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Chapter 14
Relating a Reified Adaptive Network's Emerging Behaviour Based on Hebbian Learning to Its Reified Network Structure

Abstract In this chapter another challenge is analysed for how emerging behaviour of an adaptive network can be related to characteristics of the adaptive network's structure. By applying network reification, the adaptation structure is modeled itself as a network too: as a subnetwork of the reified network extending the base network. In particular, this time the challenge is addressed for mental networks with adaptive connection weights based on Hebbian learning. To this end relevant properties of the network and the adaptation principle that have been identified are discussed. Using network reification for modeling of the adaptation principle, a central role is played by the combination function specifying the aggregation for the reification states of the connection weights, and in particular, identified mathematical properties of this combination function. As one of the results it has been found that under some conditions in an achieved equilibrium state the value of a connection weight has a functional relation to the values of the connected states that can be identified.

Keywords Reified adaptive network · Hebbian learning · Analysis of behaviour

14.1 Introduction

As for Chap. 13 the challenging issue addressed here is to predict what patterns of behaviour will emerge, and how their emergence depends on the structure of the model, in particular for adaptive network models. Here adaptive behaviour depends in some way on the reified network structure, defined by network characteristics such as connections and their weights, and the aggregation of multiple connections to one node. When adaptive networks are considered, where the network characteristics also change over time, according to certain adaptation principles this poses extra challenges. Such adaptation principles themselves depend on certain adaptation characteristics. It is this latter issue what is the topic of the current chapter: how does emerging behaviour of adaptive networks relate to the characteristics of the network and of the adaptation principles used. More in particular, the

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focus is on adaptive Mental Networks based on Hebbian learning (Bi and Poo 2001; Gerstner and Kistler 2002; Hebb 1949; Keysers and Perrett 2004; Keysers and Gazzola 2014; Kuriscak et al. 2015). Hebbian learning is, roughly stated, based on the principle 'neurons that fire together, wire together' from Neuroscience.

To address the issue, as a vehicle the Network-Oriented Modeling approach based on temporal-causal networks (Treur 2016) is used together with the notion of network reification from Chap. 3 and (Treur 2018a). For temporal-causal networks, characteristics of the network structure Connectivity, Aggregation, and Timing are represented by connection weights, combination functions and speed factors. For the type of adaptive networks considered, the connection weights are dynamic based on Hebbian learning, so they are actually not part of the characteristics of the (static) network structure anymore. Instead, by applying network reification the base network is extended by reification states that represent the network characteristics such as in this case the Connectivity characteristics indicated by connection weights and their dynamics. In the reified network the dynamics of these reification states is defined by the standard concepts for temporal-causal networks: connection weights for Connectivity, speed factors for Timing, and combination functions for Aggregation of the reification states. In particular, the focus here is on the mathematical properties of these combination functions for the reification states as they play a main role in the specification of an adaptation principle; e.g., see Chap. 3 or (Treur 2018a).

Based on the chosen approach, characteristics of Hebbian learning have been identified that play an important role in the emerging behaviour; these characteristics indeed have been expressed as mathematical properties of combination functions for the reification states for connection weights.

In Fig. 14.1 the basic relation between structure and dynamics of a reified network model is indicated by the horizontal arrow in the lower part representing the base level. For a reified network these also apply to the reification states. The properties of network structure focus on properties of the adaptation principle based on Hebbian learning represented by the reification states at the reification level; they are first discussed in a general setting in Sect. 14.4. Properties of the behaviour are addressed in Sect. 14.4 as well and related to the network structure properties (the horizontal arrow in the upper part of Fig. 14.1). In Sect. 14.5 these results are refined by introducing an extra assumption on variable separation.

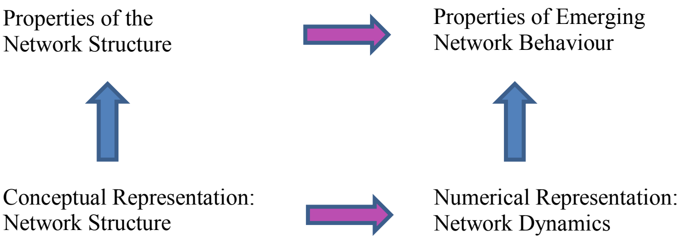

Fig. 14.1 Bottom layer: the conceptual representation of a reified network model defines the numerical representation. Top layer: properties of reified network structure entail properties of emerging reified network behaviour

In this chapter, results will be discussed that have been proven mathematically in this way for this relation between structure and behavior for such reified adaptive network models, in particular, for the equilibrium values of Hebbian learning in relation to equilibrium values of the connected network states. These results have been proven not for one specific model or function, but for classes of combination functions that fulfill certain properties. More specifically, as one of the results it has been found how for the classes of functions considered within an emerging equilibrium state the connection weight and the connected states satisfy a fixed functional relation that can be expressed mathematically.

In this chapter, in Sect. 14.2 the temporal-causal networks that are used as vehicle are briefly introduced. Section 14.3 briefly introduces Hebbian learning and how it can be modeled by a reified network. In Sect. 14.4 the properties of Hebbian learning functions are introduced that define the adaptation principle of the network. Section 14.5 focuses in particular on the class of functions for which a form of variable separation can be applied, In Sect. 14.6 a number of examples are discussed. Finally, Sect. 14.7 is a discussion.

14.2 Temporal-Causal Networks and Network Reification

For the perspective on networks used in the current chapter, the interpretation of connections based on causality and dynamics forms a basis of the structure and semantics of the considered networks. More specifically, the nodes in a network are interpreted here as states (or state variables) that vary over time, and the connections are interpreted as causal relations that define how each state can affect other states over time. This type of network has been called a *temporal-causal network* (Treur 2016). A conceptual representation of a temporal-causal network model by a *labeled graph* provides a fundamental basis. Such a conceptual representation includes representing in a declarative manner states and connections between them that represent (causal) impacts of states on each other. This part of a conceptual representation is often depicted in a *conceptual picture* by a graph with nodes and directed connections. However, a *complete conceptual representation* of a temporal-causal network model also includes a number of labels for such a graph, representing network characteristics such as Connectivity, Aggregation and Timing. A notion of *strength of a connection* is used as a label for Connectivity, some way for *Aggregation of multiple causal impacts* on a state is used, and a notion of *speed of change* of a state is used for Timing of the processes. These three notions, called connection weight, combination function, and speed factor, make the graph of states and connections a labeled graph, and form the defining structure of a temporal-causal network model in the form of a conceptual representation; see Table 14.1, first 5 rows.

There are many different approaches possible to address the issue of combining multiple impacts. To provide sufficient flexibility, the Network-Oriented Modelling approach based on temporal-causal networks incorporates for each state a way to specify how multiple causal impacts on this state are aggregated by a combination

Table 14.1 Concepts of conceptual and numerical representations of a temporal-causal network

Concepts	Notation	Explanation
States and connections	X, Y, $X \rightarrow Y$	Describes the nodes and links of a network structure (e.g., in graphical or matrix format)
Connection weight	$\boldsymbol{\omega}_{X,Y}$	The *connection weight* $\boldsymbol{\omega}_{X,Y} \in [-1, 1]$ represents the strength of the causal impact of state X on state Y through connection $X \rightarrow Y$
Aggregating multiple impacts	$\mathbf{c}_Y(..)$	For each state Y (a reference to) a *combination function* $\mathbf{c}_Y(..)$ is chosen to combine the causal impacts of other states on state Y
Timing of the causal effect	$\boldsymbol{\eta}_Y$	For each state Y a *speed factor* $\boldsymbol{\eta}_Y \geq 0$ is used to represent how fast a state is changing upon causal impact

Concepts	Numerical representation	Explanation
State values over time t	$Y(t)$	At each time point t each state Y in the model has a real number value, usually in [0, 1]
Single causal impact	$\mathbf{impact}_{X,Y}(t) = \boldsymbol{\omega}_{X,Y} X(t)$	At t state X with connection to state Y has an impact on Y, using weight $\boldsymbol{\omega}_{X,Y}$
Aggregating multiple impacts	$\mathbf{aggimpact}_Y(t)$ $= \mathbf{c}_Y(\mathbf{impact}_{X_1,Y}(t), \ldots, \mathbf{impact}_{X_k,Y}(t))$ $= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t))$	The aggregated causal impact of multiple states X_i on Y at t, is determined using combination function $\mathbf{c}_Y(..)$
Timing of the causal effect	$Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{aggimpact}_Y(t) - Y(t)]\Delta t$ $= Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)) - Y(t)]\Delta t$	The causal impact on Y is exerted over time gradually, using speed factor $\boldsymbol{\eta}_Y$

function. For this aggregation a library with a number of standard combination functions are available as options, but also own-defined functions can be added.

Next, this conceptual interpretation is expressed in a formal-numerical way, thus associating semantics to any temporal-causal network specification in a detailed numerical-mathematically defined manner.

This is done by showing how a conceptual representation based on states and connections enriched with labels for connection weights, combination functions and speed factors, can get an associated numerical representation (Treur 2016), Chap. 2; see Table 14.1, last five rows. The difference equations in the last row in Table 14.1 constitute the numerical representation of the temporal-causal network model and can be used for simulation and mathematical analysis; it can also be written in differential equation format:

$$\begin{aligned} Y(t+\Delta t) &= Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t \\ \mathbf{d}Y(t)/\mathbf{d}t &= \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)] \end{aligned} \tag{14.1}$$

In adaptive networks some of the network structure characteristics such as connection weights are dynamic and actually should be treated more in the same way as states. The concept of network reification provides a neat definition for doing this. By introducing additional network states $\mathbf{W}_{X,Y}$ representing them, called reification states, network reification avoids that connection weights $\boldsymbol{\omega}_{X,Y}$ get an ambiguous status. The thus extended network is a reified network. In Sect. 14.3 this will be discussed for Hebbian learning in particular.

14.3 Reified Adaptive Networks for Hebbian Learning

In Sect. 14.3 the basics of Hebbian learning and reification for it are briefly summarized. Next, in Sect. 14.4 relevant properties for Hebbian learning combination functions are discussed and how they imply certain behaviour. Recall from Chap. 1, Fig. 1.4 the way in which Hebbian learning can be modeled by network reification; see also Fig. 14.2 here. Following this reification approach, the adaptation principle and the dynamics it entails gets a specification in the standard form of a temporal-causal network structure, as a subnetwork of the reified network. In particular, the combination function for the reification state and its mathematical properties play a main role.

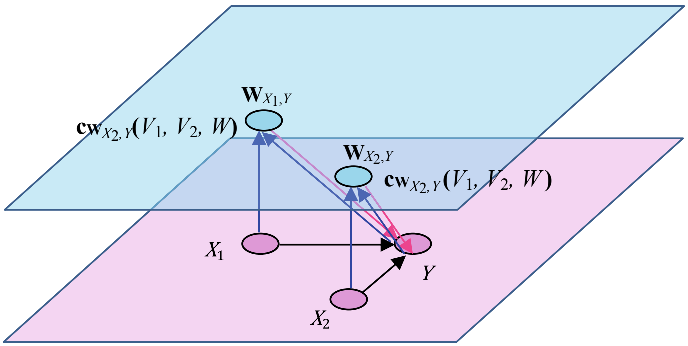

Fig. 14.2 Hebbian learning combination functions as labels in a reified network

14.3.1 Reification States for Hebbian Learning and Their Hebbian Learning Combination Functions

The Hebbian learning principle for the connection between two mental states is sometimes formulated as 'neurons that fire together, wire together'; e.g., (Bi and Poo 2001; Gerstner and Kistler 2002; Hebb 1949; Keysers and Perrett 2004; Keysers and Gazzola 2014; Kuriscak et al. 2015; Zenke et al. 2017). This can be modelled by using the activation values the two mental states $X(t)$ and $Y(t)$ have at time t. Then the reification state $\mathbf{W}_{X,Y}$ for the weight $\boldsymbol{\omega}_{X,Y}$ of the connection from X to Y is changing over time dynamically, depending on these levels $X(t)$ and $Y(t)$, but also on the value of $\mathbf{W}_{X,Y}$ itself. Therefore these *Hebbian learning combination functions* $\mathbf{c}(V_1, V_2, W)$ have suitable arguments refering to the relevant states: V_1 refers to the state value $X(t)$ of X, V_2 to the state value $Y(t)$ of Y and W to the state value $\mathbf{W}_{X,Y}(t)$ of $\mathbf{W}_{X,Y}$. In Fig. 14.2, the Hebbian learning combination functions $\mathbf{cw}_{X_1,Y}$ (V_1, V_2, W) and $\mathbf{cw}_{X_2,Y}$ (V_1, V_2, W) are indicated as labels for the reification states $\mathbf{W}_{X_1,Y}$ and $\mathbf{W}_{X_2,Y}$. Similarly, labels for adaptation speed (speed factors indicating the learning rate) can be added, and labels for the incoming connections for the reification states.

Thus in the standard way for temporal-causal networks based on such Hebbian learning combination functions the following difference and differential equations are obtained for the reification states; these define the adaptive dynamics of Hebbian learning:

$$
\begin{aligned}
&\mathbf{W}_{X,Y}(t+\Delta t) = \mathbf{W}_{X,Y}(t) + \boldsymbol{\eta}_{\mathbf{W}_{X,Y}}\left[\mathbf{c}(X(t), Y(t), \mathbf{W}_{X,Y}(t)) - \mathbf{W}_{X,Y}(t)\right]\Delta t \\
&\mathbf{d}\mathbf{W}_{X,Y}(t)/\mathbf{d}t = \boldsymbol{\eta}_{\mathbf{W}_{X,Y}}\left[\mathbf{c}(X(t), Y(t), \mathbf{W}_{X,Y}(t)) - \mathbf{W}_{X,Y}(t)\right]
\end{aligned}
\quad (14.2)
$$

Here indeed the speed factor $\boldsymbol{\eta}_{\mathbf{W}X,Y}$ now can be interpreted as learning rate for the connection weight.

14.3.2 An Example Reified Network Model with Multiple Hebbian Learning Reification States

An example of an adaptive mental network model using Hebbian learning is shown in Fig. 14.3; see also (Treur and Umair 2011) or (Treur 2016), Chap. 6, p. 163. Here ws_w are world states, ss_w sensor states, srs_w and srs_{e_i} sensory representations states for stimulus w and action effect e_i, ps_{a_i} preparation states for a_i, fs_{e_i} feeling states for action effect e_i, and es_{a_i} execution states for a_i.

It describes adaptive decision making as affected by three different adaptive connections (the green arrows in the base plane in Fig. 14.3) both for direct triggering of decision options a_i and emotion-related valuing of the options by an as-if prediction loop:

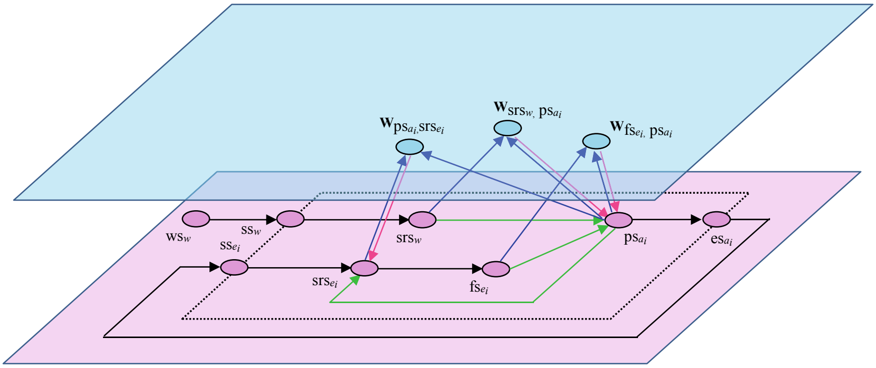

Fig. 14.3 Reified temporal-causal network model for adaptive rational decision making based on emotions

- stimulus-response connection from srs_w to action option preparation ps_{a_i}
- action effect prediction link from ps_{a_i} to effect representation srs_{e_i}
- emotion-related valuing of the action by the connection from feeling state fs_{e_i} to ps_{a_i}

Each of these connections can use Hebbian learning. A relatively simple example, also used in (Treur 2016) in a number of applications (including in Chap. 6 there for the model shown in Fig. 14.3), is the following Hebbian learning combination function:

$$\mathbf{c}_{\mathbf{W}_{X,Y}}(V_1, V_2, W) = V_1 V_2 (1 - W) + \boldsymbol{\mu} W$$
$$\text{or} \qquad (14.3)$$
$$\mathbf{c}_{\mathbf{W}_{X,Y}}(X(t), Y(t), \mathbf{W}_{X,Y}(t)) = X(t)Y(t)(1 - \mathbf{W}_{X,Y}(t)) + \boldsymbol{\mu}\mathbf{W}_{X,Y}(t)$$

Here $\boldsymbol{\mu}$ is a persistence parameter. In an emerging equilibrium state it turns out that the equilibrium value for $\mathbf{W}_{X,Y}$ functionally depends on the equilibrium values of X and Y according to a specific formula that has been determined for this case in (Treur 2016), Chap. 12, Sect. 12.5.2. For some example patterns, see Fig. 14.4.

It is shown that when the equilibrium values of X and Y are 1, the equilibrium value for $\mathbf{W}_{X,Y}$ is 0.83 (top row), when the equilibrium values of X and Y are 0.6, the equilibrium value for $\mathbf{W}_{X,Y}$ is 0.64 (middle row), and when the equilibrium values of X and Y are 0, the equilibrium value for $\mathbf{W}_{X,Y}$ is 0 (bottom row). This equilibrium value of $\mathbf{W}_{X,Y}$ is always attracting. The three different rows in Fig. 14.4 illustrate how the equilibrium value of $\mathbf{W}_{X,Y}$ varies with the equilibrium values of X and Y. It is this relation that is analysed in a more general setting in some depth in this chapter. In (Treur 2016), Chap. 12 a mathematical analysis was made for the equilibria of the specific example combination function above (although written in the slightly different but equivalent format as discussed in Chap. 15, Sect. 15.2). In the current chapter a much more general analysis is made which applies to a wide

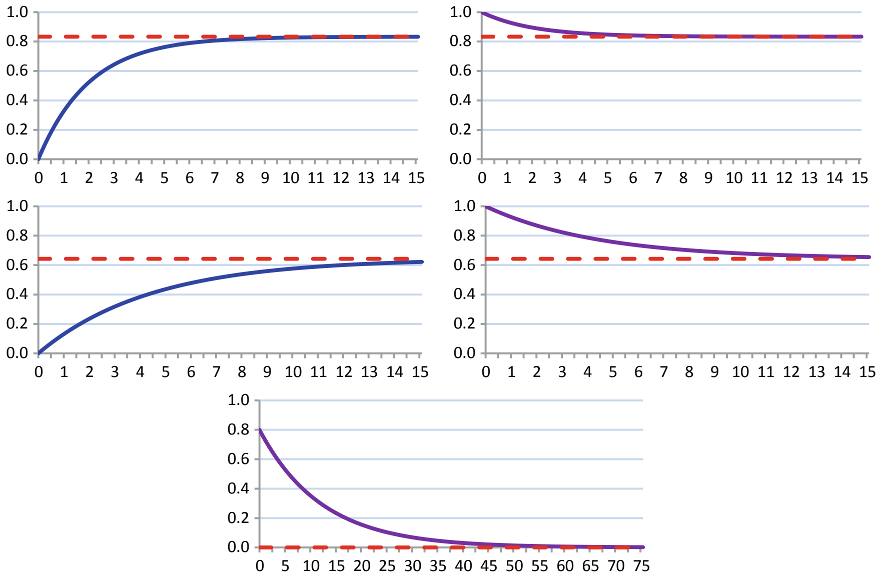

Fig. 14.4 Hebbian learning **η** = 0.4, **μ** = 0.8, Δt = 0.1; adopted from (Treur 2016), pp. 339–340. **a** Top row: activation levels X_1 = 1 and X_2 = 1; equilibrium value 0.83. **b** Middle row activation levels X_1 = 0.6 and X_2 = 0.6; equilibrium value 0.64. **c** Bottom row: activation levels $X_1 = X_2 = 0$; equilibrium value 0 (pure extinction)

class of functions. In Example 1 in Sect. 14.6 below, the above case is obtained as a special case of the more general results found, and more precise numbers will be derived for the equilibrium values.

14.4 Relevant Aggregation Characteristics for Hebbian Learning and the Implied Behaviour

In this section, it is discussed how in a reified network aggregation for the connection weight reification state can be defined by a specific class of combination functions for Hebbian learning, and it will be analysed what equilibrium values can emerge for the learnt connections.

14.4.1 Relevant Aggregation Characteristics for Hebbian Learning

First a basic definition; see also (Brauer and Nohel 1969; Hirsch 1984; Lotka 1956).

Definition 1 (stationary point and equilibrium)
A state Y has a *stationary point* at t if $\mathbf{d}Y(t)/\mathbf{d}t = 0$. The network is in *equilibrium* a t if every state Y of the model has a stationary point at t. A state Y is increasing at t if $\mathbf{d}Y(t)/\mathbf{d}t > 0$; it is decreasing at t if $\mathbf{d}Y(t)/\mathbf{d}t < 0$.

Considering the specific type of differential equation for a temporal-causal network model, and assuming a nonzero speed factor, from (14.1) and (14.2) more specific criteria can be found:

Lemma 1 (Criteria for a stationary, increasing and decreasing)
Let Y be a state and X_1, ..., X_k the states from which state Y gets its incoming connections. Then

$$
\begin{aligned}
&Y \text{ has a stationary point at } t && \Leftrightarrow \quad \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = Y(t)\\
&Y \text{ is increasing at } t && \Leftrightarrow \quad \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) > Y(t)\\
&Y \text{ is decreasing at } t && \Leftrightarrow \quad \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) < Y(t)
\end{aligned}
$$

These criteria can also be applied to adaptive connection weights based on Hebbian learning combination functions $\mathbf{c}(V_1, V_2, W)$ for the reification states $\mathbf{W}_{X,Y}$:

$$
\begin{aligned}
&\mathbf{W}_{X,Y}(t) \text{ has a stationary point at } t && \Leftrightarrow \quad \mathbf{c}(V_1, V_2, W) = W\\
&\mathbf{W}_{X,Y}(t) \text{ is increasing at } t && \Leftrightarrow \quad \mathbf{c}(V_1, V_2, W) > W\\
&\mathbf{W}_{X,Y}(t) \text{ is decreasing at } t && \Leftrightarrow \quad \mathbf{c}(V_1, V_2, W) < W
\end{aligned}
$$

The following plausible assumptions are made for the Hebbian learning functions used as combination function to specify aggregation of the reification state in the reified network: one set for fully persistent Hebbian learning and one set for hebbian learning with extinction described by a persistence parameter $\boldsymbol{\mu}$; here again V_1 is the argument of the function $c_Y(..)$ used for $X(t)$, V_2 for $Y(t)$, and W for $\mathbf{W}_{X,Y}(t)$.

Definition 2 (Hebbian Learning Function)
A function c: $[0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *fully persistent Hebbian learning function* if the following hold:

(a) $c(V_1, V_2, W)$ is a monotonically increasing function of V_1 and V_2
(b) $c(V_1, V_2, W) - W$ is a monotonically decreasing function of W
(c) $c(V_1, V_2, W) \geq W$
(d) $c(V_1, V_2, W) = W$ if and only if one of V_1 and V_2 is 0 (or both), or $W = 1$

A function c: $[0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *Hebbian learning function* with *persistence parameter* $\boldsymbol{\mu}$ if the following hold:

(a) $c(V_1, V_2, W)$ is a monotonically increasing function of V_1 and V_2
(b) $c(V_1, V_2, W) - \boldsymbol{\mu}W$ is a monotonically decreasing function of W
(c) $c(V_1, V_2, W) \geq \boldsymbol{\mu}W$
(d) $c(V_1, V_2, W) = \boldsymbol{\mu}W$ if and only if one of V_1 and V_2 is 0 (or both), or $W = 1$

Note that for $\boldsymbol{\mu} = 1$ the function is fully persistent.

14.4.2 Functional Relation for the Equilibrium Value of a Hebbian Learning Reification State

The following proposition shows that for any Hebbian learning function with persistence parameter $\boldsymbol{\mu}$ there exists a monotonically increasing function $f_{\boldsymbol{\mu}}(V_1, V_2)$ which is implicitly defined for given V_1, V_2 by the equation $c(V_1, V_2, W) = W$ in W. When applied to an equilibrium state of an adaptive temporal-causal network, the existence of this function $f_{\boldsymbol{\mu}}(V_1, V_2)$ reveals that in equilibrium states there is a direct and monotonically increasing functional relation of the equilibrium value $\underline{\mathbf{W}}_{X,Y}$ of $\mathbf{W}_{X,Y}$ with the equilibrium values $\underline{X}$, $\underline{Y}$ of the states X and Y. This is described in Theorem 1 below. Proposition 1 describes the functional relation needed for that. For proofs of Propositions 1 and 2, see Chap. 15, Sect. 15.9.

Proposition 1 (functional relation for *W*)
Suppose that $c(V_1, V_2, W)$ is a Hebbian learning function with persistence parameter $\boldsymbol{\mu}$.

(a) Suppose $\boldsymbol{\mu} < 1$. Then the following hold:
(i) The function $W \to c(V_1, V_2, W) - W$ on [0, 1] is strictly monotonically decreasing
(ii) There is a unique function $f_{\boldsymbol{\mu}}$: [0, 1] x [0, 1] → [0, 1] such for any V_1, V_2 it holds

$$c(V_1, V_2, f_{\boldsymbol{\mu}}(V_1, V_2)) = f_{\boldsymbol{\mu}}(V_1, V_2)$$

This function $f_{\boldsymbol{\mu}}$ is a monotonically increasing function of V_1, V_2, and is implicitly defined by the above equation. Its maximal value is $f_{\boldsymbol{\mu}}(1, 1)$ and minimum $f_{\boldsymbol{\mu}}(0, 0) = 0$.

(b) Suppose $\boldsymbol{\mu} = 1$. Then there is a unique function f_1: (0, 1] × (0, 1] → [0, 1], such for any V_1, V_2 it holds

$$c(V_1, V_2, f_1(V_1, V_2)) = f_1(V_1, V_2)$$

This function f_1 is a constant function of V_1, V_2 with $f_1(V_1, V_2) = 1$ for all V_1, $V_2 > 0$ and is implicitly defined on (0, 1] × (0, 1] by the above equation.
If one of V_1, V_2 is 0, then any value of W satisfies the equation $c(V_1, V_2, W) = W$, so no unique function value for $f_1(V_1, V_2)$ can be defined then.

When applied to an equilibrium state of a reified adaptive temporal-causal network, this proposition entails the following Theorem 1. For $\boldsymbol{\mu} < 1$ this follows from Proposition 1 (a) applied to the function c(..). From (a)(i) it follows that the equilibrium value $\underline{\mathbf{W}}_{X,Y}$ is attracting: suppose $\mathbf{W}_{X,Y}(t) < \underline{\mathbf{W}}_{X,Y}$, then from $c(V_1, V_2, \underline{\mathbf{W}}_{X,Y}) - \underline{\mathbf{W}}_{X,Y} = 0$ and the decreasing monotonicity of $W \to c(V_1, V_2, W) - W$ it follows that $c(V_1, V_2, \mathbf{W}_{X,Y}(t)) - \mathbf{W}_{X,Y}(t) > 0$, and therefore by Lemma 1 $\mathbf{W}_{X,Y}(t)$ is increasing. Similarly, when $\mathbf{W}_{X,Y}(t) > \underline{\mathbf{W}}_{X,Y}$, it is decreasing. For $\boldsymbol{\mu} = 1$ the statement follows from Proposition 1 (b) applied to the function c(..).

Theorem 1 (functional relation for equilibrium values of $\mathbf{W}_{X,Y}$) Suppose in a reified temporal-causal network, $\mathrm{c}(V_1, V_2, W)$ is the combination function for reification state $\boldsymbol{W}_{X,Y}$ for connection weight $\boldsymbol{\omega}_{X,Y}$ and is a Hebbian learning function with persistence parameter $\boldsymbol{\mu}$, with $\mathrm{f}_{\boldsymbol{\mu}}$ the function defined by Proposition 1. In an achieved equilibrium state the following hold.

(a) Suppose $\boldsymbol{\mu} < 1$. For any equilibrium values $\boldsymbol{X}, \boldsymbol{Y} \in [0, 1]$ of states X and Y the value $\mathrm{f}_{\boldsymbol{\mu}}(\boldsymbol{X}, \boldsymbol{Y})$ provides the unique equilibrium value $\underline{\boldsymbol{W}}_{X,Y}$ for $\boldsymbol{W}_{X,Y}$. This $\underline{\boldsymbol{W}}_{X,Y}$ monotonically depends on $\boldsymbol{X}, \boldsymbol{Y}$: it is higher when $\boldsymbol{X}, \boldsymbol{Y}$ are higher. The maximal equilibrium value $\underline{\boldsymbol{W}}_{X,Y}$ of $\boldsymbol{W}_{X,Y}$ is $\mathrm{f}_{\boldsymbol{\mu}}(1, 1)$ and the minimal equilibrium value is 0. Moreover, the equilibrium value $\underline{\boldsymbol{W}}_{X,Y}$ is attracting.

(b) Suppose $\boldsymbol{\mu} = 1$. If for the equilibrium values $\boldsymbol{X}, \boldsymbol{Y} \in [0, 1]$ of states X and Y it holds $\boldsymbol{X}, \boldsymbol{Y} > 0$, then $\underline{\boldsymbol{W}}_{X,Y} = 1$. If one of $\boldsymbol{X}, \boldsymbol{Y}$ is 0, then $\underline{\boldsymbol{W}}_{X,Y}$ can be any value in [0, 1]: it does not depend on $\boldsymbol{X}, \boldsymbol{Y}$. So, for $\boldsymbol{\mu} = 1$ the maximal value of $\underline{\boldsymbol{W}}_{X,Y}$ in an equilibrium state is 1 and the minimal value is 0.

14.5 Variable Separation for Hebbian Learning Functions and the Implied Behaviour

There is a specific subclass of Hebbian learning functions that is often used. For this subclass the implied behaviour can be determined more explicitly by obtaining certain algebraic formulae for the function $\mathrm{f}_{\boldsymbol{\mu}}$ in Theorem 1.

14.5.1 Hebbian Learning Combination Functions with Variable Separation

Relatively simple functions $\mathrm{c}(V_1, V_2, W)$ that satisfy the requirements from Definition 2 are obtained when the state arguments V_1 and V_2 and the connection argument W can be separated as follows.

Definition 3 (variable separation)
The Hebbian learning function $\mathrm{c}(V_1, V_2, W)$ with persistence parameter $\boldsymbol{\mu}$ *enables variable separation* by functions cs: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ monotonically increasing and cs: $[0, 1] \rightarrow [0, 1]$ monotonically decreasing if

$$\mathrm{c}(V_1, V_2, W) = \mathrm{cs}(V_1, V_2)\mathrm{cc}(W) + \boldsymbol{\mu} W$$

where $\mathrm{cs}(V_1, V_2) = 0$ if and only if one of V_1, V_2 is 0, and $\mathrm{cc}(1) = 0$ and $\mathrm{cc}(W) > 0$ when $W < 1$.

The function $\mathrm{cs}(V_1, V_2)$ is called the *states factor* and the function $\mathrm{cc}(W)$ the *connection factor*.

Note that the s in cs stands for states and the second c in cc for connection. When variable separation holds, the following proposition can be obtained. For this type of function the indicated functional relation can be defined.

Proposition 2 (functional relation for *W* based on variable separation) Assume the Hebbian function $\mathrm{c}(V_1, V_2, W)$ with persistence parameter $\boldsymbol{\mu}$ enables variable separation by the two functions $\mathrm{cs}(V_1, V_2)$ monotonically increasing and $\mathrm{cc}(W)$ monotonically decreasing:

$$\mathrm{c}(V_1, V_2, W) = \mathrm{cs}(V_1, V_2)\mathrm{cc}(W) + \boldsymbol{\mu} W$$

Let $\mathrm{h}_{\boldsymbol{\mu}}(\mathrm{W})$ be the function defined for $W \in [0, 1)$ by

$$\mathrm{h}_{\boldsymbol{\mu}}(W) = \frac{(1-\boldsymbol{\mu})W}{\mathrm{cc}(W)}$$

Then the following hold.

(a) When $\boldsymbol{\mu} < 1$ the function $\mathrm{h}_{\boldsymbol{\mu}}(W)$ is strictly monotonically increasing, and has a strictly monotonically increasing inverse $\mathrm{g}_{\boldsymbol{\mu}}$ on the range $\mathrm{h}_{\boldsymbol{\mu}}([0, 1))$ of $\mathrm{h}_{\boldsymbol{\mu}}$ with $W = \mathrm{g}_{\boldsymbol{\mu}}(\mathrm{h}_{\boldsymbol{\mu}}(W))$ for all $W \in [0, 1)$.

(b) When $\boldsymbol{\mu} < 1$ and $\mathrm{c}(V_1, V_2, W) = W$, then $\mathrm{g}_{\boldsymbol{\mu}}(\mathrm{cs}(V_1, V_2)) < 1$ and $W < 1$, and it holds

$$\mathrm{h}_{\boldsymbol{\mu}}(W) = \mathrm{cs}(V_1, V_2)$$
$$W = g_{\boldsymbol{\mu}}(\mathrm{cs}(V_1, V_2))$$

So, in this case the function $\mathrm{f}_{\boldsymbol{\mu}}$ from Theorem is the function composition $\mathrm{g}_{\boldsymbol{\mu}}$ o cs of cs followed by $\mathrm{g}_{\boldsymbol{\mu}}$; it holds:

$$\mathrm{f}_{\boldsymbol{\mu}}(V_1, V_2) = g_{\boldsymbol{\mu}}(\mathrm{cs}(V_1, V_2))$$

(c) For $\boldsymbol{\mu} = 1$ it holds $\mathrm{c}(V_1, V_2, W) = W$ if and only if $V_1 = 0$ or $V_2 = 0$ or $W = 1$.

(d) For $\boldsymbol{\mu} < 1$ the maximal value W with $\mathrm{c}(V_1, V_2, W) = W$ is $\mathrm{g}_{\boldsymbol{\mu}}(\mathrm{cs}(1, 1))$, and the minimal equilibrium value W is 0. For $\boldsymbol{\mu} = 1$ the maximal value W is 1 (always when $V_1, V_2 > 0$ holds) and the minimal value is 0 (occurs when one of V_1, V_2 is 0).

Note that by Proposition 2 the function $\mathrm{f}_{\boldsymbol{\mu}}(V_1, V_2)$ can be determined by inverting the function $\mathrm{h}_{\boldsymbol{\mu}}(W) = (1-\boldsymbol{\mu})W/\mathrm{cc}(W)$ to find $\mathrm{g}_{\boldsymbol{\mu}}$ and composing the inverse with the function $\mathrm{cs}(V_1, V_2)$. This will be shown below for some cases.

14.5.2 Functional Relation for the Equilibrium Value of a Hebbian Learning Reification State: The Variable Separation Case

For the case of an equilibrium state of a reified adaptive temporal network model Proposition 2 entails Theorem 2.

Theorem 2 (functional relation for equilibrium values of $\mathbf{W}_{X,Y}$: variable separation)
Assume in a reified temporal-causal network the Hebbian learning combination function $c(V_1, V_2, W)$ with persistence parameter $\boldsymbol{\mu}$ for $\mathbf{W}_{X,Y}$ enables variable separation by the two functions $cs(V_1, V_2)$ monotonically increasing and $cc(W)$ monotonically decreasing, and the functions $f_{\boldsymbol{\mu}}$ and $g_{\boldsymbol{\mu}}$ is defined as in Propositions 1 and 2. Then the following hold.

(a) When $\boldsymbol{\mu} < 1$ in an achieved equilibrium state with equilibrium values $\mathbf{X}$, $\mathbf{Y}$ for states X and Y and $\mathbf{W}_{X,Y}$ for $\mathbf{W}_{X,Y}$ it holds

$$\underline{\mathrm{W}}_{X,Y} = f_{\boldsymbol{\mu}}(\mathbf{X}, \mathbf{Y}) = g_{\boldsymbol{\mu}}(cs(\mathbf{X}, \mathbf{Y})) < 1$$

(b) For $\boldsymbol{\mu} = 1$ in an equilibrium state with equilibrium values $\mathbf{X}$, $\mathbf{Y}$ for states X and Y and $\mathbf{W}_{X,Y}$ for $\mathbf{W}_{X,Y}$ it holds $\mathbf{X} = 0$ or $\mathbf{Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$.

(c) For $\boldsymbol{\mu} < 1$ in an equilibrium state the maximal equilibrium value $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ is $g_{\boldsymbol{\mu}}(cs(1, 1)) < 1$, and the minimal equilibrium value $\underline{\mathbf{W}}_{X,Y}$ is 0. For $\boldsymbol{\mu} = 1$ the maximal value is 1 (always when $\mathbf{X}, \mathbf{Y} > 0$ holds for the equilibrium values for the states X and Y) and the minimal value is 0 (which occurs when one of $\mathbf{X}$, $\mathbf{Y}$ is 0).

14.6 Implications for Different Classes of Hebbian Learning Functions

In this section some cases are analysed as corollaries of Theorem 2.

14.6.1 Hebbian Learning Functions with Variable Separation and Linear Connection Factor

First the specific class of Hebbian learning functions enabling variable separation with $cc(W) = 1 - W$ is considered. Then

$$\mathrm{h}_{\boldsymbol{\mu}}(W) = \frac{(1-\boldsymbol{\mu})W}{\mathrm{cc}(W)} = \frac{(1-\boldsymbol{\mu})W}{1-W} \tag{14.4}$$

and the inverse $\mathrm{g}_{\boldsymbol{\mu}}(W')$ of $\mathrm{h}_{\boldsymbol{\mu}}(W)$ can be determined from (14.4) algebraically as shown in Box 14.1.

Box 14.1 Inverting $\mathrm{h}_{\boldsymbol{\mu}}(W)$ for linear connection factor $\mathrm{cc}(W) = 1-W$

$$\begin{aligned}
W' &= \mathrm{h}_{\boldsymbol{\mu}}(W) = \tfrac{(1-\boldsymbol{\mu})W}{1-W} \\
W'(1-W) &= (1-\boldsymbol{\mu})W \\
W' - W'W &= (1-\boldsymbol{\mu})W \\
W' &= (W' + (1-\boldsymbol{\mu}))W \\
W &= \tfrac{W'}{W' + (1-\boldsymbol{\mu})} \\
\mathrm{g}_{\boldsymbol{\mu}}(W') &= \tfrac{W'}{W' + (1-\boldsymbol{\mu})}
\end{aligned}$$

So it has been found that

$$\mathrm{g}_{\boldsymbol{\mu}}(W') = \frac{W'}{W' + (1-\boldsymbol{\mu})} \tag{14.5}$$

Substitute $W' = \mathrm{cs}(V_1, V_2)$ in (14.5) and it is obtained:

$$\mathrm{f}_{\boldsymbol{\mu}}(V_1, V_2) = \mathrm{g}_{\boldsymbol{\mu}}(\mathrm{cs}(V_1, V_2)) = \frac{\mathrm{cs}(V_1, V_2)}{(1-\boldsymbol{\mu}) + \mathrm{cs}(V_1, V_2)} \tag{14.6}$$

and this is less than 1 because $1-\boldsymbol{\mu} > 0$. From this and Theorem 2 (b) and (c) it follows.

Corollary 1 (cases for connection factor cc(*W*) = 1 − *W*)
Assume in a reified temporal-causal network the Hebbian learning combination function $\mathrm{c}(V_1, V_2, W)$ for $\mathbf{W}_{X,Y}$ with persistence parameter $\boldsymbol{\mu}$ enables variable separation by the two functions $\mathrm{cs}(V_1, V_2)$ monotonically increasing and $\mathrm{cc}(W)$ monotonically decreasing, where for the connection factor it hols $\mathrm{cc}(W) = 1 - W$. Then the following hold.

(a) When $\boldsymbol{\mu} < 1$ in an equilibrium state with equilibrium values $\mathbf{X}$, $\mathbf{Y}$ for states X and Y and $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ it holds

$$\underline{\mathrm{W}}_{X,Y} = \mathrm{f}_{\boldsymbol{\mu}}(\mathbf{X}, \mathbf{Y}) = \frac{\mathrm{cs}(\mathbf{X}, \mathbf{Y})}{(1-\boldsymbol{\mu}) + \mathrm{cs}(\mathbf{X}, \mathbf{Y})} < 1$$

Table 14.2 Special cases for variable separation

cc(W)	cs(V_1, V_2)	Equilibrium value $\underline{\mathbf{W}}_{X,Y} = f_{\mu}(\mathbf{X}, \mathbf{Y})$	Maximal equilibrium value $\underline{\mathbf{W}}_{X,Y}$
$1 - W$	V_1V_2	$\mathbf{X}\,\mathbf{Y}/[(1-\boldsymbol{\mu}) + \mathbf{X}\,\mathbf{Y}]$	$\frac{1}{2-\boldsymbol{\mu}}$
	$\sqrt{V_1V_2}$	$\sqrt{\mathbf{XY}}/[(1-\boldsymbol{\mu}) + \sqrt{\mathbf{XY}}]$	$\frac{1}{2-\boldsymbol{\mu}}$
	$V_1V_2(V_1 + V_2)$	$\frac{\mathbf{XY}(\mathbf{X}+\mathbf{Y})}{(1-\boldsymbol{\mu})+\mathbf{XY}(\mathbf{X}+\mathbf{Y})}$	$\frac{2}{3-\boldsymbol{\mu}}$
$1 - W^2$	$V_1V_2(V_1 + V_2)$	$\frac{-(1-\boldsymbol{\mu})+\sqrt{(1-\boldsymbol{\mu})^2+4(\mathbf{XY}(\mathbf{X}+\mathbf{Y}))^2}}{2\mathbf{XY}(\mathbf{X}+\mathbf{Y})}$	$\frac{-(1-\boldsymbol{\mu})+\sqrt{(1-\boldsymbol{\mu})^2+16}}{4}$

(b) For $\boldsymbol{\mu} = 1$ in an equilibrium state with equilibrium values $\mathbf{X}$, $\mathbf{Y}$ for states X and Y and $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ it holds $\mathbf{X} = 0$ or $\mathbf{Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$.

(c) For $\boldsymbol{\mu} < 1$ in an equilibrium state the maximal equilibrium value $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ is

$$\frac{\mathrm{cs}(1,1)}{(1-\boldsymbol{\mu})+\mathrm{cs}(1,1)} < 1$$

and the minimal equilibrium value $\mathbf{W}_{X,Y}$ is 0. For $\boldsymbol{\mu} = 1$ the maximal value is 1 (when $\mathbf{X}$, $\mathbf{Y} > 0$ holds for the equilibrium values for the states X and Y) and the minimal value is 0 (which occurs when one of $\mathbf{X}$, $\mathbf{Y}$ is 0).

Corollary 1 is illustrated in the three examples shown in Box 14.2. Note that Table 14.2 summarizes these results.

Box 14.2 Different examples of Hebbian learning functions with variable separation with linear connection factor cc(W) = 1 − W

Example 1 c(V_1, V_2, W) = $V_1V_2(1-W) + \boldsymbol{\mu}W$

$$\mathrm{cs}(V_1, V_2) = V_1V_2 \quad \mathrm{cc}(W) = 1 - W$$

This is the example shown in Fig. 14.4

$$f_{\boldsymbol{\mu}}(V_1, V_2) = \frac{\mathrm{cs}(V_1, V_2)}{(1-\boldsymbol{\mu}) + \mathrm{cs}(V_1, V_2)} \tag{14.7}$$

Substitute cs(V_1, V_2) = V_1V_2 in (14.7) then $f_{\boldsymbol{\mu}}(V_1, V_2) = \frac{V_1V_2}{(1-\boldsymbol{\mu})+V_1V_2}$
Maximal W is $W_{\max} = f_{\boldsymbol{\mu}}(1,1) = \frac{1}{2-\boldsymbol{\mu}}$, which for $\boldsymbol{\mu} = 1$ is 1; minimal W is 0.
The equilibrium values shown in Fig. 14.4 can immediately be derived from this (recall $\boldsymbol{\mu} = 0.8$):

Figure 14.4, top row $V_1 = 1$ $V_2 = 1$, then $f_{\mu}(1,1) = \frac{1}{2-\mu} = 0.833333$
Figure 14.4, middle row $V_1 = 0.6$, $V_2 = 0.6$, then
$f_{\mu}(0.6, 0.6) = 0.36/[(1 - 0.8) + 0.36] = 0.642857$
Figure 14.4, bottom row $V_1 = 0$, $V_2 = 0$, then $f_{\mu}(0, 0) = 0$

Example 2 $c(V_1, V_2, W) = \sqrt{V_1V_2}(1-W) + \mu W$

$$\begin{aligned} &\mathrm{cs}(V_1,V_2) = \sqrt{V_1V_2} \quad \mathrm{cc}(W) = 1 - W \\ &f_{\mu}(V_1,V_2) = \mathrm{cs}(V_1,V_2)/[(1-\mu) + \mathrm{cs}(V_1,V_2)] \end{aligned} \tag{14.8}$$

Substitute $\mathrm{cs}(V_1,V_2) = \sqrt{V_1V_2}$ in (14.8) to obtain

$$f_{\mu}(V_1,V_2) = \frac{\sqrt{V_1V_2}}{(1-\mu) + \sqrt{V_1V_2}} \tag{14.9}$$

Maximal W is $W_{max} = f_{\mu}(1,1) = \frac{1}{2-\mu}$, which for $\mu = 1$ is 1; minimal W is 0.

In a similar case as in Fig. 14.4, but now using this function, the following equilibrium values would be found

Top row $V_1 = 1$, $V_2 = 1$, then $f_{\mu}(1,1) = \frac{1}{2-\mu} = 0.833333$
Middle row $V_1 = 0.6$, $V_2 = 0.6$, then
$f_{\mu}(0.6, 0.6) = 0.6/[(1 - 0.8) + 0.6] = 0.75$
Bottom row $V_1 = 0$, $V_2 = 0$, then $f_{\mu}(0, 0) = 0$

Example 3 $c(V_1, V_2, W) = V_1V_2(V_1 + V_2)(1 - W) + \mu W$

$$\mathrm{cs}(V_1,V_2) = V_1V_2(V_1+V_2) \quad \mathrm{cc}(W) = 1 - W$$

$$f_{\mu}(V_1,V_2) = \frac{\mathrm{cs}(V_1,V_2)}{(1-\mu) + \mathrm{cs}(V_1,V_2)} \tag{14.10}$$

Substitute $\mathrm{cs}(V_1, V_2) = V_1V_2(V_1 + V_2)$ in (14.10) to obtain

$$f_{\mu}(V_1,V_2) = \frac{V_1V_2(V_1+V_2)}{(1-\mu) + V_1V_2(V_1+V_2)} \tag{14.11}$$

Maximal W is $W_{max} = f_{\mu}(1,1) = \frac{2}{(1-\mu)+2} = \frac{2}{3-\mu}$, which for $\mu = 1$ is 1; minimal W is 0.

In a similar case as in Fig. 14.4, but using this function the following equilibrium values would be found

Top row $V_1 = 1$, $V_2 = 1$, then $f_{\mu}(1,1) = \frac{2}{3-\mu} = 0.909090$

Middle row $V_1 = 0.6$, $V_2 = 0.6$, then $f_{\mu}(0.6, 0.6) = 0.36 *1.2/[(1 - 0.8) + 0.36 *1.2] = 0.632$

Bottom row $V_1 = 0$, $V_2 = 0$, then $f_{\mu}(0, 0) = 0$

14.6.2 Hebbian Learning Functions with Variable Separation and Quadratic Connection Factor

Next the specific class of Hebbian learning functions enabling variable separation with $cc(W) = 1 - W^2$ is considered. In that case it holds

$$h_{\mu}(W) = \frac{(1-\mu)W}{cc(W)} = \frac{(1-\mu)W}{1-W^2} \tag{14.12}$$

and the inverse of h_{μ} can be determined algebraically as shown in Corollary 2. Inverting $h_{\mu}(W)$ to get inverse $g_{\mu}(W')$ now can be done as shown in Box 14.3:

Box 14.3 Inverting $h_{\mu}(W)$ for quadratic connection factor $cc(W) = 1 - W^2$

$$W' = h_{\mu}(W) = \frac{(1-\mu)W}{1-W^2}$$
$$(1-W^2)W' = (1-\mu)W$$
$$-W' + (1-\mu)W + W^2W' = 0$$

This is a quadratic equation in W:

$$W'W^2 + (1-\mu)W - W' = 0$$

As $W \geq 0$ the solution is

$$W = \frac{-(1-\mu) + \sqrt{(1-\mu)^2 + 4W'^2}}{2W'}$$

Therefore

$$g_{\mu}(W') = \frac{-(1-\mu) + \sqrt{(1-\mu)^2 + 4W'^2}}{2W'}$$

So, from Box 14.3:

$$g_{\mu}(W') = \frac{-(1-\mu) + \sqrt{(1-\mu)^2 + 4W'^2}}{2W'} \tag{14.13}$$

By substituting $W' = \mathrm{cs}(V_1, V_2)$ it follows

$$\mathrm{f}_{\boldsymbol{\mu}}(V_1, V_2) = \mathrm{g}_{\boldsymbol{\mu}}(\mathrm{cs}(V_1, V_2)) = \frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 4\mathrm{cs}(V_1, V_2)^2}}{2\mathrm{cs}(V_1, V_2)} \tag{14.14}$$

All this is summarised in the following:

Corollary 2 (cases for quadratic connection factor cc(*W*) = 1− *W*²)
Assume in a reified temporal-causal network the Hebbian learning combination function $\mathrm{c}(V_1, V_2, W)$ for $\mathbf{W}_{X,Y}$ with persistence parameter $\boldsymbol{\mu}$ enables variable separation by the two functions $\mathrm{cs}(V_1, V_2)$ monotonically increasing and cc (W) monotonically decreasing, where for the connection factor it holds cc (W) = $1 - W^2$. Then the following hold.

(a) When $\boldsymbol{\mu} < 1$ in an equilibrium state with equilibrium values **X**, **Y** for states *X* and *Y* and $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ *it holds*

$$\underline{\mathrm{W}}_{X,Y} = \mathrm{f}_{\boldsymbol{\mu}}(\mathbf{X}, \mathbf{Y}) = \frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 4\mathrm{cs}(\mathbf{X}, \mathbf{Y})^2}}{2\mathrm{cs}(\mathbf{X}, \mathbf{Y})} < 1$$

(b) For $\boldsymbol{\mu} = 1$ in an equilibrium state with equilibrium values **X**, **Y** for states *X* and *Y* and $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ it holds $\mathbf{X} = 0$ or $\mathbf{Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$.
(c) For $\boldsymbol{\mu} < 1$ in an equilibrium state the maximal equilibrium value $\underline{\mathbf{W}}_{X,Y}$ for $\mathbf{W}_{X,Y}$ *is*

$$\frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 4\mathrm{cs}(1,1)^2}}{2\mathrm{cs}(1,1)} < 1$$

and the minimal equilibrium value $\underline{\mathbf{W}}_{X,Y}$ is 0. For $\boldsymbol{\mu} = 1$ the maximal value is 1 (when **X**, **Y** > 0 holds for the equilibrium values for the states *X* and *Y*) and the minimal value is 0 (which occurs when one of **X**, **Y** is 0).

Corollary 2 is illustrated in Example 4 in Box 14.4.

Box 14.4 Example of a Hebbian learning function with variable separation with quadratic connection factor $\mathrm{cc}(W) = 1 - W^2$
Example 4 $\mathrm{c}(V_1, V_2, W) = V_1V_2(V_1 + V_2)(1 - W^2) + \boldsymbol{\mu}W$

$$\mathrm{cs}(V_1, V_2) = V_1V_2(V_1 + V_2) \quad \mathrm{cc}(W) = 1 - W^2$$

$$\mathrm{f}_{\boldsymbol{\mu}}(V_1, V_2) = \frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 4\mathrm{cs}(V_1, V_2)^2}}{2\mathrm{cs}(V_1, V_2)} \tag{14.15}$$

Substitute cs(**X**, **Y**) = **XY** (**X** + **Y**)

$$f_{\boldsymbol{\mu}}(\mathbf{X},\mathbf{Y}) = \frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 4(\mathbf{XY}(\mathbf{X}+\mathbf{Y}))^2}}{2\mathbf{XY}(\mathbf{X}+\mathbf{Y})} \tag{14.16}$$

Maximal *W* is

$$W_{\max} = f_{\boldsymbol{\mu}}(1,1) = \frac{-(1-\boldsymbol{\mu}) + \sqrt{(1-\boldsymbol{\mu})^2 + 16}}{4}$$

which for $\boldsymbol{\mu} = 1$ is 1; minimal *W* is 0. In a similar case as in Fig. 14.4, using this function the equilibrium values can be found by applying (18).

14.7 Discussion

In this chapter it was analysed how emerging behaviour of an adaptive network can be related to characteristics of reified network structure addressing adaptation principles. Parts of this chapter were adopted from (Treur 2018b). In particular this was addressed for an adaptive Mental Network based on Hebbian learning (Bi and Poo 2001; Gerstner and Kistler 2002; Hebb 1949; Keysers and Perrett 2004; Keysers and Gazzola 2014; Kuriscak et al. 2015; Zenke et al. 2017). The approach followed is based on network reification applied to Connectivity characteristics of a network expressed by connection weights; see Chap. 3. This makes that the base network is extended by reification states representing the adaptive connection weights. Applying the standard temporal-causal network structure characteristics, these reification states get their own combination functions assigned to define aggregation of the incoming impact. Such combination functions can be used to define certain types of Hebbian learning. Given these, relevant properties of the combination functions defining variants of the Hebbian adaptation principle have been identified together with their implied behaviour.

For different classes of Hebbian learning combination functions, emerging equilibrium values for the connection weight have been expressed as a function of the emerging equilibrium values of the connected states. The presented results do not concern results for just one type of network or function, as more often is found, but were formulated and proven at a more general level and therefore can be applied not just to specific networks but to classes of networks satisfying the identified relevant properties of reified network structure including the adaptation characteristics as specified by the Hebbian learning combination function.

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