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EDUCATIONAL FORMS OF INITIATION IN MATHEMATICAL CULTURE

"Seule l'histoire peut nous débarrasser de l'histoire"
Pierre Bourdieu (1982), **Leçon sur la leçon** (p.9)

ABSTRACT. A review of literature shows that during the history of mathematics education at school the answer of what counts as 'real mathematics' varies. An argument will be given here that defines as 'real mathematics' any activity of participating in a mathematical practice. The acknowledgement of the discursive nature of school practices requires an in-depth analysis of the notion of classroom discourse. For a further analysis of this problem Bakhtin's notion of speech genre is used. The genre particularly functions as a means for the interlocutors for evaluating utterances as a legitimate part of an ongoing mathematical discourse. The notion of speech genre brings a cultural historical dimension in the discourse that is supposed to be acted out by the teacher who demonstrates the tools, rules, and norms that are passed on by a mathematical community. This has several consequences for the role of the teacher. His or her mathematical attitude acts out tendencies emerging from the history of the mathematical community (like systemacy, non-contradiction etc.) that subsequently can be imitated and appropriated by pupils in a discourse. Mathematical attitude is the link between the cultural historical dimension of mathematical practices and individual *mathematical* thinking.

KEY WORDS: activity, tool, discourse, participation, genre, attitude

1. WHAT IS REALLY MATHEMATICAL?

'Math' is widely acknowledged as an undisputed part of the school curriculum. Over the past fifty years the classroom approach to mathematics has changed radically from a drill-and-practice affair to a more insight-based problem oriented approach. Every form of mathematics education makes assumptions about what the subject matter of mathematics really is, and – consequently – how the learning individual should relate to other members of the wider culture in order to appropriate this allegedly 'real mathematics', or to put it more directly, to appropriate what is taken to be mathematics in a given community. Part of a school's responsibility is to induct students into communities of knowledge and the teaching of mathematics can be seen as a process of initiating students in the culture of the mathematical community. In fact, students are from the be-

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ginning of their life a member of a community that extensively employs embodiments of mathematical knowledge. The school focuses attention on these embodiments and their underlying insights, and by so doing draws young children into a new world of understanding, with new conventions, rules and tools. So, basically, here is a process of reacculturation in which a student is assisted to switch membership from one culture to another. Buffee's (1993) insightful analysis of this process describes reacculturation as mostly a complex and usually even painful process: "Reacculturation involves giving up, modifying, or renegotiating the language, values, knowledge, mores and so on that are constructed, established, and maintained by the community one is coming from, and becoming fluent instead in the language and so on of another community" (Buffee, 1993, p. 225).

Educational history teaches us that schools have tried to support this reacculturation process in a variety of ways. Underlying these approaches there are different assumptions concerning the nature of mathematics in the classroom, and concerning the way teachers should communicate with their pupils in the classroom. In this article I will try to apply Bakhtin's approach to the discourse in a mathematics classroom, especially focusing on the question of how the participants in this classroom are linked together and what common *background* is to be constructed in order to constitute a way of speaking and interacting that will be acknowledged as a *mathematical* discourse. The final aim is to find a way of describing some of the conditions that must be fulfilled in order to ascertain that the classroom's activity can really count as 'mathematical'. There is, however, no direct empirical way of achieving this just by observing a great number of existing classroom practices and describing the events in Bakhtinian terms. When we view the discipline of 'mathematics' as a "socially conventionalized discursive frame of understanding" (Steinbring, 1998, p. 364), we must also acknowledge – as Steinbring does – that not only factual technical mathematical operations are involved in mathematical activities in classrooms, but epistemological constraints and social conventions are also part of the process. The application of the Bakhtinian jargon requires that the hidden assumptions be brought into the open as they presumably co-determine the style and the course of the discursive process, and the authority and power relationships that are involved.

One of the values that are implicitly or explicitly applied in every mathematics classroom is an idea about what really counts as mathematical. On the basis of these notions mathematics education researchers, curriculum developers and teachers decide what is relevant or even compulsory for taking into account in the mathematics classes and courses. On the basis of their mathematical epistemology, teachers make observations of pupils' activities and select some actions as relevant or not, they value certain actions as 'good' or assess others as false or insignificant (van Oers, 2000b). Obviously, there is some normative idea at stake here about what mathematics really is, or – more modestly formulated – a norm that helps in deciding whether a particular action or utterance may count as 'mathematical' or not: one teacher focuses on number and numerals, another one on structures, while a third may stress the importance of problem solving. Introducing children in one way or another into the world of mathematics and its according speech genre probably implies teaching them the presumptions for identifying what is really mathematical and what isn't.

The idea of what mathematics really is, is of course not just an educational problem. Much of the engagement of the philosophy of mathematics is based on this very same query (see for example Rotman, 1988). Although there is probably often a relationship between the epistemological positions that can be taken with respect to mathematics as an intellectual discipline and one's view on mathematics education, I will directly focus here on the ideas about mathematics in education (school, curriculum).

As Bourdieu (1982) has already argued, education has a very important role to play in the institutionalization of a discipline through implicitly (hidden in the routines or habits of a particular community) or explicitly signaled values that create distinctions between people, and consequently mark some of them as (say) mathematicians or not, mathematically educated or not, etc. In a similar vein I shall argue here that the notion of what is mathematical and what not is developed in education, and the mastery of this value marks significantly those who will be acknowledged as mathematically educated (e.g. who may pass the exams) and who can't. Hence it is essential to find out what kind of conception of mathematics is used, and what the implications are for the relationship between teacher and pupils, as well as for the organization of the classroom discourse in mathematics. Presumably this notion of what is really mathematical in the classroom is one of the basic values that constitutes the speech genre of the mathematical classroom.

2. Views on Mathematics as Subject Matter in Schools

There exist a number of different conceptions about what the mathematical subject matter really is. The real mathematics manifests itself with different faces in the classroom, having different implications for the relationship between teacher (as a representative of culture) and pupils, and *a fortiori,* for the conception of communicating in the mathematics classroom.

As far as mathematics education is concerned we can distinguish different views on what counts as real mathematics in the classroom.

2.1. *'Mathematics' as a school subject matter is really about arithmetical operations*

This is the classical view, which used to be very common in arithmetic education in schools in the past. Children are considered to be involved in real mathematics when they are mechanically practicing counting or sums. The focus is on mastery of arithmetical operations. This is what real mathematics is supposed to be like. This view is related to the Platonic idea of eternal mathematical truths that can be discovered with honest toil. In educational practice it is not considered useful to let all children discover mathematics for themselves. As mathematical knowledge is assumed to be constituted of fixed entities, it is also believed that the elements of mathematical knowledge can be transmitted to children. The main communicational style of this approach follows the sender-receiver model that states that direct instructive language is needed to prescribe for children what to do with numbers. This point of view inevitably implies a special authoritarian relationship of a teacher towards his pupils. The teacher (as the one who knows) transmits pieces of mathematical knowledge to pupils (who don't know yet). Public discourse on mathematics in schools still follows mostly this point of view.

2.2. *'Mathematics' as a subject matter is really about structures*

The subject matter of mathematics is here conceived as essentially dealing with abstract structures that have to be applied to concrete situations and problems. The teacher or curriculum developer who subscribes to this view believes that children are really getting involved with mathematics when they are dealing with abstract structures for the organization of practical situations or for the solution of quantitative and spatial problems. It is generally believed that the basic abstract structures can already be seen in young children's play activities (see Picard, 1970; Dienes and Golding, 1966, 1967a and b), from which these structures can be elevated and further developed into explicitly reflected mathematical structures. Both Piaget (1966) and Davydov (1972) evidently endorse such a view on mathematics in school, although their view on the essence of structures is definitely different. In their argumentation for the basic structures they both refer to the French collective of mathematicians, *Bourbaki,* who tried to write a definitive history of mathematics on the basis of a few basic mother structures that engender new, more specific embedded structures, until all mathematical knowledge can be classified as an element in one structured whole (see for example Piaget, 1969a, p. 70–71). For Piaget, however, the basic structures were a consequence of the architecture of human logical thinking; for Davydov these structures were conceived as the best historical product of human thinking for structuring the whole body of mathematical knowledge. Despite their fundamental differences, however, both Piaget and Davydov defended a view on real mathematical activity that emphasizes the importance of structures. And again, despite their theoretical differences, authors committed to this point of view all propagate active methods of learning (see Picard, 1970, p. 15; Piaget, 1969b; Davydov, 1972, 1988), in which exploration or communication may play a prominent role. The so-called 'mother structures' are taken as the real objects of mathematical teaching and communication.

From their work it is evident that no one of these educators would ever propagate a direct transmission kind of teaching. Instead, the required structures are offered in situations and problems, so that the child can step by step – with more or less help – construct the basic structures and apply these subsequently in new problem situations. The child that is constructing and applying such structures is considered to be engaged in 'real mathematical activity'.

2.3. *'Mathematics' as a subject matter is really about problem solving activity with symbolic tools*

In this view the real subject matter of mathematics in the classroom is about problem solving with the help of self-invented tools in the context of realistic situations that make sense to the pupils. The seminal work of Freudenthal is important here. In many of his books he explains his view on mathematics as a human activity of problem solving with the help of tools that are invented to organize fields of experience in a schematic way (Freudenthal, 1973, 1978, 1991). In Freudenthal's view all mathematical conceptions, structures and ideas must be conceived in relation to the phenomena for which they were created in the first place (Freudenthal, 1984, p. 9). This brings him to the position of conceiving mathematical concepts and structures always as functional and contextualized tools for the solution of problems, but they are always to be conceived in relation to the context in which they originated. Structures, then, can never be seen as eternally fixed. Structures are just temporarily stabilized ways of approaching a problem. Mathematical activity in school – in order to be realistic – should focus above all on the processes of *structuring* instead of the mastery of fixed and prescribed structures. This difference between the emphasis on structures vs the emphasis on structuring is exactly the core of Freudenthal's critique on Davydov and Piaget.

This variant of real mathematics indeed fosters active learning and communication in heterogeneous groups. Hence discussion is an important element in this approach. Freudenthal's emphasis on the real life usefulness of mathematics ("If it were not useful, mathematics would not exist", Freudenthal, 1973, p. 16) has often been interpreted as emphasizing the real-life character of the contexts from which mathematical thinking should originate. The realism of mathematics then is seen in the applicability of self-invented mathematics in a meaningful problem, and for many people this seems to mean a real-life problem. For Freudenthal this included also interactive problem solving in heterogeneous groups of pupils. The teacher follows the process from a safe distance. This view is very popular at the moment in the Netherlands, where most of the schools use a realistic maths curriculum based on the ideas of Freudenthal. Realism with regard to mathematical activity then consists in a view of constructive problem solving of an individual in the context of meaningful problems and with the help of self-invented, socially evaluated tools.

Despite the enormous innovation this view could produce in the content and activities of the mathematics classrooms, it entails a serious danger by focusing too exclusively on the real life quality of the contexts from which the mathematical thinking originates. It is inconceivable how the higher, abstract levels of mathematical thinking can be based on real life situations. How could a child ever discover that he or she is doing mathematics, let alone what mathematical argumentation, proof or systemacy implies, by just getting involved with (real life) problems? How should children ever select from their endless alternatives those actions that have mathematical relevance? Indeed, dialogues between pupils can have a selective function as to the utterances or actions that eventually may be selected as acceptable. But still, there is no basis for assuming that children in their dialogue should select *per se* the mathematically relevant propositions. Dialogues between actually present non-expert pupils lack the criteria to link their own actions to the meanings of the cultural (mathematical) practice. Such dialogues are important and necessary, but obviously not sufficient. By lack of a clear and consistent solution for this problem, teachers then tend to fall into other approaches to 'real maths' (structure-oriented or operation oriented). Of course, it is possible to stretch the meaning of the notion of 'reality-based' and let it cover every meaningful context (including personally meaningful abstract problems). Similarly, one may also accept the necessity of a teacher defining the domain of mathematics for the child and telling the child after its explorations what is mathematically acceptable or not, but this is clearly not 'realistic' in Freudenthal's sense of the word. The approach, however, does not give a clear conceptual answer to this question. Such an answer would lead us to an analysis of the problem of sense and meaning. It is unclear how these are integrated conceptually into the framework of Freudenthal's didactical phenomenology.

Broader and more liberal interpretations of Freudenthal's notion of realistic mathematics have been proposed by Gravemeijer (1994, 1997a). Individual inventions (like a method for solving multiplication problems, or geometrical problems) are seen as social products that may develop into still higher levels of abstraction and constantly feed back into the community and foster the development of the community as well. As such, the individual and the community co-develop (see for example Gravemeijer, 1997b). Gravemeijer's view justifiably draws attention to the reciprocal process of communication itself and to the ways of negotiating meanings and symbolic tools in a mathematics classroom.

3. THE DISCURSIVE APPROACH IN (MATHEMATICS) EDUCATION

In the wake of the Vygotskian storm drifting over the world today, the notion of discursivity nowadays has acquired a great deal of pertinence in discussions about education. As the classical (Platonic) model of education and teaching, based on obedience and power, gradually turned out to fail, the more the required results of our Western education called for insight, understanding and interest. The once strong conception of knowledge as objective units of thought that can be transported from one person to another, or from one situation to another, led people – on the one hand – to conceiving education as a literal *transmission* of pieces of knowledge and abilities from a teacher to pupils, and on the other hand, to believing that instructional success was best measured in terms of *transfer* (applying elements of thought in new situations). Especially in situations where asymmetry exists between two people as to their ability and expertise (like in education), it was generally seen as unavoidable that the more knowledgeable one hands over his or her knowledge and abilities to the other.

But in practice, the transmission models of teaching mathematics turned out to be disappointing. Due to the disappointing outcomes of both the transmission model of education, and the transfer model of learning, people began scrutinizing the assumptions behind these models (Lave and Wenger, 1991; Greeno, 1997). As a result many teachers and researchers have gradually become aware of the basically reciprocal, communicative nature of human education (Bruner, 1996; Wertsch, 1985; Wells, 1999). However, although the history of the construction of this idea of the social mind is long (see Valsiner and van der Veer, 2000), we have only recently begun to envisage its compelling implications.

One of the intriguing and far-reaching questions to be raised here concerns the view of the relationship between the participants in the discourse, especially with regards to their differences in expertise. With the refutation of the transmission model and its assumptions about objective meanings, the related communication model based on a sender-receiver idea was also heavily questioned. Hence, the old idea of one person being dependent on the information given by another could not be accepted anymore as a valid description of the relationship between a person and a more knowledgeable other in an educational setting. But how to handle the asymmetry between people with respect to their expertise, without falling back into a sender-receiver transmission model? Especially in mathematics education the differences in expertise and authority between teacher and pupil were traditionally felt as a legitimization for a transmission kind of education in which the teacher demonstrates the operations and the pupils spend all their efforts in mastering these operations by intensive practicing. Developments in the last 25 years with regard to mathematics education, however, reinforced the call for a more discursive approach, taking into account the pupils' own understandings of a mathematical problem (see Cobb et al., 1993; Forman, 1996; Gravemeijer, 1994), as well as doing justice to the fact that mathematics is a cultural activity that emerges out of sociocultural practices of a community (Bishop, 1988: Saxe, 1991). Hence the study of the interrelations between the role of the community and actual communication processes for establishing common mathematical solutions is one of the major items on the future agenda of investigators of mathematics education (see Bower, 2000).

In addressing this very same problem, we will have to deal with the question of how classroom communication is turned into a *mathematical* one. Obviously, the interlocutors in a mathematical discourse must share some values or meta-rules (Sfard, 2000) in order to be able to acknowledge utterances as mathematically relevant and to discuss them at all from the given perspective. A preliminary reflection on the notion of discourse and its prerequisites is now necessary.

4. From Vygotsky to Bakhtin

Since the early 20th century the work of Vygotsky has opened a window on human functioning and development that helped scholars of human development with reconceptualizing education as a process of co-reconstruction of meanings. Essentially, for Vygotsky, this process starts with the pupil's own actions and meanings. Therefore he writes in his 'Educational Psychology' (1926/1991, p. 82/1997: 48):

"The traditional European school system, which always reduced the process of education and instruction to a passive apprehension by the student of a teacher's lessons and outlines, was the ultimate of psychological nonsense. The educational process must be based on the student's individual activity, and the art of education should be nothing more than guiding and monitoring this activity."

Vygotsky emphasizes the importance of the student's own activity in the teaching-learning process, but he immediately hastens to add that this does not mean that the role of the teacher is minimized! The teacher should fulfill a guiding role by introducing students in significant sociocultural practices. Quite appropriately Davydov, in his introduction to a new edition of this work of Vygotsky, summarizes Vygotsky's position by saying that "the teacher may educate students in a deliberate fashion only by constantly collaborating with them, with their environment, with their desires and willingness to cooperate with the teacher" (Davydov, 1991, p. 9/cfr. 1997, p. xxiii). For Vygotsky, according to Davydov's summary, "mental functions are essentially seen as not rooted in the individual, but in the communication [obščenie] between individuals, in their relationships between each other and in their relationships with the objects created by people" (Davydov, 1991, p. 14–15/cfr. 1997, p. xxix).

Obviously, communication for Vygotsky is now more and more taken as referring to what it originally meant: sharing communalities and constructively dealing with the meanings people seem to have in common[1]. Communication is a collaborative endeavor on publicly pooled meanings.

Despite Vygotsky's undeniable merits in opening this window on human development, recent analyses of Vygotsky's ideas have also demonstrated their limitations. In his descriptions of the process of communication, Vygotsky's picture always turns out to be a neat and orderly process of meanings improving each other for the better. In-depth analyses of communication processes often demonstrate that the exchange and negotiation of meaning is a much more complicated process, pervaded by conflicts, misunderstandings, obscurities, and ambiguities. Hence, the French psychologist Clot states outspokenly about the theory of meaning that Vygotsky unfolds in his 'Thinking and Speech': "[It] is insufficiently related to the social process of intersignification that is taking place in discourses, or to the polyphony of sociodiscursive settings. Hence, it cannot improve the theory of psychological tools that remains basically a-social. The concept of 'genre' as proposed by Bakhtin, may be more helpful here as it is a tool for action that is inherently social" (Clot, 1999, p. 174).

This view on communication and its consequences for our understanding of human consciousness was deeply understood by Bakhtin (and his collaborators Voloshinov and Medvedev[2]). For Bakhtin – like Vygotsky – it was impossible to think of human consciousness as an isolated entity. Human consciousness is basically taken as a dialogical, meaning creating process and this creative activity can only emerge at the borderline of continuous interaction between individual consciousness and the outer social world, manifested in sign producing consciousnesses (see Morris, 1994, *Introduction*; Clark and Holquist, 1984). The individual and the social reflexively constitute each other in dialogue. The one can never exist without the other. This reflexive constitutive relationship is particularly manifest in human communication: every utterance is directed to an addressee, and actually anticipates the addressee's expected reactions. "Any utterance", writes Voloshinov/Bakhtin, "no matter how weighty and complete in and of itself is only a moment in the continuous process of verbal communication. But that continuous verbal communication is, in turn, itself a moment in the continuous, all-inclusive, generative process of a given collective" (Voloshinov, 1929 in Morris, 1994, p. 59). In this quotation, it is clear how Bakhtin and his group conceive of the multiple embeddedness of human 'individual' development: on the one hand human thinking is dependent on direct dialogues with social others; on the other hand this form of interacting itself is embedded in a broader cultural process of evolution of the communicating complex as a whole. What I call here 'the communicating complex' is for Bakhtin basically a historically organized institution of persons, or what he calls a "sign community" (Voloshinov, 1929 in Morris, 1994, p. 55), i.e. "[a] community which is the totality of users of the same set of signs for ideological communication". With regard to the production of signs he writes: "the forms of signs are conditioned above all by the social organization of the participants involved and also by the immediate conditions of their interaction" (Voloshinov, op.cit.). In a more modern language we would say that people's utterances in a communication process are not only regulated by the processes that occur in direct interaction, but also *by the historically developed style of communicating in that particular community of practice.* This is a very important insight of Bakhtin with regard to the question of how the individual and the social are related. Not only do communicating participants constitute each other by anticipation and mutual regulation, but their existence as a communicating unit is also deeply determined historically by others. Without this historical context this communication unit would not be possible, neither would participants be able to recognize that they have more in common (as communicators) than the incidental and ephemeral events of that actual situation. It is through this 'sign community' that people can recognize themselves as members unified in a same practice, as basically showing some shared identity and background. It is via this connection with the evolving history of a mathematical community that 'mathematics' as such can be re-invented at all.

Bakhtin applied his dialogical point of view mainly on general cultural practices like literary practices or general philosophy of the humanities. A valuable application of these ideas in the present time requires a specification of these ideas for particular areas of culture or communities of practice. In the present article I will take Bakhtin's thinking as a starting point for the further analysis of the relationship of individuals in a community of mathematical practice, especially in those cases where people have adopted an educational intention of initiating newcomers into this community of practice. Hence, I intend to focus here on mathematical education from a Bakhtinian/sociocultural point of view.

Many scholars who have been inspired by Bakhtin's work already took up the notion of speech genre as a way of analyzing the mathematical vernacular. It must be clear that for Bakhtin a genre is not just or not even primarily a thesaurus of technical terms or rules of behavior or discourse. The genre is primarily a social tool of a sign community for organizing a discourse in advance and often even unwittingly. It is a style of speaking embodied in a community's cultural inheritance, which is passed to members of that community in the same way as grammar is passed on. A genre is not so much a strict and fixed social norm, but it is a generic system of changing variants and possible utterances that fit into a community's practices; it is some kind of arena or forge where new variants of utterances are created and valued, that contribute to the essential polyphony and dissonances of meaning and discourse. Bakhtin writes:

> "Speech genres organize our speech in almost the same way as grammatical (syntactical) forms do. We learn to cast our speech in generic forms and, when hearing others' speech, we guess its genre from the very first words; we predict a certain length (that is, the approximate length of the speech whole) and a certain compositional structure; we anticipate the end; that is, from the very beginning we have a sense of the speech whole, which is only later differentiated in the course of the speech process. If speech genres did not exist and we had not mastered them, if we had to originate them during the speech process and construct each utterance at will for the very first time, speech communication would be almost impossible" (Bakhtin, 1986, pp. 271–272; see also Morris, 1994, p. 84).

Although the phenomenon of the speech genre still is not completely understood in linguistics and psychology, Bakhtin's general notion is now widely accepted as an explanation of the fact that people seem to understand each others' utterances from a wider context than is actually given in the discursive situation. According to Bakhtin, any participant always values the utterances of the discourse against a broader background of implicit, tacit, ideological knowledge. Moreover, any participant in a discourse actually expects the other participants to act in a certain way and to abide by some basic values. "Each speech genre in an area of speech communication", he writes, "has its own typical conception of the addressee, and this defines it as a genre" (Bakhtin, 1986; in Morris, 1994, p. 87). It is important to note here, that for Bakhtin the speech genre intrinsically links the interlocutors to each other, despite their possible differences in expertise (or their asymmetry in positions). The interlocutors can effectively communicate *because of* their basic alliance in the speech genre that they share. A similar position is taken by Rommetveit, when he writes: "The speaker monitors what he is saying in accordance with what he assumes to be the listener's outlook and *background information,* whereas the latter makes sense of what he is hearing by adopting what he believes to be the speakers perspective" (Rommetveit, 1985, p. 189–190, *italics added*). The speaker incorporates anticipated reactions and qualities of the listener and vice versa. Hence speaker and listener share a common background that enables them to value and interpret each other's utterances.

Thus, basically, the Bakhtinian approach to discourse focuses on the communalities of participants and on how they collaboratively fashion the heterogeneity of meanings. The asymmetry that was so evident in the sender-receiver model of communication is now made into a core element of the discursive process: heterogeneity is fundamental to the discursive process and the best result can be a consensus about the meanings that the participants are willing to take as shared. Authority, moreover, is an indispensable position in an activity for linking the actual to the historical.

In order to really value Bakhtin's contribution to the deeper understanding of mathematical processes in the classroom, a further exploration is needed that tries to apply some of the elements of Bakhtin's thinking. Bakhtin's notion of speech genre implies that utterances of the interlocutors in the discourse are not just assessed in terms of their literal meaning, but also valued from a generic background that provides meta-rules and norms which help in defining the utterances involved as mathematical or not. "No utterance can be put together without value judgment. Every utterance is above all an evaluative orientation. Therefore, each element in a living utterance not only has a meaning but also has a value" (Voloshinov, 1929, in Morris, 1994, p. 37).

5. Mathematics Education as Improvement of Participation in a Mathematical Community of Practice

When using this perspective for the analysis or description of actual mathematical practices in classrooms, it is important to first clarify the notions of activity, practice, and discourse in their mutual relationships. 'Activity' is taken here as a concept referring to any motivated and object-oriented human enterprise, having its roots in cultural history, and depending for its actual occurrence on specific goal-oriented actions. Any activity can be accomplished in a variety of ways, and it depends on the community in which the activity is carried out how much variety (or which variant) is accepted as valid. In this I follow Leont'ev's activity theory (Leont'ev, 1975; van Oers, 1987). *Mathematical activity* can then be seen as an abstract way of referring to those ways of acting that human beings have developed for dealing with the quantitative and spatial relationships of their cultural and physical environment. When we specify the activity with the values, rules and tools adopted in a specific cultural community we tend to speak of a '*mathematical practice*'. Any practice contains *performative actions* and operations that just carry out certain tasks which have mathematical meaning within that community (like performing long division). On the other hand, practices also comprise *conversational actions* that intend to communicate about the mathematical operations or even about the mathematical utterances themselves. Cobb et al. (1993) made a similar distinction between 'talk about mathematics' and 'talk about talk about mathematics'. A community committed to a particular style of accomplishing conversational actions with regard to a special category of objects can be named a community of discourse. Hence, in my view a community of practice and a community of discourse refer to slightly different concepts. A community of mathematical practice also includes people making calculations (in their own idiosyncratic ways), e.g. in the super market (see Lave, 1988), while a community of mathematical discourse mainly includes persons interested in reflectively understanding mathematical actions. This is consistent with the more general formulation of a discursive practice as "the repeated and orderly use of some sign system, where uses are intentional, that is, directed to something" (Harré and Gillett, 1994, p. 28).

'Real mathematical activity' can now be defined as the activity that is accomplished when one legitimately participates in a mathematical practice, either by acting mathematically in an acceptable way, or by discussing mathematical or discursive mathematical actions. Hence, it is not the link with meaningful problem situations as such that defines the nature of 'real' mathematics, but the observance of particular rules, the use of particular concepts and tools, the engagement with certain values that define whether one is doing mathematics or not. So the basis of the realism is the participation in mathematical *activity.* Like in Freudenthal's definition the focus is here on problem solving, tool use, and contextuality, but their relevance is rooted in the commitments to a certain type of historically rooted activity. The context of human (mathematical) action, then, is not the meaningful situation but the culturally developed activity itself (cf. van Oers, 1998). In the case of mathematical activity, certain ways of doing and talking have developed during cultural history. Real mathematics in the classroom is actually participating in this mathematical practice.

It is the function of education to initiate children in this practice, and get them involved in the mathematical speech genre. This should give them a sense of what 'real mathematics' is like. Participation in mathematical practices (like in the case of the Brazilian street vendors, see Saxe, 1991; Nunes, et al., 1993) does not automatically lead to abilities of participating in mathematical discourses. In most cases it takes (formal or informal) education to develop these discursive abilities. In the school context, doing and learning mathematics means improving one's abilities to participate in mathematical practice, both the operational part (the symbolic technology of mathematics) and the discursive part.

In the following sections of this article I shall elaborate this latter view a bit more, by focusing especially on mathematical discourse, in order to clarify how this speech genre is passed on to new generations, how pupils may get 'infected' by this view on 'real mathematics' and what is needed to strengthen their participantship in this mathematical practice.

6. THE POLYLOGICAL CHARACTER OF A COMMUNITY OF MATHEMATICAL DISCOURSE

Having explained mathematics as a historically developed practice, dealing with certain types of objects, tools and rules, it is a logical next step to reflect a bit longer on the nature of this practice and how children are enticed to become autonomous and reflective participants in this practice.

The interpretation of cultural practices in terms of activity theory raises the question of how the dynamical elements of this activity (object, motive, actions, tools) can be defined. Mathematical practice as it has been invented and developed in our culture implies an activity that is based on the construction of mental *objects* that model the numerical and spatial aspects of physical and cultural reality. As Bishop (1988) has argued, the symbolic technology (*tools*) that resulted from these constructions during cultural history has been invented and elaborated in the context of general cultural activities that had to do with cultural *key-activities* like counting, locating, measuring, designing, playing, and explaining. In the context of such activities people encountered several problems that they tried to solve (*goals*) in many different ways, but in any way it is almost certain that some kind of symbolic representation (mostly with the help of language and drawing) was invented. While struggling with these problems, people also gradually discovered the relevance of certain values to be observed. Bishop (1988) discusses several values that have played a role in the development of mathematics as a cultural practice. Those values are intrinsic to several everyday practices and as such they offer guidelines that participants of the practice are particularly supposed to obey. According to Bishop, these values are not fixed in the history of mathematics. They have changed during history and are often sensitive to circumstantial, personal, and temporal influences. In many cultural periods these values can be found in twins that have a contrary relationship (Bishop, p. 60–83): objectism vs rationalism (as the twin ideologies of mathematics), control vs progress (as the attitudinal values of mathematics), and openness vs mystery (as values that define potential ownership of mathematics). Mathematical activity, according to Bishop (1988, p. 95), accomplishes the association of a particular symbolic technology developed by the key-activities, with the values that are articulated in a certain historical period. Both the development of the technology and the reflection on the values involved is part of the responsibilities of the participants in the mathematical practice. Real mathematical activities imply both elements.

Introducing children into the culture of a mathematical practice is basically a social process, that can be described in terms of apprenticeship learning (Rogoff, 1990), or gradual progress from a legitimate peripheral participant in that practice towards a more and more extended form of participation (see Lave and Wenger, 1991, for a general description of this model of initiation in cultural practices). In the context of the present article it is important to explain how communication takes place in such a community of practice, particularly when communication aims at improvement of the participatory abilities and qualities of the participants, both with regard to the technology, and with regard to intrinsic values and norms. I will come back to that question in the next section. First it is important to clarify who should be accepted as legitimate participants in this process. In my commentary on the Freudenthal definition of realism, I already pointed out that direct dialogues between actually present pupils might not be sufficient as an explanation for the *mathematical* content. As mathematics is a historical practice, representatives of the history of mathematics always take a part in the communication within that practice. Most of the time the teacher may be considered as a representative of the cultural history of mathematics and in that quality the teacher should take part in the discourse in the classroom: not just as a guide when the process goes astray, but also as a real participant, suggesting possible solutions, strategies, concepts etc. To use a Bakhtinian terminology, one could say that the teacher represents all absent and historical voices that essentially have a say of what should be taken as 'mathematical'. Thus, instead of a dialogue among pupils, the discourse in a mathematical community is essentially a *polylogue,* a polyphonic discourse among all possible voices that helped to create the history of that community of practice (see Davydov, 1983). The implications of this point of view might look overwhelming at first for regular school practice. They probably are, but one of the first and realizable consequences is that the teacher takes a substantial (and not just a distanced guiding) role in the classroom discourse: the teacher is a serious partner in the classroom activity and discourse, suggesting serious solutions, possibilities, questions, objections. It is exactly the teacher in this role who should introduce a cultural-historical voice in the classroom discourse, a voice that can help pupils in defining 'the mathematical' in accordance with the cultural history of that practice. A similar and even more detailed analysis of this very same viewpoint is given by Sfard (2000). She convincingly argues for the notion of meta-discursive rules that regulate participation in a practice or discourse. According to Sfard, reform of mathematics education should take the appropriation of these rules more seriously in order to help children getting access to mathematical practices.

Sure enough, this requires a radical innovation in many school practices, not only in those which still practice a transmission style of frontal teaching, but also in those who have introduced forms of cooperative learning in which the core of the activity is trusted to the pupils in dialogue. Fortunately, there are already a number of experimental classrooms that have demonstrated that teachers can indeed realize parts of this ideal. The work of Cobb and his colleagues is a good illustration of how in a mathematical classroom both the technical-conceptual development and the sociomathematical norms can be put on the agenda. This is a very important starting point for getting pupils involved in the definition of their mathematical practice, taking account of the general cultural meaning of mathematical practices.

The 'Dialogue of cultures'-schools in Russia are another example that demonstrate that a discourse of 'everybody with everybody' can be practiced in an elementary school practice. The idea of the 'Dialogue of cultures' as developed by Bibler – in line with Vygotsky and Bakhtin – is that every pupil represents a multiplicity of voices, hence is a microculture in itself. The learning processes in school, according to Bibler, should be focused on developing the pupils' own cultures in dialogue with all the other cultures available (including the teacher's). Therefore, this dialogue of cultures is basically what we called previously a polylogue (Bibler, 1992). In a report on the experimental implementation of the 'Dialogue of Cultures' Berljand and Kurganov also emphasize the importance of the participation of the teacher's culture in the mathematics classroom discourse. They write about the role of the teacher in the following:

> "On the one hand, the teacher acts as one of the participants in his own right, proposing his own hypotheses and assumptions. On the other hand, the teacher directs the process in a general but very cautious way, permitting sometimes far going digressions from the original plans and intentions. (...). *Another important function for the teacher is to canalize the discussion when something new or unexpected comes up, which might not be recognized by the pupils as significant.* Sometimes a thought is unclear for a pupil or he cannot formulate it in a way that is comprehensible for the other pupils. In those cases the teacher also helps the pupils in formulating the idea" (Berljand and Kurganov, 1993, p. 37, italics added).

From a historically advanced point of view, the teacher's responsibility, according to Berljand and Kurganov, is one of introducing new cultural elements in the discourse that could never be put forward by the pupils themselves. By so doing, the teacher not only provides new unexpected information, but also demonstrates a strategy of critically and systematically evaluating and elaborating a received result with the help of new points of view. This strategy of always asking new questions, critically looking at your results from another perspective is a strategic element of a mathematical rationality that is developed through the mathematical discourse with the teacher.

What Berljand and Kurganov were describing with respect to the teacher's activity is similar to what O'Connor and Michaels (1996) called revoicing. In the act of revoicing the teacher uses his or her own background knowledge of mathematics and the values involved. The teacher's selection of concepts, and style of phrasing is colored by his or her historical knowledge. This is one legitimate way of introducing cultural history in the process. Of course this revoicing should not impose definite knowledge onto pupils. The revoiced proposition is not *a priori* better or worse than any other input in the discourse and is, consequently, open for discussion and evaluation. Revoicing, thus, is one technique for putting cultural history at work in the classroom discourse, creating a public value position from which the pupils can learn what is counted as mathematical in this community's speech genre.

7. IMPROVING PARTICIPATION IN MATHEMATICAL PRACTICES

The polylogic character of the classroom discourse articulates the heterogeneous nature of this communicative activity. It should be clear that this couldn't easily be dealt with by a sender-receiver model of communication. In this alternative Bakhtinian communication model all participants are constructors of meaning, sharing a topic that they elaborate by adding new information ('predicates' in the sense of Vygotsky, 1987, ch.7; see also Van Oers, 2000a). These predicates in fact reveal something new about the topic at hand and distinguish that topic from other topics. An example of this process can be found in the following situation:

Two 6/7 year old girls have been building a farm with blocks, and they have been playing with it for a while. The teacher starts a conversation with these girls asking about the number of blocks that the girls used for their farm. The teacher shows interest in that aspect of their work and she (implicitly) introduces a mathematical point of view by asking '*can you count them for me?*' The teacher explains that she wants to know how many blocks are needed for making such a beautiful farm in case other children at a later moment might be willing to construct something like that. She then also invites the children to fill out a graph for her so that she can immediately see how many blocks are used in this farm (she provides a big sheet of paper with a number of columns with drawings of different types of blocks at the bottom – see example in Figure 1).

Two observations are relevant here: the teacher introduces a mathematical point of view by her questions, and kind of 'defines' the situation as a counting situation. This is a first predicate that characterizes the situation in this specific way and distinguishes it from other possible perspectives on the situation (esthetic: 'how beautiful'; physical: 'how did you do it?' etc). Moreover, by providing this tool for recording their counts, the teacher structures the children's actions in a histogram-like form. This introduces a tactical element in the children's activity if not with regard to the appropriation of histograms, then possibly in a more general way regarding the fact that counts can be recorded in a structural form. So it is not purely numbers that the teacher introduces, but also more general ways of doing, either by providing specific predicates, or by providing tools (that often lead to specific predicates). Her style of acting in this case demonstrates *ritualistic* elements from the *genre* of mathematical activity.

In their activity of counting, the children encountered different practical problems (e.g. walls tumbling down), which cause them to restart their counting several times. So after repeatedly counting the blocks of the farms the counting girl suddenly shouts: "This is the table of three!" (referring to the wall with piles of three blocks). Now in fact she predicates the situation herself in a new way and makes it different from all other situations or interpretations. Her partner knows what she is talking about and starts

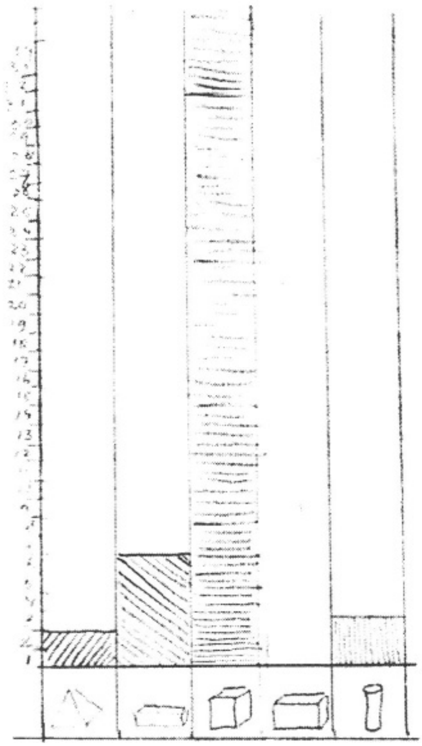

Figure 1. Histogram of 6–7 year old girls.

checking if she was right (checking systematically and answering the 'Are-you-sure?'-question is another typical element of the mathematical genre, cfr. van Oers, 1996). Actually, the other child starts evaluating this particular predicate and continues this line of reasoning by adding still other predicates (for example transforming the counting result to a *score* on the paper). Basically, this is a collaborative construction of a (mathematical) text, that is the beginning of all discursive (mathematical) thinking, and that opens the possibility of intertextual confrontation with other (historical) texts (Bibler, 1989; see also Carpay and van Oers, 1999). Finally they end up with the diagram above of the situation, which is of course a product both of the children's actions and the cultural tool provided by the teacher. It is essentially a product of a polylogic process.

Constructing meaning and negotiating meaning by constructing and evaluating new predicates is a way of talking about the processes that take place in a mathematical discourse. The diagram is one possible tool of structuring the discourse, and integrating the different (real or virtual) voices that take part in the discourse. It is clear that a multiplicity of propositions is possible all the time. The selection of propositions/predicates is a task of the community in discourse. There is, however, no universal selection principle that helps participants to decide unequivocally in advance which mathematical propositions should be used in a given situation. Within the practice, it is possible that sub-communities arise on the basis of intentional communalities among groups of participants in the practice. In general the mathematical practice comprises different groups of legitimate participants who are willing to deal with number, number relations, and spatial relations according to accepted values in the community, and above all who are willing to pursue the quest for certainty, to apply the norms of non-contradiction, systemacity, generalization, modeling etc., in short: who demonstrate *the mathematical attitude.* Hence both lay persons in the supermarket and highbrow mathematicians are to be accepted as legitimate participants in the community of mathematical practice. There is a well-known tendency to monopolize the participation in mathematical practices for the group of professional mathematicians. This is primarily an ideological struggle within our culture (and perhaps even within the community linked to mathematical practices), but the Bakhtinian theory of communication doesn't provide any principled reason why practices should be monopolized by specialist groups (experts).

What is more interesting here is the question of how participants of a mathematical practice can assist each other in order to improve their abilities for participation. There is no room here to summarize extensively the growing amount of literature that is consistent with the approach outlined here. Cobb and his colleagues have demonstrated possible ways of how pupils' mathematical understanding can be promoted through a classroom discourse. On the basis of their classroom discourse data they argue that an individual pupil's development and the development of the classroom community's understanding are reflexively related, co-existent processes. In detailed analyses they demonstrated how the development of pupils' understanding might be conceived of as a construction of a chain of signification (Cobb et al., 1997). These data provide an empirical basis for the assumption that the individual and the community are reflexively related in their discourse-based development and demonstrate what kind of processes partly constitute this development. Studies of Forman and her colleagues contributed to a further understanding of the processes of individual development in a community by analyzing the process of argumentation among participants in the discourse. It is clear from these studies that any argument always is based on common resources in the community to make up a collective argument (Forman, Larreamendy-Joerns, Stein and Brown, 1998). It is also interesting for the present argument that these authors could demonstrate the important role of the teacher in making explicit the implicit background knowledge (Forman and Larreamendy-Joerns, 1998). This probably also contributes to the emergence of the 'real' mathematical speech genre in the classroom. Both Cobb's and Forman's findings demonstrate parts of the dynamics of the development of mathematical thinking with regard to meaning development. But it is equally important to invest in building a mathematical sense in pupils. On the basis of Leont'ev's activity theory we must assume, however, that any activity always also depends on the dimension of sense, i.e. the motive-related valuation of actions and utterances. It is important to know how a person creates a chain of signification, how he or she builds arguments, but it is equally important to know *why* constructing new topic-predicate relations, chains of signification, or arguments do indeed make sense to that individual, *why* he wants to be engaged in these kind of enterprises. Basically, according to Leon'tev (1975), the development of an activity always depends on a dialectic between meaning and sense (between the 'what/how?' and the 'why?'). As 'sense' is always intrinsically related to a person's motives for acting, there is a close relationship as well with the goals that person wants to pursue. Saxe's interesting studies (see for example Saxe and Guberman, 1998) also demonstrate that the emergence of new goals (and, thus, new sources for giving sense to future actions) is dependent on collective processes. New goals emerge in a collective activity and obviously are not 'private'. The public status of newly emerging goals constitutes one of the essential elements of a shared background for communication: though individually appropriated, they provide the points for joint attention that defines part of the speech genre that may be going to be recognized by the participants in the discourse. But indeed, the mystery remains how pupils come to select the mathematically relevant goals and actions in the middle of the many possible alternatives?

All these studies, however, focus mainly on the public, goal-directed processes and qualities in a community for the development of a successful mathematical understanding in the participants in the discourse. It is becoming more and more clear that participation in a mathematical discourse presupposes the observance of a set of meta-rules (see also Sfard, 2000; Bishop, 1988) that regulate the discourse and the practice in general. These rules are culture-bound, intersubjective entities that continue to exist in the individual members of the community, that are passed on from one generation to the other, but at the same time these rules are not an authentic product of any one of them. The participation *per se* in mathematical activities with others (more mathematically advanced) covertly contributes to the development of a mathematical sense as well. It contributes to the gradual appropriation of this tacit normative background (with its included norms and meta-rules) from which students in due time start to make personal decisions about the kind of actions and goals that are assumed to be relevant in a mathematical practice. This sense cannot be instructed in a direct way. ‘Sense’ is formed by educative interaction (Leont’ev, 1975, p. 286).

This sense creates the personal stance that manifests itself as an attitude in a discourse. For a mathematical speech genre to arise it must be assumed now – at least theoretically – that mere mastery of mathematical meanings (knowledge and skills) is not enough. For participating autonomously in a community of mathematical discourse some conditions must be fulfilled at the personal level as well, in order to be able to value the real mathematical in the discourse. At least one of the persons involved must have the *attitude* of acting according to the meta-rules, of operating systematically, critically, non-contradictorily, and of looking for proofs and for forms of symbolization. In fact, this mathematical attitude is the *interiorized tendency of the meta personal dynamics of the mathematical speech genre* that has developed in the history of a particular community. As Billig (1986) already has extensively argued, attitudes represent *positions* taken in matters of controversy, having their roots in discursive processes. In the discourse the historical tendencies of mathematics (to be systematic, non-contradictory, to construct symbolic technology etc.) are introduced – either implicitly or explicitly – by those participants who have interiorized these historical tendencies as personal stances in matters of discourse regarding spatial and quantitative problems. As I argued elsewhere, for instance, the characteristic feature of ‘abstractness’, which is generally seen as a hallmark of mathematical thinking, can be interpreted as a habit of progressively focusing on imbedded relationships and assuming increasingly specific points of view (see van Oers, 2001). As I demonstrated in this latter argument ‘abstraction’ is also a product of discourse, intrinsically related to assuming points of view that have been shown relevant during cultural history. It is the teacher’s task to help children in appropriating this habit and at the same time help them in appropriating an attitude that is generally seen as essential for mathematical thinking. There is no other way of understanding how this view on ‘what mathematics really should be’ finds its way into an actual discourse than by assuming that at least one of the participants convincingly demonstrates this *mathematical attitude* and regulates the discourse accordingly. The mathematical attitude is the essential link between the mathematical community’s history and the development of understandings at the personal level that will be acknowledged as ‘really mathematical’.

From a genetical point of view, I hypothesise that the emergence of this mathematical attitude starts out from the teacher's demonstrations of a specific type of behavior and, consequently, from her/his mathematics-related *expectations* about the pupils' activity. It seems plausible that these expectations and the pupils' ways of digesting these in actions, play a significant role in the development of mathematical sense and attitude. Further study of this theoretical hypothesis should be given top-priority on the researchers' agenda in the near future.

8. Conclusion

For the educational agenda we may conclude now that the further improvement of mathematics education requires that pupils be enticed by the teacher to take part in a mathematical practice and especially in mathematical discourse within that practice. More attention therefore should be given to the development of the mathematical *genre* (rather than just to the register clarifying the concepts, rules, tools and operations). In the interaction with the teacher, pupils can get access to the specific mathematical genre (including the meta-discursive rules, Sfard, 2000). As a result of this discourse children may interiorize the rules according to which those discourses are supposed to be regulated. This is how a mathematical sense emerges in shared practice and how a mathematical attitude can be appropriated from this. Such attitude is necessary for becoming an autonomous, critical and authentic participant in mathematical practice.

Needless to say, the provision of mathematical tools and rules is in itself not enough for developing full participation in a mathematical discourse. The tool does indeed structure the participants' actions according to implicit mathematical rules, but these rules can only be fully mastered when the participants' attention is drawn explicitly to them. So at best the tool is a starting point for discourse, and again it is the teacher who should create conditions for focusing on the hidden rules and assumptions in the tools. Recent research has provided interesting evidence in favor of such critical discourses that create the necessity for co-construction of new personalized versions of the provided tool in the pupils' community (see for example Cobb, 1999). In our own research (see van Dijk et al., 1999, 2000) we could provide evidence that the co-constructive creation of mathematical models leads to different ways of problem solving in students (as compared to transmission-based teaching). More importantly, students from a co-constructive classroom, where more exploratory and problem solving discourse took place, performed better on tasks that were relatively new for them, than students who just got ready-made models and who were involved in discourses that were primarily focused on correct application of the provided models. Hence, discursive forms of initiation lead to better performances of students on a variety of complex mathematical tasks. If these students indeed also acquired the new personal quality that we referred to as 'mathematical attitude', this is something to be investigated in the future, but the start is already there, providing a mathematical culture in the classroom with opportunities of model-based structuring, invention of symbolic tools, and creating the right atmosphere for experiencing the expectations of the mathematically more advanced partners.

A fundamental requirement for achieving this attitudinal outcome is the innovation of the teacher-pupil relationships into a form of long-lasting collaborative inquiry of mathematical actions, in the context of a shared discursive activity, in which the teacher fulfills the role of a historical resource for the pupils. It is in these conditions that they are likely to experience the historically founded, mathematics based expectations that give them a window on what it means to act mathematically.

NOTES

1. It is interesting to note that Davydov also used the word 'obščenie' that has a similar etymological root as the latin word 'communicatio' (communication), referring to what is 'common'. The translation of this word as 'intercourse' (see Davydov, 1997, p. xxix; compare my translation of this quote above) is not wrong, but hides this important connotation.
2. For more information about the somewhat enigmatic relationship between Bakhtin, Voloshinov and Medvedev see Clark and Holquist (1984). For reasons of simplicity, in my descriptions of the approach I shall take Bakhtin as the main spokesperson, even when I will also quote from sources that are officially attributed to Voloshinov or Medvedev.

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