Classical no-cloning theorem

Daffertshofer, A.; Plastino, A.R.; Plastino, A.

published in
Physical Review Letters
2002

DOI (link to publisher)
10.1103/PhysRevLett.88.210601

document version
Publisher's PDF, also known as Version of record

Classical No-Cloning Theorem

A. Daffertshofer*
Faculty of Human Movement Sciences, Vrije Universiteit, van der Boechorststraat 9, 1081 BT, Amsterdam, The Netherlands

A. R. Plastino†
Faculty of Astronomy and Geophysics, National University La Plata, C.C. 727, (1900) La Plata, Argentina‡

A. Plastino§
Department of Physics, National University La Plata, C.C. 727, (1900) La Plata, Argentina‡
(Received 17 December 2001; published 9 May 2002)

A classical version of the no-cloning theorem is discussed. We show that an arbitrary probability distribution associated with a (source) system cannot be copied onto another (target) system while leaving the original distribution of the source system unperturbed. For classical dynamical systems such a perfect cloning process is not permitted by the Liouvillian (ensemble) evolution associated with the joint probability distribution of the composite source-target-copying machine system.

DOI: 10.1103/PhysRevLett.88.210601 PACS numbers: 05.20.–y, 03.67.–a, 89.70.+c

The physics of information has been investigated intensively in recent years [1–12]. Much of the current interest in the field is due to novel and counterintuitive ways of processing and transmitting information that are allowed by the laws of quantum mechanics [5–7]. Concepts from quantum information theory have been shown to provide new insights into profound topics such as the connection between quantum mechanics and the second law of thermodynamics [10]. An interesting recent development is the identification of a classical analog of one of the key ingredients underlying these information-related processes: entanglement [11]. Quantum entanglement, however, is not the only aspect of quantum physics being relevant for the processing of information that has been shown to admit of a classical counterpart. For instance, in [13,14] one can find that non-Boolean logics can arise within classical physics.

A hallmark feature of quantum information is that it cannot be cloned: an unknown quantum state of a given (source) system cannot be perfectly duplicated while leaving the state of the source system unperturbed [15,16]. In fact, an ideal copying process would read

$$|\Sigma\rangle \otimes |\phi\rangle_s \otimes |0\rangle_t \rightarrow |\Sigma_{|\phi\rangle}\rangle \otimes |\phi\rangle_s \otimes |\phi\rangle_t \,, \qquad (1)$$

where the first ket denotes the state of the copying machine, the second one corresponds to the unknown quantum state to be copied (source), and the third describes the system to which the unknown state of the source shall be copied (target). According to the no-cloning theorem no unitary (quantum mechanical) transformation exists that can perform the process (1) for arbitrary source states $|\phi\rangle_s$ [17,18]. The enormous impact of this theorem is reflected by several studies that focused on different aspects of the nonclonability of quantum information [19–23]. For example, the no-cloning theorem yields a new formulation of the quantum uncertainty principle that applies to individual systems [19]. Furthermore, universal cloning machines have been proven to exist, which can produce approximate copies of an unknown qubit with a fidelity that does not depend on the input qubit [20,21]. In analogy with the case of quantum entanglement, in this Letter we show that the quantum no-cloning theorem possesses a classical counterpart. Below we prove that universal perfect classical cloning machines violate the Liouville dynamics governing the evolution of statistical ensembles. This kind of copying process is in conflict with the conservation of the Kullback-Leibler information distance [24,25] and with the linearity of the Liouville dynamics.

Consider a general classical deterministic dynamical system, whose evolution is governed by the equations of motion

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}), \qquad \text{with } \boldsymbol{x}, \boldsymbol{v} \in R^N, \qquad (2)$$

where $\boldsymbol{x}$ denotes a point in the concomitant N-dimensional phase space [26]. A statistical ensemble of systems evolving according to (2) can be depicted by its probability distribution $\mathcal{P}(\boldsymbol{x}, t)$. Then, the well-known Liouville equation,

$$\frac{\partial}{\partial t} \mathcal{P} + \nabla \cdot (\boldsymbol{v}\mathcal{P}) = 0\,, \qquad (3)$$

describes the dynamics of this distribution [27]. For this type of evolution the Kullback-Leibler information measure [24,25],

$$K(\mathcal{P}_1, \mathcal{P}_2) = \int d\boldsymbol{x}\, \mathcal{P}_1(\boldsymbol{x}) \ln \frac{\mathcal{P}_1(\boldsymbol{x})}{\mathcal{P}_2(\boldsymbol{x})}\,, \qquad (4)$$

provides a convenient way to measure the distance between two distinct probability distributions $\mathcal{P}_1$ and $\mathcal{P}_2$ because, remarkably, it is invariant under dynamical changes prescribed by the Liouville equation (3). Indeed, after combining (4) and (3) one directly finds by integration by parts

$$\begin{aligned}\frac{dK}{dt} &= \int d\boldsymbol{x}\left[\left\{1+\ln\frac{\mathcal{P}_1}{\mathcal{P}_2}\right\}\frac{\partial \mathcal{P}_1}{\partial t}-\frac{\mathcal{P}_1}{\mathcal{P}_2}\frac{\partial \mathcal{P}_2}{\partial t}\right] \overset{(3)}{=} -\int d\boldsymbol{x}\left[\left\{1+\ln\frac{\mathcal{P}_1}{\mathcal{P}_2}\right\}\nabla\cdot(\boldsymbol{v}\mathcal{P}_1)-\frac{\mathcal{P}_1}{\mathcal{P}_2}\nabla\cdot(\boldsymbol{v}\mathcal{P}_2)\right] \\ &= \int d\boldsymbol{x}\left[\left(\nabla\ln\frac{\mathcal{P}_1}{\mathcal{P}_2}\right)\mathcal{P}_1-\left(\nabla\frac{\mathcal{P}_1}{\mathcal{P}_2}\right)\mathcal{P}_2\right]\cdot\boldsymbol{v}-\overline{\boldsymbol{v}}\mathcal{P}_1\ln\frac{\mathcal{P}_1}{\mathcal{P}_2}\Bigg| = 0,\end{aligned} \tag{5}$$

when we assume that eventual boundary terms vanish; $\overline{\boldsymbol{v}}$ denotes the sum of the components of $\boldsymbol{v}$.

To explicitly define a copying process we consider a composite system constituted by three distinct classical subsystems: the copying machine (m), the source (s), and the target (t). The corresponding phase space is equal to the Cartesian product of the spaces associated with each subsystem. Thus, coordinates $\boldsymbol{x}$ describing the entire composite system can be cast in the form

$$\boldsymbol{x} = (\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(s)}, \boldsymbol{x}^{(t)}), \tag{6}$$

where $\boldsymbol{x}^{(m)}$, $\boldsymbol{x}^{(s)}$, and $\boldsymbol{x}^{(t)}$ denote the phase space coordinates describing the state of the copying machine, the source system, and the target system, respectively. Accordingly, the volume element $d\boldsymbol{x}$ factorizes like $d\boldsymbol{x} = d\boldsymbol{x}^{(m)}d\boldsymbol{x}^{(s)}d\boldsymbol{x}^{(t)}$. For the sake of simplicity we further assume that the joint probability distribution of the initial state of the composite system can be factorized by means of

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{P}^{(m)}_{\text{start}}(\boldsymbol{x}^{(m)})\mathcal{P}^{(s)}(\boldsymbol{x}^{(s)})\mathcal{P}^{(t)}_{\text{blank}}(\boldsymbol{x}^{(t)}). \tag{7}$$

The initial distributions $\mathcal{P}^{(m)}_{\text{start}}(\boldsymbol{x}^{(m)})$ and $\mathcal{P}^{(t)}_{\text{blank}}(\boldsymbol{x}^{(t)})$ represent the starting state of the copying machine and the blank state of the target system, respectively—both are assumed to be always identical. Conversely, the initial distribution of the source system $\mathcal{P}^{(s)}(\boldsymbol{x}^{(s)})$ can be an *arbitrary distribution* that we want to copy onto the target system.

The Kullback-Leibler distance between two different states of the copying process associated with two distinct initial states $\mathcal{P}^{(s)}_{1,2}(\boldsymbol{x}^{(s)})$ of the source system (that is, two different initial states of the source that we want to clone) can be obtained by inserting (7) into (4):

$$\begin{aligned}K(\mathcal{P}_1,\mathcal{P}_2) &= \int d\boldsymbol{x}\,\mathcal{P}_1(\boldsymbol{x})\ln[\mathcal{P}_1(\boldsymbol{x})/\mathcal{P}_2(\boldsymbol{x})] = \int d\boldsymbol{x}\,\mathcal{P}_1(\boldsymbol{x})\ln[\mathcal{P}^{(s)}_1(\boldsymbol{x}^{(s)})/\mathcal{P}^{(s)}_2(\boldsymbol{x}^{(s)})] \\ &= \int d\boldsymbol{x}^{(s)}\,\mathcal{P}^{(s)}_1\ln[\mathcal{P}^{(s)}_1/\mathcal{P}^{(s)}_2] = K(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2).\end{aligned} \tag{8}$$

That is, the information distance between two different initial states is exclusively given by the distance between the two source systems.

The joint probability distribution describing the final state is denoted as

$$\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q}(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(s)}, \boldsymbol{x}^{(t)}), \tag{9}$$

so that the aforementioned conservation of the Kullback-Leibler distance formally reads

$$K(\mathcal{P}_1,\mathcal{P}_2) \overset{!}{=} K(\mathcal{Q}_1,\mathcal{Q}_2). \tag{10}$$

To show its violation under classical cloning we consider two different assumptions with respect to the final states or, more precisely, to the degree of statistical independence of the final (sub-)systems.

First, let the final state be entirely factorizable in terms of

$$\mathcal{Q}(\boldsymbol{x}) = \mathcal{Q}^{(m)}(\boldsymbol{x}^{(m)})\mathcal{P}^{(s)}(\boldsymbol{x}^{(s)})\mathcal{P}^{(s)}(\boldsymbol{x}^{(t)}). \tag{11}$$

Then, in analogy with (8), we find for the final states

$$\begin{aligned}K(\mathcal{Q}_1,\mathcal{Q}_2) &= \sum_{j\in\{m,s,t\}} K(\mathcal{Q}^{(j)}_1,\mathcal{Q}^{(j)}_2) \\ &= K(\mathcal{Q}^{(m)}_1,\mathcal{Q}^{(m)}_2) + 2K(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2).\end{aligned} \tag{12}$$

Inserting (12) and (8) into (10) leads to

$$K(\mathcal{P}^{(s)}_1,\mathcal{P}^{(s)}_2) \overset{!}{=} -K(\mathcal{Q}^{(m)}_1,\mathcal{Q}^{(m)}_2), \tag{13}$$

which clearly constitutes a contradiction because $K(\cdot,\cdot)$ is positive definite unless the two probability distributions are identical for the two initial states and, thus, the source systems must be identical. Hence, classical cloning disagrees with (10).

Second, in the more general situation we do not assume that final states can be factorized and focus our attention on marginal probabilities (of the final state). The source is being copied whenever the two marginal distributions describing source and target systems are equal to the original distribution associated with the source system; that is,

$$\begin{aligned}\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(t)}\,\mathcal{Q}(\boldsymbol{x}) &= \mathcal{P}^{(s)}(\boldsymbol{x}^{(s)}), \\ \int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(s)}\,\mathcal{Q}(\boldsymbol{x}) &= \mathcal{P}^{(s)}(\boldsymbol{x}^{(t)}).\end{aligned} \tag{14}$$

These forms are presumably the least restrictive constraints for a copying process. By recourse to the well-known inequality,

$$\int dy\,q_1(y)\ln\frac{q_1(y)}{q_2(y)} \geq \left[\int dy\,q_1(y)\right]\ln\frac{\int dy\,q_1(y)}{\int dy\,q_2(y)}, \tag{15}$$

which is verified by any pair of non-negative functions $q_1(y)$ and $q_2(y)$ [24,25], we can establish a lower bound for the Kullback-Leibler distance between the final distributions both fulfilling (14):

$$K(\mathcal{Q}_1,\mathcal{Q}_2) = \int d\boldsymbol{x}\,\mathcal{Q}_1(\boldsymbol{x})\ln\frac{\mathcal{Q}_1(\boldsymbol{x})}{\mathcal{Q}_2(\boldsymbol{x})} = \int d\boldsymbol{x}^{(s)}\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(t)}\,\mathcal{Q}_1(\boldsymbol{x})\ln\frac{\mathcal{Q}_1(\boldsymbol{x})}{\mathcal{Q}_2(\boldsymbol{x})}$$
$$\geq \int d\boldsymbol{x}^{(s)}\left[\int\!\!\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(t)}\,\mathcal{Q}_1(\boldsymbol{x})\right]\ln\frac{\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(t)}\,\mathcal{Q}_1(\boldsymbol{x})}{\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(t)}\,\mathcal{Q}_2(\boldsymbol{x})} = \int d\boldsymbol{x}^{(s)}\,\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(s)})\ln\frac{\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(s)})}{\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(s)})} = K(\mathcal{P}_1,\mathcal{P}_2)\,. \tag{16}$$

Notice that this derivation is solely based on the first of the two equations in (14). Because of symmetry, however, we can alternatively derive (16) utilizing the second equation in (14). Replacing $\boldsymbol{x}^{(s)} \leftrightarrow \boldsymbol{x}^{(t)}$ we get

$$K(\mathcal{Q}_1,\mathcal{Q}_2) = \int d\boldsymbol{x}^{(t)}\int d\boldsymbol{x}^{(m)}\,d\boldsymbol{x}^{(s)}\,\mathcal{Q}_1(\boldsymbol{x})\ln\frac{\mathcal{Q}_1(\boldsymbol{x})}{\mathcal{Q}_2(\boldsymbol{x})} \geq \int d\boldsymbol{x}^{(t)}\,\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(t)})\ln\frac{\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(t)})}{\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(t)})} = K(\mathcal{P}_1,\mathcal{P}_2)\,. \tag{17}$$

Because the identity in Eq. (15) holds if and only if $q_1(y)/q_2(y) = C = \text{const}$, we find an identity in (16) only if the final probability distributions are of the form

$$\mathcal{Q}_k(\boldsymbol{x}) = \mathcal{F}_k(\boldsymbol{x}^{(s)})\mathcal{V}(\boldsymbol{x}^{(m)},\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}), \qquad (k = 1,2)\,, \tag{18}$$

or, when considering (17), if

$$\mathcal{Q}_k(\boldsymbol{x}) = \mathcal{G}_k(\boldsymbol{x}^{(t)})\mathcal{W}(\boldsymbol{x}^{(m)},\boldsymbol{x}^{(s)},\boldsymbol{x}^{(t)}), \qquad (k = 1,2) \tag{19}$$

holds. Combining Eqs. (18) and (19) leads to

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2} = \frac{\mathcal{F}_1(\boldsymbol{x}^{(s)})}{\mathcal{F}_2(\boldsymbol{x}^{(s)})} = \frac{\mathcal{G}_1(\boldsymbol{x}^{(t)})}{\mathcal{G}_2(\boldsymbol{x}^{(t)})}\,. \tag{20}$$

Obviously, the second and third terms in Eq. (20) depend on different variables—$\boldsymbol{x}^{(s)}$ vs $\boldsymbol{x}^{(t)}$—and, hence, all ratios need to be constant. That constant has to be equal to 1 because both probability distributions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are normalized. Consequently, (20) yields $\mathcal{Q}_1 = \mathcal{Q}_2 \Rightarrow \mathcal{P}_1^{(s)} = \mathcal{P}_2^{(s)}$, while the nontrivial case, $\mathcal{P}_1^{(s)} \neq \mathcal{P}_2^{(s)}$, always leads to strict inequalities in both (16) and (17). Similar to (13) this implies the nonpreservation of the Kullback-Leibler distance if classical cloning is rendered possible.

To stress this contradiction from a slightly different perspective we finally consider a more constrained type of copying process, in which the final marginal probability distribution jointly describing the source and target systems is factorizable—note that the distribution associated with the entire system, as in (11), does not possess such a property. The linearity of the Liouville evolution has important consequences in such an instance. Indeed, the case

$$\int d\boldsymbol{x}^{(m)}\,\mathcal{Q}(\boldsymbol{x}) = \mathcal{P}^{(s)}(\boldsymbol{x}^{(s)})\mathcal{P}^{(s)}(\boldsymbol{x}^{(t)}) \tag{21}$$

requires a copying process [28] that contravenes the linearity of the evolution described by the Liouville equation (3). This violation can be detected once (21) holds for two distinct initial distributions $\mathcal{P}_1$ and $\mathcal{P}_2$ that we combine linearly by means of $\mathcal{P} = \alpha\mathcal{P}_1 + \beta\mathcal{P}_2$. With (21) we directly obtain for the corresponding final state

$$\int d\boldsymbol{x}^{(m)}\,\mathcal{Q}(\boldsymbol{x}) = \int d\boldsymbol{x}^{(m)}[\alpha\,\mathcal{Q}_1(\boldsymbol{x}) + \beta\,\mathcal{Q}_2(\boldsymbol{x})] = \alpha\,\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(s)})\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(t)}) + \beta\,\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(s)})\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(t)})\,, \tag{22}$$

whereas the (source/target) marginal probability associated with the copying process should read

$$\int d\boldsymbol{x}^{(m)}\,\mathcal{Q}(\boldsymbol{x}) = [\alpha\,\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(s)}) + \beta\,\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(s)})] \times [\alpha\,\mathcal{P}_1^{(s)}(\boldsymbol{x}^{(t)}) + \beta\,\mathcal{P}_2^{(s)}(\boldsymbol{x}^{(t)})]\,. \tag{23}$$

In other words, a copying process preserving the statistical independence of source and target states, as expressed by the factorizations in (7) and (21), is not compatible with the linearity of the Liouville dynamics.

Of course, copying processes constitute an everyday task within classical computers that must be reliably performed—just think of the millions of bits being copied every day in just a single computer. Is there anything special in these kinds of cloning processes (provided that our description of the dynamics is applicable [29])? Obviously, our line of argumentation is valid only if the fundamental Kullback-Leibler distance between $\mathcal{P}_1^{(s)}$ and $\mathcal{P}_2^{(s)}$ is always well defined. For example, probability distributions involving δ distributions have to be excluded from our considerations [30]. Thus, the clonability of states of the form

$$\mathcal{P}_i^{(s)}(\boldsymbol{x}^{(s)}) = \delta(\boldsymbol{x}^{(s)} - \boldsymbol{x}_i^{(s)})\,, \tag{24}$$

which do not exhibit any (intrinsic) uncertainty, cannot be ruled out in general. Actually, this situation is by no means far-fetched as one can immediately find explicit examples of dynamical systems that are able to perform copying processes for just this kind of distribution. Consider, for instance, a trivial source dynamics and a damped linear oscillator whose equations of motion for the (one-dimensional) target system reads

$$\begin{aligned}\dot{x}^{(s)} &= 0\,,\\ m\ddot{x}^{(t)} - k\dot{x}^{(t)} + \omega^2 x^{(t)} &= x^{(s)}\omega^2.\end{aligned} \tag{25}$$

Certainly, any initial source state described by a δ distribution (24) will be perfectly cloned into the target system. Note that in this example one could interpret the oscillator's momentum $p = m\dot{x}$ as the state variable $x^{(m)}$ of the "copying machine."

A distinct property of δ distributions like (24) is that they do not overlap. Put differently, two (different) distributions always obey $\mathcal{P}_1 \mathcal{P}_2 = 0$. Hence, it seems likely that two nonoverlapping distributions are always clonable by recourse to an appropriate classical copying machine. Yet, we have not been able to prove this in general, not the least because the Kullback-Leibler distance is not well defined—$\mathcal{P}_2$ needs to vanish at certain points at which $\mathcal{P}_1$ is finite; cf. (4).

In short, a universal copying machine preserving the statistical independence of the source and target systems, as expressed by the factorizations in (7) and (21), is ruled out by the linearity of the Liouville dynamics. Universal cloning machines (even if the statistical independence of the source and target systems is not required) are incompatible with the conservation of the Kullback-Leibler information associated with pairs of solutions of the Liouville equation. Only in special circumstances, as when dealing with δ distributions, the copying process becomes possible.

*Electronic address: marlow@fbw.vu.nl
†Electronic address: plastino@sinectis.com.ar
‡National Research Council (CONICET), C.C. 727, (1900) La Plata, Argentina.
§Electronic address: plastino@venus.fisica.unlp.edu.ar

[1] *Maxwell's Demon: Entropy, Information, Computing,* edited by H. S. Leff and A. F. Rex (Princeton University Press, Princeton, NJ, 1990).
[2] *Complexity, Entropy, and the Physics of Information,* edited by W. H. Zurek (Addison-Wesley, Redwood City, CA, 1990).
[3] C. Beck and F. Schlogl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1993).
[4] B. R. Frieden, *Physics from Fisher Information* (Cambridge University Press, Cambridge, 1998).
[5] *Introduction to Quantum Computation and Information,* edited by H.-K. Lo, S. Popescu, and T. Spiller (World Scientific, Singapore, 1998).
[6] *Quantum Computing and Quantum Communications,* edited by C. P. Williams (Springer, Berlin, 1999).
[7] *The Physics of Quantum Information,* edited by D. Bouwmeester, A. Ekert, and A. Zeilinger (Springer, Berlin, 2000).
[8] P. T. Landsberg and V. Vedral, Phys. Lett. A **247**, 211 (1998).
[9] P. T. Landsberg, *Seeking Ultimates* (Institute of Physics Publishing, Bristol and Philadelphia, 2001).
[10] J. Gemmer, A. Otte, and G. Mahler, Phys. Rev. Lett. **86**, 1927 (2001).
[11] D. Collins and S. Popescu, quant-ph/0107082.
[12] M. J. W. Hall, Phys. Rev. A **59**, 2602 (1999).
[13] M. D. Westmoreland and B. W. Schumacher, Phys. Rev. A **48**, 977 (1993).
[14] M. Westmoreland, J. Krone, and B. Schumacher, Physica (Amsterdam) **120D**, 236 (1998).
[15] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[16] D. Dieks, Phys. Lett. **92A**, 271 (1982).
[17] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, The Netherlands, 1993).
[18] *Quantum Information Theory—an Invitation,* edited by G. Alber, T. Beth, P. Horodecki, R. Horodecki, M. Röttler, H. Weinfurter, R. F. Werner, and A. Zeilinger, Springer Tracts in Modern Physics Vol. 173 (Springer, Berlin, 2001), Chap. 2.
[19] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. **68**, 557 (1992).
[20] V. Buzek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[21] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[22] T. Mor, Phys. Rev. Lett. **80**, 3137 (1998).
[23] N. Gisin and B. Huttner, Phys. Lett. A **228**, 13 (1997).
[24] S. Kullback, *Information Theory and Statistics* (Dover, New York, 1997).
[25] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley-Interscience, New York, 1991).
[26] General classical Hamiltonian systems constitute particular instances of the dynamics (2) for any conceivable interaction. In detail, for the Hamiltonian case with n degrees of freedom we have $N = 2n$, $\boldsymbol{x} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, $v_i = \partial H/\partial p_i$ $(i = 1, \ldots, n)$, and $v_i = -\partial H/\partial q_i$ $(i = n + 1, \ldots, 2n)$, where the q_i and p_i represent generalized coordinates and momenta, respectively. Recall that the Hamiltonian dynamics has the important feature of being divergence-free: $\nabla \cdot \boldsymbol{v} = 0$. The present discussion, however, is neither restricted to Hamiltonian systems nor to divergence-free flows.
[27] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[28] Analogous to (16) and (17) in this case a (stronger) lower bound can be found as $K(\mathcal{Q}_1, \mathcal{Q}_2) \geq 2K(\mathcal{P}_1, \mathcal{P}_2)$, which is again incompatible with the conservation of the Kullback-Leibler distance.
[29] Assuming one has complete knowledge about the dynamics of the (closed) system in question, the evolution of its pertinent probability distribution can sufficiently be described in terms of the Liouville equation. This description may fail, however, once the system is under external influences. Although possibly relevant for practical implementations of the copying process, such a restraint of the Liouville approach does not conceptually affect our results because, here, the surroundings can always be regarded as part of the copying machine.
[30] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1996), Chap. 6.1.