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# $S_4$ as a natural flavor symmetry for lepton mixing

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Group theoretical arguments seem to indicate the discrete symmetry $S_4$ as the minimal flavor symmetry compatible with tribimaximal neutrino mixing. We prove in a model-independent way that indeed $S_4$ can realize exact tribimaximal mixing through different symmetry breaking patterns. We present two models in which lepton tribimaximal mixing is realized in different ways and for each one we discuss the superpotential that leads to the correct breaking of the flavor symmetry.

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## I. INTRODUCTION

Harrison, Perkins, and Scott (HPS) [1] proposed the so-called tribimaximal mixing matrix

$$U_{\mathrm{TB}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \tag{1}$$

This matrix keeps in surprising agreement with experimental data [2]. Lots of theoretical models have been done to explain the mixing matrix of Eq. (1) by means of non-Abelian flavor symmetry, such as $S_3$ [3–13], $A_4$ [14–29], $T'$ [30–34], $S_4$ [35–39], and $\Delta$ (27) [40–43]. The non-Abelian discrete groups have irreducible representations of dimension bigger than one [44]. The most interesting case arises when the group contains a triplet as irreducible representation, allowing one to embed the observed three generations of fermions.

Let us consider a group $G$ and one of its subgroups $G'$. Then an irreducible representation $r$ of $G$ decomposes into the irreducible representations of $G'$ as $r = r_1 + r_2 + \cdots$. We define $U_{G'}^{r_i}$ the projector of $r$ into $r_i$ and we denote with $U_{G'}$ the collection of projectors that decomposes the representations of $G$ according to $G'$.

When a non-Abelian discrete group $G$ is broken to one of its subgroup $G'$ the projector $U_{G'}$ that decomposes the representations of $G$ according to $G'$ can be fixed and are completely model independent. This is the case, for example, of $A_4$ broken to $Z_3$: the triplet representation of $A_4$ is sent to the one-dimensional representations of $Z_3$, $1$, $1'$, $1''$. In fact, if we choose for $A_4$ the basis in which the generator $T$ of its $Z_3$ subgroup is given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2}$$

the $A_4$ triplet representation decomposes into the singlet representations of $Z_3$ through the matrix $U_\omega$ defined as

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \tag{3}$$

On the contrary, the one-dimensional representations of $A_4$ coincide with the corresponding ones of $Z_3$. A good candidate to give a tribimaximal (TBM) is a discrete group $G$ that has a triplet representation, at least two subgroups, $G'$ that decomposes according to $U_{G'}$, and $G''$ that decomposes according to $U_{G''}$. It is necessary to have at least two different subgroups of $G$ to obtain a lepton mixing matrix different from the identity: if $G$ were broken to the same subgroup $G'$ both in the charged lepton and in the neutrino sector the lepton mixing matrix would be given by $U_{\mathrm{lep}} = U_{G'}^\dagger U_{G'} = \mathcal{I}$. On the other hand when $G$ is broken in two different ways in the charged and neutral lepton sectors, such misalignment gives large angles in the lepton mixing matrix.

*A priori* $A_4$ seems to be a good candidate because it is the smallest discrete group that contains a triplet as irreducible representation. Furthermore it has two different subgroups, $Z_3$ and $Z_2$. However, while the transformation associated to $Z_3$ is fixed and model independent, the one associated to $Z_2$ is model dependent [45]. A similar analysis done with the discrete symmetry $T'$ leads to the same conclusion [30]. This means that $A_4$ and $T'$ yield exact or approximate TBM only assuming a fine-tuning in the parameters of the Yukawa Lagrangian or a particular model realization. We mention that by assuming further constraints, also models based on $S_3$ can yield an approximate TBM, although its largest irreducible representation is a doublet and not a triplet.

It has been recently claimed [46] that the minimal flavor symmetry naturally related to the tribimaximal mixing is $S_4$, the permutation symmetry of four objects. The author of [46] proved this through group theoretical arguments without entering into the details of a concrete model realization. In this paper we reconsidered $S_4$ and its subgroups. We have found that $S_4$ is able to reproduce TBM following two different symmetry breaking patterns. We have built two different models that realize TBM through the two patterns dictated by the group analysis considerations. Finally we discuss the possible superpotential that can break $S_4$ in the correct way.

## II. THE DISCRETE SYMMETRY GROUP $S_4$ AS THE ORIGIN OF TBM

### A. The group $S_4$

The discrete group $S_4$ is given by the permutations of four objects and it is composed of 24 elements. It can be defined by two generators $S$ and $T$ that satisfy

$$S^4 = T^3 = 1, \qquad ST^2S = T. \tag{4}$$

The 24 elements of $S_4$ belong to five classes

$$\begin{aligned}
&\mathcal{C}_1\colon I;\\
&\mathcal{C}_2\colon S^2, TS^2T^2, S^2TS^2T^2;\\
&\mathcal{C}_3\colon T, T^2, S^2T, S^2T^2, STST^2, STS, S^2TS^2, S^3TS;\\
&\mathcal{C}_4\colon ST^2, T^2S, TST, TSTS^2, STS^2, S^2TS;\\
&\mathcal{C}_5\colon S, TST^2, ST, TS, S^3, S^3T^2.
\end{aligned} \tag{5}$$

The elements of $\mathcal{C}_{2,4}$ define two different sets of $Z_2$ subgroups of $S_4$, the ones of the class $\mathcal{C}_3$, a set of $Z_3$ Abelian discrete symmetries, and those belonging to $\mathcal{C}_5$, a set of $Z_4$ Abelian discrete symmetries. The $S_4$ irreducible representations are two singlets, $1_1$, $1_2$, one doublet, 2, and two triplets, $3_1$ and $3_2$. We adopt the following basis:

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \tag{6}$$

for the doublet representation and

$$S_{+,-} = \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7}$$

for the triplet representations. Clearly the generators $(S_+, T)$ and $(S_-, T)$ define the two triplet representations $3_1$, $3_2$ respectively. All the product rules can be straightforwardly derived. We refer the reader to the product rules reported in [36].

### B. $S_4$ symmetry breaking patterns: generic case

We have seen in the introduction that given a discrete non-Abelian group $G$ a predictive lepton mixing matrix may be obtained if $G$ is broken to one of its subgroups, with the subgroup preserved in the charged lepton sector different from the subgroup preserved in the neutrino sector.

We disregard therefore the case when $S_4$ is completely broken in one of the two sectors. At the same time, if the left-handed leptons transform nontrivially under $S_4$, the case of $S_4$ unbroken in one sector is ruled out. Indeed in this case we could choose for lepton families the $S_4$ triplet representation or the singlet plus doublet representations. With these choices the requirement of unbroken $S_4$ would lead to a diagonal mass matrix with at least two degenerate states—namely the doublet.

As consequence if $S_4$ is broken to its subgroups $G'$ in the charged lepton sector, in the neutrino sector it has to be broken to another subgroup $G'' \neq G'$. The couple $(G', G'')$ identifies a possible symmetry breaking pattern. In this notation the lepton mixing matrix is given by

$$U_{\rm lep} = U_l^\dagger U_\nu = U_{G'}^\dagger U_{G''}, \tag{8}$$

with $U_{G'}$, $U_{G''}$ being the projectors that decompose the representations of $S_4$ into the representations of $G'$, $G''$ respectively.

$S_4$ contains a non-Abelian subgroup $S_3$, the permutation group of three objects composed by six elements. The elements of $S_4$ that belong to $S_3$ correspond to $C_1$, $T$, and $T^2$ of $C_3$ and $TSTS^2$, $STS^2$, $S^2TS$ of $C_4$. Furthermore $S_4$ contains the Abelian subgroups $Z_2$, $Z_3$, $Z_4$ corresponding to the elements of the classes $\mathcal{C}_{2,4}$, $\mathcal{C}_3$, and $\mathcal{C}_5$ respectively. The only representation that breaks $S_4$ to $S_3$ is the triplet $3_1$. The reason is the following. The six elements that define $S_3$ belonging to $S_4$ are $I$, $T$, $T^2$, $TSTS^2$, $STS^2$, $S^2TS$, where $S$ and $T$ are defined in Eq. (7). When a triplet $\phi_1 \sim 3_1$ develops vacuum expectation value (VEV) as (1, 1, 1), all the $S_3$ elements above are preserved. On the contrary, when a triplet $\phi_2 \sim 3_2$ develops VEV as (1, 1, 1), only the three elements that define $Z_3$ are preserved—$I$, $T$, $T^2$—while $TSTS^2$, $STS^2$, $S^2TS$ built according to Eq. (7) are broken.

The representations of $S_3$ are two singlet, $1_1$ and $1_2$, and a doublet, 2. In general if $S_4$ is broken to $S_3$ the representations of $S_4$ would transform under $S_3$ according to

$$\begin{gathered} 3_1 \to 1_1 + 2, \qquad 3_2 \to 1_2 + 2, \qquad 2 \to 2, \\ 1_1 \to 1_1, \qquad 1_2 \to 1_2. \end{gathered} \tag{9}$$

When $S_4$ is broken to $S_3$, a triplet of $S_4$, $F \sim (F_1, F_2, F_3) \sim 3_1$, will decompose under $S_3$ as a singlet plus a doublet $F(3_1) \to \psi_0(1_1) + \psi(2)$.

The eigenvector that identifies the singlet is given by

$$\frac{1}{\sqrt{3}}(1, 1, 1) \to \psi_0 = \frac{1}{\sqrt{3}}(F_1 + F_2 + F_3).$$

Since $S_3$ is not broken the doublet components are degenerate and the corresponding eigenvectors are identified up to an arbitrary rotation. This arbitrariness reflects the arbitrary freedom we have in fixing the doublet basis in an independent way with respect to the triplet basis. Indeed in the basis we have chosen the doublet reads as

$$\psi = \begin{pmatrix} (F_3 - F_2)/\sqrt{2} \\ (2F_1 - F_2 - F_3)/\sqrt{6} \end{pmatrix}, \tag{10}$$

and therefore we can rewrite

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_2 \end{pmatrix}, \qquad (11)$$

where $\theta$ is the arbitrary rotation in the doublet component space $\psi \sim (\psi_1, \psi_2)$ and where we have assumed that the second eigenvector corresponds to the singlet. The reason for the choice of this particular basis is very simple: in the limit in which $\theta$ goes to zero we recover the TBM matrix. However $\theta$ is undetermined as long as $S_3$ is unbroken. In the basis given by Eq. (11) the generic mass matrix for $\Psi \sim (\psi_1, \psi_0, \psi_2)$ is given by

$$M_\Psi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}, \qquad (12)$$

with $\psi_1$ and $\psi_2$ degenerate as expected. If we now assume that $S_3$ is broken only in the doublet subspace $M_\Psi$ becomes

$$M_\Psi = \begin{pmatrix} x_1 & 0 & x_3 \\ 0 & y & 0 \\ x_3 & 0 & x_2 \end{pmatrix}. \qquad (13)$$

It is clear that if $x_3 = 0$, then $\theta = 0$ and we are left with three nondegenerate eigenstates and the relation between the original $S_4$ triplet and the mass eigenstates is given by

$$F = U_{\mathrm{TB}} \cdot \Psi. \qquad (14)$$

$x_3 = 0$ is realized by requiring that $S_3$ in the doublet subspace is broken to $Z_2$ identified in the specific basis we have chosen by the $S$ generator.[1]

### C. $S_4$ symmetry breaking patterns: realizing exact TBM

We now assume that $F \sim L$ with $L$ being the left-handed lepton doublets and for the moment we leave undetermined the transformation properties under $S_4$ of the electroweak $SU(2)$ singlets.

The first case we consider is the symmetry breaking pattern $(S_3, G'')$, that means that we start by breaking $S_4$ into $S_3$ in the charged lepton sector while we still do not know which is its corresponding $S_4$ subgroup in the neutrino sector. Applying the general results obtained in Sec. II B we conclude that $M_l M_l^\dagger$ is diagonalized by $U_{S_3}$ given by

$$U_{S_3} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad (15)$$

that would lead to the wrong relation $m_e = m_\tau$. If we now break this degeneracy as sketched in Sec. II B, from Eq. (14) we would obtain $U_l = U_{\mathrm{TBM}}$ and $U_{\mathrm{lep}} = U_{\mathrm{TBM}}^T U_\nu$.

To cure this problem we could require that the neutrino mass matrix were diagonalized by $U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ in order to reproduce the TBM through $U_{\mathrm{lep}} = U_{\mathrm{TBM}}^T U_{\mathrm{TBM}} U_{\mathrm{TBM}} = U_{\mathrm{TBM}}$. However there is no $G''$ subgroup of $S_4$ that yields $U_{G''} = U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ and therefore exact TBM cannot be obtained according to Eq. (8). In fact the most general neutrino Majorana mass matrix diagonalized by $U_{\mathrm{TBM}} U_{\mathrm{TBM}}$ should take the following form:

$$\begin{aligned} &(U_{\mathrm{TBM}} U_{\mathrm{TBM}})^T m_\nu U_{\mathrm{TBM}} U_{\mathrm{TBM}} \\ &= m_\nu^{\mathrm{diag}} \longrightarrow m_\nu \\ &\sim \begin{pmatrix} a & b & c \\ b & a + \beta_1 b + \gamma_1 c & a + \beta_2 b + \gamma_2 c \\ c & a + \beta_2 b + \gamma_2 c & a + \beta_3 b + \gamma_3 c \end{pmatrix}, \end{aligned} \qquad (16)$$

where $\beta_{1,2,3}$ and $\gamma_{1,2,3}$ are fixed coefficients. By applying all the elements of $S_4$, excluding the identity, according to $G^T m_\nu G$ we discover that for all of them it holds $G^T m_\nu G \neq m_\nu$. This means there is no subgroup of $S_4$ that leads to Eq. (16) in the basis we have chosen.

On the other hand we could require one to break the degeneracy $m_e = m_\tau$ breaking $S_3$ into $Z_2$ not only in the doublet subspace but also in the singlet one. This would mean that $M_l M_l^\dagger$ after the $U_{S_3}$ rotation and the breaking $S_3 \to Z_2$ would read as

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & y & x_3 \\ 0 & x_3 & x_2 \end{pmatrix}. \qquad (17)$$

It is clear that in this case the final $U_l$ would depend on the mass parameters, and therefore the correct lepton mixing could be obtained only through a fit. We conclude that the symmetry breaking pattern with $S_4$ broken into $S_3$ in the charged lepton sector is ruled out.

We now analyze what happens considering the breaking pattern $(Z_3, G'')$. As in the previous case the subgroup $G''$, corresponding to the neutrino sector, is undetermined. As already said in $S_4$ the breaking into $Z_3$ is realized when a triplet $3_2$ develops a VEV in the direction (1, 1, 1) We expect that if we break $S_4$ into $Z_3$ in the charged lepton sector the charged lepton mixing matrix will send the $S_4$ triplet $(L_1, L_2, L_3)$ in the $Z_3$ eigenstates, 1, 1′, 1″. Indeed the mixing matrix responsible for this rotation is the $U_\omega$

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[1] From the point of view of model realization the assumption that $S_3$ is broken only in the doublet component space is not different from assuming that $S_4$ is broken to different subgroups in the charged lepton sector and in the neutrino one. Indeed we will see in Sec. III B how singlet and doublet sectors can be easily separated.

defined in Eq. (3). Given $U_\omega$ the correct TBM can be reproduced if the $U_{G''}$ of Eq. (8) is given by

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix}, \tag{18}$$

or in other words if the neutrino mass matrix $m^\nu$ is diagonalized by $U_\nu$ and it has the following form:

$$m^\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix}. \tag{19}$$

The matrix form of Eq. (19) is recovered by requiring the invariance of $m^\nu$ under the $G'' = Z_2 \times Z_2$ subgroup of $S_4$ associated to the element $TST$ of the class $\mathcal{C}_4$ and to the element $S^2$ of the class $C_2$ respectively. This breaking pattern corresponds to the usual one used in models based on $A_4$—where the breaking is given by $A_4 \to Z_2$. However we stress that in the context of $S_4$ we have obtained TBM only according to group theory considerations.

If we consider now the case $(Z_2, G'')$ we discover that $S_4$ behaves exactly as $A_4$ and exact TBM cannot be recovered. For a detailed analysis we refer the reader to the appendix of [45].

In the case $(Z_4, G'')$ we found that the charged lepton mass matrix $M_l M_l^\dagger$ has a maximal angle. Taking, for example, the $Z_4$ associated to the $S$ generators of $S_4$ we have that

$$S^T(M_l M_l^\dagger)S = (M_l M_l^\dagger) \to M_l M_l^\dagger = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & ix_3 \\ 0 & -ix_3 & x_2 \end{pmatrix}, \tag{20}$$

that gives rise to a maximal $\theta_{23}^l$ and three distinct eigenvalues $(x_1,\ x_2 - x_3,\ x_2 + x_3)$. In this case exact TBM would be recovered if $U_\nu$ would be given by a rotation in the plane $\theta_{12}$ characterized by $\tan\theta_{12}^2 = 1/2$ or in other words if $m_\nu$ would present the following form:

$$m_\nu = \begin{pmatrix} a & b & 0 \\ b & a + b/\sqrt{2} & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{21}$$

However it is easy to check that in the basis we are considering there is no $G'$ subgroup of $S_4$ that is left invariant by a $m_\nu$ of the form given in Eq. (21). Therefore we conclude that even the breaking of $S_4$ into $Z_4$ in the charged lepton sector does not lead to exact TBM.

So far we have considered all the possible cases in which the subgroup fixed in the charged lepton sector gives rise to a nondiagonal structure to the charged lepton mass matrix $M_l$. We could ask if there is any way to realize a diagonal $M_l$ with three different mass eigenvalues. Indeed this is easily realized breaking $S_4$ to $Z_2 \times Z_2$ corresponding to the elements $S^2$ and $T^2S^2T$ of the class $\mathcal{C}_2$. It is obvious that if the charged lepton mass matrix is diagonal all the mixing structure arises by the neutrino sector. Therefore the last symmetry breaking pattern we are going to consider is $(Z_2 \times Z_2, S_3)$ that means that we break $S_4$ into $S_3$ in the neutrino sector and then $S_3$ into $Z_2$ to have three nondegenerate mass eigenstates, as we have seen in the generic discussion in sec. II B that leaded to Eq. (14).

Contrary to what happened in the charged lepton sector, the breaking pattern $S_4 \to S_3 \to Z_2$ in the neutrino sector leads to exact TBM. Indeed from Eqs. (11)–(14) we have that $U_{\text{lep}} = U_{\text{TBM}}$ being the charged lepton mass matrix diagonal.

In conclusion of this first model-independent part, we have seen that on the basis of theoretical considerations based on the subgroups of $S_4$, the flavor symmetry $S_4$ has two symmetry breaking patterns giving exact TBM in the lepton sector. In the next section we will present a model realization for each breaking pattern. In the last section we build the corresponding superpotential responsible for the correct $S_4$ symmetry breaking patterns.

## III. MODEL REALIZATION

In the standard model (SM), the most general way to introduce Majorana neutrino mass terms is by means of five-dimension Weinberg operators. These operators could arise from type-I as well as type-II or type-III seesaw mechanism. The first mechanism is based on the exchange of a heavy fermion $SU(2)$ singlet. In the second mechanism the neutrino Majorana mass term arises from the exchange of $SU(2)$ Higgs triplet while in the third mechanism the heavy particle integrated out is an isotriplet fermion. In principle using an effective operator approach we should consider all the mechanisms together. In fact ultraviolet realizations, like minimal $SO(10)$ grand unified theory (GUT) models, give rise for examples both to type-I and type-II contributions. Other nonminimal GUT scenarios could give rise to the type-III contribution as well. On the other hand we lack experimental motivations to extend the SM both with right-handed neutrinos, $SU(2)$ scalar triplets, and $SU(2)$ fermion triplets and therefore models with just one seesaw type contribution are allowed. Below we will study two models with, respectively, type-II and type-I plus type-II seesaw. As we will see in Sec. III B in the second model we present, the only type-I contribution arising by a $S_4$ doublet of right-handed neutrinos cannot fit the correct $\Delta m^2_{\text{sol}}$. Therefore a further contribution is needed to the neutrino masses and in the model studied this is obtained from type-II seesaw. From a point of view of model realization, type-III seesaw is very similar to the type-I, so our model can be equivalently described making use of type-III plus type-II seesaw. For simplicity we have chosen the case with type-I plus type-II seesaw without entering into the details of the type-III version. It is worth emphasizing that type-I and type-III realizations present different phenomenological implications, whose study

TABLE I. Matter and scalar content of model I. The lepton mixing matrix is TBM.

| | $\hat{L}$ | $\hat{E}^c$ | $\hat{H}^d$ | $\hat{\Phi}$ | $\hat{\sigma}$ | $\hat{\phi}_1$ | $\hat{\phi}_2$ | $\hat{\Delta}$ |
|---|---|---|---|---|---|---|---|---|
| $SU(2)$ | 2 | 1 | 2 | 3 | 1 | 1 | 1 | 1 |
| $S_4$ | $3_1$ | $3_1$ | 1 | 1 | 1 | $3_1$ | $3_2$ | $3_1$ |
| $Z_5$ | 1 | $\omega_5^4$ | 1 | 1 | $\omega_5$ | $\omega_5$ | $\omega_5$ | 1 |

goes beyond the scope of this paper and could be studied elsewhere.

### A. Model I: $S_4 \rightarrow Z_3$ & $S_4 \rightarrow Z_2$

The first model we consider reproduces TBM through the breaking of $S_4$ into $Z_3$ and $Z_2$ in the charged lepton and neutrino sector, respectively. We assume our model to be supersymmetric. Matter and scalar supermultiplets are reported in Table I. The scalar supermultiplets charged under $S_4$, that in the following we will identify as flavons, are electroweak $SU(2) \times U(1)$ singlets. Therefore the Yukawa superpotential $\mathcal{W}_Y$ of Eq. (22) includes effective operators of dimension 5. $\Lambda$ is the cutoff of the model and an extra $Z_5$ symmetry has been introduced to separate the charged lepton sector from the neutrino one.

In Table I we have omitted the supermultiplets $\hat{H}^u$ and $\hat{\bar{\Phi}}$, doublet and triplet of $SU(2)$, respectively, necessary to give mass to the up-quarks and to cancel anomalies in a realistic model.

The full leading order $S_4 \times Z_5$ Yukawa superpotential $\mathcal{W}_Y$ is given by

$$\begin{aligned}\mathcal{W}_Y = {} & \frac{1}{\Lambda} y_0 (\hat{L}\hat{E}^c)_1 \hat{\sigma} \hat{H}^d + \frac{1}{\Lambda} y_s (\hat{L}\hat{E}^c)_{3_1} \hat{\phi}_1 \hat{H}^d \\ & + \frac{1}{\Lambda} y_a (\hat{L}\hat{E}^c)_{3_2} \hat{\phi}_2 \hat{H}^d + y_1^\nu (\hat{L}\,\hat{L})_1 \hat{\Phi} \\ & + \frac{1}{\Lambda} y_2^\nu (\hat{L}\,\hat{L})_{3_1} \hat{\Delta}\,\hat{\Phi}. \end{aligned} \tag{22}$$

When the $S_4$ triplet and doublet flavons align as

$$\langle \phi_1 \rangle \sim \langle \phi_2 \rangle \sim (1, 1, 1) \qquad \langle \Delta \rangle \sim (1, 0, 0), \tag{23}$$

the charged lepton and neutrino mass matrices present the usual forms

$$M_l = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_2 & h_0 & h_1 \\ h_1 & h_2 & h_0 \end{pmatrix} \qquad m^\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \tag{24}$$

that satisfy

$$U_\omega M_l U_\omega^\dagger = M_l^{\text{diag}}, \qquad U_\nu^T m_\nu U_\nu = m_\nu^{\text{diag}}, \tag{25}$$

with $U_\omega$ and $U_\nu$ given in Eq. (3) and (18) respectively. TBM is obtained as usual by $U_{\text{TB}} = U_\omega U_\nu$. The mass eigenvalues for the charged lepton are given by

$$\begin{gathered} m_e = h_0 + h_1 + h_2, \qquad m_\mu = h_0 + h_1\omega^2 + h_2\omega, \\ m_\tau = h_0 + h_1\omega + h_2\omega^2, \end{gathered} \tag{26}$$

and for the neutrino by $(a + b,\ a,\ b - a)$). By assuming that the flavon VEVs are of order $\sim \lambda^2 \Lambda$ with $\lambda$ the Cabibbo angle, the deviations from TBM induced by the next-to-leading order corrections to the Yukawa superpotential slightly modify lepton mixing keeping it still in agreement with neutrino data. Notice that the VEV alignments

$$\langle \phi_1 \rangle \sim \langle \phi_2 \rangle \sim (1, 1, 1) \tag{27}$$

preserve the $Z_3$ subgroup of $S_4$ associated to the element $T$ because $\phi_2 \sim 3_2$—as we have already said $\phi_1 \sim 3_1$ alone with VEV alignment (1, 1, 1) preserves the $S_3$ subgroup of $S_4$, while the $3_2$ triplet representation does not. On the contrary the VEV alignments

$$\langle \varphi \rangle \sim (0, 1) \qquad \langle \Delta \rangle \sim (1, 0, 0), \tag{28}$$

preserve the $Z_2 \times Z_2$, where the first $Z_2$ is associated to the element $TST$ while the second $Z_2$ to the element $S^2$ that in the doublet and triplet representation reads, respectively, as

$$TST = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad TST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{29}$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{30}$$

### B. Model II: $S_4 \rightarrow Z_2 \times Z_2$ and $S_4 \rightarrow S_3$

The second model we describe realizes TBM through the sequential breaking of $S_4$ into $S_3$ and then into $Z_2$ in the neutrino sector and the breaking of $S_4$ into two different $Z_2 \times Z_2$ in the charged lepton sector. The step through $S_3$ is crucial: if we broke $S_4$ directly into $Z_2$ in the neutrino sector we would find a generic neutrino mass matrix $\mu - \tau$ invariant not diagonalized by TBM. On the contrary, in the model that we present, the step through $S_3$ leads to a neutrino mass matrix $m^\nu$ which is $\mu - \tau$ invariant and satisfies the relation $m_{11}^\nu = m_{22}^\nu + m_{23}^\nu - m_{13}^\nu$ that ensures TBM diagonalization. We will see that the key ingredient in building the correct $m^\nu$ is the introduction of the right-handed neutrinos transforming as a doublet of $S_4$. The reason is very simple. Recovering Eqs. (12) and (13) in Sec. II Bwe have that the neutrino mass matrix obtained when $S_4$ is broken to $S_3$ has to be given by

$$m_{S_3}^\nu \sim \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}, \tag{31}$$

and then when $S_3$ is broken to $Z_2$ only in the doublet subspace $m_{S_3}^\nu$ becomes

$$m^{\nu}_{\rm diag} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x+z \end{pmatrix}. \tag{32}$$

If we now write $m^{\nu} \sim U_{\rm TBM} m^{\nu}_{\rm diag} U^{T}_{\rm TBM}$ we see that the $z$ contribution has to have the following structure:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & z/2 & -z/2 \\ 0 & -z/2 & z/2 \end{pmatrix} \sim z/2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot (0, 1, -1). \tag{33}$$

In the $S_4$ basis we are working, the vector $(0, 1, -1)^T$ can be obtained through the coupling $(3_1 \cdot 2_H)_{3_1}$ with $\langle 2_H \rangle \sim (1, 0)$. This means that the contribution of Eq. (33) may be obtained by an effective operator

$$\frac{1}{\Lambda}(L\phi)_{3_1}(L\phi)_{3_1} \tag{34}$$

with $L$, $\phi$ transforming under $S_4$ as $L \sim 3_1$ and $\phi \sim 2$. This effective operator is obtained by integrating out a $S_4$ doublet of right-handed neutrinos. Therefore their introduction in our model realization is crucial.

As in the case of the model presented in Sec. III A we assume our model to be supersymmetric and the flavon supermultiplets electroweak singlets. Matter and scalar supermultiplets are reported in Table II. As done in Sec. III A we have omitted the supermultiplet $\hat{\tilde{\Phi}}$, triplet of $SU(2)$, necessary to cancel anomalies. Two extra discrete Abelian symmetries, $Z_3$ and $Z_5$, have been introduced in order to avoid interferences between the sectors.

The full leading order $S_4 \times Z_3 \times Z_5$ invariant Yukawa superpotential is given by

$$\begin{aligned} \mathcal{W}_Y = {}& \frac{1}{\Lambda} y_s (\hat{L}\hat{l}^c)_1 \hat{\sigma} \hat{H}^d + \frac{1}{\Lambda} y_d (\hat{L}\hat{l}^c)_2 \hat{\phi} \hat{H}^d + y_1 (\hat{L}\,\hat{L})_1 \hat{\Phi} \\ & + \frac{1}{\Lambda} y_2 (\hat{L}\,\hat{\Delta})_2 \hat{N}^c \hat{H}^u + M_d \hat{N}^c \hat{N}^c + \tilde{y}_N \hat{\varphi} \hat{N}^c \hat{N}^c, \end{aligned} \tag{35}$$

TABLE II. Matter and scalar content of model II. The lepton mixing matrix is TBM.

| | $\hat{L}$ | $\hat{l}^c$ | $\hat{N}^c$ | $\hat{H}^u$ | $\hat{H}^d$ | $\hat{\Phi}$ | $\hat{\Delta}$ | $\hat{\sigma}$ | $\hat{\phi}$ | $\hat{\varphi}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $SU(2)$ | 2 | 1 | 1 | 2 | 2 | 3 | 1 | 1 | 1 | 1 |
| $S_4$ | $3_1$ | $3_1$ | 2 | 1 | 1 | 1 | $3_1$ | 1 | 2 | 2 |
| $Z_3$ | $\omega^2$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega$ | 1 | 1 | 1 |
| $Z_5$ | 1 | $\omega_5^3$ | 1 | 1 | 1 | 1 | 1 | $\omega_5^2$ | $\omega_5^2$ | 1 |

where as usual $\Lambda$ is the cutoff of the model and all the Yukawa terms are of order 4 with the exception of the ones involving right-handed neutrinos. We assume that the flavons $\Delta$ and $\varphi$, triplet and doublet under $S_4$ respectively, align as

$$\langle \Delta \rangle \sim (1, 1, 1) \qquad \langle \varphi \rangle \sim (0, 1). \tag{36}$$

The VEV $\langle \Delta \rangle$ preserves $S_3$ as has been already discussed in Sec. II B. The VEV $\langle \varphi \rangle$ preserves the $S$ generators of $S_3$ that coincides with the $S$ generator of $S_4$ of the doublet representation—Eq. (6).

The doublet $\phi$ does not align and develops VEV as $\langle \phi \rangle \sim (v_1, v_2)$—this means that $S_4$ is broken to $Z_2 \times Z_2$ corresponding to the elements $S^2$ and $TS^2T^2$ of $\mathcal{C}_2$ that in the $3_1$ triplet representation read as

$$S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad TS^2T^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{37}$$

For the charged lepton sector we have

$$M_l = \frac{1}{\Lambda} v^d \begin{pmatrix} y'_s v_\sigma - 2y''_d v^{\phi}_2 & 0 & 0 \\ 0 & y'_s v_\sigma + y'_d v^{\phi}_1 + y''_d v^{\phi}_2 & 0 \\ 0 & 0 & y'_s v_\sigma - y'_d v^{\phi}_1 + y''_d v^{\phi}_2 \end{pmatrix}, \tag{38}$$

with $v_\sigma = \langle \sigma \rangle$, $v^{\phi}_{1,2} = \langle \phi_{1,2} \rangle$, $v^d = \langle H^d_0 \rangle$, and the product factors absorbed in $y'_s$ and $y'_d$, $y''_d$. The neutrino mass matrix gets contributions both from type-I and type-II seesaw

$$m^{\nu} = m_{LL} - m_D \cdot \frac{1}{M_N} \cdot m^{T}_{D}, \tag{39}$$

where $m_{LL} = y_1 v_\Phi \cdot I$ with $v_\Phi = \langle \Phi \rangle$ and

$$\begin{aligned} m_D &= y_2 \frac{v^{\Delta}}{\Lambda} v^u \begin{pmatrix} 0 & -2\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}, \\ M_N &= \begin{pmatrix} M_d + V_\varphi & 0 \\ 0 & M_d - V_\varphi \end{pmatrix}, \end{aligned} \tag{40}$$

with $v^u = \langle H^u_0 \rangle$, $v^{\Delta}_{1,2,3} = v^{\Delta}$, and $V_\varphi = \tilde{y}_N \langle \varphi_2 \rangle / \sqrt{2}$. After the usual seesaw mechanism the Majorana neutrino mass matrix is given by

$$m^\nu = \begin{pmatrix} a + \frac{2}{3}b & -\frac{1}{3}b & -\frac{1}{3}b \\ -\frac{1}{3}b & a + \frac{1}{6}b + \frac{1}{2}c & \frac{1}{6}b - \frac{1}{2}c \\ -\frac{1}{6}b & \frac{1}{6}b - \frac{1}{2}c & a + \frac{1}{6}b + \frac{1}{2}c \end{pmatrix}, \tag{41}$$

with

$$\begin{aligned} a &= y_1 v_\Phi, \qquad b = -y_2^2\left(\frac{v^\Delta}{\Lambda}\right)^2 \frac{(v^u)^2}{M_d - V_\varphi}, \\ c &= -y_2^2\left(\frac{v^\Delta}{\Lambda}\right)^2 \frac{(v^u)^2}{M_d + V_\varphi}. \end{aligned} \tag{42}$$

The neutrino mass matrix $m^\nu$ is diagonalized by TBM and its eigenvalues are $(a + b, a, a + c)$ that can accommodate experimental neutrino mass splitting data being expressed in terms of three independent combinations of the parameters of the model. As in the model discussed in Sec. III A by assuming the flavon VEVs of order $\sim\lambda^2\Lambda$ next to leading order corrections to the Yukawa superpotential produce small deviations from TBM that are still compatible with neutrino data.

## IV. REALIZING THE CORRECT VACUUM CONFIGURATIONS IN $S_4$

In the context of a flavor model based on non-Abelian discrete symmetry the lepton TBM is obtained thanks to specific alignments of the flavons. The so-called alignment problem in $A_4$ and $T'$ has been extensively discussed in [18,21,25]. Different strategies have been used: the introduction of a soft breaking term of the flavor symmetry [25], the use of a continuous $U(1)_R$ symmetry [21] preserved by the scalar potential, and the promotion of the model to a fifth dimension [18]. In the context of $S_4$ in [39] the flavon superpotential was softly broken to guarantee the desired vacuum configuration.

In $S_4$ as well as in $A_4$ and $T'$ it is impossible to build a flavon superpotential that guarantees the alignments needed. In the next sections we will show that the extra discrete Abelian symmetries introduced in Sec. III to separate the two lepton sectors are sufficient to give the correct vacuum configurations at leading order. In general next-to-leading (NLO) order contributions will shift the vacuum alignments of an amount of order $v_i/\Lambda$ with $v_i$ the generic flavon VEV. It has been shown that in order not to destroy lepton mixing, $v_i/\Lambda \sim \lambda^2$ with $\lambda$ being the Cabibbo angle [18]. However a complete analysis of the NLO corrections is above the purposes of this work. Our motivation was to show two different $S_4$ based model realizations that at leading order (LO) provide exact TBM and to furnish two examples of the scalar potentials that could give the correct LO vacuum alignments.

TABLE III. Scalar content of model I including the flavons that contribute to the mass matrix structures and the ones the drive the correct vacuum alignments, the driving fields.

| | $\hat\sigma$ | $\hat\phi_1$ | $\hat\phi_2$ | $\hat\Delta$ | $\hat\varphi$ | $\hat\xi$ | $\hat\eta$ |
|---|---|---|---|---|---|---|---|
| $SU(2)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $S_4$ | 1 | $3_1$ | $3_2$ | $3_1$ | 2 | 2 | 2 |
| $Z_5$ | $\omega_5$ | $\omega_5$ | $\omega_5$ | 1 | 1 | $\omega_5^3$ | $\omega_5^2$ |

### A. Model I: Minimization of the potential

The complete flavor scalar content of the model is given in Table III. Thus the flavon potential is obtained by the following part of the full $S_4 \times Z_5$ superpotential

$$\begin{aligned} \mathcal{W}_Y = {} & M_{\xi\eta}\hat\xi\,\hat\eta + \lambda_{\xi\eta}\hat\xi\,\hat\eta\,\hat\varphi + \lambda_{\sigma\eta}\hat\sigma\,\hat\eta\,\hat\eta + \lambda_{\xi\phi 1}\hat\xi\hat\phi_1\hat\phi_1 \\ & + \lambda_{\xi\phi 2}\hat\xi\hat\phi_2\hat\phi_2 + \lambda_{\xi\phi 12}\hat\xi\hat\phi_1\hat\phi_2 + M_\Delta\hat\Delta\,\hat\Delta \\ & + M_\varphi\hat\varphi\,\hat\varphi + \lambda_{\varphi\Delta}\hat\Delta\,\hat\Delta\,\hat\varphi + \lambda_\varphi\hat\varphi\,\hat\varphi\,\hat\varphi + \lambda_\Delta\hat\Delta\,\hat\Delta\,\hat\Delta\,. \end{aligned} \tag{43}$$

We assume that the flavor symmetry is broken in the supersymmetry (SUSY) limit and therefore the vacuum configuration is obtained solving the system $\partial\mathcal{W}_Y/\partial f_i = 0$, where $f_i$ are the $f$ components of the supermultiplets entering in Eq. (43) and $i$ runs on all the supermultiplets. By assuming the general vacuum configuration

$$\begin{aligned} &\langle\Delta\rangle = (v_1^\Delta, v_2^\Delta, v_3^\Delta), \qquad \langle\varphi\rangle = (v_1^\varphi, v_2^\varphi), \\ &\langle\phi_1\rangle = (v_1^\phi, v_2^\phi, v_3^\phi), \qquad \langle\phi_2\rangle = (u_1^\phi, u_2^\phi, u_3^\phi), \\ &\langle\xi\rangle = (u_1^\xi, u_2^\xi), \qquad \langle\eta\rangle = (z^\eta, z^\eta) \qquad \langle\sigma\rangle = v_\sigma, \end{aligned} \tag{44}$$

the set of equations is given by

$$\partial W/\partial f_1^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_1^\Delta - \frac{2}{\sqrt{3}}\lambda_{\Delta\varphi}v_1^\Delta v_2^\varphi + 2\lambda_\Delta v_2^\Delta v_3^\Delta = 0 \tag{45a}$$

$$\partial W/\partial f_2^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_2^\Delta + \frac{1}{\sqrt{3}}\lambda_{\Delta\varphi}v_2^\Delta(v_2^\varphi + \sqrt{3}v_1^\varphi) + 2\lambda_\Delta v_1^\Delta v_3^\Delta = 0 \tag{45b}$$

$$\partial W/\partial f_3^\Delta = \frac{2}{\sqrt{3}}M_\Delta v_3^\Delta + \frac{1}{\sqrt{3}}\lambda_{\Delta\varphi}v_3^\Delta(v_2^\varphi - \sqrt{3}v_1^\varphi) + 2\lambda_\Delta v_1^\Delta v_2^\Delta = 0 \tag{45c}$$

$$\partial W/\partial f_1^{\varphi} = \sqrt{2}M_{\varphi}v_1^{\varphi} + \frac{\lambda_{\xi\eta}}{2}(u_2^{\xi}z_1^{\eta} + u_1^{\xi}z_2^{\eta}) + \frac{\lambda_{\Delta}}{2}[(v_2^{\Delta})^2 - (v_3^{\Delta})^2] = 0 \tag{45d}$$

$$\partial W\partial f_2^{\varphi} = \sqrt{2}M_{\varphi}v_2^{\varphi} + \frac{\lambda_{\xi\eta}}{2}(u_1^{\xi}z_1^{\eta} - u_2^{\xi}z_2^{\eta}) + \frac{\lambda_{\Delta}}{2\sqrt{3}}[-2(v_1^{\Delta})^2 + (v_2^{\Delta})^2 + (v_3^{\Delta})^2] = 0 \tag{45e}$$

$$\partial W/\partial f_1^{\eta} = \frac{M_{\xi\eta}}{\sqrt{2}}u_1^{\xi} + \frac{\lambda_{\xi\eta}}{2}(v_1^{\varphi}u_2^{\xi} + v_2^{\varphi}u_1^{\xi}) + \sqrt{2}\lambda_{\sigma\eta}v_{\sigma}z_1^{\eta} = 0 \tag{45f}$$

$$\partial W/\partial f_2^{\eta} = \frac{M_{\xi\eta}}{\sqrt{2}}u_2^{\xi} + \frac{\lambda_{\xi\eta}}{2}(v_1^{\varphi}u_1^{\xi} - v_2^{\varphi}u_2^{\xi}) + \sqrt{2}\lambda_{\sigma\eta}v_{\sigma}z_2^{\eta} = 0 \tag{45g}$$

$$\partial W/\partial f^{\sigma} = \frac{\lambda_{\sigma\eta}}{\sqrt{2}}[(z_1^{\eta})^2 + (z_2^{\eta})^2] = 0 \tag{45h}$$

$$\begin{aligned}\partial W/\partial f_1^{\xi} = {}& \frac{1}{\sqrt{2}}M_{\xi\eta}z_1^{\eta} + \frac{1}{2}\lambda_{\xi\eta}(z_1^{\eta}v_2^{\varphi} + z_2^{\eta}v_1^{\varphi}) + \frac{1}{2}\lambda_{\xi\phi1}[(v_2^{\phi})^2 - (v_3^{\phi})^2] + \frac{1}{2}\lambda_{\xi\phi2}[(u_2^{\phi})^2 - (u_3^{\phi})^2] \\ & + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi12}(2v_1^{\phi}u_1^{\phi} - v_2^{\phi}u_2^{\phi} - v_3^{\phi}u_3^{\phi}) = 0\end{aligned} \tag{45i}$$

$$\begin{aligned}\partial W/\partial f_2^{\xi} = {}& \frac{1}{\sqrt{2}}M_{\xi\eta}z_2^{\eta} + \frac{1}{2}\lambda_{\xi\eta}(z_1^{\eta}v_1^{\varphi} - z_2^{\eta}v_2^{\varphi}) + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi1}[-2(v_1^{\phi})^2 + (v_2^{\phi})^2 + (v_3^{\phi})^2] \\ & + \frac{1}{2\sqrt{3}}\lambda_{\xi\phi2}[-2(u_1^{\phi})^2 + (u_2^{\phi})^2 + (u_3^{\phi})^2] + \frac{1}{2}\lambda_{\xi\phi12}(v_2^{\phi}u_2^{\phi} - v_3^{\phi}u_3^{\phi}) = 0\end{aligned} \tag{45j}$$

$$\partial W/\partial f_1^{\phi_1} = \frac{1}{\sqrt{3}}(\lambda_{\xi\phi12}u_1^{\phi}u_1^{\xi} - 2\lambda_{\xi\phi1}u_2^{\xi}v_1^{\phi}) = 0 \tag{45k}$$

$$\partial W/\partial f_2^{\phi_1} = u_1^{\xi}\left(\lambda_{\xi\phi1}v_2^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi12}u_2^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi1}}{\sqrt{3}}v_2^{\phi} + \frac{1}{2}\lambda_{\xi\phi12}u_2^{\phi}\right) = 0 \tag{45l}$$

$$\partial W/\partial f_3^{\phi_1} = u_1^{\xi}\left(-\lambda_{\xi\phi1}v_3^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi12}u_3^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi1}}{\sqrt{3}}v_2^{\phi} - \frac{1}{2}\lambda_{\xi\phi12}u_2^{\phi}\right) = 0 \tag{45m}$$

$$\partial W/\partial f_1^{\phi_2} = \frac{1}{\sqrt{3}}(\lambda_{\xi\phi12}v_1^{\phi}u_1^{\xi} - 2\lambda_{\xi\phi2}u_2^{\xi}u_1^{\phi}) = 0 \tag{45n}$$

$$\partial W/\partial f_2^{\phi_2} = u_1^{\xi}\left(\lambda_{\xi\phi2}u_2^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi12}v_2^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi1}}{\sqrt{3}}u_2^{\phi} + \frac{1}{2}\lambda_{\xi\phi12}v_2^{\phi}\right) = 0 \tag{45o}$$

$$\partial W/\partial f_3^{\phi_2} = u_1^{\xi}\left(-\lambda_{\xi\phi2}u_3^{\phi} - \frac{1}{2\sqrt{3}}\lambda_{\xi\phi12}v_3^{\phi}\right) + u_2^{\xi}\left(\frac{\lambda_{\xi\phi2}}{\sqrt{3}}u_2^{\phi} - \frac{1}{2}\lambda_{\xi\phi12}v_2^{\phi}\right) = 0. \tag{45p}$$

By assuming not spontaneous breaking of *CP*, Eq. (45h) implies $z_{1,2}^{\eta} = 0$. As first consequence we have that a possible solution of Eqs. (45f) and (45g) and Eqs. (45k)–(45p) is given by

$$(u_1^{\xi}, u_2^{\xi}) = (0, 0) \quad \text{and} \quad v_{\sigma} \neq 0. \tag{46}$$

By substituting $(z_1^{\eta}, z_2^{\eta}) = (0, 0)$, $(u_1^{\xi}, u_2^{\xi}) = (0, 0)$, and $v_{\sigma} \neq 0$ in the equations not yet solved it is easy to check that a possible solution for Eqs. (45a)–(45e) is given by the vacuum configuration

$$\begin{gathered}(v_1^{\varphi}, v_2^{\varphi}) = (0, v^{\varphi}) \quad \text{with} \quad v^{\varphi} = \frac{M_{\Delta}}{\lambda_{\Delta}} \\ (v_1^{\Delta}, v_2^{\Delta}, v_3^{\Delta}) = (v^{\Delta}, 0, 0) \quad \text{with} \quad v^{\Delta} = 6^{1/4}\frac{\sqrt{M_{\varphi}M_{\Delta}}}{\lambda_{\Delta}}.\end{gathered} \tag{47}$$

Finally Eqs. (45i) and (45j) are solved by the vacuum configuration

$$\begin{gathered}(v_1^{\phi}, v_2^{\phi}, v_3^{\phi}) = v^{\phi}(1, 1, 1) \quad \text{and} \\ (u_1^{\phi}, u_2^{\phi}, u_3^{\phi}) = u^{\phi}(1, 1, 1).\end{gathered} \tag{48}$$

The solution found is not unique but can be stabilized once

TABLE IV. Scalar content of model II including both flavon and the driving field supermultiplets.

| | $\hat{\Delta}$ | $\hat{\sigma}$ | $\hat{\phi}$ | $\hat{\varphi}$ | $\hat{\bar{\sigma}}$ | $\hat{\xi}$ | $\hat{\eta}$ |
|---|---|---|---|---|---|---|---|
| $SU(2)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $S_4$ | $3_1$ | 1 | 2 | 2 | 1 | 1 | 1 |
| $Z_3$ | $\omega$ | 1 | 1 | 1 | 1 | $\omega$ | $\omega^2$ |
| $Z_5$ | 1 | $\omega_5^2$ | $\omega_5^2$ | 1 | $\omega_5$ | 1 | 1 |

we add apposite SUSY soft breaking terms. In Sec. III A we have assumed that the flavon VEVs is of order $\lambda^2\Lambda$. Therefore the next-to-leading order corrections to the Yukawa superpotential induced by the driving fields are sufficiently suppressed.

### B. Model II: Minimization of the potential

The complete flavor scalar content of the model is given in Table IV. Thus the flavon potential is obtained by the following part of the full superpotential

$$\begin{aligned}\mathcal{W} = {}& \lambda_{\Delta\xi}\hat{\xi}\,\hat{\Delta}\,\hat{\Delta} + \lambda_{\Delta}\hat{\Delta}\,\hat{\Delta}\,\hat{\Delta} + M_{\xi}\hat{\xi}\,\hat{\eta} + \lambda_{\xi}\hat{\xi}\,\hat{\xi}\,\hat{\xi} \\ & + \lambda_{\eta}\hat{\eta}\,\hat{\eta}\,\hat{\eta} + M_{\varphi}\hat{\varphi}\,\hat{\varphi} + \lambda_{\varphi}\hat{\varphi}\,\hat{\varphi}\,\hat{\varphi} + \lambda_{\phi}\hat{\bar{\sigma}}\,\hat{\phi}\,\hat{\phi} \\ & + \lambda_{\sigma}\hat{\bar{\sigma}}\,\hat{\sigma}\,\hat{\sigma}. \end{aligned} \tag{49}$$

By assuming the general vacuum configuration

$$\begin{gathered}\langle\Delta\rangle = (v_1^{\Delta}, v_2^{\Delta}, v_3^{\Delta}), \qquad \langle\varphi\rangle = (v_1^{\varphi}, v_2^{\varphi}), \\ \langle\phi\rangle = (v_1^{\phi}, v_2^{\phi}), \qquad \langle\xi\rangle = v_{\xi}, \qquad \langle\eta\rangle = v_{\eta} \\ \langle\sigma\rangle = v_{\sigma}, \qquad \langle\bar{\sigma}\rangle = v_{\bar{\sigma}}, \end{gathered} \tag{50}$$

the minimization of the scalar potential obtained in the SUSY limit gives the following set of equations

$$\begin{aligned}\partial\mathcal{W}_Y/\partial f_1^{\Delta} &= \sqrt{2}\lambda_{\Delta\xi}v_{\xi}v_1^{\Delta} + \sqrt{3}3\lambda_{\Delta}v_2^{\Delta}v_3^{\Delta} = 0 \\ \partial\mathcal{W}_Y/\partial f_2^{\Delta} &= \sqrt{2}\lambda_{\Delta\xi}v_{\xi}v_2^{\Delta} + \sqrt{3}\lambda_{\Delta}v_1^{\Delta}v_3^{\Delta} = 0 \\ \partial\mathcal{W}_Y/\partial f_3^{\Delta} &= \sqrt{2}\lambda_{\Delta\xi}v_{\xi}v_3^{\Delta} + \sqrt{3}\lambda_{\Delta}v_1^{\Delta}v_2^{\Delta} = 0 \\ \partial\mathcal{W}_Y/\partial f^{\xi} &= \sqrt{3}\lambda_{\Delta\xi}[(v_1^{\Delta})^2 + (v_2^{\Delta})^2 + (v_3^{\Delta})^2] \\ &\qquad + M_{\xi}v_{\eta} + 3\lambda_{\xi}v_{\xi}^2 = 0 \\ \partial\mathcal{W}_Y/\partial f^{\eta} &= M_{\xi}v_{\xi} + 3\lambda_{\eta}v_{\eta}^2 = 0 \\ \partial\mathcal{W}_Y/\partial f_1^{\varphi} &= \sqrt{2}M_{\varphi}v_1^{\varphi} + 3\lambda_{\varphi}v_1^{\varphi}v_2^{\varphi} = 0 \\ \partial\mathcal{W}_Y/\partial f_2^{\varphi} &= \sqrt{2}M_{\varphi}v_2^{\varphi} + \frac{3}{2}\lambda_{\varphi}[(v_1^{\varphi})^2 - (v_2^{\varphi})^2] = 0 \\ \partial\mathcal{W}_Y/\partial f_1^{\phi} &= \sqrt{2}\lambda_{\phi}v_1^{\phi}v_{\bar{\sigma}} = 0 \\ \partial\mathcal{W}_Y/\partial f_2^{\phi} &= \sqrt{2}\lambda_{\phi}v_2^{\phi}v_{\bar{\sigma}} = 0 \\ \partial\mathcal{W}_Y/\partial f^{\sigma} &= 2\lambda_{\bar{\sigma}}v_{\sigma}v_{\bar{\sigma}} = 0 \\ \partial\mathcal{W}_Y/\partial f^{\bar{\sigma}} &= \frac{1}{\sqrt{2}}\lambda_{\phi}[(v_1^{\phi})^2 + (v_2^{\phi})^2] + \lambda_{\bar{\sigma}}v_{\sigma}^2 = 0. \end{aligned} \tag{51}$$

Discarding for the triplet and the doublets the trivial solutions that do not break $S_4$, the solution of the system of Eqs. (51) is given by the following vacuum configuration:

$$\begin{aligned} v_1^{\Delta} = v_2^{\Delta} = v_3^{\Delta} = v^{\Delta} \quad &\text{with} \quad v^{\Delta} = \sqrt{2}\frac{\lambda_{\Delta\xi}\lambda_{\eta}}{\lambda_{\Delta}}\frac{v_{\eta}^2}{M_{\xi}} \\ v_{\xi} = -3\lambda_{\eta}\frac{v_{\eta}^2}{M_{\xi}} \quad &\text{with} \quad v_{\eta}^3 = -M_{\xi}^3\frac{\lambda_{\Delta}^2}{\lambda_{\eta}^2(2\sqrt{3}\lambda_{\Delta\xi}^3 + 27\lambda_{\xi}\lambda_{\Delta}^2)} \\ (v_1^{\varphi}, v_2^{\varphi}) \neq (0,0) \quad &\text{with} \quad \begin{cases}(0, \frac{2\sqrt{2}}{3}\frac{M_{\varphi}}{\lambda_{\varphi}}) \\ (\sqrt{\frac{2}{3}}\frac{M_{\varphi}}{\lambda_{\varphi}}, -\frac{\sqrt{2}}{3}\frac{M_{\varphi}}{\lambda_{\varphi}}) \\ (-\sqrt{\frac{2}{3}}\frac{M_{\varphi}}{\lambda_{\varphi}}, -\frac{\sqrt{2}}{3}\frac{M_{\varphi}}{\lambda_{\varphi}})\end{cases} \\ v_{\sigma}^2 = -\frac{1}{\sqrt{2}}\frac{\lambda_{\phi}}{\lambda_{\sigma}}[(v_1^{\phi})^2 + (v_2^{\phi})^2] \neq 0 \quad &\text{and} \quad v_{\bar{\sigma}} = 0. \end{aligned} \tag{52}$$

The three solutions corresponding to $\langle\varphi\rangle$ are degenerate and corresponding to the breaking of $S_3$ to its 3 different $Z_2$ subgroups. Through appropriate choices of soft terms that break the discrete Abelian symmetry $Z_3$ and $Z_5$ and not $S_4$ we can stabilize as absolute minimum the vacuum configuration $\langle\varphi\rangle \sim (0, 1)$.

## V. CONCLUSION

In this paper we have discussed the idea that $S_4$ is the minimal discrete non-Abelian group naturally related to TBM in the lepton sector. We have shown that $S_4$ can yield exact TBM according to a general group theory analysis and we have presented two explicit model realizations of how TBM can be obtained in $S_4$ once the basis of its generators is fixed. In addition we have provided a detailed study of the corresponding scalar potentials. The two models require two triplets with different VEV alignments. For each model we have built a potential that in the SUSY limit contains the minimum required. The problem of the triplet and doublet alignments is solved in a more economical way than in models based on $A_4$ [14–29]. To separate the charged lepton sector from the neutrino one we have introduced extra Abelian symmetries. The construction of the potentials has not required additional symmetries than such extra Abelian symmetries, but just the addition of "driving" fields that do not enter in the Yukawa part. We have studied neither the quark sector nor the possibility to embed such a model in a grand unified theory nor the next leading order corrections. We leave these subjects for a future publication. It is worth mentioning that in $S_4$ there is more freedom to generate the mixing in the quark sector than in $A_4$. Indeed the doublet irreducible representation could play an important role as happens in $T'$ [30].

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[1] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B **530**, 167 (2002).

[2] M. Maltoni, T. Schwetz, M. A. Tortola, and J. W. F. Valle, New J. Phys. **6**, 122 (2004).

[3] F. Feruglio and Y. Lin, Nucl. Phys. **B800**, 77 (2008).

[4] W. Grimus and L. Lavoura, J. High Energy Phys. 08 (2005) 013.

[5] M. Mitra and S. Choubey, Phys. Rev. D **78**, 115014 (2008).

[6] A. Mondragon, M. Mondragon, and E. Peinado, AIP Conf. Proc. **1026**, 164 (2008).

[7] C.-Y. Chen and L. Wolfenstein, Phys. Rev. D **77**, 093009 (2008).

[8] Y. Koide, Eur. Phys. J. C **50**, 809 (2007).

[9] R. N. Mohapatra, S. Nasri, and H.-B. Yu, Phys. Lett. B **639**, 318 (2006).

[10] S. Morisi and M. Picariello, Int. J. Theor. Phys. **45**, 1267 (2006).

[11] F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234.

[12] P. F. Harrison and W. G. Scott, Phys. Lett. B **557**, 76 (2003).

[13] S.-L. Chen, M. Frigerio, and E. Ma, Phys. Rev. D **70**, 073008 (2004).

[14] E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001).

[15] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003).

[16] M. Hirsch *et al.*, Phys. Rev. D **69**, 093006 (2004).

[17] E. Ma, Phys. Rev. D **70**, 031901 (2004).

[18] G. Altarelli and F. Feruglio, Nucl. Phys. **B720**, 64 (2005).

[19] S.-L. Chen, M. Frigerio, and E. Ma, Nucl. Phys. **B724**, 423 (2005).

[20] A. Zee, Phys. Lett. B **630**, 58 (2005).

[21] G. Altarelli and F. Feruglio, Nucl. Phys. **B741**, 215 (2006).

[22] B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma, and M. K. Parida, Phys. Lett. B **638**, 345 (2006).

[23] J. W. F. Valle, J. Phys. Conf. Ser. **53**, 473 (2006).

[24] B. Adhikary and A. Ghosal, Phys. Rev. D **75**, 073020 (2007).

[25] E. Ma, Mod. Phys. Lett. A **22**, 101 (2007).

[26] G. Altarelli, F. Feruglio, and Y. Lin, Nucl. Phys. **B775**, 31 (2007).

[27] M. Hirsch, A. S. Joshipura, S. Kaneko, and J. W. F. Valle, Phys. Rev. Lett. **99**, 151802 (2007).

[28] G. Altarelli, arXiv:0705.0860.

[29] F. Bazzocchi, S. Kaneko, and S. Morisi, J. High Energy Phys. 03 (2008) 063.

[30] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo, Nucl. Phys. **B775**, 120 (2007).

[31] A. Aranda, Phys. Rev. D **76**, 111301 (2007).

[32] D. A. Eby, P. H. Frampton, and S. Matsuzaki, Phys. Lett. B **671**, 386 (2009).

[33] P. H. Frampton and T. W. Kephart, Int. J. Mod. Phys. A **10**, 4689 (1995).

[34] G. J. Ding, Phys. Rev. D **78**, 036011 (2008).

[35] E. Ma, Phys. Lett. B **632**, 352 (2006).

[36] C. Hagedorn, M. Lindner, and R. N. Mohapatra, J. High Energy Phys. 06 (2006) 042.

[37] Y. Cai and H.-B. Yu, Phys. Rev. D **74**, 115005 (2006).

[38] H. Zhang, Phys. Lett. B **655**, 132 (2007).

[39] Y. Koide, J. High Energy Phys. 08 (2007) 086.

[40] C. Luhn, S. Nasri, and P. Ramond, J. Math. Phys. (N.Y.) **48**, 073501 (2007).

[41] I. de Medeiros Varzielas, S. F. King, and G. G. Ross, Phys. Lett. B **648**, 201 (2007).

[42] E. Ma, Phys. Lett. B **660**, 505 (2008).

[43] E. Ma, Mod. Phys. Lett. A **21**, 1917 (2006).

[44] P. H. Frampton and T. W. Kephart, Phys. Rev. D **64**, 086007 (2001).

[45] F. Bazzocchi, M. Frigerio, and S. Morisi, Phys. Rev. D **78**, 116018 (2008).

[46] C. S. Lam, Phys. Rev. D **78**, 073015 (2008).