H-cobordism Seminar Talk 4:
Elementary cobordisms and surgery.

Recall. Every triad $(w_j, v, v')$ with Morse function $f: w \to \mathbb{R}$ possesses a gradient-like vector field $\xi$ for $f$, i.e.

1) $\xi(f) > 0$ on the complement of critical points
2) Around each critical point $p$ of $f$, there exist coordinates $(x_1, \ldots, x_n)$ such that $f = f(p) - ||x||^2 + ||\eta||^2$ and $\xi = (-x_1, \ldots, -x_2, x'_3, \ldots, x'_n)$.

Thm. If $(w_j, v_0, v_1)$ has Morse number zero, then $(w_j, v_0, v_1) \cong (v_0 \times I; v_0 \times 0, v_0 \times 1)$.

Goal: Study cobordisms with Morse number one.
Def. An elementary cobordism is a triad \((W, V_1, V_2)\) which possesses a Morse function with exactly 1 critical point.

\[ \mathcal{E}(x \times y) = (\mathcal{E}x \times y) \cup (x \times \mathcal{E}y) \]

\[ \mathcal{E}(S^{2-1} \times D^{n-2}) = S^{2-1} \times S^{n-2-1} = \mathcal{E}(D^2 \times S^{n-2-1}) \]

Def. Given a manifold \(V^{n-1}\) and an embedding \(\varphi: S^{2-1} \times D^{n-2} \hookrightarrow V\) with \(2+1+n-2 = n-1 = \dim V\),

let \(x(V, \varphi)\) denote the quotient manifold

\[ V \setminus \varphi(S^{2-1} \times \{0\}) \cup_{\varphi} D^2 \times S^{n-2-1} \]

More specifically:

\[ \varphi(u, \theta v) = (\theta u, v) \quad \text{for} \quad u \in S^{k-1}, \theta \in S^1 \]

\[ u \in S^{k-1} = \partial D^2 \]
If $V' \cong \kappa(V, \varphi)$, then $V'$ is said to be obtained from $V$ by a surgery of type $(2, n-2)$.

Examples.

$q: S^0 \times D' \to S^1$

Glue back in $D' \times S^0$
Example \[ \varphi: S^1 \times D^2 \to S^2 \]

Glue in \[ D^2 \times S^0 \]
Example

\[ \varphi : S^0 \times D^2 \to S^2 \]

Glue in \( D' \times S^1 \)

Torus

Klein bottle
**Theorem.** If $V' = \chi(V, \phi)$ can be obtained from $V$ by a surgery of type $(\lambda, n - \lambda)$, then there exists an elementary cobordism $(W; V, V')$ and a Morse function $f : W \to \mathbb{R}$ with exactly one critical point of index $\lambda$.

**Proof.** Let $L_2$ denote the set of points $(x, \varphi) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ which satisfy

$$-1 \leq -|\xi|^2 + |\eta|^2 \leq 1$$

and

$$|\xi| \cdot |\eta| < \sinh 1 \cdot \cosh 1$$

In $\mathbb{R}^3$, $L_2$:

We have the mapping:

$$S^{n-1} \times \mathbb{R}^{n-1} \to \left\{ -|\xi|^2 + |\eta|^2 = -1 \right\} \subset \text{left boundary}$$

$$(u, \theta v) \mapsto (u \cosh \theta, v \sinh \theta)$$

$$\cosh^2 - \sinh^2 = 1$$
\[ D^3 \times S^{n-2} \rightarrow \{ -|x|^2 + |y|^2 = c \} \]
\[(\theta u, v) \mapsto (u \sinh \theta, v \cosh \theta)\]

For each hypersurface \(-|x|^2 + |y|^2 = c\) there are orthogonal trajectories, corresponding to flow lines of \(\nabla f = -2x \frac{\partial f}{\partial x}\).

The trajectory passing through \((x,y)\) has parametrisation \(c(t) = (e^{-t}x, e^{t}y)\)

\[ \dot{c}(t) = \nabla f(c(t)) \]
\[ (-2 e^{-t}x, 2e^{t}y) \]

If either \(x\) or \(y\) are 0, this leads to/from the origin, otherwise the trajectory is a hyperbola connecting \((u \cosh \theta, v \sinh \theta)\) to \((u \sinh \theta, v \cosh \theta)\)
Now construct an n-manifold $W = W(V, \varphi)$ as follows:

Start with disjoint union

$$(U \setminus \varphi(S^{2-1} \times [0,1]) \times [-\frac{1}{2}, \frac{1}{2}]) \cup \mathbb{L}_2$$

For every $u \in S^{2-1}, v \in S^{n-2-1}, 0 < \theta, \phi < 1, c \in [-1,1]$, identify $(\varphi(u, tv), c)$ with the unique point $(x, y) \in \mathbb{L}_2$ s.t.

1) $-|x|^2 + |y|^2 = c$

2) $(x, y)$ lies on the trajectory of the flow line through $(u \cos \theta, v \sin \theta)$ in the left body.
Picture:

\[ \varphi : S^0 \times D^1 \to S^1 \]

\[ \circ \quad \to \quad \circ \circ \quad \to \quad \circ \circ \quad \circ \circ \]

\[ S' \setminus \varphi(S^0 \times \{0\}) \times \{0\} \]

\[ L_1 : \]

\[ \sqrt{1 + \|x\|^2} = -1 \]

\[ \lambda \to \mathbb{Z}_{n-\lambda} \]
It can be checked that we have a diffeo

\[ \varphi(S^{d-1} \times (D^{n-1} \setminus \{0\})) \times [1,1] \cong \mathbb{R}^n (\mathbb{R}^d \times \mathbb{R}^{d-1}) \]

\[ \cong \omega(v, \varphi) \text{ in a smooth mfd.} \]

- \( V \) corresponds to \( c = -1 \)

- \( V' \) corresponds to \( c = 1 \)

\[ f: \omega(v, \varphi) \to \mathbb{R} \]

\[ \begin{cases} f(x, c) = c & \text{for } x \in V \setminus \varphi(S^{d-1} \times 0) \\ \tilde{f}(x, \tilde{y}) = -|\tilde{x}|^2 + 1\tilde{y}^2 & \end{cases} \]
Characteristic embedding

Let $(W_j, v_j)$ be a triad w Morse function $f$ and a gradient-like vector field $g$ for $f$.

Suppose $pt_0$ is a critical point, and $c_1 \neq c_2$ are such that $f(pt_0)$ is the only critical value in $[c_1, c_2]$.

Set $V_0 = f'(c_0)$, $V_1 = f'(c_1)$.

Let $U \ni pt_0$ be a nbhd and $g: D^n \to U$ s.t.

\[ \log(\bar{z}, \bar{y}) = c - |\bar{z}|^2 + |\bar{y}|^2 \]

and $g$ has coordinates $(-\bar{z}, \bar{y})$ for some $2 \leq n$ and some $c$. Set $V_{-\bar{z}} = f'(f(p) - \delta^2)$, $V_{\bar{z}} = f'(f(p) + \delta^2)$.

Sketch:

\[ \begin{array}{c}
  & & U \\
  & | & \\
  & | & \\
  & | & \\
  & & pt_0 \\
  | & | & | \\
 V & V_0 & V_{-\bar{z}} & V_{\bar{z}} \\
  | & | & | \\
 V & V_1 & V_1 & V_1
\end{array} \]
The characteristic embedding \( \varphi_L : S^{2n-1} \times D^{n-1} \to V_0 \)

is obtained by first defining

\[ q : S^{2n-1} \times D^{n-1} \to V_0 \]

\[ (u, \Theta \sigma) \mapsto q(\varepsilon u \cosh \theta, \varepsilon \sigma \sinh \theta) \]

flow down the unique integral curves through \( q(u, \Theta \sigma) \) to obtain a point in \( V_0 \), denoted \( q_L(u, \Theta \sigma) \).

\[ S_L := q_L(S^{2n-1} \times \{0\}) \]

\[ D_L := \text{union of integral curves used above} \]

Diagram:

- \( S_L \)
- \( S_R \)
- \( D_L \)
- \( D_R \)
- \( W \)
- \( V \)
- \( V' \)
**Theorem.** Let \((W; V, V')\) be an elementary cobordism with characteristic embedding \(\phi_L : S^{\lambda-1} \times OD^{n-\lambda} \to V\). Then \((W; V, V')\) is diffeomorphic to the triad \((\omega(V, \phi_L); V, \chi(V, \phi_L))\).

**Proof.**

\[
(W; V, V') \xrightarrow{\cong} (W; V, V)
\]

\[
(W(V_{\phi_L}); V, \chi(V_{\phi_L})) \cong (W(V_{\phi_L}), V_{\phi_L}, \chi(V_{\phi_L}))
\]

Define a good map (ship for now) \(\square\)
**Theorem.** Let \((W; V, V')\) be an elementary cobordism possessing a Morse function with (one) critical point of index \(\lambda\), and let \(D_L\) denote the left-hand disk associated to a fixed gradient-like vector field. Then \(V \cup D_L\) is a deformation retract of \(W\).

Proof: Similar to sketch from [Ref. 1]

**Corollary.** \(H_\ast(W \cup V) \cong \begin{cases} \mathbb{Z} & \ast = \lambda \\ 0 & \text{else} \end{cases}\)

Proof: \(H_\ast(W \cup V) \cong H_\ast(V \cup D_L \cup V)\)

Excision\(\Rightarrow \ H_\ast(D_L \cup S_L)\)

\(\cong \begin{cases} \mathbb{Z} & \ast = \lambda \\ 0 & \text{else} \end{cases}\)

\(\Rightarrow\) Elementary cobordisms have Morse number 1.