Q. When is $cc'$ (i.e., elementary, index 2.14) a product cobordism?

Setting $f$ Morse on $(W^n; N_0, V_1)$ at crit pts $p, p'$ index 1, s.t. $\frac{1}{3} < f(p) < f(p')$. A glv $S$ for $f$ determines a right hand sphere $S_R$ for $p$ in $V = f^{-1}(\frac{1}{2})$ and the $S_L'$ for $p'$.

Def $M^m, N^n \subset V$ intersect transversely $M \cap N$ if $\forall p \in M \cap N$, we have $T_p V = T_p M + T_p N$.

Thm. If $S_R \pitchfork S_L' = S_R \pitchfork S_L'$, then $cc'$ is product.

Thm. We can choose $S, S_1, S_R \pitchfork S_L'$ in $V$.

"Pt1" Lemma 1: If $M \subset V$ has a product nbhd, then $f$ differs $h: V \cap V$ smoothly isotopic to id, s.t. $h(M) \pitchfork N$. Hence $h(S_R) \pitchfork S_L'$ and by a lemma, one can choose $S', S_1'$, s.t. $h(S_R)$ is $h(S_R)$ and $S_L'$ is unchanged. QED
Note \( \dim S_g + \dim S_g' = n-k-1 + k = n-1 = \dim V \) and so the intersection is a finite set of points.

In fact, we can call \( S \) in a weak of \( T \) yielding \( S', \) s. t. \( S' \) is nowhere 0, all its trajs go from \( U_0 \) to \( U_1 \) and \( f' \) Morse function on \( W \) s. t. \( S' \) is glid for \( f' \) and it has no crit. pts. and equals \( f \) near \( \partial W \)
Given $U \cap T$ open nbhd of $T$ (single traj $p \rightarrow p'$), we can find $U' \subset U$ st. no traj leads from $U'$ outside $U$, back to $U'$.

**Proof.** Assume not. Then there are trajectories $T_0, T_1, \ldots$ from $T_0$ through $S_T$ outside of $U$, to $T_U$ and $S_T$, $T_U$ converge to $T$.

and $S_T \rightarrow s \in W \setminus U$. Now the traj $\Psi(t, s)$ through $s$ must come from $V_0$. Since $T$ depends cont. on the $A$, $s$, all traj. through $s'$ near $s$ come from $V_0$. The traj. from $V_0$ to $s'$, $t$, $\Psi_0$ is cont and so its least distance to $T$ exists and is non-zero. This distance depends cont on $s'$. But $T_U \rightarrow T$.
We can alter \( \mathcal{E} \) on a cpt nbhd of \( U' \), yielding \( \mathcal{E}' \) nowhere 0 at every traj of \( \mathcal{E}' \) through a point in \( U \) was outside of \( U \) for \( t < 0 \) and will be outside of \( U \) for \( t > 0 \).

**Proof:**

Assume \( \exists \mathcal{E}' \in \mathcal{U} \) st. \( \exists \) chart

\[
g: U_+ \to \mathbb{R}^n
\]

1. \( g(p) = 0, g(p') = (1, 0, \ldots) \)
2. \( g \circ \mathcal{E}(\varphi) = g(x) = (v(x_1), -x_2, \ldots, -x_{2+\epsilon}, x_{i+2}, \ldots, x_n) \)
   \[
   \text{Let } x = g(\varphi)
   \]
3. \( v(x_1) \) is smooth and

Then replace \( g(x) \) by

\[
g'(x) = (v'(x, P(x)), -x_2, \ldots, x_n)
\]

where

\[
P(x) = \Pi(x_2, \ldots, x_n) \quad \text{with} \quad v'(x, P(x)) = v(x_1) \text{ outside a cpt nbhd of } g(\varphi) \text{ and } g(\varphi) \text{ nowhere 0 at } 0, \text{ so } v'(x, 0) < 0
\]

This defines \( \mathcal{E}' \) which is nowhere zero.
In local coord. the traj. satisfy
\[ \dot{x}_1 = v'(x_1, p(x)) \quad \dot{x}_2 = -x_2, \ldots \]
\[ \dot{x}_{n+2} = x_2 + x_3 + \ldots \]

Let \( x(u) \) be an int. curve with initial cond
\[ x^0 = (x_1^0, \ldots, x_n^0). \]
Then
If a) one of \( x_{2+1}, \ldots, x_n^0 \) is non-zero, then
\[ x_n(\tau) = x_n^0 e^\tau \] increases exponentially
leaving \( g(u) \).

b) \[ x_1^0 = \ldots = x_n^0 = 0, \quad p(x(\tau)) = p(x_0) e^{-\tau} \]
decreases exp. If it remains in \( g(u) \)
then since \( v'(x_1, p(x)) \geq 0 \) on the \( x_1 \)-axis
there is \( \delta > 0 \) s.t. \( x \in g(u) \) at \( p(x) \leq \delta \)
\[ v'(x_1, p(x)) \leq 0. \]
Hence at some point
\[ \dot{x}_1 \leq 0 \] and \( x(\tau) \) leaves \( g(u) \) after all.
A3. Every traj. of $S'$ goes from $V_0$ to $V_1$.

If a traj. is in $P'$, it leaves $U$ by $A_2$, and will follow a traj. of $S$. It cannot come back to $U$ (A1), hence it follows a traj. of $S$ to $V_1$. Sim. it comes from $V_0$. 
All \( \xi \) determines a diffeo \\
\[ \phi : (\mathbb{R}, \mathbb{R} \times V_0 : 0 \times V_0, 1 \times V_0) \to (\mathbb{W}, V_0, V_1) \]

Let \( 4(C, \xi) \) be an int. curve of \( \xi \). Since \( \xi \) is not tangent \( \mathbb{W} \), the int. curve tells us that \( T_1 : \mathbb{W} \to \mathbb{R} \) is the time at which \( 4(C, \xi) \) reaches \( V_1 \). Let \( \pi_0 : \mathbb{W} \to V_0 \)

\[ q \mapsto \pi_0(C, \xi, q) \in V_0 \text{ smooth.} \]

Now \( \phi : \mathbb{R}, \mathbb{R} \times V_0 \to \mathbb{W}, (t, q) \to 4(t + q) \) and \( \phi^{-1}(q) = (\pi_0(q), \pi(q)) \) 

\[ \text{smooth.} \] \[ \square \]
A5 $\xi$ is a glit for a $t'$ Morse on $W$ which equals $f$ near $\partial W$ and has no crit. pts.

By A4, only need to find Morse $f' : [0,1] \times V_0 \to f_t$ $f'_t$ equals $f \circ \phi$ near $0 \times V_0 \cup 1 \times V_0$ and $f' > 0$.

Now $\exists \delta > 0 \forall t, (f_t)'(\xi ) > 0$ for $t < \delta$ or $t > 1 - \delta$.

Let $l : \Sigma_0 \to \Sigma_3$ be as in the picture in $W_0$.

Let $k(q) = c \int_0^1 (1 - S_0 x(s))(f_t \phi)(5,q)(ds)$.

Let $\delta$ be small enough for $k_{q_0} > 0$.

Then $f'(t,q_0) = \int_0^1 A(6x(t, \phi)(5,q_0) + (1 - A(\theta)k(q_0)ds$.
Assumptions (*) can be made when
\[ S_R + S_L' = 3 \text{ pt}^3. \]