Optimal Quality of Service Control in Communication Systems

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2014

document version
Publisher's PDF, also known as Version of record

Link to publication in VU Research Portal

citation for published version (APA)
Bosman, J. W. (2014). *Optimal Quality of Service Control in Communication Systems*. [PhD-Thesis - Research and graduation internal, Vrije Universiteit Amsterdam].

A Spectral Theory Approach for Extreme Value Analysis in a Tandem of Fluid Queues

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In Chapter 2 we studied a model for a constant bit-rate video stream over an IP network with a play-out buffer at the client side. The network is modeled as a Markov Modulated fluid queue in which a CTMC determines the actual transmission rate through the network. For the play-out buffer we derived an initial buffer level b_{init} such that the probability that the video will stall during play-out will not exceed an agreed service level probability p_{empty}. We demonstrated that the probability of this event corresponds to the event of the maximum congestion level $M(t) := \sup_{0 \leq s \leq t} X(s)$ exceeding the initial buffer level b_{init}. The analysis was focused on the case with two states.

This chapter extends the result of Chapter 2 to an arbitrary number of states. To this end we extend results on the maximum level in a busy period from Section 2.3.1, the conditional busy cycle duration, the expected busy cycle duration from and extreme value theorem asymptotic result in Section 2.3.2. Our analysis was particularly motivated by the cited papers of Berman [13] and Iglehart [65]. For a literature review we refer to Section 2.1. We further refer to Asmussen [5] for an excellent survey on extreme-value theory for queues.

For the case with more than two states there is not always an explicit expression for the initial level of the play-out-buffer. However we provide an explicit recipe to calculate the asymptotic behavior of the maximum level in the Markov Modulated fluid queue. This recipe contains results that were derived using spectral theory analysis on the fluid model equations. The result can directly be applied to dimension the initial play-out buffer size.

The organization of the remainder of this chapter is as follows. In Section 3.1 we lay out the modifications to our model and the extension to the analysis described in Sections 2.2 and 2.3. We also describe how the dimensioning rule for the initial buffer level extends to the more general case.

In Section 3.2, we provide a numerical validation of the proposed dimensioning rule by means of simulations. Section 3.3 contains a discussion of the results and looks out to future work.

[3] This chapter is based on [22] and [20].

3.1 Analysis

This section extends our analysis of Chapter 2. For the model formulation and required preliminaries for this section we refer to Sections 2.2 and 2.3.

We show that the expression for the distribution of the maximum in a busy cycle has an exponential tail. Moreover we can derive an explicit expression for the asymptotic tail. In the expression for $\Psi(x)$ from (2.11) function $C(x)$ is an $n_{\uparrow} \times n_{\downarrow}$ matrix and $A(x)$ is an $n_{\downarrow} \times n_{\downarrow}$ matrix. For the case $n_{\downarrow} > 1$ we have to take the inverse of a matrix that contains exponential terms with exponents corresponding to the eigenvalues of Q. Using Sylvester's formula [38, Page 87] the matrix exponential e^{Qt} can be decomposed as:

$$e^{Qx} = e^{\lambda_1 x}\widetilde{Q}_1 + \ldots + e^{\lambda_{n_{\downarrow}+n_{\uparrow}} x}\widetilde{Q}_{n_{\downarrow}+n_{\uparrow}}, \tag{3.1}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_{n_{\downarrow}+n_{\uparrow}}$ of Q are the solution of

$$\det[Q - \lambda I] = 0, \tag{3.2}$$

and the matrices $\widetilde{Q}_i, \quad i = 1, \ldots, n_{\downarrow} + n_{\uparrow}$ are the Frobenius covariants. Let ϕ_i^l and ϕ_i^r be the normalised left and right eigenvector corresponding to eigenvalue λ_i:

$$\phi_i^l Q = \lambda_i \phi_i^l \quad \text{and} \tag{3.3}$$

$$Q\phi_i^r = \lambda_i \phi_i^r, \tag{3.4}$$

respectively. The corresponding Frobenius covariants are given by $\widetilde{Q}_i = \phi_i^r \phi_i^l$. These describe how the exponentials $e^{\lambda_i x}$ with corresponding eigenvalues λ_i contribute to the matrix exponential e^{Qx}. If we consider the partitioning of e^{Qx} in Equation (2.10) then $C(x)$ and $A(x)$ can be represented as:

$$A(x) = e^{\lambda_1 x}\widetilde{A}_1 + \ldots + e^{\lambda_{n_{\downarrow}+n_{\uparrow}} x}\widetilde{A}_{n_{\downarrow}+n_{\uparrow}} \quad \text{and} \tag{3.5}$$

$$C(x) = e^{\lambda_1 x}\widetilde{C}_1 + \ldots + e^{\lambda_{n_{\downarrow}+n_{\uparrow}} x}\widetilde{C}_{n_{\downarrow}+n_{\uparrow}}. \tag{3.6}$$

Now we decompose (2.11) into:

$$\Psi(x) = \frac{C(x)\,\text{adj}\left[A(x)\right]}{\det\left[A(x)\right]}. \tag{3.7}$$

As $A(x)$ is $n_\downarrow \times n_\downarrow$ both the determinant of $A(x)$ and the product $C(x)\,\mathrm{adj}\left[A(x)\right]$ will contain terms that are products of $n_\downarrow$ exponentials. The resulting exponential terms have exponents that are sums of $n_\downarrow$ eigenvalues.

3.1.1. Definition. Let c be a vector with n elements. In summations we denote with

$$\sum_{k\in c} := \sum_{\substack{k=c_i,\\ i=1,\dots,n}}$$

that we iterate k over the elements from vector $c = (c_1, \dots, c_n)$.

3.1.2. Lemma. *Let A be an $n \times n$ matrix and $m \geq 1$:*

$$A = \sum_{k=1}^{m} b_k A_k,$$

with

$$A_k = \boldsymbol{r}_k^T \boldsymbol{c}_k = \begin{bmatrix} r_{k,1}c_{k,1} & \cdots & r_{k,1}c_{k,n} \\ \vdots & \ddots & \vdots \\ r_{k,n}c_{k,1} & \cdots & r_{k,n}c_{k,n} \end{bmatrix}.$$

Then the following holds:

$$\mathrm{adj}\,[A] = \sum_{c\in\mathcal{C}} \left(\prod_{k\in c} b_k\right) \mathrm{adj}\left[\sum_{k\in c} A_k\right],$$

where $\mathcal{C}$ is the set with all combinations of length $n-1$ from the set $\{1, 2, \dots, m\}$.

Proof. For the proof we refer to Appendix 3.A. ∎

3.1.3. Lemma. *Let*

$$A(x) = e^{\lambda_1 x}\widetilde{A}_1 + \dots + e^{\lambda_{n_\downarrow+n_\uparrow} x}\widetilde{A}_{n_\downarrow+n_\uparrow} \quad \textit{and}$$
$$C(x) = e^{\lambda_1 x}\widetilde{C}_1 + \dots + e^{\lambda_{n_\downarrow+n_\uparrow} x}\widetilde{C}_{n_\downarrow+n_\uparrow},$$

with

$$\widetilde{Q}_k = \phi^r_k \phi^l_k = \begin{bmatrix} \phi^r_{k,1}\phi^l_{k,1} & \cdots & \phi^r_{k,1}\phi^l_{k,n_\downarrow+n_\uparrow} \\ \vdots & \ddots & \vdots \\ \phi^r_{k,n_\downarrow+n_\uparrow}\phi^l_{k,1} & \cdots & \phi^r_{k,n_\downarrow+n_\uparrow}\phi^l_{k,n_\downarrow+n_\uparrow} \end{bmatrix}, \\ = \begin{bmatrix} \widetilde{A}_k & \widetilde{B}_k \\ \widetilde{C}_k & \widetilde{D}_k \end{bmatrix},$$

where $\widetilde{A}_k$ *is* $n_\downarrow \times n_\downarrow$, $\widetilde{B}_k$ *is* $n_\downarrow \times n_\uparrow$, $\widetilde{C}_k$ *is* $n_\uparrow \times n_\downarrow$ *and* $\widetilde{D}_k$ *is* $n_\uparrow \times n_\uparrow$. *Furthermore, let* $\mathcal{C}$ *be the set of combinations of length* n *from the set* $\{1, \dots, n\}$. *Then the following holds:*

$$C(x)\,\mathrm{adj}\,[A(x)] = \sum_{c\in\mathcal{C}} \left(\prod_{k\in c} e^{\lambda_k x}\right) \sum_{k\in c} \widetilde{C}_k\,\mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right]. \tag{3.8}$$

Proof. By applying Lemma 3.1.2 we obtain:

$$C(x)\,\mathrm{adj}\,[A(x)] = \sum_{j=1}^{n_\downarrow+n_\uparrow} e^{\lambda_j x}\widetilde{C}_j \sum_{c\in\overline{\mathcal{C}}} \left(\prod_{k\in c} e^{\lambda_k x}\right) \mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right],$$

with $\overline{\mathcal{C}}$ the set of combinations of $n_\downarrow - 1$ elements from the set $\{1, \dots, n_\downarrow + n_\uparrow\}$. From (2.10) and Observation 2.3.6 we find that both $\widetilde{A}_k$ and $\widetilde{C}_k$ share the same row vector. As all sums of $n_\downarrow - 1$ matrices $\widetilde{A}_k$ have rank $n_\downarrow - 1$ the following is true:

$$\sum_{k\in c} \widetilde{C}_k\,\mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right] = 0, \qquad \forall c \in \overline{\mathcal{C}}. \tag{3.9}$$

Using (3.9) we can rewrite:

$$
\begin{aligned}
C(x)\,\mathrm{adj}\,[A(x)] &= \sum_{j=1}^{n_\downarrow+n_\uparrow} e^{\lambda_j x}\widetilde{C}_j \sum_{c\in\overline{\mathcal{C}}}\left(\prod_{k\in c} e^{\lambda_k x}\right)\mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right],\\
&= \sum_{c\in\overline{\mathcal{C}}}\sum_{j\notin c} e^{\lambda_j x}\widetilde{C}_j\left(\prod_{k\in c} e^{\lambda_k x}\right)\mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right],\\
&= \sum_{c\in\mathcal{C}}\sum_{j\in c} e^{\lambda_j x}\widetilde{C}_j\left(\prod_{k\in c\setminus j} e^{\lambda_k x}\right)\mathrm{adj}\left[\sum_{k\in c\setminus j}\widetilde{A}_k\right],\\
&= \sum_{c\in\mathcal{C}}\left(\prod_{k\in c} e^{\lambda_k x}\right)\sum_{j\in c}\widetilde{C}_j\,\mathrm{adj}\left[\sum_{k\in c\setminus j}\widetilde{A}_k\right].
\end{aligned}
$$

By using again (3.9) we obtain:

$$
\begin{aligned}
&\sum_{c\in\mathcal{C}}\left(\prod_{k\in c} e^{\lambda_k x}\right)\sum_{j\in c}\widetilde{C}_j\,\mathrm{adj}\left[\sum_{k\in c\setminus j}\widetilde{A}_k\right],\\
&=\sum_{c\in\mathcal{C}}\left(\prod_{k\in c} e^{\lambda_k x}\right)\sum_{k\in c}\widetilde{C}_k\,\mathrm{adj}\left[\sum_{k\in c}\widetilde{A}_k\right].
\end{aligned}
$$

■

3.1.4. Observation. There are $m = \binom{n_\downarrow+n_\uparrow}{n_\downarrow}$ unique combinations of $n_\downarrow$ eigenvalues from $n_\downarrow+n_\uparrow$ eigenvalues. Let $c\in\mathcal{C}$ be the set of combinations of $n_\downarrow$ indices from the set $\{1, 2, \dots, n_\downarrow+n_\uparrow\}$. We define the sums of eigenvalues $\lambda_{k_1}, \dots, \lambda_{k_{n_\downarrow}}$ corresponding to combination c_k with index k by:

$$
\widehat{\lambda}_k := \sum_{j\in c_k}\lambda_j, \quad k = 1, \dots, m, \;\; c_k\in\mathcal{C},
$$

where the set $\mathcal{C}$ is ordered in decreasing order according to the real parts of $\hat{\lambda}_k$ such that:

$$
\Re\big(\widehat{\lambda}_1\big) \geq \Re\big(\widehat{\lambda}_2\big) \geq \dots \geq \Re\big(\widehat{\lambda}_m\big).
$$

3.1.5. Lemma. *Equation* (3.7) *can be rewritten as:*

$$
\Psi(x) = \frac{\widehat{C}_1 e^{\widehat{\lambda}_1} + \widehat{C}_2 e^{\widehat{\lambda}_2} + \dots + \widehat{C}_m e^{\widehat{\lambda}_m}}{\widehat{A}_1 e^{\widehat{\lambda}_1} + \widehat{A}_2 e^{\widehat{\lambda}_2} + \dots + \widehat{A}_m e^{\widehat{\lambda}_m}}, \tag{3.10}
$$

with values $\hat{\lambda}_k$ as defined in Observation 3.1.4, and

$$\widehat{C}_k := \sum_{j \in c_k} \widetilde{C}_j \operatorname{adj}\Big[\sum_{j \in c_k} \widetilde{A}_j\Big], \qquad c_k \in \mathcal{C},$$

and

$$\widehat{A}_k := \det\Big[\sum_{j \in c_k} \widetilde{A}_j\Big], \qquad c_k \in \mathcal{C},$$

where the elements c_k from set $\mathcal{C}$ are ordered according to Observation 3.1.4 such that:

$$\Re(\widehat{\lambda}_1) \geq \Re(\widehat{\lambda}_2) \geq \dots \geq \Re(\widehat{\lambda}_m).$$

Proof. Due to the determinant and adjoint matrix in Equation (3.7), there will be exponential terms in both numerator and denominator that result from products of $n_\downarrow$ exponentials $e^{\lambda_i x}$ with eigenvalues $\lambda_i,\ i \in \mathcal{S}$. First consider the terms in the denominator. Remember that the Frobenius covariants $\widetilde{Q}_i$ (and also $\widetilde{A}_i$, $\widetilde{C}_i$) have rank 1. Therefore only linear combinations of $n_\downarrow$ distinct Frobenius covariants, defined by $c_k \in \mathcal{C}$, will result in positive determinants. Combination $c_k \in \mathcal{C}$ is element of the set containing al combinations of length $n_\downarrow$ from the set $\{1, \dots, n_\downarrow + n_\uparrow\}$ as defined in Observation 3.1.4. Considering the numerator, the adjoint matrix of a linear combination of Frobenius covariants $\widetilde{A}_i$ will only have positive entries when it is a linear combination of $n_\downarrow - 1$ distinct Frobenius covariants as the adjoint matrix contains minors of degree $n_\downarrow - 1$. By applying Lemma 3.1.3 we observe that only remaining exponential terms in the numerator are those that correspond to sums over combinations $c_k \in \mathcal{C}$ of $n_\downarrow$ eigenvalues. ■

As $\widehat{\lambda}_k$ is ordered in decreasing order the leading exponential term is $e^{\widehat{\lambda}_1}$. Considering (3.7) the limiting distribution Ψ^∞ becomes:

$$\Psi^\infty := \lim_{x \to \infty} \Psi(x) = \frac{\widehat{C}_1}{\widehat{A}_1}. \tag{3.11}$$

Using this we can derive the tail behaviour of $\Psi(x)$:

3.1.6. Lemma. $\Psi(x)$ *has an exponential tail that behaves as*

$$\Psi^\infty - \Psi(x) \to G e^{-\kappa x}, \qquad x \to \infty, \tag{3.12}$$

where

$$\kappa = \lambda_{n_\uparrow}, \quad G = \frac{\widehat{C}_1\widehat{A}_2 - \widehat{A}_1\widehat{C}_2}{\widehat{A}_1^2}$$

and κ is the maximal (least) negative eigenvalue of Q.

Proof. Subtracting Ψ^∞ from the expression of $\Psi(x)$ in Lemma 3.1.5 gives:

$$\begin{aligned}\Psi^\infty - \Psi(x) &= \frac{\widehat{C}_1}{\widehat{A}_1} - \frac{\widehat{C}_1 e^{\widehat{\lambda}_1} + \widehat{C}_2 e^{\widehat{\lambda}_2} + ... + \widehat{C}_m e^{\widehat{\lambda}_m}}{\widehat{A}_1 e^{\widehat{\lambda}_1} + \widehat{A}_2 e^{\widehat{\lambda}_2} + ... + \widehat{A}_m e^{\widehat{\lambda}_m}}, \\ &= \frac{[\widehat{C}_1\widehat{A}_2 - \widehat{A}_1\widehat{C}_2]e^{\widehat{\lambda}_2} + ... + [\widehat{C}_1\widehat{A}_m - \widehat{A}_1\widehat{C}_m]e^{\widehat{\lambda}_m}}{\widehat{A}_1[\widehat{A}_1 e^{\widehat{\lambda}_1} + \widehat{A}_2 e^{\widehat{\lambda}_2} + ... + \widehat{A}_m e^{\widehat{\lambda}_m}]}.\end{aligned}$$

When $x \to \infty$ the two leading exponential terms $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ remain:

$$\Psi^\infty - \Psi(x) \to \frac{\widehat{C}_1\widehat{A}_2 - \widehat{A}_1\widehat{C}_2}{\widehat{A}_1^2} e^{\widehat{\lambda}_2 - \widehat{\lambda}_1}, \qquad x \to \infty. \tag{3.13}$$

According to Kulkarni [74, Theorem 11.5] the eigenvalues of Q, resulting from $\det[R - \lambda T] = 0$ can be ordered as follows:

$$\Re(\lambda_1) \le \Re(\lambda_2) \le ... \le \Re(\lambda_{n_\uparrow}) < 0 < \Re(\lambda_{n_\uparrow+2}) \le \cdots \le \Re(\lambda_{n_\uparrow+n_\downarrow}). \tag{3.14}$$

there are $n_\uparrow$ eigenvalues with negative real part, one eigenvalue is equal to zero and there are $n_\downarrow - 1$ eigenvalues with positive real part. In Definition 3.1.4 we defined $\widehat{\lambda}_k$ as the sum of $n_\downarrow$ unique eigenvalues. Consider $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$:

$$\begin{aligned}\widehat{\lambda}_1 &= 0 \quad + \lambda_{n_\uparrow+2} + ... + \lambda_{n_\uparrow+n_\downarrow}, \\ \widehat{\lambda}_2 &= \lambda_{n_\uparrow} + \lambda_{n_\uparrow+2} + ... + \lambda_{n_\uparrow+n_\downarrow}.\end{aligned}$$

Observe that $\widehat{\lambda}_1$ consists of $n_\downarrow - 1$ eigenvalues with positive real part and one eigenvalue equal to zero. The next $\widehat{\lambda}_2$ is obtained by replacing the eigenvalues equal to zero with the eigenvalue with least negative real part $\lambda_{n_\uparrow}$. Therefore

$$\widehat{\lambda}_2 - \widehat{\lambda}_1 = \max_{i \in \{i : \lambda_i < 0\}} \lambda_i = \lambda_{n_\uparrow}.$$

Plugging this in (3.13) gives:

$$\Psi^\infty - \Psi(x) \to Ge^{\kappa x}, \qquad x \to \infty,$$

with

$$G := \frac{\widehat{C}_1 \widehat{A}_2 - \widehat{A}_1 \widehat{C}_2}{\widehat{A}_1^2}$$

and

$$\kappa := \max_{i \in \{i : \lambda_i < 0\}} \lambda_i = \lambda_{n_\uparrow}.$$

■

From Lemma 3.1.6 we established that $\Psi(x)$ has an exponential tail $Ge^{-\kappa x}$. Here G is a matrix while we are interested in the general case averaging over all transitions. Therefore we define the following transition matrices:

3.1.7. Definition.

$$P_{BI} := \Psi^\infty = \frac{\widehat{C}_1}{\widehat{A}_1}, \tag{3.15}$$

$$P_{IB} := U = -\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \begin{pmatrix} T_{\downarrow\uparrow} \\ T_{0\uparrow} \end{pmatrix}, \tag{3.16}$$

$$P_{BB} := P_{BI} P_{IB} = -\begin{pmatrix} \Psi^\infty & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \begin{pmatrix} T_{\downarrow\uparrow} \\ T_{0\uparrow} \end{pmatrix}, \tag{3.17}$$

where P_{BI} is the transition matrix from a busy to an idle period, P_{IB} is the transition matrix from an idle to a busy period and P_{BB} is the transition matrix between states that initiate busy cycles. In P_{BI}, Ψ^∞ is transition matrix from a state that initiates a busy period to the state that terminates the busy period. Recall that U is the transition matrix from Definition 2.3.4 for transitions from idle period states to busy period initiating states.

We use transition matrix P_{BB} for calculating the stationary distribution π_B over states $(i \in \mathcal{S}_\uparrow)$ that initiate a busy period. The stationary distribution π_B is the solution of:

$$\pi_B P_{BB} = \pi_B, \tag{3.18}$$
$$\sum \pi_B = 1.$$

3.1.8. Corollary. *The overall expected tail of the distribution on the maximum is given by:*

$$\mathbb{P}\{M_+ > z\} \to be^{-\kappa z}, \qquad x \to \infty, \tag{3.19}$$

where $b = \pi_B \frac{\widehat{C}_1\widehat{A}_2 - \widehat{A}_1\widehat{C}_2}{\widehat{A}_1^2}\overline{e}$ *and* $\kappa = \lambda_{n_\uparrow}$.

Proof. The stationary distribution of states that initiate a busy period is given by π_B. The marginal distribution of the maximum in a busy period is given by $\Psi(x)$ and is conditioned on the states $i \in \mathcal{S}_\uparrow$ that initiate a busy period. The overall distribution of the maximum is given by:

$$\pi_B \Psi(x)\overline{e}.$$

We have to add the rows and weight the sums according to the stationary distribution π_B. The same holds for the exponential tail parameter G from Lemma 3.1.6:

$$b := \pi_B G \overline{e}. \tag{3.20}$$

We define the maximum of an arbitrary busy cycle by:

$$\mathbb{P}\{M_+ \leq x\},$$

where M_+ represents the stochastic variable corresponding to the maximum of the busy cycle. Similar to Iglehart [65, Lemma 1] we obtain an expression for

$$\mathbb{P}\{M_+ > z\} \to be^{-\kappa z}, \qquad x \to \infty.$$

In our case $b = \pi_B \frac{\widehat{C}_1\widehat{A}_2 - \widehat{A}_1\widehat{C}_2}{\widehat{A}_1^2}\overline{e}$ and $\kappa = \lambda_{n_\uparrow}$. ■

3.1.9. Lemma. *Let $M_+(k)$ be the maximum of the kth busy cycle. Then the following holds:*

$$\lim_{n\to\infty} \mathbb{P}\{\kappa \max_{1\leq k\leq n} M_+(k) - \log(bn) \leq x\} = \Lambda(x), \tag{3.21}$$

where

$$\Lambda(x) = \exp[-e^{-x}]. \tag{3.22}$$

Proof. In Corollary 3.1.8 we showed that the maximum of a busy cycle has an exponential tail according to:

$$\mathbb{P}\{M_+ > z\} \to be^{-\kappa z}, \qquad x \to \infty.$$

Using the same arguments as in Iglehart [65, Lemma 2] we can derive that

$$\lim_{n\to\infty} \mathbb{P}\{\kappa \max_{1\le k\le n} M_+(k) - \log(bn) \le x\} = \Lambda(x).$$

The following extreme value theorem argument can be used:

$$\begin{aligned}&\mathbb{P}\{\max_{1\le k\le n} M_+(k) \le \frac{x+\log(bn)}{\kappa}\} = \mathbb{P}^n\{M_+(1) \le \frac{x+\log(bn)}{\kappa}\},\\ &= \left[1 - b\exp[-(x+\log(x+bn))] + o(\exp[-(x+\log(n))])\right]^n.\end{aligned}$$

■

3.1.1 Maximum with respect to time

Rather than the asymptotics for the busy cycles, we are interested in the evolution of the maximum over time.

For this we use a result in Kulkarni and Tzenova [75], who derive an expression for the joint mean first passage time in a Markov Modulated fluid queue:

$$\begin{aligned}&\mathbb{E}[\tau_{\mathcal{S}_\downarrow} \mid X(0) = x, \varphi(0) = i], && i \in \mathcal{S},\\ &\tau_{\mathcal{S}_\downarrow} := \inf\{t > 0 : X(t) = 0, \varphi(t) \in \mathcal{S}_\downarrow\}.\end{aligned}$$

The joint mean first passage time will be represented by function $f_i(x)$:

$$f_i(x) := \mathbb{E}[\tau_{\mathcal{S}_\downarrow} \mid X(0) = x, \varphi(0) = i]. \qquad i \in \mathcal{S}. \tag{3.23}$$

An expression for the joint mean first passage time can be obtained by solving a system of differential equations:

$$R\frac{\mathrm{d}f(x)}{\mathrm{d}x} + Tf(x) + \overline{e} = 0. \tag{3.24}$$

with boundary condition:

$$f_i(x) = 0, \qquad \forall i \in \mathcal{S}_\downarrow. \tag{3.25}$$

where $R = \text{diag}(r_1, \ldots, r_n)$ is the diagonal matrix with rates of change, T is the generating matrix and where $\bar{e}$ is a column vector of ones. Here eigenvalues λ_j as the solution to

$$\det[R - \lambda T] = 0, \tag{3.26}$$

and corresponding right eigenvectors ϕ_j^r for which holds:

$$\lambda_i R \phi_j^r = T \phi_j^r. \tag{3.27}$$

Note that the eigenvalues are equal to the eigenvalues obtained in (3.2). Recall that the eigenvalues of Q, are ordered in increasing order (Lemma 3.1.6, Equation (3.14)), and have the following property:

$$\Re(\lambda_1) \leq \Re(\lambda_2) \leq \cdots \leq \Re(\lambda_{n_\uparrow}) < 0 < \Re(\lambda_{n_\uparrow+2}) \leq \cdots \leq \Re(\lambda_{n_\uparrow+n_\downarrow}).$$

In Kulkarni and Tzenova [75, Theorem 4.2] the solution for (3.24) is given by:

$$f(x) = \sum_{j=1+n_\uparrow}^{n_\downarrow+n_\uparrow} a_j \phi_j^r e^{-\lambda_j x} - \frac{\bar{e}x}{d} + g. \tag{3.28}$$

In this expression g a solution of

$$Tg = -(cR + I)\bar{e}. \tag{3.29}$$

Note that rank $(T) = n - 1$ therefore we have one free variable in g and fix $g_n = 0$ in order to get a solution to (3.29). Coefficients a_j are obtained from the solution to:

$$\sum_{j=1+n_\uparrow}^{n_\downarrow+n_\uparrow} a_j \phi_{ij}^r + g_i = 0, \qquad \forall i \in \mathcal{S}_\downarrow,\ r_i < 0, \tag{3.30}$$

where ϕ_{ij}^r is the ith entry of eigenvector ϕ_j^r.

In Section 2.3.2 we directly use [74, Example 1]. We now extend this for the case where $n_\downarrow \geq 1, \quad n_\uparrow \geq 1$ and $n_0 \geq 0$:

Resulting from the Equation (3.27) we obtain eigenvectors that are partitioned into:

$$\phi^r = \begin{pmatrix} \phi^r_\downarrow \\ \phi^r_0 \\ \phi^r_\uparrow \end{pmatrix}.$$

For the sake of readability we omit the index j in his expression. There are n_0 states with $r_i = 0$ therefore we write:

$$\lambda \begin{pmatrix} R_\downarrow & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_\uparrow \end{pmatrix} \begin{pmatrix} \phi^r_\downarrow \\ \phi^r_0 \\ \phi^r_\uparrow \end{pmatrix} = \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} & T_{\downarrow\uparrow} \\ T_{0\downarrow} & T_{00} & T_{0\uparrow} \\ T_{\uparrow\downarrow} & T_{\uparrow 0} & T_{\uparrow\uparrow} \end{pmatrix} \begin{pmatrix} \phi^r_\downarrow \\ \phi^r_0 \\ \phi^r_\uparrow \end{pmatrix},$$

and obtain:

$$\phi^r_0 = -T_{00}^{-1} T_{0\downarrow} \phi^r_\downarrow - T_{00}^{-1} T_{0\uparrow} \phi^r_\uparrow. \tag{3.31}$$

Plugging in (3.31) gives:

$$\lambda \begin{pmatrix} R_\downarrow & 0 \\ 0 & R_\uparrow \end{pmatrix} \begin{pmatrix} \phi^r_\downarrow \\ \phi^r_\uparrow \end{pmatrix} = \begin{pmatrix} T_{\downarrow\downarrow} - T_{\downarrow 0} T_{00}^{-1} T_{0\downarrow} & T_{\downarrow\uparrow} - T_{\downarrow 0} T_{00}^{-1} T_{0\uparrow} \\ T_{\uparrow\downarrow} - T_{\uparrow 0} T_{00}^{-1} T_{0\downarrow} & T_{\uparrow\uparrow} - T_{\uparrow 0} T_{00}^{-1} T_{0\uparrow} \end{pmatrix} \begin{pmatrix} \phi^r_\downarrow \\ \phi^r_\uparrow \end{pmatrix}.$$

The resulting eigenvectors will become:

$$\phi^r = \begin{pmatrix} \phi^r_\downarrow \\ -T_{00}^{-1} \left[T_{0\downarrow} \phi^r_\downarrow + T_{0\uparrow} \phi^r_\uparrow \right] \\ \phi^r_\uparrow \end{pmatrix}. \tag{3.32}$$

Observe that this is equivalent using the eigenvalues and vectors from matrix Q (see Equations 3.2–3.4) and plugging this into (3.32).

In order to have a valid solution only positive eigenvalues can contribute to (3.28). Let Φ be the matrix consisting of all right-eigenvectors ordered according to all corresponding eigenvalues with non negative real parts $\Re(\lambda_{n_\uparrow+1}) = 0 \leq \cdots \leq \Re(\lambda_{n_\downarrow+n_\uparrow})$. We now partition matrix Φ into

$$\Phi = \begin{pmatrix} \Phi_\downarrow \\ \Phi_0 \\ \Phi_\uparrow, \end{pmatrix}, \tag{3.33}$$

where $\Phi_\downarrow$ is $n_\downarrow \times n_\downarrow$, Φ_0 is $n_0 \times n_\downarrow$ and $\Phi_\uparrow$ is $n_\uparrow \times n_\downarrow$.

3.1.10. Definition. We define the conditional expected duration of a busy period and idle period by:

$$\mathbb{E}[C_B] := \left(\mathbb{E}[\tau_{\mathcal{S}_\downarrow} \mid X(0) = 0, \varphi(0) = i],\ i \in \mathcal{S}_\uparrow\right), \tag{3.34}$$

$$\tau_{\mathcal{S}_\downarrow} := \inf\{t > 0 : X(t) = 0, \varphi(t) \in \mathcal{S}_\downarrow\},$$

$$\mathbb{E}[C_I] := \left(\mathbb{E}[\tau_{\mathcal{S}_\uparrow} \mid X(0) = 0, \varphi(0) = i],\ i \in \mathcal{S}_\downarrow\right), \tag{3.35}$$

$$\tau_{\mathcal{S}_\uparrow} := \inf\{t > 0 : \varphi(t) \in \mathcal{S}_\uparrow\}.$$

3.1.11. Lemma. *The mean duration of a busy period starting in state $i \in \mathcal{S}_\uparrow$ is given by:*

$$\mathbb{E}[C_B] = \Phi_\uparrow \Phi_\downarrow^{-1} g_\downarrow + g_\uparrow, \tag{3.36}$$

where Φ is the block partitioned matrix with right eigen vectors from (3.33) *corresponding to non negative eigenvalues, g is the solution to:*

$$Tg = -(cR + I)\overline{e},$$

with vector g partitioned in

$$g = \begin{pmatrix} g_\downarrow \\ g_0 \\ g_\uparrow \end{pmatrix}.$$

Proof. The solution for (3.24) is given by:

$$f(x) = \sum_{j=1+n_\uparrow}^{n_\downarrow + n_\uparrow} a_j \Phi_j e^{-\lambda_j x} - \frac{\overline{e}x}{d} + g. \tag{3.37}$$

Coefficients a_j are obtained from the solution to:

$$\sum_{j=1+n_\uparrow}^{n_\downarrow + n_\uparrow} a_j \Phi_{ij} + g_i = 0, \qquad \forall i \in \mathcal{S}_\downarrow,\ r_i < 0, \tag{3.38}$$

where Φ_{ij} is the ith entry of jth eigenvector Φ_j in eigenvector matrix Φ. We are interested in the mean first passage time for a busy period started at $x = 0$. Therefore we take $f(0)$:

$$f(0) = \sum_{j=1+n_\uparrow}^{n_\downarrow+n_\uparrow} a_j \Phi_j + g_\uparrow.$$

Switching to matrix notation gives:

$$f(0) = \Phi_\uparrow a + g, \tag{3.39}$$

where

$$\Phi_\downarrow a + g_\downarrow = 0.$$

Matrix $\Phi_\downarrow$ is invertible, therefore we can write:

$$a = -\Phi_\downarrow^{-1} g_\downarrow. \tag{3.40}$$

Plugging (3.40) into (3.39) gives:

$$f(0) = \Phi_\uparrow \Phi_\downarrow^{-1} g_\downarrow + g_\uparrow. \tag{3.41}$$

■

3.1.12. Definition. We define the expected busy cycle time conditioned on starting in a state $i \in \mathcal{S}_\uparrow$ by:

$$\begin{aligned} \mathbb{E}[C_{BB}] &:= \mathbb{E}[\tau_B \mid \varphi(0) = i, X(0) = 0], \qquad\qquad i \in \mathcal{S}_\uparrow, \\ \tau_B &:= \inf\{t > \tau_{\mathcal{S}_\downarrow} : \varphi(t) \in \mathcal{S}_\uparrow\}, \\ \tau_{\mathcal{S}_\downarrow} &:= \inf\{t > 0 : X(t) = 0, \varphi(t) \in \mathcal{S}_\downarrow\}. \end{aligned}$$

3.1.13. Lemma. *The overall mean expected busy cycle length is given by:*

$$\mathbb{E}[C] = \pi_B \Big[\mathbb{E}[C_B] + P_{BI}\mathbb{E}[C_I]\Big],$$

where

$$\mathbb{E}[C_I] = -\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \bar{e}, \tag{3.42}$$

resulting in

$$\mathbb{E}[C] = \pi_B \left[\Phi_{\uparrow\downarrow}\Phi_{\downarrow\downarrow}^{-1} g_\downarrow + g_\uparrow - \begin{pmatrix}\Psi^\infty & 0\end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \overline{e} \right],$$

with $\Psi^\infty = \frac{\widehat{C}_1}{\widehat{A}_1}$ *as defined in* (3.11).

Proof. In Lemma 3.1.11 we obtained an expression for the mean busy period. For the idle period using standard first passage time calculations for a CTMC we obtain:

$$\mathbb{E}[C_I] = -\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1}.$$

Using $\mathbb{E}[C_B]$ and $\mathbb{E}[C_B]$ the expected cycle time can be obtained:

$$\mathbb{E}[C_{BB}] = \mathbb{E}[C_B] + P_{BI}\mathbb{E}[C_I].$$

When a busy period is initiated for a given initiating state $i \in \mathcal{S}_\uparrow$ the expected passage time is given by $\mathbb{E}[C_B]$. Remember that in Definition 3.1.7, Equation (3.15) we defined $P_{BI} = \Psi^\infty$. From this the expected idle time after a busy period that has been initiated by state $i \in \mathcal{S}_\uparrow$ is obtained:

$$\begin{aligned} &\mathbb{E}[\tau_{\mathcal{S}_\uparrow} - \tau_0 \mid X(0) = 0, \varphi(0) = i] = P_{BI}\mathbb{E}[C_I], \qquad i \in \mathcal{S}_\uparrow, \qquad (3.43) \\ &\tau_{\mathcal{S}_\uparrow} := \inf\{t > \tau_0 : \varphi(t) \in \mathcal{S}_\uparrow\}, \\ &\tau_0 := \inf\{t > 0 : X(t) = 0\}. \end{aligned}$$

This corresponds to taking the expectation over $\mathbb{E}[C_I]$ with respect to the transition matrix P_{BI}. Combining (3.42) and (3.43) gives the expected cycle time given the busy cycle started in state $i \in \mathcal{S}_\uparrow$:

$$\mathbb{E}[C_{BB}] = \mathbb{E}[C_B] + P_{BI}\mathbb{E}[C_I], \qquad i \in \mathcal{S}_\uparrow.$$

In (3.18) we defined the distribution π_B of states that initiate a busy period. The mean cycle time becomes:

$$\begin{aligned} \mathbb{E}[C] &= \pi_B \Big[\mathbb{E}[C_B] + P_{BI}\mathbb{E}[C_I] \Big], \\ &= \pi_B \left[\Phi_{\uparrow\downarrow}\Phi_{\downarrow\downarrow}^{-1} g_\downarrow + g_\uparrow - \begin{pmatrix}\Psi^\infty & 0\end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \overline{e} \right]. \end{aligned}$$

■

3.1.14. Theorem. *Let* $M^*(t) := \sup_{0 \le s \le t} \{M(s)\}$. *The limiting distribution of* $M^*(t)$ *is given by*

$$\lim_{t\to\infty} \mathbb{P}\{\kappa M^*(t) - \log(bt) \le x\} = \Lambda^{\frac{1}{\mathbb{E}[C]}}(x), \tag{3.44}$$

where

$$b = \pi_B \frac{\widehat{C}_1 \widehat{A}_2 - \widehat{A}_1 \widehat{C}_2}{\widehat{A}_1^2} \overline{e},$$

$$\mathbb{E}[C] = \pi_B \left[\Phi_{\uparrow\downarrow} \Phi_{\downarrow\downarrow}^{-1} g_\downarrow + g_\uparrow - \begin{pmatrix} \Psi^\infty & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \overline{e} \right]$$

and

$$\kappa = \lambda_{n_\uparrow}.$$

Proof. The proof is similar to that of Iglehart [65, Theorem 3]. In Lemma 3.1.9 we showed that:

$$\lim_{n\to\infty} \mathbb{P}\{\kappa \max_{1 \le k \le n} M_+(k) - \log(bn) \le x\} = \Lambda(x).$$

Define $\{c(t) : t \ge 0\}$ as the renewal process associated with the length of busy cycles. Then $M^*(t)$ satisfies:

$$\lim_{t\to\infty} \mathbb{P}\{\max_{0 \le k \le c(t)} M_+(k) \le x\} \le M^*(t) \le$$
$$\lim_{t\to\infty} \mathbb{P}\{\max_{0 \le k \le c(t)+1} M_+(k) \le x\}. \tag{3.45}$$

From Lemma 3.1.13 we know that:

$$\mathbb{E}[C] = \pi_B \left[\Phi_{\uparrow\downarrow} \Phi_{\downarrow\downarrow}^{-1} g_\downarrow + g_\uparrow - \begin{pmatrix} \Psi^\infty & 0 \end{pmatrix} \begin{pmatrix} T_{\downarrow\downarrow} & T_{\downarrow 0} \\ T_{0\downarrow} & T_{00} \end{pmatrix}^{-1} \overline{e} \right].$$

Applying the weak law of large numbers for $c(t)$ we obtain:

$$\frac{c(t)}{t} \to \frac{1}{\mathbb{E}[C]}, \qquad t \to \infty. \tag{3.46}$$

Using Berman [13, Theorem 3.2] and Lemma 3.1.9 the limiting distribution becomes:

$$\lim_{t\to\infty} \mathbb{P}\{\kappa M^*(t) - \log(bt) \le x\} = \Lambda^{\frac{1}{\mathbb{E}[C]}}(x). \tag{3.47}$$

The term $\frac{1}{\mathbb{E}[C]}$ from (3.46) represents the expected number of busy cycles per time unit and corresponds to the c in Berman [13, Theorem 3.2]. ■

From Theorem 3.1.14 the expression for the asymptotic distribution for the maximum of fluid queue

$$\mathbb{P}\{M^*(t) > b_{init}\} < p_{empty} \tag{3.48}$$

can now be used to approximate the tail probabilities:

$$\mathbb{P}\{\kappa M^*(t) - \log(bt) > x\} \approx 1 - \Lambda^{\frac{1}{\mathbb{E}[C]}}(x), \tag{3.49}$$

$$\mathbb{P}\{M^*(t) > b_{init}\} \approx 1 - \Lambda^{\frac{1}{\mathbb{E}[C]}}(\kappa b_{init} - \log(bt)), \tag{3.50}$$

whenever we have a sufficiently large b_{init} such that at least $b_{init} > \frac{\log(bt)}{\kappa}$.

Define p_{empty} as the maximal allowed probability that a buffer, with initial contents b_{init}, will become empty during play-out of a video stream of length $t = T_{play}$. Given p_{empty}, that represents the maximum probability a video is disturbed during T_{play}, the initial buffer size b_{init} should be chosen such that

$$b_{init} > \frac{-\log[-\frac{\mathbb{E}[C]}{bT_{play}} \log(1 - p_{empty})]}{\kappa}. \tag{3.51}$$

This holds when we have T_{play} sufficiently large such that

$$T_{play} > -\log(1 - p_{empty})\frac{\mathbb{E}[C]}{b}.$$

Furthermore $M^*(T_{play})$ represents the limiting distribution on the maximum congestion over time. Then if we consider $M^*(T_{play})$ it should hold that:

$$\mathbb{P}\{M^*(T_{play}) > b_{init}\} < p_{empty}. \tag{3.52}$$

Using the fact that when $t \to \infty$ the maximum M^* converges to a Gumbel distribution the following asymptotic expectation of the maximum level can be derived:

$$\mathbb{E}[M^*(t)] \to \frac{\log\left(\frac{bt}{\mathbb{E}[C]}\right) + \gamma}{\kappa}, \qquad t \to \infty, \tag{3.53}$$

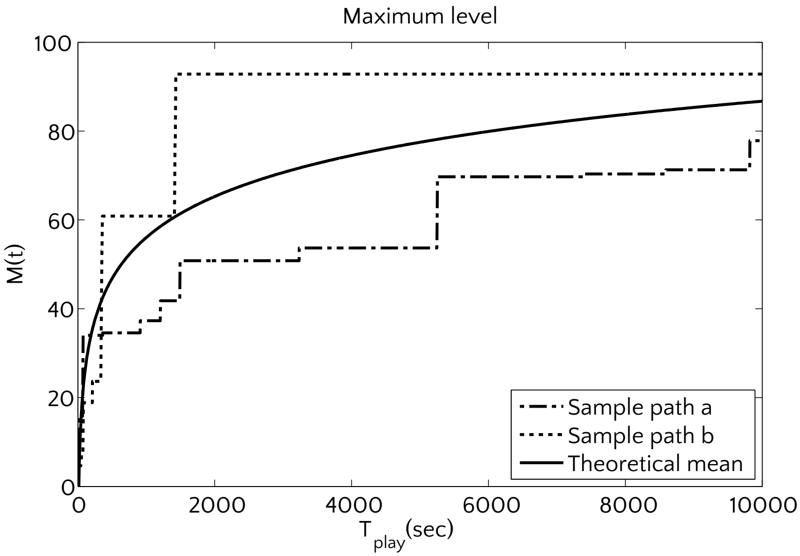
where $\gamma \approx 0.577215665$ is the Euler-Mascheroni constant. Observe that $\mathbb{E}[M^*(t)]$ grows logarithmically over time with logarithmic slope $\frac{1}{\kappa}$.

3.2 Numerical experiments

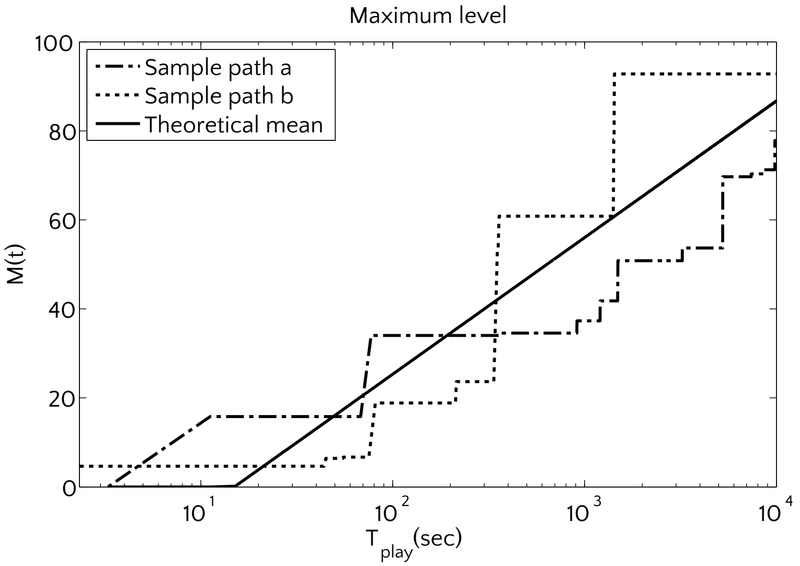
In Section 2.3 we derived that the combined buffer contents, that is congested and in the play-out buffer $X(t) + Y(t) = M^*(t)$, equals the maximum of the congestion process $X(t)$. Moreover the distribution $M^*(t)$ can be approximated by an extreme value distribution for sufficiently large t. From this we derived a mapping from the maximum buffer under-run probability p_{empty} and streaming video duration T_{play} to minimal initial buffer level b_{init}. We will now run simulations in order to evaluate the accuracy of our mapping. Our parameter setting is as follows: $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $s_1 = 8Mbps$, $s_2 = 2Mbps$, $R_{play} = 4Mbps$, $r_1 = -4$, $r_2 = 2$, $R = \text{diag}\left(\begin{bmatrix} r_1 & r_2 \end{bmatrix}\right)$ and

$$T = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix}.$$

The simulation consists of $1,000,000$ sample paths. Examples of realizations of sample paths are represented in Figure 3.1. In these figures we observe that the sample paths follow the asymptotic mean quite well. Figure 3.1b is the logarithmic time scale variant of Figure 3.1a. On the logarithmic time scale in Figure 3.1b the logarithmic growth behavior of the sample paths with respect to time t can be observed.



(a) *Linear time scale.*



(b) *Logarithmic time scale.*

Figure 3.1: *Sample paths of $M^*(t)$ compared to asymptotic mean as expressed in* (3.53). *Sample paths a and b correspond to realisations of $X(t) + Y(t)$ from the fluid model simulation.*

In Figure 3.2 simulations ran for different values of T_{play} for fixed R_{play}. On the vertical axis the required buffer (in seconds) is matched with corresponding p_{empty} on the horizontal axis. With Figure 3.3 the required buffer from simulation is compared to the required buffer using our asymptotic result. Here we observe that for reasonably long T_{play} (minutes) the asymptotic result gives a good handle for determining the required buffer time.

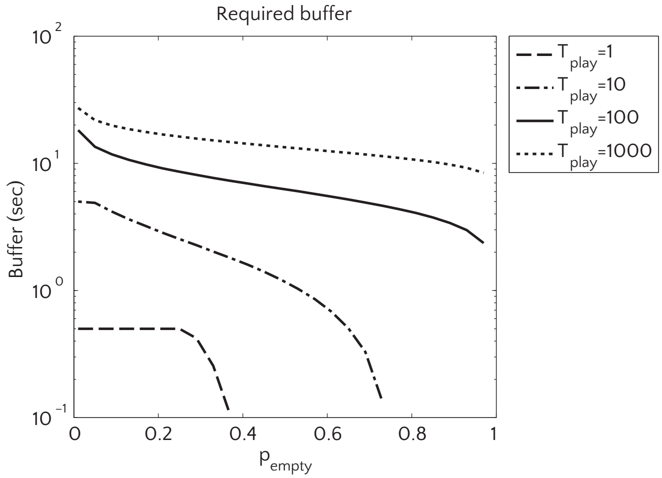


Figure 3.2: *Required buffer (from simulations) for given T_{play} and p_{empty}.*

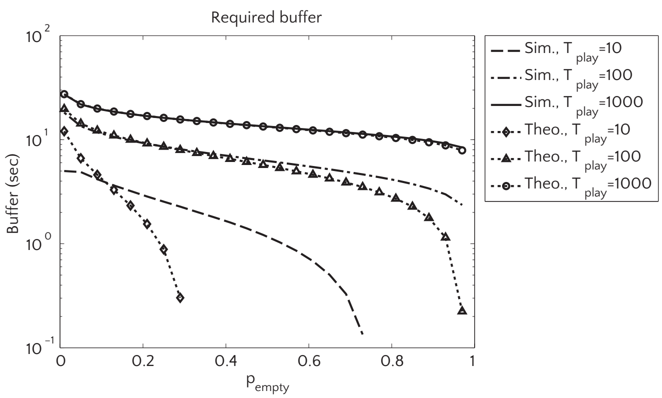
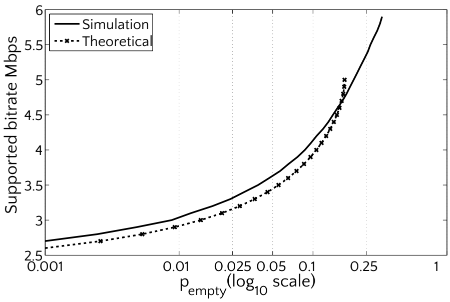
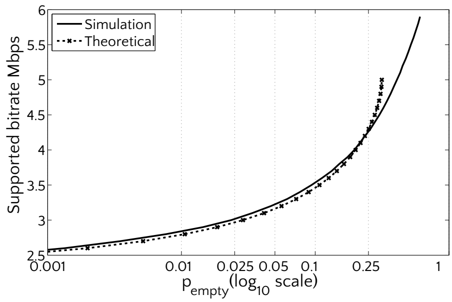


Figure 3.3: *Required buffer (from simulations) for given T_{play} and p_{empty} compared to theoretical required level given by* (3.51)*.*

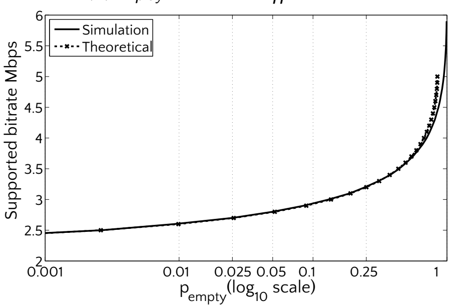
In Figures 3.4a–3.5e the buffer time is fixed while the maximum supported video bitrate is determined. In this setting the network parameters remain fixed while the parameter R_{play} is varied from 2.1 to 5.9. This range is determined by the fact that for the given parameters a minimal bit rate of 2 Mbps is achieved and the average bit rate is equal to 6 Mbps. The maximum supported level determined by simulation is compared to the theoretical maximum supported bit rate. Using (3.50) for given parameters (including R_{play}) the empty buffer probability p_{empty} can be approximated. Note that κ, b and $\mathbb{E}[C]$ all depend on R_{play}. Finding a supported R_{play} using (3.50) is done by applying a search method.



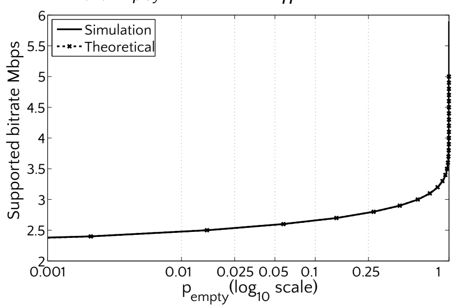
(a) $T_{play} = 60s$, *buffer*$= 10s$.



(b) $T_{play} = 120s$, *buffer*$= 10s$.

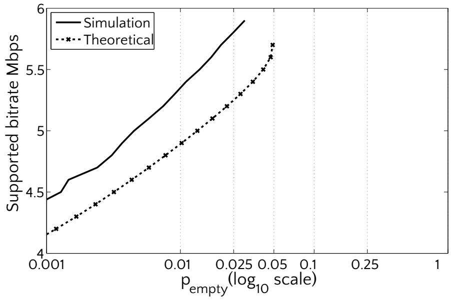


(c) $T_{play} = 600s$, *buffer*$= 10s$.

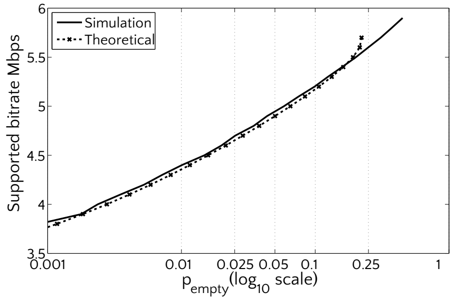


(d) $T_{play} = 3600s$, *buffer*$= 10s$.

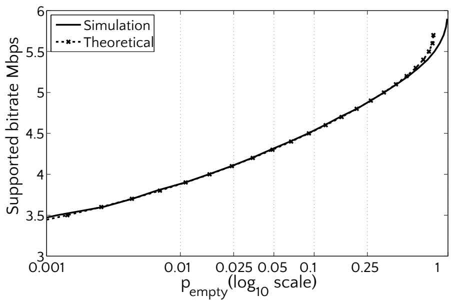
Figure 3.4: *Supported bitrate for given T_{play} and initial buffer level (in seconds) with respect to p_{empty}.*



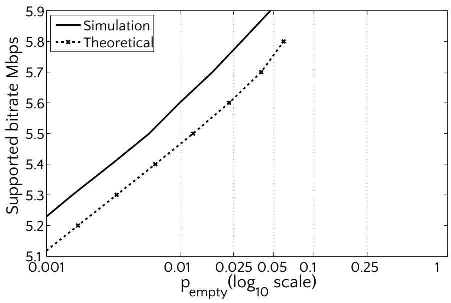
(a) $T_{play} = 120s$, *buffer*$= 30s$.



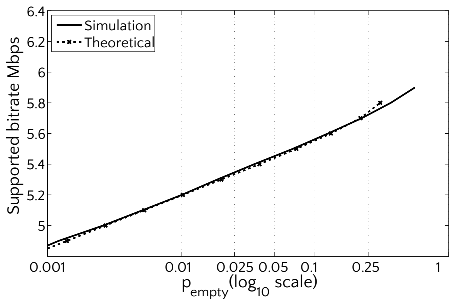
(b) $T_{play} = 600s$, *buffer*$= 30s$.



(c) $T_{play} = 3600s$, *buffer*$= 30s$.



(d) $T_{play} = 600s$, *buffer*$= 60s$.



(e) $T_{play} = 3600s$, *buffer*$= 60s$.

Figure 3.5: *Supported bitrate for given T_{play} and initial buffer level (in seconds) with respect to p_{empty}.*

3.3 Discussion

We extended our model in Chapter 2 such that it supports more than two rates in the Markov Modulated fluid model. In this case there is not always an explicit expression for the initial level of the play-out-buffer. A complicating factor is that the inverse is needed of a complicated matrix expression from the fluid model equations in Equation 2.11, which resulted in a spectral theory analysis approach. However using the Binet-Cauchy formula on minors [47] (Page 12) we were able to provide an explicit recipe to calculate the asymptotic behavior of the maximum level in the Markov Modulated fluid queue. The result can directly be plugged into Equation 2.39 from Section 2.4 to dimension the initial play-out buffer size. From the simulation results we observe that for reasonably long T_{play} the asymptotic result gives a good handle on the required buffer time. The longer the video stream the more accurately the asymptotic distribution of the maximum corresponds to the real distribution of the maximum.

The speed of convergence to the extreme value distribution depends on the rate in which transitions (of the CTMC that models throughput) occur. In the examples we observe that for small timescale the model is less accurate. An improvement would be adding an approximation for the behavior on shorter time scale. We know that when $t \approx 0$ the distribution quantiles grow linearly with respect to transmission rate and initial distribution. We expect a mix of the small timescale linear behavior model and the long time scale extreme value model to become more accurate.

Appendix 3.A Proof of Lemma 3.1.2

3.A.1. Definition. Let A be a $n \times m$ matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}.$$

Then any order-p minor of A will be denoted as:

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} := \det \left[\begin{pmatrix} a_{i_1,k_1} & a_{i_1,k_2} & \cdots & a_{i_1,k_p} \\ a_{i_2,k_1} & a_{i_2,k_2} & \cdots & a_{i_2,k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_p,k_p} & a_{i_p,k_2} & \cdots & a_{i_p,k_p} \end{pmatrix} \right],$$

provided that

$$1 \le i_1 < i_2 < \cdots < i_p \le m,$$
$$1 \le k_1 < k_2 < \cdots < k_p \le n,$$
$$p \le m, n.$$

The Binet-Cauchy formula on minors [47] (Page 12):
Let A be an $m \times n$ matrix, B be a $n \times q$ matrix and C be an $m \times q$ matrix and $C = AB$. Then any minor of C of order p is the sum of the products of all possible minors of A with order p and corresponding minors of the same order of B:

$$C\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} = \sum_{1 \le k_1 < k_2 < \cdots < k_m \le n} A\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}.$$

3.A.2. Lemma. *Let A be a $n \times n$ matrix:*

$$A = \sum_{k=1}^{m} A_k,$$

with:

$$A_k = \begin{bmatrix} a_{k,1,1} & \cdots & a_{k,1,n} \\ \vdots & \ddots & \vdots \\ a_{k,n,1} & \cdots & a_{k,n,n} \end{bmatrix}$$

Define $\mathbf{A}$ *as a $n \times mn$ matrix with:*

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix},$$

and $\mathbf{I}$ *is a $mn \times n$ matrix (consisting of m $n \times n$ identity matrices I_n) defined by:*

$$\mathbf{I} = \begin{bmatrix} I_n & I_n & \cdots & I_n \end{bmatrix}^T.$$

Let $\mathcal{V}$ be the set of subsets with exactly $n-1$ elements from the set $\{1, 2, \ldots, mn\}$ which is defined by:

$$\mathcal{V} = \{(k_1, k_2, \ldots, k_{n-1}) : 1 \le k_1 < k_2 < \cdots < k_{n-1} \le mn\}.$$

Then the following holds:

$$\mathrm{adj}(A) = \sum_{v \in \mathcal{V}} \mathrm{adj}\left(F_C(\mathbf{A}, v) F_R(\mathbf{I}, v)\right),$$

with operators:

$$F_C(\mathbf{A}, v) = \begin{bmatrix} \mathbf{a}_{1,v_1} & \cdots & \mathbf{a}_{1,v_{n-1}} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n,v_1} & \cdots & \mathbf{a}_{n,v_{n-1}} \end{bmatrix},$$

$$F_R(\mathbf{I}, v) = \begin{bmatrix} \mathbf{i}_{v_1,1} & \cdots & \mathbf{i}_{v_1,n} \\ \vdots & \ddots & \vdots \\ \mathbf{i}_{v_{n-1},1} & \cdots & \mathbf{i}_{v_{n-1},n} \end{bmatrix},$$

Operator $F_C(\mathbf{A}, v)$ selects the columns from $\mathbf{A}$ according to vector v, while operator $F_R(\mathbf{I}, v)$ selects rows from $\mathbf{I}$ according to vector v.

Proof. We write $\sum_{k=1}^{m} A_k = \mathbf{AI} = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix} \begin{bmatrix} I_n & I_n & \cdots & I_n \end{bmatrix}^T$. Using the Binet-Cauchy formula on minors, this can be rewritten to:

$$\begin{aligned}
\operatorname{adj}[A] = \operatorname{adj}[\mathbf{AI}] &= \begin{pmatrix} \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{1,1}(v) & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{1,2}(v) & \cdots & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{1,n}(v) \\ \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{2,1}(v) & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{2,2}(v) & \cdots & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{2,n}(v) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{n,1}(v) & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{n,2}(v) & \cdots & \sum_{v\in\mathcal{V}} \bar{\mathbf{a}}_{n,n}(v) \end{pmatrix} \\
&= \sum_{v\in\mathcal{V}} \begin{pmatrix} \bar{\mathbf{a}}_{1,1}(v) & \bar{\mathbf{a}}_{1,2}(v) & \cdots & \bar{\mathbf{a}}_{1,n}(v) \\ \bar{\mathbf{a}}_{2,1}(v) & \bar{\mathbf{a}}_{2,2}(v) & \cdots & \bar{\mathbf{a}}_{2,n}(v) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{a}}_{n,1}(v) & \bar{\mathbf{a}}_{n,2}(v) & \cdots & \bar{\mathbf{a}}_{n,n}(v) \end{pmatrix} \\
&= \sum_{v\in\mathcal{V}} \operatorname{adj}\Big(F_C(\mathbf{A}, v) F_R(\mathbf{I}, v)\Big),
\end{aligned}$$

with

$$\bar{\mathbf{a}}_{i,j}(v) = \mathbf{A}\begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ v_1 & \cdots & v_{i-1} & v_i & \cdots & v_{n-1} \end{pmatrix} \mathbf{I}\begin{pmatrix} v_1 & \cdots & v_{j-1} & v_j & \cdots & v_{n-1} \\ 1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix}.$$

■

We are now ready to finalize the proof of Lemma 3.1.2.

Proof. Let $\mathcal{V}$ be the set of subsets with exactly $n-1$ elements from the set $\{1, 2, \dots, mn\}$ which is defined by:

$$\mathcal{V} = \{(k_1, k_2, \dots, k_{n-1}) : 1 \leq k_1 < k_2 < \cdots < k_{n-1} \leq mn\}.$$

We define $\mathcal{P}$ as the set containing all k-permutations of $n-1$ elements from the set $\{1, \dots, n\}$. Furthermore we define $\mathcal{C}$ as the set with all combinations of $n-1$ elements from the set $\{1, \dots, m\}$. For each combination $c \in \mathcal{C}$ we define:

$$\mathbf{A}_c = \begin{bmatrix} A_{c_1} & A_{c_2} & \cdots & A_{c_{n-1}} \end{bmatrix},$$

$$\mathbf{I}_c = \begin{bmatrix} I_n & I_n & \cdots & I_n \end{bmatrix}^T,$$

and thus:

$$\mathbf{A}_c \mathbf{I}_c = \sum_{k \in c} A_k.$$

Next we apply Lemma 3.A.2:

$$\text{adj}[A] = \sum_{v \in \mathcal{V}} \left(\prod_{k \in v} b_{\lceil k/n \rceil} \right) \text{adj} \left(F_C(\mathbf{A}, v) F_R(\mathbf{I}, v) \right),$$

with

$$\mathbf{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix},$$

and

$$\mathbf{I} = \begin{bmatrix} I_n & I_n & \cdots & I_n \end{bmatrix}^T.$$

Because all matrices A_k have rank 1 the only adjugates that remain are those where there are $n-1$ columns, at $n-1$ different positions, from $n-1$ different A_k matrices. All other combinations of columns result in a matrix with rank $< n-1$ for which the minors of order $n-1$ are zero. Thus the only elements from $\mathcal{V}$ that contribute are those that correspond to any k-permutation of $n-1$ columns from the set $\{1, \dots n\}$ where each column is selected from a distinct matrix A_k, $k = 1, \dots, m$. Note that each selected column remains exactly on its originating column position in the A_k

matrix. As the only combinations consisting of $n-1$ columns at unique positions from $n-1$ unique matrices contribute to non-zero minors it holds that:

$$\begin{aligned}
&\sum_{v\in\mathcal{V}}\left(\prod_{k\in v} b_{\lceil k/n\rceil}\right)\operatorname{adj}\Big(F_C(\mathbf{A},v)F_R(\mathbf{I},v)\Big)\\
&=\sum_{c\in\mathcal{C}}\sum_{p\in\mathcal{P}}\left(\prod_{k\in c} b_k\right)\operatorname{adj}\Big(F_C(\mathbf{A}_c,v_p)F_R(\mathbf{I}_p,v_p)\Big),
\end{aligned}$$

where v_p is the vector that selects the p_ith column from matrix A_{c_i}:

$$(v_p)_i := n(i-1)+p_i, \qquad\qquad i\in\{1,\dots,n-1\},\ p\in\mathcal{P}.$$

We now define $\mathcal{V}_c$ as be the set of subsets with exactly $n-1$ elements from the set $\{1,2,\dots,n(n-1)\}$ which is defined by:

$$\mathcal{V}_c=\{(k_1,k_2,\dots,k_{n-1}):1\le k_1<k_2<\cdots<k_{n-1}\le n(n-1)\}.$$

For each combination $c\in\mathcal{C}$ we can do the opposite: add again the terms (corresponding to zero valued minors) from the set $\mathcal{V}_c$ corresponding to columns of $\mathbf{A}_c=\begin{bmatrix}A_{c_1} & \cdots & A_{c_{n-1}}\end{bmatrix}$:

$$\begin{aligned}
&\sum_{c\in\mathcal{C}}\sum_{p\in\mathcal{P}}\left(\prod_{k\in c} b_k\right)\operatorname{adj}\Big(F_C(\mathbf{A}_c,v_p)F_R(\mathbf{I}_p,v_p)\Big),\\
&=\sum_{c\in\mathcal{C}}\left(\prod_{k\in c} b_k\right)\sum_{v\in\mathcal{V}_c}\operatorname{adj}\Big(F_C(\mathbf{A}_c,v)F_R(\mathbf{I}_c,v)\Big),\\
&=\sum_{c\in\mathcal{C}}\left(\prod_{k\in c} b_k\right)\operatorname{adj}\left[\sum_{k\in c} A_k\right].
\end{aligned}$$

■