

VU Research Portal

Enumeration of local and global étale algebras applied to generalized Fermat equations

Putz, Piet Hein Casper

2024

DOI (link to publisher)

[10.5463/thesis.832](https://doi.org/10.5463/thesis.832)

document version

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

citation for published version (APA)

Putz, P. H. C. (2024). *Enumeration of local and global étale algebras applied to generalized Fermat equations*. [PhD-Thesis - Research and graduation internal, Vrije Universiteit Amsterdam]. <https://doi.org/10.5463/thesis.832>

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

E-mail address:

vuresearchportal.ub@vu.nl

Contents

Introduction	1
Notation and conventions	7
1 Coverings and generalized Fermat equations	11
1.1 Darmon and Granville’s approach	11
1.1.1 Covering sets	13
1.1.2 Eliminating global algebras	15
1.2 Summary of the method	15
1.3 Belyi maps	16
1.4 Local solutions to generalized Fermat equations	17
1.5 Prerequisites	19
1.5.1 ϕ -Newton polygons	19
1.5.2 Étale algebras	21
1.5.3 Resultants and discriminants	22
1.5.4 The Hasse derivative	24
2 Families of local étale algebras	27
2.1 Krasner’s lemma	29
2.2 Separants	31
2.3 A partial algorithm	34
2.4 Structural stability of $f^k - sg$	40
2.4.1 Notation	40
2.4.2 The root structure of $f^k - sg$	40
2.4.3 The separant of $f^k - sg$	42
2.4.4 Finite F -stable covers around 0	47
2.5 Structural stability of linear families	57
2.5.1 A standard form	60
2.5.2 Unramified étale algebras	65
2.6 A full algorithm	69
2.6.1 Generalizations	71
2.7 Implementation in Magma	76
2.7.1 Étale algebras	76

2.7.2	p -adic computations	77
2.7.3	Computing separants	78
2.7.4	Example 1	81
2.7.5	Example 2	82
2.7.6	Example 3	84
3	Targeted Hunter searching	87
3.1	Hunter's method	89
3.2	Improved Archimedean bounds	92
3.2.1	Elementary estimates	92
3.2.2	Lagrange multipliers	94
3.3	Local restrictions	97
3.3.1	Tschirnhaus transformations over local fields	98
3.3.2	Search ideals	101
3.3.3	Local reduction factors	102
3.4	Theoretical algorithm	104
3.4.1	Basic algorithm	104
3.4.2	Computing the discriminant	106
3.4.3	Using local information	109
3.4.4	The last coefficient	111
3.5	Implementation	111
3.5.1	Summary	112
3.5.2	Parallel computing	112
3.5.3	Order of enumeration	112
3.5.4	Updating the discriminant	113
3.5.5	Updating the bounds	113
3.5.6	Number of primes used	113
3.5.7	Arbitrary precision arithmetic	114
3.6	Relative Hunter searching	114
3.7	Results	115
3.7.1	Targeted Hunter searches for signature $(5, 2, 7)$	115
3.7.2	Targeted Hunter searches for signature $(5, 3, 7)$	118
4	The equation $x^5 + y^2 = z^7$	121
4.1	The Belyi map	122
4.2	Local algebras	123
4.3	Global algebras	130
4.3.1	Degrees $(6, 2)$ and $(6, 1, 1)$	132
4.3.2	Degrees $(7, 1)$	133
4.3.3	Degrees $(5, 3)$	137
4.3.4	Degree 8	139
4.4	Results	141
4.4.1	Algebras arising from trivial solutions	145
4.5	Potential strategies to eliminate the final case	146
4.5.1	Selmer group Chabauty	147

4.5.2	Reducing to hyperelliptic curves over a number field . . .	150
4.5.3	A different Belyi map	153
4.6	A full-rank subgroup of $E(L)$	156
4.6.1	The conjectural rank	156
4.6.2	Descent on E_1/L	158
4.6.3	Prime sieve	159
4.6.4	Results of the search	160
5	Some generalized Fermat equations of signature (5, 3, 11)	163
5.1	Discriminant bounds	164
5.2	Local algebras	165
5.3	Global algebras	169
6	Discussion	175
6.1	Computational complexity	175
6.2	Reliability	176
6.3	Further applications	176
A	Targeted Hunter searches	179
A.1	Octic number fields for (5, 2, 7)	179
A.1.1	Generating input	179
A.1.2	Running the program	181
A.1.3	Individual searches	181
A.2	Septic number fields for (5, 2, 7)	185
A.3	Sextic number fields for (5, 2, 7)	186
A.4	Septic number fields for (5, 3, 7)	186
A.5	Pseudocode for targeted Hunter search algorithm	188
B	Defining polynomials over finite fields	191
B.1	Polynomials with a given factorization pattern	191
B.2	Factorization patterns in linear families	194
C	Local covering set computations using ad-hoc methods	197
Summary		203
Samenvatting		205
Acknowledgements		207
Bibliography		209