. Using Definitions 2.7 and 3.1, we get that: for any $X \in \text{Min}_i((\mathbb{M}^{\mathcal{E}})^+)_\downarrow$,

$$(P_i^{\mathcal{E}})^+(X) = \sum_{(s,e) \in X} \frac{P_i(e) \cdot P_i(s) \cdot \text{pre}(e|s)}{\sum_{(s',e') \sim_i(s,e)} P_i(e') \cdot P_i(s') \cdot \text{pre}(e'|s')}.$$

By using Definitions 3.1 and 4.28, we get that: for any $[g] \in \text{Min}_i((\mathbb{M}^+)^{\mathbb{E}})_\downarrow$,

$$(P_i^+)^{\mathbb{E}_{\mathcal{E}}}([g]) = \frac{\sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(g(e)) \cdot \overline{\text{pre}}(e | \llbracket \phi \rrbracket)}{\sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(f(e)) \cdot \overline{\text{pre}}(e | \llbracket \phi \rrbracket)}.$$

Let $X \in \text{Min}_i((\mathbb{M}^{\mathcal{E}})^+)_\downarrow$. Following the notation introduced in the proof of Proposition 4.21, let $[g] \in \text{Min}_i((\mathbb{M}^+)^{\mathbb{E}})_\downarrow$ be the map such that

$$\begin{aligned} g : E &\rightarrow \mathcal{P}(S) \\ e &\mapsto X_e := \{s \in S \mid (s, e) \in X\}. \end{aligned}$$

Notice that X is a subset of one of the i -equivalence classes of $(\mathbb{M}^{\mathcal{E}})^+$, hence $g = g \wedge \overline{\text{pre}}$ and $[g] \leq [f]$ for some $[f] \in \text{Min}_i((\mathbb{M}^+)^{\mathbb{E}})_\downarrow$. Let

$$[X]_i := \{(s, e) \mid \exists (s', e') \in X, (s, e) \sim_i (s', e')\}.$$

We can easily see that $([X]_i)_e = f(e)$ where f is the canonical representative of $[f]$. We have:

$$\begin{aligned}
& (P_i^\varepsilon)^+(X) \\
&= \sum_{(s,e) \in X} \frac{P_i(e) \cdot P_i(s) \cdot \text{pre}(e \mid s)}{\sum_{(s',e') \sim_i (s,e)} P_i(e') \cdot P_i(s') \cdot \text{pre}(e' \mid s')} \\
&= \frac{\sum_{(s,e) \in X} P_i(e) \cdot P_i(s) \cdot \text{pre}(e \mid s)}{\sum_{(s',e') \in [X]_i} P_i(e') \cdot P_i(s') \cdot \text{pre}(e' \mid s')} \quad (X \text{ is a subset of the equivalence classes } [X]_i) \\
&= \frac{\sum_{e \in E} P_i(e) \cdot \sum_{s \in X_e} P_i(s) \cdot \text{pre}(e \mid s)}{\sum_{e' \in E} P_i(e') \cdot \sum_{s' \in f(e')} P_i(s') \cdot \text{pre}(e' \mid s')} \quad (([X]_i)_e = f(e)) \\
&= \frac{\sum_{e \in E} P_i(e) \cdot \sum_{\phi \in \Phi} \text{pre}(e \mid \phi) \cdot \sum_{s \in g(e) \cap \llbracket \phi \rrbracket} P_i(s)}{\sum_{e' \in E} P_i(e') \cdot \sum_{\phi \in \Phi} \text{pre}(e' \mid \phi) \cdot \sum_{s' \in f(e') \cap \llbracket \phi \rrbracket} P_i(s')} \\
& \quad \text{(In the classical case, } \Phi \text{ gives a partition of } S^\varepsilon) \\
&= \frac{\sum_{e \in E} P_i(e) \cdot \sum_{\phi \in \Phi} \text{pre}(e \mid \phi) \cdot (P_i^+)(g(e) \cap \llbracket \phi \rrbracket)}{\sum_{e' \in E} P_i(e') \cdot \sum_{\phi \in \Phi} \text{pre}(e' \mid \phi) \cdot (P_i^+)(f(e') \cap \llbracket \phi \rrbracket)} \quad (\text{Definition 4.31}) \\
&= \frac{\sum_{e \in E} P_i(e) \cdot \sum_{\phi \in \Phi} \text{pre}(e \mid \phi) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(g(e))}{\sum_{e' \in E} P_i(e') \cdot \sum_{\phi \in \Phi} \text{pre}(e' \mid \phi) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(f(e'))} \quad (\text{Remark 10 : } \text{mb}(\llbracket \phi \rrbracket) = \emptyset) \\
&= \frac{\sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(g(e)) \cdot \overline{\text{pre}}(e \mid \llbracket \phi \rrbracket)}{\sum_{e \in E} \sum_{\phi \in \Phi} P_i(e) \cdot (P_i^+)^{\llbracket \phi \rrbracket}(f(e)) \cdot \overline{\text{pre}}(e \mid \llbracket \phi \rrbracket)} \\
&= (P_i^+)^{\mathbb{E}\varepsilon}(\llbracket g \rrbracket). \quad \square
\end{aligned}$$

B PROOF OF THE SOUNDNESS OF IPDEL

PROPOSITION B.1 (SOUNDNESS). *The axiomatization for IPDEL given in Table 2 is sound w.r.t. APE-models.*

By definition, the underlying structure of an APE-structure is an epistemic Heyting algebra. Hence, it satisfies the axioms of intuitionistic propositional logic and the axioms M1–M7 and E for static modalities.

Axioms for Inequalities

As discussed in Remark 5, it is the case that $\bigvee \text{Min}_i(\mathbb{A}) = \top$ for every epistemic Heyting algebra \mathbb{A} . This implies that axioms N0 and N5 are satisfied in every APE-model. Axioms N1, N2, N3, N4, and N6 are also satisfied, because if the valuation of their antecedent is above any i -minimal element a , then so will be the valuation of their succedent.

Axioms for Probabilities

The fact that axioms P1–P3 are satisfied in every APE-model is shown similarly as axiom N0. Since $\diamond_i \mathbb{A}$ is a subalgebra of \mathbb{A} for every epistemic Heyting algebra \mathbb{A} , it is the case that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in \diamond_i \mathbb{A}$ for every i -probability formula φ and every APE-model based on \mathbb{A} . Hence, Lemma 4.4 implies the satisfiability of P5.

Finally, let us show that P4 is satisfied in every APE-model based on \mathbb{A} . For the right-to-left direction, as discussed in Remark 5, every element of $\diamond_i \mathbb{A}$ can be written as a union of i -minimal elements, and therefore $\llbracket \Box_i(\varphi \leftrightarrow \psi) \rrbracket = \bigvee \{a \in \text{Min}_i(\mathbb{A}) \mid a \wedge \llbracket \varphi \rrbracket = a \wedge \llbracket \psi \rrbracket\}$. This of course implies that $\bigvee \{a \in \text{Min}_i(\mathbb{A}) \mid a \wedge \llbracket \varphi \rrbracket = a \wedge \llbracket \psi \rrbracket\} \leq \llbracket \mu_i(\varphi) = \mu_i(\psi) \rrbracket$. As for the left-to-right direction, we have that $\llbracket \Box_i(\varphi \rightarrow \psi) \wedge (\mu(\varphi) = \mu(\psi)) \rrbracket = \bigvee \{a \in \text{Min}_i(\mathbb{A}) \mid a \wedge \llbracket \varphi \rrbracket \leq a \wedge \llbracket \psi \rrbracket\}$

and $\mu_i(a \wedge \llbracket \varphi \rrbracket) = \mu_i(a \wedge \llbracket \psi \rrbracket)$. By the strict monotonicity of the i -measure μ_i , the following holds, as required.

$$\llbracket \Box_i(\varphi \rightarrow \psi) \wedge (\mu(\varphi) = \mu(\psi)) \rrbracket \leq \bigvee \{a \in \text{Min}_i(\mathbb{A}) \mid a \wedge \llbracket \varphi \rrbracket = a \wedge \llbracket \psi \rrbracket\} = \llbracket \Box_i(\varphi \leftrightarrow \psi) \rrbracket.$$

Reduction Axioms

In this section, we aim at proving the soundness of the reduction axioms as stated in Lemma B.1. To do so, we need to define two maps F and f as follows:

Preliminary Results. Throughout this section, we let \mathbb{A} denote the complex algebra of a model \mathcal{M} and \mathcal{E} denote an event structure. Recall the definition of the event structure $\mathbb{E}_{\mathcal{E}}$ (cf. Definition 5.4). Then, we define a map $F : \mathcal{L} \rightarrow \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A}$ that associates an element in $\prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A}$ to each formula. We want F (Definition B.3) to be the map such that

$$\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} = [F(\psi)].$$

$\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}}$ is the evaluation of the formula ψ in the updated algebra $\mathbb{A}^{\mathbb{E}_{\mathcal{E}}}$ corresponding to the updated model $\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}$. Hence, $F(\psi)$ is a representative of the equivalence class $\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}}$ in the product algebra \mathbb{A}^{Π} .

Since $F(\psi) \in \mathbb{A}^{\Pi}$, $F(\psi)$ is a tuple of elements of the algebra \mathbb{A} . To aid the computation, we define the map $f : \mathcal{L} \times E \rightarrow \mathcal{L}$ (see Definition B.2) such that $F(\psi)(e) = \llbracket f(\psi, e) \rrbracket_{\mathcal{M}}$. This means that $f(\psi, e)$ is a formula such that its evaluation $\llbracket f(\psi, e) \rrbracket_{\mathcal{M}}$ in the algebra \mathbb{A} is equal to the e th coordinate of the tuple $F(\psi)$. We first prove that the maps F and f have the desired properties in Lemma B.4. Then, we prove the key lemma B.5 that we will use to prove the reduction axioms (see Section B).

Definition B.2. The map $f : \mathcal{L} \times E \rightarrow \mathcal{L}$ is defined by recursion as follows: for every $\psi \in \mathcal{L}$ and $e \in E$,

$$\begin{aligned} f(p, e) &= \text{sub}(e, p), \\ f(\perp, e) &= \perp, \\ f(\top, e) &= \top, \\ f(\psi_1 \wedge \psi_2, e) &= f(\psi_1, e) \wedge f(\psi_2, e), \\ f(\psi_1 \vee \psi_2, e) &= f(\psi_1, e) \vee f(\psi_2, e), \\ f(\psi_1 \rightarrow \psi_2, e) &= f(\psi_1, e) \rightarrow f(\psi_2, e), \\ f(\Diamond_i \psi, e) &= \bigvee_{e' \sim_i e} \Diamond_i(f(\psi, e') \wedge \text{pre}(e')), \\ f(\Box_i \psi, e) &= \bigwedge_{e' \sim_i e} \Box_i(\text{pre}(e') \rightarrow f(\psi, e')), \\ f(\langle \mathcal{E}', e' \rangle \psi, e) &= f(\text{pre}(e') \wedge f(\psi, e'), e), \\ f([\mathcal{E}', e'] \psi, e) &= f(\text{pre}(e') \rightarrow f(\psi, e'), e), \\ f(\alpha \mu_i(\psi) \geq \beta, e) &= \alpha \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^\phi(f(\psi, e')) \cdot P_i(e') \cdot \text{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \mu_i^\phi(\top) P_i(e') \text{pre}(e' \mid \phi) \geq 0. \end{aligned}$$

Definition B.3. Let us define the map $F_{\mathbb{E}_{\mathcal{E}}} : \mathcal{L} \rightarrow \mathbb{A}^{\Pi}$ such that for every $e \in E$, the e th coordinate of $F_{\mathbb{E}_{\mathcal{E}}}(\psi)$ is equal to $\llbracket f(\psi, e) \rrbracket_{\mathcal{M}}$.

For the sake of readability, we will omit the subscript when it causes no confusion.

LEMMA B.4. For \mathcal{M} and \mathcal{E} as above,

$$\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} = [F(\psi)],$$

where $F(\psi)(e) = \llbracket f(\psi, e) \rrbracket_{\mathcal{M}}$.

PROOF. The proof is by induction on the complexity of ψ with $(\text{IH}_{\psi}) : \llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} = [F(\psi)]$. The statement is trivially true in the base cases and if the main connectives are \wedge , \vee , or \rightarrow .

If $\psi = \diamond_i \psi'$, then

$$\begin{aligned} \llbracket \diamond_i \psi' \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} &= \diamond_i^{\mathbb{E}_{\mathcal{E}}} \llbracket \psi' \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} \\ &= \diamond_i^{\mathbb{E}_{\mathcal{E}}} [F(\psi')] && (\text{IH}_{\psi'}) \\ &= [\diamond_i^{\Pi} (F(\psi') \wedge \overline{\text{pre}}_{\mathcal{M}})] \end{aligned}$$

and

$$\begin{aligned} \diamond_i^{\Pi} (F(\psi') \wedge \overline{\text{pre}}_{\mathcal{M}})(e) &= \bigvee_{e' \sim_i e} \{ \diamond_i (F(\psi')(e') \wedge \overline{\text{pre}}(e')) \} && (\text{Definition 4.12}) \\ &= \bigvee_{e' \sim_i e} \{ \diamond_i (\llbracket f(\psi', e') \rrbracket_{\mathcal{M}} \wedge \overline{\text{pre}}(e')) \} \\ &= \llbracket \bigvee_{e' \sim_i e} \diamond_i (f(\psi', e') \wedge \text{pre}(e')) \rrbracket_{\mathcal{M}} \\ &= \llbracket f(\diamond_i \psi', e) \rrbracket_{\mathcal{M}} && (\text{Definition B.2}) \\ &= F(\diamond_i \psi')(e). \end{aligned}$$

If $\psi = \square_i \psi'$, then

$$\begin{aligned} \llbracket \square_i \psi' \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} &= \square_i^{\mathbb{E}_{\mathcal{E}}} \llbracket \psi' \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} \\ &= \square_i^{\mathbb{E}_{\mathcal{E}}} [F(\psi')] && (\text{IH}_{\psi'}) \\ &= [\square_i^{\Pi} (\overline{\text{pre}}_{\mathcal{M}} \rightarrow F(\psi'))] \end{aligned}$$

and

$$\begin{aligned} \square_i^{\Pi} (\overline{\text{pre}}_{\mathcal{M}} \rightarrow F(\psi'))(e) &= \bigwedge_{e' \sim_i e} \{ \square_i (\overline{\text{pre}}(e') \rightarrow F(\psi')(e')) \} && (\text{Definition 4.12}) \\ &= \bigwedge_{e' \sim_i e} \{ \square_i (\overline{\text{pre}}(e') \rightarrow \llbracket f(\psi', e') \rrbracket_{\mathcal{M}}) \} \\ &= \llbracket \bigwedge_{e' \sim_i e} \square_i (\text{pre}(e') \rightarrow f(\psi', e')) \rrbracket_{\mathcal{M}} \\ &= \llbracket f(\square_i \psi', e) \rrbracket_{\mathcal{M}} && (\text{Definition B.2}) \\ &= F(\square_i \psi')(e). \end{aligned}$$

If $\psi = \alpha \mu_i(\psi') \geq \beta$, then

$$\begin{aligned} &\llbracket \alpha \mu_i(\psi') \geq \beta \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} \\ &= \bigvee \{ [f_{e,a}] \mid \alpha \mu_i^{\mathbb{E}_{\mathcal{E}}} (\llbracket \psi' \rrbracket_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}} \wedge [f_{e,a}]) \geq \beta \} \\ &= \bigvee \{ [f_{e,a}] \mid \alpha \mu_i^{\mathbb{E}_{\mathcal{E}}} ([F(\psi')] \wedge [f_{e,a}]) \geq \beta \} && (\text{IH}_{\psi'}) \\ &= \bigvee \left\{ [f_{e,a}] \mid \alpha \frac{\mu_i'(F(\psi') \wedge f_{e,a})}{\mu_i'(f_{e,a})} \geq \beta \right\} && (\text{Definition 4.28}) \end{aligned}$$

$$\begin{aligned}
 &= \left[\bigvee \left\{ f_{e,a} \mid \alpha \frac{\mu'_i(F(\psi') \wedge f_{e,a})}{\mu'_i(f_{e,a})} \geq \beta \right\} \right] \\
 &= \left[\bigvee \{ f_{e,a} \mid \alpha \mu'_i(F(\psi') \wedge f_{e,a}) - \beta \mu'_i(f_{e,a}) \geq 0 \} \right] \\
 &= \left[\bigvee \left\{ f_{e,a} \mid \alpha \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^\phi(F(\psi')(e') \wedge a) \cdot P_i(e') \cdot \text{pre}(e' \mid \phi) \right. \right. \\
 &\quad \left. \left. + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \mu_i^\phi(a) \cdot P_i(e') \cdot \text{pre}(e' \mid \phi) \geq 0 \right\} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 &\left(\bigvee \left\{ f_{e,a} \mid \alpha \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^\phi(F(\psi')(e') \wedge a) P_i(e') \text{pre}(e' \mid \phi) \right. \right. \\
 &\quad \left. \left. + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \mu_i^\phi(a) P_i(e') \text{pre}(e' \mid \phi) \geq 0 \right\} (d) \right) \\
 &= \left(\bigvee \left\{ f_{e,a} \mid \alpha \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^\phi(\llbracket f(\psi', e') \rrbracket_{\mathcal{M}} \wedge a) P_i(e') \text{pre}(e' \mid \phi) \right. \right. \\
 &\quad \left. \left. + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta \mu_i^\phi(a) P_i(e') \text{pre}(e' \mid \phi) \geq 0 \right\} (d) \right) \\
 &= \bigvee \left\{ a \mid \alpha \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} \mu_i^\phi(\llbracket f(\psi', e') \rrbracket_{\mathcal{M}} \wedge a) P_i(e') \text{pre}(e' \mid \phi) \right. \\
 &\quad \left. + \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} -\beta \mu_i^\phi(a) P_i(e') \text{pre}(e' \mid \phi) \geq 0 \right\} \\
 &= \left[\left[\alpha \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} \mu_i^\phi(f(\psi', e')) P_i(e') \text{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} -\beta \mu_i^\phi(\tau) P_i(e') \text{pre}(e' \mid \phi) \geq 0 \right]_{\mathcal{M}} \right] \\
 &= \llbracket f(\alpha \mu_i(\psi') \geq \beta, d) \rrbracket_{\mathcal{M}} \quad \text{(Definition B.2)} \\
 &= F(\alpha \mu_i(\psi') \geq \beta)(d).
 \end{aligned}$$

If $\psi = \langle \mathcal{E}', e' \rangle \psi'$ and $\mathcal{N} = \mathcal{M}^{\mathbb{E}\varepsilon}$, then

$$\begin{aligned}
\llbracket \langle \mathcal{E}', e' \rangle \psi' \rrbracket_{\mathcal{N}} &= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge \pi_{e'} \circ i'(\llbracket \psi' \rrbracket_{\mathcal{N}^{\mathbb{E}\varepsilon'}}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge \pi_{e'} \circ i'(\llbracket F(\psi') \rrbracket) && (\text{IH}_{\psi'}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge \pi_{e'}(F(\psi') \wedge \overline{\text{pre}}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge F(\psi')(e') \wedge \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge \llbracket f(\psi', e') \rrbracket_{\mathcal{N}} \wedge \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \wedge \llbracket f(\psi', e') \rrbracket_{\mathcal{N}} \\
&= \llbracket \text{pre}(e') \wedge f(\psi', e') \rrbracket_{\mathcal{N}} \\
&= \llbracket F(\text{pre}(e') \wedge f(\psi', e')) \rrbracket_{\mathcal{N}} && (\text{IH}_{\text{pre}(e') \wedge f(\psi', e')}}
\end{aligned}$$

and

$$\begin{aligned}
F(\text{pre}(e') \wedge f(\psi', e'))(e) &= \llbracket f(\text{pre}(e') \wedge f(\psi', e'), e) \rrbracket_{\mathcal{M}} \\
&= \llbracket f(\langle \mathcal{E}', e' \rangle \psi', e) \rrbracket_{\mathcal{M}} && (\text{Definition B.2}) \\
&= F(\langle \mathcal{E}', e' \rangle \psi')(e).
\end{aligned}$$

Finally, if $\psi = [\mathcal{E}', e'] \psi'$ and $\mathcal{N} = \mathcal{M}^{\mathbb{E}\varepsilon}$, then

$$\begin{aligned}
\llbracket [\mathcal{E}', e'] \psi' \rrbracket_{\mathcal{N}} &= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow \pi_{e'} \circ i'(\llbracket \psi' \rrbracket_{\mathcal{N}^{\mathbb{E}\varepsilon'}}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow \pi_{e'} \circ i'(\llbracket F(\psi') \rrbracket) && (\text{IH}_{\psi'}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow \pi_{e'}(F(\psi') \wedge \overline{\text{pre}}) \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow F(\psi')(e') \wedge \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow \llbracket f(\psi', e') \rrbracket_{\mathcal{N}} \wedge \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \\
&= \llbracket \text{pre}(e') \rrbracket_{\mathcal{N}} \rightarrow \llbracket f(\psi', e') \rrbracket_{\mathcal{N}} && (a \rightarrow (a \wedge b) = a \rightarrow b) \\
&= \llbracket \text{pre}(e') \rightarrow f(\psi', e') \rrbracket_{\mathcal{N}} \\
&= \llbracket F(\text{pre}(e') \rightarrow f(\psi', e')) \rrbracket_{\mathcal{N}} && (\text{IH}_{\text{pre}(e') \rightarrow f(\psi', e')}}
\end{aligned}$$

and

$$\begin{aligned}
F(\text{pre}(e') \rightarrow f(\psi', e'))(e) &= \llbracket f(\text{pre}(e') \rightarrow f(\psi', e'), e) \rrbracket_{\mathcal{M}} \\
&= \llbracket f([\mathcal{E}', e'] \psi', e) \rrbracket_{\mathcal{M}} && (\text{Definition B.2}) \\
&= F([\mathcal{E}', e'] \psi')(e).
\end{aligned}$$

□

LEMMA B.5. For every \mathcal{M} , \mathcal{E} , e , and ψ ,

$$\llbracket \langle \mathcal{E}, e \rangle \psi \rrbracket_{\mathcal{M}} = \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi, e) \rrbracket_{\mathcal{M}} \quad \text{and} \quad \llbracket [\mathcal{E}, e] \psi \rrbracket_{\mathcal{M}} = \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi, e) \rrbracket_{\mathcal{M}}.$$

PROOF. We have

$$\begin{aligned}
\llbracket \langle \mathcal{E}, e \rangle \psi \rrbracket_{\mathcal{M}} &= \llbracket \text{pre}(e) \rrbracket_{\mathcal{M}} \wedge \pi_e \circ i'(\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}\varepsilon}}) \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \pi_e \circ i'(\llbracket F(\psi) \rrbracket) && (\text{Lemma B.4}) \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \pi_e(F(\psi) \wedge \overline{\text{pre}}_{\mathcal{M}}) \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge F(\psi)(e) \wedge \overline{\text{pre}}_{\mathcal{M}}(e) \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi, e) \rrbracket_{\mathcal{M}}
\end{aligned}$$

and

$$\begin{aligned}
 \llbracket [\mathcal{E}, e] \psi \rrbracket_{\mathcal{M}} &= \llbracket pre(e) \rrbracket_{\mathcal{M}} \rightarrow \pi_e \circ i'(\llbracket \psi \rrbracket_{\mathcal{M}^{\mathbb{E}_e}}) \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \pi_e \circ i'(\llbracket F(\psi) \rrbracket) && \text{(Lemma B.4)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \pi_e(F(\psi) \wedge \overline{pre}_{\mathcal{M}}) \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow F(\psi)(e) \wedge \overline{pre}_{\mathcal{M}}(e) \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi, e) \rrbracket_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e) \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi, e) \rrbracket_{\mathcal{M}}. && (a \rightarrow (a \wedge b) = a \rightarrow b)
 \end{aligned}$$

□

Proof of the Soundness of the Reduction Axioms.

Axiom I1. $[\mathcal{E}, e]p = pre(e) \rightarrow \mathbf{sub}(e, p)$.

$$\begin{aligned}
 \llbracket [\mathcal{E}, e]p \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(p, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \mathbf{sub}(e, p) \rrbracket_{\mathcal{M}}.
 \end{aligned}$$

Axiom I2. $\langle \mathcal{E}, e \rangle p = pre(e) \wedge \mathbf{sub}(e, p)$.

$$\begin{aligned}
 \llbracket \langle \mathcal{E}, e \rangle p \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(p, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket \mathbf{sub}(e, p) \rrbracket_{\mathcal{M}}.
 \end{aligned}$$

Axiom I3. $[\mathcal{E}, e] \top = \top$.

$$\begin{aligned}
 \llbracket [\mathcal{E}, e] \top \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\top, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \top \rrbracket_{\mathcal{M}} \\
 &= \llbracket \top \rrbracket_{\mathcal{M}}.
 \end{aligned}$$

Axiom I4. $\langle \mathcal{E}, e \rangle \top = pre(e)$.

$$\begin{aligned}
 \llbracket \langle \mathcal{E}, e \rangle \top \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\top, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket \top \rrbracket_{\mathcal{M}} \\
 &= \overline{pre}_{\mathcal{M}}(e).
 \end{aligned}$$

Axiom I5. $[\mathcal{E}, e] \perp = \neg pre(e)$.

$$\begin{aligned}
 \llbracket [\mathcal{E}, e] \perp \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\perp, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \perp \rrbracket_{\mathcal{M}} \\
 &= \llbracket \neg pre(e) \rrbracket_{\mathcal{M}}.
 \end{aligned}$$

Axiom I6. $\langle \mathcal{E}, e \rangle \perp = \perp$.

$$\begin{aligned}
 \llbracket \langle \mathcal{E}, e \rangle \perp \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\perp, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket \perp \rrbracket_{\mathcal{M}} \\
 &= \perp.
 \end{aligned}$$

Axiom I7. $[\mathcal{E}, e](\psi_1 \wedge \psi_2) = [\mathcal{E}, e]\psi_1 \wedge [\mathcal{E}, e]\psi_2.$

$$\begin{aligned}
\llbracket [\mathcal{E}, e](\psi_1 \wedge \psi_2) \rrbracket_{\mathcal{M}} &= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1 \wedge \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1, e) \wedge f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \wedge \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}} \\
&= (\overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}}) \wedge (\overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
& && (a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c)) \\
&= \llbracket [\mathcal{E}, e]\psi_1 \rrbracket_{\mathcal{M}} \wedge \llbracket [\mathcal{E}, e]\psi_2 \rrbracket_{\mathcal{M}}. && \text{(Lemma B.5)}
\end{aligned}$$

Axiom I8. $\langle \mathcal{E}, e \rangle(\psi_1 \wedge \psi_2) = \langle \mathcal{E}, e \rangle\psi_1 \wedge \langle \mathcal{E}, e \rangle\psi_2.$

$$\begin{aligned}
\llbracket \langle \mathcal{E}, e \rangle(\psi_1 \wedge \psi_2) \rrbracket_{\mathcal{M}} &= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1 \wedge \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \wedge f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \wedge \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}} \\
&= \llbracket \langle \mathcal{E}, e \rangle\psi_1 \rrbracket_{\mathcal{M}} \wedge \llbracket \langle \mathcal{E}, e \rangle\psi_2 \rrbracket_{\mathcal{M}}. && \text{(Lemma B.5)}
\end{aligned}$$

Axiom I9. $[\mathcal{E}, e](\psi_1 \vee \psi_2) = \text{pre}(e) \rightarrow \langle \mathcal{E}, e \rangle\psi_1 \vee \langle \mathcal{E}, e \rangle\psi_2.$

$$\begin{aligned}
\llbracket [\mathcal{E}, e](\psi_1 \vee \psi_2) \rrbracket_{\mathcal{M}} &= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1 \vee \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1, e) \vee f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \overline{\text{pre}}_{\mathcal{M}}(e) \wedge (\llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \vee \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
& && (a \rightarrow b = a \rightarrow a \wedge b) \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow (\overline{\text{pre}}_{\mathcal{M}} \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}}) \vee (\overline{\text{pre}}_{\mathcal{M}} \wedge \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
& && \text{(distributivity)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \rightarrow \llbracket \langle \mathcal{E}, e \rangle\psi_1 \rrbracket_{\mathcal{M}} \vee \llbracket \langle \mathcal{E}, e \rangle\psi_2 \rrbracket_{\mathcal{M}}. && \text{(Lemma B.5)}
\end{aligned}$$

Axiom I10. $\langle \mathcal{E}, e \rangle(\psi_1 \vee \psi_2) = \langle \mathcal{E}, e \rangle\psi_1 \vee \langle \mathcal{E}, e \rangle\psi_2.$

$$\begin{aligned}
\llbracket \langle \mathcal{E}, e \rangle(\psi_1 \vee \psi_2) \rrbracket_{\mathcal{M}} &= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1 \vee \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
&= \overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \vee f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
&= (\overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}}) \vee (\overline{\text{pre}}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) && \text{(distributivity)} \\
&= \llbracket \langle \mathcal{E}, e \rangle\psi_1 \rrbracket_{\mathcal{M}} \vee \llbracket \langle \mathcal{E}, e \rangle\psi_2 \rrbracket_{\mathcal{M}}. && \text{(Lemma B.5)}
\end{aligned}$$

Axiom I11. $[\mathcal{E}, e](\psi_1 \rightarrow \psi_2) = \langle \mathcal{E}, e \rangle \psi_1 \rightarrow \langle \mathcal{E}, e \rangle \psi_2$.

$$\begin{aligned}
 \llbracket [\mathcal{E}, e](\psi_1 \rightarrow \psi_2) \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1 \rightarrow \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\psi_1, e) \rightarrow f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow (\llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}} && (a \rightarrow (b \rightarrow c) = a \wedge b \rightarrow c) \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \wedge \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}} \\
 & && (b \rightarrow c = b \rightarrow b \wedge c) \\
 &= \llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \rightarrow \llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \wedge \llbracket \langle \mathcal{E}, e \rangle \psi_2 \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \rightarrow \llbracket \langle \mathcal{E}, e \rangle \psi_2 \rrbracket_{\mathcal{M}} && (b \rightarrow c = b \rightarrow b \wedge c).
 \end{aligned}$$

Axiom I12. $\langle \mathcal{E}, e \rangle (\psi_1 \rightarrow \psi_2) = pre(e) \wedge (\langle \mathcal{E}, e \rangle \psi_1 \rightarrow \langle \mathcal{E}, e \rangle \psi_2)$.

$$\begin{aligned}
 \llbracket \langle \mathcal{E}, e \rangle (\psi_1 \rightarrow \psi_2) \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1 \rightarrow \psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rightarrow f(\psi_2, e) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge (\llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge (\overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}}) \\
 & && (a \wedge (b \rightarrow c) = a \wedge (a \wedge b \rightarrow c)) \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge (\overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}} \rightarrow \llbracket f(\psi_2, e) \rrbracket_{\mathcal{M}} \\
 & \quad \wedge \overline{pre}_{\mathcal{M}}(e) \wedge \llbracket f(\psi_1, e) \rrbracket_{\mathcal{M}}) && (b \rightarrow c = b \rightarrow b \wedge c) \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge (\llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \rightarrow \llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \wedge \llbracket \langle \mathcal{E}, e \rangle \psi_2 \rrbracket_{\mathcal{M}}) && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \wedge (\llbracket \langle \mathcal{E}, e \rangle \psi_1 \rrbracket_{\mathcal{M}} \rightarrow \llbracket \langle \mathcal{E}, e \rangle \psi_2 \rrbracket_{\mathcal{M}}) && (b \rightarrow c = b \rightarrow b \wedge c).
 \end{aligned}$$

Axiom I13 $[\mathcal{E}, e] \diamond_i \psi = pre(e) \rightarrow \bigvee_{e' \sim_i e} \diamond_i (\langle \mathcal{E}, e' \rangle \psi)$.

$$\begin{aligned}
 \llbracket [\mathcal{E}, e] \diamond_i \psi \rrbracket_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\diamond_i \psi, e) \rrbracket_{\mathcal{M}} && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \bigvee_{e' \sim_i e} \diamond_i (f(\psi, e') \wedge pre(e')) \rrbracket_{\mathcal{M}} && \text{(Definition B.2)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee_{e' \sim_i e} \diamond_i (\llbracket f(\psi, e') \rrbracket_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e')) \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee_{e' \sim_i e} \diamond_i (\llbracket \langle \mathcal{E}, e' \rangle \psi \rrbracket_{\mathcal{M}}) && \text{(Lemma B.5)} \\
 &= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \bigvee_{e' \sim_i e} \diamond_i (\langle \mathcal{E}, e' \rangle \psi) \rrbracket_{\mathcal{M}}.
 \end{aligned}$$

Axiom I14. $\langle \mathcal{E}, e \rangle \diamond_i \psi = pre(e) \wedge \bigvee_{e' \sim_i e} \diamond_i (\langle \mathcal{E}, e' \rangle \psi)$.

$$\begin{aligned}
\llbracket \langle \mathcal{E}, e \rangle \diamond_i \psi \rrbracket_M &= \overline{pre}_M(e) \wedge \llbracket f(\diamond_i \psi, e) \rrbracket_M && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \wedge \llbracket \bigvee_{e' \sim_i e} \diamond_i (f(\psi, e') \wedge pre(e')) \rrbracket_M && \text{(Definition B.2)} \\
&= \overline{pre}_M(e) \wedge \bigvee_{e' \sim_i e} \diamond_i (\llbracket f(\psi, e') \rrbracket_M \wedge \overline{pre}_M(e')) \\
&= \overline{pre}_M(e) \wedge \bigvee_{e' \sim_i e} \diamond_i (\llbracket \langle \mathcal{E}, e' \rangle \psi \rrbracket_M) && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \wedge \llbracket \bigvee_{e' \sim_i e} \diamond_i (\langle \mathcal{E}, e' \rangle \psi) \rrbracket_M.
\end{aligned}$$

Axiom I15. $[\mathcal{E}, e] \square_i \psi = pre(e) \rightarrow \bigwedge_{e' \sim_i e} \square_i ([\mathcal{E}, e'] \psi)$.

$$\begin{aligned}
\llbracket [\mathcal{E}, e] \square_i \psi \rrbracket_M &= \overline{pre}_M(e) \rightarrow \llbracket f(\square_i \psi, e) \rrbracket_M && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \rightarrow \llbracket \bigwedge_{e' \sim_i e} \square_i (pre(e') \rightarrow f(\psi, e')) \rrbracket_M && \text{(Definition B.2)} \\
&= \overline{pre}_M(e) \rightarrow \bigwedge_{e' \sim_i e} \square_i (\overline{pre}_M(e') \rightarrow \llbracket f(\psi, e') \rrbracket_M) \\
&= \overline{pre}_M(e) \rightarrow \bigwedge_{e' \sim_i e} \square_i (\llbracket [\mathcal{E}, e'] \psi \rrbracket_M) && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \rightarrow \llbracket \bigwedge_{e' \sim_i e} \square_i ([\mathcal{E}, e'] \psi) \rrbracket_M.
\end{aligned}$$

Axiom I16. $\langle \mathcal{E}, e \rangle \square_i \psi = pre(e) \wedge \bigwedge_{e' \sim_i e} \square_i ([\mathcal{E}, e'] \psi)$.

$$\begin{aligned}
\llbracket \langle \mathcal{E}, e \rangle \square_i \psi \rrbracket_M &= \overline{pre}_M(e) \wedge \llbracket f(\square_i \psi, e) \rrbracket_M && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \wedge \llbracket \bigwedge_{e' \sim_i e} \square_i (pre(e') \rightarrow f(\psi, e')) \rrbracket_M && \text{(Definition B.2)} \\
&= \overline{pre}_M(e) \wedge \bigwedge_{e' \sim_i e} \square_i (\overline{pre}_M(e') \rightarrow \llbracket f(\psi, e') \rrbracket_M) \\
&= \overline{pre}_M(e) \wedge \bigwedge_{e' \sim_i e} \square_i (\llbracket [\mathcal{E}, e'] \psi \rrbracket_M) && \text{(Lemma B.5)} \\
&= \overline{pre}_M(e) \wedge \llbracket \bigwedge_{e' \sim_i e} \square_i ([\mathcal{E}, e'] \psi) \rrbracket_M.
\end{aligned}$$

$$\begin{aligned}
 \text{Axiom I17. } [\mathcal{E}, e](\alpha\mu_i(\psi) \geq \beta) &= pre(e) \rightarrow \sum_{\phi \in \Phi} e' \sim_i e \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi([\mathcal{E}, e']\psi) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(\top) \geq 0 \\
 \llbracket [\mathcal{E}, e](\alpha\mu_i(\psi) \geq \beta) \rrbracket_M &= \overline{pre}_M(e) \rightarrow \llbracket f(\alpha\mu_i(\psi) \geq \beta, e) \rrbracket_M \\
 = \overline{pre}_M(e) \rightarrow \llbracket \alpha \sum_{\phi \in \Phi} \mu_i^\phi(f(\psi, e')) P_i(e') pre(e' \mid \phi) - \beta \sum_{\phi \in \Phi} \mu_i^\phi(\top) P_i(e') pre(e' \mid \phi) \geq 0 \rrbracket_M & \quad (\text{Definition B.2}) \\
 = \overline{pre}_M(e) \rightarrow \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi(\llbracket f(\psi, e') \rrbracket_M \wedge a) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
 = \overline{pre}_M(e) \rightarrow \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi(\llbracket f(\psi, e') \rrbracket_M \wedge \overline{pre}_M(e') \wedge a) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
 & \quad (\llbracket \phi \rrbracket_M \leq \overline{pre}_M(e') \text{ if } pre(e' \mid \phi) \neq 0, \text{ cf. Proposition 4.16}) \\
 = \overline{pre}_M(e) \rightarrow \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi(\overline{pre}_M(e') \rightarrow \llbracket f(\psi, e') \rrbracket_M) \wedge \overline{pre}_M(e') \wedge a + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
 & \quad (a \wedge (a \rightarrow b) = a \wedge b) \\
 = \overline{pre}_M(e) \rightarrow \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi(\llbracket [\mathcal{E}, e']\psi \rrbracket_M \wedge \overline{pre}_M(e') \wedge a) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
 & \quad (\text{Lemma B.5}) \\
 = \overline{pre}_M(e) \rightarrow \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi(\llbracket [\mathcal{E}, e']\psi \rrbracket_M \wedge a) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
 = \overline{pre}_M(e) \rightarrow \llbracket \sum_{\phi \in \Phi} \alpha P_i(e') pre(e' \mid \phi) \mu_i^\phi([\mathcal{E}, e']\psi) + \sum_{\phi \in \Phi} e' \sim_i e -\beta P_i(e') pre(e' \mid \phi) \mu_i^\phi(\top) \geq 0 \rrbracket_M \\
 & \quad (\llbracket \phi \rrbracket_M \leq \overline{pre}_M(e') \text{ if } pre(e' \mid \phi) \neq 0)
 \end{aligned}$$

$$\begin{aligned}
\mathbf{Axiom\ I18.} \quad & \langle \mathcal{E}, e \rangle (\alpha \mu_i(\psi) \geq \beta) = \text{pre}(e) \wedge \sum_{\phi \in \Phi} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\langle \mathcal{E}, e' \rangle \psi) + \sum_{\phi \in \Phi} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\top) \geq 0 \\
& \llbracket \langle \mathcal{E}, e \rangle (\alpha \mu_i(\psi) \geq \beta) \rrbracket_M \\
& = \overline{\text{pre}}_M(e) \wedge \llbracket f(\alpha \mu_i(\psi) \geq \beta, e) \rrbracket_M \\
& = \overline{\text{pre}}_M(e) \wedge \llbracket \alpha \sum_{\phi \in \Phi} \mu_i^\phi(f(\psi, e')) P_i(e') \text{pre}(e' \mid \phi) - \beta \sum_{\phi \in \Phi} \mu_i^\phi(\top) P_i(e') \text{pre}(e' \mid \phi) \geq 0 \rrbracket_M \\
& \hspace{10em} \text{(Definition B.2)} \\
& = \overline{\text{pre}}_M(e) \wedge \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\llbracket f(\psi, e') \rrbracket_M \wedge a) + \sum_{\phi \in \Phi} -\beta P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
& = \overline{\text{pre}}_M(e) \wedge \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\llbracket f(\psi, e') \rrbracket_M \wedge \overline{\text{pre}}_M(e') \wedge a) + \sum_{\phi \in \Phi} -\beta P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
& \hspace{10em} (\llbracket \varphi \rrbracket_M \leq \overline{\text{pre}}_M(e') \text{ if } \text{pre}(e' \mid \varphi) \neq 0, \text{ cf. Proposition 4.16}) \\
& = \overline{\text{pre}}_M(e) \wedge \bigvee_{\phi \in \Phi} \left\{ a \sum_{\phi \in \Phi} \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\llbracket \langle \mathcal{E}, e' \rangle \psi \rrbracket_M \wedge a) + \sum_{\phi \in \Phi} -\beta P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(a) \geq 0 \right\} \\
& \hspace{10em} \text{(Lemma B.5)} \\
& = \overline{\text{pre}}_M(e) \wedge \llbracket \alpha P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\langle \mathcal{E}, e' \rangle \psi) + \sum_{\phi \in \Phi} -\beta P_i(e') \text{pre}(e' \mid \phi) \mu_i^\phi(\top) \geq 0 \rrbracket_M
\end{aligned}$$

C PROOF OF THE COMPLETENESS OF IPDEL

In the present section, we prove the weak completeness of IPDEL w.r.t. APE-models. Recall that a calculus is *weakly complete* w.r.t. a semantics if it provides a proof for every validity; namely, for any formula ϕ , if $\models \phi$, then $\vdash \phi$. Similarly to akin logical systems (cf. References [10, 34, 36, 41] [2, 7, 14]), the proof relies on a reduction procedure of IPDEL-formulas to formulas of the static fragment of IPDEL (referred to in what follows as IPEL), which preserves provable equivalence. This reduction procedure is effected using the interaction axioms and the rule of substitution of equivalent formulas. We omit the details, since this procedure is standard (see for instance References [9, 10, 43] for details). In the remainder of the present section, we prove the weak completeness of IPEL w.r.t. APE-models, i.e., we show that every APE-validity in the language of IPEL is a theorem of IPEL. By contraposition, this is equivalent to proving that for any IPEL-formula φ that is not an IPEL-theorem there exists an APE-model \mathcal{M} that does not satisfy φ in the sense that $\llbracket \varphi \rrbracket_{\mathcal{M}} \neq \top$.

The proof will proceed as follows: In Section C.1, we extract a finite sublattice of the Lindenbaum-Tarski algebra of the logic that contains φ , and we prove that it is an Epistemic Heyting Algebra satisfying certain properties akin to those described in Reference [22]. Then, in Section C.2, following ideas from Reference [21] adapted to the algebraic setting, we define appropriate i -measures over the finite Epistemic Heyting Algebra to turn it into an APE-model that does not satisfy φ .

C.1 The Epistemic Heyting Algebra \mathbb{A}_φ^\star

In this subsection, we construct the finite epistemic Heyting algebra on which the counter-model for φ is based. The construction consists of a number of steps, starting with the Lindenbaum-Tarski algebra of \mathcal{L} and restricting it accordingly.

Henceforth, we let

$$\mathbb{A} = (A, \top_{\mathbb{A}}, \perp_{\mathbb{A}}, \vee_{\mathbb{A}}, \wedge_{\mathbb{A}}, \rightarrow_{\mathbb{A}}, (\diamond_i)_{i \in \text{Ag}}, (\square_i)_{i \in \text{Ag}}) \quad (\text{C.1})$$

denote the Lindenbaum-Tarski algebra of IPEL. We will use $\neg_{\mathbb{A}}(\bullet)$ as shorthand for $\bullet \rightarrow_{\mathbb{A}} \perp_{\mathbb{A}}$. For any agent i , we define:

$$\diamond_i \mathbb{A} := \{\diamond_i a \in \mathbb{A} \mid a \in \mathbb{A}\}.$$

For any formula $\sigma \in \mathcal{L}_{\text{IPEL}}$, we let $\sigma^{\mathbb{A}} \in \mathbb{A}$ denote the equivalence class of σ modulo provable equivalence in IPEL. Let

$$\mathbb{B} := (B, \top_{\mathbb{B}}, \perp_{\mathbb{B}}, \vee_{\mathbb{B}}, \wedge_{\mathbb{B}}, \neg_{\mathbb{B}})$$

be the Boolean extension of the Heyting algebra reduct of \mathbb{A} (see Reference [37, Section 13, page 450]).¹¹ To enhance readability, we identify \mathbb{A} with its image through the embedding $A \hookrightarrow B$. Recall that \mathbb{A} is a sublattice of \mathbb{B} . Henceforth, we will use \vee and \wedge and \top and \perp ambiguously to denote the operations on both algebras. Since $\diamond_i \mathbb{A}$ is a Boolean algebra (see Lemma 4.4) and, in every Boolean algebra, negation is unique, we have that $\neg_{\mathbb{A}} a = \neg_{\mathbb{B}} a$ for every $a \in \diamond_i \mathbb{A}$ and for every agent $i \in \text{Ag}$.

Let φ be an IPEL-formula that is not a theorem. Let

$$S_\varphi := \{\sigma^{\mathbb{A}} \mid \sigma \text{ is a subformula of } \varphi\},$$

let Ag_φ be the set of agents that appear in φ and let $S_\varphi^\diamond \supseteq S_\varphi$ be

$$S_\varphi^\diamond := S_\varphi \cup \{(\diamond_i \sigma)^{\mathbb{A}}, (\neg \diamond_i \sigma)^{\mathbb{A}} \mid \sigma \in S_\varphi \text{ and } i \in \text{Ag}_\varphi\}.$$

¹¹The Boolean extension of \mathbb{A} can be identified with the algebra of clopens of the Esakia space dual to \mathbb{A} . Notice that this is exactly the same construction semantically underlying the Gödel-Tarski translation (cf. Reference [18, Section 3] for an expanded discussion).

Notice that the sets S_φ and S_φ^\diamond are finite. Now, let $\mathbb{B}_\varphi \subseteq \mathbb{B}$ be the Boolean subalgebra of \mathbb{B} generated by S_φ^\diamond . Since S_φ^\diamond is finite, so will be the domain of \mathbb{B}_φ (which we denote with B_φ). In addition, since \mathbb{B}_φ is a sub-lattice of the non-trivial Boolean algebra \mathbb{B} , it contains at least two elements and has at least one atom. Plus, it follows that \mathbb{B}_φ is generated by its atoms. In view of what will follow, let us endow \mathbb{B}_φ with a measure $\mu_{\mathbb{B}}$ as follows: Let $n \geq 1$ be the number of atoms of \mathbb{B}_φ . For every $a \in \mathbb{B}_\varphi$ that is above exactly m atoms, let

$$\mu_{\mathbb{B}}(a) = \frac{m}{n}. \quad (\text{C.2})$$

Now, let

$$\mathbb{A}_\varphi := (A_\varphi, \top, \perp, \wedge, \vee)$$

with $A_\varphi := A \cap B_\varphi$. Notice that, since both \mathbb{A} and \mathbb{B}_φ are distributive lattices, so is \mathbb{A}_φ . For every agent $i \in \text{Ag}_\varphi$, we define

$$A_\varphi^{\diamond i} := \{a \in \mathbb{A}_\varphi \mid \text{there exists } \sigma \in \mathcal{L} \text{ such that } \diamond_i \sigma \in a\} = A_\varphi \cap \diamond_i \mathbb{A}.$$

Notice that, if $a \in A_\varphi^{\diamond i}$, then $\neg_{\mathbb{A}} a \in A_\varphi^{\diamond i}$ as well (since $\neg_{\mathbb{B}} a \in B_\varphi$ and $\neg_{\mathbb{B}} a = \neg_{\mathbb{A}} a$). Hence, for every agent $i \in \text{Ag}_\varphi$, $(A_\varphi^{\diamond i}, \top, \perp, \wedge, \vee, \neg_{\mathbb{A}})$ is a Boolean subalgebra of \mathbb{A}_φ . We are now ready to endow \mathbb{A}_φ with an epistemic Heyting algebra structure.

Definition C.1. Let

$$\mathbb{A}_\varphi^\star := (\mathbb{A}_\varphi, \rightarrow^\star, (\diamond_i^\star)_{i \in \text{Ag}}, (\square_i^\star)_{i \in \text{Ag}}),$$

where, for all $a, b \in \mathbb{A}_\varphi$,

$$a \rightarrow^\star b := \bigvee \{c \in \mathbb{A}_\varphi \mid c \leq a \rightarrow_{\mathbb{A}} b\} = \bigvee \{c \in \mathbb{A}_\varphi \mid c \wedge a \leq b\},$$

for all $i \in \text{Ag}_\varphi$ and $a \in \mathbb{A}_\varphi$,

$$\diamond_i^\star a := \bigwedge \{b \in A_\varphi^{\diamond i} \mid a \leq b\} \quad \text{and} \quad \square_i^\star a := \bigvee \{b \in A_\varphi^{\diamond i} \mid b \leq a\},$$

for all $i \notin \text{Ag}_\varphi$ and $a \in \mathbb{A}_\varphi$,

$$\diamond_i^\star a := \begin{cases} \top & \text{if } a \neq \perp, \\ \perp & \text{if } a = \perp, \end{cases} \quad \text{and} \quad \square_i^\star a := \begin{cases} \perp & \text{if } a \neq \top, \\ \top & \text{if } a = \top. \end{cases}$$

The operations above are well defined, since \mathbb{A}_φ is a finite distributive lattice, and hence all the joins and meets exist.

LEMMA C.2. For every $i \in \text{Ag}_\varphi$, the algebra \mathbb{A}_φ^\star satisfies the following properties:

- (1) $\diamond_i^\star \mathbb{A}_\varphi^\star = \{\diamond_i^\star a \mid a \in \mathbb{A}_\varphi^\star\} \subseteq A_\varphi^{\diamond i}$;
- (2) $\diamond_i^\star \mathbb{A}_\varphi^\star = \square_i^\star \mathbb{A}_\varphi^\star$;
- (3) for all $a \in A_\varphi^{\diamond i}$, it holds that $\diamond_i^\star a = a$ and $\square_i^\star a = a$;
- (4) for all $a, b \in \mathbb{A}_\varphi^\star$, if $a \rightarrow_{\mathbb{A}} b \in \mathbb{A}_\varphi^\star$, then $a \rightarrow^\star b = a \rightarrow_{\mathbb{A}} b$;
- (5) for all $a \in \mathbb{A}_\varphi^\star$, if $\diamond_i a \in \mathbb{A}_\varphi^\star$ (resp. $\square_i a \in \mathbb{A}_\varphi^\star$), then $\diamond_i^\star a = \diamond_i a$ (resp. $\square_i^\star a = \square_i a$);
- (6) for all formulas $\psi, \varphi \in \mathcal{L}$, if $(\diamond_i \psi)^\mathbb{A} \in S_\varphi^\diamond$ (resp. $(\square_i \psi)^\mathbb{A} \in S_\varphi^\diamond$ or $(\psi \rightarrow \chi)^\mathbb{A} \in S_\varphi^\diamond$), then $\diamond_i^\star \psi^\mathbb{A} = \diamond_i \psi^\mathbb{A}$ (resp. $\square_i^\star \psi^\mathbb{A} = \square_i \psi^\mathbb{A}$ or $\psi^\mathbb{A} \rightarrow^\star \chi^\mathbb{A} = \psi^\mathbb{A} \rightarrow_{\mathbb{A}} \chi^\mathbb{A}$);
- (7) $\diamond_i^\star \mathbb{A}_\varphi^\star = \{\diamond_i^\star a \mid a \in \mathbb{A}_\varphi^\star\} = A_\varphi^{\diamond i}$.

PROOF. The first five items follow immediately from the definition of \diamond_i^* and \square_i^* . Item (6) is an application of items (4) and (5). Item (7) follows from items (1) and (3). \square

LEMMA C.3. *The algebra \mathbb{A}_φ^* is an epistemic Heyting algebra.*

PROOF. As mentioned early on, \mathbb{A}_φ is a distributive lattice. Moreover, by definition, \rightarrow^* is the right residual of \wedge in \mathbb{A}_φ . This shows that \mathbb{A}_φ^* is a Heyting algebra. To prove that \mathbb{A}_φ^* is an epistemic Heyting algebra, it remains to show that \mathbb{A}_φ^* satisfies the following axioms (c.f. Definition 4.1 and Definition 4.3):

$$a \leq \diamond_i a \tag{M1}$$

$$\square_i a \leq a \tag{M2}$$

$$\diamond_i(a \vee b) \leq \diamond_i a \vee \diamond_i b \tag{M3}$$

$$\square_i(a \rightarrow b) \leq \square_i a \rightarrow \square_i b \tag{M4}$$

$$\diamond_i a \leq \square_i \diamond_i a \tag{M5}$$

$$\diamond_i \square_i a \leq \square_i a \tag{M6}$$

$$\square_i(a \rightarrow b) \leq \diamond_i a \rightarrow \diamond_i b \tag{M7}$$

$$\diamond_i \perp \leq \perp \tag{M8}$$

$$\top \leq \square_i \top \tag{M9}$$

$$\diamond_i a \vee \neg \diamond_i a = \top. \tag{E}$$

Let $i \in \text{Ag}_\varphi$. By definition, it immediately follows that \diamond_i^* and \square_i^* verify axioms M1 and M2. Axiom M3 holds because $\diamond_i^* a \vee \diamond_i^* b \in \diamond_i \mathbb{A}_\varphi$ and $a \vee b \leq \diamond_i^* a \vee \diamond_i^* b$ (and similarly for axiom M4).

As for axioms M5 and M6, since $\diamond_i^* a, \square_i^* a \in \diamond_i \mathbb{A}_\varphi$, by item (3) of Lemma C.2, we obtain that $\diamond_i^* \square_i^* a = \square_i^* a$ and $\diamond_i^* a = \square_i^* \diamond_i^* a$, which imply the axioms.

In the context of axioms M1 through M6, axiom M7 is equivalent to $\diamond_i(\diamond_i p \rightarrow \diamond_i q) \rightarrow (\diamond_i p \rightarrow \diamond_i q)$ (see Reference [11, Lemma 2]), so let us show that \mathbb{A}_φ^* satisfies $\diamond_i(\diamond_i p \rightarrow \diamond_i q) \rightarrow (\diamond_i p \rightarrow \diamond_i q)$. Observe that for all $a, b \in \mathbb{A}_\varphi^*$, since $\diamond_i^* a, \diamond_i^* b \in A_\varphi^{\diamond_i}$ and $A_\varphi^{\diamond_i}$ is a Boolean algebra (and hence contains $\neg_{\mathbb{A}} \diamond_i^* a$), we have that

$$\diamond_i^* a \rightarrow_{\mathbb{A}} \diamond_i^* b = \neg_{\mathbb{A}} \diamond_i^* a \vee \diamond_i^* b \in A_\varphi^{\diamond_i},$$

which implies by item (4) of Lemma C.2 that

$$\diamond_i^* a \rightarrow^* \diamond_i^* b = \diamond_i^* a \rightarrow_{\mathbb{A}} \diamond_i^* b. \tag{C.3}$$

Now, by item (3) of Lemma C.2, we have that

$$\diamond_i^*(\diamond_i^* a \rightarrow_{\mathbb{A}} \diamond_i^* b) = \diamond_i^* a \rightarrow_{\mathbb{A}} \diamond_i^* b,$$

which by the Equation (C.3) is equivalent to

$$\diamond_i^*(\diamond_i^* a \rightarrow^* \diamond_i^* b) = \diamond_i^* a \rightarrow^* \diamond_i^* b;$$

that is, \mathbb{A}_φ^* satisfies $\diamond_i(\diamond_i p \rightarrow \diamond_i q) \rightarrow (\diamond_i p \rightarrow \diamond_i q)$.

Axioms M8 and M9 follow from the fact that $\top, \perp \in A_\varphi \cap \diamond_i \mathbb{A}$ and item (3) of Lemma C.2.

Finally, axiom E follows immediately from item (4) of Lemma C.2 and from the fact that $A_\varphi^{\diamond_i}$ is a Boolean algebra. Hence, if $a \in A_\varphi^{\diamond_i}$, then $(a \rightarrow_{\mathbb{A}} \perp_{\mathbb{A}}) \in A_\varphi^{\diamond_i}$. \square

C.2 Measures on \mathbb{A}_φ^\star

In this section, for each agent $i \in \text{Ag}_\varphi$, we will define an i -measure on the algebra \mathbb{A}_φ^\star and a valuation on \mathbb{A}_φ^\star , so as to define an APE-model \mathcal{M}_φ such that $\llbracket \sigma \rrbracket_{\mathcal{M}_\varphi} = \sigma^{\mathbb{A}}$ for every subformula σ of φ . Before defining the measures, we will state some auxiliary results.

LEMMA C.4. *The system IPEL proves all classical truths about linear inequalities.*

PROOF. See Reference [21] for an explanation of why axioms N0 to N6 are enough. Notice that, even though the result is proven for classical logic, it still holds for IPEL. Indeed, the fragment of the logic involving inequalities is classical because of the axiom N5: $(\tau \geq \beta) \vee (\neg \tau \geq \beta)$. \square

LEMMA C.5. *The formulas*

$$\left(\diamond_i \psi \wedge \left(\sum_m \alpha_m \mu_i(\phi_m) \geq \beta \right) \right) \rightarrow \left(\sum_m \alpha_m \mu_i(\phi_m \wedge \diamond_i \psi) \geq \beta \right) \quad (\text{C.4})$$

and

$$\left(\diamond_i \psi \wedge \left(\sum_m \alpha_m \mu_i(\phi_m) < \beta \right) \right) \rightarrow \left(\sum_m \alpha_m \mu_i(\phi_m \wedge \diamond_i \psi) < \beta \right) \quad (\text{C.5})$$

are provable in IPEL.

PROOF. We only prove (C.4), the proof of (C.5) being almost verbatim. Early on, we observed (see Lemma 5.8) that axiom P4 implies the validity of $\Box_i \varphi \leftrightarrow (\mu_i(\varphi) = 1)$. This and axiom M5 (i.e., $\diamond_i \psi \leftrightarrow \Box_i \diamond_i \psi$) imply

$$\vdash_{\text{IPEL}} \diamond_i \psi \leftrightarrow (\mu_i(\diamond_i \psi) = 1). \quad (\text{C.6})$$

Since $\vdash_{\text{IPEL}} \diamond_i \psi \rightarrow (\diamond_i \psi \vee \phi_m)$ for every $\phi_m \in \mathcal{L}$, by rule Sub_μ , we obtain

$$\vdash_{\text{IPEL}} \mu_i(\diamond_i \psi) \leq \mu_i(\diamond_i \psi \vee \phi_m). \quad (\text{C.7})$$

From (C.7) and Lemma C.4, we deduce that

$$\vdash_{\text{IPEL}} \mu_i(\diamond_i \psi) = 1 \rightarrow \mu_i(\phi_m \vee \diamond_i \psi) = 1. \quad (\text{C.8})$$

Lemma C.4 and axiom P3 (i.e., $\mu_i(\phi_m) = \mu_i(\phi_m \vee \diamond_i \psi) + \mu_i(\phi_m \wedge \diamond_i \psi) - \mu_i(\diamond_i \psi)$) entail

$$\vdash_{\text{IPEL}} \left(\sum_m \alpha_m \mu_i(\phi_m) \geq \beta \right) \leftrightarrow \left(\sum_m \alpha_m \left(\mu_i(\phi_m \wedge \diamond_i \psi) + \mu_i(\phi_m \vee \diamond_i \psi) - \mu_i(\diamond_i \psi) \right) \geq \beta \right). \quad (\text{C.9})$$

Combining (C.6), (C.8), and (C.9), we obtain

$$\vdash_{\text{IPEL}} \left(\diamond_i \psi \wedge A \right) \rightarrow \left((\mu_i(\diamond_i \psi) = 1) \wedge \bigwedge_m (\mu_i(\phi_m \vee \diamond_i \psi) = 1) \wedge B \right) \quad (\text{C.10})$$

with

$$A := \sum_m \alpha_m \mu_i(\phi_m) \geq \beta,$$

and

$$B := \sum_m \alpha_m (\mu_i(\phi_m \wedge \diamond_i \psi) + \mu_i(\phi_m \vee \diamond_i \psi) - \mu_i(\diamond_i \psi)) \geq \beta.$$

Again, by using Lemma C.4, we obtain that

$$\vdash_{\text{IPEL}} \left((\mu_i(\diamond_i \psi) = 1) \wedge \bigwedge_m (\mu_i(\phi_m \vee \diamond_i \psi) = 1) \wedge B \right) \rightarrow D \quad (\text{C.11})$$

with

$$D := \sum_m \alpha_m \mu_i(\phi_m \wedge \diamond_i \psi) \geq \beta.$$

Putting (C.10) and (C.11) together, we finally get:

$$\vdash_{\text{IPEL}} \left(\diamond_i \psi \wedge \sum_m \alpha_m \mu_i(\phi_m) \geq \beta \right) \rightarrow \left(\sum_m \alpha_m \mu_i(\phi_m \wedge \diamond_i \psi) \geq \beta \right),$$

as desired. \square

Observe that for any agent $i \in \text{Ag}_\varphi$, since \mathbb{A}_φ^\star is finite and $\diamond_i^\star \mathbb{A}_\varphi^\star = \mathbb{A}_\varphi^{\diamond_i}$ is a Boolean algebra, it is the case that the i -minimal elements are the atoms of this Boolean algebra and every element of $\mathbb{A}_\varphi^{\diamond_i}$ can be written as the union of some of these i -minimal elements. Let n_i be the number of i -minimal elements of \mathbb{A}_φ^\star . Let us call a_k^i , for $1 \leq k \leq n_i$, the i -minimal elements of \mathbb{A}_φ^\star . Now, for each i -probability formula σ with $\sigma^\mathbb{A} \in S_\varphi^\diamond$, we have that $\sigma^\mathbb{A} \in \mathbb{A}_\varphi^{\diamond_i}$. Hence, we have that $(\neg\sigma)^\mathbb{A} \in \mathbb{A}_\varphi^{\diamond_i}$. This implies that there exists a function $f_\sigma : \{1, 2, \dots, n_i\} \rightarrow \{0, 1\}$ such that

$$\sigma^\mathbb{A} = \bigvee_{f_\sigma(k)=1} a_k^i \quad \text{and} \quad (\neg\sigma)^\mathbb{A} = \bigvee_{f_\sigma(k)=0} a_k^i.$$

It should be stressed that, since \vee and \wedge in \mathbb{A}_φ^\star are inherited by \mathbb{A} , these equalities hold in \mathbb{A} as well.

Now, let us fix $i \in \text{Ag}_\varphi$. For every $k \in n_i$, we define a system of equations $E_{a_k^i}$, with variables x_b for every $b \leq a_k^i$ as follows¹²:

$$E_{a_k^i} := \left(\begin{array}{ll} \sum \alpha_m \cdot x_{\psi_m^\mathbb{A} \wedge a_k^i}^i \geq \beta, & \text{for all } \sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta) \text{ with } \sigma^\mathbb{A} \in S_\varphi^\diamond \text{ and } f_\sigma(k) = 1 \\ \sum \alpha_m \cdot x_{\psi_m^\mathbb{A} \wedge a_k^i}^i < \beta, & \text{for all } \sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta) \text{ with } \sigma^\mathbb{A} \in S_\varphi^\diamond \text{ and } f_\sigma(k) = 0 \\ x_b^i \geq 0 \text{ and } x_b \leq 1, & \text{for all } b \in \mathbb{A}_\varphi^\star \text{ with } b \leq a_k^i \\ x_b^i + x_c^i = x_{b \wedge c}^i + x_{b \vee c}^i, & \text{for all } b, c \in \mathbb{A}_\varphi^\star \text{ with } b, c \leq a_k^i \\ x_b^i \leq x_c^i, & \text{for all } b, c \in \mathbb{A}_\varphi^\star \text{ with } b \leq c \leq a_k^i \\ x_{a_k^i}^\perp = 0 \\ x_{a_k^i}^i = 1. \end{array} \right).$$

For a solution s of the above system, we denote with $(x_b^i)^s$ the solution according to s of x_b^i .

Notice that the system is designed in such a way that any particular solution (cf. Lemma C.8) provides an i -measure on \mathbb{A}_φ^\star that guarantees that the valuation of an i -probability formula σ is $\sigma^\mathbb{A}$. Indeed, the first two types of inequalities in the system will guarantee that exactly the i -minimal elements of \mathbb{A}_φ^\star below $\sigma^\mathbb{A}$ will constitute $\llbracket \sigma \rrbracket$ (see Definition 5.6). The rest of the inequalities will guarantee that the solution satisfies the basic properties of i -measures.

Observe that, for every $b \leq a_k^i$, there exists a formula τ_b such that $b = \tau_b^\mathbb{A}$ and if $b \leq c$, then $\vdash_{\text{IPEL}} \tau_b \rightarrow \tau_c$. Let $E_{a_k^i}^\tau$ be the system of equations where each x_b^i is replaced by $\mu_i(\tau_b)$. Since a_k^i is i -minimal, we can assume w.l.o.g that $\tau_{a_k^i}$ is of the form $\diamond_i \tau'$. Furthermore, let $PS_i \subseteq S_\varphi^\diamond$ be the set of i -probability formulas that are subformulas of φ . For every $\sigma^\mathbb{A} \in PS_i$ such that $\sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta)$, let $\sigma[a_k^i]$ be the formula $\sum \alpha_m \cdot \mu_i(\psi_m \wedge \tau_{a_k^i}) \geq \beta$.

LEMMA C.6. *For every $k \in n_i$, the system $E_{a_k^i}$ has a solution.*

¹²The sums in the system of equations $E_{a_k^i}$ range over m .

PROOF. Notice that all but the first two types of inequalities in $E_{a_k^i}^\tau$ are provable in IPEL, as they are immediate consequences of axioms P1, P2, P3, and the rule Sub_μ . Heading towards a contradiction, let us first assume that $E_{a_k^i}$ does not have a solution at all. This is a truth about linear inequalities of rational numbers, hence, by Lemma C.4, it is provable in IPDEL. As mentioned above, since some inequalities are provable, this is tantamount to saying that

$$\vdash_{\text{IPEL}} \neg \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a_k^i] \right) \right). \quad (\text{C.12})$$

Notice that, by Lemma C.5, we have: for every $\sigma^{\mathbb{A}} \in PS_i$,

$$\vdash_{\text{IPEL}} (\sigma \wedge \tau_{a_k^i}) \rightarrow \sigma[a_k^i]$$

and

$$\vdash_{\text{IPEL}} (\neg\sigma \wedge \tau_{a_k^i}) \rightarrow \neg\sigma[a_k^i].$$

Therefore,

$$\vdash_{\text{IPEL}} \left(\left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma \right) \right) \wedge \tau_{a_k^i} \right) \rightarrow \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a_k^i] \right) \right). \quad (\text{C.13})$$

Since one direction of contraposition is provable in intuitionistic logic we obtain that:

$$\vdash_{\text{IPEL}} \left(\neg \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a_k^i] \right) \right) \right) \rightarrow \neg \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma \right) \right) \wedge \tau_{a_k^i}. \quad (\text{C.14})$$

(C.12) and (C.14) imply that

$$\vdash_{\text{IPEL}} \neg \left(\left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma \right) \right) \wedge \tau_{a_k^i} \right). \quad (\text{C.15})$$

In addition, \mathbb{A}_φ^* inherits the order from \mathbb{A} and by construction $a_k^i \leq \sigma^{\mathbb{A}}$ when $f_\sigma(k) = 1$ and $a_k^i \leq (\neg\sigma)^{\mathbb{A}}$ when $f_\sigma(k) = 0$. Hence, we have that, for all $\sigma \in PS_i$, if $f_\sigma(k) = 1$, then $\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \sigma$ and if $f_\sigma(k) = 0$, then $\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \neg\sigma$. Therefore, we have

$$\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma \right) \right).$$

Hence,

$$\vdash_{\text{IPEL}} \neg \tau_{a_k^i} \leftrightarrow \neg \left(\left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg \sigma \right) \right) \wedge \tau_{a_k^i} \right)$$

and by (C.15)

$$\vdash_{\text{IPEL}} \neg \tau_{a_k^i}.$$

We have reached a contradiction, because a_k^i is an element of \mathbb{A} different from \perp , and hence each formula corresponding to it is consistent. Therefore $E_{a_k^i}$ has a solution. \square

The next two lemmas are key to show that measures on \mathbb{A}_φ^* satisfy the form of strict monotonicity of item (6) of Definition 4.6. This condition generalizes to the intuitionistic setting the requirement that probability distributions in the classical canonical model constructions (cf. References [1, 42]) be strictly positive. As far as we know, this is the most detailed presentation of this fact, and its projection to the classical setting accounts for the details missing there.

LEMMA C.7. *For every $k \in n_i$ and every $b < c \leq a_k^i$, the system $E_{a_k^i}$ has a solution $s_{b,c}$ such that $(x_b^i)^{s_{b,c}} < (x_c^i)^{s_{b,c}}$.*

PROOF. Heading towards a contradiction, let $b < c \leq a_k^i$ such that, for every solution s of $E_{a_k^i}$, we have $(x_b^i)^s = (x_c^i)^s$. This is a fact of inequalities of real numbers and, therefore, by Lemma C.4, it is provable in IPEL. Since all but the first two types of inequalities in $E_{a_k^i}$ are provable in IPEL, we have that

$$\vdash_{\text{IPEL}} \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg \sigma[a_k^i] \right) \right) \rightarrow \mu_i(\tau_b) = \mu_i(\tau_c).$$

Since $\vdash_{\text{IPEL}} \tau_b \rightarrow \tau_c$, necessitation implies $\vdash_{\text{IPEL}} \Box_i(\tau_b \rightarrow \tau_c)$. Using axiom P4

$$\left((\Box_i(\phi \rightarrow \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi)) \right) \leftrightarrow \Box_i(\psi \leftrightarrow \phi),$$

we obtain that

$$\vdash_{\text{IPEL}} \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg \sigma[a_k^i] \right) \right) \rightarrow \Box_i(\tau_c \rightarrow \tau_b). \quad (\text{C.16})$$

Recall that¹³

$$\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg \sigma \right) \right). \quad (\text{C.17})$$

¹³see proof of Lemma C.6.

Using Lemma C.5 and (C.17) (cf. (C.13)), we get that

$$\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \left(\left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in \text{PS}_i \\ f_{\sigma}(k)=1}} \sigma[a_k^i] \right) \wedge \left(\bigwedge_{\substack{\sigma^{\mathbb{A}} \in \text{PS}_i \\ f_{\sigma}(k)=0}} \neg \sigma[a_k^i] \right) \right). \quad (\text{C.18})$$

From (C.16) and (C.18), we deduce that

$$\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow \Box_i(\tau_c \rightarrow \tau_b).$$

By axiom M2 ($\Box_i p \rightarrow p$), we have

$$\vdash_{\text{IPEL}} \tau_{a_k^i} \rightarrow (\tau_c \rightarrow \tau_b),$$

which is equivalent to

$$\vdash_{\text{IPEL}} (\tau_{a_k^i} \wedge \tau_c) \rightarrow \tau_b.$$

Since $\vdash_{\text{IPEL}} \tau_c \rightarrow \tau_{a_k^i}$, the equation above implies that

$$\vdash_{\text{IPEL}} \tau_c \rightarrow \tau_b.$$

This last equation is a contradiction, since in \mathbb{A} , the Lindenbaum-Tarski algebra of IPEL, we have that $c \not\leq b$. Therefore, for every such pair $b < c \leq a_k^i$, there exists a solution $s_{b,c}$ of $E_{a_k^i}$ such that $(x_b^i)^{s_{b,c}} < (x_c^i)^{s_{b,c}}$. \square

LEMMA C.8. *For every $k \in n_i$, the system $E_{a_k^i}$ has a solution s , such that $(x_b^i)^s < (x_c^i)^s$ for all $b, c \leq a_k^i$ with $b < c$.*

PROOF. By Lemma C.7, for every pair $b < c \leq a_k^i$ there exists a solution $s_{b,c}$ of $E_{a_k^i}$ such that $(x_b^i)^{s_{b,c}} < (x_c^i)^{s_{b,c}}$. Notice that the solution space of $E_{a_k^i}$ is a convex subspace of \mathbb{R}^l , for some natural number l . Indeed, it is immediate that the solutions of each linear inequality define a convex space and the intersection of convex spaces is a convex space (cf. Reference [35, Chapter 12]). Let n be the number of aforementioned solutions. Then it is the case that $\sum_{b < c \leq a_k^i} \frac{1}{n} s_{b,c}$ is also a solution of $E_{a_k^i}$ (see, e.g., Reference [35, Chapter 12, Theorem 1.2]). Let us call this solution s and show that if $d < e$, then $(x_d^i)^s < (x_e^i)^s$.

Let $d < e$. Notice that, for every $s_{b,c}$, it is the case that $(x_d^i)^{s_{b,c}} \leq (x_e^i)^{s_{b,c}}$ by the restraints of the system $E_{a_k^i}$. Moreover, we have $(x_d^i)^{s_{d,e}} < (x_e^i)^{s_{d,e}}$. Hence,

$$(x_d^i)^s = \sum_{b < c} \frac{1}{n} (x_d^i)^{s_{b,c}} < \sum_{b < c} \frac{1}{n} (x_e^i)^{s_{b,c}} = (x_e^i)^s.$$

Therefore, we have that, for every pair $d < e \leq a_k^i$, we have $(x_d^i)^s < (x_e^i)^s$, as required. \square

For every agent $i \in \text{Ag}_{\varphi}$ and every system $E_{a_k^i}$, pick a solution s satisfying the conditions of Lemma C.8 and define $\mu_i(b) = (x_b^i)^s$, for every $b \in \text{Min}(\mathbb{A}_{\varphi}^{\star}) \downarrow$. For agents $j \notin \text{Ag}_{\varphi}$, let $\mu_j(b) = \mu_{\mathbb{B}}(b)$ (see (C.2)). Now, we define an APE-model

$$\mathcal{M}_{\varphi} = \langle \mathbb{A}_{\varphi}^{\star}, (\mu_i)_{i \in \text{Ag}_{\varphi}}, v \rangle \quad (\text{C.19})$$

such that, for every $p \in \text{AtProp} \cap S_{\varphi}^{\diamond}$, it holds that $v(p) = p^{\mathbb{A}}$.

LEMMA C.9. *The model \mathcal{M}_{φ} is an APE-model.*

PROOF. For any $i \in \text{Ag}_{\varphi}$, the restrictions imposed by the systems of inequalities and the conditions of Lemma C.8 immediately yield that μ_i is an i -measure. For $j \notin \text{Ag}_{\varphi}$, the only j -minimal

element is \top . Furthermore, $\mu_{\mathbb{B}}$ satisfies the restrictions of j -measures by definition. Hence, each μ_i is an i -measure, and by Lemma C.3 and Definition 4.7, we have that \mathcal{M}_φ is an APE-model. \square

LEMMA C.10 (TRUTH LEMMA). *For every $\psi \in \mathcal{L}$ such that $\psi^{\mathbb{A}} \in S_\varphi^\diamond$, it is the case that*

$$\llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = \psi^{\mathbb{A}}.$$

PROOF. By definition, S_φ^\diamond is closed under subformulas. The proof proceeds by induction on the complexity of ψ . For the atomic variables, this follows immediately from the definition of v . For formulas of the form $\psi \wedge \tau$ and $\psi \vee \tau$, this follows from the fact that \mathbb{A}_φ^\star inherits \vee and \wedge from \mathbb{A} . For formulas of the form $\psi \rightarrow \tau$, $\diamond_i \psi$ and $\Box_i \psi$, it follows from item (6) of Lemma C.2. Finally, for probability formulas of the form $\sigma := \sum \alpha_m \mu_i(\psi_m) \geq \beta$, notice that, by the choice of μ_i as particular solutions of the systems $E_{a_k^i}$, exactly the i -minimal elements $a_k^i \leq \sigma^{\mathbb{A}}$ are such that $\sum \alpha_m \mu_i(\llbracket \psi_m \rrbracket_{\mathcal{M}_\varphi} \wedge a_k^i) \leq \beta$. Hence, $\llbracket \sigma \rrbracket_{\mathcal{M}_\varphi} = \sigma^{\mathbb{A}}$ by definition (cf. Definition 5.6). This concludes the proof. \square

PROPOSITION C.11 (COMPLETENESS). *The axiomatization for IPDEL given in Table 2 is weakly complete w.r.t. APE-models.*

PROOF. As discussed in the beginning of this section, the problem is reduced to proving the weak completeness of IPEL. Let φ be an IPEL formula that is not a theorem. This means that $\varphi^{\mathbb{A}} \neq \top^{\mathbb{A}}$, where \mathbb{A} is the Lindembaum-Tarski algebra of IPEL (see (C.1)). By Lemma C.9, the model \mathcal{M}_φ based on the algebra \mathbb{A}_φ^\star defined in (C.19) is an APE-model. By Lemma C.10, $\llbracket \varphi \rrbracket_{\mathcal{M}_\varphi} = \varphi^{\mathbb{A}}$. Since $\top^{\mathbb{A}_\varphi^\star} = \top^{\mathbb{A}}$, this shows that $\llbracket \varphi \rrbracket_{\mathcal{M}_\varphi} \neq \top^{\mathbb{A}_\varphi^\star}$, which means that \mathcal{M}_φ does not satisfy φ , as required. \square

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This article is dedicated to the memory of Meijer Salomon Wijnberg (1883–1966), in celebration of his extraordinary decision-making abilities.

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