Setting: \( f \) Morse on \( (W^n, V_0, V_1) \) with cist. points \( p, p' \) of index \( 2, 2' \), e.t. \( f(p) < \frac{1}{2} < f(p') \).

A GLVF \( \xi \) for \( f \) deteminis a right hand where \( S_R \) for \( p \) in \( V = f^{-1}(1/2) \) and left hand where \( S_L \) for \( p' \).

Then, \( S_R \triangle S_L' = \xi \) pt \( 3 \) then we have a product coloring.
Last week: Pávlo insisted we can find \( \frac{\partial}{\partial x_i} \) among our technical hypothesis.

Hypothesis: There is a bundle \( UT \) of \( T: p \rightarrow p' \) and a chart \( g : UT \rightarrow \mathbb{R}^n \), e.g.

1) \( p, p' \) correspond to \((0, \ldots, 0)\) and \((1, \ldots, 0)\) respectively.

2) \( g \circ \xi(g) = \hat{x} (\hat{x}) = (v(x_1), -x_2, \ldots, -x_{2+1}, x_{2+2}, \ldots, x_n) \)
   where \( \xi(g) = \hat{x} \), and where

3) \( v(x_1) \) is smooth, positive on \((0, 1)\),
   0 at \( x_1 = 0, 1 \) and negative elsewhere, and
   also \( 1 \frac{\partial v}{\partial x_1} (x_1) \leq 1 \) near \( x_1 = 0, 1 \)

\[ y = v(x_1) \]

\[ y = v'(x_1) \]
Thus (Amater 6 last week): We may choose a new $GLVF \tilde{z}'$ such that the hypothesis holds.

Proof steps: 1) Reduce to a technical lemma (on $\mathbb{R}^n$)
2) Prove this lemma.

Reduction: Let $\tilde{z}(\tilde{x})$ be a VF on $\mathbb{R}^n$, so:

$$\tilde{z}(\tilde{x}) = (v(x_1), -x_2, ..., -x_{2n}, x_{2n+1}, ..., x_n),$$

where:

$v(x_1)$ smooth, positive on $(0, 1)$,

0 at $x_1 = 0$, 1 and negative elsewhere, and

also $\left| \frac{dv}{dx_1} (x_1) \right| = 1$ near $x_1 = 0$, 1.

$\Rightarrow$ Then $\tilde{z}$ is a $GLVF$ on $\mathbb{R}^n$, namely w.r.t.

$$F(\tilde{x}) = f(p) + 2 \int_0^{x_1} v(t) dt - x_1^2 - ... - x_{2n}^2 + x_{2n+1}^2 + ... + x_n^2.$$ 

$\Rightarrow$ Writing $e := (1, ..., 0)$, we can show $V$

such that $F(e) := 2 \int_0^1 v(t) dt + f(p) = f(p')$.

So on $\mathbb{R}^n$ it is okay. We left to a nullset of $T$. 
We can choose \( f(p) < b_1 < b_2 < f(p') \) and closed, disjoint neighborhoods \( L_1 \ni 0, L_2 \ni 0 \) with diffeos \( L_1 \rightarrow U(p) \), \( L_2 \rightarrow U(p') \).

\[ \text{c.t.} \quad 1) \quad g_1, g_2 \text{ vary} \quad \overset{\rightarrow}{\rightarrow} \text{ to} \quad \varepsilon \]

\[ -\overset{\rightarrow}{\rightarrow} \text{ to} \quad f \]

\[ -[0, e] \text{ to} \quad T \]

2) Writing \( p_c := T \circ f^{-1}(b_c) \), we have that

\[ -g_1(L_1) \leq f^{-1}([b(p), b_1]) \] is a

\[ \text{valled of} \quad [p, p_1] \leq T \]

\[ -g_2(L_2) \leq f^{-1}([b_2, f(p')]) \] is a

\[ \text{valled of} \quad [p_2, p'] \leq T. \]

**Recall:** Around critical points \( p, p' \) \( f, \varepsilon \) look like:

\[ f \sim \pm x_1^2 + \ldots + \pm x_n^2 \]

\[ \varepsilon \sim (\pm x_1, \ldots, \pm x_n) \]

for some system \( x_1, \ldots, x_n \) of coordinates.
What does stuff in the middle? $[b_1, b_2]$.

In a nbhd $U_1 \ni b_1$, it takes pairs to a nbhd $U_2 \ni b_2$, differentiably to $U$.

Concluding all these pairs and their trajectories, we get...
a null set $L_0$ of $[b_1, b_2]$ differenti to $U_x \times [0,1]$

$L_0 \cup L_0$ is a null set of $[0,1]$.

We can extend $g_1 : L \to U_x(p)$ to

$\tilde{g}_1 : L \cup L_0 \to W$ (embedding)

again take $F$ to $f$ on $\tilde{\eta}$ to $3$ (in terms of trajectories)

Suppose $\tilde{g}_1$ agrees with $g_2$ on some null null set of $b_2$ in $U_2$.

Gluing $\tilde{g}_1$ to $g_2$ gives a differ $\tilde{g}$ of a null set $V$ of $[0, e_3]$ onto a null set of $T$

preserving limits and trajectories.

For $l$ $\tilde{\eta} \to 3$

There is a null set positive for $k : \tilde{g}(V) \to \mathbb{R}$

such that

$\tilde{g} \cdot \tilde{\eta} = k \cdot \tilde{\eta}$ on $\tilde{g}(V)$

By maybe taking $V$ a bit, $k$ can be extended positively to $W$

Setting $\bar{z} = k \cdot \tilde{\eta}$ gives the required CLVF.
When does \( \bar{g}_1 \) agree with \( g_1 \) on some null winding of \( b_2 \) in \( U_2 \)?

\( \Rightarrow \) \( \bar{g}_1 \) gives a diffeo \( h : f^{-1}(b_1) \to f^{-1}(b_2) \)

\( \Rightarrow \) \( \bar{g}_1 \) gives a diffeo \( h : U_1 \to U_2 \)

\( \Rightarrow \) \( (*) \) holds iff \( h \) coincides with \( h_0 := g_2 \cdot h \cdot g_1^{-1} \) near \( p_1 \).

\( \Rightarrow \) Any diffeo isotopic to \( h \) corresponds to a GLVF only differing from \( f \) in \( f^{-1}(\{b_1, b_2\}) \) (Lemma 4.7).

\( \Rightarrow \) So we need to deform \( h \) to \( \bar{h} \) which agrees with \( h_0 \) near \( p_1 \), and for which:

\[ \bar{h}(S_{p_1}(b_1)) \cap S_{p_2}(b_2) = \{ p_2 \} \]

\( \Rightarrow \) Equivalently, we give an isotopy of \( h_0 \cdot h \)
on a null of \( p_1 \) that deforms \( h_0 \cdot h \) to the identity on (perhaps small) null of \( p_1 \).
We can choose \( g_2 : L^2 \rightarrow H_2(p') \) such that \( h_0^{-1} h \)
is orientation preserving at \( p' \) and \( h_0^{-1} h \) and \( S_K(b_i), S_L(b_i) \)have the same intersection number with \( S_L(b_i) \) at \( p' \).

**Def.** Let \( M, M' \in V \). \( M \) orientable, \( V(M') \) non-orientable. M & M' at \( p' \). Choose a pair of intersecting hypersurfaces \( T_M \). This gives a basis for \( V(M') \) at \( p' \), intersecting under an orientation of this basis.

We are done if the following holds:

**Thus.** Let \( a, b \in \mathbb{N}, a + b = n \). View \( \mathbb{R}^n = \mathbb{R}^a + \mathbb{R}^b \)

\[
(u, v) = (u, 0) + (0, v)
\]

Suppose \( h \) is an orientation preserving embedding of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) such that:

1) \( h(0) = 0 \)

2) \( h(\mathbb{R}^a) \cap \mathbb{R}^b = \{0\} \) with int. number + 1

**Then:** For any ruled \( N \) of \( \{0\} \), there is a smooth isotopy \( h' : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( t \in [0, 1] \) with \( h'_0 = h \) such that
$\text{(I) } h_t^\epsilon(x) = x \text{ for } x = 0 \text{ or } x \neq N$

$\text{(II) } h_t'(x) = x \text{ for } x \text{ in neighborhood of } 0.$

$\text{(III) } h_t'(R^a) \cap R^b = \emptyset.$

**Proof**

$\exists h_t \text{ such that } h_t = I \text{ on } R^a$ and $h_t(R^a) \cap R^b = \emptyset.$
Choose $E \supseteq \mathbb{E} \subseteq E$ and set:

$$h_t = \begin{cases} h_t(x) \quad x \in \mathbb{E} \\ h(x) \quad x \not\in \mathbb{E} \end{cases}$$

which is an interior only on $\overline{E} \cup (\mathbb{R}^n \setminus E)$.

How to extend to $\mathbb{R}^n$?

$h_t$ converges to the VF: $\frac{\partial}{\partial t} (t, y) = 1$, $\frac{\partial h_t}{\partial t}(h_t^{-1}(y))$ on $[0, 1] \times \mathbb{R}^n$. 

\[ \ldots \]
Extend \( \tilde{h}(t, y) \) to \([0, 1] \times \mathbb{R}^n\), resulting in an immersion \( \tilde{h}_t \):

For \( t_0 \) small enough, \( \tilde{h}_{t_0} \) has no new intersection with \( \mathbb{R}^b \) (well in well for \( t \in [0, t_0 + c] \))

Now repeat the above process starting at \( h_{t_0} \)

Yields an isotopy which is obeyed on \([0, t_0 + c] \)
By a compactness argument, only finitely many steps needed.