Further counterexamples to a conjecture of Beilinson

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by

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Abstract

We give stronger counterexamples to a conjecture of Beilinson.

Key Words: K-theory, flat proper model, Beilinson conjecture

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In [1, Conjecture 2.4.2.1] (see also [10, §3]) Beilinson posed the following conjecture.

Conjecture 1 Let $X/Q$ be a smooth, projective (but not necessarily geometrically irreducible) variety, and let $X/Z$ be a flat and proper model of $X/Q$. Then the image of the localization map

$$K'_*(X) \otimes \mathbb{Q} \to K'_*(X) \otimes \mathbb{Q} = K_*(X) \otimes \mathbb{Q}$$

is independent of $X$.

The goal of this conjecture was to have a canonical subspace of $K_*(X) \otimes \mathbb{Q}$ that plays an important role in further conjectures by Beilinson; see [10, §5]. In [3] the author gave a counterexample to Conjecture 1, but a suitable canonical subspace of $K_*(X) \otimes \mathbb{Q}$ was constructed in a different way in [11, §1] using alterations, as we shall recall below. (In fact, in loc. cit. a canonical subspace is defined for motives with coefficients in a field of characteristic zero. We refer the reader to the original source for the corresponding details.)

More precisely, let $X$ be as in Conjecture 1 and let $K_q(X) \otimes \mathbb{Q} = \bigoplus K_q^{(n)}(X)$ be the decomposition into Adams eigenspaces (see [6, Propositions 5 and 9]). Then the canonical subspace decomposes accordingly and it suffices to describe it for each $K_q^{(n)}(X)$. If $\mathcal{X}/\mathbb{Z}$ is a flat, projective model of $X/Q$ that is regular then the desired subspace $K_q^{(n)}(X/\mathbb{Z}) \subseteq K_q^{(n)}(X)$ is the image of the composition of the localization and projection maps,

$$K'_q(\mathcal{X}) \otimes \mathbb{Q} \to K'_q(X) \otimes \mathbb{Q} = K_q(X) \otimes \mathbb{Q} \to K_q^{(n)}(X),$$

this image being independent of $\mathcal{X}$ ([11, Theorem 1.1.6]; cf. [10, p. 13]). Such $\mathcal{X}$ is not known to exist in general, but by the theory of alterations (see [4] or [5])
there exists a regular $\mathcal{Y}$, projective and flat over $\mathbb{Z}$, for which its generic fibre $Y$ admits a surjective, generically finite morphism $\phi : Y \to X$ (cf. [11, p. 475]). Then $K_{q}^{(n)}(X/\mathbb{Z})$ equals $\phi_{*}(K_{q}^{(n)}(Y/\mathbb{Z})) \subseteq K_{q}^{(n)}(X)$, which is independent of $\mathcal{Y}$ and $\phi$ ([11, Theorem 1.1.6]). Here $\phi_{*}$ is the composition

$$K_{q}^{(n)}(Y) \sim Gr_{\mathcal{Y}}^{n}K_{q}(Y) \otimes \mathbb{Q} \to Gr_{\mathcal{Y}}^{n}K_{q}(X) \otimes \mathbb{Q} \leftarrow K_{q}^{(n)}(X).$$

with the isomorphisms coming from the Chern character (see [12, §1.5]) and the map in the centre from the (proper) pushforward

$$K_{q}(Y) \sim K_{q}(X) \to K_{q}(X) \leftarrow K_{q}(X)$$

(see [12, §1, (2.5)]), which induces a map $Gr_{\mathcal{Y}}^{n}K_{q}(Y) \otimes \mathbb{Q} \to Gr_{\mathcal{Y}}^{n}K_{q}(X) \otimes \mathbb{Q}$ since $X$ and $Y$ have the same dimension [12, §3, Theorem 1.1].

The counterexample to Conjecture 1 in [3] was for $K_{2}$ of certain elliptic curves $E = \mathbb{Q}$, and was related to the original discovery of the need for a canonical subspace in [2]. However, for an elliptic curve $E = \mathbb{Q}$ the rank of $K_{2}(E)/\text{torsion}$ is expected to be at most the number of primes of bad (or, more precisely, split multiplicative) reduction of $E$, plus 1. The goal of the present paper is to give an easier construction where, for fixed $X$, the image of $K_{1}'(\mathcal{X})$ in $K_{1}(X)$ is finitely generated but of arbitrarily large rank for suitably chosen flat and proper models $\mathcal{X}$. Clearly this shows that the image of $K_{1}'(\mathcal{X}) \otimes \mathbb{Q}$ in $K_{1}(X) \otimes \mathbb{Q}$ is not independent of $\mathcal{X}$, and can be arbitrarily big.

If $\mathcal{O}$ is the ring of algebraic integers in a fixed number field $F$ and $R = \mathbb{Z} + N\mathcal{O}$ for some positive integer $N$ then $R$ is a subring of $\mathcal{O}$ and $R \otimes \mathbb{Q} \cong F$. Clearly $R$ is a finite $\mathbb{Z}$-algebra, so it is Noetherian and the map $\text{Spec}(R) \to \text{Spec}(\mathbb{Z})$ is finite, hence proper. Also, $\text{Spec}(R)$ is flat over $\text{Spec}(\mathbb{Z})$ because $R$ is a free $\mathbb{Z}$-module. Therefore $\mathcal{X} = \text{Spec}(R)$ is a flat and proper model over $\mathbb{Z}$ of $X = \text{Spec}(F)$ over $\mathbb{Q}$, and we shall see that the rank of the image of $K_{1}'(\mathcal{X}) \to K_{1}(X)$, which depends on $N$, can be arbitrarily large if $F \neq \mathbb{Q}$.

The localization sequence for $\mathcal{O} \to F$ gives us the top row in the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K_{1}(\mathcal{O}) & \longrightarrow & K_{1}(F) & \longrightarrow & \coprod_{\mathcal{P}} K_{0}(\mathcal{O}/\mathcal{P}) & \longrightarrow & K_{0}(\mathcal{O}) & \longrightarrow & \cdots \\
0 & \longrightarrow & K_{1}'(R) & \longrightarrow & K_{1}(F) & \longrightarrow & \coprod_{\mathcal{P}} K_{0}(R/P) & \longrightarrow & K_{0}'(R) & \longrightarrow & \cdots .
\end{array}
$$

The coproduct in that row is over all non-zero prime ideals $\mathcal{P}$ of $\mathcal{O}$, and we also used that $K_{2}'(\mathcal{O}) = K_{2}(\mathcal{O})$ since $\mathcal{O}$ is a regular ring (see [9, §4, Corollary 2]), as well as that $\mathcal{O}^{*} \cong K_{1}(\mathcal{O}) \to K_{1}(F) \cong F^{*}$ is injective (see [7, page 159]).
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The localization sequence for $R \to F$ gives us the bottom row in this diagram, where the coproduct is over all non-zero prime ideals $P$ of $R$, but we have to justify the zero on the left. For this we note that $\text{Spec}(O) \to \text{Spec}(R)$ is also proper and preserves the codimension filtration. Therefore there is a pushforward that gives us a map from the localization sequence for $O$ to the one for $R$. This gives us the commutative diagram as above, but with the zero in the top row replaced with $\bigsqcup_P K_1(O/P)$ and the zero in the bottom row replaced with $\bigsqcup_P K_1(R/P)$.

However, the map $\bigsqcup_P K_1(O/P) \to \bigsqcup_P K_1(R/P)$ is surjective because above each $P$ in $R$ there is a $\mathcal{P}$ in $O$ and the map $K_1(O/\mathcal{P}) \to K_1(R/\mathcal{P})$, corresponding to the norm map $(O/\mathcal{P})^* \to (R/\mathcal{P})^*$, is surjective since the fields involved are finite. This, together with the injectivity of $K_1(O) \to K_1(F)$, implies that $K'_1(R) \to K_1(F)$ is injective as well.

If we let $S = O[\frac{1}{N}] = R[\frac{1}{N}]$, a regular ring, then the localization sequences and pushforward in this case yield the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K_1(O) & \longrightarrow & K_1(S) & \longrightarrow & \bigsqcup_{\mathcal{P}|N} K_0(O/\mathcal{P}) & \longrightarrow & K_0(O) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'_1(R) & \longrightarrow & K_1(S) & \longrightarrow & \bigsqcup_{\mathcal{P}|N} K_0(R/\mathcal{P}) & \longrightarrow & K'_0(R) & \longrightarrow & \cdots.
\end{array}
$$

Here $K_1(O) \to K_1(S)$ is injective because its composition with $K_1(S) \to K_1(F)$ is injective, and the injectivity of $K'_1(R) \to K_1(S)$ follows similarly. Since $K_1(S)$ is finitely generated, this implies that $K'_1(R)$ is also finitely generated.

From the last diagram we see that the cokernel of $K_1(S) \to \bigsqcup_{\mathcal{P}|N} K_0(O/\mathcal{P})$ injects into $K_0(O) \cong \mathbb{Z} \oplus \text{Cl}(F)$ where $\text{Cl}(F)$ is the class group of $F$ (see [7, Corollary 1.11]). Because $\text{Cl}(F)$ is the kernel of the composition of the localizations $K_0(O) \to K_0(S) \to K_0(F) \cong \mathbb{Z}$, this cokernel is contained in $\text{Cl}(F)$, hence is finite. Also, if $\mathcal{P}$ is a non-zero prime ideal of $O$ lying above a given non-zero prime ideal $P$ of $R$ then the map $K_0(O/\mathcal{P}) \to K_0(R/\mathcal{P})$ is surjective since it corresponds to viewing a finitely generated $O/\mathcal{P}$-vector space as a finitely generated $R/P$-vector space. Under the identifications $K_0(O/\mathcal{P}) \cong \mathbb{Z}$ and $K_0(R/\mathcal{P}) \cong \mathbb{Z}$ the image of $K_0(O/\mathcal{P})$ in $K_0(R/\mathcal{P})$ is given by $[O/\mathcal{P} : R/P] \cdot \mathbb{Z}$.

Let $I_1$ be the image of $K_1(S)$ in $\bigsqcup_{\mathcal{P}|N} K_0(O/\mathcal{P})$ and $I_2$ the image of $K_1(S)$ in $\bigsqcup_{\mathcal{P}|N} K_0(R/\mathcal{P})$, so that we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K_1(O) & \longrightarrow & K_1(S) & \longrightarrow & I_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'_1(R) & \longrightarrow & K_1(S) & \longrightarrow & I_2 & \longrightarrow & 0.
\end{array}
$$
Note that \( K_1(\mathcal{O}) \to K'_1(R) \) is injective, \( I_1 \to I_2 \) is surjective, and

\[
\frac{K'_1(R)}{K_1(\mathcal{O})} \cong \text{Ker}(I_1 \to I_2).
\]

Both \( I_1 \) and \( I_2 \) are free \( \mathbb{Z} \)-modules, and, by our arguments above, \( I_1 \) has the same rank as \( \bigsqcup_{P|N\mathcal{O}} K_0(\mathcal{O}/P) \) and \( I_2 \) has the same rank as \( \bigsqcup_{P|NR} K_0(R/P) \).

Let us now determine the rank of \( I_2 \) by determining the number of (non-zero) prime ideals of \( R \) that lie above \( p\mathbb{Z} \) for each prime factor of \( N \). Let \( P \) be a prime ideal of \( R \) lying above \( p\mathbb{Z} \) where \( p \) is a prime number dividing \( N \). If \( a \) is in \( \mathcal{O} \), then \((Na)^2 = N(Na^2) \) lies in \( pR \subseteq P \). So \( P \) contains \( NO \) and hence \( p\mathbb{Z} + NO \). Since this is a maximal ideal of \( R \) it must be equal to \( P \), and \( P \) is unique. Hence the rank of \( I_2 \) is equal to the number of distinct prime numbers dividing \( N \).

By applying a corollary of the Chebotarov density theorem (see [8, Corollary 6.5]) to the normal closure of \( F/\mathbb{Q} \) we see that, for any \( n \geq 1 \), we can take \( n \) distinct prime numbers \( p_1, \ldots, p_n \) such that each \( p_j\mathbb{Z} \) splits completely in \( \mathcal{O} \). We let \( N = p_1 \cdots p_n \) so that, with \( d = [F : \mathbb{Q}] \), above each \( p_j\mathbb{Z} \) there are \( d \) primes \( \mathcal{P} \) of \( \mathcal{O} \) but only one prime \( P \) of \( R = \mathbb{Z} + NO \). Then \( I_1 \) has rank \( nd \), \( I_2 \) has rank \( n \), and

\[
\frac{K'_n(R)}{K_n(\mathcal{O})} \cong \text{Ker}(I_1 \to I_2) \cong \mathbb{Z}^{n(d-1)}
\]

because \( I_1 \to I_2 \) is surjective. Since this holds for arbitrary \( n \geq 1 \) this means that the rank of \( K'_1(R) \) can be arbitrarily large as long as \( F \neq \mathbb{Q} \).

As a very explicit example let us take \( F = \mathbb{Q}(i) \) so that \( \mathcal{O} = \mathbb{Z}[i] \). Then each prime \( p \) congruent to 1 modulo 4 splits completely in \( \mathbb{Z}[i] \). So for \( N = p_1 \cdots p_n \) a product of \( n \geq 1 \) distinct such primes, and \( R = \mathbb{Z} + NO[i] = \mathbb{Z}[Ni] \), we find that \( K'_1(\mathbb{Z}[Ni])/K_1(\mathbb{Z}[i]) \cong \mathbb{Z}^n \), which implies that \( K'_1(\mathbb{Z}[Ni]) \cong \mathbb{Z}^n \times \mathbb{Z}/4\mathbb{Z} \).

**Remark 2** In the situation of a number field \( F \) with ring of algebraic integers \( \mathcal{O} \) and \( R = \mathbb{Z} + NO \), the method we employed does not work for the image of \( K_1(R) \) in \( K_1(F) \) rather than that of \( K'_1(R) \). In fact, for such \( R \) one cannot get any larger image than that of \( K_1(\mathcal{O}) \cong O^* \) because the localisation map \( K_1(R) \to K_1(F) \) factors through \( K_1(\mathcal{O}) \to K_1(F) \). And with \( k = |(\mathcal{O}/NO)^*| \), if \( u \) is in \( O^* \) then \( u^k \) lies in \( 1 + NO \subset R \) so that the image of \( K_1(R) \) in \( K_1(\mathcal{O}) \) is of finite index. This shows that \( K_1(R) \otimes \mathbb{Q} \) and \( K_1(\mathcal{O}) \otimes \mathbb{Q} \) always have the same image in \( K_1(F) \otimes \mathbb{Q} \).

It seems that, for general \( X/\mathbb{Q} \) smooth and projective, and a flat and proper model \( \mathcal{X}/\mathbb{Z} \) of \( X/\mathbb{Q} \), the question if the image of \( K_*(\mathcal{X}) \otimes \mathbb{Q} \to K_*(X) \otimes \mathbb{Q} \) is independent of \( \mathcal{X} \) is open. Given the above this question seems more natural than Conjecture 1.
Further counterexamples to a conjecture of Beilinson.

References


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