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AN EXAMPLE CONCERNING THE MENGER-URYSOHN FORMULA

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ABSTRACT. We construct subsets A, B of the Euclidean space $\mathbb{R}^4$ such that $\dim(A\cup B) > \dim(A\times B)+1$. This provides a counterexample to a conjecture by E. Ščepin for subspaces of $\mathbb{R}^4$.

1. Introduction

In 1991, E. Ščepin conjectured ([D], the end of section 6, and [D-D], the beginning of section 5, [L-L], page 74) that the classical Menger-Urysohn formula $\dim(A\cup B) \leq \dim A + \dim B + 1$ can be improved by the formula $\dim(A\cup B) \leq \dim(A\times B)+1$. Since $\dim(A\times B) \leq \dim A + \dim B$, a positive answer to Ščepin's conjecture would indeed be an improvement. The question was repeated recently by V. Chatyrko [C], Question 17. After this paper was completed, we were informed by M. Levin that from the results in [D] one can derive an example of a 5-dimensional compactum $Z = A\cup B$ with $\dim(A\times B) \leq 3$.

Ščepin's conjecture is true in $\mathbb{R}^6$ if the union of A and B is σ-compact. Indeed, if $X = A\cup B$ is a σ-compact set in $\mathbb{R}^6$ and $\dim X \geq 5$ this follows from a deep theorem of A. Dranishnikov [D], Theorem B in section 6, as $\dim(X\times X) = 2\dim X$; cf. [F], Ch. 5, §2 and §4. Also, as was pointed out by M. Levin, if $\dim X \leq 4$, then the inequality $\dim X \leq \dim(A\cup B) + 1$ can be derived from [D], Proposition 6.3 (even if the σ-compact set X does not embed in $\mathbb{R}^6$).

The aim of this paper is to show that Ščepin's conjecture is not always true if the union of A and B is a G_δ-subset of $\mathbb{R}^4$.

Example 1.1. There is a 3-dimensional G_δ-set X in $\mathbb{R}^4$ which can be split into two sets, $X = A\cup B$, such that each finite product of the free union $A\oplus B$ is 1-dimensional. In particular,

$$\dim(A\cup B) > \dim[(A\times B)^m] + 1$$

for all m. Moreover, A is a G_δ-set in X (and hence, B is an F_σ-subset of X).

To get the example, we put together some ideas and results from our earlier papers [vM-P1] and [vM-P2] concerning weakly n-dimensional sets; cf. section 2.

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2. Preliminaries

We shall deal exclusively with subspaces of the Euclidean spaces $\mathbb{R}^n$. Our terminology follows [K]. Given a space X, we denote by $X_{(n)}$ the set of points in X that have arbitrarily small neighbourhoods with at most $(n-1)$-dimensional boundaries; cf. [K], [vM]. We say that X is weakly n-dimensional if $\dim X = n$ and $\dim(X \setminus X_{(n-1)}) \leq n-1$, $n \geq 1$.

We shall use the following result; cf. [vM], Corollary 3.11.12 and Exercise 2 for §3.11.

Theorem 2.1. *The product of a countable family of weakly* 1*-dimensional spaces is* 1*-dimensional.*

Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate and let $C \subset \mathbb{R}$ be the Cantor set. We shall need the following fact; cf. Lelek [L], Example, p. 80; Rubin, Schori and Walsh [R-S-W], Example 4.5; or [vM], proof of Theorem 3.9.3.

Proposition 2.2. *For each n there is a compact set $K_n \subset \mathbb{R}^{n+1}$ such that $\pi(K_n) = C$ and each set $M \subset K_n$ with $\pi(M) = C$ is n-dimensional.*

We end this section with a handy observation, used in [vM-P2] and [vM], proof of Theorem 3.11.8.

Lemma 2.3. *Let $f : E \to T$ be a perfect map from an m-dimensional space onto a zero-dimensional space. Then there exists a G_δ-set F in E such that $f(F) = T$ and $\dim(F \setminus F_{(0)}) \leq m-1$.*

Let us sketch a justification of this fact. We set $T_0 = \{t \in T : \dim f^{-1}(t) = 0\}$ and choose a zero-dimensional F_σ-set M in E such that $\dim(E \setminus M) \leq m-1$; cf. [vM], Lemma 3.11.6. Then $F = f^{-1}(T_0) \cup (E \setminus M)$ has the required properties.

3. Construction of Example 1.1

We adopt the notation introduced in section 2. Let us denote by K the compact set K_3 described in Proposition 2.2, and let $p = \pi \mid K : K \to C$ be the restriction of the projection, mapping K onto the Cantor set C.

The surjection p has the following property:

(1) if $M \subset K$ and $p(M) = C$, then $\dim M = 3$.

We let

(2) $S = \{t \in C :\ \dim p^{-1}(t) = 0\}, \quad T = \{t \in C :\ 1 \leq \dim p^{-1}(t) \leq 2\}$.

Then (cf. [vM], Lemma 3.11.7),

(3) $p^{-1}(S) \subset K_{(0)}$.

Let $\mathbb{P} \subset \mathbb{R}$ be the irrationals, and let

(4) $A = p^{-1}(S) \cup \big((C \times \mathbb{P}^3) \cap K\big)$.

Then, by (3) and [K], §45, IV,

(5) $\dim(A \setminus A_{(0)}) \leq 0$; hence $\dim A \leq 1$, and A is a G_δ-set in $\mathbb{R}^4$.

Let us check that (cf. (2))

(6) $p(A) \supset C \setminus T$.

Indeed, if $t \in C \setminus (S \cup T)$, then by (2), $p^{-1}(t)$ is a 3-dimensional set in $\{t\} \times \mathbb{R}^3$. Hence it has nonempty interior in this section (cf. [vM], Theorem 3.7.1) and so there is $u \in \mathbb{P}^3$ such that $(t, u) \in p^{-1}(t) \cap K$, i.e., $t \in p(A)$; cf. (4).

Let us now consider the set $Y = p^{-1}(T)$. Then, by (2), $\dim Y \leq 2$ (cf. [vM], Lemma 3.6.10), and let us choose a zero-dimensional G_δ-set G in Y such that $\dim(Y \setminus G) \leq 1$. By (2), each fiber $p^{-1}(t)$ with $t \in T$ has positive dimension; hence $p^{-1}(t) \setminus G \neq \emptyset$. It follows that $f(Y \setminus G) = T$. Now, one can find countably many subsets of $Y \setminus G$ that are closed in Y and whose images under p are pairwise disjoint and cover T; cf. [vM-P3], Lemma 2.1 (or [vM-P1], section 4). In effect, we have sets $E_i \subset Y$ such that

(7) E_i is closed in $p^{-1}(T)$, $\dim E_i \leq 1$,

(8) $p(E_i) \cap p(E_j) = \emptyset$ for $i \neq j$, $\bigcup_i p(E_i) = T$.

We use Lemma 2.3 to get sets F_i such that

(9) F_i is a G_δ-set in E_i, $p(F_i) = p(E_i)$,

(10) $\dim(F_i \setminus (F_i)_{(0)}) \leq 0$.

Finally, we set (cf. (4))

(11) $B = \bigcup_i F_i$ and $X = A \cup B$.

Since the sets $p(E_i)$ are closed in T (cf. (7)), from (8), (9) and the countable sum theorem, we infer that

(12) F_i is closed in B, $\dim B \leq 1$, and B is a G_δ-set in $p^{-1}(T)$.

By [K], §45, IV, $p^{-1}(S \cup T)$ is a G_δ-set in $\mathbb{R}^4$; hence (4), (5) and (12) yield that

(13) X is a G_δ-set in $\mathbb{R}^4$.

By (8), (9) and (11), we also have $p(B) = T$, and hence $p(X) = C$; (cf. (6)). Therefore, by (1),

(14) $\dim X = 3$.

In particular, since by (5) and (12) the sets A and B are at most 1-dimensional, the Menger-Urysohn formula shows that they both have positive dimension. By (5),

(15) A is weakly 1-dimensional.

By (12) and the countable sum theorem, one of the sets F_i, say F_1, is 1-dimensional, and considering the sets $F_1 \cup F_i$ we conclude from (10) that

(16) B is covered by countably many closed weakly 1-dimensional sets.

It follows from (15) and (16) that any product $(A \oplus B)^m$ of the free union of A and B is covered by countably many closed sets, each of which is a product of weakly 1-dimensional spaces. By Theorem 2.1 and the countable sum theorem, $\dim((A \oplus B)^m) = 1$.

4. A comment

We can repeat the construction described in section 3 for any $n \geq 3$, starting from the compact set $K_n \subset \mathbb{R}^{n+1}$ described in Proposition 2.2. Then we get as a result an n-dimensional G_δ-set $X \subset \mathbb{R}^{n+1}$ and a decomposition $X = A \cup B$ such that A is a weakly 1-dimensional G_δ-set in $\mathbb{R}^{n+1}$ and B is a countable union of closed weakly $(n-2)$-dimensional sets. Using a theorem of Tomaszewski [T], [vM-P2], it follows that $\dim(A \times B) \leq 1 + (n-2) - 1 = n - 2$. In effect, for the n-dimensional space X we have $\dim X > \dim(A \times B) + 1 = n - 1$.

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