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Hydra Games and Tree Ordinals

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Abstract. Hydra games were introduced by Kirby and Paris, for the formulation of a result which is independent from Peano arithmetic but depends on the transfinite structure of ϵ_0. Tree ordinals are a well-known simple way to represent countable ordinals. In this paper we study the relation between these concepts; an ordinal less than ϵ_0 is canonically translated into both a hydra and a tree ordinal term, and the reduction graph of the hydra and the normal form of the term syntactically correspond to each other.

1 Introduction

Tree ordinals are one of the simplest ways known for representing countable ordinals. The main concept of tree ordinals is to represent limit ordinals by fundamental sequences themselves. The notation of tree ordinals naturally fits to Dedekind's TRS [2], which gives a simple definition for basic arithmetics for ordinals — addition, multiplication, and exponentiation. With these arithmetic functions, we can represent ordinals up to ϵ_0 by finite expressions.

On the other hand, Hydra games were invented by Kirby and Paris [4] for the formulation of an undecidable statement. The termination of hydra games cannot be proved within Peano arithmetic, but under the assumption that the ordinal ϵ_0 is well-ordered. There, nondeterministic behaviour of hydrae takes place to represent infiniteness of transfinite ordinals.

In the present work we translate infinite sequences in tree ordinals and nondeterministic reductions in hydra games into each other. Thereby we will see a syntactical correspondence between tree ordinals and hydra games.

Overview

In Section 2 we present Dedekind's TRS on tree ordinals. In Section 3 we recall the definition of hydra games. Section 4 shows the correspondence between hydra games and tree ordinals. Section 5 gives a concluding remark.

2 Tree Ordinals

Definition 1. The set $\mathcal{T}$ of *tree ordinal terms* consists of the possibly infinite terms generated by the following symbols

D. Leivant and R. de Queiroz (Eds.): WoLLIC 2007, LNCS 4576, pp. 238–247, 2007.

arity	symbol(s)
0	$\mathsf{0}, \boldsymbol{\omega}$
1	$\mathsf{succ}, \mathsf{nats}$
2	$\mathsf{cons}, \mathsf{Add}, \mathsf{Mul}, \mathsf{Exp}$

with the rewrite rules

$$\begin{aligned}
\mathsf{Add}(x, \mathsf{0}) &\to x \\
\mathsf{Add}(x, \mathsf{succ}(y)) &\to \mathsf{succ}(\mathsf{Add}(x, y)) \\
\mathsf{Add}(x, \mathsf{cons}(y, z)) &\to \mathsf{cons}(\mathsf{Add}(x, y), \mathsf{Add}(x, z)) \\
\mathsf{Mul}(x, \mathsf{0}) &\to \mathsf{0} \\
\mathsf{Mul}(x, \mathsf{succ}(y)) &\to \mathsf{Add}(\mathsf{Mul}(x, y), x) \\
\mathsf{Mul}(x, \mathsf{cons}(y, z)) &\to \mathsf{cons}(\mathsf{Mul}(x, y), \mathsf{Mul}(x, z)) \\
\mathsf{Exp}(x, \mathsf{0}) &\to \mathsf{succ}(\mathsf{0}) \\
\mathsf{Exp}(x, \mathsf{succ}(y)) &\to \mathsf{Mul}(\mathsf{Exp}(x, y), x) \\
\mathsf{Exp}(x, \mathsf{cons}(y, z)) &\to \mathsf{cons}(\mathsf{Exp}(x, y), \mathsf{Exp}(x, z)) \\
\boldsymbol{\omega} &\to \mathsf{nats}(\mathsf{0}) \\
\mathsf{nats}(x) &\to x : \mathsf{nats}(\mathsf{succ}(x)).
\end{aligned}$$

Observe that the rules are orthogonal and that there exists only one collapsing rule. The system is thus CR^∞ and UN^∞ (See [7] or [5]). We write $nf(t)$ to denote the (possibly infinite) normal form of t, if it exists.

We write $t : s$ for $\mathsf{cons}(t, s)$, where ':' is right-associative; $t_0 : t_1 : t_2 : \ldots$ represents a infinite sequence of terms $t_0, t_1, t_2, \ldots$.

Definition 2. The set $\mathcal{TO} \subset \mathcal{T}$ of *tree ordinals* is given as the smallest set that satisfies the following conditions:

$$\begin{aligned}
\mathsf{0} &\in \mathcal{TO} \\
\mathsf{succ}(t) &\in \mathcal{TO} && (t \in \mathcal{TO}) \\
t_0 : t_1 : t_2 : \ldots &\in \mathcal{TO} && (t_0, t_1, t_2, \cdots \in \mathcal{TO})
\end{aligned}$$

with *semantics* $[\![-]\!] : \mathcal{TO} \to \mathbf{\Omega}$ inductively given by

$$\begin{aligned}
[\![\mathsf{0}]\!] &= 0 \\
[\![\mathsf{succ}(t)]\!] &= [\![t]\!] + 1 \\
[\![t_0 : t_1 : t_2 : \ldots]\!] &= \limsup_{i \in \mathbb{N}} [\![t_i]\!]
\end{aligned}$$

where $\mathbf{\Omega}$ denotes the set of countable ordinals.

Definition 3. We define a certain subset $\mathcal{E}_0$ of $\mathcal{T}$ given by the following BNF:

$$\mathcal{E}_0 ::= \mathsf{0} \mid \mathsf{Add}(\mathcal{E}_0, \mathcal{E}_0) \mid \mathsf{Exp}(\boldsymbol{\omega}, \mathcal{E}_0).$$

Let $\sim$ be the equivalence relation generated by

$$\begin{aligned} \mathsf{Add}(\mathsf{Add}(t,s),u) &\sim \mathsf{Add}(t,\mathsf{Add}(s,u)) \\ \mathsf{Add}(t,\mathsf{0}) &\sim t \\ \mathsf{Add}(\mathsf{0},t) &\sim t \end{aligned}$$

and let $[\mathcal{E}_0]$ be the set of equivalence classes of $\mathcal{E}_0$ modulo $\sim$.

We write $\sum_{i=1}^{n} t_i$ and $\boldsymbol{\omega}^t$ for $\mathsf{Add}(t_1,\ldots \mathsf{Add}(t_{n-1},t_n))$ and $\mathsf{Exp}(\boldsymbol{\omega},t)$, respectively. We regard $\sum_{i=1}^{0} t_i$ as $\mathsf{0}$.

Proposition 4 (Induction on $[\mathcal{E}_0]$). *For any $t \in \mathcal{E}_0$, there exist $n \in \mathbb{N}$ and $t_1,\ldots t_n \in \mathcal{E}_0$ such that $t \sim \sum_{i=1}^{n} \boldsymbol{\omega}^{t_i}$. Moreover, the equivalence classes in $[\mathcal{E}_0]$ are enumerated by this construction. Namely, the following induction principle holds: Let $P \subset [\mathcal{E}_0]$. If $\sum_{i=1}^{n} \boldsymbol{\omega}^{t_i} \in P$ for all $n \in \mathbb{N}$ and $t_1,\ldots t_n \in P$, then $P = [\mathcal{E}_0]$.* □

In the remainder of this section we state some propositions on the productivity of our system, without proofs. Since the system forms a subclass of the system presented in [6], please see ibid. for the proofs.

Lemma 5. *For any $t \in \mathcal{E}_0$, $nf(t)$ does exist.* □

Lemma 6. *If $t \sim s$, then $nf(t) = nf(s)$. The function $nf : [\mathcal{E}_0] \to \mathcal{T}$ is thus canonically defined.* □

Let $\mathcal{TO}_{\mathcal{E}_0}$ be the image of $[\mathcal{E}_0]$ via nf.

Proposition 7. *We have $\mathcal{TO}_{\mathcal{E}_0} \subset \mathcal{TO}$.* □

Theorem 8. *Let $t, s \in \mathcal{E}_0$. Then the following equations hold:*

$$\begin{aligned} [\![nf(\mathsf{0})]\!] &= 0 \\ [\![nf(\mathsf{Add}(t,s))]\!] &= [\![nf(t)]\!] + [\![nf(s)]\!] \\ [\![nf(\mathsf{Exp}(\boldsymbol{\omega},t))]\!] &= \omega^{[\![nf(t)]\!]}. \end{aligned}$$

And therefore the image of $\mathcal{TO}_{\mathcal{E}_0}$ via $[\![-]\!]$ is ϵ_0. □

Definition 9. For any α in ϵ_0, the *nested Cantor normal form* of α, written $ncn(\alpha)$, is inductively defined by

(i) if $\alpha = 0$, then $ncn(\alpha) = \mathsf{0}$.
(ii) if $\alpha = \beta + 1$, then $ncn(\alpha) = ncn(\beta) + \mathsf{Exp}(\boldsymbol{\omega},\mathsf{0})$.
(iii) if α is a limit ordinal, then there exist unique $\beta, \gamma < \alpha$ such that $\alpha = \omega^{\beta} + \gamma$. Let $ncn(\alpha) = \mathsf{Exp}(\boldsymbol{\omega}, ncn(\beta)) + ncn(\gamma)$.

For example, $ncn(\omega \times 2)$ is computed as follows:

$$\begin{aligned} ncn(\omega \times 2) &= ncn(\omega^1 + \omega^1) \\ &= \boldsymbol{\omega}^{ncn(1)} + ncn(\omega^1) \\ &= \boldsymbol{\omega}^{ncn(0+1)} + \boldsymbol{\omega}^{ncn(0+1)} \\ &= \boldsymbol{\omega}^{ncn(0)+\boldsymbol{\omega}^0} + \boldsymbol{\omega}^{ncn(0)+\boldsymbol{\omega}^0)} \\ &\sim \boldsymbol{\omega}^{\boldsymbol{\omega}^0} + \boldsymbol{\omega}^{\boldsymbol{\omega}^0}. \end{aligned}$$

Proposition 10. *We have* $[\![-]\!] \circ nf \circ ncn = \mathbf{id}_{\epsilon_0}$*. Thus, the system* $[\mathcal{E}_0]$ *can compute every ordinal less than* ϵ_0*.* □

3 Hydra Games

In this section, we present hydra games [4] with a minor change.

Definition 11. A *hydra* is an unlabeled finite tree with arbitrary finite branches. As a hydra, a leaf node is called a *head*. A head is *short* if the immediate parent of the head is the root; *long* if it is neither short nor the root (See Figure 1). The empty hydra is called *dead* and written ◯.

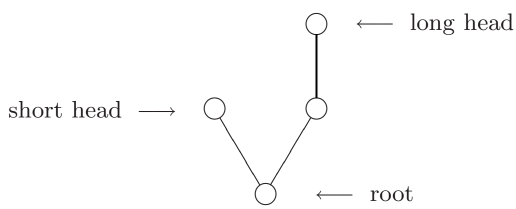

Fig. 1. A hydra

Thus, formally, the set $\mathcal{H}$ of hydrae is inductively defined as follows:

$$(h_1, \ldots, h_n) \in \mathcal{H} \quad \text{if} \quad h_1, \ldots, h_n \in \mathcal{H}$$

where (), in the case $n = 0$, is regarded as ◯.

As in the original paper, Herakles chops a head of a hydra. However, in this paper Herakles chops only the rightmost-head of the hydra. Hence, we are mainly interested in the rightmost structure of a hydra so that we take the notation

$$(\boldsymbol{h_1}(\boldsymbol{h_2}(\ldots(\boldsymbol{h_n}\bigcirc))))$$

to describe a given hydra, where $\boldsymbol{h_i}$ denotes the juxtaposition of n_i many hydras $h_{i1} \ldots h_{in_i}$. This hydra is dipicted by Figure 2. Observe that, for any hydra, there exists a unique representation of this form.

Now we define the *chopping-relation* on the hydrae.

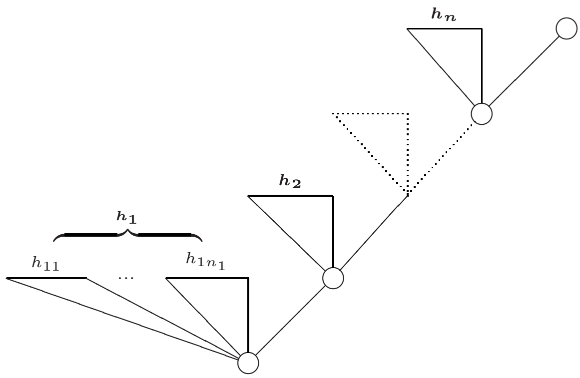

Fig. 2. The hydra $(\boldsymbol{h_1}(\boldsymbol{h_2}(\ldots(\boldsymbol{h_n}\bigcirc))))$

Definition 12. Let h be a hydra.

(i) (short-head chopping) As in Figure 3, if the rightmost-head of h is short, i.e. $h = (\boldsymbol{h_1}\bigcirc)$, then we write $h \overset{!}{\rightarrow} h'$ where $h' = (\boldsymbol{h_1})$.
(ii) (long-head chopping) As in Figure 4, if the rightmost-head of h is long, i.e. $h = (\ldots(\boldsymbol{h_{n-1}}(\boldsymbol{h_n}\bigcirc)))$, then we write $h \overset{m}{\rightarrow} h'_m$ where

$$h'_m = (\ldots(\boldsymbol{h_{n-1}}(\overbrace{\boldsymbol{h_n}\ldots\boldsymbol{h_n}}^{m \text{ times}}))).$$

for any $m \in \mathbb{N}$.

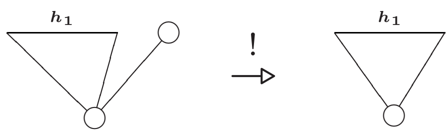

Fig. 3. Short-Head chopping

Remark 13. As proved in [4], any hydra will die after finite many times chopping; note that you can choose any m for any long-head chopping step.

Thus, given a hydra h, the *head-chopping reduction graph* of h is uniquely decided. Moreover, by the above remark, no graph has any infinite path in it.

Definition 14. The function $gr : \mathcal{H} \rightarrow \mathcal{TO}$ is defined via the head-chopping reduction graph of a hydra. Let h be a hydra. Then $gr(h)$ is inductively given by the following case analysis:

(i) if h has no chopping-reduction, i.e. $h = \bigcirc$, then $gr(h) = \mathsf{0}$.
(ii) if h has a short-head chopping $h \overset{!}{\rightarrow} h'$, then $gr(h) = \mathsf{succ}(gr(h'))$.

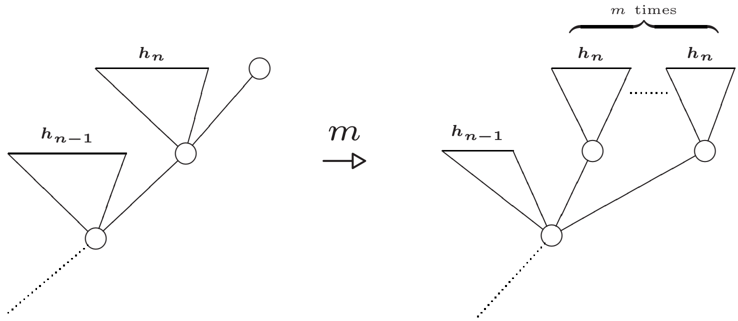

Fig. 4. Long-Head chopping

(iii) if h has a long-head chopping $h \overset{m}{\twoheadrightarrow} h_m$ indexed by m, then

$$gr(h) = gr(h_0) : gr(h_1) : gr(h_2) : \ldots .$$

Observe that one and only one of the above case is met for each h. Notice also that by the above remark gr is well-defined.

4 The Correspondence

We give a bijection between $\mathcal{H}$ and $[\mathcal{E}_0]$.

Definition 15. The translation $\overline{(-)} : \mathcal{H} \to [\mathcal{E}_0]$ is inductively defined by

$$\overline{(h_1 \ldots h_n)} = \sum_{i=1}^{n} \boldsymbol{\omega}^{\overline{h_i}}$$

and the translation $\widehat{(-)} : [\mathcal{E}_0] \to \mathcal{H}$ is inductively defined by

$$\widehat{\left(\sum_{i=1}^{n} \boldsymbol{\omega}^{t_i}\right)} = (\widehat{t_1} \ldots \widehat{t_n}).$$

Proposition 16. *We have* $\widehat{(-)} \circ \overline{(-)} = \mathbf{id}_{\mathcal{H}}$ *and* $\overline{(-)} \circ \widehat{(-)} = \mathbf{id}_{[\mathcal{E}_0]}$. □

For an example of the above proposition, we compute $\widehat{\overline{(\bigcirc((\bigcirc)\bigcirc))}}$. We have

$$\begin{aligned}
\overline{(\bigcirc((\bigcirc)\bigcirc))} &= \boldsymbol{\omega}^{\overline{\bigcirc}} + \boldsymbol{\omega}^{\overline{((\bigcirc)\bigcirc)}} \\
&= \boldsymbol{\omega}^{0} + \boldsymbol{\omega}^{\boldsymbol{\omega}^{\overline{(\bigcirc)}} + \boldsymbol{\omega}^{\overline{\bigcirc}}} \\
&= \boldsymbol{\omega}^{0} + \boldsymbol{\omega}^{\boldsymbol{\omega}^{\boldsymbol{\omega}^{\overline{\bigcirc}}} + \boldsymbol{\omega}^{0}} \\
&= \boldsymbol{\omega}^{0} + \boldsymbol{\omega}^{\boldsymbol{\omega}^{\boldsymbol{\omega}^{0}} + \boldsymbol{\omega}^{0}}
\end{aligned}$$

and

$$\begin{aligned}\left(\boldsymbol{\omega}^{0} + \widehat{\boldsymbol{\omega}^{\boldsymbol{\omega}^{\boldsymbol{\omega}^{0}}} + \boldsymbol{\omega}^{0}}\right) &= \left(\widehat{0}\ \left(\widehat{\boldsymbol{\omega}^{\boldsymbol{\omega}^{0}} + \boldsymbol{\omega}^{z}}\right)\right) \\ &= \left(\bigcirc(\widehat{\boldsymbol{\omega}^{z}}\ \widehat{0})\right) \\ &= (\bigcirc((\widehat{0})\bigcirc)) \\ &= (\bigcirc((\bigcirc)\bigcirc)).\end{aligned}$$

Before we state our main result, we show some lemmas.

Lemma 17. *The following propositions hold:*

(i) Let $t, s \in \mathcal{E}_0$. Then $nf(t+s) = nf(s)[0 := nf(t)]$.
(ii) Let $h_1, \dots h_n \in \mathcal{H}$ where $n > 0$. Then we have

$$gr(h_1 \dots h_n) = gr(h_n)[0 := gr(h_1 \dots h_{n-1})].$$

Proof. (i) By induction on $nf(s)$.
(ii) Notice that for any $h \in \mathcal{H}$ such that $h \overset{m}{\twoheadrightarrow} h'$ where $m \in \{!\} \cup \mathbb{N}$, we have

$$(h_1 \dots h_{n-1} h) \overset{m}{\twoheadrightarrow} (h_1 \dots h_{n-1} h').$$

Thus, the claim follows by induction on $gr(h_n)$.

Lemma 18. *Let $t \in [\mathcal{E}_0]$. Then the following holds:*

(i) If $[\![nf(t)]\!] = 0$, then $t = 0$.
(ii) If $[\![nf(t)]\!]$ is a successor ordinal, then t is of the form $\sum_{i=1}^{n} \boldsymbol{\omega}^{t_i}$ where $t_n = 0$.

Proof. (i) Suppose $t = \sum_{i=1}^{n} \boldsymbol{\omega}^{t_i}$. Then from Theorem 8 we have

$$[\![nf(t)]\!] = \omega^{[\![nf(t_1)]\!]} + \dots + \omega^{[\![nf(t_n)]\!]}.$$

Since $\omega^{\alpha} > 0$ for every ordinal α, we have $n = 0$.
(ii) Suppose $t = \sum_{i=1}^{n} \boldsymbol{\omega}^{t_i}$. Similarly, we have $n > 0$ and that $\omega^{[\![nf(t_n)]\!]}$ is a successor ordinal. Hence, $[\![nf(t_n)]\!] = 0$. From the above result, we have $t_n = 0$.

Lemma 19. *Let $t \in [\mathcal{E}_0]$. Then one of the following conditions holds:*

(i) $[\![nf(t)]\!] = 0$.
(ii) There exists $t' \in \mathcal{TO}_{\mathcal{E}_0}$ such that $nf(t) = \mathsf{succ}(t')$ and $[\![nf(t)]\!] = [\![nf(t')]\!] + 1$.
(iii) There exist $t_0, t_1, \dots \in \mathcal{TO}_{\mathcal{E}_0}$ such that $nf(t) = t_0 : t_1 : \dots$ and $[\![t_i]\!] < [\![nf(t)]\!]$ for all $i \in \mathbb{N}$.

Proof. Using Theorem 8, it easily follows by induction on t.

Now we set $P = \{t \in [\mathcal{E}_0] \mid nf(t) = gr(\widehat{t})\}$.

Lemma 20. *Let $t_1, \ldots, t_n \in \mathcal{E}_0$. If $t_i \in P$ for all applicable i, then $\sum_{i=1}^{n} t_i \in P$.*

Proof. By mathematical induction on n. It is trivial for the case $n = 0$. For insuction step, we assume $\sum_{i=1}^{n-1} t_i \in P$. We have

$$nf\left(\sum_{i=1}^{n} t_i\right) = nf(t_n)\left[\mathsf{0} := nf\left(\sum_{i=1}^{n-1}\right)\right]$$

and

$$gr\left(\widehat{\sum_{i=1}^{n} t_i}\right) = gr(\widehat{t_n})\left[\mathsf{0} := gr\left(\widehat{\sum_{i=1}^{n-1} t_i}\right)\right]$$

by Lemma 17. Thus, by induction hypothesis, we have

$$nf\left(\sum_{i=1}^{n}\right) = gr\left(\widehat{\sum_{i=1}^{n} t_i}\right),$$

implying $\sum_{i=1}^{n} \in P$.

Lemma 21. *If $t \in P$, then $\boldsymbol{\omega}^t \in P$.*

Proof. Transfinite induction on $[\![nf(t)]\!]$. We suppose that $[\![nf(s)]\!] < [\![nf(t)]\!]$ implies $s \in P$. Case analysis by Lemma 19.

(i) If $[\![nf(t)]\!] = 0$, then from Lemma 18 we have $t = \mathsf{0}$ and thus

$$nf(\boldsymbol{\omega}^t) = gr(\widehat{t}) = \mathsf{succ}(\mathsf{0}).$$

Hence, $\boldsymbol{\omega}^t \in P$.

(ii) If $nf(t) = \mathsf{succ}(s)$ where $[\![nf(t)]\!] = [\![s]\!] + 1$, then from Lemma 18 we have $t = \sum_{i=1}^{n-1} \boldsymbol{\omega}^{t_i} + \boldsymbol{\omega}^{\mathsf{0}}$ with $nf\left(\sum_{i=1}^{n-1}\right) = s$. Let $t' = \sum_{i=1}^{n-1}$. Then

$$\begin{aligned} nf(\boldsymbol{\omega}^t) &= nf(\boldsymbol{\omega}^{t'+\mathsf{succ}(\mathsf{0})}) \\ &= nf(\mathsf{Mul}(\boldsymbol{\omega}^{t'}, \boldsymbol{\omega})) \\ &= \mathsf{0} : nf(\boldsymbol{\omega}^{t'}) : nf(\mathsf{Add}(\boldsymbol{\omega}^{t'} + \boldsymbol{\omega}^{t'})) : \ldots \end{aligned}$$

and

$$\begin{aligned} gr(\widehat{\boldsymbol{\omega}^t}) &= gr(\widehat{t'}\ (\bigcirc)) \\ &= gr(\bigcirc) : gr(\widehat{t'}) : gr(\widehat{t'}\ \widehat{t'}) : \ldots. \end{aligned}$$

We have $nf(\sum_{i=1}^{m} t') = gr(\widehat{\sum_{i=1}^{m} t'}) = gr(\overbrace{\widehat{t'} \ldots \widehat{t'}}^{m \text{ times}})$ for all $m \in \mathbb{N}$, by induction hypothesis and Lemma 20. Therefore, we have $nf(\boldsymbol{\omega}^t) = gr(\widehat{\boldsymbol{\omega}^t})$, implying $\boldsymbol{\omega}^t \in P$x.

Theorem 22. *We have $nf \circ \overline{(-)} = gr$ and $gr \circ \widehat{(-)} = nf$.*

Proof. The latter equation follows from Proposition 4 and the above two lemmas. The other one then immediately follows, using Propositon 16.

5 Conclusion

We have presented the syntactical correspondence between hydra games and a certain subclass of tree ordinals. The contribution of the paper is dipicted by Figure 5, where the upper-left triangle commutes.

Since Buchholz [1] gives a variation of hydra games which is related to a much larger ordinal, it is expected that there exists an extension of this TRS which is related to Buchholz's hydra games.

In addition, a visualisation tool for the original hydra games due to Kirby and Paris is available at [3]; Figure 6 is a screenshot of the tool.

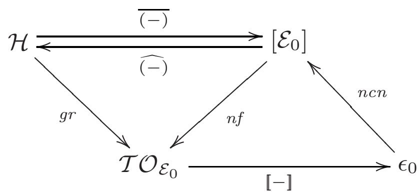

Fig. 5. The diagram which relates hydra games and tree ordinals

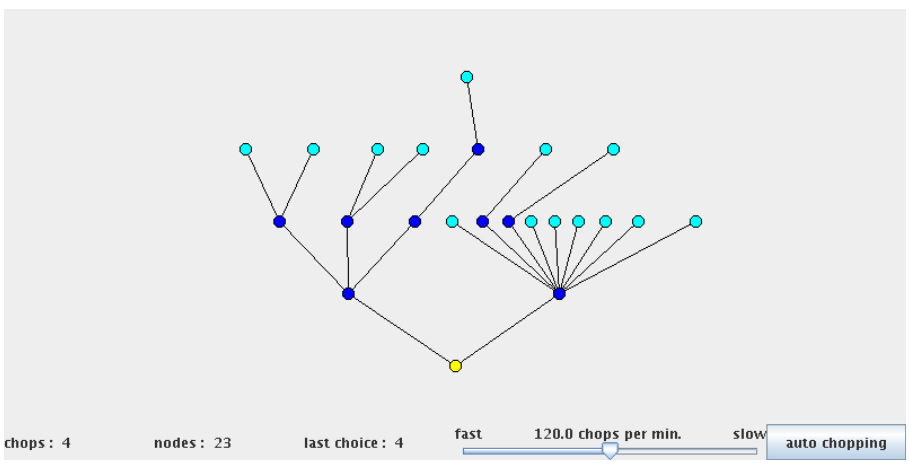

Fig. 6. Screenshot of hydra (JavaApplet)

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References

1. Buchholz, W.: An independence result for (Π^1_1-CA) + BI. Annals of Pure and Applied Logic 33(2), 131–155 (1987)
2. Dedekind, R.: Was sind und was sollen die Zahlen? Vieweg (1888)
3. Isihara, A.: hydra (JavaApplet). Available at `http://www.few.vu.nl/~ariya/app/hydra/hydra.html`
4. Kirby, L., Paris, J.: Accessible independence results for Peano Arithmetic. Bulletin of the London Mathematical Society 14, 285–293 (1982)
5. Klop, J.W., de Vrijer, R.: Infinitary normalization. In: Artemov, S., Barringer, H., d'Avila Garcez, A., Lamb, L., Woods, J. (eds.) We Will Show Them: Essays in Honour of Dov Gabbay, vol. 2, pp. 169–192. College Publications (2005)
6. Kwiatkowski, M.: Ordinal arithmetic through infinitary term rewriting. Master's thesis, Vrije Universiteit, Amsterdam, The Netherlands (2006)
7. Terese,: Term Rewriting Systems. Cambridge University Press, Cambridge (2003)