The Kac-Ward approach to the Ising Model

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Summary

The Ising model is one of the fundamental models of statistical physics. It was proposed as a model for spontaneous magnetization in ferromagnets. Ising [35] proved that the one dimensional case does not account for the existence of this phenomenon and asserted that the same should hold in higher dimensions. His conclusion was later proved wrong by Peierls [48] who established that in dimensions higher than one the Ising model does exhibit spontaneous magnetization at low temperatures. The critical point, i.e. the temperature at which the phase transition occurs, for the Ising model defined on the square lattice was first identified by Kramers and Wannier [41] as the fixed point of a certain duality transformation. The first rigorous proof of criticality of the self-dual point was given by Onsager [47], who explicitly computed the free energy density of the model.

In search for a solution simpler than the famously complicated algebraic method of Onsager, Kac and Ward [36] proposed a combinatorial approach, where the partition function of the Ising model is given by the determinant of a certain geometrically defined operator (called the Kac–Ward operator). However, their arguments were of heuristic nature and a key assumption turned out to be incorrect. The first fully rigorous account of the Kac–Ward method was given much later by Dolbilin et al. [21].

In this thesis, we revisit the approach of Kac–Ward. In Chapter 2 which is based on joint work with Wouter Kager and Ronald Meester [38], we go through the combinatorial details of the method by following the steps of Vdovichenko [55]. We focus on the loop expansions of the determinant of the Kac–Ward operator. We derive new formulas for the correlation functions and the free energy density of the Ising model in terms of loops with signed weights. By analyzing the spectrum of the Kac–Ward operator, we provide an optimal bound on the growth rate of the signed loops on the square lattice. This allows us to give a new proof of criticality of the self-dual point of Kramers and Wannier. We also show how to obtain Onsager's formula for the free energy density. In Chapter 3, we extend the methods of Chapter 2 to multi-point correlation functions and correlation functions with changed boundary conditions. As a byproduct, we obtain that the boundary multi-point correlations are expressed in terms of the two-point functions. Hence, we rederive a result of Boel, Groeneveld and Kasteleyn [7] .

In Chapter 4 which is based on the article [45], we show how the Kac–Ward method can be used to obtain bounds on the critical temperature of the Ising model defined

on general planar graphs. We obtain a bound on the eigenvalues of the Kac–Ward operator which allows us to identify regions in the complex plane where the free energy density limits are analytic functions of the inverse temperature. The bound turns out to be optimal in the case of isoradial graphs, i.e. it yields criticality of the self-dual Z-invariant coupling constants introduced by Baxter [4]. The class of self-dual Z-invariant Ising models contains, for example, the critical homogeneous models on the square, triangular and hexagonal lattice. To the best of our knowledge, our result is the first of this type which also applies to aperiodic graphs.

Finally, in Chapter 5 which is based on the article [44], we point out a connection between the Kac–Ward method and the recent discrete holomorphic approach of Smirnov et al. [13, 14, 32, 34, 53]. We show that the critical Kac–Ward operator on isoradial graphs acts in a certain sense as the operator of s-holomorphicity. This notion of strong discrete holomorphicity is central to the discrete holomorphic approach to the Ising model. Moreover, we prove that the inverse Kac–Ward operator can be identified with the Green's function of a discrete Riemann–Hilbert boundary value problem. Furthermore, using results from Chapter 4, we provide a general picture of the non-backtracking walk representation of the critical inverse Kac–Ward operators on isoradial graphs. As a consequence, the solution to the discrete Riemann–Hilbert boundary value problem looks similar to the random walk represention of the solution to the discrete Dirichlet boundary value problem for harmonic functions. The important difference is that the non-backtracking walks have complex weights which do not yield a probability measure on the space of walks.