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Bader, M.K.M.

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A The elastodynamic wave equation in the fk-domain

Here the conversion of the representation of the elastodynamic wave equation from the time-space domain to the frequency-wave number (fk) domain is derived for the example of a plane wave that is propagating in the \( x_1 x_3 \)-plane in Cartesian coordinates. It is based on the argumentation found in [118]; similar derivations hold for spherical coordinates and for arbitrary wave propagation.

The elastodynamic wave equation in the time-space domain (Eq. (3.1.11)) is

\[
\sum_{j=1}^{3} \left( (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_j \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right) = \rho \frac{\partial^2 u_i}{\partial t^2},
\]

where \( \vec{u} = (u_1, u_2, u_3) \) denotes the displacement field. For a wave vector in the \( x_1 x_3 \)-plane and a displacement field that is constant in \( x_2 \), this corresponds to three equations

\[
\begin{bmatrix}
(\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} + (\lambda + \mu) \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \mu \frac{\partial^2 u_3}{\partial x_3^2} \\
\mu \frac{\partial^2 u_1}{\partial x_1^2} + \mu \frac{\partial^2 u_3}{\partial x_3^2} \\
\mu \frac{\partial^2 u_3}{\partial x_3^2} - (\lambda + \mu) \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + (\lambda + 2\mu) \frac{\partial^2 u_3}{\partial x_3^2}
\end{bmatrix} = \rho \begin{bmatrix}
\frac{\partial^2 u_1}{\partial x_1^2} \\
0 \\
\frac{\partial^2 u_3}{\partial x_3^2}
\end{bmatrix}.
\]

As a next step it is common to modify the displacement vector to \( \vec{u} \rightarrow \vec{u}(u_1, u_2, -i u_3) \) so that \( u_1 = u_1, \ u_2 = u_2, \ u_3 = i u_3 \). This will ensure in the later calculations that the stiffness matrix is symmetric. As a result Eq. (A.2) becomes

\[
\begin{bmatrix}
(\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} + i(\lambda + \mu) \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \mu \frac{\partial^2 u_3}{\partial x_3^2} \\
\mu \frac{\partial^2 u_1}{\partial x_1^2} + \mu \frac{\partial^2 u_3}{\partial x_3^2} \\
\mu \frac{\partial^2 u_3}{\partial x_3^2} - i(\lambda + \mu) \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + (\lambda + 2\mu) \frac{\partial^2 u_3}{\partial x_3^2}
\end{bmatrix} = \rho \begin{bmatrix}
\frac{\partial^2 u_1}{\partial x_1^2} \\
0 \\
\frac{\partial^2 u_3}{\partial x_3^2}
\end{bmatrix}.
\]

In the next step Eq. (A.3) is transformed from the space-time domain to the wave number-frequency domain, making use of Eq. (3.1.32) and Eq. (3.1.34). This means that every derivative with respect to time is replaced by a factor \( i \omega \) and every derivative with respect to \( x_1 \) is replaced by a factor \(-i k_{x_1}\). For simplicity, the same letter \( \vec{u} \) for the displacement is used in the following, but now it refers to the frequency-wave number representation of the displacement. Equation (A.3) then becomes

\[
\begin{bmatrix}
-k_{x_1}^2 (\lambda + 2\mu) u_1 - k_{x_1} (\lambda + \mu) \frac{\partial u_3}{\partial x_3} + \mu \frac{\partial^2 u_3}{\partial x_3^2} \\
-k_{x_1}^2 \mu u_2 + \mu \frac{\partial^2 u_2}{\partial x_3^2} \\
-k_{x_1}^2 \mu u_3 + k_{x_1} (\lambda + \mu) \frac{\partial u_1}{\partial x_3} + (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_3^2}
\end{bmatrix} = -\omega^2 \rho \begin{bmatrix}
u_1 \\
0 \\
u_3
\end{bmatrix}.
\]

And this can be further re-expressed in matrix notation as

\[
-k_{x_1}^2 \mathbf{A} \vec{u} - k_{x_1} \mathbf{B} \frac{\partial u}{\partial x_3} + \mathbf{C} \frac{\partial^2 \vec{u}}{\partial x_3^2} = -\omega^2 \rho \vec{u}, \quad \text{where}
\]

\[
\mathbf{A} = \begin{bmatrix}
\lambda + 2\mu & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu
\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}
0 & 0 & -(\lambda + \mu) \\
0 & 0 & 0 \\
\lambda + \mu & 0 & 0
\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}
\mu & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \lambda + 2\mu
\end{bmatrix}.
\]
B Numerical integration with Gaussian quadrature

The numerical integration method employed of this work approximates the solution to the definite integral

\[ \int_{-1}^{1} f(x) \, dx \approx \sum_{j=0}^{N-1} w_j f(x_j), \quad (B.1) \]

where \( f(x) \) is a continuous function across the interval \([-1, 1]\), as a weighted sum of \( N \) values of \( f(x_j) \), evaluated at the abscissas \( x_j \) and multiplied with the corresponding weights \( w_j \). In the literature, this method is referred to as Gaussian quadrature [177]. The abscissa values \( x_0, \ldots, x_{N-1} \) in Eq. (B.1) can be understood as the zero-crossings of a polynomial of order \( N \), that is part of a set of orthonormal polynomials \( p_0, \ldots, p_N \) in the interval \([a, b]\). When the polynomial is known, then first zero-crossings can are calculated and in the next step the weights are derived by solving

\[ \int_{-1}^{1} p_j(x) \, dx = \sum_{i=0}^{N-1} w_i p_j(x_i) \quad (B.2) \]

for the \( j = 0, \ldots, N \) polynomials. These weights and abscissa values can now be used to approximate any function \( f(x) \) with Eq. (B.1). It can be shown that with those values Eq. (B.1) is exact for all polynomials of degree \( 2N - 1 \) or lower [177]. Furthermore, this numerical integration method converges exponentially with the order \( N \).

A standard method utilizes Gauss-Legendre polynomials for the Gaussian quadrature routine, where a polynomial of order \( j \) is defined recursively as

\[ (j + 1) \cdot p_{j+1} = (2j + 1) \cdot x \cdot p_j - j \cdot p_{j-1}, \quad (B.3) \]

where \( p_0(x) = 1 \) and \( p_1(x) = x \), and where \(-1 \leq x \leq 1\). The polynomial of order \( j \) has \( j \) zero-crossings which are labeled with \( x_j \) and the corresponding weights are then calculated as

\[ w_j = \frac{2}{(1 - x_j^2) \cdot p'_N(x_j)}, \quad (B.4) \]

where \( p'_N(x_j) \) is the derivative of the polynomial at its zero-crossing \( x_j \). For an integral over \([a, b]\) Eq. (B.1) can be rewritten as

\[ \int_{a}^{b} f(x) \, dx \approx \frac{b-a}{2} \sum_{j=0}^{N-1} w_j f\left(\frac{b-a}{2} x_j + \frac{b+a}{2}\right). \quad (B.5) \]

Multi-dimensional integrals, like surface or volume integrals with \( f(r, \theta, \phi) \), can be evaluated as nested loops over several dimensions as

\[ \int_{V} r^2 \sin \theta f(r, \theta, \phi) \, dV = \sum_{k} w_k \left( \sum_{i} w_{i, r_i^2} \left( \sum_{j} w_j \sin(\theta_j) f(r_i, \theta_j, \phi_k) \right) \right) \quad (B.6) \]

where \((r, \theta, \phi)\) are the variables of spherical coordinate representations and where corresponding \( i, j, k \) are the indices of the Gauss-Legendre polynomials.
C Saulson’s analytic seismic Newtonian noise model

This section summarizes the derivation of Saulson’s analytical model for seismic Newtonian noise as presented in [124], based on remarks elaborated in [179].

Volume contribution

A test mass at \( h = 0 \) is assumed to be located on the surface of a homogeneous half-space with density \( \rho \). According to Newton’s second law, a density fluctuation \( \Delta M \) near the test mass translates to a force on the test mass as

\[
\vec{F} = G \frac{\Delta M}{r^2} \hat{r}.
\]  

(C.1)

The component parallel to one detector arm is expressed as

\[
\frac{F_x}{m} = G \frac{\Delta M}{r^2} \cos \theta = G \frac{\Delta M}{r^2} \cos \phi \sin \gamma,
\]  

(C.2)

where \( \theta \) is the angle between \( \hat{r} \) and the \( x \)-axis, \( \phi \) is the azimuth angle of the orthogonal projection of \( \hat{r} \) in the \( xy \)-plane and \( \gamma \) is the polar angle (see Fig. 1).

![Figure 1: Schematic of the geometry. The angle \( \theta \) is between \( \hat{r} \) and the \( x \)-axis, \( \phi \) is the azimuth angle of the orthogonal projection of \( \hat{r} \) in the \( xy \)-plane and \( \gamma \) is the polar angle.](image)

The half-space is assumed to be filled with patches of size \( \lambda/2 \) that are fluctuating independent of each other as a result of constant P-waves. The force on the test mass from all these patches is then expressed by the quadratic sum over the individual contributions as

\[
\left(\omega_0^2 - \omega^2\right)^2 + \frac{\omega}{\tau^2} |\Delta x|^2 = G^2 |\Delta M|^2 \sum_i \frac{\cos^2 \phi_i \sin^2 \gamma_i}{r_i^4},
\]  

(C.3)

where the force on the test mass \( m \) on the left side of the equation has been replaced by the equation of motion of a harmonic oscillator with resonant frequency \( \omega_0 \), damping time \( \tau \) and displacement \( |\Delta x| \) of the test mass. The sum in Eq. (C.3) can be evaluated by approximating it as an integral, with a lower cutoff radius of \( \lambda/4 \) to avoid a singularity, as

\[
\sum_i \frac{\cos^2 \phi_i \sin^2 \gamma_i}{r_i^4} dV_i / dV \simeq \int dV \cos^2 \phi \sin^2 \gamma \frac{1}{(\lambda/2)^3} \\
= \frac{8}{\lambda^3} \int_{r=\lambda/4}^{R} \int_{\phi=0}^{2\pi} \int_{\gamma=0}^{\pi/2} \frac{\cos^2 \phi \sin^2 \gamma}{r^4} \sin \gamma \, dr \, d\phi \, d\gamma \\
= \frac{64\pi}{3\lambda^4} = \frac{64\pi}{3v^4 \left( \frac{\omega}{2\pi} \right)^4}
\]  

(C.4)

where \( \lambda = 2\pi v / \omega \), with \( v \) being the velocity of the wave, and Eq. (5) in [124] deviates from the last expression by a factor \( \pi \).
Next, we would like to find the connection between mass fluctuation $\Delta M$ and seismic displacement $\Delta X$. Assuming that the displacement corresponds to the maximum amplitude of the wave, the total amount of mass flowing in- and out of the volume element can be expressed as

$$|\Delta M|^2 = \left| 2\Delta X \left( \frac{\lambda}{2} \right)^2 \lambda^2 \rho \right|^2 = \rho \lambda^2 4 |\Delta X|^2,$$

(C.5)

which differs by a factor $\frac{\pi}{4} \approx 0.79$ from Eq. (11) in [124]. Other assumptions such as $\Delta X$ corresponding to the RMS value or to the average displacement across the patch result in an even larger difference by a factor $\frac{\pi}{2} \approx 1.57$ and $\frac{\pi}{16} \approx 1.94$. In the following we will therefore assume that Eq. (C.5) and Eq. (11) are sufficiently equivalent and continue with the expression for $\Delta M$ as in [124].

Since the seismic wavelength is short in comparison to the baseline of the interferometer we can approximate

$$\left( (\omega^2_0 - \omega^2)^2 + \frac{\omega}{\tau^2} \right) |\Delta x|^2 \approx \omega_0^4 |\Delta x|^2,$$

(C.6)

and using this in Eq. (C.3) together with Eq. (C.4) and multiplying an additional factor 4 to account for the four independent test masses, then allows to express the displacement Newtonian noise due seismic density fluctuations as

$$|\Delta x|^2 = \frac{16\pi^2 G^2 \rho^2}{3 \omega^4} |\Delta X|^2 \text{ [m}^2/\text{Hz]},$$

(C.7)

which is equivalent to Eq. (21) in [124].

**Remark concerning the surface contribution**

Furthermore, vertical displacement $\Delta Z$ is assumed to have an effect on horizontal displacement at the interface between soil and air. It is derived by casting the sum to an integral over the surface as

$$\sum_i \frac{\cos^2 \phi_i \sin^2 \gamma_i}{r_i^4} dA_i \approx \int dA \frac{\cos^2 \phi}{r^4} \frac{1}{(\lambda/2)^2}$$

$$= \frac{4}{\lambda^2} \int_{r=\lambda/4}^{R} \int_{\phi=0}^{2\pi} \frac{\cos^2 \phi}{r^4} d\phi dr$$

$$= \frac{32\pi}{\lambda^4}$$

(C.8)

where $\sin \gamma_i = 1$ and which differs by a factor 4 from Eq. (13) in [124]. The resulting displacement of the test mass is then calculated as previously and is obtained to be

$$|\Delta x|^2 = 2\pi \frac{G^2 \rho^2}{\omega^4} |\Delta Z|^2.$$  

(C.9)

Due to the additional factor $\pi$ in comparison to Eq. (16) in [124], Eq. C.9 is only about a factor 3 smaller than Eq. (C.7) instead of an order of magnitude, as estimated in [124]. This means that the Newtonian noise due to density fluctuations on the interface between soil and air may be underestimated in [124].