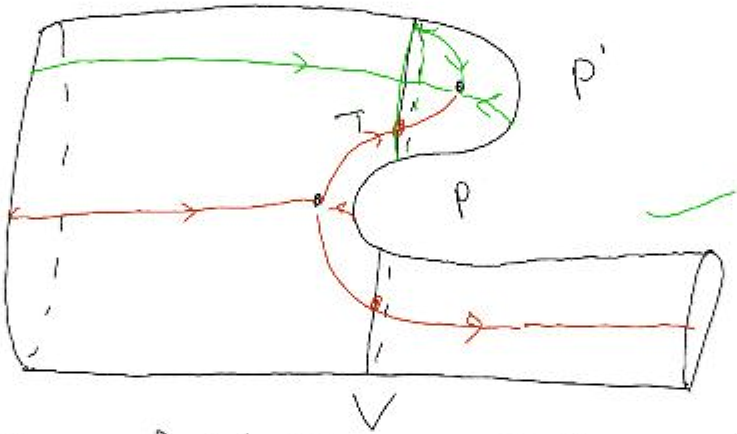
Q When is cc' (c, c' elementary, index $\lambda, \lambda+1$) a product cobordism?



Setting f Morse on $(W^n; N_0, V_1)$ w/ crit pts p, p' index $\lambda, \lambda+1$ s.t. $f(p) < \frac{1}{2} < f(p')$. A glvf ξ for f determines a right hand sphere S_R for p in $V = f^{-1}(\frac{1}{2})$ and lhs S_L' for p'.

Def $M^m, N^n \subset V^v$ intersect transversely $M \pitchfork N$ if $\forall p \in M \cap N$, we have $T_pV = T_pM + T_pN$

Thm If $S_R \pitchfork S_L' = \{pt\}$, then cc' is product

Thm We can choose ξ s.t. $S_R \pitchfork S_L'$ in V.

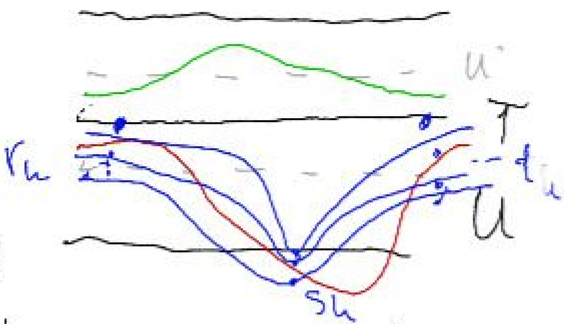
"Pf" Lemma If $M \subset V$ has a product nbhd, then $\exists$ diffeo $h: V \to V$ smoothly isotopic to id, s.t. $h(M) \pitchfork N$. ▨ Hence $h(S_R) \pitchfork S_L'$ and by a lemma, can choose ξ' s.t. its S_R is $h(S_R)$ and S_L' is unchanged. □

Note $\dim S_R + \dim S_L' = n-\lambda-1+\lambda = n-1 = \dim V$ and so the intersection is a finite set of points.

In fact, we can alter ξ in a nbhd of T yielding ξ', s.t. ξ'^{T} is nowhere 0, all its trajs go from V_0 to V_1 and $\exists f'$ Morse function on W s.t. ξ' is glvf for f' and f' has no crit. pts. and equals f near ∂W

<u>A1</u> Given $U \supset T$ open nbhd of T (single traj. p to p'). we can find $U' \subset U$ s.t. no traj. leads from U' outside U back to U'

<u>Pf</u> Assume not. Then there are trajectories $T_0, T_1, \ldots$ from r_k through s_k outside of U, to t_k and r_k, t_k converge to T. and $s_k \to s \in W \setminus U$. Now the traj $\psi(t,s)$ through s must come from V_0.

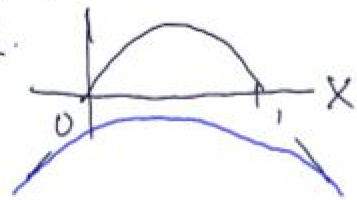


Since ψ depends cont. on the pt s, all traj. through s' near s come from V_0. The traj. from V_0 to s' &, say T_δ, is cpt and so its least distance to T exists and is non-zero. This distance depends cont on s'. But $r_k \to r \in T$ ↯ □

A2 We can alter ξ on a cpt nbhd of U', yielding ξ' nowhere 0 s.t. every traj of ξ' through a point in U was outside of U for $t'<0$ and will be outside of U for $t''>0$.

Pf Assume $\exists\, U_T \supset \overline{U}$ s.t. $\exists$ chart $g: U_T \to \mathbb{R}^n$ s.t.

(*)
1) $g(p)=0$. $g(p')=(1,0,\dots)$
2) $g_* \xi(q) = \eta(x) = (v(x_1), -x_2, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots x_n)$
   $\hookrightarrow x = g(q)$
3) $v(x_1)$ is smth and >0 on $(0,1)$, $=0$ at $0,1$, <0 other, deriv ± 1 near $0,1$



Then replace $\eta(x)$ by

$\eta'(x) = (v'(x_1, \rho(x)), -x_2, \dots, x_n)$ where $\rho(x) = \|(x_2, \dots, x_n)\|$, $v'(x_1, \rho(x)) = v(x_1)$ outside a cpt nbhd of $g(T)$ and s.t. $v'(x_1, 0) < 0$

this defines ξ' which is nowhere zero.

In local coord. the traj satisfy

$$\dot{x}_1 = v'(x_1, \rho(x)) \qquad \dot{x}_2 = -x_2, \ldots$$
$$\dot{x}_{\lambda+2} = x_{\lambda+2}, \ldots$$

Let $x(t)$ be an int. curve with initial cond $x^0 = (x^0_1, \ldots, x^0_n)$. Then

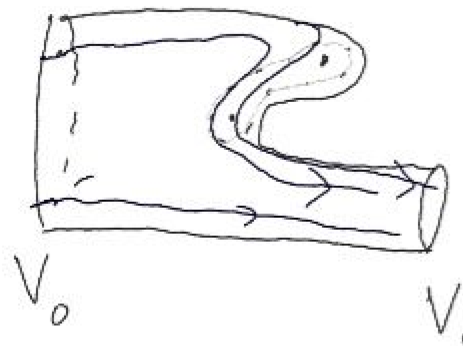
If a) one of $x^0_{\lambda+2}, \ldots, x^0_n$ is non-zero, then $x_n(t) = x^0_n e^t$ increases exponentially leaving $g(u)$.

b) $x^0_{\lambda+2} = \cdots = x^0_n = 0$, $\rho(x(t)) = \rho(x^0)e^{-t}$ decreases exp. If it remains in $g(u)$ then since $v'(x, \rho(x)) < 0$ on the x_1-axis there is $\delta > 0$ s.t. $\forall x \in g(u)$ w/ $\rho(x) \leq \delta$ $v'(x_1, \rho(x)) < 0$. Hence at some point $\dot{x}_1 < 0$ and $x(t)$ leaves $g(u)$ after all.

$\square$

A3 Every traj. of ξ' goes from V_0 to V_1.

Pf If a traj is in **Pf** it leaves U by **A2**, and will follow a



traj of ξ. It cannot come back to U (**A1**) hence it follows a traj of ξ to V_1.
Sim. it comes from V_0.

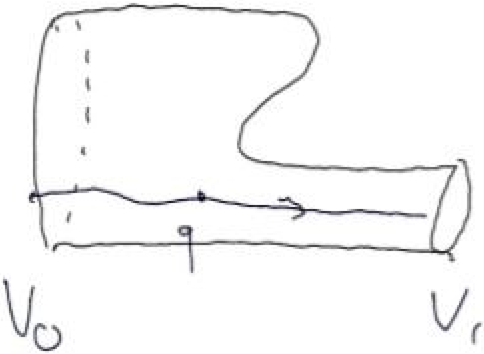
$\square$

<u>A4</u> ξ' determines a diffeo

$$\phi : ([0,1] \times V_0 : 0 \times V_0,\ 1 \times V_0) \to (W; V_0, V_1)$$

<u>Pf</u> Let $\psi(t,q)$ be an int. curve of ξ'. Since ξ' is not tangent ∂W, the ift tells us that $\tau_1 : W \to [0,1] : q \mapsto$ time at which $\psi(t,q)$ reaches V_1. (τ_0, to-time to reach V_0) is smooth. Let $\pi : W \to V_0$

$q \mapsto \pi\psi(-\tau_0(q), q) \in V_0$ smth.



Now $\phi : [0,1] \times V_0 \to W : (t, q_0) \mapsto \psi(t, q_0)$ and $\phi^{-1}(q) = (\tau_0(q), \pi(q))$ smth.

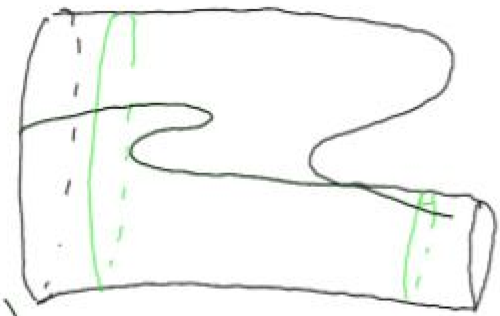
$\square$

A5 ξ' is a glvf for a f' Morse on W which equals f near ∂W and has no crit. pts.

Pf By **A4** only need to find ~~Morse~~ $f' : [0,1]\times V_0 \to [0,1]$ s.t. f' equals $f\circ\phi$ near $0\times V_0 \cup 1\times V_0$ and $\dot{f}' > 0$.

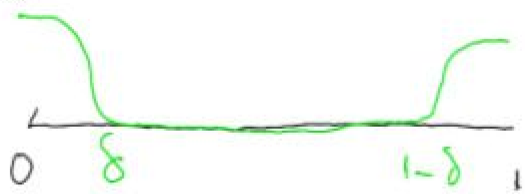
Now $\exists\, \delta>0$ s.t. $(\dot{f\circ\phi})(t) > 0$ for $t<\delta$ or $t>1-\delta$

Let $\lambda : [0,1] \circlearrowleft$ be as in the picture



Let $k(q_0) = c\left(1 - \int_0^1 \lambda(s)(\dot{f\circ\phi})(s,q_0)\,ds\right)$ (q_0 in V_0)

Let δ small enough s.t. $k(q_0) > 0$.



Then $f'(t,q_0) = \int_0^t \lambda(s)(\dot{f\circ\phi})(s,q_0) + (1-\lambda(s))k(q_0)\,ds$

A6 Assumptions (*) can be made when

$S_R \pitchfork S_L' = \{pt\}$.

$\square$