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published in

Communications in Statistics - Theory and Methods
2024

DOI (link to publisher)

[10.1080/03610926.2023.2173976](https://doi.org/10.1080/03610926.2023.2173976)

document version

Publisher's PDF, also known as Version of record

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citation for published version (APA)

De Luca, G., & Magnus, J. R. (2024). Shrinkage efficiency bounds: An extension. *Communications in Statistics - Theory and Methods*, 53(11), 4147-4152. <https://doi.org/10.1080/03610926.2023.2173976>

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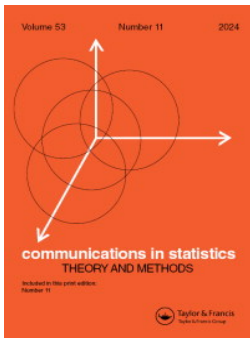
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To cite this article: Giuseppe De Luca & Jan R. Magnus (2024) Shrinkage efficiency bounds: An extension, Communications in Statistics - Theory and Methods, 53:11, 4147-4152, DOI: [10.1080/03610926.2023.2173976](https://doi.org/10.1080/03610926.2023.2173976)

To link to this article: <https://doi.org/10.1080/03610926.2023.2173976>



Published online: 15 Feb 2023.



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Shrinkage efficiency bounds: An extension

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ABSTRACT

Hansen (2005) obtained the efficiency bound (the lowest achievable risk) in the p -dimensional normal location model when $p \geq 3$, generalizing an earlier result of Magnus (2002) for the one-dimensional case ($p = 1$). The classes of estimators considered are, however, different in the two cases. We provide an alternative bound to Hansen's which is a more natural generalization of the one-dimensional case, and we compare the classes and the bounds.

ARTICLE HISTORY

Received 22 July 2022
Accepted 23 Jan 2023

KEYWORDS

Normal location model; risk; lower bound; shrinkage estimators

2020 MATHEMATICS

SUBJECT CLASSIFICATION
62C20; 62H12; 62J07

1. Introduction

Consider p independent normally distributed observations x_1, x_2, \dots, x_p where x_i has an unknown mean θ_i and variance 1, in other words $x \sim N(\theta, I_p)$, where $x = (x_1, \dots, x_p)'$, $\theta = (\theta_1, \dots, \theta_p)'$, and I_p denotes the identity matrix of order p . This is the so-called (multivariate) normal location model.

Suppose we wish to estimate θ by an estimator of the form

$$t(x) = \lambda(w_p)x, \quad (1)$$

where λ depends on x only through $w_p = x'x$. The mean squared error of $t(x)$ is the positive semidefinite $p \times p$ matrix

$$\text{MSE}(t(x)) = E[(t(x) - \theta)(t(x) - \theta)'] \quad (2)$$

and its trace is called the *risk* of the estimator, which we denote by $R(\theta, \lambda)$. For a given λ function, the risk depends only on p and on $\alpha = \theta'\theta/p$ (Hansen 2015). Given a class \mathcal{L} of λ functions, the *efficiency bound* $R_E(p, \alpha)$ is defined as the the lowest possible risk at given values of p and α , where we minimize over all λ in \mathcal{L} .

If we replace x by $y = Sx$ where S is an orthogonal matrix, then $y'y = x'x$ so that λ remains the same. Hence λ is orthogonally invariant, and the class of estimators defined by (1) is called the class of *orthogonally invariant estimators*.

Within this class, we have

$$\sup_{\theta} R(\theta, \lambda) \geq p \quad (3)$$

for all λ (Van der Vaart 1998, Proposition 8.6). When equality occurs in (3), then $t(x)$ is called *minimax*. For $p = 1$ and $p = 2$, there is a unique minimax estimator, namely the

maximum-likelihood estimator $t(x) = x$, sometimes called the “usual” estimator. But for $p \geq 3$, there are many minimax estimators.

The following two questions now arise. First, how do we choose between these minimax estimators? And second, should we also allow estimators that are not minimax but satisfy some other optimality criterion?

In the case $p = 1$, there is only one minimax estimator, namely $t(x) = x$ with risk 1, but there are many shrinkage estimators of the form $t(x) = \lambda(x)x$ whose risk is sometimes larger and sometimes smaller than 1. These estimators are not minimax, but they have other desirable (arguably more desirable) properties. One possibility to distinguish between these estimators is through the minimax regret criterium. To use this criterium, we need a lower bound for the class of estimators we are interested in. Magnus (2002) defined this class (for $p = 1$) by the following four restrictions:

- (a) $0 \leq \lambda(x) \leq 1$ for all x ,
- (b) $\lambda(-x) = \lambda(x)$ for all x ,
- (c) λ is nondecreasing on $[0, \infty)$, and
- (d) λ is continuous except possibly on a set of measure zero,

and proved that the lower bound is equal to $\theta^2/(1 + \theta^2)$ (Magnus 2002, Appendix, Theorem A.7). The resulting minimax regret solution is obviously not minimax.

Hansen takes a different route. He considers only minimax estimators and obtains an efficiency bound within this class (Hansen 2015, Theorem 2). This implies *inter alia* that all estimators with unbounded risk, such as $t(x) = x/2$, are excluded.

In order to derive these minimax regret estimators, Hansen needs the efficiency bound (the lowest achievable risk) for the class of minimax estimators of the form (1). When $p = 1$ or $p = 2$ the efficiency bound is equal to p , because there is only one λ which leads to a minimax estimator, namely $\lambda \equiv 1$. But when $p \geq 3$, then Hansen’s (2015, Theorem 2) efficiency bound is a rather intricate formula which we denote by $R_H(p, \alpha)$.

Hansen considers his bound $R_H(p, \alpha)$ to be a generalization of the efficiency bound derived in Magnus (2002) for the case $p = 1$. But the class of estimators in Magnus (2002) is quite different to the class of minimax estimators considered by Hansen. Magnus purposely considers estimators that are *not* minimax and asks whether any of them have better properties than $t(x) = x$, the minimax estimator. Following the latter strategy, we shall prove the following result.

Proposition (efficiency bound). *Under the assumption that λ is nondecreasing, the efficiency bound for the risk in the class of estimators defined by (1) is given by*

$$R_M(p, \alpha) = \frac{\alpha p}{1 + \alpha},$$

where $\alpha = \theta^2/p$ denotes the average noncentrality.

The proposition contains the bound obtained in Magnus (2002) as a special case ($p = 1$). Note that Hansen (2015) does not require that λ is nondecreasing. Hence, there are estimators in Hansen’s class that are not in our class and there are estimators in our class (in particular, $\lambda(w) = \lambda$ for some constant $0 \leq \lambda < 1$) that are not in his class.

2. Why should λ be nondecreasing

Before we prove the proposition, we argue that it does not make much sense to consider λ functions that are not nondecreasing. To see this, we consider the simple regression model where y is generated by

$$y = X\beta + u = X_1\beta_1 + X_2\beta_2 + u, \quad u \sim N(0, \sigma^2 I_n). \tag{4}$$

where the regressor matrices X_1 and X_2 have dimensions $n \times k_1$ and $n \times k_2$, respectively.

We are interested in estimating β_1 and consider two models: the unrestricted model (4) and the restricted model where $\beta_2 = 0$. We denote the least-squares estimators of β_1 and β_2 from the unrestricted model by $\hat{\beta}_{1u}$ and $\hat{\beta}_{2u}$, and the least-squares estimator of β_1 from the restricted model by $\hat{\beta}_{1r}$.

Let $M_1 = I_n - X_1(X_1'X_1)^{-1}X_1'$ and define the parameter vector θ and its estimator $\hat{\theta}$ by

$$\theta = (X_2'M_1X_2)^{1/2}\beta_2, \quad \hat{\theta} = (X_2'M_1X_2)^{1/2}\hat{\beta}_{2u}. \tag{5}$$

Then $\hat{\theta} \sim N(\theta, \sigma^2 I_{k_2})$. The estimation of β_1 and the estimation of θ are intimately connected, as we shall see shortly.

Let s^2 be the least-squares estimator of σ^2 in the unrestricted model. Then $F = \hat{\theta}'\hat{\theta}/(k_2s^2)$ is the F statistic for testing $\beta_2 = 0$, and the typical pretest estimator of β_1 takes the form

$$\hat{\beta}_{1p} = \begin{cases} \hat{\beta}_{1r} & \text{if } F \leq c, \\ \hat{\beta}_{1u} & \text{if } F > c \end{cases} \tag{6}$$

for some $c > 0$. The continuous version of the pretest estimator is the model-averaging estimator

$$\hat{\beta}_{1m} = \lambda\hat{\beta}_{1u} + (1 - \lambda)\hat{\beta}_{1r}, \quad \lambda = \lambda(\hat{\theta}, s^2). \tag{7}$$

If we take λ to be a function of F , then the larger is F the more weight we should put on $\hat{\beta}_{1u}$, so that λ should be nondecreasing in F .

The “equivalence theorem” of Magnus and Durbin (1999) shows that

$$\text{MSE}(\hat{\beta}_{1m}) = \sigma^2(X_1'X_1)^{-1} + Q \text{MSE}(\lambda\hat{\theta})Q', \tag{8}$$

where

$$Q = (X_1'X_1)^{-1}X_1'X_2(X_2'M_1X_2)^{-1/2}. \tag{9}$$

A good estimator (in the mean squared error sense) $\lambda\hat{\theta}$ of θ thus automatically implies a good estimator $\hat{\beta}_{1m}$ of β_1 , and hence, if λ is a nondecreasing function of F in the regression setup, then it should also be a nondecreasing function of F in the setup with $\hat{\theta} \sim N(\theta, \sigma^2 I_{k_2})$.

3. Proof of the proposition

The proof is in two steps. We first consider the simplest case, namely the case where λ is a constant. Then we consider the general case.

In the special case where λ is a constant, we write

$$\lambda x - \theta = \lambda(x - \theta) - \theta(1 - \lambda), \tag{10}$$

and obtain the risk

$$R(\theta, \lambda) = E(\lambda x - \theta)'(\lambda x - \theta) = p\lambda^2 + \theta'\theta(1 - \lambda)^2. \tag{11}$$

For given θ , the risk is minimized when

$$0 = \frac{\partial R(\theta, \lambda)}{\partial \lambda} = 2p\lambda - 2\theta'\theta(1 - \lambda), \tag{12}$$

which leads to $\lambda^* = \alpha/(1 + \alpha)$, and thus shows that

$$R(\theta, \lambda) \geq R(\theta, \lambda^*) = \frac{\alpha p}{1 + \alpha} = R_M(p, \alpha) \tag{13}$$

for all (constant) λ with equality if and only if $\lambda = \alpha/(1 + \alpha)$.

The previous argument shows that, in the class of constant λ functions, the bound in (13) is not just a bound but in fact the efficiency bound of the risk $R(\theta, \lambda)$. Hence, if we can show that $R(\theta, \lambda) \geq \alpha p/(1 + \alpha)$ in a larger class of λ functions, then this will also be the efficiency bound (and not just a bound) in this larger class. This is what we shall prove next.

In the general case, we have $t(x) = \lambda(w_p)x$, where $w_p = x'x$ follows a noncentral χ^2 distribution with p degrees of freedom and noncentrality $\theta'\theta$. Following De Luca and Magnus (2021, Proposition 2), we write

$$\begin{aligned} R(\theta, \lambda) &= E[(t(x) - \theta)'(t(x) - \theta)] \\ &= E[\lambda^2(w_p)x'x] - 2\theta' E[\lambda(w_p)x] + \theta'\theta \\ &= p E \lambda^2(w_{p+2}) + \theta'\theta E \lambda^2(w_{p+4}) - 2\theta'\theta E \lambda(w_{p+2}) + \theta'\theta, \end{aligned} \tag{14}$$

where the last equality follows from Bock (1975, Theorems A and B), who showed that

$$E[\lambda(w_p)x_i] = \theta_i E[\lambda(w_{p+2})] \tag{15}$$

and

$$E[\lambda^2(w_p)x_i^2] = E[\lambda^2(w_{p+2})] + \theta_i^2 E[\lambda^2(w_{p+4})] \tag{16}$$

for any function $\lambda : [0, \infty) \rightarrow (-\infty, \infty)$. Note that w_p, w_{p+2} , and w_{p+4} denote noncentral χ^2 random variables with *the same* noncentrality $\theta'\theta$ and degrees of freedom $p, p + 2$, and $p + 4$, respectively.

Let $r = \alpha/(1 + \alpha)$. Then we can write (14) as

$$\begin{aligned} R(\theta, \lambda)/p &= E \lambda^2(w_{p+2}) + \alpha E \lambda^2(w_{p+4}) - 2\alpha E \lambda(w_{p+2}) + \alpha \\ &= (1 + \alpha) E \lambda^2(w_{p+2}) - 2\alpha E \lambda(w_{p+2}) + \alpha + \alpha[E \lambda^2(w_{p+4}) - E \lambda^2(w_{p+2})] \\ &= r + (1 + \alpha) E[\lambda(w_{p+2}) - r]^2 + \alpha[E \lambda^2(w_{p+4}) - E \lambda^2(w_{p+2})]. \end{aligned} \tag{17}$$

Hence, if we can show that $E \lambda^2(w_{p+4}) \geq E \lambda^2(w_{p+2})$, then it will follow that $R(\theta, \lambda)/p \geq r$, and the proof is complete. To prove this, let $f_p(w; \theta'\theta)$ denote the noncentral χ^2 distribution with p degrees of freedom and noncentrality $\theta'\theta$. Then,

$$\begin{aligned} E \lambda^2(w_{p+4}) &= \int \lambda^2(w) f_{p+4}(w; \theta'\theta) dw \\ &= \iint \lambda^2(w + z) f_{p+2}(w; \theta'\theta) f_2(z; 0) dw dz \\ &\geq \iint \lambda^2(w) f_{p+2}(w; \theta'\theta) f_2(z; 0) dw dz \\ &= \int \lambda^2(w) f_{p+2}(w; \theta'\theta) dw = E \lambda^2(w_{p+2}), \end{aligned} \tag{18}$$

since λ^2 is nondecreasing on $(0, \infty)$.

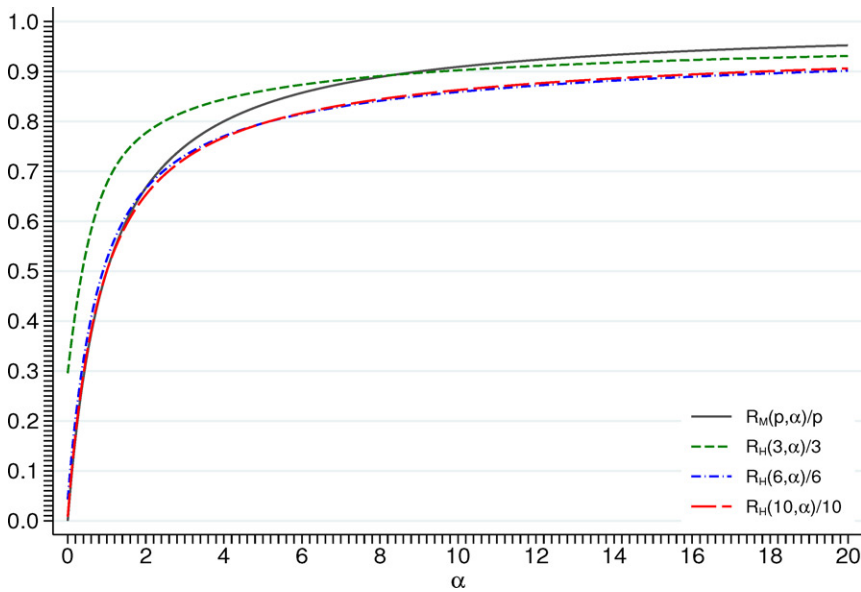


Figure 1. Efficiency bounds $R_H(p, \alpha)/p$ and $R_M(p, \alpha)/p$.

4. Comparison with Hansen's bound

Let us compare Hansen's efficiency bound with ours. In [Figure 1](#) we draw our bound $R_M(p, \alpha)/p$ which is equal to $\alpha/(1 + \alpha)$ by [Proposition 1](#), and thus does not depend on p . In contrast, Hansen's $R_H(p, \alpha)/p$ does depend on p and we have drawn it here for three values of p : 3, 6, and 10. We see that the bounds are similar but not the same, and that R_H is sometimes larger than R_M (for small values of α) and sometimes smaller (for large values of α). The larger is p , the smaller is the value of α where $R_H = R_M$. The “silly” estimator $t(x) = 0$ defined by $\lambda = 0$ has risk αp and hence it has zero risk at $\alpha = 0$. This estimator is included in our class but not in Hansen's class, which is why our efficiency bound is zero at $\alpha = 0$.

Acknowledgements

The authors thank the two referees for their positive and constructive comments.

Funding

Giuseppe De Luca acknowledges financial support from the MIUR PRIN PRJ-0324.

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Disclosure

The authors report there are no competing interests to declare.

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