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published in
Information and Computation
2022

DOI (link to publisher)
10.1016/j.ic.2021.104756

document version
Publisher's PDF, also known as Version of record

Information and Computation 287 (2022) 104756

Contents lists available at ScienceDirect

Information and Computation

www.elsevier.com/locate/yinco

Non-normal modal logics and conditional logics: Semantic analysis and proof theory

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ARTICLE INFO

Article history:
Available online 5 May 2021

Keywords:
Monotonic modal logic
Conditional logic
Proper display calculi

ABSTRACT

We introduce proper display calculi for basic monotonic modal logic, the conditional logic CK and a number of their axiomatic extensions. These calculi are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal applies the multi-type methodology in the design of proper display calculi, starting from a semantic analysis which motivates syntactic translations from single-type non-normal modal logics to multi-type normal poly-modal logics.

1. Introduction

By *non-normal logics* we understand those propositional logics algebraically captured by varieties of *(general, distributive or Boolean) lattice expansions*, i.e. algebras $\mathbb{A} = (\mathbb{B}, \mathscr{F}^{\mathbb{A}}, \mathscr{G}^{\mathbb{A}})$ such that $\mathbb{B}$ is a (general, distributive, or Boolean) lattice, and $\mathscr{F}^{\mathbb{A}}$ and $\mathscr{G}^{\mathbb{A}}$ are finite, possibly empty families of operations on $\mathbb{B}$ in which, in contrast to the corresponding connectives of logics such as normal modal logic and the Lambek calculus, the requirement that each operation in $\mathscr{F}^{\mathbb{A}}$ be finitely join-preserving or meet-reversing in each coordinate and each operation in $\mathscr{G}^{\mathbb{A}}$ be finitely meet-preserving or join-reversing in each coordinate is omitted. Well known examples of non-normal logics are *monotonic modal logic* [9] and *conditional logic* [57,8], which have been extensively investigated, since they capture key aspects of agents' reasoning, such as the epistemic [64], strategic [62,61], and hypothetical [26,51].

Non-normal logics have been investigated both with model-theoretic tools [38,9] and with proof-theoretic tools [56,58, 29]. Specific to proof theory, the main challenge is to endow non-normal logics with analytic calculi which can be modularly expanded with additional rules so as to uniformly capture wide classes of axiomatic extensions of the basic frameworks, while preserving key properties such as cut elimination. In this paper, which builds and expands on [10], we propose to achieve this goal by applying a method which proved successful in very diverse logical contexts, each of which presented its own specific challenges [21,22,24,32,34,37,33,63]. We illustrate this method by specializing it to the case studies of monotonic modal logic and conditional logic.

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1 The research of the first author is supported by the National Social Science Foundation of China under Grant No. 20&ZD047.
2 The research of the second author is supported by the NWO grant KIVI.2019.001.

https://doi.org/10.1016/j.ic.2021.104756
0890-5401/

Our approach is based on (semantically motivated) translations of the languages of monotonic modal logic and conditional logic into suitable poly-modal signatures in which all connectives are *normal*. Both validity (cf. Propositions 14 and 17 for algebraic and relational semantics, respectively) and derivability (cf. Section 8.2) are preserved by these translations. Thanks to these translations, non-normal connectives can be captured as compositions of normal connectives.

Via these translations, monotonic modal logic and conditional logic can be endowed with *proper (multi-type) display calculi* (see Proposition 18 and Theorem 24),[3] a general format of analytic calculi characterized by a "division of labour" between introduction rules and structural rules (cf. [23, Subsection 2.2] and in particular the so-called *Došen's principle* in [67, Subsection 1.5]). Specifically, in proper display calculi, the rules introducing logical connectives encode the minimal properties of each connective (namely, its arity and tonicity), while the (analytic) structural rules capture the additional properties of the connectives, including their relations with each other. Together with the defining features of analytic structural rules, this division of labour makes it possible to endow large classes of axiomatic extensions of a given base logic with analytic calculi uniformly and in full generality, simply by adding analytic structural rules to the calculus for the base logic, while preserving cut elimination. Finally, if (the translation of) the axioms defining a given axiomatic extension of a logic which is captured by a proper display calculus are of a certain syntactic shape (namely, the *analytic inductive* shape [35, Definition 55]), the analytic structural rules corresponding to these axioms can be algorithmically generated (cf. [35, Proposition 59], [11, Lemma 4.8]).

Our starting point for defining the translations mentioned above is the observation, very well-known e.g. from [9,27,38], that, under the interpretation of the modal connective of monotonic modal logic in monotone neighbourhood frames $\mathbb{F} = (W, \nu)$, the monotonic 'box' operation can be understood as the composition of a *normal* (i.e. finitely join-preserving) semantic diamond $\langle\nu\rangle$ and a *normal* (i.e. finitely meet-preserving) semantic box $[\ni]$. The binary relations R_ν and $R_\ni$ corresponding to these normal operators are not defined on one and the same domain, but span over two domains. Namely, $R_\nu \subseteq W \times \mathscr{P}(W)$ is such that $wR_\nu X$ iff $X \in \nu(w)$, and $R_\ni \subseteq \mathscr{P}(W) \times W$ is such that $XR_\ni w$ iff $w \in X$ (cf. [38, Definition 5.7] and [44,27]).

In the present paper, these relations and their associated *heterogeneous* semantic normal modal operators $\langle\nu\rangle : \mathscr{P}\mathscr{P}(W) \to \mathscr{P}(W)$ and $[\ni] : \mathscr{P}(W) \to \mathscr{P}\mathscr{P}(W)$ become core elements in the definition of the *multi-type*, in this case, two-sorted, (algebraic and relational) semantics interpreting the target languages of the translations, and the observations above are further refined and expanded so as to: (a) introduce a semantic environment of two-sorted Kripke frames (cf. Definition 4) and their heterogeneous algebras (cf. Definition 5) for monotonic modal logic and conditional logic; (b) outline a network of discrete dualities and correspondences among these semantic structures and the algebras and frames for monotonic modal logic and conditional logic (cf. Propositions 2, 10, 14, 17); (c) based on these semantic relationships, introduce multi-type *normal* logics into which the original non-normal logics can be embedded via the semantically motivated translations discussed above (cf. Section 4); (d) retrieve well-known dual characterization results for axiomatic extensions of monotonic modal logic and conditional logics as instances of general algorithmic correspondence theory for normal (multi-type) LE-logics applied to the translated axioms (cf. Section Appendix A); (e) extract analytic structural rules from the computations of the first-order correspondents of the translated axioms, so that, again by general results on proper display calculi [35] (which, as discussed in [5], can be applied also to multi-type logical frameworks) the resulting calculi are sound, complete, conservative, and enjoy cut elimination and subformula property.

Besides allowing for the principled design of proper display calculi for the two non-normal logics considered in the present paper and an infinite class of their axiomatic extensions (cf. Definition 23), the equivalent multi-type presentations of monotonic modal logic and conditional logic and their semantics are interesting both per se, and also because they introduce new possibilities to the conceptual understanding of non-normal logics. Indeed, firstly, by making it possible to consider states and neighbourhoods as entities of different types, non-normal logics can be regarded as logics describing and reasoning about the specific behaviour of each type as well as their interaction. For instance, if states are interpreted as 'states of affairs' and neighbourhoods as 'pieces of evidence', then by translating the formula $\nabla\varphi$ as $\langle\nu\rangle[\ni]\varphi$ we access a formal language in which it is possible to unpack the meaning of e.g. φ being 'definitely true' at a given state, and reformulate it in terms of the availability of a piece of evidence accessible at that state and supporting the truth of φ. Secondly, the translation we have just discussed is not the only one suggested by the multi-type reformulation of neighbourhood semantics; in fact, another is possible which translates $\nabla\varphi$ as $[\nu^c]\langle\not\ni\rangle\varphi$, and which, when used in combination with the first, makes it possible to obtain analytic translations for a large class of axioms (cf. Remark 2). Besides being technically useful, this translation has the potential to support different interpretations of the two types and their interaction. Thirdly, the multi-type reformulation of non-normal logics facilitates establishing connections with other areas of investigation in logic and neighbouring fields in which logical frameworks connecting entities of different types have already been studied and exploited. One such area is structural control, which has given rise to a rich literature both in substructural logic [30,36,42,16,65] and in formal linguistics [53,45,54,55,39,3,66].

[3] It is perhaps worth stressing that, by their definition, the (interpretation of) non-normal connectives lack the minimum order-theoretic properties necessary for any non-normal logic to be *properly displayable*, i.e. amenable to be equivalently captured by a proper display calculus, in its original presentation. A characterization of properly displayable logics is given in [35]. This explains why translations are necessary components of our proposed method. This situation is common to all the logical frameworks to which this methodology was successfully applied, none of which is properly displayable in its original presentation for different reasons.

Related work. In what follows, without claiming to be exhaustive, we briefly review the literature on proof systems for non-normal logics developed in the context of labelled sequent calculi [29,15,56], sequent calculi [48,40,41,60,49], and nested sequent calculi [1,50].

In [29], labelled sequent systems are introduced for the classical cube of non-normal modal logics, i.e. the basic non-monotonic modal logic E (also called congruential logic) and its axiomatic extensions with the axioms N, C, or the rule M and their combinations. The approach captures all logics in the cube by extending the basic system via so-called *systems of rules* still preserving cut elimination. The approach of [29] is similar to the approach followed in the present paper both because it makes use of a poly-modal translation of the original signature, and because a preliminary analysis (i.e. a syntactic characterization) of the first order correspondents of axiomatic extensions is key to the generation of equivalent rules. In [15], analytic and modular labelled sequent systems are introduced for the same non-normal modal logics treated in [29]. While no syntactic translation of the formulas of the original language intervenes in these calculi, the distinction between worlds and neighbourhoods is encoded in the label language. The approach of [15] relies on the methodology introduced in [56], the distinctive feature of which is the introduction of so-called *bi-neighbourhood semantics*, i.e. each world is associated with (sets of) pairs of neighbourhoods rather than single neighbourhoods (this is also reflected in the richer label language). As soon as the logic satisfies the rule M, the bi-neighbourhood semantics collapses onto the standard one.

In [48], complete sequent systems with cut elimination are introduced for the classical cube of non-normal modal logics, which are used to prove finite model property and provide bounds on the cardinality of countermodels. In [40], (resp. [41]), sequent calculi are introduced for monotonic modal logic (resp. congruential logic E with or without axiom N) and its axiomatic extensions with all combinations of axioms D, T, 4, B and 5. In order to capture the axiomatic extensions, the rules introducing the modal connectives differ from one calculus to another. As a consequence, cut elimination, which is proved for most of these calculi by a standard Gentzen argument, is not shown uniformly for all calculi, but has to be proved separately for each of them.

In [60], cut-free sequent calculi are introduced for the minimal conditional logic extended with axioms CEM, MP and their combination, and for the minimal conditional logic extended with axiom ID. Cut elimination is shown via a meta-theorem called *generic cut-elimination*, i.e. an argument that holds for an entire class of sequent calculi satisfying certain local conditions. In [49], the generic cut elimination method, also referred to as *cut elimination by saturation*, is extended to logics over an intuitionistic propositional base and to logics introduced by axioms of arbitrary modal rank, and in particular, to a cut-free sequent calculus introduced for Lewis' conditional logic VA.

In [1], cut-free nested sequent calculi are introduced for the basic conditional logic CK and its axiomatic extensions with ID, MP and CEM and their combinations, with the exception of CK + MP + CEM (+ID). These calculi are all *internal*, i.e. every sequent can be translated into a formula of the original language of CK. In [50], (linear) nested sequent calculi are introduced for a large class of logics which includes the classical non-normal modal cube and various extensions with axioms from P, D, T, 4, and 5.

Structure of the paper. In Section 2, we collect well-known definitions and facts about monotonic modal logic and conditional logic, their algebraic and state-based semantics, and the connection between the two. In Section 3, we introduce the multi-type environment (both in the form of heterogeneous algebras and of multi-type Kripke frames) which will provide the semantic justification for the two-sorted modal logics introduced in Section 4, as well as for the syntactic translation of the original languages of monotonic modal logic and conditional logic into suitable (multi-type) normal modal languages. In Section 5 we specify the definitions of inductive and analytic inductive inequalities/sequents for the multi-type language of monotonic modal logic and conditional logic. In Section 6, the theory of unified correspondence is applied to this two-sorted environment to establish a Sahlqvist-type correspondence framework for monotonic modal logic and conditional logic which encompasses and extends the extant correspondence-theoretic results for these logics. In Section 7, proper (multi-type) display calculi are introduced for the basic two sorted normal modal languages and for some of their best known extensions. The main properties of these calculi are discussed in Section 8. Conclusions and further directions are discussed in Section 9.

2. Preliminaries

Notation. Throughout the paper, the superscript $(\cdot)^c$ denotes the relative complement of the subset of a given set. When the given set is a singleton $\{x\}$, we will write x^c instead of $\{x\}^c$. For any binary relation $R \subseteq S \times T$, let $R^{-1} \subseteq T \times S$ be the *converse relation* of R, i.e. $tR^{-1}s$ iff sRt. For any $S' \subseteq S$ and $T' \subseteq T$, we let $R[S'] := \{t \in T \mid (s,t) \in R \text{ for some } s \in S'\}$ and $R^{-1}[T'] := \{s \in S \mid (s,t) \in R \text{ for some } t \in T'\}$. As usual, we write $R[s]$ and $R^{-1}[t]$ in place of $R[\{s\}]$ and $R^{-1}[\{t\}]$, respectively. For any ternary relation $R \subseteq S \times T \times U$ and subsets $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$, we also let

- $R^{(0)}[T', U'] = \{s \in S \mid \exists t \exists u (R(s,t,u) \ \& \ t \in T' \ \& \ u \in U')\}$,
- $R^{(1)}[S', U'] = \{t \in T \mid \exists s \exists u (R(s,t,u) \ \& \ s \in S' \ \& \ u \in U')\}$,
- $R^{(2)}[S', T'] = \{u \in U \mid \exists s \exists t (R(s,t,u) \ \& \ s \in S' \ \& \ t \in T')\}$.

Any binary relation $R \subseteq S \times T$ gives rise to the *modal operators* $\langle R\rangle, [R], [R\rangle, \langle R] : \mathscr{P}(T) \to \mathscr{P}(S)$ s.t. for any $T' \subseteq T$

- $\langle R\rangle T' := R^{-1}[T'] = \{s \in S \mid \exists t(sRt \ \& \ t \in T')\}$;
- $[R]T' := (R^{-1}[T'^c])^c = \{s \in S \mid \forall t(sRt \ \Rightarrow \ t \in T')\}$;
- $[R\rangle T' := (R^{-1}[T'])^c = \{s \in S \mid \forall t(sRt \ \Rightarrow \ t \notin T')\}$;
- $\langle R] T' := R^{-1}[T'^c] = \{s \in S \mid \exists t(sRt \ \& \ t \notin T')\}$.

By construction, these modal operators are normal. Specifically, $\langle R\rangle$ is completely join-preserving, $[R]$ is completely meet-preserving, $[R\rangle$ is completely join-reversing and $\langle R]$ is completely meet-reversing.[4] Hence, their adjoint maps exist and coincide with $[R^{-1}]\langle R^{-1}\rangle, [R^{-1}\rangle, \langle R^{-1}] : \mathscr{P}(S) \to \mathscr{P}(T)$, respectively. That is, for any $T' \subseteq T$ and $S' \subseteq S$,

$$\begin{array}{lll} \langle R\rangle T' \subseteq S' & \text{iff} & T' \subseteq [R^{-1}]S', \\ S' \subseteq [R]T' & \text{iff} & \langle R^{-1}\rangle S' \subseteq T', \\ S' \subseteq [R\rangle T' & \text{iff} & T' \subseteq [R^{-1}\rangle S' \\ \langle R]T' \subseteq S' & \text{iff} & \langle R^{-1}]S' \subseteq T'. \end{array}$$

Any ternary relation $R \subseteq S \times T \times U$ gives rise to binary modal operators

$$\rhd_R : \mathscr{P}(T) \times \mathscr{P}(U) \to \mathscr{P}(S) \quad \blacktriangle_R : \mathscr{P}(T) \times \mathscr{P}(S) \to \mathscr{P}(U) \quad \blacktriangleright_R : \mathscr{P}(S) \times \mathscr{P}(U) \to \mathscr{P}(T)$$

s.t. for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$,

- $T' \rhd_R U' := (R^{(0)}[T', U'^c])^c = \{s \in S \mid \ \forall t \forall u(R(s,t,u) \ \& \ t \in T' \Rightarrow u \in U')\}$;
- $T' \blacktriangle_R S' := R^{(2)}[T', S'] = \{u \in U \mid \ \exists t \exists s(R(s,t,u) \ \& \ t \in T' \ \& \ s \in S')\}$;
- $S' \blacktriangleright_R U' := (R^{(1)}[S', U'^c])^c = \{t \in T \mid \ \forall s \forall u(R(s,t,u) \ \& \ s \in S' \Rightarrow u \in U')\}$.

The stipulations above guarantee that these modal operators are normal. In particular, $\rhd_R$ and $\blacktriangleright_R$ are completely join-reversing in their first coordinate and completely meet-preserving in their second coordinate, and $\blacktriangle_R$ is completely join-preserving in both coordinates. These three maps are residual to each other, i.e. for any $S' \subseteq S$, $T' \subseteq T$, and $U' \subseteq U$,

$$S' \subseteq T' \rhd_R U' \quad \text{iff} \quad T' \blacktriangle_R S' \subseteq U' \quad \text{iff} \quad T' \subseteq S' \blacktriangleright_R U'.$$

2.1. Basic monotonic modal logic and conditional logic

In this section, we collect the necessary preliminaries on the logical frameworks considered in this paper, and introduce the notation that will be used throughout the paper. An overview of monotonic modal logic and conditional logic can be found in [9].

Syntax. For a countable set of propositional variables Prop, the languages $\mathscr{L}_\nabla$ and $\mathscr{L}_>$ of monotonic modal logic and conditional logic over Prop are defined as follows:

$$\mathscr{L}_\nabla \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \nabla\varphi \qquad \mathscr{L}_> \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi > \varphi.$$

The connectives $\top, \wedge, \vee, \to$ and $\leftrightarrow$ are defined as usual.

The *basic monotonic modal logic* $\mathbf{L}_\nabla$ (resp. *basic conditional logic* $\mathbf{L}_>$) is a set of $\mathscr{L}_\nabla$-formulas (resp. $\mathscr{L}_>$-formulas) containing the axioms of classical propositional logic and closed under modus ponens, uniform substitution and the following rule(s) M (resp. $RCEA$ and RCK_n for all $n \geq 0$):

$$\text{M}\ \frac{\varphi \to \psi}{\nabla\varphi \to \nabla\psi} \qquad \text{RCEA}\ \frac{\varphi \leftrightarrow \psi}{(\varphi > \chi) \leftrightarrow (\psi > \chi)} \qquad \text{RCK}_n\ \frac{\varphi_1 \wedge \ldots \wedge \varphi_n \to \psi}{(\chi > \varphi_1) \wedge \ldots \wedge (\chi > \varphi_n) \to (\chi > \psi)}$$

Algebraic semantics. A *monotone Boolean algebra expansion*, abbreviated as *m-algebra* (resp. *conditional algebra*, abbreviated as *c-algebra*) is a pair $\mathbb{A} = (\mathbb{B}, \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathbb{B}, >^{\mathbb{A}})$) s.t. $\mathbb{B}$ is a Boolean algebra and $\nabla^{\mathbb{A}}$ is a unary monotone operation on $\mathbb{B}$ (resp. $>^{\mathbb{A}}$ is a binary operation on $\mathbb{B}$ which is finitely meet-preserving in its second coordinate). Such an m-algebra (resp. c-algebra) is *perfect* if $\mathbb{B}$ is a complete and atomic Boolean algebra (and, for c-algebras, $>^{\mathbb{A}}$ is completely meet-preserving in its second coordinate). Hence, the underlying Boolean algebra of any perfect m-algebra (resp. c-algebra) can be identified with the powerset algebra $\mathscr{P}(W)$ for some set W.

Interpretation of formulas in algebras under assignments $h : \mathscr{L}_\nabla \to \mathbb{A}$ (resp. $h : \mathscr{L}_> \to \mathbb{A}$) and validity of formulas in algebras (in symbols: $\mathbb{A} \models \varphi$) are defined as usual. By a routine Lindenbaum-Tarski construction one can show that $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) is sound and complete w.r.t. the class of m-algebras V_m (resp. c-algebras V_c).

[4] That is, $\langle R\rangle \bigcup_{i\in I} T_i = \bigcup_{i\in I} \langle R\rangle T_i$, $[R]\bigcap_{i\in I} T_i = \bigcap_{i\in I}[R]T_i$, $[R\rangle \bigcup_{i\in I} T_i = \bigcap_{i\in I}[R\rangle T_i$ and $\langle R]\bigcap_{i\in I} T_i = \bigcup_{i\in I}\langle R]T_i$. For a general overview of normal logics and the order-theoretic properties characterizing their algebraic semantics, see e.g. [13].

Canonical extensions. The *canonical extension* of an m-algebra (resp. c-algebra) $\mathbb{A}$ is $\mathbb{A}^\delta := (\mathbb{B}^\delta, \nabla^\sigma)$ (resp. $\mathbb{A}^\delta := (\mathbb{B}^\delta, >^\pi)$), where $\mathbb{B}^\delta \cong \mathscr{P}(Ult(\mathbb{B}))$, with $Ult(\mathbb{B})$ denoting the set of the ultrafilters of $\mathbb{B}$, is the canonical extension of $\mathbb{B}$ [43], and ∇^σ (resp. $>^\pi$) is the σ-extension of $\nabla^{\mathbb{A}}$ (resp. the π-extension of $>^{\mathbb{A}}$). Let us recall that for all $u, u_1, u_2 \in \mathbb{B}^\delta$,

$$\nabla^\sigma u := \bigvee\{\bigwedge\{\nabla a \mid a \in \mathbb{B} \text{ and } k \leq a\} \mid k \in K(\mathbb{B}^\delta) \text{ and } k \leq u\},$$

$$u_1 >^\pi u_2 := \bigwedge\{\bigvee\{a_1 > a_2 \mid a_i \in \mathbb{B} \text{ and } o_i \leq a_i \leq k_i\} \mid k_i \in K(\mathbb{B}^\delta), o_i \in O(\mathbb{B}^\delta) \text{ and } k_i \leq u_i \leq o_i\},$$

where $K(\mathbb{B}^\delta)$ and $O(\mathbb{B}^\delta)$ respectively denote the join-closure and the meet-closure of $\mathbb{B}$ in $\mathbb{B}^\delta$ under the canonical embedding, mapping each $a \in \mathbb{B}$ to $\{U \in Ult(\mathbb{B}) \mid a \in U\}$.

By definition and general results on canonical extensions of maps (cf. [28]), the canonical extension of an m-algebra (resp. c-algebra) as above is a perfect m-algebra (resp. c-algebra).

Frames and models. A *neighbourhood frame*, abbreviated as *n-frame* (resp. *conditional frame*, abbreviated as *c-frame*) is a pair $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) s.t. W is a non-empty set and $\nu : W \to \mathscr{P}(\mathscr{P}(W))$ is a *neighbourhood function* ($f : W \times \mathscr{P}(W) \to \mathscr{P}(W)$ is a *selection function*). In the remainder of the paper, even if it is not explicitly indicated, we will assume that n-frames are *monotone*, i.e. s.t. for every $w \in W$, if $X \in \nu(w)$ and $X \subseteq Y$, then $Y \in \nu(w)$. For any n-frame (resp. c-frame) $\mathbb{F}$, the *complex algebra* of $\mathbb{F}$ is $\mathbb{F}^* := (\mathscr{P}(W), \nabla^{\mathbb{F}^*})$ (resp. $\mathbb{F}^* := (\mathscr{P}(W), >^{\mathbb{F}^*})$) s.t. for all $X, Y \in \mathscr{P}(W)$,

$$\nabla^{\mathbb{F}^*} X := \{w \mid X \in \nu(w)\} \qquad X >^{\mathbb{F}^*} Y := \{w \mid f(w, X) \subseteq Y\}.$$

Proposition 1. *If $\mathbb{F}$ is an n-frame (resp. a c-frame), then $\mathbb{F}^*$ is a perfect m-algebra (resp. c-algebra).*

Proof. Let $\mathbb{F} = (W, \nu)$ be an n-frame. Recall that, by definition, $\nu(w)$ is an upward-closed collection of subsets of W. To show that $\mathbb{F}^*$ is a perfect m-algebra, it is enough to show that $\nabla^{\mathbb{F}^*}$ is monotone. Let $w \in W$ and $X \subseteq Y \subseteq W$. Since $\nu(w)$ is upward-closed, $X \in \nu(w)$ implies that $Y \in \nu(w)$. Hence, $\nabla^{\mathbb{F}^*} X = \{w \mid X \in \nu(w)\} \subseteq \{w \mid Y \in \nu(w)\} = \nabla^{\mathbb{F}^*} Y$.

Let $\mathbb{F} = (W, f)$ be a c-frame. To show that $\mathbb{F}^*$ is a perfect c-algebra, it is enough to show that $>^{\mathbb{F}^*}$ is completely meet-preserving in its second coordinate. For any $X \subseteq W$,

$$X >^{\mathbb{F}^*} \top^{\mathbb{F}^*} = X >^{\mathbb{F}^*} W = \{w \mid f(w, X) \subseteq W\} = W = \top^{\mathbb{F}^*},$$

and for every $\mathscr{X} \subseteq \mathscr{P}(W)$,

$$\begin{aligned} X >^{\mathbb{F}^*} \bigcap \mathscr{X} &= \{w \in W \mid f(w, X) \subseteq \bigcap \mathscr{X}\} \\ &= \{w \in W \mid f(w, X) \subseteq Y \text{ for any } Y \in \mathscr{X}\} \\ &= \bigcap\{(X >^{\mathbb{F}^*} Y) \mid Y \in \mathscr{X}\}. \quad \square \end{aligned}$$

Models are pairs $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is a frame and $V : \mathscr{L} \to \mathbb{F}^*$ is a homomorphism of the appropriate type. Hence, the truth of formulas at states in models is defined as $\mathbb{M}, w \Vdash \varphi$ iff $w \in V(\varphi)$, and unravelling this stipulation for ∇- and $>$-formulas, we get:

$$\mathbb{M}, w \Vdash \nabla\varphi \quad \text{iff} \quad V(\varphi) \in \nu(w) \qquad \mathbb{M}, w \Vdash \varphi > \psi \quad \text{iff} \quad f(w, V(\varphi)) \subseteq V(\psi).$$

Local validity (notation: $\mathbb{F}, w \Vdash \varphi$) is defined as local satisfaction for every valuation V. Global satisfaction (notation: $\mathbb{M} \Vdash \varphi$) and frame validity (notation: $\mathbb{F} \Vdash \varphi$) are defined in the usual way as local satisfaction/validity at every state. Thus, by definition, $\mathbb{F} \Vdash \varphi$ iff $\mathbb{F}^* \models \varphi$, from which the soundness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) w.r.t. the corresponding class of frames immediately follows from the algebraic soundness. Completeness follows from algebraic completeness, by observing that (a) the canonical extension of any algebra refuting φ will also refute φ; (b) canonical extensions are perfect algebras; (c) perfect m-algebras (resp. c-algebras) can be associated with n-frames (resp. c-frames) as follows: for any $\mathbb{A} = (\mathscr{P}(W), \nabla^{\mathbb{A}})$ (resp. $\mathbb{A} = (\mathscr{P}(W), >^{\mathbb{A}})$) let $\mathbb{A}_* := (W, \nu_{\nabla^{\mathbb{A}}})$ (resp. $\mathbb{A}_* := (W, f_{>^{\mathbb{A}}})$) s.t. for all $w \in W$ and $X \subseteq W$,

$$\nu_{\nabla^{\mathbb{A}}}(w) := \{X \subseteq W \mid w \in \nabla^{\mathbb{A}} X\} \qquad f_{>^{\mathbb{A}}}(w, X) := \bigcap\{Y \subseteq W \mid w \in X >^{\mathbb{A}} Y\}.$$

That $\mathbb{A}_*$ is a monotone n-frame can be proved as follows: if $X \in \nu_\nabla(w)$ and $X \subseteq Y$, then the monotonicity of $\nabla^{\mathbb{A}}$ implies that $\nabla^{\mathbb{A}} X \subseteq \nabla^{\mathbb{A}} Y$ and hence $Y \in \nu_{\nabla^{\mathbb{A}}}(w)$, as required.

Let $\varphi \in \mathscr{L}_\nabla$ (resp. $\varphi \in \mathscr{L}_>$). It can be shown by a straightforward induction on φ that $w \in V(\varphi)$ iff $(\mathbb{A}_*, V), w \Vdash \varphi$ for any perfect algebra $\mathbb{A}$ and assignment V. Then, $\mathbb{A} \models \varphi$ iff $\mathbb{A}_* \Vdash \varphi$. This completes the argument deriving the frame completeness of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) from its algebraic completeness.

Proposition 2. *If $\mathbb{A}$ is a perfect m-algebra (resp. c-algebra) and $\mathbb{F}$ is an n-frame (resp. c-frame), then $(\mathbb{F}^*)_* \cong \mathbb{F}$ and $(\mathbb{A}_*)^* \cong \mathbb{A}$.*

Proof. Let $\mathbb{F} = (W, \nu)$ be an n-frame. By definition, $(\mathbb{F}^*)_* = (W, \nu_{\nabla^{\mathbb{F}^*}})$, where, for every $w \in W$,

$$\begin{aligned}\nu_{\nabla^{\mathbb{F}^*}}(w) &= \{X \subseteq W \mid w \in \nabla^{\mathbb{F}^*} X\}\\ &= \{X \subseteq W \mid w \in \{u \mid X \in \nu(u)\}\}\\ &= \{X \subseteq W \mid X \in \nu(w)\}\\ &= \nu(w),\end{aligned}$$

which shows that $(\mathbb{F}^*)_* = \mathbb{F}$, as required. Let $\mathbb{F} = (W, f)$ be a c-frame. By definition, $(\mathbb{F}^*)_* = (W, f_{>^{\mathbb{F}^*}})$, where, for every $w \in W$ and $X \subseteq W$,

$$\begin{aligned}f_{>^{\mathbb{F}^*}}(w, X) &= \bigcap\{Y \subseteq W \mid w \in\ X >^{\mathbb{F}^*} Y\}\\ &= \bigcap\{Y \subseteq W \mid w \in\ \{u \in W \mid f(u, X) \subseteq Y\}\}\\ &= \bigcap\{Y \subseteq W \mid f(w, X) \subseteq Y\}\\ &= f(w, X),\end{aligned}$$

which shows that $(\mathbb{F}^*)_* = \mathbb{F}$, as required. Let $\mathbb{A} = (\mathscr{P}(W), \nabla^{\mathbb{A}})$ be a perfect m-algebra (up to isomorphism). Then $(\mathbb{A}_*)^* = (\mathscr{P}(W), \nabla^{(\mathbb{A}_*)^*})$, where for every $X \subseteq W$,

$$\begin{aligned}\nabla^{(\mathbb{A}_*)^*} X &= \{w \in W \mid X \in \nu_{\nabla^{\mathbb{A}}}(w)\}\\ &= \{w \in W \mid X \in \{Y \subseteq W \mid w \in \nabla^{\mathbb{A}} Y\}\}\\ &= \{w \in W \mid w \in \nabla^{\mathbb{A}} X\}\\ &= \nabla^{\mathbb{A}} X,\end{aligned}$$

which shows that $(\mathbb{A}_*)^* \cong \mathbb{A}$, as required. Let $\mathbb{A} = (\mathscr{P}(W), >^{\mathbb{A}})$ be a perfect c-algebra (up to isomorphism). Then $(\mathbb{A}_*)^* = (\mathscr{P}(W), >^{(\mathbb{A}_*)^*})$, where for all $X, Y \subseteq W$,

$$\begin{aligned}X >^{(\mathbb{A}_*)^*} Y &= \{w \in W \mid f_{>^{\mathbb{A}}}(w, X) \subseteq Y\}\\ &= \{w \in W \mid \bigcap\{Z \subseteq W \mid w \in\ X >^{\mathbb{A}} Z\} \subseteq Y\}\\ &= X >^{\mathbb{A}} Y.\end{aligned}$$

Let us show the last equality. If $w \in\ X >^{\mathbb{A}} Y$, then $Y \in \{Z \subseteq W \mid w \in\ X >^{\mathbb{A}} Z\}$, and hence $\bigcap\{Z \subseteq W \mid w \in\ X >^{\mathbb{A}} Z\} \subseteq Y$. Conversely, let $w \in W$ be s.t. $\bigcap\{Z \subseteq W \mid w \in\ X >^{\mathbb{A}} Z\} \subseteq Y$. Since $>^{\mathbb{A}}$ is completely meet-preserving in the second coordinate, this implies that

$$w \in \bigcap\{X >^{\mathbb{A}} Z \mid Z \subseteq W \text{ and } w \in\ X >^{\mathbb{A}} Z\} = X >^{\mathbb{A}} \bigcap\{Z \subseteq W \mid w \in X >^{\mathbb{A}} Z\} \subseteq X >^{\mathbb{A}} Y,$$

as required. This completes the proof that $(\mathbb{A}_*)^* \cong \mathbb{A}$. □

Axiomatic extensions. A *monotonic modal logic* (resp. a *conditional logic*) is any extension of $\mathbf{L}_\nabla$ (resp. $\mathbf{L}_>$) with $\mathscr{L}_\nabla$-axioms (resp. $\mathscr{L}_>$-axioms). The correspondence results collected in the theorem below mostly concern well known axioms and are well known from the literature (cf. [38, Theorem 5.1] [58]).[5] The axiom CN below is inspired by the Connex axiom of V-logics presented in [51].

Theorem 3. *For every n-frame (resp. c-frame)* $\mathbb{F}$,

[5] The scope of applicability of the methodology presented in this paper is specified in Definition 23, Example 1 and Theorem 24.

N	$\mathbb{F} \Vdash \nabla\top$	*iff*	$\mathbb{F} \models \forall w[W \in \nu(w)]$
P	$\mathbb{F} \Vdash \neg\nabla\bot$	*iff*	$\mathbb{F} \models \forall w[\varnothing \notin \nu(w)]$
C	$\mathbb{F} \Vdash \nabla p \wedge \nabla q \to \nabla(p \wedge q)$	*iff*	$\mathbb{F} \models \forall w\forall X\forall Y[(X \in \nu(w)\ \&\ Y \in \nu(w)) \Rightarrow X \cap Y \in \nu(w)]$
T	$\mathbb{F} \Vdash \nabla p \to p$	*iff*	$\mathbb{F} \models \forall w\forall X[X \in \nu(w) \Rightarrow w \in X]$
4	$\mathbb{F} \Vdash \nabla\nabla p \to \nabla p$	*iff*	$\mathbb{F} \models \forall w\forall Y X[(X \in \nu(w)\ \&\ \forall x(x \in X \Rightarrow Y \in \nu(x))) \Rightarrow Y \in \nu(w)]$
4'	$\mathbb{F} \Vdash \nabla p \to \nabla\nabla p$	*iff*	$\mathbb{F} \models \forall w\forall X[X \in \nu(w) \Rightarrow \{y \mid X \in \nu(y)\} \in \nu(w)]$
5	$\mathbb{F} \Vdash \neg\nabla\neg p \to \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w\forall X[X \notin \nu(w) \Rightarrow \{y \mid X \in \nu(y)\}^c \in \nu(w)]$
B	$\mathbb{F} \Vdash p \to \nabla\neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w\forall X[w \in X \Rightarrow \{y \mid X^c \in \nu(y)\}^c \in \nu(w)]$
D	$\mathbb{F} \Vdash \nabla p \to \neg\nabla\neg p$	*iff*	$\mathbb{F} \models \forall w\forall X[X \in \nu(w) \Rightarrow X^c \notin \nu(w)]$
CS	$\mathbb{F} \Vdash (p \wedge q) \to (p > q)$	*iff*	$\mathbb{F} \models \forall x\forall Z[x \in Z \Rightarrow f(x, Z) \subseteq \{x\}]$
CEM	$\mathbb{F} \Vdash (p > q) \vee (p > \neg q)$	*iff*	$\mathbb{F} \models \forall X\forall y[\lvert f(y, X)\rvert \leq 1]$
ID	$\mathbb{F} \Vdash p > p$	*iff*	$\mathbb{F} \models \forall x\forall Z[f(x, Z) \subseteq Z]$.
CN	$\mathbb{F} \Vdash (p > q) \vee (q > p)$	*iff*	$\mathbb{F} \models \forall X\forall Y\forall z[(f(z, X) \subseteq Y)$ *or* $(f(z, Y) \subseteq X)]$.

In the following section we introduce a semantic environment thanks to which the correspondence results above can be obtained as instances of a suitable multi-type version of unified correspondence theory [12,13]. This environment also motivates the introduction of proper display calculi for the logics axiomatised by those axioms—among the ones listed above—the translation of which is analytic inductive (cf. Section 4).

3. Semantic analysis

3.1. Two-sorted Kripke frames and their discrete duality

Structures similar to those below are considered implicitly in [38], and explicitly in [25].

Definition 4. A *two-sorted n-frame* (resp. *c-frame*) is a structure $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{K} := (X, Y, R_\ni, R_{\not\ni}, T_f)$) such that X and Y are nonempty sets, $R_\ni, R_{\not\ni} \subseteq Y \times X$ and $R_\nu, R_{\nu^c} \subseteq X \times Y$ and $T_f \subseteq X \times Y \times X$. Such an n-frame is *supported* if for every $D \subseteq X$,

$$R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c. \tag{1}$$

For any two-sorted n-frame (resp. c-frame) $\mathbb{K}$, the *complex algebra* of $\mathbb{K}$ is

$$\mathbb{K}^+ := (\mathscr{P}(X), \mathscr{P}(Y), [\ni]^{\mathbb{K}^+}, \langle\not\ni\rangle^{\mathbb{K}^+}, \langle\nu\rangle^{\mathbb{K}^+}, [\nu^c]^{\mathbb{K}^+}) \text{ (resp. } \mathbb{K}^+ := (\mathscr{P}(X), \mathscr{P}(Y), [\ni]^{\mathbb{K}^+}, [\not\ni\rangle^{\mathbb{K}^+}, \rhd^{\mathbb{K}^+})\text{), s.t.}$$

$$\begin{aligned} \langle\nu\rangle^{\mathbb{K}^+} : \mathscr{P}(Y) &\to \mathscr{P}(X) & [\ni]^{\mathbb{K}^+} : \mathscr{P}(X) &\to \mathscr{P}(Y) & \langle\not\ni\rangle^{\mathbb{K}^+} : \mathscr{P}(X) &\to \mathscr{P}(Y) \\ U &\mapsto R_\nu^{-1}[U] & D &\mapsto (R_\ni^{-1}[D^c])^c & D &\mapsto R_{\not\ni}^{-1}[D] \end{aligned}$$

$$\begin{aligned} [\nu^c]^{\mathbb{K}^+} : \mathscr{P}(Y) &\to \mathscr{P}(X) & [\not\ni\rangle^{\mathbb{K}^+} : \mathscr{P}(X) &\to \mathscr{P}(Y) & \rhd^{\mathbb{K}^+} : \mathscr{P}(Y) \times \mathscr{P}(X) &\to \mathscr{P}(X) \\ U &\mapsto (R_{\nu^c}^{-1}[U^c])^c & D &\mapsto (R_{\not\ni}^{-1}[D])^c & (U, D) &\mapsto (T_f^{(0)}[U, D^c])^c \end{aligned}$$

The adjoints and residuals of the maps above (cf. Section 2) are defined as follows:

$$\begin{aligned} [\rotatebox[origin=c]{180}{ν}]^{\mathbb{K}^+} : \mathscr{P}(X) &\to \mathscr{P}(Y) & \langle\in\rangle^{\mathbb{K}^+} : \mathscr{P}(Y) &\to \mathscr{P}(X) & [\notin]^{\mathbb{K}^+} : \mathscr{P}(Y) &\to \mathscr{P}(X) \\ D &\mapsto (R_\nu[D^c])^c & U &\mapsto R_\ni[U] & U &\mapsto (R_{\not\ni}[U^c])^c \end{aligned}$$

$$\begin{aligned} \langle\rotatebox[origin=c]{180}{ν}^c\rangle^{\mathbb{K}^+} : \mathscr{P}(X) &\to \mathscr{P}(Y) & [\notin\rangle^{\mathbb{K}^+} : \mathscr{P}(Y) &\to \mathscr{P}(X) & \blacktriangleright^{\mathbb{K}^+} : \mathscr{P}(X) \times \mathscr{P}(X) &\to \mathscr{P}(Y) \\ D &\mapsto R_{\nu^c}[D] & U &\mapsto (R_{\not\ni}[U])^c & (C, D) &\mapsto (T_f^{(1)}[C, D^c])^c \end{aligned}$$

$$\begin{aligned} \blacktriangle^{\mathbb{K}^+} : \mathscr{P}(Y) \times \mathscr{P}(X) &\to \mathscr{P}(X) \\ (U, D) &\mapsto T_f^{(2)}[U, D] \end{aligned}$$

Complex algebras of two-sorted frames can be recognized as perfect heterogeneous algebras (cf. [6]) of the following kind:

Definition 5. A *heterogeneous m-algebra* (resp. *c-algebra*) is a structure

$$\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}) \qquad \text{(resp. } \mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})\text{)}$$

such that $\mathbb{A}$ and $\mathbb{B}$ are Boolean algebras, $\langle\nu\rangle^{\mathbb{H}}, [\nu^c] : \mathbb{B} \to \mathbb{A}$ are finitely join-preserving and finitely meet-preserving respectively, $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}} : \mathbb{A} \to \mathbb{B}$ are finitely meet-preserving, finitely join-reversing, and finitely join-preserving respectively, and $\rhd^{\mathbb{H}} : \mathbb{B} \times \mathbb{A} \to \mathbb{A}$ is finitely join-reversing in its first coordinate and finitely meet-preserving in its second coordinate. Such an $\mathbb{H}$ is *complete* if $\mathbb{A}$ and $\mathbb{B}$ are complete Boolean algebras and the operations above enjoy the complete versions of the finite preservation properties indicated above, and is *perfect* if it is complete and $\mathbb{A}$ and $\mathbb{B}$ are perfect. The *canonical extension* of a heterogeneous m-algebra (resp. c-algebra) $\mathbb{H}$ is $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, \langle\not\ni\rangle^{\mathbb{H}^\delta}, \langle\nu\rangle^{\mathbb{H}^\delta}, [\nu^c]^{\mathbb{H}^\delta})$ (resp. $\mathbb{H}^\delta := (\mathbb{A}^\delta, \mathbb{B}^\delta, [\ni]^{\mathbb{H}^\delta}, [\not\ni\rangle^{\mathbb{H}^\delta}, \rhd^{\mathbb{H}^\delta}))$, where $\mathbb{A}^\delta$ and $\mathbb{B}^\delta$ are the canonical extensions of $\mathbb{A}$ and $\mathbb{B}$ respectively [43], moreover $[\ni]^{\mathbb{H}^\delta}$, $[\not\ni\rangle^{\mathbb{H}^\delta}$, $[\nu^c]^{\mathbb{H}^\delta}$, $\rhd^{\mathbb{H}^\delta}$ are the π-extensions of $[\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}}, \rhd^{\mathbb{H}}$ respectively, and $\langle\nu\rangle^{\mathbb{H}^\delta}$, $\langle\not\ni\rangle^{\mathbb{H}^\delta}$ are the σ-extensions of $\langle\nu\rangle^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}$ respectively.

Definition 6. A heterogeneous m-algebra $\mathbb{H} := (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is *supported* if $\langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}a = [\nu^c]^{\mathbb{H}}\ \langle\not\ni\rangle^{\mathbb{H}}a$ for every $a \in \mathbb{A}$.

It immediately follows from the definitions that

Lemma 7. *The complex algebra of a supported two-sorted n-frame is a perfect heterogeneous supported m-algebra.*

Proof. Let $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ be a supported two-sorted n-frame. Then its complex algebra is $\mathbb{K}^+ = (\mathscr{P}(X), \mathscr{P}(Y), [\ni]^{\mathbb{K}^+}, \langle\not\ni\rangle^{\mathbb{K}^+}, \langle\nu\rangle^{\mathbb{K}^+}, [\nu^c]^{\mathbb{K}^+})$, which is clearly perfect. Since $\mathbb{K}$ is also supported, $R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c$ for any $D \subseteq \mathbb{K}$. Hence,

$$\langle\nu\rangle^{\mathbb{K}^+}[\ni]^{\mathbb{K}^+}D = R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c = [\nu^c]^{\mathbb{K}^+}\langle\not\ni\rangle^{\mathbb{K}^+}D. \quad \square$$

Definition 8. If $\mathbb{H} = (\mathscr{P}(X), \mathscr{P}(Y), [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ is a perfect heterogeneous m-algebra (resp. $\mathbb{H} = (\mathscr{P}(X), \mathscr{P}(Y), [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$ is a perfect heterogeneous c-algebra), its associated two-sorted n-frame (resp. c-frame) is

$$\mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c}) \qquad \text{(resp. } \mathbb{H}_+ := (X, Y, R_\ni, R_{\not\ni}, T_f)\text{), s.t.}$$

- $R_\ni \subseteq Y \times X$ is defined by $yR_\ni x$ iff $y \notin [\ni]^{\mathbb{H}}x^c$,
- $R_{\not\ni} \subseteq Y \times X$ is defined by $xR_{\not\ni}y$ iff $y \in \langle\not\ni\rangle^{\mathbb{H}}\{x\}$ (resp. $y \notin [\not\ni\rangle^{\mathbb{H}}\{x\}$),
- $R_\nu \subseteq X \times Y$ is defined by $xR_\nu y$ iff $x \in \langle\nu\rangle^{\mathbb{H}}\{y\}$,
- $R_{\nu^c} \subseteq X \times Y$ is defined by $xR_{\nu^c}y$ iff $x \notin [\nu^c]^{\mathbb{H}}y^c$,
- $T_f \subseteq X \times Y \times X$ is defined by $(x', y, x) \in T_f$ iff $x' \notin \{y\} \rhd^{\mathbb{H}} x^c$.

Lemma 9. *If $\mathbb{H}$ is a perfect supported heterogeneous m-algebra, then $\mathbb{H}_+$ is a supported two-sorted n-frame.*

Proof. To show that $\mathbb{H}_+$ is supported, for every $D \subseteq X$,

$$R_\nu^{-1}[(R_\ni^{-1}[D^c])^c] = \langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}D = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}}D = (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c. \quad \square$$

The duality between perfect BAOs and Kripke frames can be readily extended to the present two-sorted case. The following proposition collects these well-known facts, the proofs of which are analogous to those of the single-sorted case, hence are omitted.

Proposition 10. *For every heterogeneous m-algebra (resp. c-algebra) $\mathbb{H}$ and every two-sorted n-frame (resp. c-frame) $\mathbb{K}$,*

1. $\mathbb{K}^+$ *is a perfect heterogeneous m-algebra (resp. c-algebra);*
2. $(\mathbb{K}^+)_+ \cong \mathbb{K}$, *and if $\mathbb{H}$ is perfect, then $(\mathbb{H}_+)^+ \cong \mathbb{H}$.*

3.2. Equivalent representation of m-algebras and c-algebras

Every supported heterogeneous m-algebra (resp. c-algebra) can be associated with an m-algebra (resp. a c-algebra) as follows:

Definition 11. For every supported heterogeneous m-algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, \langle\not\ni\rangle^{\mathbb{H}}, \langle\nu\rangle^{\mathbb{H}}, [\nu^c]^{\mathbb{H}})$ (resp. c-algebra $\mathbb{H} = (\mathbb{A}, \mathbb{B}, [\ni]^{\mathbb{H}}, [\not\ni\rangle^{\mathbb{H}}, \rhd^{\mathbb{H}})$), let $\mathbb{H}_\bullet := (\mathbb{A}, \nabla^{\mathbb{H}_\bullet})$ (resp. $\mathbb{H}_\bullet := (\mathbb{A}, >^{\mathbb{H}_\bullet})$), where for every $a \in \mathbb{A}$ (resp. $a, b \in \mathbb{A}$),

$$\nabla^{\mathbb{H}_\bullet}a = \langle\nu\rangle^{\mathbb{H}}[\ni]^{\mathbb{H}}a = [\nu^c]^{\mathbb{H}}\langle\not\ni\rangle^{\mathbb{H}}a \qquad \text{(resp. } a >^{\mathbb{H}_\bullet} b := ([\ni]^{\mathbb{H}}a \wedge [\not\ni\rangle^{\mathbb{H}}a) \rhd^{\mathbb{H}} b\text{)}.$$

It immediately follows from the stipulations above that $\nabla^{\mathbb{H}_\bullet}$ is a monotone map (resp. $>^{\mathbb{H}_\bullet}$ is finitely meet-preserving in its second coordinate), and hence $\mathbb{H}_\bullet$ is an m-algebra (resp. a c-algebra). Conversely, every complete m-algebra (resp. c-algebra) can be associated with a complete supported heterogeneous m-algebra (resp. a c-algebra) as follows:

Definition 12. For every complete m-algebra $\mathbb{C} = (\mathbb{A}, \nabla^{\mathbb{C}})$ (resp. complete c-algebra $\mathbb{C} = (\mathbb{A}, >^{\mathbb{C}})$), let $\mathbb{C}^\bullet := (\mathbb{A}, \mathscr{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, \langle\not\ni\rangle^{\mathbb{C}^\bullet}, \langle\nu\rangle^{\mathbb{C}^\bullet}, [\nu^c]^{\mathbb{C}^\bullet})$ (resp. $\mathbb{C}^\bullet := (\mathbb{A}, \mathscr{P}(\mathbb{A}), [\ni]^{\mathbb{C}^\bullet}, [\not\ni\rangle^{\mathbb{C}^\bullet}, \rhd^{\mathbb{C}^\bullet})$), where for every $a \in \mathbb{A}$ and $B \in \mathscr{P}(\mathbb{A})$,

$$[\ni]^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid b \leq a\} \qquad \langle\nu\rangle^{\mathbb{C}^\bullet} B := \bigvee\{\nabla^{\mathbb{C}} b \mid b \in B\} \qquad [\not\ni\rangle^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid a \leq b\}$$

$$[\nu^c]^{\mathbb{C}^\bullet} B := \bigwedge\{\nabla^{\mathbb{C}} b \mid b \notin B\} \quad B \rhd^{\mathbb{C}^\bullet} a := \bigwedge\{b >^{\mathbb{C}} a \mid b \in B\} \quad \langle\not\ni\rangle^{\mathbb{C}^\bullet} a := \{b \in \mathbb{A} \mid a \not\leq b\}.$$

Lemma 13. *If $\mathbb{C}$ is a complete m-algebra (resp. complete c-algebra), then $\mathbb{C}^\bullet$ is a complete supported heterogeneous m-algebra (resp. c-algebra).*

Proof. Let $\mathbb{C} = (\mathbb{A}, \nabla^{\mathbb{C}})$ be a complete m-algebra. First we show that $\mathbb{C}^\bullet$ is a complete heterogeneous m-algebra. For $X \subseteq \mathbb{A}$ and $\Gamma \subseteq \mathscr{P}(\mathbb{A})$,

$$\begin{aligned}
[\ni]^{\mathbb{C}^\bullet} \bigwedge X &= \{b \in \mathbb{A} \mid b \leq \bigwedge X\} = \bigcap_{x\in X}\{b \in \mathbb{A} \mid b \leq x\} = \bigcap_{x\in X}[\ni]^{\mathbb{C}^\bullet} x \\
\langle\not\ni\rangle^{\mathbb{C}^\bullet} \bigvee X &= \{b \in \mathbb{A} \mid \bigvee X \not\leq b\} = \bigcup_{x\in X}\{b \in \mathbb{A} \mid x \not\leq b\} = \bigcup_{x\in X}\langle\not\ni\rangle^{\mathbb{C}^\bullet} x \\
\langle\nu\rangle^{\mathbb{C}^\bullet} \bigcup \Gamma &= \bigvee\{\nabla^{\mathbb{C}} b \mid b \in \bigcup \Gamma\} = \bigvee_{Y\in\Gamma}\bigvee\{\nabla^{\mathbb{C}} b \mid b \in Y\} = \bigvee_{Y\in\Gamma}\langle\nu\rangle^{\mathbb{C}^\bullet} Y \\
[\nu^c]^{\mathbb{C}^\bullet} \bigcap \Gamma &= \bigwedge\{\nabla^{\mathbb{C}} b \mid b \notin \bigcap \Gamma\} = \bigcap_{Y\in\Gamma}\bigwedge\{\nabla^{\mathbb{C}} b \mid b \notin Y\} = \bigcap_{Y\in\Gamma}[\nu^c]^{\mathbb{C}^\bullet} Y.
\end{aligned}$$

Let us show that $\mathbb{C}^\bullet$ is supported. For every $a \in \mathbb{A}$,

$$\begin{aligned}
\langle\nu\rangle^{\mathbb{C}^\bullet}[\ni]^{\mathbb{C}^\bullet} a &= \langle\nu\rangle^{\mathbb{C}^\bullet}\{b \in \mathbb{A} \mid b \leq a\} = \bigvee\{\nabla^{\mathbb{C}} b \mid b \leq a\} = \nabla^{\mathbb{C}} a, \\
[\nu^c]^{\mathbb{C}^\bullet}\langle\not\ni\rangle^{\mathbb{C}^\bullet} a &= [\nu^c]^{\mathbb{C}^\bullet}\{b \in \mathbb{A} \mid a \not\leq b\} = \bigwedge\{\nabla^{\mathbb{C}} b \mid a \leq b\} = \nabla^{\mathbb{C}} a.
\end{aligned}$$

Hence, $\langle\nu\rangle^{\mathbb{C}^\bullet}[\ni]^{\mathbb{C}^\bullet} a = [\nu^c]^{\mathbb{C}^\bullet}\langle\not\ni\rangle^{\mathbb{C}^\bullet} a$.

Let $\mathbb{C} = (\mathbb{A}, >^{\mathbb{C}})$ be a complete c-algebra. That $[\ni]^{\mathbb{C}^\bullet}$ is completely join preserving can be proved as shown above. As to the remaining connectives, for any $X \subseteq \mathbb{A}$ and $\Gamma \subseteq \mathscr{P}$,

$$\begin{aligned}
[\not\ni\rangle^{\mathbb{C}^\bullet} \bigvee X &= \{b \in \mathbb{A} \mid \bigvee X \leq b\} = \bigcap_{x\in X}\{b \in \mathbb{A} \mid x \leq b\} = \bigcap_{x\in X}[\not\ni\rangle^{\mathbb{C}^\bullet} x \\
\bigcup \Gamma \rhd^{\mathbb{C}^\bullet} a &= \bigwedge\{b >^{\mathbb{C}} a \mid b \in \bigcup \Gamma\} = \bigwedge_{Y\in\Gamma}\bigwedge\{b >^{\mathbb{C}} a \mid b \in Y\} = \bigwedge_{Y\in\Gamma}(Y \rhd^{\mathbb{C}^\bullet} a) \\
B \rhd^{\mathbb{C}^\bullet} \bigwedge X &= \bigwedge\{b >^{\mathbb{C}} \bigwedge X \mid b \in B\} = \bigwedge_{x\in X}\bigwedge\{b >^{\mathbb{C}} x \mid b \in B\} = \bigwedge_{x\in X}(B \rhd^{\mathbb{C}^\bullet} x). \quad \square
\end{aligned}$$

Proposition 14. *If $\mathbb{C}$ is a complete m-algebra (resp. c-algebra), then $\mathbb{C} \cong (\mathbb{C}^\bullet)_\bullet$. Moreover, if $\mathbb{H}$ is a complete supported heterogeneous m-algebra (resp. c-algebra), then $\mathbb{H} \cong \mathbb{C}^\bullet$ for some complete m-algebra (resp. c-algebra) $\mathbb{C}$ iff $\mathbb{H} \cong (\mathbb{H}_\bullet)^\bullet$.*

Proof. For the first part of the statement, by definition, $\mathbb{C}$ and $(\mathbb{C}^\bullet)_\bullet$ have the same underlying Boolean algebra. Moreover, $\nabla^{(\mathbb{C}^\bullet)_\bullet} a = \langle\nu\rangle^{\mathbb{C}^\bullet}[\ni]^{\mathbb{C}^\bullet} a = \nabla^{\mathbb{C}} a$ for every $a \in \mathbb{C}$, the first identity holding by definition, the second one being shown in the proof of Lemma 13.

As to the second part, for the left to right direction, assume that $\mathbb{H} \cong \mathbb{C}^\bullet$ for some complete m-algebra (resp. c-algebra) $\mathbb{C}$. From the first part of the proposition we know that $\mathbb{C} \cong (\mathbb{C}^\bullet)_\bullet$. Then $\mathbb{H} \cong \mathbb{C}^\bullet \cong ((\mathbb{C}^\bullet)_\bullet)^\bullet \cong (\mathbb{H}_\bullet)^\bullet$. For the right to left direction, $\mathbb{H}_\bullet$ is the required complete m-algebra (resp. c-algebra). $\square$

The proposition above characterizes up to isomorphism the supported heterogeneous m-algebras (resp. c-algebras) which arise from single-type m-algebras (resp. c-algebras).

3.3. Representing n-frames and c-frames as two-sorted Kripke frames

Thanks to the discrete dualities discussed in Sections 2.1 and 3.1, we can transfer the algebraic characterization of Proposition 14 to the side of frames, as detailed in this subsection.

Definition 15. For any n-frame (resp. c-frame) $\mathbb{F}$, we let $\mathbb{F}^\star := ((\mathbb{F}^\circledast)^\bullet)_+$, and for every supported two-sorted n-frame (resp. c-frame) $\mathbb{K}$, we let $\mathbb{K}_\star := ((\mathbb{K}^+)_\bullet)_\circledast$.

Spelling out the definition above, if $\mathbb{F} = (W, \nu)$ (resp. $\mathbb{F} = (W, f)$) then $\mathbb{F}^\star = (W, \mathscr{P}(W), R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{F}^\star = (W, \mathscr{P}(W), R_{\not\ni}, R_\ni, T_f)$) where:

- $R_\nu \subseteq W \times \mathscr{P}(W)$ is defined as $xR_\nu Z$ iff $Z \in \nu(x)$;
- $R_{\nu^c} \subseteq W \times \mathscr{P}(W)$ is defined as $xR_{\nu^c} Z$ iff $Z \notin \nu(x)$;
- $R_\ni \subseteq \mathscr{P}(W) \times W$ is defined as $ZR_\ni x$ iff $x \in Z$;
- $R_{\not\ni} \subseteq \mathscr{P}(W) \times W$ is defined as $ZR_{\not\ni} x$ iff $x \notin Z$;
- $T_f \subseteq W \times \mathscr{P}(W) \times W$ is defined as $T_f(x, Z, x')$ iff $x' \in f(x, Z)$.

Moreover, if $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$), then $\mathbb{K}_\star = (X, \nu_\star)$ (resp. $\mathbb{K}_\star = (X, f_\star)$) where:

- $\nu_\star(x) = \{D \subseteq X \mid x \in R_\nu^{-1}[(R_\ni^{-1}[D^c])^c]\} = \{D \subseteq X \mid x \in (R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c\}$;
- $f_\star(x, D) = \bigcap\{C \subseteq X \mid x \in T_f^{(0)}[\{C\}, D^c]\}$.

Lemma 16. *If $\mathbb{F} = (W, \nu)$ is an n-frame, then $\mathbb{F}^\star$ is a supported two-sorted n-frame.*

Proof. By definition, $\mathbb{F}^\star$ is a two-sorted n-frame. Moreover, for any $D \subseteq W$,

$$\begin{array}{lll}(R_{\nu^c}^{-1}[(R_{\not\ni}^{-1}[D])^c])^c & = & \{w \mid \forall X(X \notin \nu(w) \Rightarrow \exists u(X \not\ni u \;\&\; u \in D))\}\\ & = & \{w \mid \forall X(X \notin \nu(w) \Rightarrow D \not\subseteq X)\}\\ & = & \{w \mid \forall X(D \subseteq X \Rightarrow X \in \nu(w))\}\\ & = & \{w \mid \exists X(X \in \nu(w) \;\&\; X \subseteq D)\} \qquad (*)\\ & = & R_\nu^{-1}[(R_\ni^{-1}[D^c])^c].\end{array}$$

To show the identity marked with $(*)$, from top to bottom, take $X := D$; conversely, if $D \subseteq Z$ then $X \subseteq Z$, and since by assumption $X \in \nu(w)$ and $\nu(w)$ is upward closed, we conclude that $Z \in \nu(w)$, as required. □

The next proposition is the frame-theoretic counterpart of Proposition 14.

Proposition 17. *If $\mathbb{F}$ is an n-frame (resp. c-frame), then $\mathbb{F} \cong (\mathbb{F}^\star)_\star$. Moreover, if $\mathbb{K}$ is a supported two-sorted n-frame (resp. c-frame), then $\mathbb{K} \cong \mathbb{F}^\star$ for some n-frame (resp. c-frame) $\mathbb{F}$ iff $\mathbb{K} \cong (\mathbb{K}_\star)^\star$.*

Proof. For the first part of the statement,

$$\begin{array}{llll}(\mathbb{F}^\star)_\star & = & (((((\mathbb{F}^\circledast)^\bullet)_+)^+)_\bullet)_\circledast & \text{definition of } (-)^\star \text{ and } (-)_\star\\ & \cong & (((\mathbb{F}^\circledast)^\bullet)_\bullet)_\circledast & \text{Proposition 10.2, } (\mathbb{F}^\circledast)^\bullet \text{ perfect heterogeneous algebra}\\ & = & (\mathbb{F}^\circledast)_\circledast & \text{Proposition 14, since } \mathbb{F}^\circledast \text{ is complete}\\ & = & \mathbb{F}. & \text{Proposition 2}\end{array}$$

As to the second part, for the left to right direction, assume that $\mathbb{K} \cong \mathbb{F}^\star$ for some m-frame (resp. c-frame) $\mathbb{F}$. From the first part of the statement we know that $\mathbb{F} \cong (\mathbb{F}^\star)_\star$. Then $\mathbb{K} \cong \mathbb{F}^\star \cong ((\mathbb{F}^\star)_\star)^\star \cong (\mathbb{K}_\star)^\star$. For the right to left direction, $\mathbb{K}_\star$ is the required m-frame (resp. c-frame). □

4. Embedding non-normal logics into two-sorted normal logics

The two-sorted frames and heterogeneous algebras discussed in the previous section serve as semantic environment for the multi-type languages defined below.

Multi-type languages. For a denumerable set Prop of atomic propositions, the language $\mathcal{L}_{MT\nabla}$ for monotonic modal logic, in types S (sets) and N (neighbourhoods) over Prop, is defined as:

$$\mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle \nu \rangle \alpha \mid [\nu^c]\alpha$$
$$\mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid \langle \not\ni \rangle \alpha$$

and the language $\mathcal{L}_{MT>}$ for conditional logic, in types S (sets) and N (neighbourhoods) over Prop, is defined as:

$$\mathsf{S} \ni A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A$$
$$\mathsf{N} \ni \alpha ::= 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cap \alpha \mid [\ni]A \mid [\not\ni\rangle A.$$

Algebraic semantics. Interpretation of $\mathcal{L}_{MT\nabla}$-formulas (resp. $\mathcal{L}_{MT>}$ formulas) in heterogeneous m-algebras (resp. c-algebras) under homomorphic assignments $h : \mathcal{L}_{MT\nabla} \to \mathbb{H}$ (resp. $h : \mathcal{L}_{MT>} \to \mathbb{H}$) and validity of formulas in heterogeneous algebras ($\mathbb{H} \models \Theta$) are defined as usual.

Frames and models. $\mathcal{L}_{MT\nabla}$*-models* (resp. $\mathcal{L}_{MT>}$*-models*) are pairs $\mathbb{N} = (\mathbb{K}, V)$ s.t. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ is a supported two-sorted n-frame (resp. $\mathbb{K} = (X, Y, R_\ni, R_{\not\ni}, T_f)$ is a two-sorted c-frame) and $V : \mathcal{L}_{MT} \to \mathbb{K}^+$ is a heterogeneous algebra homomorphism of the appropriate signature. Hence, truth of formulas at states in models is defined as $\mathbb{N}, z \Vdash \Theta$ iff $z \in V(\Theta)$ for every $z \in X \cup Y$ and $\Theta \in \mathsf{S} \cup \mathsf{N}$, and unravelling this stipulation for formulas with a modal operator as main connective, we get:

- $\mathbb{N}, x \Vdash \langle \nu \rangle \alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for some y s.t. $x R_\nu y$;
- $\mathbb{N}, x \Vdash [\nu^c]\alpha$ iff $\mathbb{N}, y \Vdash \alpha$ for all y s.t. $x R_{\nu^c} y$;
- $\mathbb{N}, y \Vdash [\ni]A$ iff $\mathbb{N}, x \Vdash A$ for all x s.t. $y R_\ni x$;
- $\mathbb{N}, y \Vdash \langle \not\ni \rangle A$ iff $\mathbb{N}, x \Vdash A$ for some x s.t. $y R_{\not\ni} x$;
- $\mathbb{N}, y \Vdash [\not\ni\rangle A$ iff $\mathbb{N}, x \not\Vdash A$ for all x s.t. $y R_{\not\ni} x$;
- $\mathbb{N}, x \Vdash \alpha \rhd A$ iff for all y and all x', if $T_f(x, y, x')$ and $\mathbb{N}, y \Vdash \alpha$ then $\mathbb{N}, x' \Vdash A$.

Global satisfaction (notation: $\mathbb{N} \Vdash \Theta$) is defined relative to the domain of the appropriate type, and frame validity (notation: $\mathbb{K} \Vdash \Theta$) is defined as usual. Thus, by definition, $\mathbb{K} \Vdash \Theta$ iff $\mathbb{K}^+ \models \Theta$, and if $\mathbb{H}$ is a perfect heterogeneous algebra, then $\mathbb{H} \models \Theta$ iff $\mathbb{H}_+ \Vdash \Theta$.

Correspondence theory for multi-type normal logics. The semantic environment introduced above supports a straightforward extension of unified correspondence theory for multi-type normal logics, which includes the definition of inductive and analytic inductive formulas and inequalities in $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ (cf. Section 5), and a corresponding version of the algorithm ALBA [13] for computing their first-order correspondents and analytic structural rules.

Translation. Correspondence theory and analytic calculi for the non-normal logics $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their analytic extensions can be then obtained 'via translation', i.e. by recursively defining translations $\tau_1, \tau_2 : \mathcal{L}_\nabla \to \mathcal{L}_{MT\nabla}$ and $(\cdot)^\tau : \mathcal{L}_> \to \mathcal{L}_{MT>}$ as follows:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\tau_1(p) & = & p & \tau_2(p) & = & p & p^\tau & = & p \\
\tau_1(\varphi \wedge \psi) & = & \tau_1(\varphi) \wedge \tau_1(\psi) & \tau_2(\varphi \wedge \psi) & = & \tau_2(\varphi) \wedge \tau_2(\psi) & (\varphi \wedge \psi)^\tau & = & \varphi^\tau \wedge \psi^\tau \\
\tau_1(\neg\varphi) & = & \neg\tau_2(\varphi) & \tau_2(\neg\varphi) & = & \neg\tau_1(\varphi) & (\neg\varphi)^\tau & = & \neg\varphi^\tau \\
\tau_1(\nabla\varphi) & = & \langle \nu \rangle [\ni] \tau_1(\varphi) & \tau_2(\nabla\varphi) & = & [\nu^c]\langle \not\ni \rangle \tau_2(\varphi) & (\varphi > \psi)^\tau & = & ([\ni]\varphi^\tau \cap [\not\ni\rangle \varphi^\tau) \rhd \psi^\tau
\end{array}$$

Let $\tau(\varphi \vdash \psi) := \varphi^\tau \vdash \psi^\tau$ if $\varphi \vdash \psi$ is an $\mathcal{L}_>$-sequent, and $\tau(\varphi \vdash \psi) := \tau_1(\varphi) \vdash \tau_2(\psi)$ if $\varphi \vdash \psi$ is an $\mathcal{L}_\nabla$-sequent.

Proposition 18. *If $\mathbb{F}$ is an n-frame (resp. c-frame) and $\varphi \vdash \psi$ is an $\mathcal{L}_\nabla$-sequent (resp. an $\mathcal{L}_>$-sequent), then $\mathbb{F} \Vdash \varphi \vdash \psi$ iff $\mathbb{F}^\star \Vdash \tau(\varphi \vdash \psi)$.*

Proof. When $\mathbb{F}$ is an n-frame, the proposition is an immediate consequence of the following claim:

$$(\mathbb{F}, V), w \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \tau_1(\varphi) \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \tau_2(\varphi),$$

which can be proved by induction on φ. We only sketch the case in which $\varphi := \nabla\psi$. In this case, $\tau_1(\nabla\psi) = \langle \nu \rangle [\ni] \tau_1(\psi)$ and $\tau_2(\nabla\psi) = [\nu^c]\langle \not\ni \rangle \tau_2(\psi)$.

$$\begin{array}{rll}
\mathbb{F}, V, w \Vdash \nabla\psi & \text{iff} & \exists D (D \in \nu(w) \ \&\ D \subseteq V(\psi)) \\
& \text{iff} & \exists D (w R_\nu D \ \&\ \forall d (D R_\ni d \Rightarrow d \in V(\psi))) \\
& \text{iff} & \exists D (w R_\nu D \ \&\ \forall d (D R_\ni d \Rightarrow d \in V(\tau_1(\psi)))) \qquad \text{Induction hypothesis} \\
& \text{iff} & \mathbb{F}^\star, V, w \Vdash \langle \nu \rangle [\ni] \tau_1(\psi)
\end{array}$$

$$
\begin{array}{rll}
\mathbb{F}, V, w \Vdash \nabla\psi & \text{iff} & \exists D(D \in \nu(w) \;\&\; D \subseteq V(\psi)) \\
& \text{iff} & \exists D(wR_\nu D \;\&\; \forall d(DR_\ni d \Rightarrow d \in V(\psi))) \\
(*) & \text{iff} & \forall D(wR_{\nu^c} D \Rightarrow \exists d(DR_{\not\ni} d \;\&\; d \in V(\psi))) \\
& \text{iff} & \forall D(wR_{\nu^c} D \Rightarrow \exists d(DR_{\not\ni} d \;\&\; d \in V(\tau_2(\psi)))) \quad \text{Induction hypothesis} \\
& \text{iff} & \mathbb{F}^\star, V, w \Vdash [\nu^c]\langle\not\ni\rangle\tau_2(\psi).
\end{array}
$$

The equivalence marked by $(*)$ follows from Lemma 16.

When $\mathbb{F}$ is a c-frame, the proposition is an immediate consequence of the following claim, which can be shown by induction on φ.

$$(\mathbb{F}, V), w \Vdash \varphi \quad \text{iff} \quad (\mathbb{F}^\star, V), w \Vdash \varphi^\tau.$$

We only sketch the case in which $\varphi := \varphi > \psi$. In this case, $(\varphi > \psi)^\tau = ([\ni]\varphi^\tau \cap [\not\ni\rangle\varphi^\tau) \rhd \psi^\tau$.

$$
\begin{array}{rl}
(\mathbb{F}, V), w \Vdash \varphi > \psi & \text{iff } f(w, V(\varphi)) \subseteq V(\psi) \\
& \text{iff } \forall x(x \in f(w, V(\varphi)) \Rightarrow x \in V(\psi)) \\
& \text{iff } \forall x \forall Y(x \in f(w, Y) \;\&\; Y = V(\varphi) \Rightarrow x \in V(\psi)) \\
& \text{iff } \forall x \forall Y(x \in f(w, Y) \;\&\; Y = V(\varphi^\tau) \Rightarrow x \in V(\psi^\tau)) \qquad \text{I.H.} \\
& \text{iff } \forall x \forall Y(T_f(w, Y, x) \;\&\; (\forall y(YR_\ni y \Rightarrow y \in V(\varphi^\tau))) \;\&\; \\
& \qquad (\forall y(YR_{\not\ni} y \Rightarrow y \notin V(\varphi^\tau))) \Rightarrow x \in V(\psi^\tau)) \\
& \text{iff } (\mathbb{F}^\star, V), w \Vdash ([\ni]\varphi^\tau \cap [\not\ni\rangle\varphi^\tau) \rhd \psi^\tau. \quad \square
\end{array}
$$

With this framework in place, we are in a position to (a) retrieve correspondence results in the setting of *non-normal* logics, such as those collected in Theorem 3, as instances of the general Sahlqvist theory for multi-type *normal* logics, and (b) recognize whether the translation of a non-normal axiom is analytic inductive, and compute its corresponding analytic structural rules (cf. Section Appendix A).

5. Analytic inductive inequalities

In the present section, we specialize the definitions of *inductive inequalities* (cf. [13, Definition 3.4]) and *analytic inductive inequalities* (cf. [35, Definition 55]) to the multi-type languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ provided in Section 4.

An *order-type* over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$. If ε is an order type, ε^∂ is its *opposite* order type; i.e. $\varepsilon^\partial(i) = 1$ iff $\varepsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_{\mathsf{S}} \cup \mathcal{F}_{\mathsf{N}} \cup \mathcal{F}_{\mathsf{MT}}$ and $\mathcal{G} := \mathcal{G}_{\mathsf{S}} \cup \mathcal{G}_{\mathsf{N}} \cup \mathcal{G}_{\mathsf{MT}}$, defined as follows:

$$
\begin{array}{ll}
\mathcal{F}_{\mathsf{S}} := \{\neg\} & \mathcal{G}_{\mathsf{S}} = \{\neg\} \\
\mathcal{F}_{\mathsf{N}} := \{\sim\} & \mathcal{G}_{\mathsf{N}} := \{\sim\} \\
\mathcal{F}_{\mathsf{MT}} := \{\langle\nu\rangle, \langle\not\ni\rangle\} & \mathcal{G}_{\mathsf{MT}} := \{[\ni], [\nu^c], \rhd, [\not\ni\rangle\}
\end{array}
$$

For any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), we let $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) denote the arity of f (resp. g), and the order-type ε_f (resp. ε_g) on n_f (resp. n_g) indicate whether the ith coordinate of f (resp. g) is positive ($\varepsilon_f(i) = 1$, $\varepsilon_g(i) = 1$) or negative ($\varepsilon_f(i) = \partial$, $\varepsilon_g(i) = \partial$).

Definition 19 *(Signed generation tree).* The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{MT}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_\ell(i) = 1$ (resp. if $\varepsilon_\ell(i) = \partial$). Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

For any term $s(p_1, \ldots p_n)$, any order type ε over n, and any $1 \leq i \leq n$, an *ε-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon(i) = 1$ or $-p_i$ with $\varepsilon(i) = \partial$. An *$\varepsilon$-critical branch* in the tree is a branch ending in an ε-critical node. For any term $s(p_1, \ldots p_n)$ and any order type ε over n, we say that $+s$ (resp. $-s$) *agrees with* ε, and write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(s') \prec *s$ (resp. $\varepsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Table 1
Skeleton and PIA nodes.

Skeleton	PIA
Δ-adjoints	SRA
$+$ $\vee$ $\cup$	$+$ $\wedge$ $\cap$ $[\ni]$ $[\nu^c]$ $\rhd$ $[\not\ni\rangle$ $\neg$ $\sim$
$-$ $\wedge$ $\cap$	$-$ $\vee$ $\cup$ $\langle\nu\rangle$ $\langle\not\ni\rangle$ $\neg$ $\sim$
SLR	SRR
$+$ $\wedge$ $\cap$ $\langle\nu\rangle$ $\langle\not\ni\rangle$ $\neg$ $\sim$	$+$ $\vee$ $\cup$
$-$ $\vee$ $\cup$ $[\ni]$ $[\nu^c]$ $\rhd$ $[\not\ni\rangle$ $\neg$ $\sim$	$-$ $\wedge$ $\cap$

Definition 20 *(Good branch).* Nodes in signed generation trees are called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+,-\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Definition 21 *(Inductive $\mathscr{L}_{MT\nabla}$- and $\mathscr{L}_{MT>}$-inequalities).* For any order type ε and any irreflexive and transitive relation $<_\Omega$ on $p_1,\dots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of an $\mathscr{L}_{MT\nabla}$-term (resp. $\mathscr{L}_{MT>}$-term) $s(p_1,\dots p_n)$ is (Ω,ε)-*inductive* if

1. for all $1 \le i \le n$, every ε-critical branch with leaf p_i is good (cf. Definition 20);
2. for all $1 \le i \le n$, every SRR-node occurring in any ε-critical branch with leaf p_i is of the form $\circledast(s,\beta)$ or $\circledast(\beta,s)$, where the critical branch goes through β and
   (a) $\varepsilon^\partial(s) \prec *s$ (cf. discussion before Definition 20), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in s and for every $1 \le k \le n$.

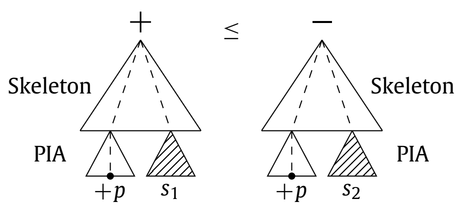
We will refer to $<_\Omega$ as the *dependency order* on the variables. An $\mathscr{L}_{MT\nabla}$-inequality (resp. $\mathscr{L}_{MT>}$-inequality) $s \le t$ is (Ω,ε)-*inductive* if the signed generation trees $+s$ and $-t$ are (Ω,ε)-inductive. An inequality $s \le t$ is *inductive* if it is (Ω,ε)-inductive for some Ω and ε.

Definition 22 *(Analytic inductive $\mathscr{L}_{MT\nabla}$- and $\mathscr{L}_{MT>}$-inequalities).* For any order type ε and any irreflexive and transitive relation Ω on the variables $p_1,\dots p_n$, the signed generation tree $*s$ ($* \in \{+,-\}$) of an $\mathscr{L}_{MT\nabla}$-term (resp. $\mathscr{L}_{MT>}$-term) $s(p_1,\dots p_n)$ is *analytic* (Ω,ε)-*inductive* if

1. $*s$ is (Ω,ε)-inductive (cf. Definition 21);
2. every branch of $*s$ is good (cf. Definition 20).

An inequality $s \le t$ is *analytic* (Ω,ε)-*inductive* if $+s$ and $-t$ are both (Ω,ε)-analytic inductive. An inequality $s \le t$ is *analytic inductive* if is (Ω,ε)-analytic inductive for some Ω and ε.

The syntactic shape of analytic inductive inequalities is illustrated by the following picture:



Remark 1. In what follows, in order to be consistent with proof-theoretic notation, we also refer to *inductive sequents* and *analytic inductive sequents*. Since inductive and analytic inductive inequalities are syntactic objects, inductive and analytic inductive sequents are obtained from the former by substituting the inequality symbol with the symbol $\vdash$.

6. Algorithmic correspondence for non-normal logics

In this section, we detail how the two-sorted environment introduced and discussed in the previous sections can be used to establish a Sahlqvist-type correspondence framework for *classes* of non-normal logics (see Theorem 24 and the discussion in Section 9) which can be specialized to the signatures of monotonic modal logic and conditional logic, encompasses and extends the well-known correspondence-theoretic results for these logics collected in Theorem 3, and brings them into the fold of unified correspondence theory [12,13]. The unified correspondence approach pivots on the order theoretic properties of the algebraic interpretation of logical connectives. As pointed out in [5], when the relevant order theoretic properties hold in a given multi-type setting such as the one introduced in Section 3, the insights, tools and results of unified correspondence theory can be straightforwardly transferred to it. As the first step of this process, specifically for the present cases of monotonic modal logic and conditional logic, we have specialized the definition of inductive and analytic inductive inequalities/sequents to the languages $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ (cf. Definitions 21 and 22); in the following table, we list the translations of the axioms of Theorem 3, for each of which, the last column of the table specifies whether its translation is analytic inductive.

	Axiom	Translation	Inductive	Analytic
N	$\nabla\top$	$\top \leq [\nu^c]\langle\not\ni\rangle\top$	✓	✓
P	$\neg\nabla\bot$	$\top \leq \neg\langle\nu\rangle[\ni]\bot$	✓	✓
C	$\nabla p \wedge \nabla q \rightarrow \nabla(p \wedge q)$	$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq [\nu^c]\langle\not\ni\rangle(p \wedge q)$	✓	✓
T	$\nabla p \rightarrow p$	$\langle\nu\rangle[\ni]p \leq p$	✓	✓
4	$\nabla\nabla p \rightarrow \nabla p$	$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle p$	✓	×
4’	$\nabla p \rightarrow \nabla\nabla p$	$\langle\nu\rangle[\ni]p \leq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$	✓	×
5	$\neg\nabla\neg p \rightarrow \nabla\neg\nabla\neg p$	$\neg[\nu^c]\langle\not\ni\rangle\neg p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
B	$p \rightarrow \nabla\neg\nabla\neg p$	$p \leq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	✓	×
D	$\nabla p \rightarrow \neg\nabla\neg p$	$\langle\nu\rangle[\ni]p \leq \neg\langle\nu\rangle[\ni]\neg p$	✓	✓
CS	$(p \wedge q) \rightarrow (p > q)$	$p \wedge q \leq ([\ni]p \cap [\not\ni\rangle p) \rhd q$	✓	✓
CEM	$(p > q) \vee (p > \neg q)$	$\top \leq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$	✓	✓
ID	$p > p$	$\top \leq ([\ni]p \cap [\not\ni\rangle p) \rhd p$	✓	✓
CN	$(p > q) \vee (q > p)$	$\top \leq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]q \cap [\not\ni\rangle q) \rhd p)$	✓	✓

Remark 2. The positional translation of $\mathcal{L}_\nabla$-axioms/sequents allows for a larger set of translated axioms to be (analytic) inductive, compared to e.g. using only τ_1. To illustrate this point, consider axioms 4 and C above; translating them using τ_1 respectively yields $\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \leq \langle\nu\rangle[\ni]p$, which is not inductive since no branch is good, and $\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq \langle\nu\rangle[\ni](p \wedge q)$, which is inductive but not analytic, since in $-\langle\nu\rangle[\ni](p \wedge q)$ some branches (in fact all) are not good (cf. Definition 20). This trick is not a panacea: also under the positional translation, nested occurrences of ∇ connectives, as in axioms 4, 4’, 5 and B, give rise to nestings of modal operators in which Skeleton nodes occur in the scope of PIA nodes, violating the ‘good branch’ requirement (cf. Definition 20). We return to this point in the discussion after Definition 23.
An analogous positional translation trick is not applicable to $\mathcal{L}_>$. The reason is that $>$ is (finitely/completely) meet-preserving in its second coordinate and arbitrary in its first coordinate, thereby forcing any *normal* connective $*$ at the root of the translated term to be (at least) *binary*, and (finitely/completely) meet-preserving in one of its coordinates. However, to occur as a positive Skeleton node, the connective $*$ would also need to be (finitely/completely) join-preserving (resp. meet-reversing) in its positive (resp. negative) coordinates and, as observed in [11, Footnote 4], all normal operations endowed with both sets of properties need to be *unary*.

The algorithm ALBA defined in [13] can straightforwardly be adapted to $\mathcal{L}_{MT\nabla}$ and $\mathcal{L}_{MT>}$ and their algebraic and relational semantics; since the translations of all the axioms listed above are inductive, by the general theory, ALBA succeeds in eliminating the propositional variables occurring in them and in equivalently transforming their validity on frames into suitable conditions expressible in the predicate languages canonically associated with n-frames (resp. c-frames). The ALBA runs on these axioms are reported in Section Appendix A.

To further expand on how the correspondence results of Theorem 3 can be obtained as instances of algorithmic correspondence on two-sorted frames and their complex algebras, let $\mathbb{F}$ be an n-frame (resp. a c-frame) and $\varphi \vdash \psi$ an $\mathcal{L}_\nabla$-sequent (resp. $\mathcal{L}_\nabla$-sequent). Let $\tau(\varphi \vdash \psi)$ denote $\tau_1(\varphi) \vdash \tau_2(\psi)$ or $\varphi^\tau \vdash \psi^\tau$ as appropriate. Let $ALBA(\tau(\varphi \vdash \psi))$ denote an output of ALBA when run on $\tau(\varphi \vdash \psi)$, and $\mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$ be its standard translation in the appropriate predicate language of n-frames (resp. c-frames). Then the following chain of equivalences holds[6]:

	$\mathbb{F} \Vdash \varphi \vdash \psi$	
iff	$\mathbb{F}^\star \Vdash \tau(\varphi \vdash \psi)$	Proposition 18
iff	$(\mathbb{F}^\star)^+ \models \tau(\varphi \vdash \psi)$	def. of validity on two sorted-frames
iff	$(\mathbb{F}^\star)^+ \models ALBA(\tau(\varphi \vdash \psi))$	two-sorted correspondence
iff	$\mathbb{F}^\star \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$	
iff	$\mathbb{F} \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$	

6 In the last equivalence the relations are interpreted according to Definition 15.

Let us concretely illustrate this proof pattern by applying it to the following axiom:

$$\nabla p \wedge \nabla q \vdash \nabla(p \wedge q). \tag{2}$$

Let $\mathbb{F} = (W, \nu)$ be a n-frame, and $\mathbb{F}^\star = (W, \mathscr{P}(W), R_\ni, R_{\not\ni}, R_\nu, R_{\nu^c})$ be its associated two-sorted n-frame, where e.g. $wR_\nu Z$ iff $Z \in \nu(w)$ and so on (full details are in Definition 15). By Proposition 18, the validity of axiom (2) on $\mathbb{F}$ is equivalent to its translation

$$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \vdash [\nu^c]\langle\not\ni\rangle(p \wedge q) \tag{3}$$

being valid on $\mathbb{F}^\star$, which, by definition of satisfaction and validity in the two-sorted environment, is equivalent to the validity of axiom (3) on the complex algebra $(\mathbb{F}^\star)^+ = (\mathscr{P}(W), \mathscr{P}\mathscr{P}(W), [\ni], \langle\not\ni\rangle, \langle\nu\rangle, [\nu^c])$.

According to Definition 22, axiom (3) is a (Ω, ε)-analytic inductive inequality for $p <_\Omega q$ and $\varepsilon(p) = \varepsilon(q) = 1$. Let us now run ALBA on axiom (3). In what follows we let $\mathbf{i}_1$ and $\mathbf{i}_2$ be nominal variables of type N and $\mathbf{m}$ be a co-nominal variable of type N. This means that $\mathbf{i}_1$ and $\mathbf{i}_2$ are interpreted as — and hence range in the set of — atoms of the second domain $\mathscr{P}\mathscr{P}(W)$ of the perfect heterogeneous c-algebra $(\mathbb{F}^\star)^+$ (i.e. singleton subsets $\{Z\}$ for $Z \subseteq W$), while $\mathbf{m}$ ranges over the set of coatoms of $\mathscr{P}\mathscr{P}(W)$, and hence is interpreted as the collection of subsets $\{Z\}^c := \{Y \subseteq W \mid Y \neq Z\}$ for an arbitrary $Z \subseteq W$.

As no preprocessing is needed, ALBA performs first approximation, which equivalently transforms

$$\forall p \forall q[\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \leq [\nu^c]\langle\not\ni\rangle(p \wedge q)]$$

into the following quasi-inequality:

$$\forall p \forall q \forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[(\mathbf{i}_1 \leq [\ni]p \,\&\, \mathbf{i}_2 \leq [\ni]q \,\&\, \langle\not\ni\rangle(p \wedge q) \leq \mathbf{m}) \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}].$$

Recall that $\langle\in\rangle$ and $[\ni]$ form a residuation pair. Hence, $\mathbf{i}_1 \leq [\ni]p$ is equivalent to $\langle\in\rangle\mathbf{i}_1 \leq p$ and $\mathbf{i}_2 \leq [\ni]q$ is equivalent to $\langle\in\rangle\mathbf{i}_2 \leq q$. Then the quasi inequality above is equivalent to the following quasi-inequality:

$$\forall p \forall q \forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[(\langle\in\rangle\mathbf{i}_1 \leq p \,\&\, \langle\in\rangle\mathbf{i}_2 \leq q \,\&\, \langle\not\ni\rangle(p \wedge q) \leq \mathbf{m}) \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}].$$

The quasi inequality above is in Ackermann shape, hence the Ackermann rule can be applied (cf. [13, Lemma 4.2]) to eliminate all occurrences of p and q, yielding the following (pure) quasi inequality in output

$$\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\not\ni\rangle(\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2) \leq \mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}],$$

which, for the sake of convenience, applying adjunction, we equivalently rewrite as

$$\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2 \leq [\notin]\mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}]. \tag{4}$$

Let $ALBA(\tau(\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)))$ denote the quasi inequality above. The soundness of ALBA on perfect heterogeneous m-algebras and the validity of (3) on $(\mathbb{F}^\star)^+$ imply that $ALBA(\tau(\nabla p \wedge \nabla q \vdash \nabla(p \wedge q)))$ holds in $(\mathbb{F}^\star)^+$. The next step is to translate this quasi-inequality into a condition on $\mathbb{F}^\star$ expressible in its appropriate correspondence language.

As discussed above, nominal and conominal variables correspond to subsets of W. Moreover, recall that the heterogeneous connectives $[\ni], \langle\not\ni\rangle, \langle\nu\rangle, [\nu^c]$ are interpreted in $(\mathbb{F}^\star)^+$ as heterogeneous operations defined by the following assignments: for any $D \in \mathscr{P}(W)$ and $U \in \mathscr{P}\mathscr{P}(W)$ (cf. Definition 4),

$$[\notin]U = (R_{\not\ni}[U^c])^c \quad \langle\in\rangle U = R_\ni[U] \quad \langle\nu\rangle U = R_\nu^{-1}[U] \quad [\nu^c]U = (R_{\nu^c}^{-1}[U^c])^c.$$

Let $Z_1, Z_2, Z_3 \subseteq W$ and $\{Z_1\}, \{Z_2\}, \{Z_3\}^c$ be the interpretations of $\mathbf{i}_1, \mathbf{i}_2, \mathbf{m}$, respectively. Then, writing $R_\circ[Z]$ for $R_\circ[\{Z\}]$ for any $\circ \in \{\ni, \not\ni, \nu, \nu^c\}$, we can translate (4) as follows:

$$\begin{aligned}
&\forall \mathbf{i}_1 \forall \mathbf{i}_2 \forall \mathbf{m}[\langle\in\rangle\mathbf{i}_1 \wedge \langle\in\rangle\mathbf{i}_2 \leq [\notin]\mathbf{m} \Rightarrow \langle\nu\rangle\mathbf{i}_1 \wedge \langle\nu\rangle\mathbf{i}_2 \leq [\nu^c]\mathbf{m}]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\} \leq [\notin]\{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \leq [\nu^c]\{Z_3\}^c]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[\{Z_3\}^{cc}])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[\{Z_3\}^{cc}])^c]\\
=&\forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[Z_3])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[Z_3])^c].
\end{aligned}$$

Thus, we have obtained

$$\mathbb{F}^\star \models \forall Z_1 \forall Z_2 \forall Z_3[R_\ni[Z_1] \cap R_\ni[Z_2] \subseteq (R_{\not\ni}[Z_3])^c \Rightarrow R_\nu^{-1}[Z_1] \cap R_\nu^{-1}[Z_2] \subseteq (R_{\nu^c}^{-1}[Z_3])^c].$$

The final step is to translate this condition into a condition on $\mathbb{F}$. Recalling the definitions of $R_{\not\ni}, R_\ni, R_\nu, R_{\nu^c}$ in Definition 15, it is easy to see that for any $Z \subseteq W$,

Table 2
Skeleton, PIA and Non-normal nodes.

Skeleton	PIA	Non-Normal
Δ-adjoints	SRA	
$+$ $\vee$	$+$ $\wedge$ $\neg$	$+$ ∇
$-$ $\wedge$	$-$ $\vee$ $\neg$	
SLR	SRR	$-$ ∇ $>$
$+$ $\wedge$ $\neg$	$+$ $\vee$	
$-$ $\vee$ $\neg$	$-$ $\wedge$	

$$R_{\ni}[Z] = Z = (R_{\not\ni}[Z])^c \quad \text{and} \quad R_{\nu}^{-1}[Z] = \{w \in W \mid Z \in \nu(w)\} = (R_{\nu^c}^{-1}[Z])^c.$$

Hence, we get:

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall Z_3 [Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow Z_3 \in \nu(x)]],$$

which, by uncurrying and then currying again, and suitably distributing quantifiers, is equivalent to

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow \forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow Z_3 \in \nu(x)]],$$

which is equivalent to

$$\mathbb{F} \models \forall Z_1 \forall Z_2 \forall x[(Z_1 \in \nu(x) \,\&\, Z_2 \in \nu(x)) \Rightarrow Z_1 \cap Z_2 \in \nu(x)] :$$

Indeed, for the top-to-bottom direction, take $Z_3 = Z_1 \cap Z_2$. Conversely, assume that $Z_1 \cap Z_2 \subseteq Z_3$, and that $Z_1 \in \nu(x)$ and $Z_2 \in \nu(x)$. Then, the assumption implies that $Z_1 \cap Z_2 \in \nu(x)$. Since $\nu(x)$ is upward-closed, $Z_1 \cap Z_2 \subseteq Z_3$ implies that $Z_3 \in \nu(x)$. This completes the algorithmic proof of item C of Theorem 3. The remaining items can be obtained by similar arguments. In Appendix A we collect the relevant ALBA runs and translations of their output.

The discussion above also motivates the following definition, aimed at identifying those $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities $\varphi \vdash \psi$ such that $\tau(\varphi \vdash \psi)$ is (analytic) inductive.

Definition 23 *((Analytic) inductive $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities).* Nodes in signed generation trees will be called *Δ-adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, *syntactically right adjoint (SRA)*, and *Non-Normal* according to the specification given in Table 2. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, with at most one Non-Normal connective in between P_1 and P_2, and such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Inductive and analytic inductive $\mathcal{L}_\nabla$- and $\mathcal{L}_>$-inequalities are defined verbatim as in Definitions 21 and 22, relative to the definition of good branch given above, with the additional restriction that a (sub)formula $\varphi > \psi$ is allowed to occur positively in signed generation trees of inductive $\mathcal{L}_>$-inequalities only if every leaf of φ is a constant.

The translations of Section 4 map good branches defined above to good branches as in Definition 20; indeed, occurrences of $+\nabla$ (resp. $-\nabla$) are translated as $+\langle\nu\rangle[\ni]$ (resp. $-[\nu^c]\langle\not\ni\rangle$). Also, occurrences of $>$ in good branches can only be negative, and conditional formulas $-\varphi > \psi$ are translated as $-([\ni]\varphi \cap [\not\ni\rangle\varphi) \rhd \psi$. In all these cases, PIA nodes occur in the scope of a Skeleton node, as required by the 'good branch' shape of the target language. Moreover, Non-Normal connectives are allowed to occur at most once in a good branch, since, as discussed in Remark 2, independently of their sign, the nested occurrence of Non-Normal connectives would create, when translated, at least one branch in which a Skeleton node occurs in the scope of a PIA node, which violates the 'good branch' requirement. Finally, the requirement that positive occurrences of $\varphi > \psi$ are allowed in signed generation trees only if φ does not contain occurrences of atomic propositions is motivated by the fact that, as discussed in Remark 2, no positional translation is available which would map branches with occurrences of $+>$ nodes to good branches of the target language. Hence, applying the translation $(\cdot)^\tau$ to $+(\varphi > \psi)$ generates Skeleton nodes in the scope of a PIA node. Therefore, the translation $+([\ni]\varphi \cap [\not\ni\rangle\varphi) \rhd \psi$ of this formula would be allowed to occur as a subformula in an (Ω, ε)-inductive inequality only if it was ε^∂-uniform. However, if φ contains atomic propositions, then by construction, the formula $([\ni]\varphi \cap [\not\ni\rangle\varphi) \rhd \psi$ is not ε^∂-uniform for any order-type ε.

Example 1. Axioms N and P give rise to the $\mathcal{L}_\nabla$-inequalities $\top \leq \nabla\top$ and $\top \leq \neg\nabla\bot$ which are trivially analytic inductive. Axioms C, T and D give rise to the $\mathcal{L}_\nabla$-inequalities $\nabla p \wedge \nabla q \leq \nabla(p \wedge q)$, $\nabla p \leq p$ and $\nabla p \leq \neg\nabla\neg p$ which are analytic (Ω, ε)-inductive for any order type ε and the empty Ω. Axioms 4', 5, and B give rise to $\nabla p \leq \nabla\nabla p$, $\neg\nabla\neg p \leq \nabla\neg\nabla\neg p$ and $p \leq \nabla\neg\nabla\neg p$ which are (Ω, ε)-inductive for the order type $\varepsilon(p) = 1$ and the empty Ω, but not for $\varepsilon(p) = \partial$, and hence are not analytic. Axiom 4 gives rise to $\nabla\nabla p \leq \nabla p$ which is (Ω, ε)-inductive for the order type $\varepsilon(p) = \partial$ and the empty Ω, but not for $\varepsilon(p) = 1$, and hence is not analytic. The $\mathcal{L}_>$-inequality $(p > q) \leq (p \to q)$, arising from Axiom MP from [58], is not inductive, since $>$ occurs positively and the atomic proposition p occurs in the scope of its first coordinate. Axioms CS, CEM,

ID, CN give rise to the $\mathscr{L}_>$-inequalities $(p \wedge q) \leq (p > q)$, $\top \leq (p > q) \vee (p > \neg q)$, $\top \leq p > p$, and $\top \leq (p > q) \vee (q > p)$, which are analytic (Ω, ε)-inductive for any order type ε and the empty Ω.
The class of inductive $\mathscr{L}_\nabla$-inequalities properly includes those arising from *KW-formulas* (cf. [38, Definition 5.13]). Indeed, it is not difficult to see that if $\varphi \to \psi$ is a KW-formula, then $\tau(\varphi \vdash \psi)$ is a Sahlqvist (and hence inductive) inequality.[7] To see that the inclusion is proper, consider e.g. $\nabla(p \vee q) \leq (\nabla\nabla p) \vee q$, which is (Ω, ε)-inductive for $\varepsilon(p, q) = (1, \partial)$ and $q <_\Omega p$, but $\nabla(p \vee q) \to ((\nabla\nabla p) \vee q)$ is not a KW-formula.

The tools of unified correspondence can be used also for computing analytic rules corresponding to analytic inductive axioms in the given two-sorted languages, so to obtain analytic calculi for some axiomatic extensions of the basic monotonic modal logic and basic conditional logic as an application of the theory developed in [35]. This treatment yields the analytic calculi defined in the next section. In particular, in the light of the general results [13, Theorem 6.1, Theorem 8.8] and [35, Proposition 59], the discussion so far yields the following

Theorem 24. *For any inductive $\mathscr{L}_\nabla$-sequent (resp. $\mathscr{L}_>$-sequent) $\varphi \vdash \psi$, ALBA successfully terminates on $\tau(\varphi \vdash \psi)$, and if $\mathbb{F}$ is an n-frame (resp. c-frame), then $\mathbb{F} \Vdash \varphi \vdash \psi$ if and only if $\mathbb{F} \models \mathsf{ST}(ALBA(\tau(\varphi \vdash \psi)))$. Furthermore, if $\varphi \vdash \psi$ is analytic inductive, then one or more analytic structural rules in the language of the display calculus D.MT∇ (resp. D.MT$>$)—cf. Section 7—can be read off from the same ALBA run of $\tau(\varphi \vdash \psi)$, and hence will be semantically equivalent to $\varphi \vdash \psi$.*

7. Proper display calculi for non-normal logics

In this section we introduce proper multi-type display calculi for $\mathbf{L}_\nabla$ and $\mathbf{L}_>$ and their axiomatic extensions generated by the analytic axioms considered in Section 6. For an introduction to (proper) display calculi we refer the reader to [4,67,31]. For a generalization to multi-type (proper, display) calculi we refer the reader to [20–22,24,32,34,37,33,63].

Languages. The language $\mathscr{L}_{DMT\nabla}$ of the calculus D.MT∇ for $\mathbf{L}_\nabla$ is defined as follows:

$$\mathsf{S}\begin{cases} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \langle\nu\rangle\alpha \mid [\nu^c]\alpha \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \hat{\wedge} X \mid X \check{\vee} X \mid \langle\hat{\nu}\rangle\Gamma \mid [\check{\nu}^c]\Gamma \mid \langle\hat{\in}\rangle\Gamma \mid [\check{\notin}]\Gamma \end{cases}$$

$$\mathsf{N}\begin{cases} \alpha ::= [\ni]A \mid \langle\not\ni\rangle A \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \hat{\cap} \Gamma \mid \Gamma \check{\cup} \Gamma \mid [\check{\ni}]X \mid \langle\hat{\not\ni}\rangle X \mid [\check{\rotatebox[origin=c]{180}{ν}}]X \mid \langle\hat{\rotatebox[origin=c]{180}{ν}}^c\rangle X \end{cases}$$

The language $\mathscr{L}_{DMT>}$ of the calculus D.MT$>$ for $\mathbf{L}_>$ is defined as follows:

$$\mathsf{S}\begin{cases} A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid \alpha \rhd A \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \tilde{\neg} X \mid X \hat{\wedge} X \mid X \check{\vee} X \mid \langle\hat{\in}\rangle\Gamma \mid \Gamma \check{\rhd} X \mid \Gamma \hat{\blacktriangle} X \mid [\check{\notin}\rangle\Gamma \end{cases}$$

$$\mathsf{N}\begin{cases} \alpha ::= [\ni]A \mid [\not\ni\rangle A \mid \alpha \cap \alpha \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\sim}\Gamma \mid \Gamma \hat{\cap} \Gamma \mid \Gamma \check{\cup} \Gamma \mid [\check{\ni}]X \mid [\check{\not\ni}\rangle X \mid X \check{\blacktriangleright} X \end{cases}$$

Multi-type display calculi. In what follows, we use X, Y, W, Z as structural S-variables, and $\Gamma, \Delta, \Sigma, \Pi$ as structural N-variables.

Propositional base. The calculi D.MT∇ and D.MT$>$ share the rules listed below.

- Identity and Cut:

$$Id_{\mathsf{S}}\ \frac{}{p \vdash p} \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\ Cut_{\mathsf{S}} \qquad \frac{\Gamma \vdash \alpha \quad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\ Cut_{\mathsf{N}}$$

- Pure S-type display rules:

$$\bot\ \frac{}{\bot \vdash \check{\bot}} \qquad \frac{}{\hat{\top} \vdash \top}\ \top$$

$$gal_{\mathsf{S}}\ \frac{\tilde{\neg} X \vdash Y}{\tilde{\neg} Y \vdash X} \qquad \frac{X \vdash \tilde{\neg} Y}{Y \vdash \tilde{\neg} X}\ gal_{\mathsf{S}} \qquad res_{\mathsf{S}}\ \frac{X \hat{\wedge} Y \vdash Z}{Y \vdash \tilde{\neg} X \check{\vee} Z} \qquad \frac{X \vdash Y \check{\vee} Z}{\tilde{\neg} Y \hat{\wedge} X \vdash Z}\ res_{\mathsf{S}}$$

[7] In fact, it is already pointed out in the proof of [38, Theorem 5.14] that the definition of KW-formulas is designed so as to target Sahlqvist formulas under a translation that is slightly different, but not in essential ways, from the one adopted in the present paper.

- Pure N-type display rules:

$$gal_{\mathsf{N}}\ \frac{\tilde{\sim}\Gamma \vdash \Delta}{\tilde{\sim}\Delta \vdash \Gamma} \qquad \frac{\Gamma \vdash \tilde{\sim}\Delta}{\Delta \vdash \tilde{\sim}\Gamma}\ gal_{\mathsf{N}} \qquad res_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Sigma}{\Delta \vdash \tilde{\sim}\Gamma \mathbin{\check{\cup}} \Sigma} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Sigma}{\tilde{\sim}\Delta \mathbin{\hat{\cap}} \Gamma \vdash \Sigma}\ res_{\mathsf{N}}$$

- Pure S-type structural rules:

$$cont_{\mathsf{S}}\ \frac{X \vdash Y}{\tilde{\neg}Y \vdash \tilde{\neg}X} \qquad \hat{\top}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} \hat{\top} \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} \check{\bot}}\ \check{\bot}$$

$$W_{\mathsf{S}}\ \frac{X \vdash Y}{X \mathbin{\hat{\wedge}} Z \vdash Y} \qquad \frac{X \vdash Y}{X \vdash Y \mathbin{\check{\vee}} Z}\ W_{\mathsf{S}} \qquad C_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Y}{X \vdash Y}\ C_{\mathsf{S}}$$

$$E_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} Y \vdash Z}{Y \mathbin{\hat{\wedge}} X \vdash Y} \qquad \frac{X \vdash Y \mathbin{\check{\vee}} Z}{X \vdash Z \mathbin{\check{\vee}} Y}\ E_{\mathsf{S}} \qquad A_{\mathsf{S}}\ \frac{X \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} Z) \vdash W}{(X \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} Z \vdash W} \qquad \frac{W \vdash X \mathbin{\hat{\wedge}} (Y \mathbin{\hat{\wedge}} Z)}{W \vdash (X \mathbin{\hat{\wedge}} Y) \mathbin{\hat{\wedge}} Z}\ A_{\mathsf{S}}$$

- Pure N-type structural rules:

$$cont_{\mathsf{N}}\ \frac{\Gamma \vdash \Delta}{\tilde{\sim}\Delta \vdash \tilde{\sim}\Gamma} \qquad \hat{1}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \hat{1} \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \check{0}}\ \check{0}$$

$$W_{\mathsf{N}}\ \frac{\Gamma \vdash \Delta}{\Gamma \mathbin{\hat{\cap}} \Pi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Pi}\ W_{\mathsf{N}} \qquad C_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Delta}{\Gamma \vdash \Delta}\ C_{\mathsf{N}}$$

$$E_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} \Delta \vdash \Pi}{\Delta \mathbin{\hat{\cap}} \Gamma \vdash \Pi} \qquad \frac{\Gamma \vdash \Delta \mathbin{\check{\cup}} \Pi}{\Gamma \vdash \Pi \mathbin{\check{\cup}} \Delta}\ E_{\mathsf{N}} \qquad A_{\mathsf{N}}\ \frac{\Gamma \mathbin{\hat{\cap}} (\Delta \mathbin{\hat{\cap}} \Pi) \vdash \Sigma}{(\Gamma \mathbin{\hat{\cap}} \Delta) \mathbin{\hat{\cap}} \Pi \vdash \Sigma} \qquad \frac{\Sigma \vdash \Gamma \mathbin{\hat{\cap}} (\Delta \mathbin{\hat{\cap}} \Pi)}{\Sigma \vdash (\Gamma \mathbin{\hat{\cap}} \Delta) \mathbin{\hat{\cap}} \Pi}\ A_{\mathsf{N}}$$

- Pure S-type logical rules:

$$\neg\ \frac{\tilde{\neg}A \vdash X}{\neg A \vdash X} \qquad \frac{X \vdash \tilde{\neg}A}{X \vdash \neg A}\ \neg \qquad \wedge\ \frac{A \mathbin{\hat{\wedge}} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \quad Y \vdash B}{X \mathbin{\hat{\wedge}} Y \vdash A \wedge B}\ \wedge$$

Monotonic modal logic. D.MT∇ also includes the rules listed below.

- Multi-type display rules:

$$\langle\hat{\nu}\rangle[\check{\text{℩}}]\ \frac{\langle\hat{\nu}\rangle\Gamma \vdash X}{\Gamma \vdash [\check{\text{℩}}]X} \qquad \langle\hat{\text{℩}}^c\rangle[\check{\nu}^c]\ \frac{\langle\hat{\text{℩}}^c\rangle X \vdash \Gamma}{X \vdash [\check{\nu}^c]\Gamma} \qquad \langle\hat{\in}\rangle[\check{\ni}]\ \frac{\langle\hat{\in}\rangle\Gamma \vdash X}{\Gamma \vdash [\check{\ni}]X} \qquad \langle\hat{\not\ni}\rangle[\check{\notin}]\ \frac{\langle\hat{\not\ni}\rangle X \vdash \Gamma}{X \vdash [\check{\notin}]\Gamma}$$

- Logical rules for multi-type connectives:

$$\langle\nu\rangle\ \frac{\langle\hat{\nu}\rangle\alpha \vdash X}{\langle\nu\rangle\alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\langle\hat{\nu}\rangle\Gamma \vdash \langle\nu\rangle\alpha}\ \langle\nu\rangle \qquad [\nu^c]\ \frac{\alpha \vdash \Gamma}{[\nu^c]\alpha \vdash [\check{\nu}^c]\Gamma} \qquad \frac{X \vdash [\check{\nu}^c]\alpha}{X \vdash [\nu^c]\alpha}\ [\nu^c]$$

$$\langle\not\ni\rangle\ \frac{\langle\hat{\not\ni}\rangle A \vdash \Gamma}{\langle\not\ni\rangle A \vdash \Gamma} \qquad \frac{X \vdash A}{\langle\hat{\not\ni}\rangle X \vdash \langle\not\ni\rangle A}\ \langle\not\ni\rangle \qquad [\ni]\ \frac{A \vdash X}{[\ni]A \vdash [\check{\ni}]X} \qquad \frac{\Gamma \vdash [\check{\ni}]A}{\Gamma \vdash [\ni]A}\ [\ni]$$

Conditional logic. D.MT$>$ includes left and right logical rules for $[\ni]$, the display postulates $\langle\hat{\in}\rangle[\check{\ni}]$ and the rules listed below.

- Multi-type display rules:

$$\hat{\blacktriangle}\ \check{\triangleright}\ \frac{X \vdash \Gamma \mathbin{\check{\triangleright}} Y}{\Gamma \mathbin{\hat{\blacktriangle}} X \vdash Y} \qquad \frac{\Gamma \vdash X \mathbin{\check{\blacktriangleright}} Y}{X \vdash \Gamma \mathbin{\check{\triangleright}} Y}\ \check{\blacktriangleright}\ \check{\triangleright} \qquad \frac{X \vdash [\check{\notin}\rangle\Gamma}{\Gamma \vdash [\check{\not\ni}\rangle X}\ [\check{\notin}\rangle[\check{\not\ni}\rangle$$

- Logical rules for multi-type connectives and pure G-type logical rules:

$$\rhd\ \frac{\Gamma \vdash \alpha \qquad A \vdash X}{\alpha \rhd A \vdash \Gamma \,\check{\rhd}\, X} \qquad \frac{X \vdash \alpha \,\check{\rhd}\, A}{X \vdash \alpha \rhd A}\ \rhd \qquad [\not\ni\rangle\ \frac{X \vdash A}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle X} \qquad \frac{\Gamma \vdash [\check{\not\ni}\rangle A}{\Gamma \vdash [\not\ni\rangle A}\ [\not\ni\rangle$$

$$\cap\ \frac{\alpha \,\hat{\cap}\, \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \,\hat{\cap}\, \Delta \vdash \alpha \cap \beta}\ \cap$$

Axiomatic extensions of monotonic modal logic. Each rule is labelled with the name of its corresponding axiom.

$$\mathrm{N}\ \frac{\langle \hat{\not\ni} \rangle \hat{\top} \vdash \Gamma}{\hat{\top} \vdash [\check{\nu}^c]\Gamma} \qquad \mathrm{C}\ \frac{\langle \hat{\not\ni} \rangle (\langle \hat{\in} \rangle \Gamma \,\hat{\wedge}\, \langle \hat{\in} \rangle \Delta) \vdash \Theta}{\langle \hat{\nu} \rangle \Gamma \,\hat{\wedge}\, \langle \hat{\nu} \rangle \Delta \vdash [\check{\nu}^c]\Theta} \qquad \mathrm{D}\ \frac{\Gamma \vdash [\check{\ni}]\, \tilde{\neg} \langle \hat{\in} \rangle \Delta}{\langle \hat{\nu} \rangle \Delta \vdash \tilde{\neg} \langle \hat{\nu} \rangle \Gamma}$$

$$\mathrm{M}\ \frac{\langle \hat{\not\ni} \rangle \langle \hat{\in} \rangle \Gamma \vdash \Delta}{\langle \hat{\alpha}^c \rangle \langle \hat{\nu} \rangle \Gamma \vdash \Delta} \qquad \mathrm{P}\ \frac{\Gamma \vdash [\check{\ni}]\check{\bot}}{\hat{\top} \vdash \tilde{\neg} \langle \hat{\nu} \rangle \Gamma} \qquad \mathrm{T}\ \frac{\Gamma \vdash [\check{\ni}]X}{\langle \hat{\nu} \rangle \Gamma \vdash X}$$

Axiomatic extensions of conditional logic. Each rule is labelled with the name of its corresponding axiom.

$$\mathrm{ID}\ \frac{\Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \qquad \langle \hat{\in} \rangle \Gamma \vdash X}{\hat{\top} \vdash (\Gamma \,\hat{\cap}\, \Delta) \,\check{\rhd}\, X} \qquad \mathrm{CS}\ \frac{\Gamma \vdash [\check{\ni}][\check{\notin}\rangle \Delta \qquad X \vdash [\check{\notin}\rangle \Delta \qquad Y \vdash Z}{X \,\hat{\wedge}\, Y \vdash (\Gamma \,\hat{\cap}\, \Delta) \,\check{\rhd}\, Z}$$

$$\mathrm{CEM}\ \frac{\Pi \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \quad \Pi \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Theta \quad \Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Gamma \quad \Delta \vdash [\check{\not\ni}\rangle \langle \hat{\in} \rangle \Theta \quad Y \vdash X}{\hat{\top} \vdash ((\Gamma \,\hat{\cap}\, \Delta) \,\check{\rhd}\, X) \,\check{\vee}\, ((\Theta \,\hat{\cap}\, \Pi) \,\check{\rhd}\, \tilde{\neg} Y)}$$

$$\mathrm{CN}\ \frac{\Gamma \vdash [\check{\ni}][\check{\notin}\rangle \Delta \quad \Gamma \vdash [\check{\ni}]Y \quad \Theta \vdash [\check{\ni}][\check{\notin}\rangle \Pi \quad \Theta \vdash [\check{\ni}]X}{\hat{\top} \vdash ((\Gamma \,\hat{\cap}\, \Delta) \,\check{\rhd}\, X) \,\check{\vee}\, ((\Theta \,\hat{\cap}\, \Pi) \,\check{\rhd}\, Y)}$$

8. Properties

In this section we discuss the properties of the display calculi presented in the section above. Proofs of the following results for the display calculi associated with the basic logics of arbitrary D.LE languages are discussed in [35, Section 4.2]. They straightforwardly apply the basic proper display calculi associated with basic logics of the multi-type languages discussed in the present paper since their proof only relies on the order theoretic properties of the interpretation of the logical connectives. Below we only expand on the properties of the calculi for the relevant axiomatic extensions.

The display calculi introduced in the section above are proper (cf. [67,35]), and hence the general theory of proper multi-type display calculi guarantees that they enjoy *cut elimination* and *subformula property* [20].[8]

In [35, Section 7], it is shown that any analytic inductive inequality can be equivalently transformed via ALBA into a set of quasi-inequalities each of which corresponds to an analytic rule of the corresponding display calculus. Instantiating this result to the calculi of Section 7, let H_m (resp. H_c) be the class of all perfect heterogeneous m-algebras (resp. perfect heterogeneous c-algebras). Given a set of analytic inductive sequents R, the extension of D.MT∇ (resp. D.MT$>$) with inference rules obtained by running ALBA on R is denoted by D.MT∇R (resp. D.MT$>R$). The subclass of H_m (resp. H_c) defined by R is denoted by $H_m(R)$ (resp. $H_c(R)$).

8.1. Soundness

To show the soundness of the rules of D.MT∇R (resp. D.MT$>R$) w.r.t. $H_m(R)$ (resp. $H_c(R)$), it suffices to show that the interpretation of each rule[9] in D.MT∇R (resp. D.MT$>R$) is valid in $H_m(R)$ (resp. $H_c(R)$). The soundness of the rules in D.MT∇ and D.MT$>$ follows from the definitions of H_m and H_c, respectively. And the soundness of the rules from R follows from the soundness of ALBA rules on members of H_m (resp. H_c), and the ALBA runs reported in the appendix. Specifically, in what follows, for any perfect m-algebra (resp. c-algebra) $\mathbb{H} := (\mathbb{A}, \mathbb{B}, \ldots)$, let x range over $\mathbb{A}$ and γ, δ, θ range over $\mathbb{B}$. Then the rules on the left-hand side of the squiggly arrows below are interpreted as the quasi-inequalities on the right-hand side:

[8] As also observed in [11], the adjective 'proper' singles out a subclass of Belnap's display calculi [4] identified by Wansing in [67, Section 4.1]. A display calculus is *proper* if every structural rule is closed under uniform substitution. This requirement strengthens Belnap's conditions C_6 and C_7. In [20], this requirement is extended to *multi-type* display calculi. A logic is *(properly) displayable* if it can be captured by some (proper) display calculus (see [35, Section 2.2]).

[9] A rule is interpreted as an implication of the inequalities that correspond to the assumptions and conclusions of the rule. For a precise definition we refer the reader to [35, Section 4.2.1].

$$\frac{\langle\hat{\in}\rangle\Gamma \vdash X}{\Gamma \vdash [\check{\ni}]X} \quad\rightsquigarrow\quad \forall\gamma\forall x[\langle\in\rangle\gamma \leq x \Leftrightarrow \gamma \leq [\ni]x]$$

$$\frac{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle\Gamma \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle\Delta) \vdash \Theta}{\langle\hat{\nu}\rangle\Gamma \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle\Delta \vdash [\check{\nu^c}]\Theta} \quad\rightsquigarrow\quad \forall\gamma\forall\delta\forall\theta[\langle\not\ni\rangle(\langle\in\rangle\gamma \wedge \langle\in\rangle\delta) \leq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \leq [\nu^c]\theta]$$

$$\frac{\Gamma \vdash [\check{\ni}]\check{\bot}}{\hat{\top} \vdash \tilde{\neg}\langle\hat{\nu}\rangle\Gamma} \quad\rightsquigarrow\quad \forall\gamma[\gamma \leq [\ni]\bot \Rightarrow \top \leq \neg\langle\nu\rangle\gamma]$$

The validity of $\forall\gamma\forall x[\langle\in\rangle\gamma \leq x \Leftrightarrow \gamma \leq [\ni]x]$ follows from the fact that $\langle\in\rangle$ and $[\ni]$ form a residuation pair in $\mathbb{H}$. The validity of the quasi-inequalities corresponding to axioms C and P in $H_m(\{C\})$ and $H_m(\{P\})$ respectively follows from the validity-preserving ALBA runs reported in the appendix. We report below on the validity-preserving ALBA run for C.

C.	$\mathbb{H} \models \nabla p \wedge \nabla q \to \nabla(p \wedge q) \;\rightsquigarrow\; \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$	
	$\mathbb{H} \models \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$	
iff	$\mathbb{H} \models \forall\gamma\forall\delta\forall\theta\forall pq[\gamma \subseteq [\ni]p \;\&\; \delta \subseteq [\ni]q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta]$	first approx.
iff	$\mathbb{H} \models \forall\gamma\forall\delta\forall\theta\forall pq[\langle\in\rangle\gamma \subseteq p \;\&\; \langle\in\rangle\delta \subseteq q \;\&\; \langle\not\ni\rangle(p \wedge q) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta]$	Residuation
iff	$\mathbb{H} \models \forall\gamma\forall\delta\forall\theta[\langle\not\ni\rangle(\langle\in\rangle\gamma \wedge \langle\in\rangle\delta) \subseteq \theta \Rightarrow \langle\nu\rangle\gamma \wedge \langle\nu\rangle\delta \subseteq [\nu^c]\theta]$	$(\star)$ Ackermann

8.2. Completeness

As discussed above, the algorithmic correspondence perspective on the theory of analytic calculi (here in their incarnation as “proper display calculi”) allows for a uniform justification of the soundness of analytic rules in terms of the soundness of the algorithm ALBA used to generate them. These benefits extend also to the uniform justification of the *completeness* of proper display calculi w.r.t. the logics they are intended to capture.

First let us show the completeness w.r.t. the basic monotonic modal logic and conditional logic. The (translations of the) rules M, RCEA and RCK_n are derivable as follows.

M. $$\frac{A \vdash B}{\nabla A \vdash \nabla B} \quad\rightsquigarrow\quad \frac{A \vdash B}{\langle\nu\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B}$$

$$\begin{array}{rc}
 & A \vdash B \\ \hline
 & [\ni]A \vdash [\check{\ni}]B \\ \hline
 & \langle\hat{\in}\rangle[\ni]A \vdash B \\ \hline
 & \langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline
\mathrm{M} & \langle\hat{\iota}^c\rangle\langle\hat{\nu}\rangle[\ni]A \vdash \langle\not\ni\rangle B \\ \hline
 & \langle\hat{\nu}\rangle[\ni]A \vdash [\check{\nu^c}]\langle\not\ni\rangle B \\ \hline
 & \langle\hat{\nu}\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B \\ \hline
 & \langle\nu\rangle[\ni]A \vdash [\nu^c]\langle\not\ni\rangle B
\end{array}$$

RCEA. $$\frac{A \leftrightarrow B}{(A > C) \leftrightarrow (B > C)} \quad\rightsquigarrow\quad \frac{A \vdash B \qquad B \vdash A}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \rhd C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash A}{[\ni]B \vdash [\check{\ni}]A}}{[\ni]B \vdash [\ni]A} \qquad \dfrac{\dfrac{A \vdash B}{[\not\ni\rangle B \vdash [\check{\not\ni}\rangle A}}{[\not\ni\rangle B \vdash [\not\ni\rangle A}}{[\ni]B \mathbin{\hat{\cap}} [\not\ni\rangle B \vdash [\ni]\varphi \cap [\not\ni\rangle A}}{[\ni]B \cap [\not\ni\rangle B \vdash [\ni]\varphi \cap [\not\ni\rangle A} \qquad C \vdash C}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \mathbin{\check{\rhd}} C}}{([\ni]A \cap [\not\ni\rangle A) \rhd C \vdash ([\ni]B \cap [\not\ni\rangle B) \rhd C}$$

RCK_n. $$\frac{A_1 \wedge \ldots \wedge A_n \to B}{(C > A_1) \wedge \ldots \wedge (C > A_n) \to (C > B)}$$

$$\rightsquigarrow\quad \frac{A_1 \wedge \ldots \wedge A_n \vdash B}{([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge \ldots \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_n \vdash ([\ni]C \cap [\not\ni\rangle C) \rhd B}$$

To show that the translation of RCK_n is derivable, let us preliminarily show that $([\ni]C \cap [\not\ni\rangle C) \mathbin{\hat{\blacktriangle}} ([\ni]C \cap [\not\ni\rangle C) \rhd A_1 \wedge ([\ni]C \cap [\not\ni\rangle C) \rhd A_2 \vdash A_1 \wedge A_2$ is derivable.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{C\vdash C}{[\ni]C\vdash[\check{\ni}]C}}{[\ni]C\vdash[\ni]C}\quad\cfrac{\cfrac{C\vdash C}{[\not\ni\rangle C\vdash[\check{\not\ni}\rangle C}}{[\not\ni\rangle C\vdash[\not\ni\rangle C}}{[\ni]C\,\hat{\cap}\,[\not\ni\rangle C\vdash[\ni]C\cap[\not\ni\rangle C}}{[\ni]C\cap[\not\ni\rangle C\vdash[\ni]C\cap[\not\ni\rangle C}\quad A_1\vdash A_1}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_1}\,W_S}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\,\hat{\wedge}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_1}}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_1}}{([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash A_1}
\quad
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{C\vdash C}{[\ni]C\vdash[\check{\ni}]C}}{[\ni]C\vdash[\ni]C}\quad\cfrac{\cfrac{C\vdash C}{[\not\ni\rangle C\vdash[\check{\not\ni}\rangle C}}{[\not\ni\rangle C\vdash[\not\ni\rangle C}}{[\ni]C\,\hat{\cap}\,[\not\ni\rangle C\vdash[\ni]C\cap[\not\ni\rangle C}}{[\ni]C\cap[\not\ni\rangle C\vdash[\ni]C\cap[\not\ni\rangle C}\quad A_2\vdash A_2}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_2}\,W_S}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\,\hat{\wedge}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_2}}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,A_2}}{([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash A_2}
$$

$$
C_S\,\cfrac{(([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2)\,\hat{\wedge}\,(([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2)\vdash A_1\wedge A_2}{([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_2\vdash A_1\wedge A_2}
$$

Iterating the previous derivation $n-1$ times (where the specific instantiation of W_S is suitably chosen so as to derive the specific instantiation of the end sequent), we obtain the left premise of the following derivation, which provides the required derivation of the conclusion of RCK_n from its premise.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,(([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge\ldots\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_n)\vdash A_1\wedge\ldots\wedge A_n}\quad A_1\wedge\ldots\wedge A_n\vdash B}{([\ni]C\cap[\not\ni\rangle C)\,\hat{\blacktriangle}\,(([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge\ldots\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_n)\vdash B}\,Cut_S}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge\ldots\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_n\vdash([\ni]C\cap[\not\ni\rangle C)\,\check{\rhd}\,B}}{([\ni]C\cap[\not\ni\rangle C)\rhd A_1\wedge\ldots\wedge([\ni]C\cap[\not\ni\rangle C)\rhd A_n\vdash([\ni]C\cap[\not\ni\rangle C)\rhd B}
$$

As for the completeness of the axiomatic extensions, in [11] an effective procedure is introduced for generating cut free derivations of the translations of each rule and analytic inductive axiom (of any normal lattice expansion signature) in the corresponding proper display calculus. Below, we illustrate this effective procedure by applying it to the analytic axioms of the present setting.

N. $\nabla\top\ \rightsquigarrow\ [\nu^c]\langle\not\ni\rangle\top$ P. $\neg\nabla\bot\ \rightsquigarrow\ \neg\langle\nu\rangle[\ni]\bot$ T. $\nabla A\to A\ \rightsquigarrow\ \langle\nu\rangle[\ni]A\vdash A$

$$
\text{N}\,\cfrac{\cfrac{\hat{\top}\vdash\top}{\langle\hat{\ni}\rangle\hat{\top}\vdash\langle\ni\rangle\top}}{\hat{\top}\vdash[\check{\nu^c}]\langle\ni\rangle\top}
\qquad
\text{P}\,\cfrac{\cfrac{\bot\vdash\check{\bot}}{[\ni]\bot\vdash[\check{\ni}]\check{\bot}}}{\hat{\top}\vdash\tilde{\neg}[\ni]\bot}
\qquad
\text{T}\,\cfrac{\cfrac{A\vdash A}{[\ni]A\vdash[\check{\ni}]A}}{\langle\hat{\nu}\rangle[\ni]A\vdash A}
$$

ID. $A>A\ \rightsquigarrow\ ([\ni]A\cap[\not\ni\rangle A)\rhd A$

$$
\text{ID}\,\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A\vdash A}{[\not\ni\rangle A\vdash[\check{\not\ni}\rangle A}}{A\vdash[\check{\notin}\rangle[\not\ni\rangle A}}{[\ni]A\vdash[\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A}}{\langle\hat{\in}\rangle[\ni]A\vdash[\check{\notin}\rangle[\not\ni\rangle A}}{[\not\ni\rangle A\vdash[\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A}\quad\cfrac{\cfrac{A\vdash A}{[\ni]A\vdash[\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A\vdash A}}{\hat{\top}\vdash([\check{\ni}]A\,\hat{\cap}\,[\check{\not\ni}\rangle A)\,\check{\rhd}\,A}
$$

CS. $(A\wedge B)\to(A>B)\ \rightsquigarrow\ A\wedge B\vdash([\ni]A\cap[\not\ni\rangle A)\rhd B$

$$
\text{CS}\,\cfrac{\cfrac{\cfrac{\cfrac{A\vdash A}{[\not\ni\rangle A\vdash[\check{\not\ni}\rangle A}}{A\vdash[\check{\notin}\rangle[\not\ni\rangle A}}{[\ni]A\vdash[\check{\ni}][\check{\notin}\rangle[\not\ni\rangle A}\quad\cfrac{\cfrac{A\vdash A}{[\not\ni\rangle A\vdash[\check{\not\ni}\rangle A}}{A\vdash[\check{\notin}\rangle[\not\ni\rangle A}\quad B\vdash B}{A\,\hat{\wedge}\,B\vdash([\check{\ni}]A\,\hat{\cap}\,[\check{\not\ni}\rangle A)\,\check{\rhd}\,B}
$$

CEM. $(A>B)\vee(A>\neg B)\ \rightsquigarrow\ ([\ni]A\cap[\not\ni\rangle A)\rhd B\vee([\ni]A\cap[\not\ni\rangle A)\rhd\neg B$

$$
\text{CEM}\ \frac{
\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A}
\quad
\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A}
\quad
\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A}
\quad
\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle\langle\hat{\in}\rangle[\ni]A}
\quad
B \vdash B
}{\hat{\top} \vdash ([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\triangleright}} B \mathbin{\check{\vee}} ([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\triangleright}} \tilde{\neg} B}
$$

C. $\nabla A \wedge \nabla B \to \nabla(A \wedge B) \rightsquigarrow \langle\nu\rangle[\ni]A \wedge \langle\nu\rangle[\ni]B \vdash [\nu^c]\langle\not\ni\rangle(A \wedge B)$

D. $\nabla A \to \neg\nabla\neg A \rightsquigarrow \langle\nu\rangle[\ni]A \vdash \neg\langle\nu\rangle[\ni]\neg A$

$$
\text{C}\ \frac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\ni]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}\quad\dfrac{\dfrac{B \vdash B}{[\ni]B \vdash [\ni]B}}{\langle\hat{\in}\rangle[\ni]B \vdash B}}{\langle\hat{\in}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]B \vdash A \wedge B}}{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\in}\rangle[\ni]B) \vdash \langle\not\ni\rangle(A \wedge B)}}{\langle\hat{\nu}\rangle[\ni]A \mathbin{\hat{\wedge}} \langle\hat{\nu}\rangle[\ni]B \vdash [\check{\nu^c}]\langle\not\ni\rangle(A \wedge B)}
\qquad
\text{D}\ \frac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}}{\langle\hat{\in}\rangle[\ni]A \vdash A}}{\neg A \vdash \tilde{\neg}\langle\hat{\in}\rangle[\ni]A}}{[\ni]\neg A \vdash [\check{\ni}]\tilde{\neg}\langle\hat{\in}\rangle[\ni]A}}{\langle\hat{\nu}\rangle[\ni]A \vdash \tilde{\neg}\langle\hat{\nu}\rangle[\ni]\neg A}
$$

CN. $(A > B) \vee (B > A) \ \rightsquigarrow\ ([\ni]A \cap [\not\ni\rangle A) \triangleright B \vee ([\ni]B \cap [\not\ni\rangle B) \triangleright A$

$$
\text{CN}\ \frac{
\dfrac{\dfrac{\dfrac{A \vdash A}{[\not\ni\rangle A \vdash [\check{\not\ni}\rangle A}}{A \vdash [\check{\not\in}\rangle[\not\ni\rangle A}}{[\ni]A \vdash [\check{\ni}][\check{\not\in}\rangle[\not\ni\rangle A}
\quad
\dfrac{A \vdash A}{[\ni]A \vdash [\check{\ni}]A}
\quad
\dfrac{\dfrac{\dfrac{B \vdash B}{[\not\ni\rangle B \vdash [\check{\not\ni}\rangle B}}{B \vdash [\check{\not\in}\rangle[\not\ni\rangle B}}{[\ni]B \vdash [\check{\ni}][\check{\not\in}\rangle[\not\ni\rangle B}
\quad
\dfrac{B \vdash B}{[\ni]B \vdash [\check{\ni}]B}
}{\hat{\top} \vdash (([\ni]A \mathbin{\hat{\cap}} [\not\ni\rangle A) \mathbin{\check{\triangleright}} B) \mathbin{\check{\vee}} (([\ni]B \mathbin{\hat{\cap}} [\not\ni\rangle B) \mathbin{\check{\triangleright}} A)}
$$

8.3. *Conservativity*

To argue that the calculi introduced in Section 7 conservatively extend their corresponding Hilbert systems, we follow the standard proof strategy discussed in [35,36]. Let $\vdash_{\mathbf{L}}$ denote the syntactic consequence relation arising from Hilbert systems presented in Section 2.1, and $\models_{\mathrm{H}}$ denote the semantic consequence relation arising from heterogeneous Kripke frames and their complex (heterogeneous) algebras. We need to show that, for all formulas A and B of the original language of the Hilbert system, if $\tau(A \vdash B)$ is derivable in a display calculus, then $A \vdash_{\mathbf{L}} B$. This claim can be proved using the following facts: (a) the rules of display calculi are sound w.r.t. heterogeneous Kripke frames and their complex (heterogeneous) algebras (cf. Section 8.1); (b) Hilbert systems are complete w.r.t. their respective class of algebras; and (c) homogenous algebras are equivalently presented as heterogeneous algebras (cf. Section 3.2), so that the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Proposition 18). Then, let $A \vdash B$ be an entailment between formulas of the language of the original Hilbert systems. If $\tau(A \vdash B)$ is derivable in a display calculus, then, by (a), $\models_{\mathrm{H}} \tau(A \vdash B)$. By (c), this implies that $A \models_V B$, where $\models_V$ denotes the semantic consequence relation arising from m-algebras or c-algebras. By (b), this implies that $A \vdash_{\mathbf{L}} B$, as required.

9. Conclusions and further directions

Present contributions. In the present paper, we have proposed a semantic analysis of two well-known non-normal logics (monotonic modal logic and conditional logic), and used it to introduce both a uniform correspondence-theoretic framework encompassing and significantly extending various well-known Sahlqvist-type results for these logics, and a proof-theoretic framework modularly capturing not only the basic logics but also an infinite class of axiomatic extensions of the basic monotonic modal logic and conditional logic. The correspondence-theoretic and the proof-theoretic frameworks are closely connected with each other, both because they stem from the same semantic analysis, and because, more fundamentally, they instantiate results, tools and insights developed at the interface of correspondence theory and structural proof theory [35]. This line of research can be naturally extended in various ways, and in what follows we list some natural further directions.

A modular framework for classical modal logic. In the present paper, we have considered monotonic modal logic and conditional logic because this choice made it possible to address a significant diversity of order-theoretic behaviour of the non-normal connectives with a minimal set of examples: namely a unary monotone operator and a binary operator which is normal (finitely meet-preserving) in its second coordinate and arbitrary in the first coordinate. A natural further direction concerns the systematic application of these techniques to wider classes of non-normal logics. Even restricting attention to the signature of $\mathscr{L}_\nabla$, a natural direction concerns developing a modular account of classical modal logic [9] and its (monotone, regular) extensions up to normal modal logic. Of course the translations employed in the present paper for monotonic modal logic do not account for classical modal logic, because monotonicity is in-built in these translations. The question is then whether one can express monotonicity as an (analytic) inductive condition under a translation similar to the one used in the non-normal coordinate of the conditional logic operator $>$.

From Boolean to distributive lattice-based non-normal logics. The semantic analysis of the present paper hinges on the embedding of well-known state-based semantics (monotone neighbourhood frames, selection functions) into two-sorted classical Kripke frames and their discrete dualities with perfect (heterogeneous) Boolean algebras. Pivoting on more general discrete dualities, such as Birkhoff's discrete duality between perfect distributive lattices and posets, one can develop the systematic theory of e.g. the *non-normal* counterparts of positive modal logic [17,7] or intuitionistic modal logics [18,19,59]. In particular, it would be interesting to investigate the applicability of the present approach for capturing the lattice of non-normal intuitionistic modal logics introduced in [14].

Neighbourhood and selection functions as formal tools for context-relativization and category-formation. We plan to investigate alternative (intuitive) interpretations of neighbourhood and conditional frames in order to expand the realm of possible applications.
A natural option would be to consider a neighbourhood as a context relativising the interpretation of a term. An obvious application would be in lexical semantics (see e.g. [2]) where the meaning of a word is often context-dependent.
A second option would be to consider neighbourhoods as categories. Again, an obvious application would be in computational linguistics (see e.g. [47]) where each word is assigned to a syntactical category depending on the role it plays in the formation of grammatically correct sentences or phrases.
Notice that a word can occur in different contexts or it can be assigned to different categories. Therefore, one may consider generalizations of the framework with multiple (weighed) neighbourhood functions or relations as a way to represent (probabilistic) distributions in a data set.
In many machine learning approaches, a system needs both positive and negative evidence. For example, a classification system needs examples for each class that it is capable of predicting; if the classification is binary (e.g. the system tries to decide whether an email is spam or not), it needs to have positive and negative examples. This generalises to multiple classes (e.g. given a music song, predict the genre of that song). Therefore, one may consider (generalisations of) bi-neighbourhood frames (see e.g. [15]), in which sets of pairs of neighbourhoods provide independent positive and negative evidence.
Finally, each neighbourhood can be endowed with additional structure in order to capture specific behaviour. This refinement would build a bridge between the literature in non-normal modal logics and the literature on so-called modal logics for structural control in linguistics and logic (see e.g. [46,52,30,36]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Algorithmic proof of Theorem 3

In what follows, we show that the correspondence results collected in Theorem 3 can be retrieved as instances of a suitable multi-type version of algorithmic correspondence for normal logics (cf. [12,13]), hinging on the usual order-theoretic properties of the algebraic interpretations of the logical connectives, while admitting nominal variables of two sorts. For the sake of enabling a swift translation into the language of m-frames and c-frames, we write nominals directly as singletons, and, abusing notation, we quantify over the elements defining these singletons. These computations also serve to prove that each analytic structural rule is sound on the heterogeneous perfect algebras validating its correspondent axiom. In the computations relative to each analytic axiom, the line marked with $(\star)$ marks the quasi-inequality that interprets the corresponding analytic rule. This computation does *not* prove the equivalence between the axiom and the rule, since the variables occurring in each starred quasi-inequality are restricted rather than arbitrary. However, the proof of soundness is completed by observing that all ALBA rules in the steps above the marked inequalities are (inverse) Ackermann and adjunction rules, and hence are sound also when arbitrary variables replace (co-)nominal variables.

N. $\mathbb{H} \models \nabla\top \rightsquigarrow \top \subseteq [\nu^c]\langle\not\ni\rangle\top$

	$\top \subseteq [\nu^c]\langle\not\ni\rangle\top$	
iff	$\forall X \forall w[\langle\not\ni\rangle\top \subseteq \{X\}^c \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c]$	$(\star)$ first. app.
iff	$\forall X \forall w[X = W \Rightarrow \{w\} \subseteq [\nu^c]\{X\}^c)$	$(\langle\ni\rangle\top = \{W\}^c)$
iff	$\forall w[\{w\} \subseteq [\nu^c]\{W\}^c]$	
iff	$\forall w[\{w\} \subseteq (R_{\nu^c}^{-1}[W])^c]$	
iff	$\forall w[\{w\} \subseteq R_{\nu}^{-1}[W]]$	
iff	$\forall w[W \in \nu(w)]$	

P. $\mathbb{H} \models \neg\nabla\bot \rightsquigarrow \top \subseteq \neg\langle\nu\rangle[\ni]\bot$

	$\top \subseteq \neg\langle\nu\rangle[\ni]\bot$	
iff	$\forall X[X \subseteq [\ni]\bot \Rightarrow T \subseteq \neg\langle\nu\rangle X]$	$(\star)$ first. app.
iff	$W \subseteq \neg\langle\nu\rangle[\ni]\emptyset$	
iff	$W \subseteq \neg\langle\nu\rangle\{\emptyset\}$	$[\ni]\emptyset = \{Z \subseteq W \mid Z \subseteq \emptyset\}$
iff	$W \subseteq \{w \in W \mid wR_{\nu}\emptyset\}^c$	
iff	$\forall w[\emptyset \notin \nu(w)]$.	

C. $\mathbb{H} \models \nabla p \wedge \nabla q \to \nabla(p \wedge q) \rightsquigarrow \langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

$\langle\nu\rangle[\ni]p \wedge \langle\nu\rangle[\ni]q \subseteq [\nu^c]\langle\not\ni\rangle(p \wedge q)$

iff $\forall Z_1 Z_2 Z_3 \forall pq[\{Z_1\} \subseteq [\ni]p \ \&\ \{Z_2\} \subseteq [\ni]q \ \&\ \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ first approx.

iff $\forall Z_1 Z_2 Z_3 \forall pq[\langle\in\rangle\{Z_1\} \subseteq p \ \&\ \langle\in\rangle\{Z_2\} \subseteq q \ \&\ \langle\not\ni\rangle(p \wedge q) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ Residuation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\langle\not\ni\rangle(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq \{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ $(\star)$ Ackermann

iff $\forall Z_1 \forall Z_2 \forall Z_3[(\langle\in\rangle\{Z_1\} \wedge \langle\in\rangle\{Z_2\}) \subseteq [\not\in]\{Z_3\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \wedge \langle\nu\rangle\{Z_2\} \subseteq [\nu^c]\{Z_3\}^c]$ Residuation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\forall x(xR_\in Z_1 \ \&\ xR_\in Z_2 \Rightarrow \neg xR_{\not\in} Z_3) \Rightarrow \forall x(xR_\nu Z_1 \ \&\ xR_\nu Z_2 \Rightarrow \neg xR_{\nu^c} Z_3)]$ Standard translation

iff $\forall Z_1 \forall Z_2 \forall Z_3[\forall x(x \in Z_1 \ \&\ x \in Z_2 \Rightarrow x \in Z_3) \Rightarrow \forall x(Z_1 \in \nu(x) \ \&\ Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$ Relations interpretation

iff $\forall Z_1 \forall Z_2 \forall Z_3[Z_1 \cap Z_2 \subseteq Z_3 \Rightarrow \forall x(Z_1 \in \nu(x) \ \&\ Z_2 \in \nu(x) \Rightarrow Z_3 \in \nu(x))]$

iff $\forall Z_1 \forall Z_2 \forall x[Z_1 \in \nu(x) \ \&\ Z_2 \in \nu(x) \Rightarrow Z_1 \cap Z_2 \in \nu(x))]$. Monotonicity

4'. $\mathbb{H} \models \nabla p \to \nabla\nabla p \rightsquigarrow \langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

$\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p$

iff $\forall Z_1 \forall x' \forall p[\{Z_1\} \subseteq [\ni]p \ \&\ [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ first approx.

iff $\forall Z_1 \forall x' \forall p[\langle\in\rangle\{Z_1\} \subseteq p \ \&\ [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle p \subseteq \{x'\}^c) \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Residuation

iff $\forall Z_1 \forall x'[[\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\} \subseteq \{x'\}^c \Rightarrow \langle\nu\rangle\{Z_1\} \subseteq \{x'\}^c]$ Ackermann

iff $\forall Z_1[\langle\nu\rangle\{Z_1\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu^c]\langle\not\ni\rangle\langle\in\rangle\{Z_1\}]$

iff $\forall Z_1 \forall x[xR_\nu Z_1 \Rightarrow \forall Z_2(xR_{\nu^c} Z_2 \Rightarrow \exists y(Z_2 R_{\not\ni} y \ \&\ \forall Z_3(yR_{\nu^c} Z_3 \Rightarrow \exists w(Z_3 R_{\not\ni} w \ \&\ wR_\in Z_1))))]$ Standard translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \ \&\ \forall Z_3(Z_2 \notin \nu(y) \Rightarrow \exists w(w \notin Z_3 \ \&\ w \in Z_1))))]$ Relations translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \forall Z_2(Z_2 \notin \nu(x) \Rightarrow \exists y(y \notin Z_2 \ \&\ \forall Z_3(Z_2 \notin \nu(y) \Rightarrow Z_1 \not\subseteq Z_3)))]$ Relations translation

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(\forall Z_3(Z_1 \subseteq Z_3 \Rightarrow Z_3 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$ Contraposition

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow (\forall Z_2(\forall y(Z_1 \in \nu(y)) \Rightarrow y \in Z_2) \Rightarrow Z_2 \in \nu(x)))]$ Monotonicity

iff $\forall Z_1 \forall x[x \in \nu(Z) \Rightarrow \{y \mid Z_1 \in \nu(y)\} \in \nu(x)]$. Monotonicity

4. $\mathbb{H} \models \nabla\nabla p \to \nabla p \rightsquigarrow \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$

$\langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \subseteq [\nu^c]\langle\not\ni\rangle p$

iff $\forall x \forall Z_1 \forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \ \&\ \langle\not\ni\rangle p \subseteq \{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ first approx.

iff $\forall x \forall Z_1 \forall p[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni]p \ \&\ p \subseteq [\not\in]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ Adjunction

iff $\forall x \forall Z_1[\{x\} \subseteq \langle\nu\rangle[\ni]\langle\nu\rangle[\ni][\not\in]\{Z_1\}^c \Rightarrow \{x\} \subseteq [\nu^c]\{Z_1\}^c]$ Ackermann

iff $\forall x \forall Z_1[(\exists Z_2(xR_\nu Z_2 \ \&\ \forall y(Z_2 R_\ni y \Rightarrow \exists Z_3(yR_\nu Z_3 \ \&\ \forall w(Z_3 R_\ni w \Rightarrow \neg wR_{\not\in} Z_1))))) \Rightarrow \neg xR_{\nu^c} Z_1]$ Standard translation

iff $\forall x \forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(\forall w \in Z_3)(w \in Z_1)) \Rightarrow Z_1 \in \nu(x)]$ Relation translation

iff $\forall x \forall Z_1[((\exists Z_2 \in \nu(x))(\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$

iff $\forall x \forall Z_1 \forall Z_2[(Z_2 \in \nu(x) \ \&\ (\forall y \in Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_1 \in \nu(x)]$

iff $\forall x \forall Z_1 \forall Z_2[(Z_2 \in \nu(x) \ \&\ (\forall y \in Z_2)(Z_1 \in \nu(y))) \Rightarrow Z_1 \in \nu(x)]$ Monotonicity

5. $\mathbb{H} \models \neg\nabla\neg p \to \nabla\neg\nabla\neg p \rightsquigarrow \neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

$\neg[\nu^c]\langle\not\ni\rangle\neg p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$

iff $\forall x \forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \ \&\ \langle\not\ni\rangle\neg p \subseteq \{Z_1\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ first approx.

iff $\forall x \forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p \subseteq \{x\}^c \ \&\ \neg[\not\in]\{Z_1\}^c \subseteq p \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ Residuation

iff $\forall x \forall Z_1[[\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\not\in]\{Z_1\}^c \subseteq \{x\}^c \Rightarrow \neg[\nu^c]\{Z\}^c \subseteq \{x\}^c]$ Ackermann

iff $\forall Z_1[\neg[\nu^c]\{Z_1\}^c \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\neg[\not\in]\{Z_1\}^c]$

iff $\forall Z_1 \forall x[xR_{\nu^c} Z_1 \Rightarrow \forall Z_2(xR_{\nu^c} Z_2 \Rightarrow \exists y(Z_2 R_{\not\ni} y \ \&\ \forall Z_3(yR_\nu Z_3 \Rightarrow \exists w(Z_3 R_\ni w \ \&\ wR_{\not\in} Z_1))))]$ Standard translation

iff $\forall Z_1 \forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(\exists w \in Z_3)(w \notin Z_1)]$ Relation translation

iff $\forall Z_1 \forall x[Z_1 \notin \nu(x) \Rightarrow (\forall Z_2 \notin \nu(x))(\exists y \notin Z_2)(\forall Z_3 \in \nu(y))(Z_3 \not\subseteq Z_1)]$

iff $\forall Z_1 \forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2(((\forall y \notin Z_2)(\exists Z_3 \in \nu(y))(Z_3 \subseteq Z_1)) \Rightarrow Z_2 \in \nu(x))]$ Contraposition

iff $\forall Z_1 \forall x[Z_1 \notin \nu(x) \Rightarrow \forall Z_2((\forall y \notin Z_2)(Z_1 \in \nu(y)) \Rightarrow Z_2 \in \nu(x))]$ Monotonicity

iff $\forall Z_1 \forall x[Z_1 \notin \nu(x) \Rightarrow \{y \mid Z_1 \in \nu(y)\}^c \in \nu(x))]$ Monotonicity

B. $\mathbb{H} \models p \to \nabla\neg\nabla\neg p \;\rightsquigarrow\; p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$		
	$p \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p$	
iff	$\forall x \forall p[\{x\} \subseteq p \Rightarrow \{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg p]$	first approx.
iff	$\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle\neg\langle\nu\rangle[\ni]\neg\{x\}]$	Ackermann
iff	$\forall x[\{x\} \subseteq [\nu^c]\langle\not\ni\rangle[\nu]\langle\ni\rangle\{x\}]$	
iff	$\forall x[\forall Z_1(xR_{\nu^c}Y \Rightarrow \exists y(YR_{\not\ni}x \;\&\; \forall Z_2(yR_\nu Z_2 \Rightarrow Z_2R_\ni x)))]$	Standard translation
iff	$\forall x[\forall Z_1(Z_1 \notin \nu(x) \Rightarrow \exists y(x \notin Z_1 \;\&\; \forall Z_2(Z_2 \in \nu(y) \Rightarrow x \in Z_2)))]$	Relations translation
iff	$\forall x[\forall Z_1(\forall y(\forall Z_2(x \notin Z_2 \Rightarrow Z_2 \notin \nu(y)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$	Contrapositive
iff	$\forall x[\forall Z_1(\forall y(\{x\}^c \notin \nu(y_1)) \Rightarrow y \in Z_1) \Rightarrow Z_1 \in \nu(x))]$	Monotonicity
iff	$\forall x[\{y \mid \{x\}^c \notin \nu(y)\} \in \nu(x))]$	Monotonicity
iff	$\forall x \forall X[x \in X \Rightarrow \{y \mid X^c \notin \nu(y)\} \in \nu(x)]$	Monotonicity

D. $\mathbb{H} \models \nabla p \to \neg\nabla\neg p \;\rightsquigarrow\; \langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$		
	$\langle\nu\rangle[\ni]p \subseteq \neg\langle\nu\rangle[\ni]\neg p$	
iff	$\forall Z \forall Z'[\{Z\} \subseteq [\ni]p \;\&\; Z' \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle Z']$	first approx.
iff	$\forall Z \forall Z'[\langle\in\rangle\{Z\} \subseteq p \;\&\; \{Z'\} \subseteq [\ni]\neg p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$	Residuation
iff	$\forall Z \forall Z'[\{Z'\} \subseteq [\ni]\neg\langle\in\rangle\{Z\} \Rightarrow \langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle\{Z'\}]$	$(\star)$ Ackermann
iff	$\forall Z[\langle\nu\rangle\{Z\} \subseteq \neg\langle\nu\rangle[\ni]\neg\langle\in\rangle\{Z\}]$	
iff	$\forall Z[\langle\nu\rangle\{Z\} \subseteq [\nu]\langle\ni\rangle\langle\in\rangle\{Z\}]$	
iff	$\forall Z \forall x[xR_\nu Z \Rightarrow \forall Y(xR_\nu Y \Rightarrow \exists w(YR_\ni w \;\&\; wR_\in Z))]$	Standard Translation
iff	$\forall Z \forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow \exists w(w \in Y \;\&\; w \in Z))]$	Relation translation
iff	$\forall Z \forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \in \nu(x) \Rightarrow Y \not\subseteq Z^c)]$	
iff	$\forall Z \forall x[Z \in \nu(x) \Rightarrow \forall Y(Y \subseteq Z^c \Rightarrow Y \notin \nu(x))]$	Contrapositive
iff	$\forall Z \forall x \forall Y[Z \in \nu(x) \Rightarrow Z^c \notin \nu(x)]$	Monotonicity

CS. $\mathbb{H} \models (p \wedge q) \to (p \succ q) \;\rightsquigarrow\; (p \wedge q) \subseteq ([\ni]p \cap [\not\ni\rangle p)\rhd q$		
	$(p \wedge q) \subseteq ([\ni]p \cap [\not\ni\rangle p)\rhd q$	
iff	$\forall x \forall Z \forall x' \forall pq[\{x\} \subseteq p \wedge q \;\&\; \{Z\} \subseteq [\ni]p \cap [\not\ni\rangle p \;\&\; q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$	first. approx.
iff	$\forall x \forall Z \forall x \forall p \forall q[\{x\} \subseteq p \;\&\; \{x\} \subseteq q \;\&\; \{Z\} \subseteq [\ni]p \;\&\; \{Z\} \subseteq [\not\ni\rangle p \;\&\; q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$	Splitting rule
iff	$\forall x \forall Z \forall x' \forall p \forall q[\{x\} \subseteq p \;\&\; \{x\} \subseteq q \;\&\; \{Z\} \subseteq [\ni]p \;\&\; p \subseteq [\notin\rangle\{Z\} \;\&\; q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$	Residuation
iff	$\forall x \forall Z \forall x' \forall q[\{x\} \subseteq [\notin\rangle\{Z\} \;\&\; \{x\} \subseteq q \;\&\; \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \;\&\; q \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$	Ackermann
iff	$\forall x \forall Z \forall x'[\{x\} \subseteq [\notin\rangle\{Z\} \;\&\; \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \;\&\; \{x\} \subseteq \{x'\}^c \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x'\}^c]$	$(\star)$ Ackermann
iff	$\forall x \forall Z[\{x\} \subseteq [\notin\rangle\{Z\} \;\&\; \{Z\} \subseteq [\ni][\notin\rangle\{Z\} \Rightarrow \{x\} \subseteq \{Z\}\rhd\{x\}]$	
iff	$\forall x \forall Z[\neg xR_\notin Z \;\&\; \forall y(ZR_\ni y \Rightarrow \neg yR_\notin Z) \Rightarrow \forall y(T_f(x, Z, y) \Rightarrow y = x)]$	Standard translation
iff	$\forall x \forall Z[x \in Z \;\&\; \forall y(y \in Z \Rightarrow Z \in y) \Rightarrow \forall y(y \in f(x, Z) \Rightarrow y = x)]$	Relation interpretation
iff	$\forall x \forall Z[x \in Z \;\Rightarrow\; \forall y(y \in f(x, Z) \Rightarrow y = x)]$	
iff	$\forall x \forall Z[x \in Z \Rightarrow f(x, Z) \subseteq \{x\}]$	

ID. $\mathbb{H} \models p \succ p \;\rightsquigarrow\; ([\ni]p \cap [\not\ni\rangle p)\rhd p$		
	$\top \subseteq ([\ni]p \cap [\not\ni\rangle p)\rhd p$	
iff	$\forall Z Z' \forall x' p[(\{Z\} \subseteq [\ni]p \;\&\; \{Z'\} \subseteq [\not\ni\rangle p \;\&\; p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\})\rhd\{x'\}^c]$	first approx.
iff	$\forall Z Z' \forall x' p[(\langle\in\rangle\{Z\} \subseteq p \;\&\; \{Z'\} \subseteq [\not\ni\rangle p \;\&\; p \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\})\rhd\{x'\}^c]$	Adjunction
iff	$\forall Z \forall Z' \forall x'[(\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \;\&\; \langle\in\rangle\{Z\} \subseteq \{x'\}^c) \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\})\rhd\{x'\}^c$	Ackermann
iff	$\forall Z \forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \;\Rightarrow \forall x'[\langle\in\rangle\{Z\} \subseteq \{x'\}^c \Rightarrow \top \subseteq (\{Z\} \cap \{Z'\})\rhd\{x'\}^c]]$	Currying
iff	$\forall Z \forall Z'[\{Z'\} \subseteq [\not\ni\rangle\langle\in\rangle\{Z\} \;\Rightarrow \top \subseteq (\{Z\} \cap \{Z'\})\rhd\langle\in\rangle\{Z\}]$	$(\star)$ Ackermann
iff	$\forall x \forall Z \forall Z'[\forall w(Z'R_{\not\ni}w \Rightarrow \neg wR_\in Z) \Rightarrow \forall y(T_f(x, Z, y) \;\&\; Z = Z' \Rightarrow y \in Z)]$	Standard Translation
iff	$\forall x \forall Z \forall Z' \forall y[\forall w(Z'R_{\not\ni}w \Rightarrow \neg wR_\in Z) \;\&\; (T_f(x, Z, y) \;\&\; Z = Z' \Rightarrow y \in Z)]$	
iff	$\forall x \forall Z \forall Z' \forall y[\forall w(w \notin Z' \Rightarrow w \notin Z) \;\&\; (y \in f(x, Z) \;\&\; Z = Z' \Rightarrow y \in Z)]$	Relation interpretation
iff	$\forall x \forall Z \forall Z' \forall y[Z \subseteq Z' \;\&\; (y \in f(x, Z) \;\&\; Z = Z' \Rightarrow y \in Z)]$	
iff	$\forall x \forall Z \forall y[(y \in f(x, Z) \Rightarrow y \in Z)]$	
iff	$\forall x \forall Z[f(x, Z) \subseteq Z]$	

T. $\mathbb{H} \models \nabla p \to p \ \rightsquigarrow\ \langle\nu\rangle[\ni]p \subseteq p$

	$\langle\nu\rangle[\ni]p \subseteq p$	
iff	$\forall x \forall Z \forall p[p \subseteq \{x\}^c \ \&\ \{Z\} \subseteq [\ni]p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c]$	first approx.
iff	$\forall x \forall Z \forall p[p \subseteq \{x\}^c \ \&\ \langle\in\rangle\{Z\} \subseteq p \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c]$	Adjunction
iff	$\forall x \forall Z[\langle\in\rangle\{Z\} \subseteq \{x\}^c \Rightarrow \langle\nu\rangle\{Z\} \subseteq \{x\}^c]$	$(\star)$ Ackermann
iff	$\forall Z[\langle\nu\rangle\{Z\} \subseteq \langle\ni\rangle\{Z\}]$	inverse approx.
iff	$\forall x \forall Z[xR_\nu Z \Rightarrow xR_\ni Z]$	Standard translation
iff	$\forall x \forall Z[Z \in \nu(x) \Rightarrow x \in Z]$.	Relation translation

CEM. $\mathbb{H} \models (p \succ q) \vee (p \succ \neg q) \ \rightsquigarrow\ (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$

	$\top \subseteq (([\ni]p \cap [\not\ni\rangle p) \rhd q) \vee (([\ni]p \cap [\not\ni\rangle p) \rhd \neg q)$	
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ \{Y\} \subseteq [\ni]p \cap [\not\ni\rangle p \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\})$	first approx.
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \ \&\ \{X\} \subseteq [\not\ni\rangle p \ \&\ \{Y\} \subseteq [\ni]p \ \&\ \{Y\} \subseteq [\not\ni\rangle p \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\})$ $(\star)$	Splitting
iff	$\forall p \forall q \forall X \forall Y \forall x \forall y(\{X\} \subseteq [\ni]p \ \&\ p \subseteq [\notin\rangle\{X\} \ \&\ \{Y\} \subseteq [\ni]p \ \&\ p \subseteq [\notin\rangle\{Y\} \ \&\ q \subseteq \{x\}^c \ \&\ \{y\} \subseteq q$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\})$	Residuation
iff	$\forall X \forall Y \forall x \forall y(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \ \&\ \{y\} \subseteq \{x\}^c$ $\Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\})$	Ackermann
iff	$\forall X \forall Y \forall x(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \Rightarrow \forall y(\{y\} \subseteq \{x\}^c \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{y\}))$	Currying
iff	$\forall X \forall Y \forall x(\{X\} \vee \{Y\} \subseteq [\ni]([\notin\rangle\{X\} \wedge [\notin\rangle\{Y\}) \Rightarrow \top \subseteq (\{X\} \rhd \{x\}^c) \vee (\{Y\} \rhd \neg\{x\}^c))$	
iff	$\forall X \forall Y \forall x[(\forall y(XR_\ni y \ \text{ or } \ YR_\ni y) \Rightarrow \neg yR_\notin X \ \&\ \neg yR_\notin Y)$ $\Rightarrow \forall y(\neg T_f(y, X, x) \ \text{ or } \ (\forall z(T_f(y, Y, z) \Rightarrow z = x)))]$	Standard translation
iff	$\forall X \forall Y \forall x[(\forall y(y \in X \text{ or } \ y \in Y) \Rightarrow y \in X \ \&\ y \in Y)$ $\Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))]$	Relation interpretation
iff	$\forall X \forall Y \forall x[(X \cup Y \subseteq X \cap Y) \ \Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))]$	
iff	$\forall X \forall Y \forall x[X = Y \Rightarrow \forall y(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, Y) \Rightarrow z = x)))]$	
iff	$\forall X \forall x \forall y[(x \notin f(y, X) \ \text{ or } \ (\forall z(z \in f(y, X) \Rightarrow z = x)))]$	
iff	$\forall X \forall x \forall y[(x \in f(y, X) \ \Rightarrow \ f(y, X) = \{x\})]$	
iff	$\forall X \forall y[\lvert f(y, X)\rvert \leq 1]$.	

CN. $\mathbb{H} \models (p > q) \vee (q > p) \ \rightsquigarrow\ ([\ni]p \wedge [\not\ni\rangle p) \rhd q) \vee (([\ni]q \wedge [\not\ni\rangle q) \rhd p$

	$\top \subseteq (([\ni]p \wedge [\not\ni\rangle p) \rhd q) \vee (([\ni]q \wedge [\not\ni\rangle q) \rhd p)$	
iff	$\forall p \forall q \forall X \forall Y(\{X\} \subseteq [\ni]p \ \&\ \{Y\} \subseteq [\ni]q \ \Rightarrow \top \subseteq ((\{X\} \cap [\not\ni\rangle p) \rhd q) \vee ((\{Y\} \cap [\not\ni\rangle q) \rhd p)$	Approx.
iff	$\forall p \forall q \forall X \forall Y(\langle\in\rangle\{X\} \subseteq p \ \&\ \langle\in\rangle\{Y\} \subseteq q \Rightarrow \top \subseteq ((\{X\} \cap [\not\ni\rangle p) \rhd q) \vee ((\{Y\} \cap [\not\ni\rangle q) \rhd p)$	Residuation
iff	$\forall X \forall Y(\top \subseteq ((\{X\} \cap [\not\ni\rangle\langle\in\rangle\{X\}) \rhd \langle\in\rangle\{Y\}) \vee ((\{Y\} \cap [\not\ni\rangle\langle\in\rangle\{Y\}) \rhd \langle\in\rangle\{X\}))$	Ackermann
iff	$\forall X \forall Y \forall z[z \in (T_f^{(0)}[\{X\} \cap (R_{\not\ni}^{-1}[R_\ni[\{X\}]])^c, (R_\ni[\{Y\}])^c])^c \cup (T_f^{(0)}[\{Y\}$ $\cap (R_{\not\ni}^{-1}[R_\ni[\{Y\}]])^c, (R_\ni[\{X\}])^c])^c]$	Standard translation
iff	$\forall X \forall Y \forall z[z \in (T_f^{(0)}[\{X\} \cap (R_{\not\ni}^{-1}[X])^c, Y^c])^c \cup (T_f^{(0)}[\{Y\} \cap (R_{\not\ni}^{-1}[Y])^c, X^c])^c]$	
iff	$\forall X \forall Y \forall z[z \in (T_f^{(0)}[\{X\} \cap \{Z' \mid X \subseteq Z'\}, Y^c])^c \cup (T_f^{(0)}[\{Y\} \cap \{Z' \mid Y \subseteq Z'\}, X^c])^c]$	
iff	$\forall X \forall Y \forall z[z \in (T_f^{(0)}[\{X\}, Y^c])^c \cup (T_f^{(0)}[\{Y\}, X^c])^c]$	
iff	$\forall X \forall Y \forall z \forall u[(u \in f(z, X) \Rightarrow u \in Y) \text{ or } (u \in f(z, Y) \Rightarrow u \in X)]$	
iff	$\forall X \forall Y \forall z[(f(z, X) \subseteq Y) \text{ or } (f(z, Y) \subseteq X)]$.	

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