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ORIGINAL PAPER

On α-constant-sum games

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Abstract

Given any $\alpha \in [0, 1]$, an α-constant-sum game (abbreviated as α-CS game) on a finite set of players, N, is a function that assigns a real number to any coalition $S \subseteq N$, such that the sum of the worth of the coalition S and the worth of its complementary coalition $N \setminus S$ is α times the worth of the grand coalition. This class contains the constant-sum games of Khmelnitskaya (Int J Game Theory 32:223–227, 2003) (for $\alpha = 1$) and games of threats of (Kohlberg and Neyman, Games Econ Behav 108:139–145, 2018) (for $\alpha = 0$) as special cases. An α-CS game may not be a classical TU cooperative game as it may fail to satisfy the condition that the worth of the empty set is 0, except when $\alpha = 1$. In this paper, we (i) extend the α-quasi-Shapley value giving the Shapley value for constant-sum games and quasi-Shapley-value for threat games to any class of α-CS games, (ii) extend the axiomatizations of Khmelnitskaya (2003) and Kohlberg and Neyman (2018) to any class of α-CS games, and (iii) introduce a new efficiency axiom which, together with other classical axioms, characterizes a

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solution that is defined by exactly the Shapley value formula for any class of α-CS games.

Keywords α-Constant-sum game · α-Quasi-Shapley value · Contribution efficiency

1 Introduction

A *cooperative game with transferable utility (TU game)* describes situations where players can earn certain payoffs by cooperating. It assigns a worth to every subset of the player set, called coalition, which represents the rewards that the players in the coalition can earn by cooperating. By definition, it assigns zero worth to the empty set. A *value* for TU games is a function that assigns a single payoff vector to each TU game. The components of this payoff vector reflect an assessment of the corresponding player's gains for participating in the game. The most widely studied value in TU games is the Shapley value (1953), which assigns to every player its expected marginal contribution assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability. In Shapley's paper, he proved that the Shapley value is the unique value satisfying efficiency, symmetry, linearity and the null player property. Young (1985) characterized the Shapley value by efficiency, symmetry and marginality, while van den Brink (2001) also provided an axiomatization of the Shapley value using efficiency, the null player property and a fairness property.

Khmelnitskaya (2003) showed that Young's axiomatization is valid on the class of *constant-sum games*, being TU games where the sum of the worths of a coalition and its complement always equals the worth of the grand coalition. Kohlberg and Neyman (2018) introduced the class of *games of threats* where the sum of the worth of any coalition and that of its complement equals zero. Consequently, these are not a subclass of TU games, since the worth of the empty set should be the negative of the worth of the grand coalition, implying that the worth of the empty set does not need to be zero. However, the structure of this class is similar to that of the class of constant-sum games.

This paper has three goals. First, we extend the classes of constant-sum games and threat games to classes of, what we call, *α-constant-sum games* (or *α-CS games* for short). These are games where the sum of the worths of any coalition and its complement equals a fraction $\alpha \in [0, 1]$ from the worth of the grand coalition. The classes of constant-sum games ($\alpha = 1$) and threat games ($\alpha = 0$) are special cases. For $\alpha \neq 1$, the α-CS games are not TU games, as they do not satisfy the condition that the worth of the empty set is zero. For any $\alpha \in [0, 1]$, we introduce the α-quasi-Shapley value, which has a similar structure as the Shapley value for TU games, and in fact equals the Shapley value considered by Khmelnitskaya (2003) for $\alpha = 1$, and the value considered by Kohlberg and Neyman (2018) for $\alpha = 0$.

Second, we show that both results mentioned above can be proven in the same way by showing that characterizations of the α-quasi-Shapley value on the corresponding class of α-CS games by classical axioms (for example Shapley's axiomatization system: efficiency, symmetry, linearity and the null player property), can be done by applying the same technique of proof to any class of α-CS games. We show this by closely following the arguments of Khmelnitskaya (2003) and Kohlberg and Neyman (2018). We also apply the axioms characterizing the Shapley value given by Young (1985) to characterize the α-quasi-Shapley value.

Third, we introduce an alternative efficiency axiom which, together with other classical axioms, characterizes a solution that is defined by exactly the Shapley value formula for any class of α-CS games. This alternative efficiency axiom, called contribution efficiency, requires that we allocate the difference between the worth of the grand coalition and the worth of the empty set, which is the contribution of the grand coalition compared to the empty coalition.

The paper is organized as follows. After discussing preliminaries in Sects. 2, 3 gives the definition of α-CS games, while Sect. 4 introduces and characterizes the α-quasi-Shapley value. In Sect. 5, we consider an alternative efficiency property, specifically for games where the worth of the empty set is nonzero. In Sect. 6, we show logical independence of the axioms in the axiomatizations in this paper. Finally, Sect. 7 concludes and develops some suggestions for future research.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the set of *players*. A subset $S \subseteq N$ is called *coalition*. In particular, N is called the *grand coalition*. We denote the size of coalition S as s.

A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each coalition $S \in 2^N$ the *worth* $v(S)$, with the convention that $v(\emptyset) = 0$. For each coalition S, the real number $v(S)$ represents the reward that coalition S can guarantee by itself without the cooperation of the other players. We denote by $\mathcal{G}^N$ the game space consisting of all TU games with player set N.

A *value* on a subclass $\mathcal{C} \subseteq \mathcal{G}^N$ is a function that assigns a single payoff vector to each TU game in $\mathcal{C}$. The most widely studied value in TU games is the Shapley value (1953), which assigns to every player its expected marginal contribution, assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability. Formally, the Shapley value Sh on $\mathcal{G}^N$ is defined by

$$Sh_i(N, v) = \sum_{S\subseteq N,\ S\ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S\backslash\{i\})], \tag{1}$$

for any $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

Khmelnitskaya (2003) considered the class of *constant-sum games* being games $\langle N, v\rangle$ satisfying $v(S) + v(N\backslash S) = v(N)$ for all $S \subseteq N$. Kohlberg and Neyman (2018) introduced the class of *games of threats*, being pairs $\langle N, d\rangle$, where $d : 2^N \to \mathbb{R}$ is a function that assigns a real number to any coalition, $S \subseteq N$, such that $d(S) = -d(N\backslash S)$, i.e., $d(S) + d(N\backslash S) = 0$. Notice that this implies that games of threats are not TU games since the empty set can have a nonzero worth. In fact, it has nonzero worth if and only if the worth of the grand coalition is nonzero. Kohlberg and Neyman

(2018) interpreted the amount $d(S)$ as the threat power of the coalition S. The condition $d(S)+d(N\backslash S)=0$ implies that the threat powers of coalition S and its complementary coalition $N\backslash S$ are contrary, and offset each other. Following this interpretation, in a constant-sum game, the sum of the threat powers of coalition S and its complementary coalition $N\backslash S$ are fixed as the threat power of the grand coalition.

3 α-constant-sum games

Given any $\alpha \in [0, 1]$, an α-*constant-sum game* (abbreviated as α-CS game) is a pair $\langle N, \mu \rangle$, where

- $N = \{1, 2, \ldots, n\}$ is a finite set of players;
- $\mu : 2^N \to \mathbb{R}$ is a function such that $\mu(S) + \mu(N\backslash S) = \alpha\mu(N)$, for all $S \subseteq N$.

We denote by $\mathcal{C}_\alpha^N$, $\alpha \in [0, 1]$, the game space consisting of all α-CS games with player set N. In particular, $\mathcal{C}_1^N$ is the class of constant-sum games, and $\mathcal{C}_0^N$ is the class of games of threats.

Example 1 Consider $\alpha = 0.8$, $N = \{1, 2, 3\}$, and game $\langle N, \mu \rangle \in \mathcal{C}_{0.8}^N$, given by

$$\mu(\emptyset) = -2,\ \mu(\{1\}) = 2,\ \mu(\{2\}) = 4,\ \mu(\{3\}) = 5,$$
$$\mu(\{1, 2\}) = 3,\ \mu(\{1, 3\}) = 4,\ \mu(\{2, 3\}) = 6,\ \mu(\{1, 2, 3\}) = 10.$$

By choosing, for every $S \subseteq N$, either S or $N\backslash S$, we can describe any $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ by means of 2^{n-1} numbers, thereby identifying $\mathcal{C}_\alpha^N$ with $\mathbb{R}^{2^{n-1}}$. A convenient choice is to take any $i \in N$, and consider $(\mu(S))_{S\ni i}$. The worths of the complementary coalitions $S \subseteq N\backslash\{i\}$ follow immediately from the definition of α-CS game.

Example 2 Given $\alpha = 0.8$, choosing $i = 1$, the above Example 1 is described as follows:

$$\mu(\{1\}) = 2,\ \mu(\{1, 2\}) = 3,\ \mu(\{1, 3\}) = 4,\ \mu(\{1, 2, 3\}) = 10.$$

The worths of the other coalitions (without player 1) are derived from the definition of α-CS game, specifically $\mu(\emptyset) = 0.8\mu(N) - \mu(N) = 8 - 10 = -2$, $\mu(\{2\}) = 0.8\mu(N) - \mu(\{1, 3\}) = 8 - 4 = 4$, $\mu(\{3\}) = 0.8\mu(N) - \mu(\{1, 2\}) = 8 - 3 = 5$ and $\mu(\{2, 3\}) = 0.8\mu(N) - \mu(\{1\}) = 8 - 2 = 6$.

4 The α-quasi-Shapley value

Consider any $\alpha \in [0, 1]$, and any α-CS game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$. Two players $i, j \in N$ are *symmetric* in $\langle N, \mu \rangle$ if, for every coalition $S \subseteq N\backslash\{i, j\}$, $\mu(S \cup \{i\}) = \mu(S \cup \{j\})$. A player $i \in N$ is a *null player* in $\langle N, \mu \rangle$ if, for every coalition $S \subseteq N\backslash\{i\}$, $\mu(S\cup\{i\}) = \mu(S)$. For any pair of α-CS games $\langle N, \mu \rangle$, $\langle N, \nu \rangle \in \mathcal{C}_\alpha^N$ and $a, b \in \mathbb{R}$, the game $a\mu + b\nu$ is given as $(a\mu + b\nu)(S) = a\mu(S) + b\nu(S)$, for all $S \subseteq N$.

A *value on* $\mathcal{C}_\alpha^N$ is a function $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ that associates with each α-CS game, a vector of payoffs $\phi(N, \mu) \in \mathbb{R}^N$ to the players. Following Shapley (1953), we consider the following properties.

- *Efficiency* For any α-CS game $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$, $\sum_{i \in N} \phi_i(N, \mu) = \mu(N)$.
- *Symmetry* For any α-CS game $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$, if players $i, j \in N$ are symmetric in $\langle N, \mu\rangle$, then $\phi_i(N, \mu) = \phi_j(N, \mu)$.
- *Linearity* For any pair of α-CS games $\langle N, \mu\rangle, \langle N, \nu\rangle \in \mathcal{C}_\alpha^N$ and $a, b \in \mathbb{R}$, $\phi(N, a\mu + b\nu) = a\phi(N, \mu) + b\phi(N, \nu)$.
- *Null player property* For any α-CS game $\langle N, v\rangle \in \mathcal{C}_\alpha^N$, if player $i \in N$ is a null player in $\langle N, \mu\rangle$, then $\phi_i(N, \mu) = 0$.

Notice that these are the usual efficiency, symmetry, linearity and null player axioms, except that they are defined on subclasses $\mathcal{C}_\alpha^N$.

It turns out that, for every $\alpha \in [0, 1]$, there exists a unique value on $\mathcal{C}_\alpha^N$ that satisfies the above defined classical axioms on this class. Before giving this result, we define the corresponding value.

Definition 1 Take any $\alpha \in [0, 1]$. The α-*quasi-Shapley value* is the value SH^α on $\mathcal{C}_\alpha^N$ defined by

$$SH_i^\alpha(N, \mu) = \sum_{S \subseteq N,\ S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(N \backslash S)] \right), \tag{2}$$

for every $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$ and $i \in N$.

Theorem 1 *Take any $\alpha \in [0, 1]$. The α-quasi-Shapley value is the unique value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry, linearity and the null player property.*

For $\alpha = 1$, the corresponding α-quasi-Shapley value is the ‘classical’ Shapley value on the class of constant-sum games $\mathcal{C}_1^N$ as characterized in Khmelnitskaya (2003). For $\alpha = 0$, the corresponding α-quasi-Shapley value is the Shapley type value on the class of games of threats $\mathcal{C}_0^N$ as characterized in Kohlberg and Neyman (2018). Notice also that, for $\alpha = 0$, Theorem 1 boils down to Theorem 1 of Kohlberg and Neyman (2018).

We stress once more, that the axioms in Theorem 1 are identical to the classical axioms for the Shapley value on the class of all TU games. It is somehow remarkable that the same set of axioms characterize a solution on any class of α-CS games.

Before proving Theorem 1, we introduce basis games for the classes of α-CS games.

Definition 2 Consider any $\alpha \in [0, 1]$ and $T \subseteq N$, $T \neq \emptyset$. The game $\langle N, u_T^\alpha\rangle \in \mathcal{C}_\alpha^N$, is defined by

$$u_T^\alpha(S) = \begin{cases} \frac{2}{2-\alpha}, & \text{if } S \supseteq T; \\ \frac{2\alpha-2}{2-\alpha}, & \text{if } S \subseteq N \backslash T; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise.} \end{cases} \tag{3}$$

For $\alpha = 1$, these are two times the type of unanimity game used in Khmelnitskaya (2003), while for $\alpha = 0$, these are $\frac{1}{|T|}$ times the type of unanimity game used by Kohlberg and Neyman (2018).

Proposition 1 *Consider any $\alpha \in [0, 1]$, and take any $i \in N$. The games $\{\langle N, u^\alpha_T \rangle\}_{T \subseteq N,\ T \ni i}$, span the class of α-CS games $\mathcal{C}^N_\alpha$, i.e., for every $\langle N, \mu \rangle \in \mathcal{C}^N_\alpha$, there exist numbers $a_T \in \mathbb{R}$, $i \in T \subseteq N$, such that $\mu = \sum_{T \subseteq N,\ T \ni i} a_T u^\alpha_T$.*

Proof Consider any $\alpha \in [0, 1]$, and let $i_0 \in N$. It is sufficient to show that the 2^{n-1} games $\{\langle N, u^\alpha_T \rangle\}_{T \subseteq N, T \ni i_0}$, are linearly independent. On the contrary, suppose that there exist numbers a_j, $j = 1, \ldots, 2^{n-1}$, such that

$$\sum_{j=1}^{2^{n-1}} a_j u^\alpha_{T_j} = 0, \tag{4}$$

where $(T_j)_{j=1,\ldots,2^{n-1}}$, is such that $i_0 \in T_j$ and $T_j \neq T_k$ for all $j, k \in \{1, \ldots, 2^{n-1}\}$, $j \neq k$.

Since, for $j \neq k$, we have $T_j \neq T_k$ and $T_j \cap T_k \supseteq \{i_0\} \neq \emptyset$, neither set is contained in the other's complement and therefore, using the fact that $T_k \supseteq T_j \Leftrightarrow N \setminus T_k \subseteq N \setminus T_j$,

$$u^\alpha_{T_j}(T_k) = \begin{cases} \frac{2}{2-\alpha}, & T_k \supseteq T_j; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise,} \end{cases} \quad \text{and} \quad u^\alpha_{T_j}(N \setminus T_k) = \begin{cases} \frac{2\alpha-2}{2-\alpha}, & T_k \supseteq T_j; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise.} \end{cases}$$

Hence,

$$u^\alpha_{T_j}(T_k) - u^\alpha_{T_j}(N \setminus T_k) = \begin{cases} 2, & T_k \supseteq T_j; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Among the T_j choose one, say T_m, with a minimum number of players among those coalitions in the sequence $(T_j)_{j=1,\ldots,2^{n-1}}$ with $a_j \neq 0$, i.e., $a_m \neq 0$ and $T_j \subset T_m$ implies that $a_j = 0$. Then for any $j \neq m$ with $a_j \neq 0$, it holds that $T_m \not\supseteq T_j$ and therefore, by Eq. (5),

$$u^\alpha_{T_j}(T_m) - u^\alpha_{T_j}(N \setminus T_m) = 0. \tag{6}$$

Thus,

$$\begin{aligned} 0 &= \sum_{j=1}^{2^{n-1}} a_j u^\alpha_{T_j}(T_m) - \sum_{j=0}^{2^{n-1}} a_j u^\alpha_{T_j}(N \setminus T_m) \\ &= \sum_{j=1}^{2^{n-1}} a_j [u^\alpha_{T_j}(T_m) - u^\alpha_{T_j}(N \setminus T_m)] \\ &= a_m [u^\alpha_{T_m}(T_m) - u^\alpha_{T_m}(N \setminus T_m)] \\ &= 2a_m, \end{aligned}$$

where the first equality follows from Eq. (4), the third equality follows from Eq. (6) and the assumption that $a_j = 0$ if $T_j \subset T_m$, and the last equality follows from Eq. (5). Thus, we have a contradiction with $a_m \neq 0$. □

Now, we can prove the main theorem.

Proof of Theorem 1 We first prove the uniqueness. Given any $\alpha \in [0, 1]$, let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be a value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry, linearity and the null player property. We prove that the value ϕ is uniquely determined on $\mathcal{C}_\alpha^N$.

Let $T \subseteq N$, $t \geq 2$. In the game $\langle N, u_T^\alpha \rangle \in \mathcal{C}_\alpha^N$, for any $i, j \in T$ and all $S \subseteq N \backslash \{i, j\}$, we have

$$u_T^\alpha(S \cup \{i\}) = u_T^\alpha(S \cup \{j\}) = \frac{\alpha}{2-\alpha},$$

showing that all players $i \in T$ are symmetric in $\langle N, u_T^\alpha \rangle$. Further, for any $i \notin T$ and all $S \subseteq N \backslash \{i\}$, we have

$$u_T^\alpha(S \cup \{i\}) = u_T^\alpha(S),$$

showing that all players $i \notin T$ are null players in $\langle N, u_T^\alpha \rangle$.

According to ϕ satisfying the null player property, the players $i \notin T$ earn zero payoff in $\langle N, u_T^\alpha \rangle$. By efficiency and symmetry of the function ϕ, and the fact that $u_T^\alpha(N) = \frac{2}{2-\alpha}$, we obtain

$$\phi_i(N, u_T^\alpha) = \begin{cases} \frac{2}{t(2-\alpha)}, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

With linearity, the function ϕ is uniquely determined on $\mathcal{C}_\alpha^N$.

For any $\alpha \in [0, 1]$, we now prove that the α-quasi-Shapley value SH^α, defined in (2), satisfies efficiency, symmetry, linearity and the null player property.

The α-quasi-Shapley value SH^α satisfying symmetry, linearity and the null player property follows directly from the following equivalent representation: for any $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$,

$$\begin{aligned} SH_i^\alpha(N, \mu) &= \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(N \backslash S)] \right) \\ &= \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(S \backslash \{i\})] \right). \end{aligned} \tag{7}$$

To prove efficiency, for any $S \subseteq N$, denote $r_n(S) := \frac{(s-1)!(n-s)!}{n!}$. It follows from (2) that

$$\begin{aligned} \sum_{i \in N} SH_i^\alpha(N, \mu) &= \sum_{i \in N} \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(N \backslash S)] \right) \\ &= \frac{1}{2-\alpha} \sum_{i \in N} \sum_{S \subseteq N, S \ni i} r_n(S)[\mu(S) - \mu(N \backslash S)] \end{aligned}$$

$$= \frac{1}{2-\alpha} \sum_{S \subseteq N} \sum_{i \in S} r_n(S)[\mu(S) - \mu(N \setminus S)]$$

$$= \frac{1}{2-\alpha} \sum_{S \subseteq N, S \neq \emptyset} s \cdot r_n(S)[\mu(S) - \mu(N \setminus S)]$$

$$= \frac{1}{2-\alpha} \sum_{S \subseteq N, S \neq \emptyset} \frac{1}{\binom{n}{s}}[\mu(S) - \mu(N \setminus S)]. \quad (8)$$

Since (i) $\emptyset \neq S \subsetneq N$, implies that $\emptyset \neq N \setminus S \subseteq N$, while (ii) $\binom{n}{s} = \binom{n}{n-s}$, (8) can be written as

$$\sum_{i \in N} SH_i^{\alpha}(N, \mu) = \frac{1}{2-\alpha}[\mu(N) - \mu(\emptyset)].$$

With $\mu(\emptyset) + \mu(N) = \alpha\mu(N)$, we conclude that

$$\sum_{i \in N} SH_i^{\alpha}(N, \mu) = \frac{1}{2-\alpha}[\mu(N) - (\alpha - 1)\mu(N)] = \mu(N).$$

This completes the proof of Theorem 1. □

Young (1985) showed that the existence and uniqueness theorem for the Shapley value on $\mathcal{G}^N$ remains valid when linearity and the null player property are replaced by marginality, which requires that the value of a player $i \in N$ in a TU game $\langle N, v \rangle$ depends only on the player's marginal contributions, $v(S \cup \{i\}) - v(S)$, for all $S \subseteq N \setminus \{i\}$.

- *Marginality* For any pair of α-CS games $\langle N, \mu \rangle, \langle N, \nu \rangle \in \mathcal{C}_{\alpha}^N$, and $i \in N$ such that $\mu(S \cup \{i\}) - \mu(S) = \nu(S \cup \{i\}) - \nu(S)$ for all $S \subseteq N \setminus \{i\}$, it holds that $\phi_i(N, \mu) = \phi_i(N, \nu)$.

The following theorem can be shown similar as Young (1985), but by induction on the number of coalitions T with nonzero coefficients a_T in the basis given by Proposition 1.[1]

Theorem 2 *Take any* $\alpha \in [0, 1]$. *The* α*-quasi-Shapley value is the unique value on* $\mathcal{C}_{\alpha}^N$ *that satisfies efficiency, symmetry and marginality.*

For $\alpha = 1$, this gives Theorem 1 of Khmelnitskaya (2003), while for $\alpha = 0$ this gives Corollary 1 of Kohlberg and Neyman (2018).

5 Contribution efficiency

In Sect. 4, we followed Khmelnitskaya (2003) and Kohlberg and Neyman (2018), taking the Shapley axioms and showing that these characterize a unique value on the

[1] The full proof can be found in the appendix.

classes of α-CS games. Specifically, we used the efficiency axiom that requires that the sum of all payoffs equals the worth of the grand coalition $\mu(N)$. However, for α-CS games, since generally the worth of the empty set is nonzero, an alternative efficiency notion is to require that the sum of all payoffs equals the payoff that all players cooperating together can generate compared to the situation where there is no cooperation at all, i.e. the total sum of payoffs equals $\mu(N) - \mu(\emptyset)$.

– *Contribution efficiency* For any α-CS game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, $\sum_{i \in N} \phi_i(N, \mu) = \mu(N) - \mu(\emptyset)$.

Notice that for classical TU games, this is equivalent to efficiency, since for those games $\mu(\emptyset) = 0$. We refer to this as 'contribution efficiency', since it reflects that the players earn the payoff that they can generate by their *ability* to cooperate.

Interestingly, applying this contribution efficiency together with symmetry, the null player property and linearity on classes of α-CS games $\mathcal{C}_\alpha^N$, characterizes a value that is given exactly by the famous Shapley value formula (without any α-term), see (1). To stress that we consider classes of α-constant-sum games, we give the definition below. We refer to this value as the *quasi-Shapley value* since it is also defined on classes where the worth of the emptyset is nonzero.

Definition 3 Take any $\alpha \in [0, 1]$. The *quasi-Shapley value* is the value SH on $\mathcal{C}_\alpha^N$ defined by

$$SH_i(N, \mu) = \sum_{S \subseteq N,\ S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot [\mu(S) - \mu(N \backslash S)] \right). \tag{9}$$

for every $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ and $i \in N$.

Theorem 3 *Take any $\alpha \in [0, 1]$. The quasi-Shapley value is the unique value on $\mathcal{C}_\alpha^N$ that satisfies contribution efficiency, symmetry, linearity and the null player property.*

Proof Uniqueness follows in a similar way as uniqueness in the proof of Theorem 1, just replacing efficiency by contribution efficiency, and therefore this proof is omitted.[2]

Let $\alpha \in [0, 1]$. Since the α-quasi-Shapley value satisfies efficiency on $\mathcal{C}_\alpha^N$, i.e. $\sum_{i \in N} SH_i^\alpha(N, \mu) = \mu(N)$, the fact that $SH(N, \mu) = (2 - \alpha) SH^\alpha(N, \mu)$, implies that $\sum_{i \in N} SH_i(N, \mu) = (2 - \alpha)\mu(N) = 2\mu(N) - (\mu(N) - \mu(\emptyset)) = \mu(N) - \mu(\emptyset)$, showing that SH satisfies contribution efficiency on $\mathcal{C}_\alpha^N$.

Whereas in Sect. 4, we showed that the classical axioms characterizing the Shapley values are valid on the classes of α-CS games, giving for each $\alpha \in [0, 1]$ a value depending on α, Theorem 3 shows that modifying efficiency to contribution efficiency (which on classes of TU games, i.e. games $\langle N, v \rangle$ with $v(\emptyset) = 0$, is also equivalent to efficiency) gives the following corollary. □

Corollary 1 *Take any $\alpha \in [0, 1]$. The unique value on $\mathcal{C}_\alpha^N$ that satisfies contribution efficiency, symmetry and marginality yields exactly the Shapley formula* (1)*, and thus does not depend on α.*

2 The proof can be obtained from the authors on request.

6 Independence of properties

The purpose of this section is to show that each of the four properties used in Theorem 1 and the three properties used in Theorem 2 is logically independent of the remaining properties.

We do this by presenting the following four alternative values, where each time the property in parentheses is the one violated by the value proposed.

1. **Efficiency** Take any $\alpha \in [0, 1]$. Let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be the value on $\mathcal{C}_\alpha^N$ defined by

$$\phi_i(N, \mu) = \frac{1}{2^{n-1}} \sum_{S \subseteq N,\ S \ni i} [\mu(S) - \mu(S \backslash \{i\})], \quad i \in N.$$

The value ϕ satisfies symmetry, linearity, the null player property and marginality, but it violates efficiency.

2. **Symmetry** Take any $\alpha \in [0, 1]$, and $\lambda \in \mathbb{R}_{++}^N$ an exogenous weight vector with positive weights $\lambda_i > 0$ for all $i \in N$ such that $\lambda_i \neq \lambda_j$ for at least one pair of players $i, j \in N$ with $i \neq j$. Let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be the value on $\mathcal{C}_\alpha^N$ defined by

$$\phi_i(N, \mu) = \sum_{S \subseteq N,\ S \ni i} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_\mu(S),$$

where the $\Delta_\mu(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \mu(T)$ are the classical Harsanyi dividends of the game. The value ϕ satisfies efficiency, linearity, the null player property and marginality, but it violates symmetry.

3. **Linearity, Marginality** Take any $\alpha \in [0, 1]$. For every $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N \to \mathbb{R}^N$, let $Null(N, \mu) = \{i \in N | i \text{ is a null player in } \langle N, \mu \rangle\}$ and t is the size of coalition $Null(N, \mu)$. Let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be the value on $\mathcal{C}_\alpha^N$ defined by

$$\phi_i(N, \mu) = \begin{cases} 0, & i \in Null(N, \mu); \\ \frac{\mu(N)}{n-t}, & i \in N \backslash Null(N, \mu). \end{cases}$$

The value ϕ satisfies efficiency, symmetry and the null player property, but it violates linearity and marginality.

4. (The null player property). Take any $\alpha \in [0, 1]$. Let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be the equal division value on $\mathcal{C}_\alpha^N$ defined by

$$\phi_i(N, \mu) = \frac{\mu(N)}{n}, \quad i \in N.$$

The value ϕ satisfies efficiency, symmetry and linearity, but it violates the null player property.

7 Conclusions

Khmelnitskaya (2003) and Kohlberg and Neyman (2018) showed that classical axiomatizations of the Shapley value hold for constant-sum games, respectively games of threats. In this paper, we showed that this can be extended to any class of α-CS games, where the sum of the worth of a coalition and its complement always equals the same fraction of the worth of the grand coalition. This is an interesting feature of these classes of games since mostly, when considering a subclass of TU games, a classical axiomatization of the Shapley value does not give uniqueness since some axioms (in particular axioms that compare different games such as linearity and marginality) have less bite when we consider subclasses of games.

Whereas in Sect. 4, we showed that the classical axioms of efficiency, symmetry, the null player property and linearity characterize the corresponding α-quasi Shapley value as a unique solution for every class of α-CS games (where the parameter α appears in the solution), we showed in Sect. 5 that replacing efficiency by contribution efficiency also yields a unique solution on every class of α-CS games, being the quasi-Shapley value which is defined by the classical Shapley value formula which does not depend on α. Contribution efficiency is an alternative efficiency, which reflects that the players allocate among themselves what they can earn with their ability to cooperate, i.e. the difference between the worths of the grand coalition and the empty set.

We showed that the games as given by (3) allow to apply various axioms that characterize the Shapley value for TU games, to the classes of α-CS games and characterize a Shapley type value. Axioms used in characterizations of some other values become void when restricted to a class of α-constant sum games. For example, van den Brink (2007) characterized the equal division value by efficiency, symmetry, linearity and the nullifying player property, the last axiom requiring that a nullifying player (i.e. a player such that every coalition containing this players earns worth zero) gets a payoff of zero. This result is generalized in van den Brink and Funaki (2015) who characterized every *discounted Shapley value* (see Joosten 1996; Driessen and Radzik 2002) on the class of all TU games by efficiency, symmetry, linearity and the δ-reducing player property, the last axiom requiring that a δ-reducing player (i.e. a player such that when this player joins any coalition, the worth of the coalition becomes $\delta \in [0, 1]$ times the worth of the coalition without this player) earns a zero payoff. It can be shown that on any class of α-CS games, the existence of a δ-reducing player for $\delta \in [0, 1)$ implies that the game is the null game where all coalitions earn worth zero.[3] Thus, efficiency and symmetry imply the δ-reducing player property for any $\delta \in [0, 1)$, and therefore, efficiency, symmetry, linearity and the δ-reducing player property for any $\delta \in [0, 1)$ do not give uniqueness.[4]

We remark that in this paper the parameter α determined a class of games, and on these classes we characterized a Shapley type value by classical (non-parameterized) axioms. In the literature on classical TU games, there exist classes of (parameterized) solutions, such as the classes of egalitarian Shapley values and discounted Shapley

[3] The proof can be obtained from the authors on request.

[4] Besides the equal division value, for example, also the α-quasi Shapley value satisfies these axioms on any class of α-CS games.

values, that can be characterized by axioms that depend on this parameter. We want to stress the difference with the underlying paper, where we only consider one solution but apply it to different classes of games. Another future goal is to consider classes of (parameterized) solutions, such as the ones mentioned above, for classes of α-CS games.

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Appendix: The Proof of Theorem 2

Consider any $\alpha \in [0, 1]$. Efficiency and symmetry follow from Theorem 1, while it is straightforward to see that the α-quasi-Shapley value satisfies marginality on $\mathcal{C}_\alpha^N$.

To prove uniqueness, suppose that $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ is a value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry and marginality.

According to Proposition 1, for any α-CS game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, there exist numbers $a_T \in \mathbb{R}$, $T \subseteq N$, $T \neq \emptyset$, such that

$$\mu = \sum_{T \subseteq N} a_T u_T^\alpha. \tag{10}$$

.

Notice that in any game $\langle N, u_T^\alpha \rangle$, $\emptyset \neq T \subseteq N$, all players $i \in T$ are symmetric players, and every player $i \notin T$ is a null player, i.e., all of his marginal contributions are equal to zero.

Let the index I of a game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ be the minimum number of terms under the summation in expression (10), i.e.,

$$\mu = \sum_{k=1}^{I} a_{T_k} u_{T_k}^\alpha,$$

where all $a_{T_k} \neq 0$. Similar as the proof of uniqueness in Young (1985), we proceed the remaining part of the proof by induction on this index I (instead of the number of nonzero Harsanyi dividends).

If $I = 0$, then $\langle N, \mu \rangle$ is a null game given by $\mu(S) = 0$ for all $S \subseteq N$. Uniqueness of $\phi(N, \mu)$ then follows directly from efficiency and symmetry of ϕ.

Assume now that $\phi(N, v)$ is uniquely determined whenever the index of $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ is at most I, and let $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ have index $I + 1$ with expression

$$\mu = \sum_{k=1}^{I+1} a_{T_k} u_{T_k}^\alpha, \quad \text{all } a_{T_k} \neq 0.$$

Let $T = \cap_{k=1}^{I+1} T_k$. For all $i, j \in T$, symmetry implies that $\phi_i(N, \mu) = \phi_j(N, \mu)$. Hence, combined with the requirement of efficiency, it is sufficient to prove that $\phi_i(N, \mu)$ is uniquely determined when $i \notin T$. Define a game

$$\mu^{(i)} = \sum_{k:i \in T_k} a_{T_k} u^{\alpha}_{T_k}.$$

Obviously, the index of $\langle N, \mu^{(i)} \rangle$ is at most I and, therefore, by the induction hypothesis, $\phi_i(N, \mu^{(i)})$ is uniquely determined. Since i's marginal contributions in the games $\langle N, \mu \rangle$ and $\langle N, \mu^{(i)} \rangle$ coincide, by marginality of ϕ, $\phi_i(N, \mu) = \phi_i(N, \mu^{(i)})$ which is uniquely determined by the induction hypothesis.

Thus, we have shown that there can be only one value that satisfies efficiency, symmetry and marginality. Since the α-quasi-Shapley value satisfies these axioms, it must be that $\phi = SH^{\alpha}$.

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