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Subordination Algebras as Semantic Environment of Input/Output Logic

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Abstract. We establish a novel connection between two research areas in non-classical logics which have been developed independently of each other so far: on the one hand, *input/output logic*, introduced within a research program developing logical formalizations of normative reasoning in philosophical logic and AI; on the other hand, *subordination algebras*, investigated in the context of a research program integrating topological, algebraic, and duality-theoretic techniques in the study of the semantics of modal logic. Specifically, we propose that the basic framework of input/output logic, as well as its extensions, can be given formal semantics on (slight generalizations of) subordination algebras. The existence of this interpretation brings benefits to both research areas: on the one hand, this connection allows for a novel conceptual understanding of subordination algebras as mathematical models of the properties and behaviour of norms; on the other hand, thanks to the well developed connection between subordination algebras and modal logic, the output operators in input/output logic can be given a new formal representation as modal operators, whose properties can be explicitly axiomatised in a suitable language, and be systematically studied by means of mathematically established and powerful tools.

Keywords: input/output logic · subordination algebra · modal logic · Sahlqvist theory

1 Introduction

Input/output logic has been introduced in [25] as a formal framework for modelling the interaction between logical inferences and other agency-related notions such as conditional obligations, goals, ideals, preferences, actions, and beliefs. Although, initially, this framework was intended "not [for] studying some kind of non-classical logic, but [as] a way of using the classical one", its generality and versatility makes it very suitable to support a range of enhancements in

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its expressiveness, such as those brought about by the addition of modal operators. Moreover, recently, there has been an interest in studying the interaction between the agency-related notions mentioned above with various forms of *non-classical* reasoning [30,34]. This interest has contextually motivated the introduction of algebraic and proof-theoretic methods in the study of input/output logic [18,20,36].

In this paper, we contribute to the latter research direction in the mathematical background of input/output logic[1] by introducing an algebraic semantics for it, based on (generalizations of) *subordination algebras* [4]. These can be defined as tuples $(A, \prec)$ such that A is a Boolean algebra and $\prec$ is a binary relation on A such that the direct (resp. inverse) image of each element $a \in A$ is a filter (resp. an ideal) of A.

Subordination algebras are equivalent presentations of pre-contact algebras [15] and quasi-modal algebras [6,7]. Since their introduction, subordination algebras have been systematically connected with various modal algebras (i.e. Boolean algebras expanded with semantic modal operators). This has made it possible to endow various modal languages with algebraic semantics based on subordination algebras, and use these languages to axiomatize the properties of these subordination algebras. In particular, Sahlqvist-type canonicity for modal and tense formulas on subordination algebras has been studied in [13] using topological techniques; in [14], using algebraic techniques, the canonicity result of [13] was strengthened and captured within the more general notion of canonicity in the context of *slanted algebras*, which was established using the tools of *unified correspondence theory* [9,11,12]. Slanted algebras are based on general lattices, and encompass variations and generalizations of subordination algebras such as those very recently introduced by Celani in [8], which are based on distributive lattices, and for which Celani develops duality-theoretic and correspondence-theoretic results.

Structure of the Paper. In Sect. 2, we collect preliminaries about the abstract logical framework in which we are going to develop our results, input/output logics as embedded in this framework, the general environment of proto-subordination algebras and their properties, canonical extensions and slanted algebras. In Sect. 3, we associate slanted algebras to proto-subordination algebras with certain properties, and characterize their further properties in terms of the validity of modal inequalities on their associated slanted algebras. In Sect. 4, we provide an axiomatic modal characterization of the output operators of input/output logic (cf. Proposition 3), and to obtain Celani's dual characterization results for subordination lattices as consequences of standard modal correspondence (cf. Proposition 4). We conclude in Sect. 5.

[1] In the present work, we focus exclusively on the framework of *unconstrained* input/output logic [26]. Extending the results to its constrained version is an avenue for future work.

2 Preliminaries

Selfextensional Logics. In what follows, we align to the literature in abstract algebraic logic [19], and understand a *logic* to be a tuple $\mathcal{L} = (\mathrm{Fm}, \vdash)$, such that Fm is the term algebra (in a given algebraic signature) over a set Prop of atomic propositions, and $\vdash$ is a *consequence relation* on Fm, i.e. $\vdash$ is a relation between sets of formulas and formulas such that, for all $\Gamma, \Delta \subseteq \mathrm{Fm}$ and all $\varphi \in \mathrm{Fm}$, (a) if $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$; (b) if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$; (c) if $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$, then $\Gamma \vdash \varphi$. Clearly, any such $\vdash$ induces a preorder on Fm, which we still denote $\vdash$, by restricting to singletons. A logic $\mathcal{L}$ is *selfextensional* (cf. [22]) if the relation $\equiv \subseteq \mathrm{Fm} \times \mathrm{Fm}$, defined by $\varphi \equiv \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$, is a congruence of Fm. In this case, the *Lindenbaum-Tarski algebra* of $\mathcal{L}$ is the partially ordered algebra $Fm = (\mathrm{Fm}/{\equiv}, \vdash)$ where, abusing notation, $\vdash$ also denotes the partial order on $\mathrm{Fm}/{\equiv}$, defined as $[\varphi]_\equiv \vdash [\psi]_\equiv$ iff $\varphi \vdash \psi$. In what follows, we will also assume that each element in the class $\mathsf{Alg}(\mathcal{L})$ of algebras canonically associated with $\mathcal{L}$ is partially ordered, and that, if φ and ψ are formulas, then $\varphi \vdash \psi$ iff $h(\varphi) \leq h(\psi)$ for every $A \in \mathsf{Alg}(\mathcal{L})$ and every homomorphism $h : \mathrm{Fm} \to A$.

For any $\Gamma \subseteq \mathrm{Fm}$, let $Cn(\Gamma) := \{\psi \mid \Gamma \vdash \psi\}$.[2] The *conjunction property* holds for $\mathcal{L}$ if a term $t(x, y) := x \wedge y$ exists such that $Cn(\varphi \wedge \psi) = Cn(\{\varphi, \psi\})$ for all $\varphi, \psi \in \mathrm{Fm}$. The *disjunction property* holds for $\mathcal{L}$ if a term $t(x, y) := x \vee y$ exists such that $Cn(\varphi \vee \psi) = Cn(\varphi) \cap Cn(\psi)$ for all $\varphi, \psi \in \mathrm{Fm}$.

Although the original framework of input/output logic takes $\mathcal{L}$ to be classical propositional logic, in the next subsection we present it in the more general framework of selfextensional logics just described.

Input/Output Logic and (Proto-)Subordination Algebras. The general theory of input/output logic aims at modelling relations generalizing inference, where inputs need not be included among outputs, and outputs need not be reusable as inputs [25].

Definition 1. *Let* $\mathcal{L} = (\mathrm{Fm}, \vdash)$ *be a logic in the sense specified above. A* normative system *is a relation* $N \subseteq \mathrm{Fm} \times \mathrm{Fm}$*, the elements* (α, φ) *of which are called* conditional norms *(or obligations). An* input/output logic *is a tuple* $\mathbb{L} = (\mathcal{L}, N)$ *s.t.* $\mathcal{L} = (\mathrm{Fm}, \vdash)$ *is a (selfextensional) logic, and* N *is a normative system on* Fm*.*

The reading of each norm $(\alpha, \varphi) \in N$ is "given α, it is obligatory that φ". The formula α is the *body* of the norm, and represents some situation or condition, while φ is the *head* and represents what is obligatory or desirable in that situation. For any $\Gamma \subseteq \mathrm{Fm}$, let $N(\Gamma) := \{\psi \mid \exists\alpha(\alpha \in \Gamma \;\&\; (\alpha, \psi) \in N)\}$.

Definition 2 (Output operations). *For any input/output logic* $\mathbb{L} = (\mathcal{L}, N)$*, and each* $1 \leq i \leq 4$*, the output operation* out_i^N *is defined as follows: for any* $\Gamma \subseteq \mathrm{Fm}$*,*

$$out_i^N(\Gamma) := N_i(\Gamma) = \{\psi \in \mathrm{Fm} \mid \exists\alpha(\alpha \in \Gamma \;\&\; (\alpha, \psi) \in N_i)\}$$

[2] In what follows, we write e.g. $Cn(\varphi)$ for $Cn(\{\varphi\})$.

where $N_i \subseteq \mathrm{Fm} \times \mathrm{Fm}$ *is the* closure *of* N *under (i.e. the smallest extension of* N *satisfying) the inference rules below, as specified in the table.*

$$\frac{}{(\top,\top)}\ (\top) \qquad \frac{(\alpha,\varphi) \quad \beta \vdash \alpha}{(\beta,\varphi)}\ (\mathrm{SI}) \qquad \frac{(\alpha,\varphi) \quad \varphi \vdash \psi}{(\alpha,\psi)}\ (\mathrm{WO})$$

$$\frac{(\alpha,\varphi) \quad (\alpha,\psi)}{(\alpha,\varphi\wedge\psi)}\ (\mathrm{AND}) \qquad \frac{(\alpha,\varphi) \quad (\beta,\varphi)}{(\alpha\vee\beta,\varphi)}\ (\mathrm{OR}) \qquad \frac{(\alpha,\varphi) \quad (\alpha\wedge\varphi,\psi)}{(\alpha,\psi)}\ (\mathrm{CT})$$

N_i	*Rules*		
N_1	*(⊤), (SI), (WO), (AND)*	N_2	*(⊤), (SI), (WO), (AND), (OR)*
N_3	*(⊤), (SI), (WO), (AND), (CT)*	N_4	*(⊤), (SI), (WO),(AND), (OR), (CT)*

Definition 3 ((Proto-)subordination algebra). *A* proto-subordination algebra *is a tuple* $\mathbb{S} = (A, \prec)$ *such that* A *is a (possibly bounded) poset (with bottom denoted* $\bot$ *and top denoted* $\top$ *when they exist), and* $\prec\ \subseteq A \times A$*. A proto-subordination algebra is named as indicated in the left-hand column in the tables below when* $\prec$ *satisfies the properties indicated in the right-hand column. In what follows, we will refer to a proto-subordination algebra* $\mathbb{S} = (A, \prec)$ *as e.g.* (distributive) lattice-based ((D)L-based)*, or* Boolean-based (B-based) *if* A *is a (distributive) lattice, a Boolean algebra, and so on. More in general, for any logic* $\mathcal{L}$*, we say that* $\mathbb{S} = (A, \prec)$ *is* $\mathsf{Alg}(\mathcal{L})$-based *if* $A \in \mathsf{Alg}(\mathcal{L})$*. The reader can safely assume that* A *is a (bounded distributive) lattice, or a Boolean algebra, although, if this is not specified, the results presented below will hold more generally. We will flag out the assumptions we need in the statements of propositions.*

(⊥) $\bot \prec \bot$ *(⊤)* $\top \prec \top$
(SI) $a \leq b \prec x \Rightarrow a \prec x$ *(WO)* $b \prec x \leq y \Rightarrow b \prec y$
(AND) $a \prec x \ \&\ a \prec y \Rightarrow a \prec x \wedge y$ *(OR)* $a \prec x \ \&\ b \prec x \Rightarrow a \vee b \prec x$
(D) $a \prec c \Rightarrow \exists b(a \prec b \ \&\ b \prec c)$ *(S6)* $a \prec b \Rightarrow \neg b \prec \neg a$
(CT) $a \prec b \ \&\ a \wedge b \prec c \Rightarrow a \prec c$ *(T)* $a \prec b \ \&\ b \prec c \Rightarrow a \prec c$
(DD) $a \prec x_1 \ \&\ a \prec x_2 \Rightarrow \exists x(a \prec x \ \&\ x \leq x_1 \ \&\ x \leq x_2)$
(UD) $a_1 \prec x \ \&\ a_2 \prec x \Rightarrow \exists a(a \prec x \ \&\ a_1 \leq a \ \&\ a_2 \leq a)$
(S9) $\exists c(c \prec b \ \&\ x \prec a \vee c) \iff \exists a'\exists b'(a' \prec a \ \&\ b' \prec b \ \&\ x \leq a' \vee b')$
(SL1) $a \prec b \vee c \Rightarrow \exists b'\exists c'(b' \prec b \ \&\ c' \prec c \ \&\ a \prec b' \vee c')$
(SL2) $b \wedge c \prec a \Rightarrow \exists b'\exists c'(b' \prec b \ \&\ c' \prec c \ \&\ b' \wedge c' \prec a)$

Name	Properties
◇-premonotone	(SI)
■-premonotone	(WO)
premonotone	(SI) (WO)
◇-directed	(WO) (DD)
■-directed	(SI) (UD)
◇-monotone	(WO) (DD) (SI)
■-monotone	(SI) (UD) (WO)

Name	Properties
directed/monotone	(SI) (WO) (UD) (DD)
◇-regular	(SI) (WO) (DD) (OR)
■-regular	(SI) (WO) (UD) (AND)
regular	(SI) (WO) (OR) (AND)
◇-normal	(SI) (WO) (DD) (OR) (⊥)
■-normal	(SI) (WO) (UD) (AND) (⊤)
subordination algebra	(SI) (WO) (OR) (AND) (⊥) (⊤)

Normative systems can be interpreted in proto-subordination algebras as follows:

Definition 4. *A* model *for an input/output logic* $\mathbb{L} = (\mathcal{L}, N)$ *is a tuple* $\mathbb{M} = (\mathbb{S}, h)$ *s.t.* $\mathbb{S} = (A, \prec)$ *is an* $\mathsf{Alg}(\mathcal{L})$*-based proto-subordination algebra (i.e.* $A \in \mathsf{Alg}(\mathcal{L})$*), and* $h : \mathrm{Fm} \to A$ *is a homomorphism s.t. for all* $\varphi, \psi \in \mathrm{Fm}$*, if* $(\varphi, \psi) \in N$*, then* $h(\varphi) \prec h(\psi)$*.*

Canonical Extensions and Slanted Algebras. In the present subsection, we adapt material from [13, Sects 2.2 and 3.1], [16, Sect. 2]. For any poset A, a subset $B \subseteq A$ is *upward closed*, or an *up-set* (resp. *downward closed*, or a *down-set*) if $\lfloor B \rfloor := \{c \in A \mid \exists b(b \in B \ \& \ b \leq c)\} \subseteq B$ (resp. $\lceil B \rceil := \{c \in A \mid \exists b(b \in B \ \& \ c \leq b)\} \subseteq B$); a subset $B \subseteq A$ is *down-directed* (resp. *up-directed*) if, for all $a, b \in B$, some $x \in B$ exists s.t. $x \leq a$ and $x \leq b$ (resp. $a \leq x$ and $b \leq x$). It is straightforward to verify that when A is a lattice, down-directed upsets and up-directed down-sets coincide with lattice filters and ideals, respectively.

Definition 5. *Let A be a subposet of a complete lattice A'.*

(i) *An element $k \in A'$ is* closed *if $k = \bigwedge F$ for some down-directed $F \subseteq A$; an element $o \in A'$ is* open *if $o = \bigvee I$ for some up-directed $I \subseteq A$;*
(ii) *A is* dense *in A' if every element of A' can be expressed both as the join of closed elements and as the meet of open elements of A.*
(iii) *A is* compact *in A' if, for all $F, I \subseteq A$ s.t. F is down-directed, I is up-directed, if $\bigwedge F \leq \bigvee I$ then $a \leq b$ for some $a \in F$ and $b \in I$.*[3]
(iv) *The* canonical extension *of a poset A is a complete lattice A^δ containing A as a dense and compact subposet.*

The canonical extension A^δ of any poset A always exists and is unique up to an isomorphism fixing A (cf. [16, Propositions 2.6 and 2.7]). The set of the closed (resp. open) elements of A^δ is denoted $K(A^\delta)$ (resp. $O(A^\delta)$). The following proposition collects well known facts used in the remainder of the paper.

Proposition 1. *For every poset A,*

(i) *if A is a distributive lattice (DL), then A^δ is completely distributive.*
(ii) *if $\neg : A \to A$ is antitone and s.t. $(A, \neg) \models \forall a \forall b(\neg a \leq b \Leftrightarrow \neg b \leq a)$, then $\neg^\sigma : A^\delta \to A^\delta$ defined as $\neg^\sigma o := \bigwedge\{\neg a \mid a \leq o\}$ for any $o \in O(A^\delta)$ and $\neg^\sigma u := \bigvee\{\neg^\sigma o \mid u \leq o\}$ for any $u \in A^\delta$ is antitone and s.t. $(A^\delta, \neg^\sigma) \models \forall u \forall v(\neg u \leq v \Leftrightarrow \neg v \leq u)$. If in addition, $(A, \neg) \models a \leq \neg\neg a$, then $(A^\delta, \neg^\sigma) \models u \leq \neg\neg u$. Hence, if $(A, \neg) \models a = \neg\neg a$ (i.e. $\neg$ is involutive), then $(A^\delta, \neg^\sigma) \models a = \neg\neg a$.*
(iii) *if $\neg : A \to A$ is antitone and s.t. $(A, \neg) \models \forall a \forall b(a \leq \neg b \Leftrightarrow b \leq \neg a)$, then $\neg^\pi : A^\delta \to A^\delta$ defined as $\neg^\pi k := \bigvee\{\neg a \mid k \leq a\}$ for any $k \in K(A^\delta)$ and $\neg^\pi u := \bigwedge\{\neg^\pi k \mid k \leq u\}$ for any $u \in A^\delta$ is antitone and s.t. $(A^\delta, \neg^\pi) \models \forall u \forall v(u \leq \neg v \Leftrightarrow v \leq \neg u)$. If in addition, $(A, \neg) \models \neg\neg a \leq a$, then $(A^\delta, \neg^\pi) \models \neg\neg u \leq u$. Hence, if $\neg$ is involutive, then so is $\neg^\pi$.*

Definition 6. *A* slanted algebra *is a triple $\mathbb{A} = (A, \Diamond, \blacksquare)$ such that A is a poset, and $\Diamond, \blacksquare : A \to A^\delta$ s.t. $\Diamond a \in K(A^\delta)$ and $\blacksquare a \in O(A^\delta)$ for every a. A slanted algebra as above is* tense *if $\Diamond a \leq b$ iff $a \leq \blacksquare b$ for all $a, b \in A$; is* monotone *if $\Diamond$ and $\blacksquare$ are monotone; is* regular *if $\Diamond$ and $\blacksquare$ are regular (i.e. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$ for all $a, b \in A$); is* normal *if $\Diamond$ and $\blacksquare$ are normal (i.e. they are regular and $\Diamond\bot = \bot$ and $\blacksquare\top = \top$).*

[3] When the poset A is a lattice, the compactness can be equivalently reformulated by dropping the requirements that F be down-directed and I be up-directed.

The following definition is framed in the context of monotone slanted algebras, but can be given for arbitrary slanted algebras, albeit at the price of complicating the definition of $\Diamond^\sigma$ and $\blacksquare^\pi$. Because we are mostly going to apply it in the monotone setting, we present the simplified version here.

Definition 7. *For any monotone slanted algebra $\mathbb{A} = (A, \Diamond, \blacksquare)$ the* canonical extension *of $\mathbb{A}$ is the (standard!) modal algebra $\mathbb{A}^\delta := (A^\delta, \Diamond^\sigma, \blacksquare^\pi)$ such that $\Diamond^\sigma, \blacksquare^\pi : A^\delta \to A^\delta$ are defined as follows: for every $k \in K(A^\delta)$, $o \in O(A^\delta)$ and $u \in A^\delta$,*

$$\Diamond^\sigma k := \bigwedge\{\Diamond a \mid a \in A \text{ and } k \leq a\}, \quad \Diamond^\sigma u := \bigvee\{\Diamond^\sigma k \mid k \in K(A^\delta) \text{ and } k \leq u\},$$

$$\blacksquare^\pi o := \bigvee\{\blacksquare a \mid a \in A \text{ and } a \leq o\}, \quad \blacksquare^\pi u := \bigwedge\{\blacksquare^\pi o \mid o \in O(A^\delta) \text{ and } u \leq o\}.$$

For any slanted algebra $\mathbb{A}$, any assignment $v : \mathsf{PROP} \to \mathbb{A}$ uniquely extends to a homomorphism $v : \mathcal{L} \to \mathbb{A}^\delta$ (abusing notation, the same symbol denotes both the assignment and its homomorphic extension). Hence,

Definition 8. *A modal inequality $\phi \leq \psi$ is* satisfied *in a slanted algebra $\mathbb{A}$ under the assignment v (notation: $(\mathbb{A}, v) \models \phi \leq \psi$) if $(\mathbb{A}^\delta, e \cdot v) \models \phi \leq \psi$ in the usual sense, where $e \cdot v$ is the assignment on $\mathbb{A}^\delta$ obtained by composing the canonical embedding $e : \mathbb{A} \to \mathbb{A}^\delta$ to the assignment $v : \mathsf{Prop} \to \mathbb{A}$.*

Moreover, $\phi \leq \psi$ is valid *in $\mathbb{A}$ (notation: $\mathbb{A} \models \phi \leq \psi$) if $(\mathbb{A}^\delta, e \cdot v) \models \phi \leq \psi$ for every assignment v into $\mathbb{A}$ (notation: $\mathbb{A}^\delta \models_{\mathbb{A}} \phi \leq \psi$).*

3 Proto-Subordination Algebras and Slanted Algebras

Let $\mathbb{S} = (A, \prec)$ be a proto-subordination algebra s.t. $\mathbb{S} \models$ (DD) + (UD). The slanted algebra associated with $\mathbb{S}$ is $\mathbb{S}^* = (A, \Diamond, \blacksquare)$ s.t. $\Diamond a := \bigwedge \prec[a]$ and $\blacksquare a := \bigvee \prec^{-1}[a]$ for any a. From $\mathbb{S} \models$ (DD) it follows that $\prec[a]$ is down-directed for every $a \in A$, hence $\Diamond a \in K(A^\delta)$. Likewise, $\mathbb{S} \models$ (UD) guarantees that $\blacksquare a \in O(A^\delta)$ for all $a \in A$.

Lemma 1. *For any proto-subordination algebra $\mathbb{S} = (A, \prec)$ and all $a, b \in A$,*

(i) $a \prec b$ implies $\Diamond a \leq b$ and $a \leq \blacksquare b$.
(ii) if $\mathbb{S} \models$ (WO) + (DD), then $\Diamond a \leq b$ iff $a \prec b$.
(iii) if $\mathbb{S} \models$ (SI) + (UD), then $a \leq \blacksquare b$ iff $a \prec b$.

Proof. (i) $a \prec b$ iff $b \in \prec[a]$ iff $a \in \prec^{-1}[b]$, hence $a \prec b$ implies $b \geq \bigwedge \prec[a] = \Diamond a$ and $a \leq \bigvee \prec^{-1}[b] = \blacksquare b$.

(ii) By (i), to complete the proof, we need to show the 'only if' direction. The assumption $\mathbb{S} \models$ (DD) implies that $\prec[a]$ is down-directed for any $a \in A$. Hence, by compactness, $\bigwedge \prec[a] = \Diamond a \leq b$ implies that $c \leq b$ for some $c \in \prec[a]$, i.e. $a \prec c \leq b$ for some $c \in A$, and by (WO), this implies that $a \prec b$, as required.

(iii) is proven similarly.

Lemma 2. *For any lattice-based proto-subordination algebra $\mathbb{S} = (A, \prec)$,*

(i) $\mathbb{S} \models$ (OR) *implies* $\mathbb{S} \models$ (UD).
(ii) $\mathbb{S} \models$ (AND) *implies* $\mathbb{S} \models$ (DD).
(iii) If $\mathbb{S} \models$ (SI), *then* $\mathbb{S} \models$ (UD) *iff* $\mathbb{S} \models$ (OR).
(iv) if $\mathbb{S} \models$ (WO), *then* $\mathbb{S} \models$ (DD) *iff* $\mathbb{S} \models$ (AND).

Proof. (i) and (ii) are straightforward. As for (iii), by (i), to complete the proof we need to show the 'only if' direction. Let $a, b, x \in A$ s.t. $a \prec x$ and $b \prec x$. By (UD), this implies that $c \prec x$ for some $c \in A$ such that $a \leq c$ and $b \leq c$. Since A is a lattice, this implies that $a \vee b \leq c \prec x$, and by (SI), this implies that $a \vee b \prec x$, as required. (vi) is proven similarly.

Lemma 3. *For every proto-subordination algebra $\mathbb{S} = (A, \prec)$,*

(i) If $\mathbb{S} \models$ (SI), *then:*
  (a) $\Diamond$ *on* $\mathbb{S}^*$ *is monotone;*
  (b) if $\mathbb{S}$ *is DL-based, then* $\mathbb{S} \models$ (AND) *implies* $\mathbb{S}^* \models \blacksquare a \wedge \blacksquare b \leq \blacksquare(a \wedge b)$*;*
  (c) if $\mathbb{S} \models$ (UD), *then* $\mathbb{S} \models$ (AND) *implies* $\mathbb{S}^* \models \blacksquare a \wedge \blacksquare b \leq \blacksquare(a \wedge b)$.
(ii) If $\mathbb{S} \models$ (WO), *then*
  (a) $\blacksquare$ *on* $\mathbb{S}^*$ *is monotone;*
  (b) if $\mathbb{S}$ *is DL-based, then* $\mathbb{S} \models$ (OR) *implies* $\mathbb{S}^* \models \Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$*;*
  (c) if $\mathbb{S} \models$ (DD), *then* $\mathbb{S} \models$ (OR) *implies* $\mathbb{S}^* \models \Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$.
(iii) If $\mathbb{S} \models (\bot)$, *then* $\mathbb{S}^* \models \Diamond\bot \leq \bot$.
(iv) If $\mathbb{S} \models (\top)$, *then* $\mathbb{S}^* \models \top \leq \blacksquare\top$.

Proof. (i) (a) Let $a, b \in A$ s.t. $a \leq b$. To show that $\Diamond a = \bigwedge {\prec}[a] \leq \bigwedge {\prec}[b] = \Diamond b$, it is enough to show that ${\prec}[b] \subseteq {\prec}[a]$, i.e. that if $x \in A$ and $b \prec x$, then $a \prec x$. Indeed, by (SI), $a \leq b \prec x$ implies $a \prec x$, as required. (ii) (a) is shown similarly.

(ii) (b) Let $a, b \in A$. By definition, $\Diamond(a \vee b) = \bigwedge {\prec}[a \vee b] = \bigwedge\{d \mid a \vee b \prec d\}$, and, since A^δ is completely distributive when A is a DL (cf. Proposition 1(i)),

$$\Diamond a \vee \Diamond b = (\bigwedge {\prec}[a]) \vee (\bigwedge {\prec}[b]) = \bigwedge\{c \vee c' \mid a \prec c \text{ and } b \prec c'\}.$$

So, to show that $\Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$, it is enough to show that $\{c \vee c' \mid a \prec c \text{ and } b \prec c'\} \subseteq \{d \mid a \vee b \prec d\}$, i.e. that for all $c, c' \in A$, if $a \prec c$ and $b \prec c'$, then $a \vee b \prec c \vee c'$. By (WO), $a \prec c \leq c \vee c'$ and $b \prec c' \leq c \vee c'$ imply that $a \prec c \vee c'$ and $b \prec c \vee c'$, which by (OR) implies that $a \vee b \prec c \vee c'$, as required. (i) (b) is argued similarly. (ii) (c) To show that $\Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$, it is enough to show that for any $x \in A$, if $\Diamond a \vee \Diamond b \leq x$, then $\Diamond(a \vee b) \leq x$. By Lemma 1 (ii) $\Diamond a \vee \Diamond b \leq x$ iff $a \prec x$ and $b \prec x$, we now apply (OR) and Lemma 1 (i) to finish the proof. (i)(c) is proven similarly. (iii) By assumption, $\bot \prec \bot$, i.e. $\bot \in {\prec}[\bot]$, which implies $\Diamond\bot = \bigwedge {\prec}[\bot] \leq \bot$, as required. (iv) is argued similarly.

The following lemma gives a converse of Lemma 3 for $\Diamond$-directed or $\blacksquare$-directed proto-subordination algebras.

Lemma 4. *For any proto-subordination algebra $\mathbb{S} = (A, \prec)$,*

(i) If $\mathbb{S} \models \text{(WO)} + \text{(DD)}$*, then:*
- *(a)* $\mathbb{S} \models \text{(SI)}$ *iff* $\Diamond$ *on* $\mathbb{S}^*$ *is monotone.*
- *(b)* $\mathbb{S} \models \text{(OR)}$ *iff* $\mathbb{S}^* \models \Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$.
- *(c)* $\mathbb{S} \models (\bot)$ *iff* $\mathbb{S}^* \models \Diamond\bot \leq \bot$.

(ii) If $\mathbb{S} \models \text{(SI)} + \text{(UD)}$*, then:*
- *(a)* $\mathbb{S} \models \text{(WO)}$ *iff* $\blacksquare$ *on* $\mathbb{S}^*$ *is monotone;*
- *(b)* $\mathbb{S} \models \text{(AND)}$ *iff* $\mathbb{S}^* \models \blacksquare a \wedge \blacksquare b \leq \blacksquare(a \wedge b)$*;*
- *(c)* $\mathbb{S} \models (\top)$ *iff* $\mathbb{S}^* \models \top \leq \blacksquare\top$.

Proof. We only show the items in (i), the proofs of those in (ii) being similar. (a) By Lemma 3 (i)(a), the proof is complete if we show the 'if' direction. Let $a, b, x \in A$ s.t. $a \leq b \prec x$. By Lemma 1 (ii), to show that $a \prec x$, it is enough to show that $\Diamond a \leq x$. Since $\Diamond$ is monotone, $a \leq b$ implies $\Diamond a \leq \Diamond b$, and, again by Lemma 1 (ii), $b \prec x$ implies that $\Diamond b \leq x$. Hence, $\Diamond a \leq x$, as required.

(b) By Lemma 3 (ii)(c), the proof is complete if we show the 'if' direction. Let $a, b, x \in A$ s.t. $a \prec x$ and $b \prec x$. By Lemma 1 (ii), to show that $a \vee b \prec x$, it is enough to show that $\Diamond(a \vee b) \leq x$, and since $\mathbb{S}^* \models \Diamond(a \vee b) \leq \Diamond a \vee \Diamond b$, it is enough to show that $\Diamond a \vee \Diamond b \leq x$, i.e. that $\Diamond a \leq x$ and $\Diamond b \leq x$. These two inequalities hold by Lemma 1 (ii), and the assumptions on a, b and x.

(c) By Lemma 1 (ii), $\bot \prec \bot$ is equivalent to $\Diamond\bot \leq \bot$, as required.

Corollary 1. *For every directed proto-subordination algebra $\mathbb{S} = (A, \prec)$,*

(i) $\mathbb{S}$ *is monotone iff* $\mathbb{S}^*$ *is monotone;*
(ii) $\mathbb{S}$ *is regular iff* $\mathbb{S}^*$ *is regular;*
(iii) $\mathbb{S}$ *is a subordination algebra iff* $\mathbb{S}^*$ *is normal.*

Lemma 5. *For any proto-subordination algebra $\mathbb{S} = (A, \prec)$, for all $a, b \in A$, $k \in K(A^\delta)$, and $o \in O(A^\delta)$, and all $D, U \subseteq A$,*

(i) if $\mathbb{S} \models \text{(SI)} + \text{(DD)} + \text{(WO)}$*, then*
- *(a) if $D \subseteq A$ is down-directed, then so is $\prec[D] := \{c \mid \exists a(a \in D \ \&\ a \prec c)\}$;*
- *(b) if $k = \bigwedge D$ for some down-directed $D \subseteq A$, then $\Diamond k = \bigwedge \prec[D] \in K(A^\delta)$;*
- *(c) $\Diamond k \leq b$ implies $a \prec b$ for some $a \in A$ s.t. $k \leq a$.*
- *(d) $\Diamond k \leq o$ implies $a \prec b$ for some $a, b \in A$ s.t. $k \leq a$ and $b \leq o$.*

(ii) if $\mathbb{S} \models \text{(WO)} + \text{(UD)} + \text{(SI)}$*, then*
- *(a) if $U \subseteq A$ is up-directed, then so is $\prec^{-1}[U] := \{c \mid \exists a(a \in U \ \&\ c \prec a)\}$;*
- *(b) if $o = \bigvee U$ for some up-directed $U \subseteq A$, then $\blacksquare o = \bigvee \prec^{-1}[U] \in O(A^\delta)$;*
- *(c) $a \leq \blacksquare o$ implies $a \prec b$ for some $b \in A$ s.t. $b \leq o$.*
- *(d) $k \leq \blacksquare o$ implies $a \prec b$ for some $a, b \in A$ s.t. $k \leq a$ and $b \leq o$.*

Proof. We only prove (i), the proof of (ii) being similar.

(a) If $c_i \in \prec[D]$ for $1 \leq i \leq 2$, then $a_i \prec c_i$ for some $a_i \in D$. Since D is down-directed, some $a \in D$ exists s.t. $a \leq a_i$ for each i. Thus, (SI) implies that $a \prec c_i$, from which the claim follows by (DD).

(b) By definition, $\Diamond k = \bigwedge\{\Diamond a \mid a \in A, k \leq a\} = \bigwedge\{c \mid \exists a(a \prec c \ \&\ k \leq a)\}$. Since $k = \bigwedge D$ for some $D \subseteq A$ down-directed, by compactness, $k \leq a$ implies $d \leq a$ for some $d \in D$, thus $\Diamond k = \bigwedge := \{c \mid \exists a(a \prec c \ \&\ k \leq a)\} = \bigwedge\{c \mid \exists a(a \in \lfloor D \rfloor \ \&\ a \prec c)\} = \bigwedge {\prec}[D] \in K(A^\delta)$, the last membership holding by (a).

(c) By (b), $\Diamond k \in K(A^\delta)$. Hence, $\Diamond k \leq b$ implies by compactness that $c \leq b$ for some $c \in A$ s.t. $a \prec c$ for some $a \in D$ (hence $k = \bigwedge D \leq a$). By (WO), this implies that $a \prec b$ for some $a \in A$ s.t. $k \leq a$, as required.

(d) By (b), $\Diamond k \in K(A^\delta)$. Since $o \in O(A^\delta)$, some updirected $U \subseteq A$ exists s.t. $o = \bigvee U$. Hence, by compactness, $\Diamond k \leq o$ implies that $a \prec b$ for some $a \in A$ s.t. $k \leq a$ and some $b \in U$ (for which $b \leq o$).

Proposition 2. *For any proto-subordination algebra $\mathbb{S} = (A, \prec)$,*

(i) $\mathbb{S} \models {\prec} \subseteq {\leq}$ *iff* $\mathbb{S}^* \models a \leq \Diamond a$ *iff* $\mathbb{S}^* \models \blacksquare a \leq a$.
(ii) If $\mathbb{S} \models$ (WO) + (DD), *then* $\mathbb{S} \models {\leq} \subseteq {\prec}$ *iff* $\mathbb{S}^* \models \Diamond a \leq a$;
(iii) if $\mathbb{S} \models$ (WO) + (DD) + (SI), *then*
  (a) $\mathbb{S} \models$ (T) *iff* $\mathbb{S}^* \models \Diamond a \leq \Diamond\Diamond a$.
  (b) $\mathbb{S} \models$ (D) *iff* $\mathbb{S}^* \models \Diamond\Diamond a \leq \Diamond a$.
(iv) if $\mathbb{S} \models$ (WO) + (DD) + (SI) *and is meet-semilattice based, then*
  (a) $\mathbb{S} \models$ (CT) *iff* $\mathbb{S}^* \models \Diamond a \leq \Diamond(a \wedge \Diamond a)$.
  (b) $\mathbb{S} \models$ (SL2) *iff* $\mathbb{S}^* \models \Diamond(\Diamond a \wedge \Diamond b) \leq \Diamond(a \wedge b)$.
(v) if $\mathbb{S} \models$ (SI), *then* $\mathbb{S} \models$ (CT) *implies* $\mathbb{S} \models$ (T).
(vi) if $\mathbb{S}$ *is directed and based on* $(A, \neg)$ *with* $\neg$ *antitone, involutive, and (left or right) self-adjoint,*
  (a) $\mathbb{S} \models$ (S6) *iff* $\mathbb{S}^* \models \neg\Diamond a = \blacksquare\neg a$, *thus* $\blacksquare a := \neg\Diamond\neg a$.
  (b) $\mathbb{S} \models$ (S6) *iff* $\mathbb{S}^* \models \Diamond\neg a = \neg\blacksquare a$, *thus* $\Diamond a := \neg\blacksquare\neg a$.
(vii) If $\mathbb{S} \models$ (SI) + (UD) + (WO) *and is join-semilattice based, then*
  (a) $\mathbb{S} \models$ (S9 $\Rightarrow$) *iff* $\mathbb{S}^* \models \blacksquare(a \vee \blacksquare b) \leq \blacksquare a \vee \blacksquare b$.
  (b) $\mathbb{S} \models$ (S9 $\Leftarrow$) *iff* $\mathbb{S}^* \models \blacksquare a \vee \blacksquare b \leq \blacksquare(a \vee \blacksquare b)$.
  (c) $\mathbb{S} \models$ (SL1) *iff* $\mathbb{S}^* \models \blacksquare(a \vee b) \leq \blacksquare(\blacksquare a \vee \blacksquare b)$.

Proof. (i) By definition, $\forall a(a \leq \Diamond a)$ iff $\forall a(a \leq \bigwedge {\prec}[a])$ iff $\forall a(a \leq \bigwedge\{b \in A \mid a \prec b\})$ iff $\forall a \forall b(a \prec b \Rightarrow a \leq b)$ iff ${\prec} \subseteq {\leq}$. The second part of the statement is proved similarly.

(ii) By Lemma 1 (i), if $a \prec a$, then $\Diamond a \leq a$. Hence, the left-to-right direction follows from the reflexivity of $\leq$ and the assumption. Conversely, $\Diamond a \leq a$ and $a \leq b$ imply $\Diamond a \leq b$, which, by Lemma 1 (ii) and $\mathbb{S} \models$ (WO) + (DD), is equivalent to $a \prec b$.

(iii) (a) From left to right,

$\Diamond\Diamond a = \bigwedge\{\Diamond b \mid \Diamond a \leq b\}$	Definition 7 applied to $\Diamond a \in K(A^\delta)$
$= \bigwedge\{\Diamond b \mid a \prec b\}$	Lemma 1 (ii) since $\mathbb{S} \models$ (WO) + (DD)
$= \bigwedge\{c \mid \exists b(a \prec b \ \&\ b \prec c)\}$	$\Diamond b = \bigwedge\{c \in A \mid b \prec c\}$

Hence, to show that $\Diamond a = \bigwedge\{c \mid a \prec c\} \leq \Diamond\Diamond a$, it is enough to show that $\{c \mid \exists b(a \prec b \ \&\ b \prec c)\} \subseteq \{c \mid a \prec c\}$, which is immediately implied by the assumption (T). Conversely, let $a, b, c \in A$ s.t. $a \prec b$ and $b \prec c$. To show that $a \prec c$, by Lemma 1 (ii), and $\mathbb{S} \models$ (WO) + (DD), it is enough to show that $\Diamond a \leq c$, and since $\Diamond a \leq \Diamond\Diamond a$, it is enough to show that $\Diamond\Diamond a \leq c$. The assumption $a \prec b$ implies $\Diamond a \leq b$ which implies $\Diamond\Diamond a \leq \Diamond b$, by the monotonicity of $\Diamond$ (which depends on (SI), cf. Lemma 3(i)(a)). Hence, combining the latter inequality with $\Diamond b \leq c$ (which is implied by $b \prec c$), by the transitivity of $\leq$, we get $\Diamond\Diamond a \leq c$, as required.

(iii) (b) From left to right, by the definitions spelled out in the proof of (ii) (b), it is enough to show that $\{c \mid a \prec c\} \subseteq \{c \mid \exists b(a \prec b \ \&\ b \prec c)\}$, which is immediately implied by the assumption (D). Conversely, let $a, c \in A$ s.t. $a \prec c$, and let us show that $a \prec b$ and $b \prec c$ for some $b \in A$. The assumption $a \prec c$ implies $\Diamond a \leq c$. Since $\Diamond\Diamond a \leq \Diamond a$, this implies $\Diamond\Diamond a \leq c$, i.e. (see discussion above) $\bigwedge\{d \in A \mid \exists b(a \prec b \ \&\ b \prec d)\} \leq c$.

We claim that $D := \{d \in A \mid \exists b(a \prec b \ \&\ b \prec d)\}$ is down-directed: indeed, if $d_1, d_2 \in A$ s.t. $\exists b_i(a \prec b_i \ \&\ b_i \prec d_i)$ for $1 \leq i \leq 2$, then by (DD), some $b \in A$ exists s.t. $a \prec b$ and $b \leq b_i$. By (SI), $b \leq b_i \prec d_i$ implies $b \prec d_i$. By (DD) again, this implies that some $d \in A$ exists s.t. $b \prec d$ and $d \leq d_i$, which concludes the proof of the claim.

By compactness, $d \leq c$ for some $d \in A$ s.t. $a \prec b$ and $b \prec d$ for some $b \in A$. To finish the proof, it is enough to show that $b \prec c$, which is immediately implied by $b \prec d \leq c$ and (SI).

(iv) (a) From left to right, Let $a \in A$. If ${\prec}[a] = \varnothing$, then $\Diamond a = \bigwedge \varnothing = \top = \Diamond(a \wedge \top) = \Diamond(a \wedge \Diamond a)$, as required. If ${\prec}[a] \neq \varnothing$, then $a \wedge \Diamond a = \bigwedge\{a \wedge e \mid a \prec e\} \in K(A^\delta)$, since, by (DD), $\{a \wedge e \mid a \prec e\}$ is down-directed. Hence,

$$\begin{aligned}\Diamond(a \wedge \Diamond a) &= \bigwedge\{\Diamond c \mid a \wedge \Diamond a \leq c\} && \text{definition of } \Diamond \text{ on } K(A^\delta)\\ &= \bigwedge\{d \mid \exists c(c \prec d \ \&\ a \wedge \Diamond a \leq c)\} && \text{definition of } \Diamond \text{ on } A\end{aligned}$$

By compactness, $\bigwedge\{a \wedge e \mid a \prec e\} = a \wedge \Diamond a \leq c$ is equivalent to $a \wedge e \leq c$ for some $e \in A$ such that $a \prec e$. Thus,

$$\begin{aligned}\Diamond(a \wedge \Diamond a) &= \bigwedge\{d \mid \exists c(c \prec d \ \&\ a \wedge \Diamond a \leq c)\}\\ &= \bigwedge\{d \mid \exists c(c \prec d \ \&\ \exists e(a \prec e \ \&\ a \wedge e \leq c))\}.\end{aligned}$$

To finish the proof that $\Diamond a = \bigwedge\{d \mid a \prec d\} \leq \Diamond(a \wedge \Diamond a)$, it is enough to show that if $d \in A$ is such that $\exists c(c \prec d \ \&\ \exists e(a \prec e \ \&\ a \wedge e \leq c))$ then $a \prec d$. Since $a \wedge e \leq c$ and $c \prec d$, by (SI), $a \wedge e \prec d$. Hence, by (CT), from $a \prec e$ and $a \wedge e \prec d$ it follows that $a \prec d$, as required.

Conversely, assume that $\Diamond a \leq \Diamond(a \wedge \Diamond a)$ holds for any a. By (WO), (DD), and Lemma 1 (ii), (CT) can be equivalently rewritten as follows:

$$\text{if } \Diamond a \leq b \quad \text{and} \quad \Diamond(a \wedge b) \leq c, \text{ then } \Diamond a \leq c.$$

Since $\wedge$ is monotone, $\Diamond a \leq b$ implies that $a \wedge \Diamond a \leq a \wedge b$, which implies, by the monotonicity of $\Diamond$ (which is implied by (SI), cf. Lemma 3(i)(a)), that $\Diamond(a \wedge \Diamond a) \leq \Diamond(a \wedge b)$. Hence, combining the latter inequality with $\Diamond a \leq \Diamond(a \wedge \Diamond a)$

and $\Diamond(a \wedge b) \leq c$, by the transitivity of $\leq$, we get $\Diamond a \leq c$, as required. The proof of (iv)(b) is dual to that of (vii)(c), and is omitted.

(v) Let $a, b, c \in A$. If $a \prec b$ and $b \prec c$, then by (SI), $a \wedge b \leq b \prec c$ implies $a \wedge b \prec c$, which, by (CT), implies $a \prec c$, as required.

(vi) By Proposition 1(ii) and (iii), both extensions $\neg^\sigma$ and $\neg^\pi$ on A^δ are involutive. Hence, the following chains of equivalences hold under both interpretations of the negation, and thus, abusing notation, we omit the superscript.

$$\begin{array}{lll} & \neg\Diamond a \leq \blacksquare\neg a & \\ \text{iff} & \neg\blacksquare\neg a \leq \Diamond a = \bigwedge\{b \mid a \prec b\} & \neg \text{ antitone and involutive} \\ \text{iff} & \neg\blacksquare\neg a \leq b \text{ for all } b \in A \text{ s.t. } a \prec b & \\ \text{iff} & \neg b \leq \blacksquare\neg a \text{ for all } b \in A \text{ s.t. } a \prec b & \neg \text{antitone and involutive} \\ \text{iff} & \neg b \prec \neg a \text{ for all } b \in A \text{ s.t. } a \prec b & \text{Lemma 1 (iii), (SI), (UD)} \end{array}$$

$$\begin{array}{lll} & \Diamond\neg b \leq \neg\blacksquare b & \\ \text{iff} & \bigvee\{a \mid a \prec b\} = \blacksquare b \leq \neg\Diamond\neg b & \neg \text{ antitone and involutive} \\ \text{iff} & a \leq \neg\Diamond\neg b \text{ for all } a \in A \text{ s.t. } a \prec b & \\ \text{iff} & \Diamond\neg b \leq \neg a \text{ for all } a \in A \text{ s.t. } a \prec b & \neg \text{ antitone and involutive} \\ \text{iff} & \neg b \prec \neg a \text{ for all } a \in A \text{ s.t. } a \prec b & \text{Lemma 1 (ii), (WO), (DD)} \end{array}$$

Since $\neg$ is involutive, condition (S6) can equivalently be rewritten as $\forall a \forall b(a \prec \neg b \Rightarrow b \prec \neg a)$ and as $\forall a \forall b(\neg a \prec b \Rightarrow \neg b \prec a)$. Hence:

$$\begin{array}{lll} & \neg\blacksquare a \leq \Diamond\neg a = \bigwedge\{b \mid \neg a \prec b\} & \\ \text{iff} & \neg\blacksquare a \leq b \text{ for all } b \in A \text{ s.t. } \neg a \prec b & \\ \text{iff} & \neg b \leq \blacksquare a \text{ for all } b \in A \text{ s.t. } a \prec b & \neg \text{ antitone and involutive} \\ \text{iff} & \neg b \prec a \text{ for all } b \in A \text{ s.t. } \neg a \prec b & \text{Lemma 1 (iii), (SI), (UD)} \end{array}$$

$$\begin{array}{lll} & \bigvee\{a \mid a \prec \neg b\} = \blacksquare\neg b \leq \neg\Diamond b & \\ \text{iff} & a \leq \neg\Diamond b \text{ for all } b \in A \text{ s.t. } a \prec \neg b & \\ \text{iff} & \Diamond b \leq \neg a \text{ for all } b \in A \text{ s.t. } a \prec \neg b & \neg \text{ antitone and involutive} \\ \text{iff} & b \prec \neg a \text{ for all } b \in A \text{ s.t. } a \prec \neg b & \text{Lemma 1 (ii), (WO), (DD)} \end{array}$$

(vii) (a) From left to right, let $a, b \in A$. If $\prec^{-1}[b] = \varnothing$, then $\blacksquare b = \bigvee\varnothing = \bot$, hence $\blacksquare(a \vee \blacksquare b) = \blacksquare(a \vee \bot) = \blacksquare a = \blacksquare a \vee \bot = \blacksquare a \vee \blacksquare b$, as required. If $\prec^{-1}[b] \neq \varnothing$, then by definition, $a \vee \blacksquare b = \bigvee\{a \vee e \mid e \prec b\} \in O(A^\delta)$ since, by (UD), $\{a \vee e \mid e \prec b\}$ is up-directed. Hence:

$$\begin{array}{rll} \blacksquare(a \vee \blacksquare b) & = \bigvee\{\blacksquare d \mid d \leq a \vee \blacksquare b\} & \text{definition of } \blacksquare \text{ on } O(A^\delta) \\ & = \bigvee\{c \mid \exists d(c \prec d \ \&\ d \leq a \vee \blacksquare b)\} & \text{definition of } \blacksquare \text{ on } A \end{array}$$

$$\begin{array}{rl} \blacksquare a \vee \blacksquare b & = \bigvee\{y \mid y \prec a\} \vee \bigvee\{z \mid z \prec b\} \\ & = \bigvee\{y \vee z \mid y \prec a \text{ and } z \prec b\}. \end{array}$$

Hence, to show that $\blacksquare(a \vee \blacksquare b) \leq \blacksquare a \vee \blacksquare b$, it is enough to show that if $c \in A$ is s.t. $\exists d(c \prec d \ \&\ d \leq a \vee \blacksquare b)$, then $c \leq y \vee z$ for some $y, z \in A$ s.t. $y \prec a$ and $z \prec b$. By compactness, $d \leq a \vee \blacksquare b$ implies that $d \leq a \vee e$ for some $e \in A$ s.t. $e \prec b$.

By (WO), $c \prec d \leq a \vee e$ implies $c \prec a \vee e$. Summing up, $\exists e(e \prec b \ \& \ c \prec a \vee e)$. Hence, by (S9 $\Rightarrow$), $\exists a' \exists b'(a' \prec a \ \& \ b' \prec b \ \& \ c \leq a' \vee b')$, which is the required condition for $y := a'$ and $z := b'$.

Conversely, let $x, a, b \in A$ s.t. $c \prec b$ and $x \prec a \vee c$ for some $c \in A$. By Lemma 1(i), $x \prec a \vee c$ implies that $x \leq \blacksquare(a \vee c)$, and $c \prec b$ implies that $c \leq \blacksquare b$. Hence, the monotonicity of $\blacksquare$ (which is guaranteed by (WO), cf. Lemma 3(ii)(a)) and the assumption imply that the following chain of inequalities holds: $x \leq \blacksquare(a \vee c) \leq \blacksquare(a \vee \blacksquare b) \leq \blacksquare a \vee \blacksquare b = \left(\bigvee \prec^{-1}[a]\right) \vee \left(\bigvee \prec^{-1}[b]\right) = \bigvee\{a' \vee b' \mid a' \prec a \text{ and } b' \prec b\}$.

We claim that $U := \{a' \vee b' \mid a' \prec a \text{ and } b' \prec b\}$ is up-directed: indeed, if $a'_i \vee b'_i \in U$ for $1 \leq i \leq 2$, then $a'_i \prec a$ and $b'_i \prec b$ imply by (UD) that $a' \prec a$ and $b' \prec b$ for some $a', b' \in A$ s.t. $a'_i \leq a'$ and $b'_i \leq b'$, hence $a'_i \vee b'_i \leq a' \vee b' \in U$. Hence, by compactness, $x \leq a' \vee b'$ for some $a', b' \in A$ s.t. $a' \prec a$ and $b' \prec b$, as required.

(vii) (b) From left to right, let $a, b \in A$. By the definitions spelled out in the proof of (vii) (a), to show that $\blacksquare a \vee \blacksquare b \leq \blacksquare(a \vee \blacksquare b)$, it is enough to show that for all $x, y \in A$, if $y \prec a$ and $z \prec b$, then $y \vee z \prec d$ for some $d \in A$ s.t. $d \leq a \vee e$ for some $e \in A$ s.t. $e \prec b$. By (S9 $\Leftarrow$), $y \vee z \prec a \vee c$ for some $c \in A$ s.t. $c \prec b$. Then the statement is verified for $d := a \vee c$ and $e := c$.

Conversely, let $a, b, x \in A$ s.t. $a' \prec a$, $b' \prec b$ and $x \leq a' \vee b'$ for some $a', b' \in A$, and let us show that $x \prec a \vee c$ for some $c \in A$ s.t. $c \prec b$. By Lemma 1 (i), the assumptions imply that the following chain of inequalities holds: $x \leq a' \vee b' \leq \blacksquare a \vee \blacksquare b \leq \blacksquare(a \vee \blacksquare b) = \bigvee\{e \mid \exists d(e \prec d \ \& \ d \leq a \vee \blacksquare b)\}$, the last identity being discussed in the proof of (vii) (a).

We claim that the set $U := \{e \mid \exists d(e \prec d \ \& \ d \leq a \vee \blacksquare b)\}$ is up-directed: indeed, if $e_i \prec d_i$ for some $d_i \in A$ s.t. $d_i \leq a \vee \blacksquare b$ where $1 \leq i \leq 2$, then by (WO), $e_i \prec d_i \leq d_1 \vee d_2$ implies $e_i \prec d_1 \vee d_2$, hence by (UD), $e \prec d_1 \vee d_2$ for some $e \in A$ s.t. $e_i \leq e$, and finally, $e \in U$, its witness being $d := d_1 \vee d_2 \leq a \vee \blacksquare b$.

Hence, by compactness, $x \leq e$ for some $e \in A$ s.t. $e \prec d$ for some $d \in A$ s.t. $d \leq a \vee \blacksquare b = \bigvee\{a \vee c \mid c \prec b\}$. Again by compactness (which is applicable because, as discussed in the proof of (vii)(a), $\{a \vee c \mid c \prec b\}$ is up-directed), $d \leq a \vee c$ for some $c \in A$ s.t. $c \prec b$. Hence, by (WO) and (SI), $x \leq e \prec d \leq a \vee c$ implies $x \prec a \vee c$, as required.

(vii)(c) From left to right, let $a, b, \in A$. By definition, $\blacksquare a \vee \blacksquare b = \bigvee\{x \vee y \mid x \prec a \text{ and } y \prec b\} \in O(A^\delta)$, since $\{x \vee y \mid x \prec a \text{ and } y \prec b\}$ is up-directed, as discussed in the proof of (vii)(a). Then,

$$\begin{aligned}
\blacksquare(\blacksquare a \vee \blacksquare b) &= \bigvee\{\blacksquare c \mid c \leq \blacksquare a \vee \blacksquare b\} && \text{definition of } \blacksquare \text{ on } O(A^\delta) \\
&= \bigvee\{d \mid \exists c(d \prec c \ \& \ c \leq \blacksquare a \vee \blacksquare b)\}. && \blacksquare c := \bigvee\{d \mid d \prec c\}
\end{aligned}$$

Hence, to prove that $\bigvee\{d \mid d \prec a \vee b\} = \blacksquare(a \vee b) \leq \blacksquare(\blacksquare a \vee \blacksquare b)$, it is enough to show that if $d \prec a \vee b$, then $d \prec c$ for some $c \in A$ s.t. $c \leq \blacksquare a \vee \blacksquare b$. By (SL1), $d \prec a \vee b$ implies that $d \prec a' \vee b'$ for some $a', b' \in A$ s.t. $a' \prec a$ and $b' \prec b$. Since $a' \leq \blacksquare a$ and $b' \leq \blacksquare b$, The statement is verified for $c := a' \vee b'$.

Conversely, let $a, b, c \in A$ s.t. $c \prec a \vee b$, and let us show that $c \prec a' \vee b'$ for some $a', b' \in A$ s.t. $a' \prec a$ and $b' \prec b$. From $c \prec a \vee b$ it follows that $c \leq \blacksquare(a \vee b) \leq \blacksquare(\blacksquare a \vee \blacksquare b) = \bigvee\{d \mid \exists e(d \prec e \ \& \ e \leq \blacksquare a \vee \blacksquare b)\}$.

We claim that $U := \{d \mid \exists e(d \prec e \;\&\; e \leq \blacksquare a \vee \blacksquare b)\}$ is up-directed: indeed, if $d_i \prec e_i$ and $e_i \leq \blacksquare a \vee \blacksquare b$, then $e_1 \vee e_2 \leq \blacksquare a \vee \blacksquare b$, and by (WO), $d_i \prec e_i \leq e_1 \vee e_2$ implies that $d_i \prec e_1 \vee e_2$, hence, by (UD), $d \prec e_1 \vee e_2$ for some $d \in A$ s.t. $d_i \leq d$, and $d \in U$, its witness being $e := e_1 \vee e_2$.

Hence, by compactness, $c \leq d$ for some $d \in A$ s.t. $d \prec e$ for some $e \in A$ s.t. $e \leq \blacksquare a \vee \blacksquare b = \bigvee\{x \vee y \mid x \prec a \text{ and } y \prec b\}$. Since, s discussed above, $\{x \vee y \mid x \prec a \text{ and } y \prec b\}$ is up-directed, by compactness, $e \leq a' \vee b'$ for some $a', b' \in A$ s.t. $a' \prec a$ and $b' \prec b$. By (SI) and (WO), $c \leq d \prec e \leq a' \vee b'$ implies $c \prec a' \vee b'$, as required.

4 Applications

In the present section, we discuss two independent but connected ways of using the characterization results of the previous section. Firstly, the output operators out_i^N for $1 \leq i \leq 4$ associated with a given input/output logic $\mathbb{L} = (\mathcal{L}, N)$ can be given semantic counterparts in the environment of proto-subordination algebras as follows: for every proto-subordination algebra $\mathbb{S} = (A, \prec)$, we let $\mathbb{S}_i := (A, \prec_i)$ where $\prec_i \subseteq A \times A$ is the smallest extension of $\prec$ which satisfies the properties indicated in the following table:

$\prec_i$	Properties
$\prec_1$	($\top$), (SI), (WO), (AND)
$\prec_2$	($\top$), (SI), (WO), (AND), (OR)
$\prec_3$	($\top$), (SI), (WO), (AND), (CT)
$\prec_4$	($\top$), (SI), (WO), (AND), (OR), (CT)

Then, for each $1 \leq i \leq 4$, and every $B \subseteq A$, if $k = \bigwedge B \in K(A^\delta)$, then[4]

$$\Diamond_i^\sigma k := \bigwedge\{\prec_i[a] \mid a \in A \text{ and } k \leq a\}$$

encodes the algebraic counterpart of $out_i^N(\Gamma)$ for any $\Gamma \subseteq \mathrm{Fm}$, and the characteristic properties of $\Diamond_i$ for each $1 \leq i \leq 4$ are those identified in Lemma 4, Corollary 1, and Proposition 2. For any directed proto-subordination algebra $\mathbb{S} = (A, \prec)$, let $\mathbb{S}_i^* := (A, \Diamond_i, \blacksquare_i)$ denote the slanted algebras associated with $\mathbb{S}_i = (A, \prec_i)$ for each $1 \leq i \leq 4$.

Proposition 3. *For any directed proto-subordination algebra* $\mathbb{S} = (A, \prec)$*,*

(i) $\Diamond_1$ *(resp.* $\Diamond_2$*) is the largest monotone (resp. regular) map dominated by* $\Diamond$ *(i.e. pointwise-smaller than or equal to* $\Diamond$*), and* $\blacksquare_1$ *(resp.* $\blacksquare_2$*) is the largest monotone (resp. regular) map dominated by* $\blacksquare$*.*

[4] When $\mathcal{L}$ does not have the conjunction property, this construction works only under the additional assumption that B is down-directed; however, in most common cases (e.g. when $\mathbb{S}$ is lattice-based) this assumption is not needed.

(ii) $\Diamond_3$ *(resp.* $\Diamond_4$*) is the largest monotone (resp. regular) map satisfying* $\Diamond a \leq \Diamond(a \wedge \Diamond a)$ *dominated by* $\Diamond$*, and* $\blacksquare_3$ *(resp.* $\blacksquare_4$*) is the largest monotone (resp. regular) map satisfying* $\blacksquare(a \vee \blacksquare a) \leq \blacksquare a$ *dominated by* $\blacksquare$*.*

Proof. By Lemma 4 and Proposition 2, the properties stated in each item of the statement hold for $\Diamond_i$ and $\blacksquare_i$. To complete the proof, we need to argue for $\Diamond_i$ being the largest such map (the proof for $\blacksquare_i$ is similar). By Lemma 1 (ii), $a \prec_i b$ iff $\Diamond_i a \leq b$ for all $a, b \in A$ and $1 \leq i \leq 4$. Any $f : A \to A^\delta$ s.t. $f(a) \in K(A^\delta)$ for every $a \in A$ induces a proto-subordination relation $\prec_f \subseteq A \times A$ defined as $a \prec_f b$ iff $f(a) \leq b$. Clearly, if $f(a) \leq f'(a)$ for every $a \in A$, then $\prec_{f'} \subseteq \prec_f$. Moreover, if $f(a) < f'(a)$, then, by denseness, $f(a) \leq b$ for some $b \in A$ s.t. $f'(a) \nleq b$, hence $\prec_{f'} \subset \prec_f$.

If $\Diamond_i$ is not the largest map endowed with the properties mentioned in the statement and dominated by $\Diamond$, then a map f exists which is endowed with these properties such that $\Diamond_i a \leq f(a) \leq \Diamond a$ for all $a \in A$, and $\Diamond_i b < f(b)$ for some $b \in A$. Then, by the argument in the previous paragraph, $\prec = \prec_\Diamond \subseteq \prec_f \subset \prec_{\Diamond_i} = \prec_i$. As f is endowed with the the properties mentioned in the statement, $\prec_f$ is an extension of $\prec$ which enjoys the required properties, and is strictly contained in $\prec_i$. Hence, $\prec_i$ is not the smallest such extension.

As to the second application, in [8], Celani introduces an expansion of Priestley's duality for bounded distributive lattices to *subordination lattices*, i.e. tuples $\mathbb{S} = (A, \prec)$ such that A is a distributive lattice and $\prec \subseteq A \times A$ is a subordination relation.[5] The dual structure of any subordination lattice $\mathbb{S} = (A, \prec)$ is referred to as the *(Priestley) subordination space* of $\mathbb{S}$, and is defined as $\mathbb{S}_* := (X(A), R_\prec)$, where $X(A)$ is (the Priestley space dual to A, based on) the set of prime filters of A, and $R_\prec \subseteq X(A) \times X(A)$ is defined as follows: for P, Q prime filters of A,

$$(P, Q) \in R_\prec \quad \text{iff} \quad \prec[P] := \{x \in A \mid \exists a(a \in P \ \&\ a \prec x)\} \subseteq Q.$$

Up to isomorphism, we can equivalently define the subordination space of $\mathbb{S}$ as follows:

Definition 9. *The* subordination space *associated with a subordination lattice* $\mathbb{S} = (A, \prec)$ *is* $\mathbb{S}_* := (J^\infty(A^\delta), R_\prec)$*, where* $J^\infty(A^\delta)$ *is the set of the completely join-irreducible elements of* A^δ*, and* $R_\prec \subseteq J^\infty(A^\delta) \times J^\infty(A^\delta)$ *such that* $(j, i) \in R_\prec$ *iff* $i \leq \Diamond j$*.*

Lemma 6. *For any subordination lattice* $\mathbb{S} = (A, \prec)$*, the subordination spaces* $\mathbb{S}_*$ *given according to the two definitions above are isomorphic.*

Proof. As is well known, in the canonical extension A^δ of any distributive lattice A, the set $J^\infty(A^\delta)$ of the completely join-irreducible elements of A^δ coincides with the set of its completely join-prime elements, which are in dual order-isomorphism with the prime filters of A. Specifically, if $P \subseteq A$ is a prime filter,

[5] In the terminology of the present paper, subordination lattices are subordination algebras based on bounded distributive lattices (cf. Definition 3).

then $j_P := \bigwedge P \in K(A^\delta)$ is a completely join-prime element of A^δ; conversely, if j is a completely join-prime element of A^δ, then $P_j := \{a \in A \mid j \leq a\}$ is a prime filter of A. Clearly, $j = \bigwedge P_j = j_{P_j}$ for any $j \in J^\infty(A^\delta)$; moreover, it is easy to show, by applying compactness, that $P_{j_P} = \{a \in A \mid \bigwedge P \leq a\} = P$ for any prime filter P of A.

To complete the proof and show that the two relations $R_\prec$ can be identified modulo the identifications above, it is enough to show that $\prec[P] \subseteq Q$ iff $\bigwedge Q \leq \bigwedge \prec[P]$ for all prime filters P and Q of A. Clearly, $\prec[P] \subseteq Q$ implies $\bigwedge Q \leq \bigwedge \prec[P]$. Conversely, if $b \in \prec[P]$, then $\bigwedge Q \leq \bigwedge \prec[P] \leq b$, hence, by compactness and Q being an up-set, $b \in Q$, as required.

In [8], some properties of subordination lattices are dually characterized in terms properties of their associated subordination spaces, including those listed in the following proposition, which can be obtained as consequences of the dual characterizations in Proposition 2, slanted canonicity [14], and correspondence theory for distributive modal logic [10].

Proposition 4 (cf. [8], Theorem 5.7). *For any subordination lattice* $\mathbb{S}$*,*

(i) $\mathbb{S} \models \prec \subseteq \leq$ *iff* $R_\prec$ *is reflexive;*
(ii) $\mathbb{S} \models$ (D) *iff* $R_\prec$ *is transitive, i.e.* $R_\prec \circ R_\prec \subseteq R_\prec$*;*
(iii) $\mathbb{S} \models$ (T) *iff* $R_\prec$ *is dense, i.e.* $R_\prec \subseteq R_\prec \circ R_\prec$*;*
(iv) $\mathbb{S} \models (a = \bot) \vee (\blacksquare a \neq \bot)$ *iff* $R_\prec$ *is proper.*

Proof. We will give proof only of (i), since the other arguments are similar. By Proposition 2 (i), $\mathbb{S} \models \prec \subseteq \leq$ iff $\mathbb{S}^* \models a \leq \Diamond a$; the inequality $a \leq \Diamond a$ is analytic inductive (cf. [21, Definition 55]), and hence *slanted* canonical by [14, Theorem 4.1]. Hence, from $\mathbb{S}^* \models a \leq \Diamond a$ it follows that $(\mathbb{S}^*)^\delta \models a \leq \Diamond a$, where $(\mathbb{S}^*)^\delta$ is a *standard* (perfect) distributive modal algebra. By algorithmic correspondence theory for distributive modal logic (cf. [10, Theorems 8.1 and 9.8]), $(\mathbb{S}^*)^\delta \models a \leq \Diamond a$ iff $(\mathbb{S}^*)^\delta \models \forall \mathbf{j}(\mathbf{j} \leq \Diamond \mathbf{j})$ where $\mathbf{j}$ ranges in the set $J^\infty((\mathbb{S}^*)^\delta)$. By Definition 9, this is equivalent to $R_\prec$ being reflexive.

Likewise, items (iv), (vi), and (vii) of Proposition 2 can be used to extend Celani's results and provide relational characterizations, on subordination spaces, of conditions (CT), (S9), (SL1), (SL2), noticing that the modal inequalities corresponding to those conditions are all analytic inductive.

Proposition 5. *For any subordination lattice* $\mathbb{S}$*,*

(i) $\mathbb{S} \models$ (CT) *iff* $jR_\prec i$ *implies* $jR_\prec k, kR_\prec i$*, for some* $k \leq j$*.*
(ii) $\mathbb{S} \models$ (S9) *iff* $i_3 R_\prec i_1, i_3 R_\prec i_2 \Leftrightarrow (\exists j \leq i_1)\ jR_\prec i_2, i_3 R_\prec j$*.*
(iii) $\mathbb{S} \models$ (SL1) *iff* $i_4 R_\prec i_1, i_4 R_\prec i_2, i_3 R_\prec i_4 \Rightarrow (\exists j \leq i_1 \wedge i_2)\ i_3 R_\prec j$*.*
(iv) $\mathbb{S} \models$ (SL2) *iff* $i_1 R_\prec i_4, i_2 R_\prec i_4, i_4 R_\prec i_3 \Rightarrow (\exists j \leq i_1 \wedge i_2)\ jR_\prec i_3$*.*

5 Conclusions and Related Works

We have established a novel connection between the research fields of subordination algebras and of input/output logics. The present paper focuses only on conditional obligations; however, similarly to the duality between box and diamond operators in modal logic, conditional permission (aka negative permission) has been introduced and analysed by Makinson and van der Torre as the dual concept of conditional obligation [27]. In future work, we will study conditional permission in the environment of pre-contact algebras [15], algebraic structures defined dually to subordination algebras. Another interesting future direction concerns extending the present framework to constrained input/output logic [26,28,31], which are used for handling contrary-to-duty reasoning and also applied in the formalization of normative systems [1] in philosophical logic and AI [26]. We will also investigate the connections between our approach and joining-systems [24], an algebraic formalization of normative reasoning alternative to input/output logic.

We have presented a bi-modal characterization of input/output logic in the context of selfextensional logics, a class of logics defined in terms of minimal properties, which are satisfied both by classical propositional logic and by the best known nonclassical logics. The present approach is different from other modal formulations of input/output logic [25,29,35], also on a non-classical propositional base, in that the output operators themselves are semantically characterized as (suitable restrictions of) modal operators, and their properties characterized in terms of modal axioms (inequalities).

Legal Informatics has recently received a lot of attention from industry and institutions due to the rise of RegTech and FinTech. The input/output logic is expressive enough to support reasoning about constitutive, regulative and defeasible rules; these notions play an important role in the legal domains [5]. For example, reified input/output logic is a suitable formalism for expressing legal statements like those in the General Data Protection Regulation, for more details see the DAPRECO knowledge base [32]. There are several active projects for implementing input/output reasoners [3,17,23,33]. One of the current challenges is scalability of legal (and I/O) reasoners. Subordination algebras and its correspondence theorems in first-order logic can be used to understand algorithmic correspondence of input/output operations in first-order logic for designing scalable I/O reasoners, for example see [2], for legal applications.

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