Chapter 3

The Gouy phase of Airy beams

This Chapter is based on

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  “The Gouy phase of Airy beams,”

Abstract
The phase behavior of Airy beams is studied, and their Gouy phase is defined. Analytic expressions for the idealized, infinite-energy type beam are derived. They are shown to be excellent approximations for finite-energy beams generated under typical experimental conditions.
3.1 Introduction

Beams that do not spread on propagation, so-called non-diffracting beams, have attracted considerable attention since they were discovered by Durnin et al. [Durnin, 1987; Durnin et al., 1987; Turunen and Friberg, 2010]. A special type of such beams are the so-called Airy beams described by Berry and Balazs in the context of quantum mechanics [Berry and Balazs, 1979]. These beams have the remarkable property that they “accelerate” away from the original direction of propagation. Airy beams are idealizations, because they carry an infinite amount of energy. Siviloglou and Christodoulides discussed how an exponentially modulated Airy function source would produce a finite-energy beam, which would retain its non-diffracting and accelerating behavior over an appreciable propagation distance [Siviloglou and Christodoulides, 2007]. After the experimental realization of such a beam [Siviloglou et al., 2007], several studies have been devoted to their properties [Bandres, 2008; Morris et al., 2009; S. Vo et al., 2010; Kaganovsky and Heyman, 2010], and a number of applications are being pursued. For instance, the “self-healing” capacity of Airy beams [Broky et al., 2008] makes them excellent candidates for optical communication through turbulent media [Gu and Gbur, 2010]. Other intriguing applications are the generation of curved plasma channels [Polynkin et al., 2009], and the manipulation of particles along bends in labs-on-a-chip [Hannappel et al., 2009].

Traditionally, the term Gouy phase describes how the phase of a monochromatic, focused field differs from that of a plane wave with the same frequency (see [Visser and Wolf, 2010] and the references therein). Recently, however, it has also been used to describe the phase of a non-diffracting Bessel beam [Martelli et al., 2010]. In this chapter we study the phase behavior of both finite-energy and infinite-energy Airy beams. By comparing their phase to that of a suitable reference field, their Gouy phase can be defined. A good understanding of the phase properties of Airy beams is of great importance in interferometric or remote sensing applications employing them.
3.2 The Schrödinger equation and the paraxial wave equation

The one-dimensional potential-free Schrödinger equation for a particle with mass \( m \) reads

\[
-\frac{\hbar}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} = \imath \hbar \frac{\partial \psi(x, t)}{\partial t}.
\]

(3.1)

A possible solution [Berry and Balazs, 1979] can be expressed as

\[
\psi(x, t) = \text{Ai}\left[\frac{B}{\hbar^{2/3}} \left( x - \frac{B^3 t^2}{4m^2} \right) \right] \exp \left[ \imath \frac{B^3 t}{2m \hbar} \left( x - \frac{B^3 t^2}{6m^2} \right) \right].
\]

(3.2)

Here \( \text{Ai} \) denotes the Airy function and \( B \) is an arbitrary constant. In this solution, the probability density \( |\psi|^2 \) propagates without distortion and with constant acceleration. The correctness of Eq. (3.2) can be verified by direct substitution, while making use of the differential property of the Airy function [Abramowitz and Stegun, 1965]

\[
\frac{d^2 \text{Ai}(z)}{dz^2} = z \text{Ai}(z).
\]

(3.3)

The one-dimensional paraxial wave equation reads [Mandel and Wolf, 1995, Sec. 5.6.1]

\[
\frac{\partial^2 \phi}{\partial x^2} + 2\imath k \frac{\partial \phi}{\partial z} = 0,
\]

(3.4)

where \( k = 2\pi/\lambda \) is the wavenumber and \((x, z)\) are the transverse and longitudinal coordinates, respectively. Comparing Eqs. (3.4) and (3.1) we find that the two equations are of the same mathematical form. We therefore try a solution for the paraxial wave equation of the type

\[
\phi(x, z) = \text{Ai}(\chi x - \epsilon z^2) \exp[\imath(\gamma x z - \eta z^3)],
\]

(3.5)

with \( \chi \) an arbitrary constant, and \( \epsilon, \gamma, \eta \) to be determined. Differentiation with respect to \( x \) yields

\[
\frac{\partial \phi}{\partial x} = (\chi \text{Ai}' + \imath \gamma z \text{Ai}) \exp[\imath(\gamma x z - \eta z^3)],
\]

(3.6)
The Schrödinger equation and the paraxial wave equation

\[ \frac{\partial^2 \phi}{\partial x^2} = (\chi^2 A'' + i2\chi\gamma z A' - \gamma^2 z^2 A) \exp[i(\gamma x z - \eta z^3)]. \]  
(3.7)

Using the differential property of the Airy function [Eq. (3.3)], we find that the previous equation can be re-written as

\[ \frac{\partial^2 \phi}{\partial x^2} = [i2\chi\gamma z A' + (\chi^3 x - \gamma^2 z^2 - \epsilon\chi^2 z^2) A] \exp[i(\gamma x z - \eta z^3)]. \]  
(3.8)

Differentiation with respect to \( z \) of Eq. (3.5) gives

\[ -i2k \frac{\partial \phi}{\partial z} = [i4\epsilon k z A' + (2\gamma k x - 6\eta k z^2) A] \exp[i(\gamma x z - \eta z^3)]. \]  
(3.9)

The terms in \( A \) and \( A' \) in Eq. (3.8) and Eq. (3.9) must be identical, and thus we obtain the relations

\[ 2\gamma k x - 6\eta k z^2 = \chi^3 x - (\gamma^2 + \epsilon\chi^2) z^2, \]  
(3.10)

\[ 4\epsilon k = 2\chi\gamma. \]  
(3.11)

Since the same kind of terms in \( x \) and \( z \) must have the same coefficients, we find the following relationships

\[ \gamma = \chi^3/(2k) = 1/(2k x_0^3), \]  
(3.12)

\[ \epsilon = \chi^4/(4k^2) = 1/(4k^2 x_0^4), \]  
(3.13)

\[ \eta = \chi^6/(12k^3) = 1/(12k^3 x_0^6), \]  
(3.14)

where we have defined \( \chi = 1/x_0 \). So a solution of the paraxial wave equation in terms of an Airy function can expressed as

\[ \phi(x, z) = Ai \left[ \frac{x}{x_0} - \left( \frac{z}{2k x_0^2} \right)^2 \right] \exp \left[ i \frac{z x}{2k x_0^3} - i \frac{1}{12} \left( \frac{z}{k x_0^2} \right)^3 \right]. \]  
(3.15)

Next we define \( s \equiv x/x_0 \), which represents a dimensionless transverse coordinate, and \( \xi \equiv z/(k x_0^2) \), a normalized propagation distance. The field envelope \( \phi \) can then be rewritten as [Siviloglou and Christodoulides, 2007]

\[ \phi(s, \xi) = Ai \left[ s - \left( \frac{\xi}{2} \right)^2 \right] \exp(is\xi/2 - i\xi^3/12). \]  
(3.16)

This expression for the envelope of an infinite-energy Airy beam will be analyzed in the succeeding sections.
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3.3 Green’s function and Hankel function

The Helmholtz equation for scalar fields reads

\[ \nabla^2 U(r, \omega) + k^2 U(r, \omega) = -4\pi \kappa(r, \omega), \quad (3.17) \]

where \( \kappa \) is the source density.

A differential equation such as Eq. (3.17) defines a local relationship between the field at given point and the source term. A Green’s function is an integral kernel that can be used to solve such an equation with certain boundary conditions. For the Helmholtz equation, the Green’s function is defined as:

\[ \nabla^2 G(r, r', \omega) + k^2 G(r, r', \omega) = -4\pi \delta(r - r'), \quad (3.18) \]

The Green’s function of Eq. (3.18) can be used to construct the solution

\[ U(r, \omega) = \int_V d^3 r' G(r, r', \omega) \kappa(r', \omega), \quad (3.19) \]

in which a homogeneous solution of Eq. (3.17) is omitted and \( V \) is the support of \( \kappa \). In three-dimensional space the Green function is of the form

\[ G(r, r', \omega) = \frac{e^{ik|r-r'|}}{|r-r'|}. \quad (3.20) \]

In two-dimensional space the Green’s function is given by the formula

\[ G(\rho, \rho', \omega) = -\frac{i}{4} H_0^{(1)}(k |\rho - \rho'|), \quad (3.21) \]

where \( H_0^{(1)} \) is the Hankel function of the first kind and zero order. The Hankel functions of order \( \alpha \), which are also known as Bessel functions of the third kind, are defined by the relations

\[ H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x), \quad (3.22a) \]
\[ H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x), \quad (3.22b) \]

where \( J_\alpha(x) \) and \( Y_\alpha(x) \) are Bessel functions of first and second kind, respectively. We will make use of Eq. (3.21) in the next section.
3.4 The Gouy phase of Airy beams

Consider a monochromatic, one-dimensional beam-like wave field $U(x, z, \omega)$ that propagates in the positive $z$-direction, and can be written as

$$U(x, z, \omega) = \phi(x, z)e^{i(kz-\omega t)}, \quad (3.23)$$

with the envelope $\phi(x, z)$ a solution of the paraxial wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + 2ik\frac{\partial \phi}{\partial z} = 0. \quad (3.24)$$

Here $k = \omega/c$ is the wavenumber associated with frequency $\omega$, $c$ denotes the speed of light, and $t$ the time. As discussed in Section 3.2, a possible solution to Eq. (3.24) is the so-called Airy beam, given by the expression [Berry and Balazs, 1979]

$$\phi(s, \xi) = \text{Ai} \left[ s - \left( \frac{\xi^2}{4} \right) \right] \exp \left[ i \left( \frac{s\xi}{2} - \frac{\xi^3}{12} \right) \right], \quad (3.25)$$

with $\text{Ai}$ the Airy function, $s = x/x_0$ a dimensionless transverse coordinate, and $\xi = z/kx_0^2$ a normalized propagation distance. In the remainder the constant $x_0$ is taken to be positive, and the time-dependent part of the wave field is suppressed. An example of the intensity distribution of an Airy beam is shown in Fig. 3.1, from which both the diffraction-free propagation and the transverse acceleration can be seen.

Because of its curved trajectory, we define the Gouy phase $\delta$ of an Airy beam as the difference between its phase $\psi$ and that of an ideal (non-diffracted) diverging cylindrical wave $U_{\text{cyl}}(x, z, \omega)$ centered on the $y$-axis and propagating into the half-space $z > 0$, i.e.

$$\delta(x, z, \omega) = \psi[U(x, z, \omega)] - \psi[U_{\text{cyl}}(x, z, \omega)], \quad (3.26)$$

with

$$U_{\text{cyl}}(x, z, \omega) = \frac{iC}{4} H_0^{(1)}(k\rho). \quad (3.27)$$

Here $C$ is a complex-valued constant, $H_0^{(1)}$ denotes a Hankel function of the first kind of order zero, and $\rho = (x^2 + z^2)^{1/2}$. The asymptotic
behavior of the cylindrical wave field is given by the expression [Arfken and Weber, 1995]

\[ U_{cyl}(x, z, \omega) \sim C \sqrt{\frac{2}{\pi k \rho}} e^{i(k \rho - \pi/4)}, \quad (k \rho \gg 1/4). \] (3.28)

We choose the constant \( C \) in Eq. (3.27) such that \( \psi[U_{cyl}(x, z, \omega)] = k \rho \).

For \( z \gg x \) this may be written as

\[ k \rho \approx k z \left[ 1 + \frac{1}{2} \left( \frac{x}{z} \right)^2 \right] = k z + \frac{1}{2} \frac{s^2}{\xi}. \] (3.29)

Thus we have from Eqs. (3.23), (3.25) and (3.29) that

\[ \delta(s, \xi, \omega) = \frac{s \xi}{2} - \frac{\xi^3}{12} - \frac{s^2}{2 \xi} + \psi_{Ai}, \] (3.30)

where \( \psi_{Ai} \) is the phase of the Airy function of Eq. (3.25). For real values of its argument the Airy function is real, and hence \( \psi_{Ai} \) equals 0 or \( \pi \). The first zero of \( \text{Ai}(x) \) (i.e. the zero with the largest value of \( x \)), occurs near \( x = -2.34 \). On making use of this in Eq. (3.25), we find that \( \psi_{Ai} = 0 \) when \( \xi < 2(s + 2.34)^{1/2} \). We first restrict our attention to this region of \( s \xi \)-space.
It is seen from Eq. (3.25) that the maximum beam intensity, $|\phi(s, \xi)|^2$, occurs on a quadratic trajectory. We therefore study the behavior of the Gouy phase on curves of the type $s = a\xi^2$, with $a$ a positive constant. On substituting this form into Eq. (3.30), it immediately follows that the Gouy phase vanishes identically along two curves, viz.

$$\delta(s, \xi, \omega) = 0, \quad \text{if} \quad s = (3 \pm 3^{1/2})\xi^2/6. \quad (3.31)$$

Similarly, it is seen that the maximum Gouy phase occurs along the curve $s = \xi^2/2$, namely

$$\delta(s, \xi, \omega) = \frac{\xi^3}{24}, \quad \text{if} \quad s = \xi^2/2. \quad (3.32)$$

The quadratic trajectory along which the intensity equals $\text{Ai}^2(0)$, (next to the maximum intensity, see Fig. 3.1) is given by the expression $s = \xi^2/4$. On substituting this form into Eq. (3.30) we find that

$$\delta(s, \xi, \omega) = \frac{\xi^3}{96}, \quad \text{if} \quad s = \xi^2/4. \quad (3.33)$$

We notice in passing that along the $\xi$-axis (i.e., the $z$-direction) the Gouy phase takes on negative values, i.e.

$$\delta(0, \xi, \omega) = -\frac{\xi^3}{12}. \quad (3.34)$$

Contours of the Gouy phase are shown in Fig. 3.2. Superposed are several quadratic curves. It is seen that the two dashed curves given by Eq. (3.31) indeed coincide with the zero contours. The curve along which the Gouy phase reaches its maximum [see Eq. (3.32)] is displayed as a solid line. The dotted curve is given by Eq. (3.33).

We next turn our attention to the region $\xi > 2(s + 2.34)^{1/2}$. Here the Airy function can take on the value zero. At such points its phase $\psi_{\text{Ai}}$ is singular, as is the Gouy phase. Both phases display a discontinuity of an amount $\pi$ at these singularities. An example of this behavior is shown in Fig. 3.3. The diagonal line that runs from the left-hand bottom to the right-hand top indicates the fifth zero of the Airy function, i.e. $\text{Ai}(s - \xi^2/4 = -7.94) = 0$. It is seen from the color-coding that the Gouy phase exhibits a $\pi$-discontinuity across this line.
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Figure 3.2: Color-coded plot of the Gouy phase of an Airy beam. Only the $s\xi$-region in which the Airy function has no zeros is shown. Along the two dashed curves, given by Eq. (3.31), the Gouy phase equals zero. Along the solid curve, given by Eq. (3.32), the Gouy phase reaches its maximum. The dotted curve is given by Eq. (3.33).

The beams we discussed so far are idealizations because the Airy function is not square integrable, i.e. a beam described by Eq. (3.25) carries an infinite amount of energy. Siviloglou and Christodoulides [Siviloglou and Christodoulides, 2007] considered an Airy beam source with an exponential envelope, i.e.

$$\phi^{(fe)}(s, 0) = \text{Ai}(s) e^{as}, \quad (3.35)$$

with the decay parameter $a > 0$ as to ensure a finite energy contribution, called $(fe)$, from the tail of the Airy function. They showed that such a beam propagates as

$$\phi^{(fe)}(s, \xi) = \text{Ai}(s - \xi^2/4 + i a \xi)e^{as - a\xi^2/2}$$

$$\times e^{i(\xi^3/12 + a^2 \xi/2 + s \xi/2)}. \quad (3.36)$$

Such a finite-energy beam still shows the characteristic acceleration and is, at least to some extent, diffraction-free. A beam of this type has
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Figure 3.3: Color-coded plot of the Gouy phase of an Airy beam. A portion of the region in which the function $\text{Ai}(x)$ has zeros is shown. The solid black line indicates the fifth zero of the Airy function. The Gouy phase jumps by an amount $\pi$ across this line.

been realized using a Gaussian beam incident on a spatial light modulator [Siviloglou et al., 2007]. It follows from Eqs. (3.26) and (3.4) that the Gouy phase for such beams is given by the expression

$$
\delta^{(fe)}(s, \xi, \omega) = \frac{s \xi}{2} - \frac{\xi^3}{12} - \frac{s^2 \xi}{2} + \frac{a^2 \xi}{2} + \psi_{\text{Ai}}. 
$$

(3.37)

It is to be noted that $\psi_{\text{Ai}}$ now pertains to the Airy function of Eq. (3.4), and is no longer restricted to the values 0 and $\pi$. In the experiment reported in [Siviloglou et al., 2007] the parameter values were $x_0 = 53 \mu m$, $a = 0.11$ and $\lambda = 488$ nm. In Fig. 3.4 intensity contours of a finite-energy Airy beam are shown and in Fig. 3.5 selected cross-sections of the corresponding beam intensity are plotted. On propagation the height of the central peak gradually decreases and the beam remains essentially diffraction-free up to $\xi \approx 5$ (corresponding to a propagation length of 18 cm), after which it rapidly spreads. However, the result expressed in Eq. (3.31), namely that the Gouy phase is zero along two quadratic curves, is still an excellent approximation under these conditions. This is shown in Fig. 3.6 in which the Gouy phase $\delta^{(fe)}(s, \xi, \omega)$ is plotted along the curves
Figure 3.4: Normalized intensity distribution of a finite-energy Airy beam propagating in the positive $\xi$-direction. In this example $x_0 = 53$ $\mu$m, $a = 0.11$ and $\lambda = 488$ nm.

$s = (3 \pm 3^{1/2})\xi^2/6$. It is seen that the actual value of the phase anomaly is always less than 2. This corresponds to a deviation of less than $\lambda/3$ from the approximate value zero after a propagation distance of 360,000 wavelengths. Along the curves of Eqs. (3.32) and (3.33) the difference between the analytic expressions pertaining to the infinite-energy beam and a numerical evaluation of Eq. (3.37) is even smaller.

In conclusion, the phase behavior of infinite-energy Airy beams has been analyzed. By comparing this behavior to that of an outgoing cylindrical wave, analytical expressions for their Gouy phase were derived. It was shown numerically that these results are excellent approximations for the Gouy phase of finite-energy Airy beams generated under typical conditions.
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Figure 3.5: Intensity of a finite-energy Airy beam in different cross-sections perpendicular to the $\xi$-axis: the source plane $\xi = 0$ (black), $\xi = 2$ (blue), $\xi = 4$ (red), and $\xi = 6$ (green).

Figure 3.6: Gouy phase of a finite-energy Airy beam along the curves $s = (3 + 3^{1/2})/6|\xi|^2$ (red), and $s = (3 - 3^{1/2})/6|\xi|^2$ (blue).