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Abstract

We consider cooperative games with transferable utility (TU-games), in which we allow for a social structure on the set of players, for instance a hierarchical ordering or a dominance relation. The social structure is utilized to refine the core of the game, being the set of payoffs to the players that cannot be improved upon by any coalition of players. For every coalition the relative strength of a player within that coalition is induced by the social structure and is measured by a power function. We call a payoff vector socially stable if at the collection of coalitions that can attain it, all players have the same power. The socially stable core of the game consists of the core elements that are socially stable. In case the social structure is such that every player in a coalition has the same power, social stability reduces to balancedness and the socially stable core coincides with the core.

We show that the socially stable core is non-empty if the game itself is socially stable. In general the socially stable core consists of a finite number of faces of the core and generically consists of a finite number of payoff vectors. Convex TU-games have a non-empty socially stable core, irrespective of the power function. When there is a clear hierarchy of players in terms of power, the socially stable core of a convex TU-game consists of exactly one element, an appropriately defined marginal vector. We demonstrate the usefulness of the concept of the socially stable core by two applications. One application concerns sequencing games and the other one the distribution of water.

JEL classification: C60, C70, D70

Key words: Transferable Utility game, Social structure, Balancedness, Core

1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. In a TU-game players only differ with respect to their position in the game. Examples of models in which players not only differ with respect to their position in the game, but also are part of some relational structure (which possibly affects the cooperation possibilities or payoff distributions) are *games in coalition structure* and *games with limited communication structure.* In games with coalition structure it is assumed that the set of players is partitioned into disjoint sets which represent social groups. For a particular player it is more easy to cooperate with players in his own group than to cooperate with players in other groups (see, e.g., Aumann and Drèze (1974), Owen (1977), Hart and Kurz (1983) and Winter (1989)). In games with limited communication structure the edges of an undirected graph on the set of players represent binary communication links between the players such that players can cooperate only if they are connected (see, e.g. Myerson (1977), Kalai, Postlewaite and Roberts (1978), Owen (1986) and Borm, Owen and Tijs (1992)).

Another line of research in the field of cooperative games are situations in which the players are part of some hierarchical structure such as the *games with a permission structure*. In such games it is assumed that players in a TU-game are part of a hierarchical organization in which there are players that need permission from other players before they are allowed to cooperate within a coalition, see for instance Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996). Related is also the model of Faigle and Kern (1992) who consider feasible rankings of the players. Players are also hierarchically ordered in the paper of van den Brink, van der Laan and Vasil'ev (2003). However, instead of restricting the cooperation possibilities, in the latter paper the hierarchical ordering directly affects the distribution of the so-called *Harsanyi dividends* (see Harsanyi (1959)) of the game amongst the players.

In this paper we consider TU-games in which there is a social structure on the set of players, for instance a hierarchical ordering or a dominance relation. Instead of restricting the cooperation possibilities or the distribution of the Harsanyi dividends, the social ordering is utilized to select certain payoff vectors within the core of the game. Therefore we assign to any coalition a *power vector*, whose components reflect the relative strengths of the individual members of the coalition. To derive the main results, we will take the power vector as exogenously given, but in some applications we will use suitable methods known from the literature to determine the strength of an individual within the particular organizational structure.

Given the power vectors, we use the concept of *socially stable core* to select a subset of the core of the game. This concept has been introduced in Herings, van der Laan and

Talman (2003) within the more general framework of Nontransferable Utility Games. For a payoff vector to be in the socially stable core, there should be neither incentives to deviate from an economic point of view, nor from a social one. A payoff vector is *economically stable* if it is feasible and undominated, i.e. when the payoff vector is in the core of the game. No player has an incentive to deviate from a core payoff vector from an economic point of view. Socially motivated deviations do not occur when all individuals are equally powerful at the proposed payoff vector. This is formalized by considering the power vectors of all coalitions that could realize the proposed payoff vector. If there is a weighted sum of these power vectors that gives all individuals the same power, then individuals are said to be equally powerful at the proposed payoff. Obviously, the socially stable core consisting of all payoff vectors that are economically and socially stable is a subset of the core. Generically, it is shown to be a proper subset of the core consisting of a finite number of elements only, and therefore it can be considered as a core selection device.

We define the property of social stability for a socially structured game and refer to games satisfying this property as socially stable games. It will be shown that a socially stable game has a non-empty socially stable core. We also show that a convex game is socially stable for any social structure and thus has a non-empty socially stable core. When there is a clear hierarchy of players in terms of power, the socially stable core of a convex TU-game consists of exactly one element. This element corresponds to the marginal vector that is consistent with the ordering of players according to their respective power.

We demonstrate the usefulness of the concept of the socially stable core by some applications in which the social structure on the players arises naturally from the characteristics of the economic situation. Amongst these examples are the water distribution problem and sequencing situations.

The structure of the paper is as follows. Section 2 gives some preliminaries. In Section 3 socially structured transferable utility games are introduced as well as the associated solution concept of the socially stable core. In Section 4 the main theorem is presented and it is proven that a convex game is socially stable. Section 5 contains examples and applications and Section 6 concludes.

2 Preliminaries

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), with $N = \{1, 2 \ldots, n\}$ a finite set of n players and $v \colon 2^N \to \mathbb{R}$ a characteristic function assigning to any coalition $S \subseteq N$ of players a real number $v(S)$ as the *worth* of coalition S with $v(\emptyset) = 0$, i.e. the members of coalition S can obtain a total payoff of

$v(S)$ by agreeing to cooperate. In this paper we assume that N is a fixed set of players, allowing to denote a game (N, v) shortly by its characteristic function v. A TU-game v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of subsets $S, T \subseteq N$ such that $S \cap T = \emptyset$. Further, a TU-game v is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. We denote the collection of all TU-games on N by $\mathcal{G}$ and the collection of all convex TU-games on N by $\mathcal{G}_c$. A solution F assigns a set $F(v) \subset \mathbb{R}^n$ of payoff vectors to every TU-game $v \in \mathcal{G}$. A well-known set-valued solution is the *Core*, assigning to every game v the (possibly empty) set

$$C(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N), \ \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N\}.$$

It is well-known that $C(v)$ is non-empty if and only if v is balanced (see Bondareva (1963)).

An important point-valued solution is the *Shapley value*. This value can be defined in several ways, for instance as a weighted sum of the so-called marginal contributions (see Shapley (1953)) or as an equal distribution of the so-called Harsanyi dividends of coalitions (see Harsanyi (1959)) amongst the players in the coalitions. Because of reasons later on in this paper, we use here the concept of marginal vector to define the Shapley value. For a permutation $\pi \colon N \to N$, assigning rank number $\pi(i) \in N$ to any player $i \in N$, define $\pi^i = \{j \in N | \pi(j) \leq \pi(i)\}$, i.e., π^i is the set of all players with rank number at most equal to the rank number of i, including i himself. Then the *marginal value vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad i \in N,$$

and thus assigns to player i his marginal contribution to the worth of the coalition consisting of all his predecessors in π. The Shapley value is equal to the average of the marginal value vectors over all permutations. When v is convex, the core of v is equal to the convex hull of all marginal value vectors and thus the Shapley value is in the core.

A game (N, v) is called permutationally convex, see Granot and Huberman (1982), if there exists a permutation π such that for all $1 \leq j < k < n$ it holds that $\max[v(S),\ v(\pi^j \cup S) - v(\pi^j)] \leq v(\pi^k \cup S) - v(\pi^k)$ for all $S \subset N \setminus \pi^k$. When (N, v) is permutationally convex with respect to the permutation π, it holds that the corresponding marginal vector $m^\pi(v)$ is in the core $C(v)$ and, hence, the core is non-empty.

3 Structured TU-Games

In this paper we assume that there is a social structure on any coalition of players. This social structure could be, for example, a structure where one agent is the leader of the coalition and makes all the decisions, while all other agents of the coalition follow him, a structure in which all members of the coalition are in an equal position to each other and decisions are made by a unanimity or majority voting rule, or a hierarchy in which the agents are ordered on several levels. We assume that the social structure is represented by a power vector reflecting the relative strengths of the individual members of the coalition within the social structure. For example, in a hierarchy the agent at the top of the hierarchy has more power within the coalition than the other coalition members, whereas within a coalition in which the members are in an equal position to each other, all members have the same power.

A structured TU-game (STG) is given by the characteristic function v and a power function p from $\mathcal{N}$ to S^n, where $\mathcal{N} = 2^N \setminus \emptyset$ is the collection of all non-empty subsets of N and $S^n = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1 \text{ and } x_i \geq 0,\ i \in N\}$ is the $(n-1)$-dimension unit simplex. For $S \in \mathcal{N}$, the nonnegative number $p_i(S)$ denotes the power of agent $i \in N$ within the social structure on S. It is assumed that every player outside the coalition S has power equal to zero, every agent within S has a nonnegative power with a strictly positive power for at least one of these players, and the total power has been normalized to one, i.e. for every $S \subset N$, we have that $p_i(S) = 0$ for all $i \in N \setminus S$, $p_i(S) \geq 0$ for all $i \in S$ and $\sum_{i \in S} p_i(S) = 1$. We now have the following definition of a structured TU-game (STG).

Definition 3.1 (Structured TU-Game)
A structured TU-game *is given by the triple* $\Gamma = (N, v, p)$.

We are interested in payoff vectors that are *socially* and *economically stable*. If an individual at a certain payoff vector has more power than another individual, then he is assumed to be able to increase his payoff at the expense of the other individual. Such a payoff vector is not socially stable. To define social stability of a payoff vector x in a structured game STG $\Gamma = (N, v, p)$ formally, we first define the set of feasible power vectors for a payoff vector x, where $x(S) = \sum_{i \in S} x_i$ denotes the total payoff to the players in S. The set of feasible power vectors for a payoff vector x is defined by

$$FP(x) = \{y \in S^n \mid y = \sum_{\{S \in \mathcal{N} \mid x(S) \leq v(S)\}} \lambda_S p(S),\ \lambda_S \geq 0 \text{ for all } S,\ \sum_{S \in \mathcal{N}} \lambda_S = 1\}$$

with the convention that $FP(x) = \emptyset$ when $x(S) > v(S)$ for all $S \in \mathcal{N}$. The set of feasible power vectors for x is equal to the set of all convex combinations of power vectors of all coalitions S for which $x(S) \leq v(S)$. Notice that the set of feasible power vectors for an arbitrary payoff vector in $\mathbb{R}^n$ is a, possibly empty, convex set and is a subset of S^n. A payoff vector is called socially stable if the vector $1/n \cdot e^N$ is contained in its set of feasible power vectors, where e^N denotes the n-dimensional vector of ones.

Definition 3.2 (Socially Stable Payoff)
For a socially structured game $\Gamma = (N, v, p)$ *a payoff vector* $x \in \mathbb{R}^n$ *is* socially stable *if* $FP(x)$ *contains the vector* $1/n \cdot e^N$.

Social stability of a payoff vector x means that nonnegative real numbers or weights can be assigned to the coalitions S for which $x(S) \leq v(S)$ in such a way that the weighted total power of every agent is equal to $1/n$ and therefore the same for every agent. It will be useful to define stability of a collection of coalitions without reference to a particular payoff vector.

Definition 3.3 (Stable Collection of Coalitions)
A collection of coalitions in $\mathcal{N}$, $\{S_1, \ldots, S_k\}$, is stable *if the system of equations*

$$\sum_{j=1}^{k} \lambda_j p(S_j) = 1/n \cdot e^N$$

has a nonnegative solution. A stable collection of coalitions in $\mathcal{N}$ is minimal *if no proper subset of it is stable.*

A socially stable payoff vector is therefore a payoff vector whose components can be achieved by every element of some stable collection of coalitions for its members. Observe that the definition of a stable collection reduces to the standard concept of balancedness when for any $S \in \mathcal{N}$ it holds that $p_i(S) = 1/|S|$, $i \in S$. In the remaining we denote this case by $p = e$.

A socially stable payoff vector may not be achieved by the grand coalition. In general, a payoff vector x is said to be feasible if it can be attained by the grand coalition. Furthermore, social stability of a payoff vector x does not imply that x is undominated, i.e. there may exist an $S \in \mathcal{N}$ and $y \in \mathbb{R}^n$ satisfying $y(S) \leq v(S)$ and $y_i > x_i$ for all $i \in S$. A feasible payoff vector which is undominated is called economically stable. Clearly a payoff vector x is economically stable if and only if x is in the core of v. The set of all socially and economically stable payoff vectors is called the socially stable core of the game.

Definition 3.4 (Socially Stable Core)
The socially stable core *of a structured TU-game $\Gamma = (N, v, p)$ consists of the set of socially and economically stable payoff vectors of Γ.*

A payoff vector x lies in the socially stable core if and only if x is in the core of the game (economic stability) and can be sustained by a socially stable collection of coalitions (social stability). For a game $\Gamma = (N, v, p)$, we denote the set of payoff vectors in the socially stable core by $SC(v, p)$. Observe that the grand coalition is a stable collection when $p = e$. Hence, $SC(v, p) = C(v)$ when $p = e$.

4 Non-emptiness of the Socially Stable Core

In this section we give sufficient conditions for the non-emptiness of the socially stable core of a structured TU-game. When $p = e$, we know from Bondareva (1963) that the socially stable core coincides with the core and hence the socially stable core is non-empty if and only if the game is balanced. For an arbitrary power function p, the socially stable core is a subset of the core and might be empty even if the core is not empty, i.e., the balancedness condition for TU-games is not sufficient for the non-emptiness of the socially stable core when $p \neq e$. However, the next definition of social stability of the game Γ generalizes the balancedness definition and provides a sufficient condition for the non-emptiness of the socially stable core. A socially structured game is called *socially stable* if every socially stable payoff vector can be achieved by the grand coalition.

Definition 4.1 (Socially Stable Game)
A structured TU-game $\Gamma = (N, v, p)$ *is* socially stable *if any socially stable payoff* x *is feasible.*

It should be observed that this social stability condition reduces to the usual balancedness condition for $p = e$. We now have the following theorem.

Theorem 4.2
A structured TU-game $\Gamma = (N, v, p)$ *has a non-empty socially stable core is socially stable.*

Proof. The proof follows from a more general theorem for socially structured NTU-games given in Herings, van der Laan and Talman (2003). Q.E.D.

The theorem above requires feasibility to be shown for any socially stable payoff vector. This may be a demanding task. The next theorem says that any convex game has a non-empty socially stable core.

Theorem 4.3
If v *is a convex game, then for any power function* p *the structured TU-game* $\Gamma = (N, v, p)$ *has a non-empty socially stable core.*

Proof. To prove non-emptiness, we first construct for any given power function p a stable collection of coalitions.

Step 1. Set $k = 1$, $S_k = N$, $q^k = 1/n \cdot e^N$, and $r_k = n$. Goto Step 2.

Step 2. Define $T_k = \{j \in S_k \,|p_j(S_k)/q^k_j = \max_{h \in S_k} p_h(S_k)/q^k_h\}$ and $t_k = |T_k|$. Define $\lambda_k = (\max_{h\in S_k} p_h(S_k)/q^k_h)^{-1}$. Goto Step 3.

Step 3. For $j \in T_k$, define $\pi(j) \in N$ such that $\{\pi(j) \mid j \in T_k\} = \{r_k,\ r_k-1, \ldots, r_k-t_k+1\}$. If $r_k = t_k$, define $k^* = k$ and stop the procedure; otherwise set $k = k+1$ and goto Step 4.

Step 4. Set $S_k = S_{k-1} \setminus T_{k-1}$, $q^k = q^{k-1} - \lambda_k p(S_{k-1})$ and $r_k = r_{k-1} - t_{k-1} = |S_k| > 0$. Return to step 2.

By construction we have that the collection $\{S_1, S_2, \ldots, S_{k^*}\}$ is a stable collection of coalitions and that $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ is a permutation of the elements of N such that for any $k = 1, \ldots, k^*$ it holds that

$$S_k = \{\pi(1), \pi(2), \ldots, \pi(\ell_k)\}, \text{ with } \ell_k = \sum_{h=k}^{k^*} t_h.$$

Next take the power vector x equal to the marginal vector $m^\pi(v)$. Then it follows for any $k = 1, \ldots, k^*$ that

$$\sum_{j\in S_k} x_j = \sum_{j=1}^{\ell_k} (v(\pi^j) - v(\pi^j \setminus \{j\})) = v(\pi^{\ell_k}) = v(S_k).$$

By construction of the sets S_k it follows that x is socially stable. Moreover, since v is convex we also have that $x \in C(v)$. Hence $x \in SC(v, p)$. Q.E.D.

Observe that the marginal vector constructed in the proof is unique if and only if $k^* = n$ and thus $|T_k| = 1$ for all k. When for some k, T_k contains multiple players, we can take in Step 3 any order of sequence of the players within T_k. So, in general there are $\Pi_k\ |T_k|$ different marginal vectors satisfying the conditions and thus being elements of $SC(v, p)$.

The assumptions in Theorems 4.2 and 4.3 are independent, i.e. socially stability of Γ does not imply convexity of v and vice versa. Clearly, when $p = e$, Γ is socially stable when v satisfies balancedness. However, balancedness of v does not imply convexity. The following example shows that convexity of v does not imply social stability of Γ.

Example 4.4 Take $N = \{1, 2, 3\}$, $v(1) = v(3) = v(1, 3) = 0$, $v(2) = v(1, 2) = v(2, 3) = v(1, 2, 3) = 1$. Take any power vector function p such that $p(1, 2) = (2, 1, 0)^\top$ and $p(2, 3) = (0, 1, 2)^\top$. Clearly this game is convex, so the socially stable core of $\Gamma = (N, v, p)$ is non-empty. We show that Γ is not socially stable. Notice that $x = (1, 0, 1)^\top$ is a socially stable payoff vector sustained by the stable collection of coalitions $\{\{1, 2\},\ \{2, 3\}\}$. However, x is not feasible and thus $\Gamma = (N, v, p)$ is not socially stable. Q.E.D.

For a collection of coalitions $\mathcal{F} \subset \mathcal{N}$, define

$$C^{\mathcal{F}}(v) = \{x \in C(v) \mid x(S) = v(S), \text{for all } S \in \mathcal{F}\},$$

i.e., $C^{\mathcal{F}}(v)$ is a (possibly empty) face of $C(v)$. When $x \in SC(v,p)$ lies in the (relative) interior of a face $C^{\mathcal{F}}(v)$ of the core, then every point of the face $C^{\mathcal{F}}(v)$ belongs to $SC(v,p)$. Hence, $SC(v,p)$ is equal to the union of a finite number of faces of $C(v)$. The following example shows that the socially stable core may consist of two or more disjoint faces.

Example 4.5 Take $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = 0$, $v(1,2) = 2$, $v(1,3) = v(2,3) = 3$, $v(1,2,3) = 6$. Clearly this game is convex. The core of this game is equal to $C(v) = \{x \in \mathbb{R}^3 | \ 0 \leq x_1, x_2 \leq 3, \ 0 \leq x_3 \leq 4, \ x_1 + x_2 + x_3 = 6\}$. Take $p(1,2) = (3,1,0)^\top$, $p(1,3) = (3,0,1)^\top$, $p(2,3) = (0,3,1)^\top$, $p(1,2,3) = (2,1,3)^\top$. For this power function the socially stable core $SC(v,p)$ consists of the two marginal vectors $(2,0,4)^\top$ and $(3,3,0)^\top$. The point $(2,0,4)^\top$ is sustained by the stable collection $B_1 = \{\{1,2,3\}, \ \{1,2\}, \ \{2\}\}$ and the point $(3,3,0)^\top$ by the stable collection $B_2 = \{\{1,3\}, \ \{2,3\}, \ \{3\}\}$. Q.E.D.

Theorem 4.3 implies that for convex games v the socially stable core is non-empty for all power functions p. When the power function p is such that for some permutation π a player i has little power in any coalition involving players from $N \setminus \pi^i$, the socially stable core can be shown to consist of a unique element given by the marginal vector $m^\pi(v)$. To make this statement more precise, we introduce the notion of π-*compatibility*.

Definition 4.6 *A power function $p : \mathcal{N} \to S^n$ is* π-compatible *for a permutation π of N, when for all players $i \in N$, for all coalitions S containing i such that $S \cap (N \setminus \pi^i) \neq \emptyset$, it holds that $p_i(S) < 1/n$.*

When a power function is π-compatible, the power of a player i in any coalition that involves another player that is ranked higher according to π, is less than $1/n$.

Theorem 4.7 *Consider a structured TU-game (N,v,p), where v is convex and p is π-compatible for some permutation π of N. Then the socially stable core contains the marginal vector $m^\pi(v)$ as its unique element.*

Proof. Without loss of generality, we may assume that the permutation π corresponds to the ordering, $\pi(i) = n+1-i$, $i = 1,\ldots,n$. Let the payoff vector x belong to $SC(v,p)$ and let $\{S_1,\ldots,S_m\}$ be a stable collection of coalitions with $(\lambda_1,\ldots,\lambda_m)$ a vector of weights such that $\sum_{j=1}^m \lambda_j p(S_j) = 1/n \cdot e^N$ and $x(S_j) \leq v(S_j)$ for $j = 1,\ldots,m$.

We define the ordering $\prec^\ell$ on $\mathcal{N}$ by $S \prec^\ell T$ if and only if the lowest ranked individual in $S \cup T$ not in $S \cap T$ belongs to S. Without loss of generality, we may choose $\{S_1,\ldots,S_m\}$ to be minimal, and we can order the coalitions such that $S_j \prec^\ell S_{j+1}$.

We claim that, for $i = 1, \ldots, m$, this stable collection satisfies $i \in S_i \subset \{i, \ldots, n\}$. Suppose that $\{S_1, \ldots, S_m\}$ does not contain the singleton coalition $\{n\}$. Since p is π-compatible, it follows that $p_n(S_j) < 1/n$, $j = 1, \ldots, m$, so

$$\frac{1}{n} = \sum_{j=1}^{m} \lambda_j p_n(S_j) < \frac{1}{n},$$

a contradiction. Consequently, $\{S_1, \ldots, S_m\}$ does contain the singleton coalition $\{n\}$, and by the properties of $\prec^\ell$ it follows that $S_m = \{n\}$.

We now use an induction argument to proceed. Assume it is true that, for some $k' \leq m$, for $k = 1, \ldots, k'$, $n - k \in S_{m-k} \subset \{n - k, \ldots, n\}$. We show that then $n - k - 1 \in S_{m-k-1} \subset \{n - k - 1, \ldots, n\}$. Obviously, it is not the case that $S_{m-k-1} \subset \{n - k, \ldots, n\}$ as this would violate the minimality of $\{S_1, \ldots, S_m\}$. Suppose it is not true that $S_{m-k-1} \subset \{n - k - 1, \ldots, n\}$, so the lowest ranked player in S_{m-k-1} is $i' < n - k - 1$. For all S_j, it holds that $p_{n-k-1}(S_j) < 1/n$, so $1/n = \sum_{j=1}^{m} \lambda_j p_{n-k-1}(S_j) < 1/n$, a contradiction. It follows that the lowest ranked player in S_{m-k-1} is $n - k - 1$. This completes the proof of the induction step, and it follows as a corollary that $m = n$.

It remains to be shown that $x = m^\pi(v)$, i.e, that for $k' = 1, \ldots, n$, $\sum_{i=n-k'}^{n}, x_i = v(\{n - k', \ldots, n\})$ holds. Since x is economically stable, it follows that, for $k' = 1, \ldots, n$, $\sum_{i=n-k'}^{n} x_i \geq v(\{n - k', \ldots, n\})$.

Obviously, it holds that $x_n = v(\{n\})$. We proceed with an induction argument. Assume it is true that, for some k', $\sum_{i=n-k'}^{n} x_i = v(\{n - k', \ldots, n\})$. Then,

$$\begin{aligned}
\sum_{i=n-k'-1}^{n} x_i &= \sum_{i=n-k'}^{n} x_i + x_{n-k'-1} \\
&= v(\{n - k', \ldots, n\}) + x_{n-k'-1} \\
&= v(\{n - k', \ldots, n\}) + v(S_{n-k'-1}) - \sum_{i \in S_{n-k'-1} \setminus \{n-k'-1\}} x_i \\
&\leq v(\{n - k' - 1, \ldots, n\}) + v(S_{n-k'-1} \setminus \{n - k' - 1\}) - \sum_{i \in S_{n-k'-1} \setminus \{n-k'-1\}} x_i \\
&\leq v(\{n - k' - 1, \ldots, n\}),
\end{aligned}$$

where the second to last inequality uses the convexity of v and the last inequality the economic stability of x. Q.E.D.

When the power function p is π-compatible, a highly ranked player is able to extract all payoffs from lower-ranked players, up to the point where the lower-ranked players could form a deviating coalition. In this case, the socially stable core consists of a unique element, corresponding to the marginal vector $m^\pi(v)$.

In general, the socially stable core need not consist of a unique element as was already demonstrated by Example 4.5. However, in general, barring exceptional cases, the socially stable core consists of a finite number of payoffs. To show this, observe that once the number of players is fixed, a structured TU-game is completely determined by the tuple of payoffs v, which can be represented by a vector in $\mathbb{R}^{2^n-1}$, and the tuple of power functions, which can be represented by a vector in $S^{(2^n-1)n}$. The standard topology and measure on $\mathbb{R}^{2^n-1} \times S^{(2^n-1)n}$ therefore induce a topology and a measure on structured TU-games.

Theorem 4.8 *Let N be the set of players. Then there is an open set of payoffs and power functions with full Lebesgue measure $V \times P$ such that for any $(v, p) \in V \times P$, the socially stable core of the structured TU-game $\Gamma = (N, v, p)$ consists of a finite number of elements.*

Proof. For $S \in \mathcal{N}$, define the vector $e(S)$ by $e_i(S) = 1$ if $i \in S$ and $e_i(S) = 0$ if $i \notin S$. We define the closed subset W of $\mathbb{R}^n$ with measure zero by

$$W = \cup_{(S_1,\ldots,S_{n-1}) \in \mathcal{N}^{n-1}} \text{span}\,[e(S_1), \ldots, e(S_{n-1})]\,.$$

Next we define the open subset V of $\mathbb{R}^{2^n-1}$ with full measure by

$$V = \{v \in \mathbb{R}^{2^n-1} \mid \forall (S_1, \ldots, S_n) \in \mathcal{N}^n \text{ with } S_j \neq S_{j'} \text{ when } j \neq j',\ (v_{S_1}, \ldots, v_{S_n}) \notin W\}.$$

Finally, we define the open subset P of $S^{(2^n-1)n}$ with full measure as the set of vectors $p = (p(S))_{S \in \mathcal{N}}$ for which it holds that any selection of n vectors from the vectors $p(S)$, $S \in \mathcal{N}$, and e^N yields an independent set of vectors.

We now examine the socially stable core for the structured TU-game $\Gamma = (N, v, p)$, where $(v, p) \in V \times P$. All socially stable core elements are obtained by considering, for all minimal stable collections of coalitions $\{S_1, \ldots, S_m\}$, the solutions to the system of equations

$$x(S_j) = v(S_j),\ \ j = 1, \ldots, m.$$

In fact, the union over all stable collections of solutions to the corresponding system is a superset of the socially stable core. Since $\{S_1, \ldots, S_m\}$ is a stable collection, there exists a vector of weights λ such that

$$\sum_{j=1}^{m} \lambda_j p(S_j) = 1/n \cdot e^N.$$

Moreover, $\{S_1, \ldots, S_m\}$ is minimal, so that the vectors $p(S_j)$ are independent, and in particular $m \leq n$. Since $p \in P$, it holds that $m = n$.

Consider the system of equations,

$$x(S_j) = v(S_j),\ \ j = 1, \ldots, n.$$

If the vectors $e(S_j)$ are all independent, it follows that this system has exactly one solution, and therefore we obtain at most one socially stable core element. When the vectors $e(S_j)$ are not all independent, it follows from our definition of V that this system of equations has no solution. Hence, there is at most one solution for each minimal stable collection. Since the number of minimal stable collections is finite, this proves the theorem. Q.E.D.

Theorem 4.8 shows that in general the socially stable core refines the core to a great extent. There is typically only a finite number of payoffs in the socially stable core.

5 Sequencing games

A one-machine sequencing situation, see e.g. Curiel (1988) or Hamers (1995) is described as a triple (N, q, c), where $N = \{1, \ldots, n\}$ is the set of jobs in a queue to be processed, $q \in \mathbb{R}^n_+$ is an n-vector with q_i the processing time of job i and $c = (c_i)_{i \in N}$ is a collection of cost functions $c_i\colon \mathbb{R}_+ \to \mathbb{R}_+$, specifying the costs $c_i(t)$ when t is the total time needed to complete job i. For an ordering ρ on N describing the positions of the jobs in the queue, the completion time of job i is given by $T_i(\rho) = \sum_{\{j \mid \rho(j) \leq \rho(i)\}} q_j$, i.e. the completion time is the sum of its waiting time and its own processing time, and the costs of processing i are given by $C_i(\rho) = c_i(T_i(\rho))$. The total costs of a coalition $S \subseteq N$ given an ordering ρ are given by $C_S(\rho) = \sum_{i \in S} C_i(\rho)$. In the sequel we assume without loss of generality that the initial positions of the jobs in the queue are given by the ordering ρ^0 with $\rho^0(i) = i$ for all $i \in N$, so that the costs of a coalition S of jobs according to ρ^0 are given by

$$C_S(\rho^0) = \sum_{i \in S} C_i(\rho^0) = \sum_{i \in S} c_i\Big(\sum_{\{j \mid j \leq i\}} q_j\Big), \quad S \subseteq N.$$

Now, each coalition S of jobs can obtain cost savings by rearranging the jobs amongst the members of S. Then the minimal cost of the grand coalition is given by

$$C_N = \min_{\rho} C_N(\rho).$$

However, members of any other coalition S can only rearrange their positions under the condition that the members of S are not allowed to 'jump' over jobs outside S. So, an ordering ρ is admissible for S if for any $j \notin S$ the set of its predecessors does not change with respect to the initial situation, i.e. if for any $j \notin S$ it holds that $\{k \in N \mid \rho(k) < \rho(j)\} = \{k \in N \mid \rho^0(k) < \rho^0(j)\} = \{k \in N \mid k < j\}$. Let $\mathcal{A}(S)$ be the set of admissible orderings for S. Then the minimal cost of S is given by

$$C_S = \min_{\rho \in \mathcal{A}(S)} C_S(\rho).$$

This gives the cost savings *sequencing* game (N, v) with N the set of jobs as the set of players and characteristic function v given by

$$v(S) = C_S(\rho^0) - C_S, \quad S \subseteq N.$$

In the following, we use the terminology players instead of jobs. Obviously, since for any S only orderings in $\mathcal{A}(S)$ are admissible, only connected coalitions (i.e. coalitions of consecutive players) can realise cost reductions. Therefore, for $i < j$, denote the set $\{i, i+1, \ldots, j\}$ of consecutive players by $[i, j]$ and let $\mathcal{L}$ denote the set of all coalitions of consecutive players, i.e.

$$\mathcal{L} = \{T \in \mathcal{N} \mid T = [i, j],\ 1 \leq i \leq j \leq n\}.$$

For ease of notation, in the following we denote $v(S)$ by $v[i, j]$ when $S = [i, j]$.

For some $S \in \mathcal{N}$, let $P(S)$ be the unique minimal partition of S in coalitions of consecutive players, i.e. $T \in \mathcal{L}$ if $T \in P(S)$ and $T_1 \cup T_2 \notin \mathcal{L}$ for any pair $T_1,\ T_2 \in P(S)$. Then it holds that

$$v(S) = \sum_{T \in P(S)} v(T), \quad S \in \mathcal{N},$$

i.e. the value of a coalition S is equal to the sum of the values of the coalitions of consecutive players in its unique minimal partition. From this property it follows immediately that the characteristic function is superadditive and also satisfies permutational convexity, implying that v has a non-empty core. In particular, let u and ℓ be the two permutations on N defined by $u(i) = i,\ i = 1, 2, \ldots, n$ and $\ell(i) = n + 1 - i,\ i = 1, 2, \ldots, n$. Further, denote $\mu(v) = m^u(v)$ and $\lambda(v) = m^\ell(v)$ as the corresponding marginal value vectors. Then it follows that v satisfies the permutational convexity conditions for the two permutations u and ℓ, implying that the two marginal vectors $\mu(v)$ and $\lambda(v)$ are in $C(v)$.

Under certain conditons on the power function we have that $\mu(v)$, respectively $\lambda(v)$, is the unique element of the socially stable core. In particular, for any power function $p\colon \mathcal{N} \to S^n$ satisfying for every $S \in \mathcal{N}$

$$p_i(S) < p_j(S) \text{ when } i < j, \quad i, j \in S, \tag{1}$$

we have the following theorem.

Theorem 5.1

For any sequencing game it holds that $\mu(v)$ is the unique element of $SC(v, p)$ when p satisfies condition (1).

Proof. We first show that $\mu(v) \in SC(v,p)$. Let I^u be the collection of coalitions given by

$$I^u = \{[1,h] \mid h = 1,2,\ldots,n\}.$$

Clearly, I^u is a socially stable collection of coalitions for any power function p satisfying (1). Moreover, I^u sustains $\mu(v)$, since $\mu(v)$ is the marginal vector of the game with respect to the permutation u, so that $\sum_{i=1}^{h} \mu_i(v) = v[1,h]$. Hence, $\mu(v)$ is in the core and is socially stable, and therefore $\mu(v) \in SC(v,p)$.

To prove that $SC(v,p)$ contains no other payoff vectors, let $\mathcal{B}$ be any collection of stable coalitions for the power function p sustaining a vector $x \in C(v)$, i.e. $x(S) \leq v(S)$ for all $S \in \mathcal{B}$. From Definition 3.3 and condition (1), it follows that for any $k \leq n-1$, there must be a coalition $S \in \mathcal{B}$, such that $k \in S$ and $k+1 \notin S$. Let S^k denote such a coalition in $\mathcal{B}$. By definition of the characteristic function v, we have that $v(S) = \sum_{T \in P(S^k)} v(T)$. Hence,

$$\sum_{T \in P(S^k)} x(T) = x(S^k) \leq v(S^k) = \sum_{T \in P(S^k)} v(T).$$

Since $x \in C(v)$, we also have that $x(T) \geq v(T)$ for all $T \in P(S^k)$. Therefore,

$$x(T) = v(T), \text{ for any } T \in P(S^k).$$

Clearly, for $k = 1$ it holds that $\{1\} \in P(S^1)$ and thus $x_1 = v(1)$. We now proceed by induction and assume that for certain $k \leq n-1$ it holds that $x(T) = v(T)$ for $T = [1,h]$, $h = 1,\ldots,k-1$. For k it holds that $[j,k] \in P(S^k)$ for some $j \leq k$ and thus $x[j,k] \leq v[j,k]$. Hence $x[j,k] = v[j,k]$, since $x \in C(v)$. So, with the induction assumption it follows that

$$x[1,k] = x[1,j-1] + x[j,k] = v[1,j-1] + v[j,k].$$

From the superadditivity it follows that $v[1,j-1]+v[j,k] \leq v[1,k]$, while $x \in C(v)$ implies that $x[1,k] \geq v[1,k]$. Hence $x[1,k] = v[1,k]$ and therefore $x(T) = v(T)$ for all $T = [1,h]$, $h = 1,\ldots,n-1$. Of course, also $x[1,n] = v[1,n]$ holds. Therefore it follows that for any k it holds that

$$x_k = x[1,k] - x[1,k-1] = v[1,k] - v[1,k-1],$$

showing that $x = \mu(v)$. Hence, $x \in SC(v,p)$ iff $x = \mu(v)$. Q.E.D.

Similarly, the vector $\lambda(v)$ is the unique element of the socially stable core for any power function $p\colon \mathcal{N} \to \mathbb{R}^n_+ \setminus \{0^n\}$ satisfying for every $S \in \mathcal{N}$

$$p_i(S) < p_j(S) \quad \text{when} \quad i > j, \quad i,j \in S. \tag{2}$$

Now, the collection of coalitions I^ℓ given by

$$I^\ell = \{[h, n] \mid h = 1, 2, \ldots, n\}$$

is a socially stable collection. Moreover, by definition of the marginal vector $\lambda(v) = m^\ell(v)$, we have that $\sum_{i=h}^{n} \lambda_i(v) = v[h, n]$. Hence, $\lambda(v)$ is in the core and is also sustained by I^ℓ, so $\lambda(v) \in SC(v, p)$. The next theorem says that it is also the unique element in $SC(v, p)$. The proof goes analogously to the proof of Theorem 5.1.

Theorem 5.2
For any sequencing game it holds that $\lambda(v)$ is the unique element of $SC(v, p)$ when p satisfies condition (2).

We now consider the special case of linear costs, i.e., $c_i(t) = \alpha_i t$ for all $t \geq 0$ with $\alpha_i > 0$. It is well-known that the characteristic function v is convex, see e.g. Curiel (1988) or Hamers (1995). Moreover, in these references it has been shown that under linear costs for each coalition $[i, j] \in \mathcal{L}$ it holds that

$$v[i, j] = \sum_{\{h,k \in [i,j] | h<k\}} g_{hk},$$

where $g_{hk} = \max(0, \alpha_k q_h - \alpha_h q_k)$ is the gain of a switch between player h and k in any ordering such that player h is directly in front of k. Now, the net-costs of player h resulting from $\mu(v)$ are given by the costs of the waiting time in the initial order minus the savings obtained from cooperation, i.e., the net-costs $c^u_h(N, q, c)$ of player $h \in N$ in the linear cost sequencing situation (N, q, c) induced by $m^u(v)$ is given by

$$\begin{aligned} c^u_h(N, q, c) &= C_h(\rho^0) - \mu_h(v) = C_h(\rho^0) - (v[1, h] - v[1, h-1]) \\ &= C_h(\rho^0) - (\sum_{\{i,j \in [1,h] | i<j\}} g_{ij} - \sum_{\{i,j \in [1,h-1] | i<j\}} g_{ij}) \\ &= C_h(\rho^0) - \sum_{i=1}^{h-1} g_{ih}. \end{aligned}$$

So, according to this solution all the savings obtained from a switch of player h with any of its predecessors goes to player h. Fernández, Borm, Hendrickx and Tijs (2002) show that this cost-assignment rule is the unique solution being stable (i.e. $c^u(N, q, c)$ is in the core of the cost-game for any linear cost sequencing situation (N, q, c)) and satisfying the so-called property of *Drop Out Monotonicity* (DOM). Clearly, the stableness property follows from the fact that $\mu(v)$ is in the core of the cost-savings game. To state DOM, let (N_{-k}, q_{-k}, c_{-k}) with player set $N_{-k} = N \setminus \{k\}$ be the $(n-1)$-player sequencing situation obtained when player k leaves the queue (i.e., job k is cancelled) and let v^{-k} be the corresponding characteristic function. Then a cost assignment rule r assigning costs $r_h(N, q, c)$ for all $h \in N$ satisfies DOM if for any linear cost situation (N, q, c) it holds that

$$r_h(N_{-k}, q_{-k}, c_{-k}) \leq r_h(N, q, c), \;\; h \in N_{-k},$$

i.e. if one of the players leaves the queue, for each of the remaining players the costs are non-increasing.

The cost rule c^u induced by $\mu(v)$ indeed satisfies DOM. In fact, it follows by straightforward calculations that

$$c^u_h(N_{-k}, q_{-k}, c_{-k}) = c_h(N, q, c), \quad h = 1, \ldots, k-1,$$

and

$$c^u_h(N_{-k}, q_{-k}, c_{-k}) = c^u_h(N, q, c) - \min(\alpha_h q_k, \alpha_k q_h) < c^u_h(N, q, c), \quad h = k+1, \ldots, n.$$

So, for the players in front of k there is no change in the net-costs, whereas for any player h after k the decrease $\alpha_h q_k$ in initial costs when k leaves the queue is bigger than the loss of the cost-savings g_{kh} (if positive) from a switch between k and h. The DOM property advocated in Fernández et al. (2002) seems to be very appealing and reasonable: when player k drops out, the players in front of k are not affected, while for the players after k the costs are decreasing.

On the other hand, let (N^j, q, c) denote the adjusted sequencing situation in which some player j refuses to cooperate with any player $k > j$. As a consequence of this refusal of j, any coalition $[i, h]$, $i \leq j < h$ cannot form and hence all the gains g_{ih} of a switch between i and h, $i \leq j < h$ cannot be realized anymore. Let v^j be the corresponding characteristic function of the cost-savings game. For $S \in \mathcal{N}$, let $P^j(S)$ be the unique minimal partition of S in coalitions of consecutive players not containing both j and $j+1$, i.e. $T \in \mathcal{L}$ if $T \in P^j(S)$ and if $T_1 \cup T_2 \in \mathcal{L}$ for some pair T_1, $T_2 \in P^j(S)$, then $j \in T_1$ and $j+1 \in T_2$ (or reversely). Then it holds that

$$v^j(S) = \sum_{T \in P^j(S)} v(T), \quad S \in \mathcal{N},$$

i.e. the value of a coalition S is equal to the sum of the values of the coalitions in its unique minimal partition $P^j(S)$. From this property it follows immediately that the net-cost according to $\mu(v^j)$ becomes equal to

$$c^u_h(N^j, q, c) = \begin{cases} C_h(\rho^0) - \sum_{i=1}^{h-1} g_{ih} = c^u_h(N, q, c), & h = 1, \ldots, j, \\ C_h(\rho^0) - \sum_{i=j+1}^{h-1} g_{ih}, & h = j+1, \ldots, n. \end{cases}$$

Comparing this with the costs $c^u_h(N, q, c)$ of the original situation, it follows that the costs do not change for the players $1, \ldots, j$, whereas a player h after j looses all the gains g_{ih}, $i \leq j$, and therefore suffers from an increase in the costs with $\sum_{i \leq j} g_{ih}$. So, the unique core outcome satisfying DOM has the serious drawback that it does not give any incentive to a player to cooperate with its successors in the queue: not cooperating does not hurt her. To make the point more clear, consider a two player sequencing situation. Of course, nothing happens if the initial order of 1 before 2 is optimal already. So, suppose it is optimal to reverse the initial order and to place 2 in front of 1 generating a decrease g_{12} of the costs. According to the payoff vector $\mu(v)$, this decrease is fully assigned to player 2. Why should player 1 be willing to cooperate by agreeing to take the second position? On the contrary, player 1 has the power to play the noncooperative ultimatum game and to offer the first place in the queue to player 2 if player 2 is willing to give all the gains of this change to player 1, i.e. player 1 is willing to sell his place against a price equal to $\alpha_1 q_2$ (the additional costs of waiting for player 1) plus the gains g_{12} of this trade.

Extending this reasoning we obtain that any player $j < n$ can decide upon whether or not to cooperate with his successors. This can be modelled by any power function $p: \mathcal{N} \to S^n$ satisfying condition (2). From Theorem 5.2 we know that for this power vector the marginal vector $\lambda(v) = m^\ell(v)$ is the unique element in $SC(v, p)$. So, the marginal vector $\lambda(v)$ is economically stable and socially stable with respect to a social structure reflecting the dominance of any player j over its successors in the sequence. The resulting costs induced by $\lambda(v)$ are given by

$$c_h^\ell(N, q, c) = C_h(\rho^0) - \lambda_h(v) = C_h(\rho^0) - \sum_{h<j} g_{hj}, \quad h \in N.$$

Of course, this cost rule does not satisfy DOM, but any player i gets the gains of switching with her successors and therefore is willing to cooperate with her successors.

6 The water distribution problem

In their paper 'Sharing a river' Ambec and Sprumont (2002) consider the problem of the optimal distribution of water to agents located along a river from upstream to downstream. Let $N = \{1, \ldots, n\}$ be the set of agents, numbered successively from upstream to downstream and let $f_i \geq 0$ be the flow of water entering the river between agent $i-1$ and i, $i = 1, \ldots, n$, with f_1 the inflow before the most upstream agent 1. Agent i, $i = 1, \ldots, n$, has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$, where t_i is a monetary compensation to agent i, x_i is the amount of water allocated to agent i, and $b^i: \mathbb{R}_+ \to \mathbb{R}$ a continuous non-decreasing function yielding the benefit $b^i(x_i)$ to agent i of the consumption x_i of water. An allocation is a pair $(x, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ of water distribution and compensation scheme, satisfying

$$\sum_{i=1}^n t_i \leq 0 \quad \text{and} \quad \sum_{i=1}^j x_i \leq \sum_{i=1}^j f_i, \quad j = 1, \ldots, n.$$

The first condition is a budget condition and says that the total amount of compensations is non-positive, i.e. the compensations only redistribute the total welfare. The second condition reflects that any agent can use the water that entered upstream, but that the water inflow downstream of some agent can not be allocated to this agent. Because of the quasi-linearity and the possibility of making money transfers, an allocation is Pareto optimal if and only if the distribution of the water streams maximizes the total benefits, i.e. the water distribution $x^* \in \mathbb{R}^n_+$ solves the following maximization problem:

$$\max_x \sum_{i=1}^n b^i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^j x_i \leq \sum_{i=1}^j f_i, \quad j = 1, \ldots, n. \tag{3}$$

A welfare distribution allocates the total benefits of an optimal water distribution x^* over the agents, i.e. it is a vector $z \in \mathbb{R}^n$ assigning utility z_i to agent i and satisfying $\sum_{i=1}^n z_i = \sum_{i=1}^n b^i(x_i^*)$. Clearly, any welfare distribution can be implemented by the allocation (x, t) with $x_i = x_i^*$ and $t_i = z_i - b^i(x_i^*)$, $i = 1, \ldots, n$.

The problem to find a reasonable welfare distribution can be modelled as a TU-game. Obviously, for any pair of players i, j with $j > i$ it holds that the water inflow entering the river before the upstream agent i can only be allocated to the downstream agent j if all agents between i and j cooperate, otherwise any agent between i and j can take the flow from i to j for its own use. Hence, only coalitions of consecutive agents are admissible. Clearly, for $S = N$, $v(N) = \sum_{i=1}^n b^i(x_i^*)$ with $x^* \in \mathbb{R}^n$ the solution of maximization problem (3). For any connected coalition $S = [i, j] \in \mathcal{L}$, its worth $v(S)$ is given by

$$v(S) = \sum_{h=i}^j b^h(x_h^{*S}),$$

where $x^{*S} = (x_h^{*S})_{h=i}^j$ solves

$$\max_{x_i, \ldots, x_j} \sum_{h=i}^j b^h(x_h) \quad \text{s.t.} \quad \sum_{k=i}^h x_k \leq \sum_{k=i}^h f_k, \quad h = i, \ldots, j. \tag{4}$$

Without loss of generality we normalize the benefit functions by taking $b^i(f_i) = 0$ implying that $v(\{i\}) = b^i(f_i) = 0$, $i = 1, \ldots, n$, so that the values $v(S)$, $|S| \geq 2$ represent the net-gains of cooperating. Again, for an arbitrary coalition $S \in \mathcal{N}$, the value $v(S)$ is equal to the sum of its consecutive parts, i.e.,

$$v(S) = \sum_{T \in P(S)} v(T), \quad S \in \mathcal{N},$$

with $P(S)$ the minimal partition as defined in the previous section. Clearly, the game v is superadditive and hence it follows from Granot and Huberman (1982) that v is permutationally convex for the permutations u and ℓ. Consequently, both marginal vectors $\mu(v)$ and $\lambda(v)$ are core solutions. In case all functions b^i are differentiable with derivative going to infinity as x_i tends to zero, strictly increasing and strictly concave, Ambec and

Sprumont (2002) have even shown that the game is convex and hence the core contains all marginal vectors.

Under the conditions for convexity, Ambec and Sprumont (2002) have shown that the marginal vector corresponding to the permutation u is the unique element in the core of the game satisfying a so-called fairness condition. This condition (quite different from the fairness condition of Myerson to characterize the Shapley value) says that any coalition S gets at most its aspiration level, being the highest utility it can obtain when it may use all the water of all the agents $1, \ldots, \widehat{s}$, where $\widehat{s} = \max\{s \mid s \in S\}$. Clearly, this implies that any coalition $[1, j]$ can get at most $v[1, j]$, $j = 1, \ldots, n$, so that it trivially follows that indeed the marginal vector $m^u(v)$ assigning $\mu_i(v) = v[1, i] - v[1, i-1]$, $i = 1, \ldots, n$, is the unique candidate in the core satisfying the aspiration requirements. For the proof that it indeed satisfies the requirements we refer to Ambec and Sprumont (2002).

As in the sequencing situation again we have that the payoff vector $\mu(v)$ has the property that when a player j does not want to cooperate, the players in front of j, including j itself are not hurt. Like in the sequencing game, this is a very counterintuitive outcome. Although any upstream coalition $[1, j]$ can prevent that coalition $[j+1, n]$ gets more than $v[j+1, n]$ by using all flows $f_1, \ldots, f_j$ by itself, all benefits from cooperating go to the coalition $[j+1, n]$. Again the outcome $\mu(v)$ has the serious drawback that it does not give any incentive to a player j to cooperate with its successors in the queue. Repeating the reasoning once more, again consider a two agent situation. In this case there is no gain of cooperation when in the optimal solution player 1 fully consumes its upstream inflow f_1. However, suppose it is optimal to allocate a part of f_1 to the second agent. According to the outcome $\mu(v)$, agent 1 is just compensated by agent 2 for its loss of utility, i.e. player 1 receives a compensation $t_1 = b^1(f_1) - b^1(x_1^*)$, giving her utility $b^1(f_1) = v(\{1\}) = 0$. So, like in the sequencing game, there is no reason for player 1 to cooperate. However, player 1 has the power to play the noncooperative ultimatum game and to pass the stream $f_1 - x_1^*$ to player 2 if this player is willing to give up all the gains of cooperation, i.e. player 1 is willing to sell this stream against a price (or compensation t_1) equal to $v[1, 2] - v(\{2\}) = v[1, 2]$. Player 2 is indifferent to accepting this offer or not and therefore is willing to accept the offer (or any slightly lower price). Also in this river game we may argue that player 1 is in control of whether or not to cooperate by letting through a part of its water inflow f_1. In general, any player $j < n$ is in control of cooperation with its successors. This can be modelled by a power function $p \colon \mathcal{N} \to \mathbb{R}^n_+ \setminus \{0^n\}$ satisfying for any pair $i, j \in T$ the condition (2). As in the sequencing situation, the collection of coalitions I^ℓ is socially stable and sustains the core outcome $\lambda(v)$. So, $\lambda(v) \in SC(v, p)$ yields an outcome reflecting to a social structure in which any player j dominates its successors in the sense that player j controls the water inflow up to j.

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