Chapter 1

Introduction

1.1 Geometrical optics and physical optics

Optics is the field of science concerned with the behavior and properties of light. Traditionally, optics is divided into two main branches: geometrical optics and physical optics. Geometrical optics describes light as rectilinear rays. These rays can be reflected and refracted at the interface between two media. Geometrical optics is governed by the eikonal equation. Physical optics describes light as a wave phenomenon. These waves can interfere with each other, and they can be diffracted by obstacles. The central formula in this approach is the wave equation. These two theories are not unrelated, in fact, geometrical optics can be regarded as an asymptotic limit of physical optics as the wavenumber \( k = \frac{2\pi}{\lambda} \) (\( \lambda \) denoting the wavelength) tends to infinity [BORN AND WOLF, 1999, Sec. 3.1]. Physical optics can be further subdivided into two branches. In the vector theory, which is based on Maxwell’s equations, the full electromagnetic field is analyzed. In the scalar theory a much more simplified picture is used, and field properties such as polarization are ignored.

In this thesis the methods of physical optics are used to analyze the phase behavior of wave fields under different circumstances.
1.2 The Gouy phase

More than 120 years ago, L.G. Gouy (see Fig. 1.1) discovered an anomalous phase behavior in a converging, diffracted spherical wave as it passes through its focus [GOUY, 1890; GOUY, 1891]. He wrote (translated from French):

“If one considers a converging wave that has passed through a focus and has then become divergent, a simple calculation shows that the vibration of that wave has advanced half a period compared to what it should be according to the distance travelled and the speed of light.”

Figure 1.1: Louis Georges Gouy (1854-1926), around the time of his discovery of the phase anomaly that now bears his name.

Gouy confirmed his theoretical analysis by an interferometric experiment. Letting the light from a point source impinge onto two mirrors,
one concave, the other plane, two beams were generated. The mirrors were positioned so that the beams were nearly parallel to each other. In any transverse plane of observation their superposition yielded a circular interference pattern, with ring-shaped fringes. The central disk was found to change from dark to bright, or vice versa, when the observation plane was moved through the focus of the converging beam. This transition confirmed the predicted $180^\circ$ phase change. Since Gouy’s original work many additional observations have been reported [Farnell, 1958; Mertz, 1959; Ruffin et al., 1999; McGowan et al., 2000; Feurer et al., 2002; Chow et al., 2004; Klaassen et al., 2004; Lamouche et al., 2004; Lindner et al., 2004; Steuernagel et al., 2005; Zhu et al., 2007; Kandpal et al., 2007; Rolland et al., 2010].

However, the origin of the phase anomaly continues to be a matter of debate, with different authors attributing it to widely differing causes. One of the earliest treatments of the Gouy phase was given by Walker [Walker, 1904], who used the principle of stationary phase to demonstrate that when a ray associated with an astigmatic wavefront passes through the two centers of curvature, there is a phase discontinuity of an amount of $\pi/2$ at each of them, in agreement with Gouy’s prediction. The first three-dimensional analysis of the phase behavior in the focal region is due to Linfoot and Wolf [Linfoot and Wolf, 1956] who examined the phase anomaly along different rays through the geometrical focus.

Boyd [Boyd, 1980] has attributed the Gouy phase to the diffraction properties of Gaussian beams. But the phase anomaly has also been associated with Berry’s phase, which is an additional geometric (or topological) phase acquired by a system after a cyclic adiabatic evolution in parameter space [Simon and Mukunda, 1993; Subbarao, 1995]. There is also an explanation based on Heisenberg’s uncertainty relations [Hariharan and Robinson, 1996; Feng and Winful, 2001], in which the lateral confinement of the field near the focus is accompanied by an increase in momentum in the longitudinal direction. The tilted wave interpretation is yet another way to explain the Gouy phase shift [Zhan, 2004a; Chen et al., 2007]. There it is related to the averaged phase retardation of the tilted plane-wave components of a Gaussian beam.

A recent paper showed that the phase anomaly can be considered as
1.2. The Gouy phase

A degenerate case of a rapid $\pi/2$ phase change that occurs at each focal line of an astigmatic pencil of rays [Visser and Wolf, 2010]. In this paper, it was pointed out that the phase anomaly near focus can be understood by considering a wave of a more general form, namely a converging wave exhibiting astigmatism. As is well-known, a geometrical optics analysis of this situation shows that the wavefront of such a field has, at each point, two principal radii of curvature and two, mutually orthogonal, focal lines [Born and Wolf, 1999, Sec. 4.6]. Geometrical optics may be regarded as the asymptotic limit of physical optics as the wavenumber $k = 2\pi/\lambda$ tends to infinity. With the help of the method of stationary phase it can be shown that in this limit the field exhibits a phase discontinuity of an amount $\pi/2$ at each focal line [Van Kampen, 1949; Stamnes, 1986]. Geometrical optics is governed by the eikonal equation, the actual wave field however, satisfies the Helmholtz equation. The solutions of the latter are well known to be continuous. Hence, according to physical optics, the two phase discontinuities have to be “smoothed out”, and become continuous but rapid phase changes. When the astigmatic wave aberration decreases to zero, i.e., when the field in the aperture becomes a converging spherical wave, the two foci coincide and the sharp phase change in the focal region is the Gouy phase change of an amount $\pi$. In this way, the phase anomaly can be understood from elementary properties of rays and from the relation between geometrical optics and physical optics.

In higher-order laser modes the Gouy phase has a more complicated behavior than in the converging spherical waves discussed so far. For a Hermite-Gaussian mode with indices $(m, n)$ it has the value $(m + n + 1)\pi$, and for a Laguerre-Gaussian mode with indices $(p, l)$ it takes on the value $(2p + l + 1)\pi$ [Siegman, 1986].

The Gouy phase is of great importance because it plays a role in so many physical systems and applications. In curved-mirror laser cavities, it determines the resonance frequencies of different transverse modes [Siegman, 1986]. For such modes, the Gouy phase also can supply quantitative information about the optical aberrations in cavities [Klaassen et al., 2004]. Utilizing the Gouy phase, one can transform a Hermite-Gaussian mode into a Laguerre-Gaussian mode and vice versa [Allen et al., 1992; Beijersbergen et al., 1993]. In nonlinear optics, the Gouy phase influences
the efficiency of higher-order harmonics generation [Boyd, 1992; Lindner et al., 2003]. It has also been used in the creation of so-called bottle beams [Arlt and Padgett, 2000] and in optical coherence tomography [Lamouche et al., 2004]. In singular optics, the Gouy phase affects the propagation of optical vortices [Hamazaki et al., 2006; Baumann et al., 2009]. In addition, the Gouy phase can be used in the interferometry of a single nanoparticle [Hwang and Moerner, 2007] and in the application of Terahertz time-domain spectroscopy [Federici et al., 2006]. In chemical reactions, the Gouy phase can be used to control the branching ratio for products formed at different total energies [Barge et al., 2006; Gordon and Barge, 2007; Barge et al., 2008]. The Gouy phase is not limited to electromagnetic waves but has also been found in acoustic fields [Holme et al., 2003; Kolomenskii et al., 2005]. Very recently, it has even been observed in matter waves [Guzzinati et al., 2013].

Although the term Gouy phase is traditionally reserved for focused wave fields, recently its meaning has been extended to apply to beam-like fields as well. In [Martelli et al., 2010] it is used to characterize the phase of a non-diffracting Bessel beam by comparing it to that of a plane wave with the same frequency.

In the next two sections we briefly review some concepts that will be used throughout this thesis.

1.3 Singular optics

Singular optics [Nye and Berry, 1974; Nye, 1999; Soskin and Vassnetsov, 2001; Karmen et al., 1997; Berry, 1998; Nye, 1998; Schouten et al., 2003; Schoonover and Visser, 2006; Dennis et al., 2009] is a branch of wave analysis concerned with the presence of singular structures in a wavefield and the topology of the wavefield around those structures. The most common singular structure is a phase singularity. Consider a complex monochromatic scalar field \( U(r,t) \) of frequency \( \omega \) which can be written as

\[
U(r,t) = A(r)e^{i\psi(r)}e^{i\omega t},
\]  

(1.1)

Here \( r \) denotes a position, and \( t \) a moment in time. A phase singularity occurs at points where the amplitude \( A(r) \) vanishes and the phase \( \psi(r) \) therefore is undefined or singular. The two key concepts of singular optics
are the *topological charge* and the *topological index* of the features. The topological charge $s$ of a phase singularity is defined as

$$s \equiv \frac{1}{2\pi} \oint_C \nabla \psi(r) \cdot dr,$$

where the path $C$ encloses the phase singularity and is traversed in a counter-clockwise direction. The topological index is defined as the topological charge of the vector field $\nabla \psi(r)$. In this field the "phase" is the orientation angle of $\nabla \psi(r)$.

In a monochromatic electromagnetic beam, the field is completely polarized at each point in space [Born and Wolf, 1999, Sec. 1.4]. The polarization ellipse is characterized by three parameters describing its eccentricity, orientation and handedness, respectively. A polarization singularity [Berry and Dennis, 2001] occurs at a point at which the polarization ellipse is degenerate. Points where the polarization is purely circular, and hence the orientation of the ellipse is undefined, are called $C$-points. At $L$-lines, where the polarization is linear, the handedness is undefined.

If the field is partially coherent, its statistical properties in the space-frequency domain are described by the *spectral degree of coherence* [Mandel and Wolf, 1995, Sec. 4.3], see also Sec. 1.4 in this Chapter. This is a complex-valued function of two spatial variables $r_1$ and $r_2$, so at pairs of points where the spectral degree of coherence vanishes, its phase is undefined and a *coherence singularity* [Gbur and Visser, 2003] occurs. In contrast to the classical singularities that are found in two or three dimensions, coherence singularities occur in a six-dimensional space.

### 1.4 Coherence theory

In optics, coherence theory is the study of the statistical properties of light. It describes optical fields in terms of correlation functions, which can be measured through interference experiments.

Consider a random, wide-sense stationary scalar wave field $V(r, t)$, which is a member of an ensemble of realizations $\{V(r, t)\}$. The correlation properties of the field can be described by the *mutual coherence function*, which is defined as ([Mandel and Wolf, 1995], Sec.4.3.1)

$$\Gamma(r_1, r_2, \tau) = \langle V^*(r_1, t) V(r_2, t + \tau) \rangle,$$
where \( \tau \) is the time difference, the asterisk indicates the complex conjugate and the angular brackets denote an ensemble average. It is convenient to normalize the mutual coherence function by defining the complex degree of coherence as

\[
\gamma(r_1, r_2, \tau) = \frac{\Gamma(r_1, r_2, \tau)}{\sqrt{I(r_1)I(r_2)}},
\]

where

\[
I(r) = \Gamma(r, r, 0),
\]
is the averaged intensity at position \( r \). In Young’s interference experiment, the value of \( |\gamma(r_1, r_2, \tau)| \) equals the visibility of fringes that are produced when two pinholes (located at position \( r_1 \) and \( r_2 \)) are illuminated with equal intensity. When \( |\gamma(r_1, r_2, \tau)| = 1 \) the light at the two pinholes is called fully coherent, resulting in a fringe pattern with maximal sharpness. When \( |\gamma(r_1, r_2, \tau)| = 0 \) the light at the two pinholes is completely incoherent and there is no visible interference pattern. For intermediate values of \( |\gamma(r_1, r_2, \tau)| \) the light is called partially coherent.

For many applications it is advantageous to work in the space-frequency domain, where the basic quantity is the cross-spectral density function \( W(r_1, r_2, \omega) \), which is the temporal Fourier transform of the mutual coherence function, i.e.

\[
W(r_1, r_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(r_1, r_2, \tau) e^{i\omega \tau} d\tau.
\]

It can be shown that, like the mutual coherence function, the cross-spectral density function \( W(r_1, r_2, \omega) \) is also a correlation function ([Mandel and Wolf, 1995], Sec.4.7.2), that is

\[
W(r_1, r_2, \omega) = \langle U^*(r_1, \omega)U(r_2, \omega) \rangle_\omega,
\]

where \( U(r, \omega) \) is a member of an ensemble of monochromatic realizations of the field. The suffix \( \omega \) on the angular brackets is to stress that the average is taken over an ensemble of space-frequency realizations. Often it is useful to consider a normalized version of \( W \), the spectral degree of coherence, which is given by the expression

\[
\mu(r_1, r_2, \omega) = \frac{W(r_1, r_2, \omega)}{\sqrt{S(r_1, \omega)S(r_2, \omega)}},
\]
where
\[ S(r, \omega) = W(r, r, \omega), \quad (1.9) \]
is the spectral density at position \( r \). Just like the complex degree of coherence, the spectral degree of coherence can also be determined by Young’s interference experiment but now with filters in front of the pin-holes [WOLF, 1983]. It can be shown that spectral degree of coherence is bounded ([MANDEL AND WOLF, 1995], Sec.4.3.2 ) by
\[ 0 \leq |\mu(r_1, r_2, \omega)| \leq 1, \quad (1.10) \]
where 0 represents complete spatial incoherence, and 1 represents full spatial coherence.

Each of these two correlation functions obeys two precise propagation laws. The mutual coherence function in free space satisfies the two wave equations [WOLF, 1955]
\[
(\nabla_1^2 - \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2}) \Gamma(r_1, r_2, \tau) = 0, \\
(\nabla_2^2 - \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2}) \Gamma(r_1, r_2, \tau) = 0, \quad (1.11)
\]
where \( \nabla_1^2 \) and \( \nabla_2^2 \) denote the Laplace operator acting on \( r_1 \) and \( r_2 \), respectively and \( c \) is the speed of light. The cross-spectral density satisfies two Helmholtz equations, namely
\[
(\nabla_1^2 + k^2)W(r_1, r_2, \omega) = 0, \\
(\nabla_2^2 + k^2)W(r_1, r_2, \omega) = 0, \quad (1.12)
\]
where \( k = \omega/c \) is the wave number corresponding to frequency \( \omega \). The two pairs of equations above imply that these two correlation functions both have a wave-like character.

Thus far we have considered scalar fields, but the concept of correlation functions can be generalized to electromagnetic beams and forms the basis of the unified theory of coherence and polarization [WOLF, 2003a; WOLF, 2003b]. Coherence describes the correlation between fluctuations at two or more points in space. Polarization, on the other hand, is a manifestation of the correlation between fluctuating components of the electric field vector at a single point. The basic quantity of the unified theory of coherence
and polarization is the \textit{electric cross-spectral density matrix} $W(r_1, r_2, \omega)$, which is defined as

$$W(r_1, r_2, \omega) = \begin{bmatrix} W_{xx}(r_1, r_2, \omega) & W_{xy}(r_1, r_2, \omega) \\ W_{yx}(r_1, r_2, \omega) & W_{yy}(r_1, r_2, \omega) \end{bmatrix},$$

(1.13)

where

$$W_{ij}(r_1, r_2, \omega) = \langle E^*_i(r_1, \omega) E_j(r_2, \omega) \rangle, \quad (i, j = x, y).$$

(1.14)

Here $E_i(r, \omega)$ is a Cartesian component of the electric field at a point specified by a position vector $r$ at frequency $\omega$, of a typical realization of the statistical ensemble representing the beam.

The coherence properties of a beam are described only by the diagonal elements of the electric cross-spectral density matrix whereas the state of polarization depends also on the off-diagonal elements. An overview is given by Wolf [Wolf, 2007]. Recently, the role of the off-diagonal matrix elements in characterizing the state of coherence has been emphasized [Setäla \textit{et al.}, 2006].

\section*{1.5 Outline of this thesis}

Nearly all the literature dealing with the Gouy phase uses the scalar theory. In a high-aperture optical system, however, the vector nature of the field can no longer be ignored. In Chapter 2, the Gouy phases of the three Cartesian components of the electric field are examined. We show that these components exhibit different phase anomalies. It is also found that the phase of the electric field exhibits singularities in all three components.

As one kind of the recently discovered non-diffracting beams, Airy beams have attracted considerable attention. Such beams have unique properties, like their “accelerating” behavior and their capacity for “self-healing”. The latter means that they are remarkably insensitive to perturbations. In Chapter 3 the Gouy phase for idealized infinite-energy Airy beams is defined, and analytical expressions for its behavior are derived. It is shown numerically that these expressions are excellent approximations for the Gouy phase of realistic finite-energy Airy beams generated under typical conditions.
1.5. Outline of this thesis

Under many practical circumstances, light is not monochromatic, but is partially coherent, and its phase is a random quantity. When such a field is focused, the Gouy phase is therefore undefined. However, the correlation functions that characterize partially coherent fields do have a well-defined phase. In Chapter 4, partially coherent fields are examined and it is demonstrated that their correlation functions exhibit a generalized Gouy phase. In the coherent limit this generalized Gouy phase reduces to the classical Gouy phase. It is also shown that this generalized Gouy phase affects the interference of focused fields, altering the fringe spacing in a non-trivial manner.

In Chapter 5 we examine the focusing of radially polarized fields. If one follows the state of polarization along an oblique ray through the focus, it is seen to vary rapidly. We show that is a manifestation of the different Gouy phases that the two electric field components undergo.

Every lens suffers from some form of wave front aberrations. In Chapter 6 we analyze the influence of primary spherical aberration on the Gouy phase. We find that the phase anomaly in front of the diffraction focus and right behind it are quite different. This coincides with a wavefront spacing that is larger than the effective wavelength on one side, and smaller than the effective wavelength on the other side. This has consequences for optical metrology in which one strives for accuracy levels of $10^{-10}$. 