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published in
Operator and Matrix Theory, Function Spaces, and Applications
2024

DOI (link to publisher)
10.1007/978-3-031-50613-0_8

document version
Publisher's PDF, also known as Version of record

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Small Rank Perturbations of H-Expansive Matrices

G. J. Groenewald, D. B. Janse van Rensburg, and A. C. M. Ran

Abstract In this paper small rank perturbations of H-expansive and H-unitary matrices are explored. Particular attention is given to the location of eigenvalues with respect to the unit circle for these classes of matrices. The canonical form (in the H-unitary case) and the simple form (in the H-expansive case) for the pair (A, H) will be the starting point.

Keywords H-unitary matrices · H-expansive matrices · Perturbation · Canonical form

Mathematics Subject Classification (2000) 47B50, 15A63, 15A18, 47A55

1 Introduction

Let $H = H^*$ be an $n \times n$ invertible complex matrix. Then a complex matrix A is called H-*expansive* when

$$A^*HA - H \geq 0.$$

This means that A is expansive in the indefinite inner product given on $\mathbb{C}^n$ by the invertible Hermitian matrix H. In particular, this class includes the class of H-

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M. Ptak et al. (eds.), *Operator and Matrix Theory, Function Spaces, and Applications*, Operator Theory: Advances and Applications 295,
https://doi.org/10.1007/978-3-031-50613-0_8

unitary matrices for which the inequality above is an equality. Note that when we consider $H = H^T$ real and A real as well, then the matrix A is called H-*orthogonal* when $A^T HA = H$.

We shall consider in this paper the effect on the eigenvalues of A of a finite rank perturbation $B = (I + UV^*)A$, where U and V are both of size $n \times k$ and of rank k. First we discuss in Sect. 2 conditions under which B is again H-expansive. Compare also [8], where rank one perturbations of H-unitary and H-orthogonal matrices are explored. The approach there is to consider rank one perturbations of the type $B = (I - \frac{2}{u^*Hu} uu^* H)A$. Eigenvalues of A on the unit circle are special ones, and in particular, even size Jordan blocks with eigenvalues ± 1 come in coupled pairs. It turns out that for generic vectors u, the matrix B will have a Jordan block with the same eigenvalue but size one larger than the original one. In addition, in the H-orthogonal case, if 1 is an eigenvalue of A but -1 is not, then -1 is always an eigenvalue of B (and vice-versa). The canonical form for the pair (A, H) in these cases plays an essential role in [8]. For the case where A is H-expansive a canonical form does not exist, however, a simple form does exist, see [6]. This will be used in the examples in Sect. 6.

Section 3 considers the relation between rank-k perturbations and k consecutive rank one perturbations.

Like in the H-unitary case, eigenvalues of A on the unit circle for an H-expansive matrix A are of special type. In particular, additional care needs to be taken for the eigenvalues 1 and -1, where in the H-unitary and H-orthogonal cases Jordan blocks of even size with these eigenvalues come in pairs. We introduce a real parameter t in the perturbation, $B(t) = (I + tUV^*)A$, to study the movement of the eigenvalues as a function of t. We are especially interested in the following questions: for which values of t is it possible that $B(t)$ has an eigenvalue on the unit circle that is not already an eigenvalue of A; if so, how do the eigenvalues cross the unit circle? This will be the topic of Sects. 4 and 5. In Sect. 4 small rank perturbations for H-expansive matrices is discussed, while the attention shifts to H-unitary matrices in Sect. 5. First steps in the direction of the results of Sect. 5 were taken in the master thesis by A. Nobakhti in 2013 (under supervision of the third author), with the title “Eigenvalues of rank one perturbations of matrices with structure in an indefinite inner product.”

2 Preliminaries

We start by considering the type of small rank perturbations to be considered. Inspired by Mehl et al. [8], where rank perturbations of H-unitary matrices were studied, we consider perturbations of the following type: $B(t) = (I + tUV^*)A$, where U and V are $n \times k$ matrices of rank k. When A is H-unitary, we would like to have that $B := B(1)$ is H-unitary as well, to stay close to [8]. We first analyse what that means for the relation between U and V.

Proposition 1 *Let A be H-unitary. Then a perturbation $B = (I + UV^*)A$, with U and V both of size $n \times k$ and rank k is also H-unitary if and only if $V = HUE$ where E is an invertible $k \times k$ matrix satisfying*

$$U^*HU = -(E^{-1} + (E^*)^{-1}) = -E^{-1}(E + E^*)(E^*)^{-1}. \tag{1}$$

Proof Assume $A^*HA = H$, and compute

$$\begin{aligned} B^*HB &= A^*(I + VU^*)H(I + UV^*)A = \\ &= A^*HA + A^*(VU^*H + HUV^* + VU^*HUV^*)A. \end{aligned}$$

This is equal to H if and only if $VU^*H + HUV^* + VU^*HUV^* = 0$, in other words, if and only if

$$V(U^*H + U^*HUV^*) = -HUV^*. \tag{2}$$

In particular, since both sides have to be of rank k, it follows that there is an invertible $k \times k$ matrix E such that $V = HUE$. To be more precise on the condition on E, insert $V = HUE$ back into (2):

$$HUE(U^*H + U^*HUE^*U^*H) = -HUE^*U^*H,$$

which, after a little rewriting gives

$$HUEU^*H + HUE^*U^*H + HUEU^*HUE^*U^*H = 0. \tag{3}$$

Now H is invertible, and by assumption there is an $k \times n$ matrix L such that $LU = I_k$, so the above equation is equivalent to

$$E + E^* + E(U^*HU)E^* = 0. \tag{4}$$

Multiplying on the left by E^{-1} and on the right by $(E^*)^{-1}$ we obtain (1). This proves the proposition. □

Turning back to the more general case of H-expansive matrices A with the parameter dependent perturbation $B(t) = (I + tUV^*)A$, we shall also in this case assume from now on that $V = HUE$ as in the proposition above. That is:

$$B(t) = (I + tUE^*U^*H)A,$$

where E is an invertible $k \times k$ matrix satisfying (1).

Remark 1 Note that one possible choice for E would be to take $E = -2(U^*HU)^{-1}$ provided that U^*HU is invertible. This will generically be the case. To be more precise, let $M = \operatorname{Im} U$. Then U^*HU is invertible if and only if $HM \cap M^{\perp} = (0)$, in other words, there is no non-zero vector in M which is H-orthogonal to all vectors in M.

In case we insist that E is Hermitian the only possible choice would be $E = -2(U^*HU)^{-1}$, see also [8]. In the general case, let us write $E^{-1} = F + G$, where $F^* = F$ and $G^* = -G$. Then according to (1) we have that $F = -\frac{1}{2}U^*HU$, and so we can write $E^{-1} = -\frac{1}{2}U^*HU + G = -\frac{1}{2}(U^*HU - 2G)$. Then $E^* = -2(U^*HU + 2G)^{-1}$, so $B(t)$ has the following form

$$B(t) = (I - 2tU(U^*HU + 2G)^{-1}U^*H)A,$$

for some skew-Hermitian G such that $U^*HU + 2G$ is invertible.

We claim that generically the latter is the case for all skew-Hermitian G. Indeed, suppose $(U^*HU + 2G)x = 0$. Since G is skew-Hermitian $0 = 2\langle Gx, x\rangle = -\langle U^*HUx, x\rangle = -\langle HUx, Ux\rangle$. Thus, if $U^*HU + 2G$ is not invertible, then the range of U contains an H-neutral vector. Because the set of H-neutral vectors is the set of vectors y for which $y^*Hy = 0$, this is the zero set of a polynomial in several variables. Hence, for generic U the range will not intersect this set other than at the zero vector. □

The final observation we make concerns the determinant of $B = B(1)$ for the real case, so the case where $H = H^T$ is invertible and A is H-orthogonal.

Proposition 2 *Let A and H be real $n \times n$ matrices, such that $H = H^T$ is invertible and A is H-orthogonal, let U be an $n \times k$ real matrix and E be an invertible $k \times k$ real matrix such that (1) holds. Let G be a skew-symmetric real $k \times k$ matrix such that $E = -2(U^THU - 2G)^{-1}$. Consider the matrix $B = (I - 2U(U^THU + 2G)^{-1}U^TH)A$. Then*

$$\det B = (-1)^k \det A.$$

Proof To show the proposition we need to show that $\det(I_n - 2U(U^THU + 2G)^{-1}U^TH) = (-1)^k$. Indeed,

$$\begin{aligned}
&\det(I_n - 2U(U^THU + 2G)^{-1}U^TH) = \det(I_k - 2(U^THU + 2G)^{-1}U^THU) \\
&= \det\Big((U^THU + 2G)^{-1}(U^THU + 2G - 2U^THU)\Big) \\
&= \det\Big((U^THU + 2G)^{-1}(-U^THU + 2G)\Big) \\
&= \frac{\det(-U^THU + 2G)}{\det(U^THU + 2G)} = (-1)^k \frac{\det(U^THU - 2G)}{\det(U^THU + 2G)}.
\end{aligned}$$

Now, since the matrices H, U and $G = -G^T$ are real, we have

$$\det(U^T HU - 2G) = \det((U^T HU - 2G)^T) = \det(U^T HU + 2G).$$

Thus, indeed $\det(I_n - 2U(U^T HU + 2G)^{-1}U^T H) = (-1)^k$. □

Recall that the set of H-orthogonal matrices has two connected components. One component consists of the H-orthogonal matrices with determinant one, the other component consists of the H-orthogonal matrices with determinant minus one. The observation above implies that if A is real and H-orthogonal, and we consider a rank k perturbation of the form B, then for odd k the H-orthogonal matrix B is not in the same connected component as the matrix A, while for even k the matrix B is in the same connected component as the matrix A.

3 Rank *k* Perturbations Versus *k* Consecutive Rank One Perturbations

Proposition 3 *Let A be H-expansive, U_1 an $n \times k$ matrix of rank k, and U_2 an $n \times l$ matrix of rank l. Let G_1 and G_2 be skew-Hermitian matrices such that $U_i^* HU_i + 2G_i$ is invertible for $i = 1, 2$. Form the rank k perturbation B_1 of A, i.e.,*

$$B_1 = (I - 2U_1(U_1^* HU_1 + 2G_1)^{-1}U_1^* H)A, \tag{5}$$

and consider the rank l perturbation B_2 of B_1 given by

$$B_2 = (I - 2U_2(U_2^* HU_2 + 2G_2)^{-1}U_2^* H)B_1. \tag{6}$$

Then B_2 can be viewed as a rank $k + l$ perturbation of A, that is,

$$B_2 = \left(I - 2\begin{bmatrix} U_2 & U_1 \end{bmatrix} \left(\begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \begin{bmatrix} U_2 & U_1 \end{bmatrix} + \begin{bmatrix} 2G_2 & U_2^* HU_1 \\ -U_1^* HU_2 & 2G_1 \end{bmatrix} \right)^{-1} \begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \right) A.$$

Proof Substituting (5) into (6) we have

$$B_2 = (I - 2U_2(U_2^* HU_2 + 2G_2)^{-1}U_2^* H)(I - 2U_1(U_1^* HU_1 + 2G_1)^{-1}U_1^* H)A.$$

Then (compare Section 2.3 in [1])

$$B_2 = \left(I - 2\begin{bmatrix} U_2 & U_1 \end{bmatrix} \begin{bmatrix} U_2^* H U_2 + 2G_2 & 2U_2^* H U_1 \\ 0 & U_1^* H U_1 + 2G_1 \end{bmatrix}^{-1} \begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \right) A$$

$$= \left(I - 2\begin{bmatrix} U_2 & U_1 \end{bmatrix} \left(\begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \begin{bmatrix} U_2 & U_1 \end{bmatrix} + \begin{bmatrix} 2G_2 & U_2^* H U_1 \\ -U_1^* H U_2 & 2G_1 \end{bmatrix} \right)^{-1} \begin{bmatrix} U_2^* \\ U_1^* \end{bmatrix} H \right) A.$$

Hence a rank k perturbation followed by a rank l perturbation may be understood as a rank $k + l$ perturbation. Note that the matrix $\begin{bmatrix} 2G_2 & U_2^* H U_1 \\ -U_1^* H U_2 & 2G_1 \end{bmatrix}$ is skew-Hermitian. □

The next proposition may be viewed as a converse of the previous one.

Proposition 4 *Let A be an $n \times n$ H-expansive matrix, and let $B = (I - 2U(U^*HU + 2G)^{-1}U^*H)A$ be a rank k perturbation of A with U an $n \times k$ matrix of rank k and G a skew-Hermitian $n \times n$ matrix such that $U^*HU + 2G$ is invertible. Then B can be written as*

$$B = \left(\overset{\curvearrowright}{\prod_{i=1}^{k}} (I - \frac{2}{u_i^* H u_i} u_i u_i^* H) \right) A,$$

where the product is understood from left to right. In other words, the rank k perturbation can be viewed as k consecutive rank one perturbations.

Proof Let

$$B = (I - 2U(U^*HU + 2G)^{-1}U^*H)A,$$

with U of rank k. We show that B can be factorized as a product of k consecutive rank one perturbations. It suffices to show that $Q := (I - 2U(U^*HU + 2G)^{-1}U^*H)$ can be factorized into factors of rank one of the same structure. According to Theorem 10.9 in [1], Q factorizes in rank one factors when a unitary W exists such that the matrices

$$W^*(U^*HU + 2G)W \text{ and } W^*(U^*HU + 2G - 2U^*HU)W = -W^*(U^*HU - 2G)W$$

can simultaneously be brought into upper, respectively lower, triangular form. By Schur's decomposition there is a unitary W such that $W^*(U^*HU + 2G)W$ is upper triangular. Let us write $W^*U^*HUW = L + D + L^*$, where D is diagonal and L is strictly lower triangular. Since $W^*(U^*HU + 2G)W$ is upper triangular it follows that $2W^*GW$, which is also skew-Hermitian, must be equal to $-L + L^*$. Then $W^*(U^*HU - 2G)W = L + D + L^* - (-L + L^*) = D + 2L$ is lower triangular as desired. Compare Theorem 17.16 in [2] to see that $(I - 2U(U^*HU + 2G)^{-1}U^*H)$ factorizes as a product of k H-unitaries of the form I + rank one. □

Remark Note that genericity has a different meaning for rank k perturbations than for consecutive rank one perturbations. What is generic as a rank l perturbation of B_1 which in turn is a generic perturbation of A, may be a different set than generic $k + l$ perturbations of A. (Compare the remarks concerning this in [3], see also [4].)

4 Small Rank Perturbations of H-Expansive Matrices

Suppose that A is H-expansive, i.e., $A^*HA - H \geq 0$, and consider $B(t) = (I + tUE^*U^*H)A$, where E satisfies (1). The first result of this section gives conditions on U and t under which $B(t)$ is also H-expansive.

Proposition 5 *Let A be H-expansive, and let $B(t) = (I + tUE^*U^*H)A$, where E satisfies* (1). *Then $B(t)$ is also H-expansive in one of the following two cases:*

*(*a.*) U^*HU is positive semidefinite and either $t \leq 0$ or $t \geq 1$,*
*(*b.*) U^*HU is negative semidefinite and $0 \leq t \leq 1$.*

Compare Proposition 1.2 in [5], which presents a similar result for H-positive real matrices.

Proof Compute $B(t)^*HB(t)$. Using (3) and (4), we have

$$\begin{aligned} B(t)^*HB(t) &= (A^* + tA^*HUEU^*)H(A + tUE^*U^*HA) \\ &= A^*HA + t\left[A^*HUEU^*HA + A^*HUE^*U^*HA\right] \\ &\quad +t^2A^*HUEU^*HUE^*U^*HA \\ &= A^*HA + t\left[A^*HUEU^*HA + A^*HUE^*U^*HA\right] \\ &\quad -t^2A^*HU(E + E^*)U^*HA \\ &= A^*HA + (t - t^2)A^*HU(E + E^*)U^*HA. \end{aligned} \tag{7}$$

Hence

$$B(t)^*HB(t) - H = A^*HA - H + (t - t^2)A^*HU(E + E^*)U^*HA.$$

In particular, since A is H-expansive

$$B(t)^*HB(t) - H \geq (t - t^2)A^*HU(E + E^*)U^*HA. \tag{8}$$

Recalling that by (1) the signatures of U^*HU and $-E-E^*$ coincide, the proposition follows from (8). □

Now let us assume that U^*HU is invertible. As observed earlier, the latter condition is satisfied generically. We shall investigate whether or not $B(t)$ can have an eigenvalue on the unit circle different from eigenvalues of A.

Proposition 6 *Let A be an $n \times n$ H-expansive matrix, and let U be an $n \times k$ matrix such that U^*HU is positive definite or negative definite. Let E be a $k \times k$ invertible matrix such that (1) is satisfied, and let $B(t) = (I + tUE^*U^*H)A$. Then the following hold.*

(a.) *In case U^*HU is positive definite $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when $0 < t \leq 1$.*
(b.) *In case U^*HU is negative definite $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when either $t < 0$ or $t \geq 1$.*
(c.) *In case either one of the conditions in Proposition 5 is satisfied, $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when $t = 1$.*

Note that these conditions are just the opposite of the conditions in Proposition 5. The choice of an invertible E for which (1) holds was discussed in Remark 1.

Proof Assume that $B(t)$ has an eigenvalue on the unit circle which is not already an eigenvalue of A. That is, suppose $B(t)x = \lambda x$ for some $\lambda \in \mathbb{T} \setminus \sigma(A)$ and some nonzero vector x. Then

$$\langle Hx, x\rangle = |\lambda|^2 \langle Hx, x\rangle = \langle HB(t)x, B(t)x\rangle,$$

so by (8) we have

$$0 = \langle HB(t)x, B(t)x\rangle - \langle Hx, x\rangle \geq \langle (t - t^2)A^*HU(E + E^*)U^*HAx, x\rangle. \tag{9}$$

To prove part a, we argue as follows. From the assumption that U^*HU is positive definite (i.e., $E + E^*$ is negative definite), we have from (8) that either $t - t^2 \geq 0$ or $U^*HAx = 0$. But, if $U^*HAx = 0$, we see that $B(t)x = Ax = \lambda x$, and so $\lambda \in \sigma(A)$, a contradiction, leaving us with the only option that $B(t)$ can only have an eigenvalue in the set $\mathbb{T} \setminus \sigma(A)$ when $0 < t \leq 1$. Similarly, part b follows.

To prove part c, if either one of the conditions in Proposition 5 is satisfied, the right hand side of (9) is also greater than or equal to zero. In particular, it follows that equality must hold throughout. Hence

$$(t - t^2)\langle (E + E^*)U^*HAx, U^*HAx\rangle = 0.$$

Since by assumption $E + E^*$ is semidefinite this implies that either $t = 0$, $t = 1$ or $(E + E^*)U^*HAx = 0$. Also by assumption U^*HU is invertible and so $E + E^*$ is invertible. Hence $U^*HAx = 0$. Now if either $t = 0$ or $U^*HAx = 0$, then $B(t)x = (I + tUE^*U^*H)Ax = Ax$, and so $Ax = \lambda x$. As the assumption is that λ

is not an eigenvalue of A, this can not happen. So, that leaves only one possibility for $B(t)$ to have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$, namely $t = 1$. □

This result may be seen as an analogue of Theorem 2.1 in [5].

Remark When we take the special choice $E = -2(U^*HU)^{-1}$, then $B(t) = (I - 2tU(U^*HU)^{-1}U^*H)A$ and in particular $B(\frac{1}{2}) = (I - U(U^*HU)^{-1}U^*H)A$. We then have $B(\frac{1}{2})$ always has zero as an eigenvalue; equivalently, $\det B(\frac{1}{2}) = 0$. Indeed,

$$\begin{aligned}\det B(\frac{1}{2}) &= \det(I_n - U(U^*HU)^{-1}U^*H)\det A \\ &= \det(I_k - (U^*HU)^{-1}U^*HU)\det A = 0.\end{aligned}$$

5 Small Rank Perturbations of H-Unitary Matrices

In this section we will specialize the results of the previous section to the case where A is H-unitary. For this case, (8) becomes an equality, i.e.,

$$B(t)^*HB(t) - H = (t - t^2)A^*HU(E + E^*)U^*HA.$$

Proposition 5 can be sharpened to the following statement.

Proposition 7 *Let A be H-unitary, and let $B(t) = (I + tUE^*U^*H)A$, where E satisfies (1). Then $B(t)$ is H-expansive if and only if:*

(a.) U^*HU is positive semidefinite and either $t \le 0$ or $t \ge 1$,
(b.) U^*HU is negative semidefinite and $0 \le t \le 1$.

Also, $B(t)$ is H-contractive if and only if:

(c.) U^*HU is positive semidefinite and $0 \le t \le 1$,
(d.) U^*HU is negative semidefinite and either $t \le 0$ or $t \ge 1$.

In all other cases $B(t)^*HB(t) - H$ is indefinite.

Proposition 6 for the H-unitary case now becomes:

Proposition 8 *Let A be an $n \times n$ H-unitary matrix, and let U be an $n \times k$ matrix such that U^*HU is positive definite or negative definite. Let E be a $k \times k$ invertible matrix such that (1) is satisfied, and let $B(t) = (I + tUE^*U^*H)A$. Then $B(t)$ can only have an eigenvalue in $\mathbb{T} \setminus \sigma(A)$ when $t = 1$.*

Proof Indeed, the inequality (9) now becomes an equation because A is H-unitary. So, if $B(t)x = \lambda x$ for some $\lambda \in \mathbb{T} \setminus \sigma(A)$, the arguments from the proof of Proposition 6 can be followed to see that this can only happen when $t = 1$. □

Example 1 In the following example we will consider a rank two perturbation. Let $A = J_3(1) \oplus J_3(1)$ and let H be given by

$$H = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 0 \\ -1 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -1 & -1/2 & 0 \end{bmatrix}.$$

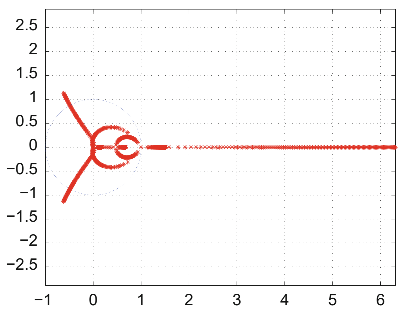
Then A is H-unitary (see [7]). Let

$$U = \begin{bmatrix} 0.5377 & 1.4090 \\ 1.8339 & 1.4172 \\ -2.2588 & 0.6715 \\ 0.8622 & -1.2075 \\ 0.3188 & 0.7172 \\ -1.3077 & 1.6302 \end{bmatrix}.$$

In this case U^*HU is positive definite. When $t = 1$ we have $B = (I - 2U(U^*HU)^{-1}U^*H)A$ with eigenvalues of B given by 5.8446, 1.4927 (outside the unit circle); 0.1711, 0.6699 (inside the unit circle) and $-0.4865 \pm 0.8737i$, which are on the unit circle.

When we consider $B(t) = (I - 2tU(U^*HU)^{-1}U^*H)A$, we have that $B(t)$ is H-contractive for $0 < t < 1$ ($B(t)^*HB(t) - H \leq 0$) and H-expansive for $t > 1$. The fact that H has four positive and two negative eigenvalues means that $B(t)$ has four eigenvalues inside the unit circle and two outside the unit circle for $0 < t < 1$ and four eigenvalues outside the unit circle and two inside the unit circle for $t > 1$.

The next figure is a plot of the eigenvalues of $B(t)$ for t running from 0 to 1.2. At $t = 1$ two eigenvalues will be on the unit circle, and will move from the inside to the outside in this example.



Example 2 In this example let $A = J_3(1) \oplus J_3(1)$ and let H be given by

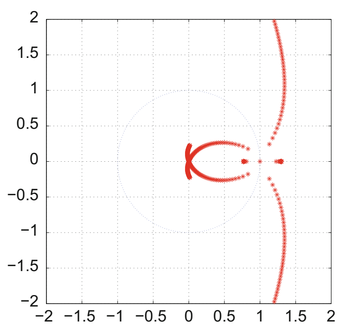
$$H = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 0 \\ -1 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \end{bmatrix}.$$

Again A is H-unitary (see [7]). Let

$$U = \begin{bmatrix} -1.0891 & 1.4090 \\ 0.0326 & 1.4172 \\ 0.5525 & 0.6715 \\ 1.1006 & -1.2075 \\ 1.5442 & 0.7172 \\ 0.0859 & 1.6302 \end{bmatrix}.$$

Now U^*HU is negative definite. The eigenvalues of $B = (I - 2U(U^*HU)^{-1} U^*H)A$ are given by $-0.0240 \pm 5.1107i$ and 1.3004 (outside the unit circle) and $-0.0009 \pm 0.1957i$ and 0.7690 (inside the unit circle).

This is also shown by the following figure, where eigenvalues of $B(t)$ are plotted for $t \in [0, 1.2]$. Originally, we start for $t = 0$ with six eigenvalues at 1 on the unit circle, while for $t > 0$ three eigenvalues move inside the unit disc, and three eigenvalues move outside the unit disc. This shows that it may happen that $B(1)$ does not have eigenvalues on the unit circle.



The three eigenvalues inside the unit disc and three outside the unit disc for all t fits with the fact that H has three positive and three negative eigenvalues, so the signature of H is zero.

Remark Suppose that the signature of H (i.e., the difference between the number of positive and negative eigenvalues of H) is non-zero. Also, let A be H-unitary, and let U^*HU be either positive definite or negative definite. Then the eigenvalues of $B(t)$ must cross the unit circle at $t = 1$. To see this, suppose first that $B(t)$ is contractive for $0 < t < 1$ and expansive for $t > 1$. Then the number of eigenvalues of $B(t)$ inside the unit disc is equal to the number of positive eigenvalues of H for $0 < t < 1$, and for $t > 1$ it is equal to the number of negative eigenvalues of H. So at $t = 1$ this number must change, and hence $B(1)$ must have at least one eigenvalue on the unit circle. A similar argument works when $B(t)$ is expansive for $0 < t < 1$ and contractive for $t > 1$.

The previous example serves to show that the condition that the signature of H is non-zero is necessary. □

The next proposition discusses the situation whenever there are eigenvalues crossing the unit circle as t increases through $t = 1$.

Proposition 9 *Let A be H-unitary, and let $B(t) = (I + tUE^*U^*H)A$, where E satisfies* (1)*. Suppose that for $t = 1$, the perturbed matrix $B(t)$ has an eigenvalue λ on the unit circle which is not an eigenvalue of A. Viewing λ as a function of t, we have the following two possibilities:*

*(*a.*) λ crosses the unit circle from outside to inside if either $E + E^* > 0$ and $\langle Hx, x\rangle > 0$ or $E + E^* < 0$ and $\langle Hx, x\rangle < 0$,*
*(*b.*) λ crosses the unit circle from inside to outside if either $E + E^* > 0$ and $\langle Hx, x\rangle < 0$ or $E + E^* < 0$ and $\langle Hx, x\rangle > 0$.*

Proof Suppose that $B(t)x = \lambda x$, using (7) we have

$$\begin{aligned}\langle HB(t)x, B(t)x\rangle &= \langle H(\lambda x), \lambda x\rangle = |\lambda|^2\langle Hx, x\rangle \\ &= \langle HAx, Ax\rangle + (t - t^2)\langle (E + E^*)U^*HAx, U^*HAx\rangle.\end{aligned}$$

This implies

$$(|\lambda|^2 - 1)\langle Hx, x\rangle = \langle (A^*HA - H)x, x\rangle + (t - t^2)\langle (E + E^*)U^*HAx, U^*HAx\rangle$$

or equivalently

$$\begin{aligned}&(t - t^2)\langle (E + E^*)U^*HAx, U^*HAx\rangle + (1 - |\lambda|^2)\langle Hx, x\rangle \\ &= -\langle (A^*HA - H)x, x\rangle. \qquad (10)\end{aligned}$$

Hence, if $\langle Hx, x\rangle \neq 0$,

$$|\lambda|^2 = 1 + \frac{\langle (A^*HA - H)x, x\rangle}{\langle Hx, x\rangle} + \frac{(t - t^2)}{\langle Hx, x\rangle}\langle (E + E^*)U^*HAx, U^*HAx\rangle. \qquad (11)$$

From (10) we see that when A is H-expansive, an eigenvalue on the unit circle can only occur when

$$(t - t^2)\langle (E + E^*)U^* HAx, U^* HAx\rangle \leq 0.$$

This happens when either $(E + E^*) \geq 0$ and $t - t^2 \leq 0$ or when $(E + E^*) \leq 0$ and $t - t^2 \geq 0$.

However, if A is H-unitary there is more we can say. Equation (10) simplifies to

$$(t - t^2)\langle (E + E^*)U^* HAx, U^* HAx\rangle + (1 - |\lambda|^2)\langle Hx, x\rangle = 0 \qquad (12)$$

so that $|\lambda| = 1$ can only happen when $t = 0$ or $t = 1$, provided that $U^* HAx \neq 0$.

Focusing on $t = 1$, we can view $|\lambda|^2$ as a function of t, and use (11) to determine $\frac{d|\lambda|^2}{dt}$ when $t = 1$ by direct calculation (or the implicit function theorem). Thus from (11) we have

$$\frac{d|\lambda|^2}{dt} = \frac{(1 - 2t)}{\langle Hx, x\rangle}\langle (E + E^*)U^* HAx, U^* HAx\rangle.$$

At $t = 1$ we have

$$\frac{d|\lambda|^2}{dt}|_{t=1} = \frac{(-1)}{\langle Hx, x\rangle}\langle (E + E^*)U^* HAx, U^* HAx\rangle.$$

Note that $B(1)$ is H-unitary, and if λ is an eigenvalue on the unit circle of multiplicity one, then $\langle Hx, x\rangle$ is the sign in the sign characteristic of the pair $(B(1), H)$ corresponding to the eigenvalue λ. There are four cases depending on the signs of $(E + E^*)$ and $\langle Hx, x\rangle$:

1. $(E + E^*) > 0$ and $x^* Hx > 0$: λ crosses the unit circle from outside to inside,
2. $(E + E^*) > 0$ and $x^* Hx < 0$: λ crosses the unit circle from inside to outside,
3. $(E + E^*) < 0$ and $x^* Hx > 0$: λ crosses the unit circle from inside to outside,
4. $(E + E^*) < 0$ and $x^* Hx < 0$: λ crosses the unit circle from outside to inside.

This is precisely the statement of the proposition. □

For eigenvalues that are not on the unit circle, (and hence for $t \neq 0, 1$), the sign of $\langle Hx, x\rangle$ is connected to whether $|\lambda| > 1$ or $|\lambda| < 1$ in the following way: there are eight cases to consider, depending on the sign of $(E + E^*)$, the value of t and the location of λ. The crucial point is whether $B(t)$ is H-expansive or $B(t)$ is H-contractive.

We summarize the conclusions of this section in the following table.

$(E + E^*)$	t	A	$B(t)$	$\langle Hx, x\rangle$	Location λ
Pos. def.	$t < 0$ or $t > 1$	Unitary	Contractive	≤ 0	Outside
Pos. def.	$t < 0$ or $t > 1$	Unitary	Contractive	≥ 0	Inside
Pos. def.	$0 < t < 1$	Expansive	Expansive	≥ 0	Outside
Pos. def.	$0 < t < 1$	Unitary	Expansive	≤ 0	Inside
Neg. def.	$t < 0$ or $t > 1$	Expansive	Expansive	≥ 0	Outside
Neg. def.	$t < 0$ or $t > 1$	Unitary	Expansive	≤ 0	Inside
Neg. def.	$0 < t < 1$	Unitary	Contractive	≤ 0	Outside
Neg. def.	$0 < t < 1$	Unitary	Contractive	≥ 0	Inside
Pos. def.	$t = 1$	Unitary	Unitary	≤ 0	On
Neg. def.	$t = 1$	Unitary	Unitary	≥ 0	On

6 Examples of Rank One Perturbations of H-Expansive Matrices

In [8] rank one perturbations of H-unitary and H-orthogonal matrices were discussed. In particular, if A is an $n \times n$ H-unitary matrix, and $u \in \mathbb{R}^n$, then

$$B = A - \frac{2}{u^T H u} u u^T H A$$

is H-orthogonal. The following surprising result was shown: suppose $1 \in \sigma(A)$, and suppose that the size p of the largest Jordan block with eigenvalue 1 is even. Then such a block has a "twin" in the canonical form of the pair (A, H); i.e., even size blocks are coupled. For generic vectors u, the corresponding matrix B then has a Jordan block with eigenvalue one of size $p + 1$ (i.e., the largest Jordan block gets larger, contrary to the unstructured case), and if -1 is not already in $\sigma(A)$, then -1 is an eigenvalue of B. The condition that -1 is not already an eigenvalue is necessary, as can be seen by considering what happens when $A = J_2(1) \oplus J_2(1) \oplus J_2(-1) \oplus J_2(-1)$.

A similar result holds for symplectic matrices, but without the new eigenvalue appearing in -1.

A series of examples will complete this article. These examples are intended to illustrate the different possibilities that may occur for coupled and uncoupled Jordan blocks with eigenvalue 1 of different block sizes for H-expansive matrices A. We will focus on rank one perturbations.

In the first example we look at what the perturbation does with coupled and uncoupled blocks.

Example 3 Let $A = J_2(1) \oplus J_2(1) \oplus J_2(1) \oplus J_2(1)$ and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then A is H-expansive, see [6], with the first two Jordan blocks "coupled" in the way that it happens for H-unitary matrices, and the last two blocks are not coupled in this way. Choosing a real u randomly generated in Matlab (u=randn(8,1)), we see that generically B will have 3 blocks of size 2 at eigenvalue 1, and two real eigenvalues not equal to one. This points to the fact that apparently, the rank one perturbation has a preference to destroy the uncoupled block.

If we would take very specific vectors u, namely those with the last four entries zero, then we get a different picture: there are two Jordan blocks with size 2 at eigenvalue 1, one Jordan block of size 3 at eigenvalue 1, and also an eigenvalue at -1. This is because we only make the perturbation to the coupled blocks.

Example 4 Let $A = J_2(1) \oplus J_2(1) \oplus J_2(1)$ and

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Again, A is H-expansive, and we see the same kind of behavior as in Example 3. With generic u there are 4 eigenvalues at 1, and two real eigenvalues, since the uncoupled block is destroyed, and replaced by two real eigenvalues. We will show next that the product of these real eigenvalues is -1. Moreover, we shall show that in the non-generic case where $u_6 = 0$, the perturbed matrix $B = (I - \frac{2}{u^T H u} u u^T H)^{-1} A$ has five eigenvalues 1 and one eigenvalue equal to -1; compare the case where A is H-orthogonal.

Take $u = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}^T$. We assume that the vector $\tilde{u} = \begin{bmatrix} u_2 & u_4 & u_6 \end{bmatrix}^T$ is not the zero vector.

First we will consider the null space of $B - I$. For this we observe that a direct computation shows that

$$(I - \frac{2}{u^T H u} u u^T H)^{-1} = I - \frac{2}{u^T H u} u u^T H.$$

If $Bx = x$, then $(I - \frac{2}{u^T H u} u u^T H) A x = x$, if and only if $Ax = (I - \frac{2}{u^T H u} u u^T H) x$. This is equivalent to $(A - I)x = -\frac{2}{u^T H u} u u^T H x$. However, the range of $A - I$ is the span of the set $\{e_1, e_3, e_5\}$. Since $\tilde{u} \neq 0$ this means two things: $(A - I)x = 0$ and

$u^T Hx = 0$. From $(A - I)x = 0$ we have $x_2 = x_4 = x_6 = 0$ and then $u^T Hx = 0$ is equivalent to

$$u_4 x_1 - u_2 x_3 + u_6 x_5 = 0,$$

which implies that $\dim\ker(B - I) = 2$.

Next, for a vector $x \in \ker(B-I)$ we consider the equation $(B-I)y = x$, in other words, we aim to construct a Jordan chain of length (at least) two corresponding to eigenvalue 1 of B starting from a given eigenvector x. Note that if this is possible for any eigenvector x, then 1 is an eigenvalue of B with multiplicity at least 4, and there are at least two Jordan chains of length at least two in a Jordan basis. Indeed, $(B - I)y = x$ if and only if $By = x + y$, so $Ay = (I - \frac{2}{u^T Hu} uu^T H)(x + y)$. Using that $u^T Hx = 0$, means $Ay = x+(I-\frac{2}{u^T Hu} uu^T H)y = x+y-\frac{2}{u^T Hu} uu^T Hy$. Hence $(A-I)y = x - \frac{2}{u^T Hu} uu^T Hy$. Again, since the range of $(A-I) = \operatorname{span}\{e_1, e_3, e_5\}$, together with the fact that also x is in the span of these vectors, means two things: $u^T Hy = 0$ and $(A - I)y = x$. So $y = \begin{bmatrix} y_1 & x_1 & y_3 & x_3 & y_5 & x_5 \end{bmatrix}^T$ and it should satisfy

$$u^T Hy = \begin{bmatrix} u_4 & -u_3 & -u_2 & u_1 & u_6 & u_5 \end{bmatrix} \begin{bmatrix} y_1 \\ x_1 \\ y_3 \\ x_3 \\ y_5 \\ x_5 \end{bmatrix} = 0.$$

The latter equation is equivalent to

$$\begin{bmatrix} u_4 & -u_2 & u_6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_3 \\ y_5 \end{bmatrix} = \begin{bmatrix} u_3 & -u_1 & -u_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix},$$

which is solvable for any given x in the null space of $B - I$, because $\tilde{u} \neq 0$.

To finish the analysis, we consider the characteristic polynomial of B:

$$\begin{aligned}
\det(\lambda I - B) &= \det(\lambda I - A + \frac{2}{u^T Hu} uu^T HA) \\
&= \det\left((\lambda I - A)(I + \frac{2}{u^T Hu}(\lambda I - A)^{-1} uu^T HA\right) \\
&= \det(\lambda I - A)\det(I + \frac{2}{u^T Hu}(\lambda I - A)^{-1} uu^T HA) \\
&= \det(\lambda I - A)(1 + \frac{2}{u^T Hu} u^T HA(\lambda I - A)^{-1} u)
\end{aligned}$$

$$
\begin{aligned}
&= \det(\lambda I - A)\frac{2}{u^T H u}(\frac{1}{2}u^T H u + u^T H A(\lambda I - A)^{-1}u) \\
&= \det(\lambda I - A)\frac{2}{u^T H u}u^T H(\frac{1}{2}I + A(\lambda I - A)^{-1})u \\
&= \det(\lambda I - A)\frac{2}{u^T H u}u^T H(\frac{1}{2}(\lambda I + A)(\lambda I - A)^{-1})u \\
&= \det(\lambda I - A)\frac{1}{u^T H u}u^T H(\lambda I + A)(\lambda I - A)^{-1}u \\
&= \frac{\det(\lambda I - A)}{m_A(\lambda)}\frac{1}{u^T H u}u^T H m_A(\lambda)(\lambda I + A)(\lambda I - A)^{-1}u.
\end{aligned}
$$

The zeroes of the polynomial

$$
p_B(\lambda) := \frac{1}{u^T H u}u^T H m_A(\lambda)(\lambda I + A)(\lambda I - A)^{-1}u \tag{13}
$$

are the eigenvalues of B which are not eigenvalues of A.

We compute this polynomial in general: denote $m_A(\lambda) = \sum_{k=0}^{\ell} m_k \lambda^k$ (since $m_A(\lambda)$ is monic we have $m_\ell = 1$, but for simplicity of the formulas it is handier to keep it as m_ℓ). Also, for $|\lambda| > \|A\|$ we have

$$
(\lambda I + A)(\lambda I - A)^{-1} = -I + 2\lambda(\lambda I - A)^{-1} = -I + 2\sum_{j=0}^{\infty}\frac{A^j}{\lambda^j} = I + 2\sum_{j=1}^{\infty}\frac{A^j}{\lambda^j}.
$$

Using that we know that $m_A(\lambda)(\lambda I - A)^{-1}$ is a polynomial in λ, we obtain

$$
\begin{aligned}
m_A(\lambda)(\lambda I + A)(\lambda I - A)^{-1} &= \sum_{k=0}^{\ell} m_k \lambda^k \cdot \left(I + 2\sum_{j=1}^{\infty}\frac{A^j}{\lambda^j} \right) \\
=&(m_\ell \lambda^\ell + m_{\ell-1}\lambda^{\ell-1} + \cdots + m_1\lambda + m_0)\left(I + \frac{2A}{\lambda} + \frac{2A^2}{\lambda^2} + \cdots \right) \\
=&\lambda^\ell m_\ell + \lambda^{\ell-1}(m_{\ell-1} + 2Am_\ell) + \lambda^{\ell-2}(m_{\ell-2} + 2Am_{\ell-1} + 2A^2 m_\ell) \\
&+ \lambda^{\ell-3}(m_{\ell-3} + 2Am_{\ell-2} + 2A^2 m_{\ell-1} + 2A^3 m_\ell) + \cdots \\
=&\lambda^\ell m_\ell + \sum_{j=0}^{\ell-1}\lambda^j(m_j + 2Am_{j+1} + 2A^2 m_{j+2} + \cdots + 2A^{\ell-j}m_\ell) \\
=&\lambda^\ell m_\ell + \sum_{j=0}^{\ell-1}\lambda^j\left(m_j + 2\sum_{\nu=1}^{\ell-j}A^\nu m_{j+\nu} \right).
\end{aligned}
$$

Then the polynomial (13) becomes

$$p_B(\lambda) = \lambda^\ell m_\ell + \sum_{j=0}^{\ell-1} \lambda^j \left(m_j + 2\sum_{\nu=1}^{\ell-j} \frac{u^T HA^\nu u}{u^T Hu} m_{j+\nu} \right).$$

For this computation, compare [10, 11].

Now, let us return to the example. In that case $m_A(\lambda) = (\lambda - 1)^2$ so $m_2 = 1, m_1 = -2, m_0 = 1$ implies

$$p_B(\lambda) = \lambda^2 + \lambda \left(-2 + 2\frac{u^T HAu}{u^T Hu} \right) + \left(1 + 2\frac{u^T HA^2 u}{u^T Hu} - 4\frac{u^T HAu}{u^T Hu} \right).$$

Write $A = I + N$, then $A^2 = I + 2N$, and inserting this into the last expression we obtain

$$\begin{aligned} p_B(\lambda) &= \lambda^2 + \lambda \left(2\frac{u^T HNu}{u^T Hu} \right) + \left(-1 + 2\frac{u^T H2Nu}{u^T Hu} - 4\frac{u^T HNu}{u^T Hu} \right) \\ &= \lambda^2 + \lambda \left(2\frac{u^T HNu}{u^T Hu} \right) - 1. \end{aligned}$$

As a result in this example there are two eigenvalues with product equal to -1. Note that

$$u^T HNu = u_6^2, \qquad u^T Hu = 2(u_1u_4 - u_2u_3 + u_5u_6).$$

If $u_6 = 0$, then $p_B(\lambda) = \lambda^2 - 1$ and in this (non-generic) case B has a quintuple eigenvalue at 1, which, under the assumption that $\tilde{u} \neq 0$ must come from a Jordan block of size two and a Jordan block of size three, and a single eigenvalue at -1. In case $u_6 \neq 0$ then B has a quadruple eigenvalue at 1 corresponding to two Jordan blocks of size two (as we discussed above) and a pair of real eigenvalues whose product is -1.

Example 5 We will now look into conditions for the general case, where the matrix A consists of two even sized Jordan blocks at eigenvalue 1. Therefore, let $A = J_l(1) \oplus J_l(1) = (I_l + N_l) \oplus (I_l + N_l) = I + N$, where N_l is a $l \times l$ nilpotent matrix with ones on the superdiagonal. Take note that $m_A(\lambda) = (\lambda - 1)^l$. Let H_0 be such that A is H_0-unitary with the matrices in the pair (A, H_0) in canonical form, see [7]. We consider the case where A is H-expansive and write H as $H = H_1 + H_0$ with H_1 such that A has no H-unitary part (see [9]). Then $A^T HA - H = A^T H_1 A - H_1 \geq 0$, since $A^T H_0 A - H_0 = 0$. As a reminder, the canonical form for H_0 is

$$H_0 = \begin{bmatrix} 0 & 0 & 0 & Q_l \\ 0 & 0 & -Q_l^T & 0 \\ 0 & -Q_l & 0 & 0 \\ Q_l^T & 0 & 0 & 0 \end{bmatrix},$$

where $Q_l = (q_{ij})_{i=1,j=1}^{\frac{l}{2},\frac{l}{2}}$ is a $\frac{l}{2} \times \frac{l}{2}$ matrix whose entries are as follows:

$$q_{ij} = 0 \quad \text{when} \quad i + j \leq \frac{l}{2},$$
$$q_{ij} = (-1)^{j-1} \binom{j-1}{\frac{l}{2} - i} \quad \text{when } i + j \geq \frac{l}{2} + 1.$$

Using the above, we can now write equation (13) with the restriction that $u^T H u \neq 0$ as

$$\begin{aligned} p_B(\lambda) &= \frac{1}{u^T H u} u^T H m_A(\lambda)(\lambda I + A)(\lambda I - A)^{-1} u \\ &= \frac{1}{u^T H u} u^T H (\lambda - 1)^l ((\lambda + 1)I + N)((\lambda - 1)I - N)^{-1} u \\ &= \frac{1}{u^T H u} u^T H ((\lambda + 1)I + N) \cdot \sum_{j=0}^{l-1} (\lambda - 1)^{l-j-1} N^j u. \end{aligned}$$

We want to know under what conditions B will have an eigenvalue 1 or -1. In case $H = H_0$ there will be an "extra" eigenvalue 1, that is, the Jordan block at eigenvalue 1 of B will have size $l + 1$. Also in this case, the matrix B will have an eigenvalue -1, see [8]. Thus, we are interested in whether or not $p_B(1) = 0$ and similarly whether or not $p_B(-1) = 0$. From the above we have

$$\begin{aligned} p_B(1) &= \frac{1}{u^T H u} u^T H (2I + N) N^{l-1} u = \frac{2}{u^T H u} u^T H N^{l-1} u \\ &= \frac{2}{u^T H u} u^T (H_1 + H_0) N^{l-1} u = \frac{2}{u^T H u} u^T H_1 N^{l-1} u. \end{aligned}$$

Here, the last equality follows from the fact that A is H_0-orthogonal, and so the term involving H_0 is zero, as is shown in [8]. Thus, $p_B(1) = 0$ if and only if $u^T H_1 N^{l-1} u = 0$. This is an algebraic condition for the entries of u. Using a similar computation, the expression for $p_B(-1)$ results in

$$p_B(-1) = \frac{1}{u^T H u} u^T H_1 \cdot \sum_{j=0}^{l-1} (-2)^{l-j-1} N^{j+1} u.$$

Hence, $p_B(-1) = 0$ is again an algebraic condition on the entries of u. It follows that for $H_1 \neq 0$ such that A has no H-unitary part, the set of u's for which both 1 and -1 are not an eigenvalue of B is a generic set.

Acknowledgments The work of the third author is based on research supported in part by the National Research Foundation of South Africa (Grant Number 145688).

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