H-cobordism seminar

F. Buccoliero

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Recall that the *boundary* of a smooth manifold W, denoted by $\operatorname{Bd} W$, is the set of points of W which do not have neighbourhoods homeomorphic to $\mathbb{R}^n$.

In this seminar, we will talk about cobordisms, which are a tool useful to "compare" two manifolds with each other via another manifold one dimension bigger. We will introduce this concept and immediately prove some nice results, for instance how the "cobordant manifold" looks like near its boundary. We will in the end define a category: the cobordism category.

1 Cobordisms with Morse number zero

We define once again the main tools of this talk.

Definition 1. *A* smooth manifold triad (W, V_0, V_1) *is a compact smooth n-dimensional manifold* W*, whose boundary is given by the disjoint union of two open and closed submanifolds* V_0 *and* V_1.

Definition 2. *Given two closed, smooth n-manifolds* M_0 *and* M_1*, a* cobordism *from* M_0 *to* M_1 *is a 5-tuple* (W, V_0, V_1, h_0, h_1)*, where* (W, V_0, V_1) *is a smooth manifold triad,* $h_i\colon V_i \to M_i$ *is a diffeomorphism for* $i = 0, 1$.

Two cobordisms (W, V_0, V_1, h_0, h_1) *and* $(W', V_0', V_1', h_0', h_1')$ *are said to be* equivalent *if there exists a diffeomorphism* $g\colon W \to W'$ *which, for* $i = 0, 1$*, carries* V_i *to* V_i' *and makes the following diagram commute*

$$\begin{array}{ccc} V_i & \xrightarrow{\;g_{|V_i}\;} & V_i' \\ & {\scriptstyle h_i}\searrow \quad \swarrow{\scriptstyle h_i'} & \\ & M_i & \end{array}$$

The easiest example of a cobordism class is the *product cobordism.*

Example 1. Let $h\colon M \to M'$ be a diffeomorphism. Then we can define a cobordism class, called *product cobordism*,

$$c_h := (M \times I, M \times 0, M \times 1, h_0, h_1),$$

where $h_0(x, 0) = x$ and $h_1(x, 1) = h(x)$ for $x \in M$.

We would like to state a necessary condition for a cobordism to be a product cobordism. We recall that last time we talked about Morse functions for triads and in particular that any smooth manifold triad has a Morse function. The next definition will be soon useful.

Definition 3. *The* Morse number μ *of a smooth manifold triad* (W, V_0, V_1) *is the minimum over all Morse functions* f *for* (W, V_0, V_1) *of the number of critical points of* f*.*

Theorem 1. *If the Morse number* μ *for the triad* (W, V_0, V_1) *is zero, then the triad is a product cobordism, meaning that* $(W, V_0, V_1, \mathrm{Id}_{V_0}, \mathrm{Id}_{V_1})$ *is equivalent to the product cobordism* $(V_0 \times I, V_0 \times 0, V_0 \times 1, h_0, h_1)$*.*

Before proving this, we need a definition and a lemma.

Definition 4. *Let* f *be a Morse function for the triad* (W, V_0, V_1)*. A vector field* $\xi \colon W \to TW$ *on* W *is said to be a* gradient-like *vector field for* f *if the following holds true.*

- $\xi(f) \colon W \to \mathbb{R}$ *is positive on the regular points of* W *(i.e. the complement of the set of critical points of* W*);*
- *given any critical point* $p \in W$*, there exist a neighbourhood* U *of* p *and coordinates on* U $(x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_n)$ *such that* $f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ *on* U *and* ξ *has coordinates* $(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$ *on* U*.*

So this means that a gradient-like vector field really has the "shape" of the gradient of f, at least on the neighbourhoods of critical points. One nice aspect of Morse functions is that they always have gradient-like vector fields.

Lemma 1. *Let* f *be a Morse function for the triad* (W, V_0, V_1)*. Then there exists a gradient-like vector field* ξ *for* f*.*

Proof. We assume that f has only one critical point in W (we can because the statement is local and critical points are isolated, so we can "split" W in submanifolds with only one critical point) and we call it p. Thanks to Morse Lemma, there exists coordinates $(x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_n)$ on a neighbourhood U_0 of p such that $f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ on U_0. Consider a neighbourhood U of p such that $\overline{U} \subset U_0$.

$p' \in W \setminus U_0$ is not a critical point for f, therefore by the Implicit Function Theorem we can find a neighbourhood U' of p' with coordinates $x_1', \ldots, x_n'$ such that $f = x_1' + \text{constant}$ in U'. As $W \setminus U_0$ is compact, we can find a finite covering $U_1, \ldots, U_k$ such that

- $W \setminus U_0 \subset U_1 \cup \cdots \cup U_k$.
- $U \cap U_i = \emptyset$ for each $i = 1, \ldots, k$.
- U_i has coordinates $x_1^i, \ldots, x_n^i$ such that $f = x_1^i + \text{constant}$ in U_i for $i = 1, \ldots, k$.

On U_0 we have a vector field with coordinates $(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$ (namely the gradient of f), while on U_i we can choose the vector field $\partial / \partial x_1^i$, with coordinates $(1, 0, \ldots, 0)$ for $i = 1, \ldots, k$. Using a partition of unity argument (subordinate to the covering $U_0, U_1 \ldots, U_k$), we can glue them together in order to get a vector field ξ on W. it is easy now to check that ξ is the gradient-like vector field that we wanted. □

Proof of Theorem. As the Morse number for the triad (W, V_0, V_1) is zero, then therre exists f Morse function with no critical points. For the Lemma there exists a gradient-like vector field ξ for f, for which $\xi(f) > 0$. Up to multiplying each point by $1/\xi(f)$ we can suppose that $\xi(f) = 1$ everywhere.

By a result of Seeley, we can extend both f and ξ to an open in the boundary of W, so that they are still smooth. Thus locally on W we can apply the fundamental existence and uniqueness theorem for ODE. Let $\varphi\colon [a,b] \to W$ be an integral curve for the vector field ξ. Then we have that

$$\frac{d(f \circ \varphi)}{dt} = \xi(f) = 1.$$

Therefore, we get that

$$f(\varphi(t)) = t + \text{constant}.$$

Perform a change of parameter, so that $\psi(s) = \varphi(s - \text{constant})$, in such a way that we obtain an integral curve ψ which satisfies

$$f(\psi(s)) = s.$$

By Escape Lemma, we can extend uniquely the integral curve ψ to its maximal interval, which is $[0,1]$ as W is compact. Thus, for any $y \in W$ we obtain a unique integral curve for ξ

$$\psi_y\colon [0,1] \to W,$$

which passes through y and such that $f(\psi_y(s)) = s$. It can also be shown that $\psi_y(s)$ is a smooth function in both y and s.

It can be checked that the diffeomorphism which makes (W, V_0, V_1) into a product cobordism is given by:

$$h\colon V_0 \times [0,1] \to W, \quad h(y_0, s) = \psi_{y_0}(s),$$

whose inverse is

$$h^{-1}\colon W \to V_0 \times [0,1], \quad h^{-1}(y) = (\psi_y(0), f(y)).$$

□

Corollary 1 (The collar neighbourhood theorem)**.** *Let W be a compact smooth manifold with boundary. Then there exists a* collar neighbourhood *U of* $\operatorname{Bd} W$ *such that U is diffeomorphic to* $\operatorname{Bd} W \times [0,1)$.

Proof. We should first recall a Lemma proved by Michael last week.

Lemma 2. *Let (W, V_0, V_1) be a triad. There exists a smooth function $f\colon W \to [0,1]$ such that $f^{-1}(0) = V_0, f^{-1}(1) = V_1$ and f has no critical points in the neighbourhood of the boundary of W.*

By the lemma, there exists $f\colon W \to [0,1]$ such that $f^{-1}(0) = \operatorname{Bd} W$ and f has no critical points on a neighbourhood U of $\operatorname{Bd} W$. Then f is a Morse function on $f^{-1}[0,\varepsilon/2]$, for $\varepsilon > 0$ such that $f^{-1}[0,\varepsilon] \subset U$. For instance, we can take ε to be a lower bound for f on the compact set $W \setminus U$.

Then we can apply the Theorem to $f^{-1}[0,\varepsilon/2)$ and so we obtain a diffeomorphism with $\operatorname{Bd} W \times [0,1)$. □

The previous corollary tells us what happens around the boundary of a triad. We would like now to understand what happens away from the boundary. In order to do so, we need first a new definition.

Definition 5. *Let W be a smooth n-manifold, with boundary* $\operatorname{Bd} W$. *The connected, closed $(n-1)$-submanifold $M \subset W \setminus \operatorname{Bd} W$ is said to be* two-sided *if there exists a neighbourhood U of M in W which is "cut" into two connected components when M is deleted, i.e. $U \setminus M$ consists of two components.*

Corollary 2 (The bicollaring theorem)**.** *Let M be a smooth submanifold of a smooth,compact manifold W. Suppose that each component of M is compact and two-sided. Then there exists a* bicollar *neighbourhood of M in W, i.e. a neighbourhood of M diffeomorphic to $M \times (-1, 1)$ in such a way that M corresponds to $M \times 0$.*

Proof. We can suppose that M has just one component, i.e. is connected, as we can otherwise repeat the same procedure for each component by simply choosing disjoint open neighbourhoods.

Let Z be a two-sided neighbourhood of M and consider $U \subset Z$ to be an open neighbourhood of M such that $\overline{U}$ is compact. Then U splits up into the union of two submanifolds U_1 and U_2 such that $U_1 \cap U_2 = M$ is their boundary.

From the proof of Lemma 2 we obtain that there exists a function $\varphi\colon U \to \mathbb{R}$ such that φ has no critical points on M, $\varphi < 0$ on $\overline{U} \setminus U_1$, $\varphi = 0$ on M and $\varphi > 0$ on $\overline{U} \setminus U_2$. Therefore, we can choose an open neighbourhood V of M such that $\overline{V} \subset U$ and φ has no critical points on V. Using a similar method to the one of the collar neighbourhood theorem, we can find $\varepsilon' < 0, \varepsilon" > 0$ such that $\varphi^{-1}[\varepsilon', \varepsilon"]$ is a compact submanifold of V with boundary $\varphi^{-1}(\varepsilon') \cup \varphi^{-1}(\varepsilon")$. Again by Lemma 2, φ is also a Morse function for $\varphi^{-1}[\varepsilon', \varepsilon"]$. Applying the Theorem to $\varphi^{-1}[\varepsilon', \varepsilon"]$, we find a *bicollar* neighbourhood to M, i.e. a neighbourhood of M which is diffeomorphic to $M \times (-1, 1)$. □

2 The cobordism category

Now we would like to make these cobordisms into a category. How to do so?

Definition 6. *We define the* cobordism category *to be the category whose objects are smooth, closed manifolds and whose arrows are equivalence classes of cobordisms.*

We will spend the end of the talk to prove that such category is actually well-defined.

In particular, we need to check that composition of cobordisms are associative and that there exists an "identity cobordism" .

We start with associativity.
Let M_0, M_1 and M_2 be three smooth,closed manifolds and take two cobordisms classes respectively from M_0 to M_1 and from M_1 to M_2, $c = (W, V_0, V_1, h_0, h_1)$ and $c' = (W', V_1', V_2, k_1, k_2)$. We define the composition of c, c' to be the cobordism class

$$c'c = (W \cup_{k_1^{-1} \circ h_1} W', V_0, V_2, h_0, k_2),$$

which goes from M_0 to M_2.

Using both the collar and bicollar neighbourhoods theorems, it is possible to prove the following result.

Theorem 2. *Let $(W, V_0, V_1), (W', V_1', V_2)$ be two smooth triads with Morse functions $f\colon W \to [0, 1], f'\colon W' \to [1, 2]$. We have also seen that there exist gradient-like vector fields ξ and ξ' respectively, normalized so that $\xi(f) = \xi'(f) = 1$ away from critical points. Let $h\colon V_1 \to V_1'$ be a diffeomorphism. Then there exists a unique smoothness structure $\mathcal{S}$ on $W \cup_h W'$ such that it is compatible with the given structures on W, W';f and f' piece together to give a Morse function on $W \cup_h W'$ and the same happens for ξ and ξ' to give a smooth vector field on $W \cup_h W'$.*

This concludes that we can compose the cobordism classes and now it is immediate from the definition to check associativity.

The following corollary is now straightforward from Theorem 2 and nicely gives us an upper bound for the Morse number of the composition of cobordisms.

Corollary 3. $\mu(W \cup_h W', V_0, V_2) \leq \mu(W, V_0, V_1) + \mu(W', V_1', V_2)$.

We show now which is the identity cobordism for a manifold M.
Let M be a smooth, closed manifold. We define the *identity cobordism* 1_M to be the following cobordism class:

$$1_M := (M \times [0,1], M \times 0, M \times 1, p_0, p_1),$$

where $p_i(x, i) = x$ for $x \in M$ and $i = 0, 1$.

For a cobordism class $c = (W, V_0, V_1, h_0, h_1)$ from M to M', we have that $c1_M = 1_M c = c$.