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VRIJE UNIVERSITEIT

Hierarchies, communication and restricted cooperation in cooperative games

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. F.A. van der Duyn Schouten,
in het openbaar te verdedigen
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door
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geboren te Amsterdam

Contents

Introduction

Game theory is about the analysis of the behaviour of agents that try to maximize their utility in situations of social interaction between these agents. Two branches of game theory can be distinguished: non-cooperative game theory and cooperative game theory. The main difference between the two lies in the fact that within cooperative game-theory agents are able to sign binding agreements with each other, whereas this is not the case within non-cooperative game theory. Being able to sign binding agreements allows agents to obtain benefit by working together and forming coalitions. Cooperative game theory uses the model of cooperative TU-games to represent cooperation between agents. In a cooperative TU-game every possible coalition of the agents is assigned a worth, representing what a coalition of agents can obtain by working together. A solution is a function that assigns to every game a payoff vector or a set of payoff vectors which components are the individual payoffs of the players. The two main questions that cooperative game theory tries to answer are what coalitions will be formed between the players and how the benefit of cooperation is divided among them. The first question is more of a strategic one, whereas the second question depends more on notions of what people think of as fair. Both questions are strongly interrelated; agents will seek to form only those coalitions benefitting them. In the literature a number of ways have been considered to determine which coalitions will be formed and what payoffs are associated with these coalitions. Instead of considering the aspects of coalition formation, one might also start by considering a number of desirable properties that a solution should satisfy and then try to find a solution that satisfies these properties. This is the axiomatic approach to cooperative game theory. One of the most famous solutions to cooperative game theory is undoubtedly the Shapley value given by Shapley (1953). This solution has a number of desirable interpretations relating to both coalition formation and distribution of benefit.

In many situations of social interaction between economic agents, it is possible that some of the agents are unable to work together. For this reason models with restricted cooperation have also been studied within the field of cooperative game theory. Myerson (1977) uses an undirected graph to model communication restrictions between agents. In this setup cooperation between agents is only possible when they are connected in the graph. Gilles, Owen and van den Brink (1992) consider games with a hierarchy. Their games with a permission structure model hierarchies by a digraph. Two approaches with respect to permission are distinguished. In the conjunctive approach coalitions are only able to cooperate fully when for players in the coalition their complete set of predecessors in the digraph is also present in the coalition. In the disjunctive approach to permission structures this condition is relaxed; a coalition is able to cooperate fully when for players

in the coalition with predecessors in the graph, at least one predecessor is also present in the coalition. Faigle and Kern (1992) use a similar model, but interpret the digraph in a different way. In their model it represents a number of precedence constraints on the players. Players can only enter a coalition according to the order determined by the digraph. For all of these games a Shapley-type value is defined and axiomatized. Many more models with cooperation restrictions have been considered within the literature. What most of these models have in common, is that they add a collection of so-called feasible coalitions to the classical model of cooperative games. Any coalition that is not feasible is somehow restricted with respect to cooperation and therefore cannot be formed. Several combinatorial structures have been used to generate collections of feasible coalitions. The games on antimatroids by Algaba, Bilbao, van den Brink and Jiménez-Losáda (2004) are obtained by letting the feasible sets be given by an antimatroid. These games have been shown to generalize the games with a permission structure and with these games it is possible to model a wider class of hierarchical situations. Another important class of games with restricted cooperation are the games on union stable systems by Algaba, Bilbao, Borm and López (2001). These represent a general class of games with restricted cooperation, generalizing both the games on antimatroids and the games with communication restrictions by Myerson (1977).

This dissertation will add to the literature on restricted cooperation between players within cooperative TU-games. Several new models are introduced and analyzed that study different situations of restricted cooperation. With the new models of restricted cooperation it will be possible to consider applications that are difficult/impossible to model by the already existing models. We also show how these new models extend already existing models from the literature and how we can relate some of the different models of restricted cooperation to each other.

Overview of the dissertation

We now give a short overview of this dissertation. The first two chapters of this dissertation have an introductory character.

In Chapter 1 an introduction is given to the field of cooperative game theory. Aside from an introduction to the technical aspects of cooperative game theory, in this chapter we also give a motivation for the different concepts within cooperative game theory.

Chapter 2 starts with some preliminaries on graph theory. After this the models of restricted cooperation relevant to this dissertation are introduced. It is shown that a general principle underlying a number of these models is that of so-called collections of feasible coalitions.

In Chapter 3 we introduce union-values for the games with a priori unions introduced by Aumann and Drèze (1974). These games partition the set of players into a number of unions. In the literature solutions for these games have been defined that assign a payoff to the players. The union-values discussed in this chapter assign a payoff to each of the unions instead of to the players. Our reasoning behind these new solutions is that a union represents exactly one decision making agent, while the players belonging to a union represent sub-agents. Union values allow us to more easily model economic situ-

ations in which a decision making agent consists of a number of sub-agents, for example a company with multiple departments or a sports team that consists of a number of players. We introduce two union-values that are both generalizations of the classical Shapley value for cooperative games. The first one is the *union-Shapley value* and considers the agents in the most unified way. The second one is the *player-Shapley value* which takes all players as units, and the payoff of a union is the sum of the payoffs over all its players. An axiomatization of both values is provided. Two collusion neutrality properties are used. The union-Shapley value is axiomatized using *player-collusion neutrality*, while for the player-Shapley value *union-collusion neutrality* is used.

In Chapter 4 we first argue that some known results on peer group games hold more generally for digraph games. It is known that peer group games, introduced by Brânzei, Fragnelli, Tijs (2002), are a special class of the restricted games obtained from a permission structure. However, peer group games are also a special class of the digraph games by van den Brink and Borm (2002). To be specific, they are digraph games in which the digraph is the transitive closure of some rooted tree. Second, we generalize both digraph games as well as conjunctive restricted games by applying a *local approach to permission structures*. In this approach every player needs permission from its predecessors to generate worth. However a player does not need its predecessors in order to give permission to its own successors to generate worth. This implies a separation between generation of worth and permission. The locally restricted games are obtained by applying this approach to games with a permission structure. We introduce and axiomatize a Shapley value type solution for these games, generalizing both the conjunctive permission value for games with a permission structure as well as the β-measure for (weighted) digraph games.

In Chapter 5 a new class of feasible set systems is introduced that has both communication and hierarchical properties. These set systems generalize both cooperation restrictions in communication networks as well as hierarchies. We introduce and analyze *accessible union stable systems* where union stability reflects the communication network and accessibility reflects the hierarchy. Special classes of these new structures are the sets of connected coalitions in a communication graph, antimatroids (and therefore also permission structures) and augmenting systems. We study how accessible union stable systems relate to these structures and we also study a number of applications that can be modelled by accessible union stable systems, but not by these other structures. A subclass of accessible union stable systems is given by adding the property of cycle-freeness. This class coincides with the class of cycle-free graphs, when all singletons are feasible. Games on accessible union stable systems are defined and an extension of the Shapley value for TU-games is characterized on the class of union stable systems by using a balanced contributions property.

In Chapter 6 we consider the hierarchical solution for games with precedence constraints by Faigle and Kern (1992). Payoff assigned by this new solution to relevant players is unaffected by the presence of irrelevant players. We show that this is not the case for the precedence Shapley value by Faigle and Kern (1992). The hierarchical solution can be seen to belong to the class of precedence power solutions. The solutions allocate the worth of a coalition relative to a power measure for acyclic digraphs. The hierarchical solutions allocates proportionally to the hierarchical measure. We give an axiomatization

of this power measure on the class of acyclic digraphs. In addition we extend the hierarchical measure to regular set systems. These feasible set systems contain the class of feasible set systems obtained from acyclic digraphs according to Faigle and Kern. Finally we consider a subclass of acyclic digraphs, given by forests and sink forests and consider the normalized version of the hierarchical measure on these subclasses as well as a number of other power measures.

In Chapter 7 we consider games with conjunctive restrictions where the permission structure is a rooted tree. The conjunctive permission value is considered for these games, as well as modifications of the Myerson value and the hierarchical outcomes for communication graph games. A fourth new solution for these games, the top value, is introduced. This solution assigns the value of the grand coalition to the top player in the tree, and assigns zero payoff to the other players. These four solutions will be compared to each other by providing comparable axiomatizations.

Chapter 1

Cooperative game theory

Game theory can be summarized as the analysis of the behaviour of agents that try to maximize their utility in situations of social interaction between these agents. Research within game theory was started by von Neumann and Morgenstern (1944). Within game theory there exist two main branches of research; non-cooperative and cooperative game theory. Non-cooperative game theory models the agents by players in a strategic game and assumes that every agent is only interested in maximizing his own utility. No binding contracts between agents can be forged and cooperation between agents is not considered. Cooperative game theory in this sense is different; it is assumed that agents can make binding agreements and cooperate with each other. Research focuses primarily on how cooperation takes shape and how to distribute the benefits obtained from cooperation. Within cooperative game theory we can distinguish between the analysis of cooperative games with transferable utitility (TU-games) and those with non transferable utility (NTU-games). For the former it holds that there is some common means of utility that is valued equally by the different agents (for example money) and that can be traded freely among those agents, where this is not the case for the latter. The games appearing in this thesis will be exclusively cooperative games with transferable utility. From here on a cooperative game will therefore refer to a cooperative game with transferable utitility.

Definition 1.0.1 *A cooperative game with transferable utility (or TU-game) is a pair (N, v), where $N \subseteq \mathbb{N}$ is a set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, that assigns to every possible subset S of N a worth $v(S)$, where $v(\emptyset) = 0$.*

The subsets of player set N are referred to as coalitions. The worth $v(S)$ represents what the players in coalition S can obtain by cooperating in game (N, v). The class of all cooperative games is denoted by $\mathcal{G}$. Given a fixed player set N the set of all cooperative games (N, v) is denoted by $\mathcal{G}^N$. The game created by considering only coalition S and its subsets is denoted by (S, v_S), where $v_S(T) = v(T)$ for all $T \subseteq S$. We refer to (S, v_S) as the subgame on S. In some chapters if there is no confusion about the player set N a game (N, v) will be referred to by its characteristic function v. The operations of $+ : \mathcal{G}^N \times \mathcal{G}^N \to \mathcal{G}^N$ and $\cdot : \mathbb{R} \times \mathcal{G}^N \to \mathcal{G}^N$ are also defined, where for every $v, w \in \mathcal{G}^N$ and $c \in \mathbb{R}$, $(v + w)(S) = v(S) + w(S)$ and $(c \cdot v)(S) = c \cdot v(S)$ for $S \subseteq N$. It is assumed that every possible coalition of players is assumed to be able to work together in some way.

In its most basic sense cooperative game theory attempts to study social interaction between agents by using cooperative games to represent these situations. We give an example that shows how a cooperative game can be used to model economic situations.

Example 1.0.2 *Consider a situation with three economic agents, where one of the agents is the seller of a good and the other two players are potential buyers. The agents are modelled by player set $N = \{1, 2, 3\}$, where player 1 represents the seller and players 2 and 3 represent the buyers. Assume that player 1 has no use for the good at all if he cannot sell it and therefore his valuation of the good is 0. Player 2 has a valuation for the good of 10 and for player 3 this is 8. The game v is constructed as follows. Any coalition not containing both a buyer and a seller does not obtain any worth, therefore $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$. Now when a buyer and a seller get together, the good can be utilized by the buyer with the highest valuation for it and therefore $v(\{1, 2\}) = 8$, $v(\{1, 3\}) = 10$ and $v(\{1, 2, 3\}) = 10$.*

It should be noted that the worth obtained by a coalition of players $v(S)$ is not affected by what any of the players outside of S are doing, ignoring a strategic aspect that plays an important part in non-cooperative game theory (how economic agents 'play' versus each other). One can imagine real-life situations of cooperation however, where the actions of these players do matter. For these situations a game (N, v) is too limited to describe all details. An extension of the classical TU-game to more accurately model these situations is given by the cooperative game in partition function form introduced by Lucas and Thrall (1963). In a partition function form game the worth obtained by a coalition also depends on how the players outside of the coalition cooperate amongst themselves. In this way externalities of the cooperation between sets of agents can also be taken into account. In this thesis however we only consider classical TU-games.

Often rather than studying the class of all games $\mathcal{G}$ only a subclass with some distinct properties on the characteristic function v is studied. We present some of the important subclasses that will appear throughout this thesis.

Definition 1.0.3 *Let $(N, v) \in \mathcal{G}$ be a cooperative TU-game.*

(i) *(N, v) is monotonic if, when $S \subseteq T \subseteq N$, it holds that $v(S) \leq v(T)$.*

(ii) *(N, v) is additive if for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, it holds that $v(S \cup T) = v(S) + v(T)$.*

(iii) *(N, v) is superadditive if for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, it holds that $v(S \cup T) \geq v(S) + v(T)$.*

(iv) *(N, v) is convex, when for all $S, T \subseteq N$ it holds that $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. Another way of describing convex games is by saying that for all coalitions $S, T \subseteq N$ where $S \subseteq T \subseteq N$ it holds that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N \backslash T$.*

(v) *(N, v) is balanced, when for every function $\alpha : 2^N \backslash \{\emptyset\} \to [0, 1]$ where for all $i \in N$ it holds that $\displaystyle\sum_{\{S \in 2^N : i \in S\}} \alpha(S) = 1$, it holds that $\displaystyle\sum_{S \in 2^N \backslash \{\emptyset\}} \alpha(S) \cdot v(S) \leq v(N)$.*

The class of all monotonic games is denoted by $\mathcal{G}_M$. The class of all additive games is denoted by $\mathcal{G}_A$. The class of all superadditive games is denoted by $\mathcal{G}_{SA}$. The class of all convex games is denoted by $\mathcal{G}_C$. Note that convex games form a subclass of the class of superadditive games. The class of all balanced games is denoted by $\mathcal{G}_B$. For fixed player set N these classes of subgames will be denoted by $\mathcal{G}_M^N, \mathcal{G}_A^N, \mathcal{G}_{SA}^N, \mathcal{G}_C^N$ and $\mathcal{G}_B^N$ respectively.

The basic model of cooperative games only associates a worth $v(S)$ with each coalition S of players. It does not say anything about what coalitions will eventually be formed and in what way worth is to be divided among the players.

Define $x \in \mathbb{R}^N$ to be a payoff or distribution vector for the players in a cooperative game (N, v). The payoff assigned to player $i \in N$ is given by x_i. A payoff vector can be interpreted as what can be obtained by the players from playing the game. A solution to a game is then defined to be a mapping that assigns to every game (N, v) either a single payoff-vector (also referred to as single-valued solutions) or a set of payoff-vectors (also referred to as set-valued solutions). Whether the payoff vector assigned to game (N, v) by a solution can be considered fair depends strongly on the worths associated with that game. In some way the worths of a game can be said to provide the players with a set of claims reflecting their bargaining position in a game. The idea is that by looking at these claims we will be able to make some conclusions on both coalition formation and reasonable payoffs to the players.

What coalitions will eventually form and the final payoff that a player is able to obtain are strongly interrelated. This is made clear by the following example:

Example 1.0.4 *Consider game (N, v) where $N = \{1, 2, 3\}$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2\}) = 1, v(\{1, 3\}) = 1, v(\{2, 3\}) = 1$ and $v(\{1, 2, 3\}) = 1$. This game is also known as a simple majority game. It is assumed that players have to decide on how to efficiently distribute the worth of 1 that can be obtained by cooperation. One might think that perhaps a two-player coalition is most likely to form, since singleton coalitions do not obtain any worth and the three-player coalition $\{1, 2, 3\}$ obtains the same worth as any of the two-player coalitions (and so a third player is redundant). Suppose therefore that players 1 and 2 decide to form coalition $\{1, 2\}$, excluding player 3. One might consider a payoff vector x given by $(x_1, x_2, x_3) = (0.5, 0.5, 0)$. However this does not represent a stable situation since both player 2 as well as player 3 can obtain higher payoffs by forming coalition $\{2, 3\}$ and for instance distributing the worth of 1 by payoff vector $x' = ((x')_1, (x')_2, (x')_3) = (0, 0.8, 0.2)$. For this example it can actually be shown that for any efficient payoff vector x there exists another efficient payoff vector x' that is preferred by 2 of the 3 players. No solution can be found such that each coalition of players obtains at least its own worth. Therefore a straightforward outcome to this game with respect to coalition formation does not seem to exist. Coalition formation with respect to simple majority games has also been studied in Bennett and van Damme (1991) and Montero (2000).*

For the simple majority game from the previous example a payoff vector such that every coalition obtains at least its own worth does not exist. Related to this concept is the set-valued solution known as the Core, introduced by Gillies (1953). For a game (N, v) the Core is the set of payoff-vectors where the total worth distributed is $v(N)$, the worth

of the grand coalition N, and every coalition S earns at least $v(S)$, the worth that it can obtain on its own.

Definition 1.0.5 *The Core is the set-valued solution on $\mathcal{G}$ given by*

$$Core(N,v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \;\; and \;\; \sum_{i \in S} x_i \geq v(S), \;\; for \; all \; S \subset N\}.$$

Since every coalition obtains at least its own worth, the Core can be considered stable in a strategic sense; no coalition can do better by breaking off from the grand coalition. It must be noted however that in the context of the class of all cooperative games $\mathcal{G}$, some games might have an empty Core (containing no payoff-vectors). Taking Example 1.0.2 we find that not all two-player coalitions can obtain their worth at the same time. Therefore in the previous example the game has an empty Core. The following theorem was proven independently by Bondareva (1963) and Shapley (1967).

Theorem 1.0.6 (Bondareva, 1963) and (Shapley, 1967)
The Core of a game (N,v) is non-empty if and only if $(N,v) \in \mathcal{G}_B$.

We remark that every convex game is balanced and therefore has a non-empty Core.

If the previous example represents a situation between economic agents, is it likely for them not to come to some kind of agreement eventually? Benefit is to be made, so at some point two players might just 'find' each other and make some binding agreement on the payoffs, not allowing them to deviate from those payoffs. In this case perhaps the assumption of rational agents is too strict. Perhaps it might be better to think less of the strategic considerations with respect to coalition formation in resolving a game. Instead of deciding what coalitions will form and then what to divide, we could start by defining a solution that assigns some payoff-vector to the players of a game on the basis of some reasonable arguments or properties that we would like a solution to have. Whether a solution can then be reasonably applied to any single game, needs to be decided for specific subclasses. This is quite different from first resolving conflicts before deciding the amount that needs to be distributed. This so-called 'axiomatic' approach to cooperative games has been the main approach in cooperative game theory since its inception and it is also the approach that will be taken with respect to cooperative games in this thesis.

Set-valued solutions like the Core possibly contain a number of different payoff vectors. In some cases it might be desirable to have a solution that assigns exactly one payoff-vector to every game. For this reason the single-valued solutions to games were introduced (sometimes also referred to as value functions). In this thesis only single-valued solutions will be considered. When there is no confusion, we will therefore simply refer to single-valued solutions as solutions. One of the most famous solutions within cooperative game theory is the Shapley value introduced by Shapley (1953) and denoted by Sh.

Definition 1.0.7 *The Shapley value is the solution on $\mathcal{G}$ given by*

$$Sh_i(N,v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S) - v(S \setminus \{i\})) \; for \; all \; i \in N.$$

Note that this shows that the Shapley value of a game (N, v) can be expressed as a weighted average of the marginal contributions $v(S) - v(S \setminus \{i\})$ of a player i to all coalitions that he is not part of. Shapley justified his solution by showing that it is uniquely characterized by a number of desirable properties or axioms that seem to adhere to general intuitions of what is fair in dividing worth.

A solution f satisfies efficiency if the total amount of payoff distributed is exactly equal to $v(N)$, the value of the grand coalition.

Efficiency For each game $(N, v) \in \mathcal{G}$ it holds that $\sum_{i \in N} f_i(N, v) = v(N)$.

A solution f satisfies additivity if it is additive over games.

Additivity For every pair of games (N, u) and $(N, w) \in \mathcal{G}$ it holds that $f(N, u + w) = f(N, u) + f(N, w)$

Call two players $i, j \in N$ symmetric in game v if their marginal contribution to any coalition is the same, meaning that for each $S \subset N$ for which $i, j \notin S$ it holds that $v(S \cup \{i\}) = v(S \cup \{j\})$. A solution f satisfies symmetry if symmetric players obtain the same payoff.

Symmetry For each game $(N, v) \in \mathcal{G}$ in which players $i, j \in N$ are symmetric it holds that $f_i(N, v) = f_j(N, v)$.

Call a player $i \in N$ a null player if for every $S \subset N$ for which $i \notin S$ it holds that $v(S \cup \{i\}) - v(S) = 0$. A solution f satisfies the null player property if null players obtain a payoff of 0.

Null player property For each game $(N, v) \in \mathcal{G}$ in which i is a null player it holds that $f_i(N, v) = 0$.

The above axioms uniquely characterize the Shapley value.[1]

Theorem 1.0.8 (Shapley, 1953)
A solution on $\mathcal{G}$ is equal to the Shapley value if and only if it satisfies efficiency, additivity, symmetry and the null player property.

Young (1985) considers the axiom of strong monotonicity. A solution f satisfies this axiom if, when the marginal contributions of a player i in game (N, v) are at least as high as those in game (N, w), then that player should obtain at least as much payoff in game (N, v) as in game (N, w).

Strong monotonicity For games (N, v) and $(N, w) \in \mathcal{G}$ if $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$ then $f_i(N, v) \geq f_i(N, w)$.

Another axiomatization of the Shapley value is obtained by replacing additivity and the null player axiom by strong monotonicity.

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[1]Shapley (1953) combines efficiency and the null player property into a carrier axiom.

Theorem 1.0.9 (Young, 1985)
A solution on $\mathcal{G}$ is equal to the Shapley value if and only if it satisfies efficiency, symmetry and strong monotonicity.

In Young's proof the property of strong monotonicity in the above axiomatization can in fact be replaced by the weaker property of marginalism. This property replaces the $\geq$ sign of strong monotonicity by an $=$ sign; the payoffs to a player i in games (N, v) and (N, w) are equal, as long as this player's marginal contributions in both games are the same.

Marginalism For games (N, v) and $(N, w) \in \mathcal{G}$ if $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$ then $f_i(N, v) = f_i(N, w)$.

We note that the above axiomatizations by Shapley (1953) and Young (1985) are in fact given for the class of games $\mathcal{G}^N$, given fixed player set N.

Myerson (1980) considers the axiom of balanced contributions. A solution f satisfies balanced contributions if the change in payoff to player j when player $i \neq j$ leaves the game, is the same as the change in payoff to player i, when player j leaves the game.

Balanced contributions For game $(N, v) \in \mathcal{G}$ it holds that $f_i(N, v) - f_i(N \setminus \{j\}, v_{N \setminus \{j\}}) = f_j(N, v) - f_j(N \setminus \{i\}, v_{N \setminus \{i\}})$

Another well known axiomatization of the Shapley value is obtained by using efficiency and balanced contributions.

Theorem 1.0.10 (Myerson, 1980)
A solution on $\mathcal{G}$ is equal to the Shapley value if and only if it satisfies efficiency and balanced contributions.

The axiomatization by Myerson cannot be given for the class of games $\mathcal{G}^N$, given fixed player set N, since balanced contributions is applied to games with different sets of players.

Haller (1994) shows that if payoffs are distributed in a game according to the Shapley value, collusive agreements between two players need not be beneficial to these players. To demonstrate this he introduces several collusion neutrality properties for TU-games. Such axioms state that the sum of the payoffs of two players does not change if they 'collude'. In particular Haller considers the situation in which two players can decide to make a so-called *proxy agreement*. In this agreement player $i \in N$ acts as a proxy for player $j \in N \setminus \{i\}$. When i acts as a proxy for player j, instead of v we consider the characteristic function $v_{ij} \in \mathcal{G}^N$ given by

$$v_{ij}(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S \cup \{j\}) & \text{if } i \in S. \end{cases} \tag{1.0.1}$$

So, when player i acts as a proxy for another player j the latter player becomes a null player, and whenever player i enters a coalition also the contribution of j is added. A

solution f satisfies the proxy agreement property by Haller if a proxy agreement between two players does not affect the sum of the payoffs to those two players.

Haller shows that the Shapley value does not satisfy the proxy agreement property and neither does it satisfy a number of other collusion properties. He shows that these properties can be used to axiomatize the (non-efficient) Banzhaf value. Later, Malawski (2002) and Casajus (2012) showed that several other collusion neutrality properties can be used.[2] Van den Brink (2012) later showed that requiring efficiency and collusion neutrality at the same time is very restrictive for TU-games. For example, there is no solution for TU-games that satisfies efficiency, association collusion neutrality and the null player property.[3]

We have seen that the Shapley value satisfies efficiency. This means that it always distributes exactly the worth of the grand coalition among the players. The Shapley value however is not always in the Core. This holds even for games (N, v), where the Core is non-empty, as the following example illustrates.

Example 1.0.11 *Consider game (N, v) where $N = \{1, 2, 3\}$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 0$ and $v(\{1, 2, 3\}) = 6$. This might be interpreted as another game where players 2 and 3 want to buy an object from player 1, where they both have a valuation for the object of 6 and player 1 has zero valuation for the object. Here there is only one payoff vector in the Core, namely $x = (6, 0, 0)$ while the Shapley value is given by $Sh(N, v) = (4, 1, 1)$.*

For the class of convex games $\mathcal{G}_C$ it holds that the Shapley value is always in the Core. This was proven by Shapley (1971).

The Shapley value will be the main solution concept to cooperative games considered in this thesis. We have seen that there exist games (N, v) with a non-empty Core, where the payoff vector assigned by the Shapley value is not in the Core. We have argued before that the Core can be considered stable in the sense that any payoff in the Core guarantees any coalition S a payoff that is at least equal to $v(S)$. Therefore why do we not consider another solution which is always in the Core when it is non-empty (for example the nucleolus defined by Schmeidler (1969))? Although the Core has a number of desirable properties, it might also be argued that in some cases there exist payoff vectors outside the Core that also make sense. In the previous example, if both potential buyers are heavily competing with each other in order to buy the good, the unique Core payoff does not seem to be unreasonable. However now consider the situation where the buyers are willing/able to make binding agreements with each other. They might decide to agree on buying the good at a given price p where $0 \le p \le 6$ and give some worth to the person that does not obtain the product. Such a strategy would allow them to obtain more worth than they get from the payoff vector in the Core. This situation shows us that to determine which solution concept is most reasonable we need more information

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[2]A characterization of the Banzhaf value with amalgamation or merge properties where two players merge in the sense that they are replaced by one player can be found in Lehrer (1988).

[3]It is also mentioned that the equal division solution, that distributes $v(N)$ equally over all players, is the only solution satisfying efficiency, collusion neutrality and symmetry.

than can be included in a cooperative game, first of all with respect to how the agents will behave. This illustrates the already mentioned idea that perhaps the best we can do in cooperative game theory, given that not all aspects from reality can fit into a cooperative game, is to use the axiomatic approach: to choose from solutions based on a set of reasonable properties. Out of all the solutions that can be obtained by applying an axiomatic approach to cooperative games, this thesis will focus mainly on (extensions of) the Shapley value.

Next let $\Pi(N)$ be the collection of all permutations $\pi \colon N \to N$ on N. For a permutation $\pi \in \Pi(N)$ let $\pi(i)$ be the position of a player i in π. The marginal vector $m^\pi(N, v) \in \mathbb{R}^N$, associated with game (N, v) and permutation $\pi \in \Pi(N)$, is given by the vector of marginal contributions, where players join to form the grand coalition N according to π:

$$m_i^\pi(N, v) = (v(\{j \in N : \pi(j) \le \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})) \text{ for all } i \in N. \quad (1.0.2)$$

Another well-known expression of the Shapley value is obtained by taking the average marginal contribution over all permutations on the player set. We obtain the following alternative definition.

Definition 1.0.12 *The Shapley value is the solution on $\mathcal{G}$ given by*

$$Sh_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} m_i^\pi(N, v) \text{ for all } i \in N.$$

Up until this point we have only considered the worths of a game v in distributing payoff to the players in N. We now consider the so-called *Harsanyi dividends* of a game, which were first introduced by Harsanyi (1959).

Definition 1.0.13 *The (unique) dividend of a coalition $T \subseteq N$ in game (N, v) is given by*

$$\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} v(S).$$

Another way of writing the Harsanyi dividends is given by $\Delta_v(\{i\}) = v(\{i\})$ for all $i \in N$ and $\Delta_v(T) = v(T) - \sum_{S \subset T, S \neq \emptyset} \Delta_v(S)$. This last expression states that the dividend of a coalition T is given by the worth of T minus the dividends obtained by its proper sub-coalitions. Therefore the dividend to a coalition T might be considered as the benefit (or loss) of cooperation exclusively generated by T. We illustrate this with the following example.

Example 1.0.14 *Consider game (N, v) where $N = \{1, 2, 3\}$ and $v(\{1\}) = 3$, $v(\{2\}) = 4$, $v(\{3\}) = 5$, $v(\{1, 2\}) = v(\{1, 3\}) = 9$, $v(\{2, 3\}) = 10$ and $v(\{1, 2, 3\}) = 20$. The dividends for singleton coalitions are given by $\Delta_v(\{1\}) = 3$, $\Delta_v(\{2\}) = 4$ and $\Delta_v(\{3\}) = 5$. For two-player coalitions $\{i, j\}$ we obtain that $\Delta_v(\{i, j\})$ is equal to $v(\{i, j\}) - \Delta_v(\{i\}) - \Delta_v(\{j\})$. This gives $\Delta_v(\{1, 2\}) = 2$, $\Delta_v(\{1, 3\}) = 1$, $\Delta_v(\{2, 3\}) = 1$. Finally $\Delta_v(\{1, 2, 3\}) = 4$. Note that if $v(\{1, 2, 3\})$ had been smaller than 16, we would have obtained $\Delta_v(\{1, 2, 3\}) < 0$.*

A special type of TU-game consists of the class of so-called unanimity games.

Definition 1.0.15 *For each $T \subseteq N$, $T \neq \emptyset$, the* unanimity *game (N, u_T) is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise.*

For every $v \in \mathcal{G}^N$, from Harsanyi (1959) we obtain that v can be written as the sum over all coalitions T of the multiplication of the unanimity game of T by $\Delta_v(T)$. This gives the following expression:

$$v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T. \qquad (1.0.3)$$

It follows that the worth $v(T)$ of a coalition T is equal to the sum of the dividends of all subsets of T, so $v(T) = \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} \Delta_v(S)$.

The Shapley value can be seen to equally divide the dividends of any coalition S among the players in that coalition, thereby adhering to the idea of a fair solution. We obtain the following alternative definition.

Definition 1.0.16 *The Shapley value is the solution on $\mathcal{G}$ given by*

$$Sh_i(N, v) = \sum_{\{S \subseteq N : i \in S\}} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N.$$

Chapter 2

Restricted cooperation

In a classical cooperative TU-game (N, v) no restrictions with respect to cooperation are inherently part of the model. However there exist situations of social interaction between decision making agents in which not all coalitions can be formed. Two agents might only be able to communicate with each other through a third intermediary agent or an agent might be part of a bigger group that is only able to cooperate with agents outside of the group as a whole. Extensions of the classical model have been devised in order to study such situations in more detail. In this chapter some of the more prominent extensions established in the literature will be discussed. What these models have in common is that they add some mathematical structure $\mathcal{C}$ (for example a set system) to a classical game (N, v), thereby creating a model of restricted cooperation. The mathematical structure is what represents the restrictions on coalition formation. Solutions for the new models of restricted cooperation can be obtained by applying solutions for classical games to restricted games deduced from these models. These new solutions are axiomatized using axioms that somehow relate to the mathematical structure underlying the restricted game.

2.1   Graph Theory

In this section some basic concepts in graph theory will be studied. Later these concepts will be applied to games.

2.1.1   Undirected Graphs

An undirected graph is a pair (N, L) where N is a set of nodes and $L \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$ is a set of unordered pairs of distinct elements of N. We assume N to be finite. In this thesis the nodes represent the players in a game (N, v). We therefore refer to these nodes as players. The elements of L are called (undirected) links or edges. We denote the set of all graphs by $\mathcal{L}$ and those on N by $\mathcal{L}^N$. In some chapters if there is no confusion about the player set N, we will write a graph $(N, L) \in \mathcal{L}^N$ just by its set of links L. For $S \subseteq N$, the graph $(S, L(S))$ with $L(S) = \{\{i, j\} \in L : i, j \in S\}$ is called the subgraph of L on S. Note that $L(N) = L$. For a graph (N, L) and a player $i \in N$ let $L_{-i} = L(N \setminus \{i\})$. Given $(N, L) \in \mathcal{L}^N$, a sequence of k different players $(i_1, ..., i_k)$ is a *path*

in $L(S)$ if $\{i_l, i_{l+1}\} \in L(S)$ for $l = 1, ..., k-1$. Two players $i, j \in S$ are called *connected* in $L(S)$ if $i = j$ or there exists a path $(i_1, ..., i_k)$ in $L(S)$ with $i_1 = i$ and $i_k = j$. A path $(i_1, \ldots, i_k)$ with $k \geq 3$ is defined to be a *cycle* in $(S, L(S))$, if also $\{i_1, i_k\} \in L$. A graph (N, L) is *cycle-free*, if it does not contain any cycle. A coalition $S \subseteq N$ is said to be a connected coalition if every two players in S are connected in $L(S)$. A graph (N, L) is said to be *connected* if any two players $i, j \in N$ are connected in L. A graph that is both cycle-free and connected is called a *tree*. The collection of trees is denoted by $\mathcal{L}_t$. The collection of trees on N is denoted by $\mathcal{L}_t^N$. Note that in a tree two players are connected by exactly one path. A coalition $K \subseteq N$ is a *component* of (N, L) if and only if (i) K is connected in L, and (ii) $K \cup \{i\}$ is not connected in L for every $i \in N \backslash K$. The set of components of $(S, L(S))$ is denoted by $C^L(S)$. Note that every player in $S \subseteq N$ that is not linked with any other player in S is a (singleton) component in $(S, L(S))$.

2.1.2 Directed Graphs

A directed graph or digraph is a pair (N, D) where N is a set of nodes and $D \subseteq \{(i, j) : i, j \in N, i \neq j\}$ is a set of ordered pairs of distinct elements of N. As with undirected graphs the nodes represent the players in a game (N, v). We refer to these nodes as players. The elements of D are called directed links or arcs. We denote the set of all directed graphs by $\mathcal{D}$ and those on N by $\mathcal{D}^N$. In some chapters if there is no confusion about the player set N, we will write a graph $(N, D) \in \mathcal{D}^N$ just by its set of arcs D. For $S \subseteq N$, the graph $(S, D(S))$ with $D(S) = \{(i, j) \in D : i, j \in S\}$ is called the subgraph of D on S. For a digraph (N, D) and a player $i \in N$ let D_{-i} be $D(N \setminus \{i\})$. For $i \in N$ the nodes in $F_D(i) := \{j \in N : (i, j) \in D\}$ are called the *followers* of i, and the nodes in $P_D(i) := \{j \in N : (j, i) \in D\}$ are called the *predecessors* of i in D. Given $(N, D) \in \mathcal{D}^N$, a sequence of k different players $(i_1, ..., i_k)$ is a (directed) *path* in D if $(i_l, i_{l+1}) \in D$ for $l = 1, ..., k-1$. Next for a digraph (N, D) define $(N, L_D) \in \mathcal{L}^N$ to be the undirected graph where $\{i, j\} \in L_D$ if $(i, j) \in D$ or $(j, i) \in D$. Two players $i, j \in N$ are connected in $D \in \mathcal{D}^N$ if they are connected in (N, L_D). A set of players $S \subseteq N$ is connected in $D \in \mathcal{D}^N$ if S is connected in (N, L_D). A *component* in a directed graph is a set of players that that forms a component in (N, L_D). We denote the set of all components in $(S, D(S))$ by $C_D(S)$.

The *transitive closure* of $(N, D) \in \mathcal{D}^N$ is the digraph $tr(D)$, where $(i, j) \in tr(D)$ for any $i, j \in N, i \neq j$, if and only there is a directed path from i to j in D. By $\widehat{F}_D(i) = F_{tr(D)}(i)$ we denote the set of followers of i in the transitive closure of D, and refer to these players as the *subordinates* of i in D. We refer to the players in $\widehat{P}_D(i) = \{j \in N : i \in \widehat{F}_D(j)\}$ as the *superiors* of i in D. A digraph $D \in \mathcal{D}^N$ is transitive if $D = tr(D)$. For a set of players $S \subseteq N$ we denote by $F_D(S) = \bigcup_{i \in S} F_D(i)$ the set of followers of players in coalition S and by $P_D(S) = \bigcup_{i \in S} P_D(i)$, the set of predecessors of players in coalition S. Also, for $S \subseteq N$, we denote $\widehat{F}_D(S) = \bigcup_{i \in S} \widehat{F}_D(i)$ and $\widehat{P}_D(S) = \bigcup_{i \in S} \widehat{P}_D(i)$. Call a player in a digraph a *top player* when it has no predecessors, meaning that $P_D(i) = \emptyset$. Denote by $TOP(N, D)$ the set of all top players in digraph (N, D).

A directed path $(i_1, \ldots, i_k)$, $k \geq 2$, in D is a *cycle* in D if $(i_k, i_1) \in D$. We call digraph D *acyclic* if it does not contain any cycle. Note that acyclicity of digraph D

implies that $TOP(N, D)$ is non-empty. Acyclicity of D does not imply that L_D is cycle-free. We denote the collection of all acyclic digraphs by $\mathcal{D}_a$ and those on N by $\mathcal{D}_a^N$. A digraph $D \in \mathcal{D}^N$ is a hierarchical digraph if and only if (i) if it is *quasi-strongly connected*, meaning there is an $i_0 \in N$ $\widehat{F}_D(i_0) = N \setminus \{i\}$, and (ii) D is acyclic. We denote the collection of all hierarchical digraphs by $\mathcal{D}_h$ and those on N by $\mathcal{D}_h^N$. A digraph $D \in \mathcal{D}^N$ is a *rooted tree* if and only if (i) there is an $i_0 \in N$ such that $P_D(i_0) = \emptyset$ and $\widehat{F}_D(i_0) = N \setminus \{i\}$, and (ii) $|P_D(i)| = 1$ for all $i \in N \setminus \{i_0\}$. Note that this implies that D is acyclic. We denote the collection of all rooted trees by $\mathcal{D}_t$ and those on N by $\mathcal{D}_t^N$. A rooted tree is also a hierarchical digraph. Note that for a rooted tree D the undirected graph L_D contains no cycles, while for D a hierarchical digraph this is possible.

2.2 Games with restricted cooperation

2.2.1 Games with a priori unions

Aumann and Drèze (1974) were one of the first to extend classical TU-games to situations with restricted cooperation. They considered the so-called *games with a priori unions*. A situation is considered where the player set N is partitioned into a number of a priori unions.

Definition 2.2.1 *A collection $P = \{P_1, ..., P_m\}$ of sets is a partition of the player set N if it satisfies the following properties:*

(i) $\bigcup_{k=1}^{m} P_k = N$

(ii) $P_k \neq \emptyset$ *for all $k \in \{1, \ldots, m\}$*

(iii) $P_k \cap P_l = \emptyset$ *for all $k, l \in \{1, \ldots, m\}$ with $k \neq l$.*

Let $\mathcal{P}^N$ be the collection of all partitions of N. For a given $P = \{P_1, ..., P_m\} \in \mathcal{P}^N$, let $M = \{1, ..., m\}$. Then $P = \{P_j : j \in M\}$ is called a system of a *priori unions* (often also reffered to as *coalition structure*), and any element P_j, $j \in M$, is called a *union* of P.

Definition 2.2.2 *A TU-game with a priori unions is a triple (N, v, P) with $(N, v) \in \mathcal{G}$ and $P \in \mathcal{P}^N$ a partition of N.*

We denote the collection of all TU-games with a priori unions by $\mathcal{G}_{APU}$. Different interpretations have been given with respect to the a priori unions.

The interpretation given to these games by Aumann and Drèze is that players are only able to work together as long as they are from the same union. A single-valued solution f on $\mathcal{G}_{APU}$ assigns a unique payoff vector $f(N, v, P) \in \mathbb{R}^N$ to every $(N, v, P) \in \mathcal{G}_{APU}$. One of the solutions Aumann and Drèze introduce in their paper is an extension of the classical Shapley value; it assigns to every player the Shapley value applied to the subgame on the union which that player is part of.

Another well-known interpretation to games with a priori unions was given by Owen (1977). He assumes that unions as a whole are freely able to work together. Subsets

of different unions cannot work together, but it is possible that a subset of (only) one union deviates from that union and works together with other (whole) unions. The Owen value can be obtained by first assigning to every union the Shapley value of the so-called *quotient* game on the unions. In this classical TU game the players are given by the unions $M = \{1, \ldots, m\}$ and every time union $k \in M$ enters a coalition, all players in P_k enter the coalition.

Definition 2.2.3 *The quotient game $v^P \in \mathcal{G}^M$ of game with a priori unions $(N, v, P) \in \mathcal{G}_{APU}$, where $P = \{P_1, ..., P_m\}$, is given by*

$$v^P(S) = v(\bigcup_{k \in S} P_k) \text{ for all } S \subseteq M.$$

The Owen value first assigns a payoff to each union $k \in M$. This payoff is given by $Sh_k(M, v^P)$. After a payoff has been assigned to a union it is then further divided among the players in that union by applying the Shapley value another time to a game between the players of that union. Recall from Definition 1.0.12 that the Shapley value assigns to the players the average over all marginal vectors associated with permutations of the player set N. The Owen value can be expressed in a similar way. Instead of taking all possible permutations of the player set N however, only a subset is taken. Let

$$\Pi_{APU}(N, P) = \left\{ \pi \in \Pi(N) : \begin{array}{l} \text{if } \pi(i) < \pi(j) < \pi(k) \text{ for } i, j, k \in N \\ \text{and } i \in P_h, j \notin P_h \text{ for } h \in M \rightarrow k \notin P_h \end{array} \right\} \quad (2.2.1)$$

be the set of permutations where the players are ordered according to their unions.

Definition 2.2.4 *The Owen value is the solution on $\mathcal{G}_{APU}$ given by*

$$Ow_i(N, v, P) = \frac{1}{|\Pi_{APU}(N, P)|} \sum_{\pi \in \Pi_{APU}(N, P)} m_i^\pi(N, v) \text{ for all } i \in N.$$

Note that for both the solution by Aumann and Drèze and the Owen value the partition in a number of a priori unions is exogenously given. Research has also been conducted on the endogenous formation of these unions, see Shenoy (1979), Hart and Kurz (1983), (1984).

Other examples of solutions for games with a priori unions are the coalitional τ-value (Casas-Méndez, García-Jurado, van den Nouweland and Vázquez-Brage, 2003), the two-step Shapley value (Kamijo, 2009) and the collective value (Kamijo, 2011).

2.2.2 Communication and games with a graph structure

Another line of research on cooperation restrictions was started by Myerson (1977). In his paper an undirected graph is used to model the communication relations between any two players. A coalition is said to be able to work together if it is connected within the graph. A restricted game is created based on the communication opportunities of the players. The Myerson value is obtained by taking the Shapley value of this restricted game. The communication situations Myerson introduces were later popularized by Jackson and

Wolinsky (1996). Economic applications of TU-games where communication is limited are given by the sequencing games (see e.g. Curiel (1988), Curiel, Potters Rajendra Prasad, Tijs and Veltman (1993, 1994) and Hamers (1995)) and the water distribution games by Ambec and Sprumont (2002). These games are all special cases of games where the communication structure is given by a line-graph, see van den Brink, van der Laan and Vasil'ev (2007). Extension to the water distribution games, where the river can be modelled by a sink-tree or a rooted tree were considered by Khmelnitskaya (2010) and van den Brink, van der Laan and Moes (2012). Note that these applications show that communication can be interpreted in different ways. In the sequencing games two players are only able to cooperate if they are next to each other in the queue, whereas in the water distribution game cooperation between two players is determined by their location along the river. Following Myerson's paper, a number of solutions to games with communication restrictions have been studied in the literature, for example the position value by Meessen (1988) and Borm, Owen, Tijs (1992) and the average tree solution by Herings, van der Laan and Talman (2008).

Definition 2.2.5 *A TU-game with a graph structure (or graph game) is a triple (N, v, L) with $(N, v) \in \mathcal{G}^N$ and $(N, L) \in \mathcal{L}^N$ a graph on N.*

Denote the collection of all TU-games with a graph structure (N, v, L) by $\mathcal{G}_G$. Given a fixed player set N the set of all TU-games with a graph structure (N, v, L) is denoted by $\mathcal{G}_G^N$. Denote the collection of all TU-games with a graph structure (N, v, L), where the graph (N, L) is cycle-free by $\mathcal{G}_{CG}$. Given a fixed player set N the set of all TU-games with a graph structure (N, v, L), where the graph (N, L) is cycle-free is denoted by $\mathcal{G}_{CG}^N$. If no confusion about the player set arises, we denote a game with a graph structure by the pair (v, L). In a game with graph structure $(N, v, L) \in \mathcal{G}_G$, a coalition S is only able to realize its worth $v(S)$ if S is connected in L. When S is not connected in L, the players in S can realize the sum of the worths of the components of the subgraph $(S, L(S))$. A classical TU-game is obtained by restriction from (N, v) using graph L.

Definition 2.2.6 *The Myerson or graph restricted game $v^L \in \mathcal{G}^N$ of graph game $(N, v, L) \in \mathcal{G}_G$ is given by*

$$v^L(S) = \sum_{T \in C^L(S)} v(T) \text{ for all } S \subseteq N.$$

A single-valued solution f on $\mathcal{G}_G$ assigns a unique payoff vector $f(N, v, L) \in \mathbb{R}^N$ to every $(N, v, L) \in \mathcal{G}_G$. The Myerson value (Myerson, 1977) for TU-games with graph structure, denoted by μ, is defined as the solution that assigns to every $(N, v, L) \in \mathcal{G}_G$ the Shapley value of the corresponding graph restricted game (N, v^L).

Definition 2.2.7 *The Myerson value is the solution on $\mathcal{G}_G$ given by*

$$\mu(N, v, L) = Sh(N, v^L).$$

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Myerson (1977) axiomatized this value for games with graph structure using the following two axioms.

A solution f satisfies component efficiency if the amount of worth it distributes among the players within a component C is exactly equal to $v(C)$.

Component efficiency For each graph game $(N, v, L) \in \mathcal{G}_G^N$ and every component $C \in C^L(N)$ it holds that $\sum_{i \in C} f_i(N, v, L) = v(C)$.

A solution f satisfies fairness if removing a link between two players i and j from the graph, has the same effect on the payoffs of both players.

Fairness For each graph game $(N, v, L) \in \mathcal{G}_G^N$ and each link $\{i, j\} \in L$ it holds that $f_i(N, v, L) - f_i(N, v, L \setminus \{\{i, j\}\}) = f_j(N, v, L) - f_j(N, v, L \setminus \{\{i, j\}\})$.

Theorem 2.2.8 (Myerson, 1977)
A solution on the class of graph games $\mathcal{G}_G^N$ is equal to the Myerson value if and only if it satisfies component efficiency and fairness.

It was later shown by Myerson (1980) that in order to axiomatize the Myerson value, a balanced contributions property for graph games can also be used.

Balanced contributions for graph games For each graph game $(N, v, L) \in \mathcal{G}_G^N$ it holds that $f_i(N, v, L) - f_i(N, v, L_{-j}) = f_j(N, v, L) - f_j(N, v, L_{-i})$.

Note that this propery is different from the balanced contributions used in Chapter 1 to axiomatize the Shapley value on $\mathcal{G}$, since now players i and j are not deleted from the game. Instead all links containing those players are removed from the graph.

Theorem 2.2.9 (Myerson, 1980)
A solution on the class of graph games $\mathcal{G}_G^N$ is equal to the Myerson value if and only if it satisfies component efficiency and balanced contributions.

The solution given by Aumann and Drèze (1974) on a game with a priori unions (N, v, P) assigns the same payoffs to players in N as the Myerson value on graph game (N, v, L), where $\{i, j\} \in L$ if and only if for players i and j there exists some union belonging to partition P such that players i and j are both in that union.

Note that the Myerson value does not always have to be in the Core of the restricted game. It inherits this property from the Shapley value. This holds even if we consider the class of games on cycle-free graphs $\mathcal{G}_{CG}$. It was proven by Kaneko and Wooders (1982) that for any graph game, where the game is superadditive and the graph cycle free, the Core is always non-empty (see also Le Breton, Owen and Weber (1994) and Demange (1994). Note that this is not normally the case for superadditive games, as was shown in the previous chapter. The fact that superaddivity implies that the Core is non-empty, does not imply that the Shapley-value is always part of the Core, it may in fact lie outside of the Core.

A family of solutions on games (N, v, L) where L is a tree, that always lie in the Core when $(N, v) \in \mathcal{G}_{SA}$, was introduced by Demange (2004). For each player $i \in N$ a digraph T^i is obtained from L. For a tree $(N, L) \in \mathcal{L}^N$ let $P_L(j, k) = \{j, ..., k\}$ be the set of players on the unique path $(j, ..., k)$ in L connecting player $j \in N$ to player $k \in N$ for $j \neq k$ and let $P_L(j, j) = \{j\}$. We obtain the following expression for T^i:

$$T^i = \{(j, k) : \{j, k\} \in L \text{ and } P_L(i, j) \subset P_L(i, k)\}. \tag{2.2.2}$$

Player i is the unique top player of this digraph. The hierarchical outcome t^i_j of a player $j \in N$ is given by its marginal contribution in the graph restricted game (N, v^L) to the coalition of its successors in T^i.

Definition 2.2.10 *The hierarchical outcome associated with player i is the solution on $\mathcal{G}_G$ given by*

$$t^i_j(N, v, L) = v(\widehat{F}_{T^i}(j) \cup \{j\}) - \sum_{h \in F_{T^i}(j)} v(\widehat{F}_{T^i}(h) \cup \{h\}) \text{ for all } j \in N.$$

The hierarchical outcome was later used in defining the average tree solution by Herings, van der Laan and Talman (2008) on games (N, v, L) with a cycle-free graph. This solution is calculated by taking the average over the hierarchical outcomes t^i for all players $i \in N$. Béal, Rémila and Solal (2010) also considered other convex combinations of hierarchical outcomes.

2.2.3 Hierarchies and games on a directed graph

A lot of situations of social interaction involve some kind of hierarchical ordering on the economic agents. Also in cooperative game theory a number of models have been proposed where a hierarchy is present. Examples are the airport games of Littlechild and Owen (1973) where aircraft landings can be ordered by the cost of the landing strip they need, the auction situations of Graham and Marshall (1987) and Graham, Marshall and Richard (1990) where players can be ordered according to their valuation of a good, the sequencing games of Curiel, Potters, Rajendra Prasad, Tijs and Veltman (1993, 1994) where jobs are ordered in an initial queue, the queueing games of Maniquet (2003) where jobs are not in an initial queue but can be ordered according to their waiting cost, the water distribution problems of Ambec and Sprumont (2002) or polluted river problems of Ni and Wang (2007) where countries are ordered by their location on a river flowing from upstream to downstream. Note that in these models the focus is not so much on the hierarchy; it is only present in an implicit way. Cooperative game theory has also proposed a number of models, where the hierarchy is itself the focus of study and is modelled as a directed graph. In this section a number of these models is discussed.

Games with a permission structure

One class of games in the study of hierarchical structures is formed by the so-called games with a permission structure in Gilles, Owen and van den Brink (1992), van den Brink and

Gilles (1996), Gilles and Owen (1994) and van den Brink (1997). These games have been applied to describe several economic organizations, such as hierarchically structured firms in e.g. van den Brink (2008) and van den Brink and Ruys (2008). A digraph (N, D) is used to represent a hierarchy. This graph is combined with a cooperative TU-game, creating the so-called games with a permission structure. Two interpretations to the graph are considered: a conjunctive approach developed by Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996) and a disjunctive approach developed by Gilles and Owen (1994) and van den Brink (1997). In the conjunctive case coalitions can only cooperate fully, if for any player in the coalition all of its predecessors are also present in the coalition. In the disjunctive case a coalition can only cooperate fully, if for every non-top player in the coalition at least one predecessor is also present in the coalition. For both interpretations a restricted game is created where the worth of a coalition is based on its maximal subset of players that is able to cooperate fully. For each approach the corresponding permission value is defined to be the Shapley value of the associated restricted game.

Definition 2.2.11 *A game with a permission structure is a triple (N, v, D) with $(N, v) \in \mathcal{G}^N$ and $(N, D) \in \mathcal{D}^N$ a digraph on N.*

Denote the collection of all TU-games with a permission structure (N, v, D) by $\mathcal{G}_D$. Given a fixed player set N the set of all games with permission structure (N, v, D) is defined by $\mathcal{G}_D^N$. In some chapters, if no confusion about the player set arises, we denote a game with a permission structure by the pair (v, D). In the *conjunctive approach* to permission structures a coalition is feasible if and only if for every player in the coalition all its predecessors are also in the coalition. A subset of 2^N is obtained from each digraph $D \in \mathcal{D}^N$, representing the coalitions that are conjunctively feasible with respect to the permission structure.

Definition 2.2.12 *The set $\Phi^c(N, D)$ of conjunctively feasible coalitions of permission structure (N, D) is given by*

$$\Phi^c(N, D) = \{S \subseteq N : P_D(i) \subseteq S \text{ for all } i \in S\}.$$

For any coalition $S \subseteq N$ there exists a unique largest conjunctively feasible subset.[1]

Definition 2.2.13 *For any coalition $S \subseteq N$ the unique largest conjunctively feasible subset in $\Phi^c(N, D)$ is given by*

$$\sigma_D^c(S) = \bigcup_{\{T \in \Phi^c(N,D) : T \subseteq S\}} T.$$

A restricted game is obtained by assigning to any coalition the worth of its unique largest conjunctively feasible subset.[2]

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[1]This follows from the fact that $\Phi^c(N, D)$ is union closed, see section 2.2.4.

[2]Note that for Myerson's graph-restricted game, the worth of a coalition S is given by the sum of the worths of its maximally connected subcoalitions. These always form a partition of S. For the conjunctively restricted game on the other hand the worth of a coalition S is given by its unique largest conjunctively feasible subset.

Definition 2.2.14 *The conjunctively restricted game $r^c_{N,v,D} \in \mathcal{G}^N$ of game with a permission structure $(N, v, D) \in \mathcal{G}_D$ is given by*

$$r^c_{N,v,D}(S) = v(\sigma^c_D(S)) \text{ for all } S \subseteq N.$$

Note that the worth of a coalition S in the restricted game depends only on the players belonging to $\sigma^c_D(S)$. This is the largest subset of players in S that are able to generate their own worth. Therefore we will also refer to $\sigma^c_D(S)$ as the (conjunctively) worth generating set of S. By definition of $\sigma^c_D(S)$ a player $i \in S$ is able to generate worth within a coalition S if also $P_D(i) \subseteq \sigma^c_D(S)$. For any player $j \in P_D(i) \cap \sigma^c_D(S)$ the same holds; j is only able to generate worth if $P_D(j) \subseteq \sigma^c_D(S)$. Continuing this line of reasoning we obtain that if all the players in a set $S \subseteq N$ are able to generate worth within S, for any player $i \in S$ it must hold that its set of superiors $\widehat{P}_D(i)$ is also a part of S. We obtain the following alternative definition of the largest conjunctively feasible subset of S in terms of players.

Definition 2.2.15 *For any coalition $S \subseteq N$ the unique largest conjunctively feasible subset in $\Phi^c(N, D)$ is given by*

$$\sigma^c_D(S) = \{i \in S : \widehat{P}_D(i) \subseteq S\}.$$

The conjunctive authorizing set of any coalition $S \subseteq N$ is the smallest feasible coalition in $\Phi^c(N, D)$ that contains S.

Definition 2.2.16 *For any coalition $S \subseteq N$ the conjunctive authorizing set of S is given by*

$$\alpha^c_D(S) = S \cup \widehat{P}_D(S).$$

A single-valued solution f on $\mathcal{G}_D$ assigns a unique payoff vector $f(N, v, D) \in \mathbb{R}^N$ to every $(N, v, D) \in \mathcal{G}_D$. The *conjunctive permission value* φ^c is the solution that assigns to every game with a permission structure the Shapley value of the conjunctively restricted game.

Definition 2.2.17 *The conjunctive permission value is the solution on $\mathcal{G}_D$ given by*

$$\varphi^c(N, v, D) = Sh(N, r^c_{N,v,D}).$$

The following axioms are used to axiomatize the conjunctive permission value.

Efficiency states that the amount of worth to be distributed is exactly equal to $v(N)$, the worth of the grand coalition.

Efficiency For every $(N, v, D) \in \mathcal{G}_D$, it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

Additivity states that a solution should be additive over games.

Additivity For every $(N, v, D), (N, w, D) \in \mathcal{G}_D$, it holds that $f(N, v+w, D) = f(N, v, D) + f(N, w, D)$.

Player $i \in N$ is *inessential* in game with permission structure (N, v, D) if i and all its subordinates are *null* players in v, i.e. if $v(S) = v(S \setminus \{j\})$ for all $S \subseteq N$ and $j \in \widehat{F}_D(i) \cup \{i\}$. The inessential player property states that these players should get a payoff of 0.

Inessential player property For every $(N, v, D) \in \mathcal{G}_D$, if $i \in N$ is an inessential player in (N, v, D) then $f_i(N, v, D) = 0$.

Player $i \in N$ is called *necessary* in game v if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. The necessary player property states that necessary players should get at least as much as all the other players, if a game is monotonic.

Necessary player property For every $(N, v, D) \in \mathcal{G}_D$ such that $(N, v) \in \mathcal{G}_M$, if $i \in N$ is a necessary player in v then $f_i(N, v, D) \geq f_j(N, v, D)$ for all $j \in N$.

The structural monotonicity property states that players should get at least as much as their subordinates if the game is monotonic.

Structural monotonicity For every $(N, v, D) \in \mathcal{G}_D$ such that $(N, v) \in \mathcal{G}_M$, if $i \in N$ and $j \in F_D(i)$ then $f_i(N, v, D) \geq f_j(N, v, D)$.

The previous five axioms characterize the conjunctive permission value.

Theorem 2.2.18 (van den Brink and Gilles, 1996)
A solution on $\mathcal{G}_D$ is equal to the conjunctive permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.

Logical independence of the five axioms stated in Theorem 2.2.18 is shown by the following alternative solutions for games with a permission structure.

(i) The solution given by $f_i(N, v, D) = 0$ for all $i \in N$ satisfies additivity, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy efficiency.

(ii) Let $In(N, v, D)$ be the set of inessential players in a game with permission structure (N, v, D). The solution $f(N, v, D)$ given by $f_i(N, v, D) = \frac{v(N)}{|N \setminus In(N, v, D)|}$ for $i \notin In(N, v, D)$ and $f_i(N, v, D) = 0$ for $i \in In(N, v, D)$ satisfies efficiency, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy additivity.

(iii) The solution $f(N, v, D)$ that assigns to any player $i \in N$ a payoff $f_i(N, v, D) = \frac{v(N)}{|N|}$, thereby equally dividing the worth of the grand coalition over all the players, satisfies efficiency, additivity, the necessary player property and structural monotonicity. It does not satisfy the inessential player property.

(iv) The solution $f(N, v, D)$ that on games (N, v, D) where $|TOP(N, D)| = 1$, is given by $f_i(N, v, D) = v(N)$ for $i \in N \cap TOP(N, D)$ and $f_i(N, v, D) = 0$ otherwise, and on games where $|TOP(N, D)| \neq 1$ is given by the permission value satisfies efficiency, additivity, the inessential player property and structural monotonicity. It does not satisfy the necessary player property.

(v) For any game (N, v, D) the Shapley value $Sh(N, v)$ satisfies efficiency, additivity, the inessential player property and the necessary player property. It does not satisfy structural monotonicity.

Denote by $\mathcal{G}_H$ the collection of all TU-games (N, v, D) with a permission structure such that $D \in \mathcal{D}_h^N$ is a hierarchical digraph. We will also refer to the games in $\mathcal{G}_H$ as games with a hierarchical permission structure. Now consider $(N, v, D) \in \mathcal{G}_H$. Define a *permission path* Q in digraph D to be a directed path from a player $j \in TOP(N, D)$ to a player $i \in \widehat{F}_D(i)$. For any permission path $Q = (i_1, ..., i_k)$ define $P_Q = \{i_1, ..., i_k\}$ to be the set of players included in Q. For $i \in N$ define Q_i to be the set of permission paths Q such that $i \in P_Q$ and $P_Q \setminus \{i\} \subseteq \widehat{P}_D(i)$. In the conjunctive approach to permission structures a player i can only generate worth within a coalition S when for all $Q \in Q_i$ it holds that $P_Q \subseteq S$.

Let a single-valued solution f on $\mathcal{G}_H$ be a function that assigns a unique payoff vector $f(N, v, D) \in \mathbb{R}^N$ to every $(N, v, D) \in \mathcal{G}_H$. Another axiomatization of the conjunctive permission value by van den Brink (1997) is given on the class $\mathcal{G}_H$. This axiomatization replaces the property of structural monotonicity by the axioms of weak structural monotonicity and conjunctive fairness. To define weak structural monotonicity the idea of complete domination within a graph is needed. Let $\overline{F}_D(i) = \{j \in \widehat{F}_D(i) : i \in P_Q$ for all $Q \in Q_j\}$ denote the set of followers of player i that he dominates completely. Let $\overline{P}_D(i) = \{j \in \widehat{P}_D(i) : i \in \overline{F}_D(j)\}$ denote the set of players that a player i gets dominated by completely.

Weak structural monotonicity states that a player should obtain at least as much as the players that he dominates completely, if the game is monotonic.

Weak structural monotonicity For every $(N, v, D) \in \mathcal{G}_H$, such that $(N, v) \in \mathcal{G}_M$, if $i \in N$ and $j \in \overline{F}_D(i)$ then $f_i(N, v, D) \geq f_j(N, v, D)$.

Conjunctive fairness states that if for a player k with $|P_D(k)| \geq 2$ we consider digraph $D \setminus \{(j, k)\}$ for any $j \in P_D(k)$ instead of digraph D the payoff to player k and any other predecessor $i \in P_D(k) \setminus \{j\}$ change by the same amount. Moreover also the payoffs of all players that completely dominate the other predecessor i change by this same amount.

Conjunctive fairness For every $(N, v, D) \in \mathcal{G}_H$ and $i, j, k \in N$ such that $i \neq j$ and $k \in F_D(i) \cap F_D(j)$, it holds that $f_l(N, v, D) - f_l(N, v, D \setminus \{(j, k)\}) = f_k(N, v, D) - f_k(N, v, D \setminus \{(j, k)\})$ for all $l \in \{i\} \cup \overline{P}_D(i)$

The axioms of efficiency, additivity, the inessential player property and the necessary player property are adapted to games with a hierarchical permission structure in a straightforward way.

Theorem 2.2.19 (van den Brink, 1999)
A solution on $\mathcal{G}_H$ is equal to the conjunctive permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.

A different approach to hierarchical permission structures is given by the *disjunctive approach* developed in Gilles and Owen (1994) and van den Brink (1997). A coalition is feasible if and only if every non-top player in the coalition has at least one predecessor who also belongs to the coalition. A subset of 2^N is obtained from each digraph $D \in \mathcal{D}^N$, representing the coalitions that are disjunctively feasible with respect to the permission structure (those sets for which all players in that set are able to fully cooperate).

Definition 2.2.20 *The set $\Phi^d(N, D)$ of disjunctively feasible coalitions of hierarchical permission structure (N, D) is given by*

$$\Phi^d(N, D) = \left\{ S \subseteq N : P_D(i) \cap S \neq \emptyset \text{ for all } i \in S \setminus TOP(N, D) \right\}.$$

For any coalition $S \subseteq N$ there also exists a unique largest disjunctively feasible subset.[3]

Definition 2.2.21 *For any coalition $S \subseteq N$ the unique largest disjunctively feasible subset in $\Phi^d(N, D)$ is given by*

$$\sigma_D^d(S) = \bigcup_{\{T \in \Phi^d(N,D) : T \subseteq S\}} T$$

A restricted game is obtained by assigning to any coalition the worth of its unique largest disjunctively feasible subset.

Definition 2.2.22 *The disjunctively restricted game $r_{N,v,D}^d \in \mathcal{G}^N$ of game with a permission structure $(N, v, D) \in \mathcal{G}_H$ is given by*

$$r_{N,v,D}^d(S) = v(\sigma_D^d(S)) \text{ for all } S \subseteq N$$

Compared to the conjunctive approach it no longer holds that the set of superiors for a player also needs to be present in a coalition in order for that player to generate worth. In terms of permission paths, it can be stated that a player i is able to generate worth within a coalition S if it holds that there is at least one $Q \in Q_i$, such that $P_Q \subseteq S$. Where the conjunctive permission value requires that all the players of all permission paths are present, the disjunctive permission approach only requires that all the players are presents of at least one permission path.

The *disjunctive permission value φ^d* is the solution that assigns to every game with a hierarchical permission structure the Shapley value of the disjunctively restricted game.

Definition 2.2.23 *The disjunctive permission value is the solution on $\mathcal{G}_H$ given by*

$$\varphi^d(N, v, D) = Sh(N, r_{N,v,D}^d).$$

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[3]Again this follows from the fact that $\Phi^d(N, D)$ is union closed, see section 2.2.4.

The disjunctive permission value can be characterized on the class of games with a hierarchical permission structure by the axioms of efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and a property called disjunctive fairness.

Disjunctive fairness states that if for a player k with $|P_D(k)| \geq 2$ we consider digraph $D \setminus \{(j,k)\}$ for any $j \in P_D(k)$ instead of digraph D the payoff to player k, j and all players $\overline{P}_D(j)$ that completely dominate j changes in the same way.

Disjunctive fairness For every $(N, v, D) \in \mathcal{G}_H$ and $j, k \in N$ such that $(j, k) \in D$ and $|P_D(k)| \geq 2$, it holds that $f_k(N, v, D) - f_k(N, v, D \setminus \{(j,k)\}) = f_l(N, v, D) - f_l(N, v, D \setminus \{(j,k)\})$ for $l \in \{j\} \cup \overline{P}_D(j)$

Theorem 2.2.24 (van den Brink, 1997)
A solution on $\mathcal{G}_H$ is equal to the disjunctive permission value φ^d if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.

Note that both the conjunctive as well as the disjunctive permission value consider permission paths within the digraph to assess whether a player i is able to generate worth. In some way this constitutes a 'global' approach to permission within the graph, since also players outside of the set of 'local' predecessors $P_D(i)$ are considered. In Chapter 4 of this thesis a local (conjunctive) approach to permission is discussed. This approach only considers the predecessors with respect to worth generation.

Weighted digraph games

Another model of games with a digraph on the set of players are the *(weighted) digraph games* introduced in van den Brink and Borm (2002). For a given (weighted) digraph, with weights assigned to the players, the associated digraph game assigns to every coalition of players the sum of the weights of all players in the coalition whose predecessors also belong to the coalition.

A *weighted directed graph*, shortly referred to as *weighted digraph*, is a triple (N, δ, D) where $N \subset \mathbb{N}$ is a set of nodes, $D \in \mathcal{D}^N$ is a digraph, and $\delta \in \mathbb{R}^{|N|}_+$ is a vector of nonnegative weights assigned to the nodes.

Definition 2.2.25 *The weighted digraph game $\overline{v}_{\delta,D} \in \mathcal{G}^N$ corresponding to weighted digraph (N, δ, D) is given by*

$$\overline{v}_{\delta,D}(S) = \sum_{\{i \in S : P_D(i) \subseteq S\}} \delta_i, S \subseteq N.$$

So, the worth of an arbitrary coalition $S \subseteq N$ is the sum of the weights of the players in that coalition for whom all predecessors belong to the coalition. In terms of unanimity games, a digraph game can be written as

$$\overline{v}_{\delta,D} = \sum_{i \in N} \delta_i u_{P_D(i) \cup \{i\}}. \tag{2.2.3}$$

Restricted cooperation

If there is no confusion about the player set we denote a weighted digraph and weighted digraph game on N as (δ, D), respectively, $\overline{v}_{\delta,D}$. Van den Brink and Borm (2002) also apply the Shapley value to (weighted) digraph games. They refer to this solution as the β-measure for (weighted) digraphs.

Consider the additive game $w^\delta \in \mathcal{G}_A$ given by

$$w^\delta(S) = \sum_{i \in S} \delta_i, \ S \subseteq N. \tag{2.2.4}$$

The weighted digraph game $\overline{v}_{\delta,D}$ may be considered a restricted game of additive game w^δ, similar to the conjunctively restricted game obtained from a game with a permission structure. The difference for weighted digraph games is that a player i is able to generate worth if all of its predecessors $P_D(i)$ in the digraph are present. For games with a permission structure a player i is only able to generate worth if all of its superiors $\widehat{P}_D(i)$ in the digraph are present. Therefore the weighted digraph games might be considered a conjunctive approach to permission on a more local scale, as discussed previously. A disjunctive approach might also be considered, where the worth of a coalition is equal to the sum of the weights of the players who have at least one predecessor present in the coalition. Such an approach has not yet been studied in the literature however.

Peer group games

The earlier mentioned auction games of Graham and Marshall (1987) and Graham, Marshall and Richard (1990), the DR-polluted river game of Ni and Wang (2007) and the dual of the airport game of Littlechild and Owen (1973) are all applications of peer group games.

Brânzei, Fragnelli and Tijs (2002) define a peer group situation as a triple (N, a, T) where $N \subset \mathbb{N}$ is a set of players, $T \in \mathcal{D}_t^N$ is a rooted tree, and $a \in \mathbb{R}_+^{|N|}$ is a vector of nonnegative weights assigned to the players. Again, when we take the player set N to be fixed, we denote a peer group situation just as a pair (a, T).

Definition 2.2.26 *The peer group game* $v_{a,T}^P \in \mathcal{G}^N$ *corresponding to peer group situation* (N, a, T) *is given by*

$$v_{a,T}^P(S) = \sum_{\substack{i \in S \\ \widehat{P}_T(i) \subseteq S}} a_i, \ S \subseteq N.$$

So, a coalition $S \subseteq N$ might be considered *feasible* if $\widehat{P}_T(i) \subseteq S$ for all $i \in S$ (i.e. when it belongs to the set of conjunctively feasible coalitions), and the worth of an arbitrary coalition $S \subseteq N$ is the sum of the weights of the players in its largest feasible subset. In terms of unanimity games a peer group game can be written as

$$v_{a,T}^P = \sum_{i \in N} a_i u_{\widehat{P}_T(i) \cup \{i\}}. \tag{2.2.5}$$

In Brânzei, Fragnelli and Tijs (2002) it is already mentioned that every peer group game corresponding to a peer group situation (a, T) corresponds to a game with a permission structure (N, w^a, T) where the permission structure T is a rooted tree and the game $w^a \in \mathcal{G}_A$ is the additive game given by

$$w^a(S) = \sum_{i \in S} a_i, \ S \subseteq N. \tag{2.2.6}$$

Since any player $i \in N$ has exactly one predecessor in the rooted tree T the conjunctive and disjunctive approach coincide. The peer group game is therefore seen to be equal to both the conjunctive as well as the disjunctively restricted game of w^a by graph T, $v_{a,T}^P = r_{N,w^a,T}^c = r_{N,w^a,T}^d$.

In Chapter 4 it will be seen that the set of peer group games is also contained within the set of weighted digraph games. To be specific, any peer group game $v_{a,T}^P$ obtained by the restriction of peer group situation (N, a, T) can be obtained by taking weighted digraph game $(N, \overline{v}_{a,T}, tr(T))$, where $tr(T)$ is the transitive closure of rooted tree T.

Games with precedence constraints

Faigle and Kern (1992) consider situations where cooperation between players is restricted by a partial order on the player set. They interpret this partial order as a precedence relation. The partial order can be represented by an acyclic digraph. A coalition can be considered feasible, if for any player in the coalition all of its successors in the digraph are also present in the coalition.

Definition 2.2.27 *The set $\Phi^p(N, D)$ of feasible coalitions according to digraph $(N, D) \in \mathcal{D}_a$ is given by*

$$\Phi^p(N, D) = \{S \subseteq N : F_D(i) \subseteq S \text{ for all } i \in S\}.$$

Faigle and Kern consider cooperative games, where for digraph $(N, D) \in \mathcal{D}_a$ the domain of the characteristic function is given by the set $\Phi^p(N, D)$.

Definition 2.2.28 *A TU-game with precedence constraints is a triple (N, v, D), where $N \subseteq \mathbb{N}$ is a set of players, $(N, D) \in \mathcal{D}_a$ is an acyclic digraph, and $v : \Phi^p(N, D) \to \mathbb{R}$ is the characteristic function, that assigns to every set S in $\Phi^p(N, D)$ a worth $v(S)$, where $v(\emptyset) = 0$.*

Denote the class of all games with precedence constraints by $\mathcal{G}_{PC}$. Whereas for games with a permission structure, the characteristic function is defined on every possible subset of the player set N, for games with precedence constraints, the characteristic function is defined only on the set of feasible coalitions $\Phi^p(N, D)$. Let $\mathcal{G}_{PC}^{(N,D)}$ be the class of games with precedence constraints on graph $(N, D) \in \mathcal{D}_a$. The game with precedence constraints obtained from $(N, v, D) \in \mathcal{G}_{PC}$ by considering only feasible coalition S and its subsets is denoted by $(S, v_S, D(S))$, where $v_S(T) = v(T)$ for all feasible $T \subseteq S$. We refer to $(S, v_S, D(S))$ as the subgame on S of (N, v, D). The operations of $+ : \mathcal{G}_{PC}^{(N,D)} \times \mathcal{G}_{PC}^{(N,D)} \to$

$\mathcal{G}_{PC}^{(N,D)}$ and $\cdot : \mathbb{R} \times \mathcal{G}_{PC}^{(N,D)} \to \mathcal{G}_{PC}^{(N,D)}$ are defined similar to these same operations for classical TU-games, where for $(N, v + w, D) \in \mathcal{G}_{PC}^{(N,D)}$, it holds that $(v + w)(S) = v(S) + w(S)$ and for $c \in \mathbb{R}$ and $(N, c \cdot v, D) \in \mathcal{G}_{PC}^{(N,D)}$ it holds that $(c \cdot v)(S) = c \cdot v(S)$ for $S \in \Phi^p(N, D)$.

The set of *admissible* permutations $\Pi_D(N)$ in D is defined by

$$\Pi_D(N) = \{\pi \in \Pi(N) : \pi(i) > \pi(j) \text{ if } (i, j) \in D\}, \tag{2.2.7}$$

i.e. it is the collection of permutations on N where successors enter before their predecessors in the digraph. Note that it holds that the set of admissible permutations in D is the same as that of its transitive closure $tr(D)$: $\Pi_D(N) = \Pi_{tr(D)}(N)$. We say that a *player j precedes a player i in permutation* $\pi \in \Pi_D(N)$ if $\pi(j) < \pi(i)$.

In the context of games with precedence constraints we say that a *player j precedes a player i in digraph* (N, D), if j belongs to the set of i's subordinates. So j precedes i in (N, D) if and only if for every $\pi \in \Pi_D(N)$ it holds that j precedes i in permutation π.

The *precedence marginal vector* $m^\pi(N, v, D) \in \mathbb{R}^N$, associated with game with precedence constraints (N, v, D) and permutation $\pi \in \Pi_D(N)$, is given by the vector of marginal contributions, where players join to form the grand coalition N according to π:

$$m_i^\pi(N, v, D) = (v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})), \ i \in N. \tag{2.2.8}$$

Recall from Definition 1.0.12 that the Shapley value assigns to the players the average over all marginal vectors associated with permutations of the player set N. The *precedence Shapley value* $H(N, v, D)$ of Faigle and Kern (1992) assigns to the players in N the average over all precedence marginal vectors of game with precedence constraints (N, v, D).

Definition 2.2.29 *The precedence Shapley value is the solution on $\mathcal{G}_{PC}$ given by*

$$H_i(N, v, D) = \frac{1}{|\Pi_D(N)|} \sum_{\pi \in \Pi_D(N)} m_i^\pi(N, v, D), \ i \in N.$$

For $(N, v, D) \in \mathcal{G}_{PC}$, all permutations in $\Pi(N)$ are admissible when $D = \emptyset$. In that case the domain of characteristic function v is given by 2^N. This allows us to consider the classical TU-game $(N, v) \in \mathcal{G}$. It is straightforward to see that, when $D = \emptyset$, the precedence Shapley value of (N, v, D) yields the Shapley value of (N, v).

Faigle and Kern (1992) also give an axiomatization of the precedence Shapley value using the following axioms.

Effiency is adapted from classical TU-games in a straightforward way.

Efficiency For each game $(N, v, D) \in \mathcal{G}_{PC}$ it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

A solution f for games with precedence constraints satisfies linearity if it is linear over games with precedence constraints.

Linearity For every pair of games (N, u, D) and $(N, w, D) \in \mathcal{G}_{PC}^{(N,D)}$ it holds that $f(N, u + w, D) = f(N, u, D) + f(N, w, D)$ and for $(N, v, D) \in \mathcal{G}_{PC}^{(N,D)}$ and $c \in \mathbb{R}$ it holds that $f(N, cv, D) = cf(N, v, D)$.

Call a player $i \in N$ a precedence null player in game with precedence constraints (N, v, D), if for every $\pi \in \Pi_D(N)$ it holds that $m_i^\pi(N, v, D) = 0$. The next axiom adapts the null player property used for the Shapley value to games with precedence constraints.[4]

Precedence null player property For each $(N, v, D) \in \mathcal{G}_{PC}$, if $i \in N$ is a precedence null player in (N, v, D), then $f_i(N, v, D) = 0$.

Besides these three axioms, an axiom is introduced that is based on the hierarchical strength of players.

First for $i \in S \in \Phi^p(N, D)$ the set of permutations $\Pi_D^i(N, S)$ is defined by

$$\Pi_D^i(N, S) = \{\pi \in \Pi_D(N) : \pi(i) > \pi(j), j \in S \setminus \{i\}\}. \tag{2.2.9}$$

This is the collection of those permutations in $\Pi_D(N)$ where i is preceded in the permutation by the players in $S \setminus \{i\}$. Note that the collection $\{\Pi_D^i(N, S)\}_{i \in S}$ is a partition of $\Pi_D(N)$.

The *absolute hierarchical strength* for $(N, D) \in \mathcal{D}_a$ and coalition $S \in \Phi^p(N, D)$, assigns a player $i \in S$ the cardinality of $\Pi_D^i(N, S)$, i.e. the number of permutations in $\Pi_D(N)$ where i is preceded in the permutation by the players in $S \setminus \{i\}$.

Definition 2.2.30 *The absolute hierarchical strength is the function h that assigns to every $(N, D) \in \mathcal{D}_a$ and coalition $S \in \Phi^p(N, D)$ the vector $h(N, D, S) \in \mathbb{R}^S$, where $h_i(N, D, S) = |\Pi_D^i(N, S)|, i \in S$.*

The *normalized hierarchical strength* for $(N, D) \in \mathcal{D}_a$ and coalition $S \in \Phi^p(N, D)$, assigns a player $i \in S$ the fraction of permutations in $\Pi_D(N)$ where i is preceded in the permutation by the players in $S \setminus \{i\}$.

Definition 2.2.31 *The normalized hierarchical strength[5] is the function $\overline{h}$ that assigns to every $(N, D) \in \mathcal{D}_a$ and a coalition $S \in \Phi^p(N, D)$ the vector $\overline{h}(N, D, S) \in \mathbb{R}^S$, where $\overline{h}_i(N, D, S) = \frac{|\Pi_D^i(N, S)|}{|\Pi_D(N)|}, i \in S$.*

Note that $\sum_{i \in S} \overline{h}_i(N, D, S) = 1$ for all $S \in \Phi^p(N, D)$.

Let unanimity games with precedence constraints be defined similar to classical unanimity TU-games.

Definition 2.2.32 *For each $T \in \Phi^p(N, D)$, $T \neq \emptyset$, the* unanimity *game with precedence constraints $(N, u_T, D) \in \mathcal{G}_{PC}$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise.*

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[4]We remark that, similar to Shapley (1953), Faigle and Kern (1992) combine efficiency and the precedence null player property into a carrier axiom.

[5]Both the absolute as well as the normalized hierarchical strength assign a value to a player $i \in N$, given $D \in \mathcal{D}_a^N$ and coalition $S \in \Phi^p(N, D)$ and are therefore more correctly denoted by $h((N, D), S)$ and $\overline{h}_i((N, D), S)$ respectively. For convenience however we will refer to these functions as $h(N, D, S)$ and $\overline{h}_i(N, D, S)$ respectively throughout this thesis.

Restricted cooperation

Faigle and Kern (1992) also consider the dividend of a coalition $S \in \Phi^p(N, D)$ in game with precedence constraints (N, v, D).

Definition 2.2.33 *The dividend of a coalition $S \in \Phi^p(N, D)$ in game with precedence constraints (N, v, D) is given by $\Delta_v^D(S) = v(S) - \sum\limits_{T \subset S, T \in \Phi^p(N,D), T \neq \emptyset} \Delta_v^D(T)$.*

For every $(N, v, D) \in \mathcal{G}_{PC}$, Faigle and Kern (1992) show that (N, v, D) can be written as the sum over all feasible coalitions T of the multiplication of the unanimity game with precedence constraints (N, u_T, D) of T by $\Delta_v^D(T)$. This gives the following expression:

$$v = \sum_{\substack{T \in \Phi^p(N,D) \\ T \neq \emptyset}} \Delta_v^D(T) u_T. \tag{2.2.10}$$

The axiom of hierarchical strength for a solution for games with precedence constraints states that in unanimity games with precedence constraints the earnings are distributed among the players in the unanimity coalition proportionally to their normalized hierarchical strength in that coalition. Of course, this is equivalent to distributing the dividends proportionally to the absolute hierarchical strength of the players.

Hierarchical strength For every $(N, D) \in \mathcal{D}_a$, every $S \in \Phi^p(N, D)$ and every $i, j \in S$, it holds that $\overline{h}_i(N, D, S) f_j(N, u_S, D) = \overline{h}_j(N, D, S) f_i(N, u_S, D)$.

The precedence Shapley value is uniquely characterized by the hierarchical strength axiom and efficiency, linearity and the precedence null player property.

Theorem 2.2.34 (Faigle and Kern, 1992)
A solution on $\mathcal{G}_{PC}$ is equal to the precedence Shapley value H if and only if it satisfies efficiency, linearity, the precedence null player property and hierarchical strength.

Alternatively, the precedence Shapley value can be defined as the solution that allocates the dividend of a coalition $S \in \Phi^p(N, D)$ proportionally to the hierarchical strength $h(N, D, S)$ of the players in S.

Definition 2.2.35 *The precedence Shapley value is the solution on $\mathcal{G}_{PC}$ given by*

$$H_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(N, D, S)}{\sum_{j \in S} h_j(N, D, S)} \Delta_v^D(S) \ \textit{for all } i \in N.$$

2.2.4   Feasible set systems

In some of the models discussed before a clear distinction can be made between coalitions that are unrestricted with respect to cooperation and those that are not. We might also refer to these coalitions unrestricted with respect to cooperation as feasible coalitions. For a coalition to be feasible we have seen that it might be required that all players in the coalition belong to some group (games with a priori unions), are able to fully communicate among themselves (communication graph games) or have full permission (games on permission structures). Now define a feasible coalition or set system to be a pair $(N, \mathcal{F}), \mathcal{F} \subseteq 2^N$. The collection $\mathcal{F}$ represents those coalitions that are feasible and are therefore assumed to be unrestricted with respect to cooperation between the players. If there is no confusion about the player set N feasible set system $(N, \mathcal{F})$ will be referred to as $\mathcal{F}$. In the literature different classes of feasible set systems have been considered. For a comprehensive summary of feasible set systems and games on these systems see also Bilbao (2000). Note that in the games discussed in the previous sections the feasible set system is implicitly present. In what follows we will see how these games fit into the literature on feasible set systems.

Antimatroids

Algaba, Bilbao, van den Brink and Jiménez-Losáda (2004) show that the conjunctively feasible and disjunctively feasible sets obtained by applying the conjunctive and disjunctive approach to hierarchical digraphs respectively are *antimatroids*. Antimatroids were introduced by Dilworth (1940) as particular examples of semimodular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte, Lovász, and Schrader (1991) and Edelman and Jamison (1985)).

Definition 2.2.36 *A set system $\mathcal{A} \subseteq 2^N$ is an antimatroid if it satisfies the following properties*

(feasible empty set) $\emptyset \in \mathcal{A}$,

(union closedness) *for $S, T \in \mathcal{A}$ we have $S \cup T \in \mathcal{A}$,*

(accessibility) *for $S \in \mathcal{A}$ with $S \neq \emptyset$, there exists $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$.*

Union closedness means that the union of two feasible coalitions is also feasible. Union closed set systems have been considered in van den Brink, Katsev and van der Laan (2011). Accessibility means that every nonempty feasible coalition has at least one player such that, after it leaves, the remaining players also form a feasible coalition.

A set system $\mathcal{F} \subseteq 2^N$ is called normal if $N = \bigcup_{S \in \mathcal{F}} S$, i.e. if every player belongs to at least one feasible coalition. Notice that if $\mathcal{A} \subseteq 2^N$ is a normal antimatroid then union closedness implies that $N \in \mathcal{A}$.

Definition 2.2.37 *A player $i \in S \in \mathcal{F} \subseteq 2^N$ such that $S \setminus \{i\} \in \mathcal{F}$ is an extreme player of coalition S in $\mathcal{F}$.*

The set of extreme players of a feasible coalition $S \in \mathcal{F}$ is denoted by $ex(S) = \{i \in S : S \backslash \{i\} \in \mathcal{F}\}$. By accessibility every nonempty feasible coalition in an antimatroid has at least one extreme player. A coalition that has exactly one extreme player is called a *path*.

Definition 2.2.38 *A coalition $S \in \mathcal{F} \subseteq 2^N$ is a path in $\mathcal{F}$ if $|ex(S)| = 1$. A path $S \in \mathcal{F}$ is a i-path if $i \in S$ is the extreme player in S.*

Let $\mathcal{A} \subseteq 2^N$ be an antimatroid and let $S, T \in \mathcal{A}$ be two feasible sets. Accessibility implies an ordering $(i_1, \ldots, i_{|T|})$, where $\bigcup_{j=1}^{|T|} \{i_j\} = T$ such that $\{i_1, \ldots, i_j\} \in \mathcal{A}$ for $j = 1, \ldots, |T|$. Let $k \in \{1, \ldots, |T|\}$ be the minimum index with $i_k \notin S$. Then $S \cup \{i_k\} = S \cup \{i_1, \ldots, i_k\} \in \mathcal{A}$ by union closedness. Therefore, the definition of an antimatroid implies the following property: If $S, T \in \mathcal{A}$ with $T \backslash S \neq \emptyset$ then there exists $i \in T \backslash S$ such that $S \cup \{i\} \in \mathcal{A}$.

The class of poset antimatroids is generated by the so-called ideals of some partially ordered set $\mathcal{P}$ (for more information on partially ordered sets and ideals, see Davey and Priestley (1990), Birkhoff (1967)). Another way of defining these poset antimatroids is by requiring that an antimatroid also satisfies the following property:

(intersection closedness) for $S, T \in \mathcal{A}$ we have $S \cap T \in \mathcal{A}$.

The poset antimatroids can also be defined as those antimatroids such that there exists a unique i-path for all $i \in N$. It is proven in Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) that the class of poset antimatroids constitute exactly the class of conjunctively feasible coalitions $\Phi^c(N, D)$ generated by applying the conjunctive approach to an acyclic digraph (N, D). They also consider the antimatroids satisfying the so-called path property.

Definition 2.2.39 *An antimatroid satisfies the path property if it satisfies the following properties:*

(i) *Every path $T \in \mathcal{A}$ has a unique feasible ordering $(i_1, \ldots, i_{|T|})$, where $\bigcup_{j=1}^{|T|} \{i_j\} = T$ such that $\{i_1, \ldots, i_j\} \in \mathcal{A}$ for $j \in \{1, \ldots, |T|\}$. The union of these orderings for all paths is a partial ordering of N.*

(ii) *If S, T and $S \backslash \{i\}$ are paths and the extreme point of T equals the extreme point of $S \backslash \{i\}$, then $T \cup \{i\} \in \mathcal{A}$.*

Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) also prove that the class of antimatroids with the path property constitute exactly the class of sets of disjunctively feasible coalitions $\Phi^d(N, D)$ generated by applying the disjunctive approach to a hierarchical digraph (N, D).

In Algaba, Bilbao, van den Brink and Jiménez-Losada (2003) games on an antimatroid are studied.

Definition 2.2.40 *A TU-game on an antimatroid is a triple $(N, v, \mathcal{A})$ with $(N, v) \in \mathcal{G}^N$ and $(N, \mathcal{A})$ an antimatroid on N.*

We denote the collection of all TU-games on antimatroids by $\mathcal{G}_{AM}$. A classical TU-game $(N, v_{\mathcal{A}})$ is obtained by restriction from (N, v) using antimatroid $\mathcal{A}$. Let the interior $int_{\mathcal{A}}$ be the function that maps any set $S \subseteq N$ to its largest subset in $\mathcal{A}$ (see also Korte, Lovász, and Schrader (1991)).

Definition 2.2.41 *The restricted game* $v_{\mathcal{A}} \in \mathcal{G}^N$ *of game on antimatroid* $(N, v, \mathcal{A}) \in \mathcal{G}_{AM}$ *is given by*

$$v_{\mathcal{A}}(S) = v(int_{\mathcal{A}}(S)) \text{ for all } S \subseteq N.$$

A single-valued solution f on $\mathcal{G}_{AM}$ assigns a unique payoff vector $f(N, v, \mathcal{A}) \in \mathbb{R}^N$ to every $(N, v, \mathcal{A}) \in \mathcal{G}_{AM}$. Algaba, Bilbao, van den Brink and Jiménez-Losada (2003) define and characterize a straightforward adaption of the Shapley value for antimatroids. This solution is obtained by applying the Shapley value to restricted game $(N, v_{\mathcal{A}})$.

Definition 2.2.42 *The restricted Shapley value is the solution* φ *on* $\mathcal{G}_{AM}$ *given by*

$$\varphi(N, v, \mathcal{A}) = Sh(N, v_{\mathcal{A}}).$$

An axiomatization of the restricted Shapley value for antimatroids is also given. The axioms of efficiency, additivity, the necessary player property, the inessential player property and weak structural monotonicity used in the axiomatization of the conjunctive and disjunctive permission value are all extended in some way to games on antimatroids. The following fairness axiom is also introduced.

Fairness For every $(N, v, \mathcal{A}) \in \mathcal{G}_{AM}$ if $S \in \mathcal{A}$ is such that $\mathcal{A} \setminus \{S\}$ is an antimatroid on N, then $f_i(v, \mathcal{A}) - f_i(v, \mathcal{A} \setminus \{S\}) = f_j(v, \mathcal{A}) - f_j(v, \mathcal{A} \setminus \{S\})$ for all $i, j \in S$.

This axiom generalizes both the conjunctive and disjunctive fairness for games with a permission structure.

All of these axioms are used to axiomatize the restricted Shapley value for antimatroids.

Theorem 2.2.43 (Algaba, Bilbao, van den Brink and Jiménez-Losada, 2003) *A solution on* $\mathcal{G}_{AM}$ *is equal to* φ *if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, structural monotonicity and fairness.*

It should be mentioned that by leaving out fairness we obtain a characterization on the class of games on poset antimatroids.

Communication feasible set systems

In the communication graph game (N, v, L) discussed in section 2.2.2, a coalition of players $S \subseteq N$ is able to communicate if S is connected in communication graph (N, L). For a communication graph (N, L) let

$$\mathcal{F}_L = \{S \subseteq N : S \text{ is connected in } (N, L)\} \tag{2.2.11}$$

be the set of connected coalitions in (N, L). The class of communication feasible set systems is now defined as follows.

Definition 2.2.44 *A set system $\mathcal{F} \subseteq 2^N$ is a communication feasible set system if there exists a communication graph (N, L) such that $\mathcal{F} = \mathcal{F}_L$.*

Since all singletons in a communication graph are connected, it follows communication feasible set systems contain the empty set and satisfy normality, i.e. every player belongs to at least one feasible coalition. Accessibility is also satisfied. Communication feasible set systems even satisfy the stronger *2-accessibility* property meaning that every feasible coalition with two or more players has at least two players that can leave the coalition leaving behind a feasible coalition. Communication feasible set systems are not union closed. However, as shown by Algaba, Bilbao, Borm and López (2001), communication feasible set systems satisfy the weaker *union stability* meaning that the union of two feasible coalitions that have a nonempty intersection is also feasible. In van den Brink (2012) it is shown that a set system is a set of connected coalitions belonging to some undirected graph if and only if it contains the empty set and satisfies normality, 2-accessibility and union stability.

Theorem 2.2.45 (van den Brink, 2012)
A set system $\mathcal{F} \subseteq 2^N$ is a communication feasible set system if it satisfies the following properties:

*(**feasible empty set**) $\emptyset \in \mathcal{F}$.*

*(**normality**) $N = \bigcup_{S \in \mathcal{F}} S$.*

*(**union stability**) $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$ implies that $S \cup T \in \mathcal{F}$.*

*(**2-accessibility**) $S \in \mathcal{F}$ with $|S| \geq 2$ implies that there exist $i, j \in S$, $i \neq j$, such that $S \setminus \{i\} \in \mathcal{F}$ and $S \setminus \{j\} \in \mathcal{F}$.*

Augmenting systems

Augmenting systems were introduced by Bilbao (2003). These systems, apart from containing the empty set, are characterized by the properties of union stability and a so-called augmentation property.

Definition 2.2.46 *A set system $\mathcal{F} \subseteq 2^N$ is an augmenting system if it satisfies the following properties:*

*(**feasible empty set**) $\emptyset \in \mathcal{F}$,*

*(**union stability**) $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, implies that $S \cup T \in \mathcal{F}$,*

*(**augmentation**) for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.*

Augmentation states that whenever one feasible coalition is contained in another feasible coalition, we can continue selecting players from the 'bigger' coalition and sequentially add those to the 'smaller' coalition, such that after each addition the added to coalition is feasible. Assuming feasibility of the grand coalition N, every feasible set is

guaranteed to belong to some sequence of feasible coalitions starting at $\emptyset$ and ending at N. Similar to Faigle and Kern (1992) admissible permutations of the player set can therefore be defined based on these sequences. Bilbao (2003) also considers games on augmenting systems and gives an extension of the precedence Shapley value by Faigle and Kern (1992) on these games by taking the average of the marginal vectors corresponding to admissible permutations. This solution was later characterized by Bilbao and Ordóñez (2008), see also Chapter 6.

Union stable systems

Algaba, Bilbao, Borm and López (2000) considered those set systems that, apart from containing the empty set, are characterized by union stability. They refer to these systems as union stable systems.

Definition 2.2.47 *A set system $\mathcal{F} \subseteq 2^N$ is a union stable system if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, implies that $S \cup T \in \mathcal{F}$.

The antimatroids, communication feasible set systems and augmenting systems discussed in this chapter all satisfy union stability. They are therefore contained within the class of union stable systems.

Next for $\mathcal{F} \subseteq 2^N$ a set system and $S \subseteq N$, let the maximal nonempty feasible subsets of S be called *components* of S. We denote the set of all components of S by $C_{\mathcal{F}}(S)$. This set is given by

$$C_{\mathcal{F}}(S) = \{T \subseteq S : R \notin \mathcal{F} \text{ for all } T \subset R \subseteq S\} \tag{2.2.12}$$

Observe that the set $C_{\mathcal{F}}(S)$ may be the empty set. As shown in Algaba, Bilbao, Borm and López (2000) a set system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, it holds for any two sets $T, U \in C_{\mathcal{F}}(S)$, that $T \cap U = \emptyset$. Therefore, if $\mathcal{F}$ is normal, the components form a partition of N.

Algaba, Bilbao, Borm and López (2000) also study games on union stable systems.

Definition 2.2.48 *A TU-game on a union stable system is a triple $(N, v, \mathcal{F})$ with $(N, v) \in \mathcal{G}$ and $(N, \mathcal{F})$ a union stable system on N.*

We denote the collection of all TU-games on union stable systems by $\mathcal{G}_{US}$. The set of all games on union stable systems on player set N is denoted by $\mathcal{G}_{US}^N$. A classical TU-game $(N, v_{\mathcal{F}})$ is obtained by restriction from (N, v) using union stable system $\mathcal{F}$.

Definition 2.2.49 *The restricted game $v_{\mathcal{F}} \in \mathcal{G}^N$ of game on union stable system $(N, v, \mathcal{F}) \in \mathcal{G}_{US}$ is given by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \text{ for all } S \subseteq N.$$

Note that for any $S \subseteq N$ such that $C_{\mathcal{F}}(S) = \emptyset$, we have $v^{\mathcal{F}}(S) = 0$. If for union stable system $(N, \mathcal{F})$ the feasible coalitions are the connected coalitions of some graph (N, L), then the game $v^{\mathcal{F}}$ is the graph-restricted game of Myerson (1977) and Owen (1986), see also Definition 2.2.6. If $(N, \mathcal{F})$ is an antimatroid then $\mathcal{F}$ is union closed, and any coalition $S \subseteq N$ has a unique component given by the interior operator $int(S) = \bigcup_{T \in \mathcal{F} : T \subseteq S} T$. Therefore $v^{\mathcal{F}}(S) = v_{\mathcal{F}}(S) = v(int_{\mathcal{F}}(S))$, which is the antimatroid restricted game used in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003), see also Definition 2.2.41.

Algaba, Bilbao, Borm and López (2000) define and characterize a straightforward adaption of the Shapley value for union stable systems. This solution is obtained by applying the Shapley value to restricted game $v^{\mathcal{F}}$. It generalizes the Myerson value for communication graph games and the restricted Shapley value for antimatroids.

Definition 2.2.50 *The restricted Shapley value is the solution φ on $\mathcal{G}^N_{US}$ given by*

$$\varphi(v, \mathcal{F}) = Sh(v^{\mathcal{F}}).$$

Chapter 3

Union values for games with a priori unions

In classical cooperative transferable utility games a decision-making agent is represented by exactly one player in the game. Benefit is assigned to the players. In this chapter we consider games with a priori unions introduced by Aumann and Drèze (1974) that were discussed in Chapter 2. We define a new type of solutions for games with a priori unions; the so-called union values. These values no longer assign benefit to the players, but instead they assign benefit to the unions. The interpretation behind these solutions is that decision making agents are modeled by unions, thereby allowing a decision making agent to be represented by all the players in that union. We introduce and discuss two of these union values, both generalizing the Shapley value for classical cooperative transferable utility games. The first is the *union-Shapley value*. It considers the unions in the most unified way: when a union enters a coalition it can only do so as a whole; with all of its players. The second is the *player-Shapley value* which takes all players in the game as units, and the payoff of a union is the sum of the payoffs over all its players.

This chapter is based on van den Brink and Dietz (2014a).

We provide axiomatic characterizations of the two solutions differing only in a collusion neutrality axiom. After that we consider two applications. First we consider the application of these solutions to airport games. We distinguish between the costs that depend on the type of airplane that uses a landing strip, and costs that do not depend on this. Second we consider the application of these solutions to voting situations.

3.1   Introduction

Recall from Chapter 1 that a *cooperative game with transferable utility*, or simply a TU-game, consists of a finite set of players and for every subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A (single-valued) solution is a function that assigns to every game a payoff vector which components are the individual payoffs of the players. One of the most applied solutions for cooperative TU-games is the *Shapley value*, see Shapley (1953).

In a TU-game a player representing an agent only has to decide on whether to

cooperate within a coalition or not. However, real life situations of cooperation are in most cases more than just a question of participation. For example, an agent might have to decide on how much money to invest in a certain project. In this case there is a range of options that the agent can actually choose from. Another example is a situation where a manager within a department has to decide on which employees to put on a certain project.

The idea behind these examples is that an agent might have multiple sub-agents or 'assets' that it may be able to call on in a situation of cooperation. Therefore, instead of a cooperation situation being modeled solely by a game on the agents, we argue that it might be modelled on the assets that the agents may be able to employ (be it money, subordinates or something else), but that benefit should still be assigned to the agent itself.

An important application of cooperative games in cost allocation is the airport game of Littlechild and Owen (1973), where different airplanes that want to use the same landing strip must pay landing fees that cover the cost of building and maintaining the landing strip, see also Littlechild and Thompson (1977). Although in these airport games the players are usually the airline movements, i.e. every player represents the landing of one airplane, the real decision making agents are the airline companies. So, the airline companies might be considered to be the 'real' decision makers whose assets are the landings of any of their airplanes. This indeed focuses on the assets that an agent is able to employ. However, it seems to be silently implied that an airline company is nothing more than the sum of its separate airline movements, without considering whether just summing is really a desirable way of answering the question of how to divide cost among airline companies. In some sense the question of how to divide benefits from different assets over the agents is ignored within classical TU-games.

This problem was partly adressed by Hsiao and Raghavan (1990), and later extended by van den Nouweland, Potters, Tijs and Zarzuelo (1995), in their papers on multi-choice TU-games. In these games the decision-making agent is assigned a different (finite) number of activity levels. Participation is no longer a yes/no decision but agents can decide on the activity level with which to enter a coalition, choosing from a discrete finite number of levels ranging from no activity to full activity. It can be argued that these different levels can be seen as assets that a decision-making agent can use in cooperating. A high level of activity (high activity asset) being present for a certain decision-making agent also implies the presence of lower levels of activity (lower activity assets), and so these games are suitable to model situations (e.g. money investment) where there exists a natural order on the assets that a decision-making agent may call on. They do however exclude a class of situations like the one with the airplane landings mentioned above, where the choice made by an airline for one airplane landing does not necessarily imply that of another airplane landing.

In this chapter we apply *games with a priori unions* introduced by Aumann and Drèze (1974) to model situations where agents can be represented by more than one player. In a game with a priori unions, the player set is partitioned into a number of disjoint unions. The union expresses that players belong to a common group (this might

for example be a family, a sports team, a firm etc.).[1] The solutions already defined for such games assign a payoff to every individual player (for examples of such solutions see Chapter 2. In our interpretation, decision-making agents (that can be represented by possibly more than one player) are modeled as the unions in such a game with a priori unions. Therefore, in our approach, a solution assigns a payoff to every union (instead of to every player) in the game. To avoid confusion with the usual interpretation we will refer to such solutions that assign payoffs to the unions as *union values* for games with a priori unions. We introduce two such union values, both being generalizations of the Shapley value for TU-games. The first is the *union-Shapley value* and considers the unions in the most unified way. It simply takes a union with all its players as one unit, and when a union enters a coalition then it enters with all its players. The other solution is called the *player-Shapley value* and takes the players as units in the cooperative situation. Here, the payoff of a union is the sum of the payoffs to its players. By considering these two solutions we consider the above mentioned question of how to divide benefit from different assets.

Interestingly, both solutions mentioned above are efficient and satisfy (different) collusion neutrality axioms.[2] It was shown by van den Brink (2012) that there exist no solutions for TU-games satisfying efficiency, the null player property and association collusion neutrality. We will see that extending the model of TU-games to allow agents to be represented by more than one player through a priori unions does allow for collusion neutrality properties that are compatible with (extensions of) efficiency and the null player property. In particular, we provide axiomatizations of the two Shapley type solutions mentioned above that differ only in the collusion neutrality axiom that is used. First, the union-Shapley value satisfies *player collusion neutrality* stating that collusion of two players belonging to the same union does not change the payoff of this union. On the other hand, the player-Shapley value satisfies *union collusion neutrality* stating that after a collusion of two unions, the sum of their payoffs does not change.

After axiomatizing the union- and player-Shapley values we apply them to airport games and voting situations. In particular, for airport games we distinguish between the costs that depend on the size of the airplanes that are using the landing strip, and costs that do not depend on this. We argue that for one type of costs the union-Shapley value is a suitable solution, while for the other type of cost the player-Shapley value is more suitable. For voting games, the union-Shapley value yields the 'traditional' Shapley value (or Shapley-Shubik index (1954)) for the weighted voting games, often used as a measure assigning voting power to the different parties in parliament. The player-Shapley value simply assigns to every party the number of members in parliament, and is often used to distribute the ministries among the parties that form the government.

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[1]Models where every union again can be partitioned into subunions (which can also be partitioned into subunions and so on), are considered in, e.g. Charnes and Littlechild (1975), Winter (1989) and Alvarez-Mozos and Tejada (2011).

[2]We refer to the collusion neutrality type of axioms that are used by Haller (1994) and Malawski(2002)

3.2 The model and solutions

In this chapter we assume that decision making agents can be represented by more than one player in a cooperative game situation. Therefore, we represent such a situation by a game with a priori unions (N, v, P) where $N \subset \mathbb{N}$ is the set of players, v is a characteristic function on the set of players N, and $P = (P_1, \ldots, P_m)$ is a partition of N (see Chapter 2). Given $P = (P_1, \ldots, P_m)$ we denote $M = \{1, \ldots, m\}$ as representing the set of unions. The idea behind the partition P is that every union $k \in M$ is a decision making agent, with the set of players P_k representing its sub-agents. If no confusion arises we refer to $k \in M$ as union P_k. We denote by $\mathcal{P}^N$ the collection of all partitions of N. Recall that the collection of all TU-games with a priori unions is defined by $\mathcal{G}_{APU}$.

We define a union-value for games with a priori unions to be a function $f : \mathcal{G}_{APU} \to \bigcup_{M \subset \mathbb{N}} \mathbb{R}^M$ such that $f(N, v, P) \in \mathbb{R}^M$ for all $(N, v, P) \in \mathcal{G}_{APU}$.

In this chapter we introduce two union-values for games with a priori unions that can be considered extreme cases. In the *union-Shapley value* the unions are considered in the most unified way. It simply takes a union with all of its players as one unit, and when a union enters a coalition it can only do so as a whole; with all of its players. Every union is therefore assigned its Shapley value in the quotient game v^P, see Definition 2.2.3.[3]

Definition 3.2.1 *The union-Shapley value is the union-value on $\mathcal{G}_{APU}$ given by*

$$Sh^u(N, v, P) = Sh(M, v^P).$$

The *player-Shapley value* takes the original game $v : 2^N \to \mathbb{R}$ on the set of players, applies the Shapley value to this game and assigns to every union the sum of the Shapley values of all its players.

Definition 3.2.2 *The player-Shapley value is the union-value on $\mathcal{G}_{APU}$ given by*

$$Sh_k^p(N, v, P) = \sum_{i \in P_k} Sh_i(N, v) \text{ for all } k \in M.$$

In the next section we provide axiomatizations of these two solutions.

3.3 Axiomatizations

We provide axiomatizations of the union- and player-Shapley values that differ only in a collusion neutrality axiom. The other four axioms are standard for solutions to classical TU-games, but here we present them in terms of union-values for games with a priori unions. The first axiom is efficiency and can be taken directly from the TU-game literature, but adding partition P.

Efficiency For every $(N, v, P) \in \mathcal{G}_{APU}$, it holds that $\sum_{k \in M} f_k(N, v, P) = v(N)$.

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[3]Note that for game with a priori unions (N, v, P) the union-Shapley value assigns a payoff to union $k \in M$ that is exactly equal to the sum of the Owen values of the players in $P_k \in P$.

In order to state symmetry, we need to identify what it means for two unions to be symmetric in a game. We say that unions $k, l \in \{1, \ldots m\}$ are *symmetric* in (N, v, P) if $|P_k| = |P_l|$ (i.e. they have the same number of players), and there exist permutations $\pi^k = (\pi^k_1, \ldots, \pi^k_{|P_k|})$ on P_k, and $\pi^l = (\pi^l_1, \ldots, \pi^l_{|P_k|})$ on P_l such that $v(S \cup \{\pi^k_i\}) = v(S \cup \{\pi^l_i\})$ for all $i \in \{1, \ldots, |P_k|\}$ and $S \subseteq N \setminus \{\pi^k_i, \pi^l_i\}$ (i.e. the players of unions k and l can be ordered such that two corresponding unions are 'symmetric' in game v). Since two symmetric unions are in some sense identical with respect to the contributions of their players, symmetry requires them to earn equal payoffs.

Symmetry For every $(N, v, P) \in \mathcal{G}_{APU}$, it holds that $f_k(N, v, P) = f_l(N, v, P)$ whenever $k, l \in M$ are symmetric unions in (N, v, P).

Recall that player $i \in N$ is a null player in game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. The null player out property of Derks and Haller (1999) states that deleting a null player from a TU-game does not change the payoffs of other players. Here we state a similar property for union values by saying that deleting a null player from any union does not change the payoffs assigned to the unions. For game (N, v) we denote the set of null players by $Null(v)$. Recall from Chapter 1 that (S, v_S) is the subgame on coalition S, so $v_S(T) = v(T)$ for all $T \subseteq S$.

Null player out property For every $(N, v, P) \in \mathcal{G}_{APU}$, it holds that $f(N, v, P) = f(N \setminus \{i\}, v_{N \setminus \{i\}}, (P \setminus \{P_k\}) \cup \{P_k \setminus \{i\}\})$ whenever $i \in P_k$ is a null player in (N, v) and $|P_k| \geq 2$.

In this axiom we require $|P_k| \geq 2$ since the union itself should still exist after one of its players is deleted. When the union consists of only one player then the player is the decision-making unit and cannot be discarded. So, although the null player out property of Derks and Haller (1999) for TU-games is an axiom that allows decision-makers (i.e. the players in the TU-game) to be discarded, in our interpretation of games with a priori unions the above axiom keeps the set of decision-makers (i.e. unions) fixed.

Next, we generalize marginalism as used by Young (1985) by saying that whenever all marginal contributions of all players of a union in game v are equal to the corresponding marginal contributions in game w, then this union earns the same in games v and w.

Marginalism For every pair of games with a priori unions $(N, v, P), (N, w, P) \in \mathcal{G}_{APU}$ and union $k \in \{1, \ldots, m\}$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $i \in P_k$ and $S \subseteq N \setminus \{i\}$, it holds that $f_k(N, v, P) = f_k(N, w, P)$.

As mentioned above (and proved later), both the union- and player Shapley value satisfy these four axioms. They differ with respect to a collusion neutrality axiom. In this chapter, we consider two types of collusion, one on the level of the players and one on the level of the unions. Considering the first one, we allow two players of the same union to collude in the sense of Haller (1994)'s *proxy agreement* without there being an effect on the payoff of that union. Recall from (1.0.1) that if player i becomes a proxy for player j in game v, the game v_{ij} generated by the proxy agreement is given by

$$v_{ij}(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S \cup \{j\}) & \text{if } i \in S. \end{cases}$$

The player collusion neutrality property states that if there is a proxy agreement between two players of the same union, the payoff to that union does not change.

Player collusion neutrality For every $(N, v, P) \in \mathcal{G}_{APU}$ with $P = \{P_1, \ldots, P_m\}$, and $i, j \in P_k$, $k \in \{1, \ldots, m\}$, it holds that $f_k(N, v, P) = f_k(N, v_{ij}, P)$.

Player collusion neutrality can be seen as a stability property implying that a union does not gain by splitting or merging some of its players. A similar argument is given by Haller (1994) for collusion neutrality of a TU-game solution. Such a stability property is even more desirable for union values for games with a priori unions than for TU-game solutions since the union itself might decide by how many players it will be represented. For TU-games there are two decision-makers (players) involved to agree on a collusion agreement.

It turns out that this type of collusion neutrality, together with the above four axioms, characterizes the union-Shapley value. Before we continue with the axiomatization, we first consider so-called co-dependent unions in games with a priori unions.

Two unions $k, l \in M$ are *co-dependent* in game with a priori unions (N, v, P) if and only if for every $T \subseteq N$ with $\Delta_v(T) \neq 0$ it holds that $[P_k \cap T \neq \emptyset]$ if and only if $[P_l \cap T \neq \emptyset]$. So two unions are co-dependent if every coalition with nonzero dividend either contains players from both unions, or does not contain any player from these two unions. This also implies that $v^P(S \cup \{k\}) = v^P(S \cup \{l\}) = v^P(S)$ for all $S \subseteq M \setminus \{k, l\}$, and so for two co-dependent unions to generate worth in the quotient game they are dependent on each other in the sense that one union cannot generate any worth without the other union.[4] This also implies that two co-dependent unions are symmetric players in the quotient game (M, v^P). They are not necessarily symmetric unions in (N, v, P) however.

Example 3.3.1 *Let (N, v, P) be a game with a priori unions, where $N = \{1, 2, 3\}$, $v = u_{\{1,3\}} + 2u_{\{2,3\}} - u_{\{1,2,3\}}$, and $P = \{P_1, P_2\}$ with $P_1 = \{1, 2\}$ and $P_2 = \{3\}$. Unions 1 (representing P_1) and 2 (representing P_2) are co-dependent in (N, v, P) and symmetric players in $v^P = u^M$. Unions 1 and 2 are not symmetric unions in (N, v, P), since already $|P_1| \neq |P_2|$.*

The next lemma states that a union value satisfying symmetry and the null player out property assigns the same payoff to two co-dependent unions k and l when both P_k and P_l contain exactly one non-null player in the game.

Lemma 3.3.2 *Let f be a solution satisfying symmetry and the null player out property, and let $k, l \in M$ be two co-dependent unions such that $|P_k \setminus Null(v)| = |P_l \setminus Null(v)| = 1$. Then $f_k(N, v, P) = f_l(N, v, P)$.*

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[4] This can be seen since $v^P(S \cup \{k\}) = v(\bigcup_{h \in S \cup \{k\}} P_h) = \sum_{T \subseteq (\bigcup_{h \in S \cup \{k\}} P_h)} \Delta_v(T) = \sum_{T \subseteq (\bigcup_{h \in S} P_h)} \Delta_v(T) + \sum_{\substack{T \subseteq (\bigcup_{h \in S \cup \{k\}} P_h) \\ T \cap P_k \neq \emptyset}} \Delta_v(T) = \sum_{T \subseteq (\bigcup_{h \in S} P_h)} \Delta_v(T) = v(\bigcup_{h \in S} P_h) = v^P(S)$, where the fourth equality follows since $\Delta_v(T) = 0$ for $T \subseteq \bigcup_{h \in S \cup \{k\}} P_h$ with $T \cap P_k \neq \emptyset$ and $P_l \cap T = \emptyset$.

PROOF
Suppose that solution f satisfies symmetry and the null player out property, and let $k, l \in M$ be two co-dependent unions such that $|P_k \setminus Null(v)| = |P_l \setminus Null(v)| = 1$. The null player out property implies that $f(N, v, P) = f(N', v, P')$, where $N' = N \setminus ((P_k \cup P_l) \cap Null(v))$ and $P' = (P_1', \ldots, P_m')$ with $P_h' = P_h \setminus Null(v)$ if $h \in \{k, l\}$, and $P_h' = P_h$ otherwise. Since k and l are also co-dependent in (N', v, P') with $|P_k'| = |P_l'| = 1$, unions k and l are symmetric unions in (N', v, P') since their only players are symmetric in (N, v). Symmetry of f then implies that $f_k(N', v, P') = f_l(N', v, P')$, and thus $f_k(N, v, P) = f_l(N, v, P)$ by the null player out property. $\square$

Now we state our axiomatization of the union-Shapley value.

Theorem 3.3.3 *A union value f on $\mathcal{G}_{APU}$ is equal to the union-Shapley value if and only if it satisfies efficiency, marginalism, symmetry, the null player out property and player collusion neutrality.*

PROOF
It is straightforward to verify that the union-Shapley value satisfies efficiency, marginalism, symmetry and the null player out property. Player collusion neutrality follows since $v^P = (v_{ij})^P$ if $i, j \in P_k$ for some $k \in \{1, \ldots, m\}$.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. Recall from (2.2.2) that every game v can be written as a (unique) linear combination of unanimity games by $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$. Let $D(N, v) = \{T \subseteq N : \Delta_v(T) \neq 0\}$, and $d(N, v) = |D(N, v)|$. Similar to Young (1985), we perform induction on $d(N, v)$.

If $d(N, v) = 0$ (i.e. (N, v) is the null game given by $v(S) = 0$ for all $S \subseteq N$) then by the null player out property we have that $f_k(N, v, P) = f_k(N', v, P')$ where $N' = \bigcup_{k \in M} P_k'$ with $P' = (P_1', \ldots, P_m')$ such that $P_k' \subseteq P_k$ and $|P_k'| = 1$ for all $k \in M$. Since all unions are symmetric in (N', v, P'), symmetry implies that all $f_k(N', v, P')$ are equal, and therefore by the null player out property all $f_k(N, v, P)$ are equal. By efficiency, $f_k(N, v, P) = 0$ for all $k \in M$.

Proceeding by induction, assume that $f(N, w, P)$ is uniquely determined whenever $d(N, w) < d(N, v)$. Define $R(N, v, P) = \{k \in M : P_k \cap T \neq \emptyset \text{ for all } T \in D(N, v)\}$ as the set of unions that have at least one player in every unanimity coalition with nonzero dividend.

If $k \in M \setminus R(N, v, P)$ then there exists a $T \in D(N, v)$ with $T \cap P_k = \emptyset$. Since the marginal contributions of all players in P_k in (N, v, P) are equal to the corresponding marginal contributions in $(N, v - \Delta_v(T) u_T, P)$, marginalism then implies that $f_k(N, v, P) = f_k(N, v - \Delta_v(T) u_T, P)$, which is uniquely determined by the induction hypothesis since $d(N, v - \Delta_v(T) u_T) = d(N, v) - 1$.

To determine the payoffs of the unions in $R(N, v, P)$, define $Q(N, v, P) = \{k \in R(N, v, P) : |P_k \setminus Null(v)| > 1\}$ as the set of unions in $R(N, v, P)$ that have more

than one non-null player, and let $q(N, v, P) = |Q(N, v, P)|$. We determine $f_k(N, v, P)$, $k \in R(N, v, P)$, by performing a second induction on $q(N, v, P)$.

If $q(N, v, P) = 0$, i.e. all unions have at most one non-null player who is in every $T \in D(N, v)$, then any two unions in $R(N, v, P)$ are co-dependent, and therefore by Lemma 3.3.2 there exists a $c^* \in \mathbb{R}$ such that $f_k(N, v, P) = c^*$ for all $k \in R(N, v, P)$. This c^* is uniquely determined by efficiency, and so is $f_k(N, v, P)$ for all $k \in R(N, v, P)$.

Proceeding by induction, suppose that $f(N, w, P)$ is uniquely determined whenever $q(N, w, P) < q(N, v, P)$. Take $k \in Q(N, v, P)$ and $i, j \in P_k \setminus Null(v)$, $i \neq j$. (Note that such i and j exist since $k \in Q(N, v, P)$.) Player collusion neutrality implies that $f_k(N, v, P) = f_k(N, v_{ij}, P)$. Note that j is a null player in v_{ij}. We can then repeatedly apply player collusion neutrality with player i and any other $h \in P_k \setminus (\{i\} \cup Null(v))$ as done for players i and j above, each time leaving the payoff of union k unchanged by player collusion neutrality. Eventually, we obtain a game where, besides player i there is one other non-null player in P_k and applying collusion neutrality (between i and this other non-null player) once more yields the game v^i given by $v^i = v + \sum_{\substack{T \subseteq N \\ T \cap P_k \neq \emptyset}} \Delta_v(T)(u_{(T \setminus P_k) \cup \{i\}} - u_T)$. By player collusion neutrality it holds that $f_k(N, v, P) = f_k(N, v^i, P)$. Note that all players $j \in P_k \setminus \{i\}$ are null players in v^i. Since player $j \in N \setminus P_k$ is a null player in v^i if and only if it is a null player in v, it holds that $q(N, v^i, P) = q(N, v, P) - 1$, and the payoff $f_k(N, v, P) = f_k(N, v^i, P)$ is uniquely determined by the induction hypothesis.

Finally, all unions $i \in R(N, v, P) \setminus Q(N, v, P)$ are symmetric in $(\overline{N}, v, \overline{P})$ where $\overline{P} = \{\overline{P}_1, \ldots, \overline{P}_m\}$ is such that $\overline{P}_k = P_k \setminus Null(v)$ if $k \in R(N, v, P) \setminus Q(N, v, P)$ and $\overline{P}_k = P_k$ otherwise, and $\overline{N} = \bigcup_{k \in M} \overline{P}_k$.[5] Therefore, by symmetry there exists a $c^* \in \mathbb{R}$ such that $f_k(\overline{N}, v, \overline{P}) = c^*$ for all $k \in R(N, v, P) \setminus Q(N, v, P)$. Since all payoffs to unions not in $R(N, v, P) \setminus Q(N, v, P)$ have been uniquely determined, c^* is uniquely determined by efficiency, and so is $f_k(\overline{N}, v, \overline{P})$ for all $k \in R(N, v, P) \setminus Q(N, v, P)$. Finally by the null player out property it holds that $f_k(N, v, P) = f_k(\overline{N}, v, \overline{P})$, for all $k \in R(N, v, P) \setminus Q(N, v, P)$. $\qquad \square$

Logical independence of the axioms in Theorem 3.3.3 is shown in the appendix of the chapter.

Instead of collusion between players of one union, we can also consider collusion between unions. In this case we do not change the characteristic function, but merge the two unions by replacing them with one union who 'controls' all players of the two original unions together. A collusion between unions $k, l \in M = \{1, \ldots, m\}$ can therefore be described by merging the sets of players of these two unions, i.e. by considering the partition $P^{kl} = (P \setminus \{P_k, P_l\}) \cup \{P_k \cup P_l\}$. Assuming without loss of generality that $k < l$, we label the elements in P^{kl} such that $P_k^{kl} = P_k \cup P_l$, and $P_h^{kl} = P_h$ for all $h \in M \setminus \{k, l\}$.

Union collusion neutrality For every $(N, v, P) \in \mathcal{G}_{APU}$, it holds that $f_k(N, v, P) + f_l(N, v, P) = f_k(N, v, P^{kl})$.

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[5]Explicitly, $\overline{N}$ is given by $\overline{N} = \left(\bigcup_{k \in R(N,v,P) \setminus Q(N,v,P)} P_k \setminus Null(v) \right) \cup \left(\bigcup_{k \in N \setminus (R(N,v,P) \setminus Q(N,v,P))} P_k \right)$.

Note that in union collusion neutrality we do not change the characteristic function, but we merge two elements of the partition. As with player collusion neutrality, union collusion neutrality can be seen as a stability property. It implies that two unions do not gain by splitting or merging. Similar to Haller (1994)'s axiom this collusion involves two decision making units, while for colluding players within a union only one decision-making agent is involved.

Replacing player collusion neutrality in Theorem 3.3.3 by this union collusion neutrality characterizes the player-Shapley value. Moreover, in this case we can do without the null player out property.

Theorem 3.3.4 *A solution f on $\mathcal{G}_{APU}$ is equal to the player-Shapley value if and only if it satisfies efficiency, marginalism, symmetry and union collusion neutrality.*

PROOF

It is straightforward to verify that the player-Shapley value satisfies efficiency, marginalism and symmetry. Union collusion neutrality follows since $P_k \cap P_l = \emptyset$, and thus $\{P_k, P_l\}$ is a partition of $P_k \cup P_l$, so $\sum_{i \in P_k} Sh_i(N, v) + \sum_{i \in P_l} Sh_i(N, v) = \sum_{i \in P_k \cup P_l} Sh_i(N, v)$.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. We perform induction on the number of unions with more than one player. As in the proof of Theorem 3.3.3, we define $D(N, v) = \{T \subseteq N : \Delta_v(T) \neq 0\}$, $d(N, v) = |D(N, v)|$, and $R(N, v, P) = \{k \in M : P_k \cap T \neq \emptyset$ for all $T \in D(N, v)\}$. Further, let $H(N, P) = \{k \in M : |P_k| \geq 2\}$ be the set of unions having at least two players, and $h(N, P) = |H(N, P)|$.

We prove uniqueness by induction on $h(N, P)$. First, suppose that $h(N, P) = 0$, i.e. $|P_k| = 1$ for all $k \in M$. Then the proof is similar to that of Young (1985). We perform a second induction on $d(N, v)$. If $d(N, v) = 0$ then, by symmetry and efficiency $f_k(N, v, P) = 0$ for all $k \in M$. Proceeding by induction, assume that $f(N, w, P)$ is uniquely determined whenever $d(N, w) < d(N, v)$. If $k \in M \setminus R(N, v, P)$ then there exists a $T \in D(N, v)$ with $T \cap P_k = \emptyset$. But then, $f_k(N, v, P) = f_k(N, v - \Delta_v(T)u_T, P)$ by marginalism. Since $d(N, v - \Delta_v(T)u_T) = d(N, v) - 1$, the payoff $f_k(N, v, P)$ is uniquely determined by the induction hypothesis. Since all $k \in R(N, v, P)$ are symmetric, their payoffs are then determined by symmetry and efficiency.

Proceeding by induction, suppose that $f(N, v, P')$ is uniquely determined whenever $h(N, P') < h(N, P)$. For $k \in H(N, P)$, with $P_k = \{k_1, \ldots, k_p\}$, consider $\widetilde{P} = (\widetilde{P}_1, \ldots, \widetilde{P}_{k-1}, \widetilde{P}_{k_1}, \ldots, \widetilde{P}_{k_p}, \widetilde{P}_{k+1}, \ldots, \widetilde{P}_m)$ with $\widetilde{P}_h = P_h$ for all $h \in M \setminus \{k\}$, and $\widetilde{P}_{k_i} = \{k_i\}$ for $k_i \in P_k$. Since $h(N, \widetilde{P}) \leq h(N, P) - 1$, the induction hypothesis implies that $f(N, v, \widetilde{P})$ is uniquely determined. But then union collusion neutrality implies that $f_k(N, v, P) = \sum_{i=1}^p f_{k_i}(N, v, \widetilde{P})$ is determined.

For unions $k \in M \setminus H(N, P)$, i.e. the unions with one player, the proof is again similar to that of Young (1985). We again perform a second induction on $d(N, v)$. If $d(N, v) = 0$ then all unions in $M \setminus H(N, P)$ are symmetric, and since we already determined the payoffs of the unions in $H(N, P)$, efficiency and symmetry determine the payoffs of the players in $M \setminus H(N, P)$. Proceeding by induction, assume that $f_k(N, w, P)$ is uniquely determined whenever $d(N, w) < d(N, v)$. If $k \in M \setminus (H(N, P) \cup R(N, v, P))$ then

there exists a $T \in D(N, v)$ with $T \cap P_k = \emptyset$. But then, $f_k(N, v, P) = f_k(N, v - \Delta_v(T) u_T, P)$ by marginalism. Thus, $f_k(N, v, P)$ is uniquely determined by the induction hypothesis. Since all $k \in (M \setminus H(N, P)) \cap R(N, v, P)$ are symmetric, and all the other payoffs are determined, also their payoffs are determined by symmetry and efficiency. $\square$

Logical independence of the axioms in Theorem 3.3.3 is also shown in the appendix of the chapter.

Now consider the game on a priori unions (N, v, P), where $P = (P_1, ..., P_m)$ consists of a partition into m unions. The quotient game is given by (M, v^P). The quotient game where $k \in M$ acts as a proxy for $l \in M$ according to Haller's proxy agreement property is given by (M, v_{kl}^P) (where $v_{kl}^P = (v^P)_{kl}$). In this game l is a null player. Now define $M' = M \setminus \{l\}$. The subgame of the colluded quotient game on M' is given by $(M', v_{kl}^P|_{M'})$. Now define quotient game $(M'', v^{P^{kl}})$ of the game on a priori unions (N, v, P^{kl}) where union k has absorbed union l. Note that M'' is equal to $M \setminus \{l\}$ and therefore $M' = M''$. We will show that $(M'', v^{P^{kl}})$ is in fact equivalent to $(M', v_{kl}^P|_{M'})$.

Proposition 3.3.5 *For game with a priori unions (N, v, P), where $P = (P_1, ..., P_m)$ and $k, l \in M$ it holds that $(M', v_{kl}^P|_{M'}) = (M', v^{P^{kl}})$ where M' is given by $M \setminus \{l\}$.*

PROOF
By definition of the quotient game and Haller's collusion neutrality $v_{kl}^P|_{M'}(S) = v_{kl}^P(S) = v^P(S) = v(\bigcup_{k \in S} P_k)$ for all $S \subseteq M'$ such that $k \notin S$. By definition of the quotient game and collusion between unions $v^{P^{kl}}(S) = v^P(S) = v(\bigcup_{k \in S} P_k)$ for all $S \subseteq M'$ such that $k \notin S$. Therefore $v_{kl}^P|_{M'}(S) = v^{P^{kl}}(S)$ for all $S \subseteq M$ such that $k \notin S$. By definition of the quotient game and Haller's collusion neutrality for $S \subseteq M'$ such that $k \in S$ it holds that $v_{kl}^P|_{M'}(S) = v_{kl}^P(S) = v^P(S \cup l) = v(\bigcup_{m \in S} P_m \cup P_l)$. By definition of the quotient game and our collusion between unions for $S \subseteq M'$ such that $k \in S$ it holds that $v^{P^{kl}}(S) = v^P(S \cup \{l\}) = v(\bigcup_{m \in S} P_m \cup P_l)$. Therefore also $v_{kl}^P|_{M'}(S) = v^{P^{kl}}(S)$ for all $S \subseteq M$ such that $k \in S$. $\square$

Similar to what van den Brink (2012) shows for collusion under an *association agreement* (see Haller (1994)), it can be shown that there is no solution for TU-games that satisfies efficiency, (proxy) collusion neutrality and the null player property. We remark that both union values introduced in this chapter satisfy the null player out property. They also satisfy a collusion neutrality property. Therefore by using games with a priori unions to represent decision making agents by unions of players instead of by players we can define solutions that satisfy extensions of efficiency, the null player property and the property that collusion between agents (the unions) does not change the sum of the payoffs of these agents. The reason for this is the distinction that is made between agents, that receive payoff, and players, that play the game. The extension to union values on games with a priori unions therefore allows for more possibilities.

Instead of marginalism we might also have used a version of strong monotonicity for union values, similar to that introduced by Young (1985) for classical TU-games. See also Chapter 1 for more details on the characterization of the Shapley value using this property.

3.4 Applications

3.4.1 Airport games

Airport games are introduced by Littlechild and Owen (1973) to allocate the building and maintenance costs of airport landing strips, see also e.g. Littlechild and Thompson (1977).[6]Suppose that a landing strip needs to facilitate n airplane movements. The n airplane movements involve airplanes of different sizes and therefore need landing strips of different size. Assume that the airplane movements are labeled so that the cost of building a landing strip for airplane movement $i \in \{1, ..., n\}$ is given by c_i, where $0 \leq c_1 \leq c_2 \leq ... \leq c_n$. The corresponding airport game (see Littlechild and Owen (1973)) is a cost game (N, v) with the set of airplane movements as the set of players and the characteristic function given by

$$v(S) = \max_{i \in S} c_i \text{ for all } S \subseteq N. \tag{3.4.1}$$

The total cost of the landing strip that can handle all the n airplane movements is therefore determined by the airplane movement involving the most expensive airplane (in terms of landing strip). We assume that larger airplanes are more expensive than smaller ones. Applying the Shapley value to allocate the total cost over the airplane movements one arrives at the famous formula

$$Sh_i(v) = \sum_{j=1}^{i} \frac{c_j - c_{j-1}}{n - j + 1} \text{ for all } i \in N, \tag{3.4.2}$$

where $c_0 = 0$, see Littlechild and Owen (1973).

Instead of allocating costs over the airplane movements, using union values for games with a priori unions we can allocate the costs over the airline companies. Consider the game with a priori unions (N, v, P) where $N = \{1, \ldots, n\}$ is the set of airplane movements, $v \colon 2^N \to \mathbb{R}$ is the airport (cost) game described above, and $P = \{P_1, \ldots, P_m\}$ is the partition of the set of airplane movements into the m airlines, such that P_k is the set of airplane movements of airline $k \in \{1, \ldots, m\}$.[7] The union-Shapley value allocates cost according to the Shapley value of the associated game

$$v^P(S) = v\left(\bigcup_{k \in S} P_k\right) = \max_{i \in \bigcup_{k \in S} P_k} c_i \text{ for all } S \subseteq M. \tag{3.4.3}$$

So, the cost of a coalition of airplane movements is fully determined by the airplane movement involving the largest airplane among the airlines in the coalition. In particular,

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[6]Although we do not discuss these games further, we remark that the class of airport games is equivalent to the class of dual auction games (see Graham, Marshall and Richard (1990)) and DR-polluted river games (see Ni and Wang (2007)).

[7]Vázquez-Brage, García-Jurado and van den Nouweland (1997) also model the airport problem as a game with a priori unions, they allocate the costs according to the Owen value and therefore they apply a player value allocating costs over airplanes. Although one might implicitly assume that the costs are allocated over the airlines by adding for every airline the cost of its airplane movements, this union value is not characterized.

$v^P(\{k\}) = \max_{i \in P_k} c_i$ is the cost of the airplane movement of airline k involving the largest airplane. Each time an airline enters a coalition, its marginal contribution is fully determined by its airplane movement involving the largest airplane. Taking the Shapley value of this game, we allocate cost over the airlines in a way similar to how the Shapley value for airport games allocates cost over the airplane movements.

Example 3.4.1 *Consider a situation, where the set of airplane movements is given by $\{1, 2, 3, 4, 5\}$ and that these airplane movements are divided over airlines 1 and 2 (the partitions). Assume that airplane movements 2 and 5 are flown by airline 1, while airplane movements 1, 3 and 4 are flown by airline 2. Let the cost for each airline movement be given by $c_i = i$ for $i \in N = \{1, \ldots, 5\}$. The associated quotient game v^P is given by $v^P(\{1\}) = 5$, $v^P(\{2\}) = 4$ and $v^P(\{1, 2\}) = 5$. The union-Shapley value is given by $Sh^u(N, v, P) = Sh(N, v^P) = (3, 2)$. The Shapley value for each airplane movement is given by $Sh(N, v) = \frac{1}{60}(12, 27, 47, 77, 137)$. The player Shapley value for each airline is obtained by adding the costs of its own airplane movements. It is given by $Sh^p(N, v, P) = \frac{1}{60}(164, 136)$. Therefore, according to the player-Shapley value more cost is allocated to airline 2 than to airline 1, while the opposite is true for the union-Shapley value.*

Considering the interpretation of the two collusion neutrality axioms to airport games, player collusion neutrality implies that when an airline decides to only use airplane movements involving its largest airplane, then its cost share does not change. On the other hand, union collusion neutrality implies that when two airlines merge in such a way that the new airline maintains their separate airplane movements, then the total share in the costs of the new merged airline equals the sum of the shares in the cost of the two previous airlines. The union-Shapley value might therefore be considered more appropiate for allocating the costs of constructing the landing strip, since these costs depend only on the largest airplane. The player-Shapley value on the other hand might be considered more appropiate for allocating the costs of airport maintenance, since these costs depend on all airplanes.

3.4.2 Voting power

Measuring power in political bodies is often done by applying TU-game solutions to simple games representing the possibilities of political parties to form majority coalitions. A TU-game is called *simple* if $v(S) \in \{0, 1\}$ for every $S \subseteq N$. In such a game a coalition is called *winning* when its worth equals one, otherwise (i.e. when its worth is zero) it is called *losing*. Usually the player set in such games is formed by the political parties that belong to the voting body (and not by the individual members of, say, the parliament). A special class of simple games that are often used to represent political bodies are *weighted majority voting games*. These games describe situations where a coalition of parties is winning if and only if this coalition in total has a number of seats (or votes) that is at least equal to some quota, which is often taken to be higher than half of the number of seats.

A weighted voting game is given by a tuple $(M; s_1, \ldots, s_m; q)$, where $s_k \in \mathbb{N}$ is the number of votes of party k, $k \in M$, and the quota $q \in \mathbb{N}$ is the number of votes needed

to win. The corresponding simple game (M, v) is given by $v(S) = 1$ when $\sum_{k \in S} s_k \geq q$ (S is winning) and $v(S) = 0$ otherwise. A simple game is *proper* if $v(S) = 1$ implies that $v(M \setminus S) = 0$. So, for a weighted majority voting game this means that $q > \frac{1}{2} \sum_{k \in M} s_k$.

One can use voting power measures or indices, such as the Banzhaf index (see Banzhaf (1965)) or the Shapley-Shubik index (see Shapley and Shubik (1954)), to measure the voting power of the political parties in a parliament. The Shapley-Shubik index is obtained by applying the Shapley value to the associated simple voting game. Both indices are based on the marginal contributions of parties in voting games. Note that the marginal contribution of a player to any coalition in a weighted majority voting game is either zero or one. It equals one if and only if the player turns a losing coalition into a winning one. When the marginal contribution of party k to coalition S equals one, then S is called a *swing* for party k.

Although the power of political parties depends heavily on the swings of that party, usually when a majority coalition forms a government, the parties in that coalition divide the number of ministries among them proportionally to the number of seats each party has, and not using the 'real' voting power of the parties. So, it seems that both the Banzhaf and Shapley-Shubik voting power indices as well as the seat distribution among parties plays a role in the formation of a government. The player- and union-Shapley values can be used to obtain both.

Consider the game with a priori unions (N, v, P) where the set N is the set of members (seats) of parliament, v is the characteristic function defined on the set of members of parliament, and $P = (P_1, \ldots, P_m)$ is the partition of the members N into the different political parties. The members in P_k are exactly those that belong to party $k \in \{1, \ldots, m\}$. Notice that $v(S) = 1$ if and only if $|S| \geq q$, even if S contains members of different parties.

The union-Shapley value of (N, v, P) is given by $Sh^u(N, v, P) = Sh(M, v^P)$. We find that (M, v^P) is similar to a 'standard' voting game on the set of parties, where $s_k = |P_k|$ is given by the number of members belonging to each party. Therefore the union-Shapley value of (N, v, P) yields the Shapley-Shubik index of the standard voting game $(M; |P_1|, \ldots, |P_m|; q)$.

All members (players) in N are symmetric in (N, v), and therefore the Shapley value of each member is given by $Sh_i(N, v) = \frac{1}{n}$ for all $i \in N$. It follows that the player-Shapley value of party (union) $k \in M$ equals the fraction of seats of party k and is given by $Sh_k^p(N, v, P) = \sum_{i \in P_k} Sh_i(N, v) = \frac{|P_k|}{|N|} = \frac{s_k}{|N|}$.

Example 3.4.2 *Consider the weighted majority voting game $(M; s_1, \ldots, s_n; q)$ with $M = \{1, 2, 3\}$, $s_1 = 20$, $s_2 = s_3 = 40$ and $q = 51$. So, there are 100 members of parliament who are divided among three parties, and decisions are made by majority voting. Assuming the seats to be labeled such that each party has a consecutive set of seats, this can be modelled by a game with a priori unions (N, v, P) with $N = \{1, \ldots, 100\}$, $P = (P_1, P_2, P_3)$ with $P_1 = \{1, \ldots, 20\}$, $P_2 = \{21, \ldots, 60\}$ and $P_3 = \{61, \ldots, 100\}$. Considering voting power, the game v^P is given by $v^P(S) = 1$ if $S \subseteq M$ with $|S| \geq 2$, and $v^P(S) = 0$ otherwise. Then all unions are symmetric in v^P, and the union-Shapley value $Sh^u(N, v, P) = Sh(M, v^P) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ which equals the Shapley-Shubik index of (N, v^P) and expresses the fact that each party is equally powerful in turning losing coalitions into winning ones.*

However, when parties form a majority coalition then ministerial posts are usually distributed proportionally to the seat distribution which is given by the player-Shapley value $Sh^p(N, v, P) = \frac{1}{100}(20, 40, 40)$ since $Sh_i(N, v) = \frac{1}{100}$ for all $i \in N$.

3.5 Concluding remarks

We used games with a priori unions to model situations where a decision-making agent can be represented by more than one player. The decision making agents are the unions in the partition. We introduced a new type of solution for games with a priori unions, called union value, that assigns payoffs to every union (instead of assigning payoffs to the players as done by the standard solutions in the literature). We introduced two such union values that both generalized the Shapley value, and axiomatized them using different collusion neutrality axioms.

Both solutions can be obtained by applying a standard solution for games with a priori unions (i.e. a solution assigning payoffs to individual players), and then obtaining the payoff of a union as the sum of the payoffs of its individual players. The union-Shapley value is obtained in this way by using the Owen value (see Definition 2.2.4), and the player-Shapley value is obtained by taking the Shapley value of the characteristic function (see Definition 1.0.7). Instead we can use any standard solution φ for games with a priori unions and define the corresponding union value f^φ by

$$f^\varphi(N, v, P) = \sum_{i \in P_k} \varphi_i(N, v, P) \text{ for all } k \in M. \tag{3.5.4}$$

An idea for future research is to extend the model and solutions discussed in this chapter by allowing restrictions in coalition formation. In the TU-game literature there are various restrictions in coalition formation arising from, e.g. restricted communication or authority relations. Applying such restrictions to games with a priori unions, we might consider restrictions in the cooperation between different unions but also restrictions with respect to the players of one union. For example, if we assume a linear ordering of the players of a union and require that 'bottom' players enter before 'top' players we obtain the multi-choice games introduced in Hsiao and Raghavan (1993).

Solutions for multi-choice games as modeled by van den Nouweland, Potters, Tijs and Zarzuelo (1995) may be considered union values for games with coalition structure where a restriction (a linear order) is applied to the set of players belonging to one decision making agent. They define a multi-choice game as a triple (N, a, v). Here N is a set of agents . The vector a is the vector of maximum activity levels for each agent, meaning every agent i can choose their activity level from 0 to a_i. Finally the characteristic function v assigns a worth to every possible combination of activity level choices made by the agents. A agent is assigned a payoff based on the game that is played on the activity levels. The activity levels are ordered, and the presence within a coalition of some activity level j of some agent also implies the presence of levels 1 to $j-1$ of this agent. Solutions to multi-choice games assign a payoff to each agent. In terms of games with a priori unions, the activity levels in a multi-choice game can be considered players and the agents in a multi-choice game the unions. We find that compared to games with a priori unions,

players are restricted by a linear order and therefore have less freedom. This is fine if we want to model a situation where there exists a natural order on the activity levels, such as an amount of money to invest. However this cannot model a situation where, for example, a decision-making agent has a number of subordinates and has to decide which of them work on some project.

Another application of union values can be the entrepreneurial model of Littlechild (1979) where the player set consists of active participants (entrepreneurs) and inactive participants. Although not exactly the same, one can consider the unions in a game with coalition structure as the active participants, where several inactive participants are represented by an active union.

Appendix: Logical independence

Logical independence of the five axioms stated in Theorem 3.3.3 is shown by the following alternative union values.

1. The union value given by $f_k(N, v, P) = 0$ for all $k \in M$ satisfies marginalism, symmetry, the null player out property and player collusion neutrality. It does not satisfy efficiency.

2. The union value given by $f(N, v, P) = Nuc(v^P)$ that assigns to every game with a priori unions the nucleolus of the game v^P, satisfies efficiency, symmetry, the null player out property and player collusion neutrality. It does not satisfy marginalism.

3. Consider exogenously given positive weights $\omega_k > 0$, assigned to all unions $k \in M = \{1, \ldots, m\}$. The union value given by $f_k^\omega(N, v, P) = \sum_{\substack{H \subseteq M \\ k \in H}} \frac{\omega_k}{\sum_{j \in H} \omega_j} \Delta_{v^P}(H)$, $k \in M$, that assigns to every game with a priori unions the weighted Shapley value of v^P corresponding to weight system ω, satisfies efficiency, marginalism, the null player out property and player collusion neutrality. It does not satisfy symmetry.

4. The union value given by f^ω with $\omega_k = |P_k|$, that applies the weighted Shapley value with weights equal to the number of players of every union, satisfies efficiency, marginalism, symmetry and player collusion neutrality. It does not satisfy the null player out property. (Note that this solution itself is not a weighted Shapley value since the weights change if the partition P changes.)

5. The player-Shapley value satisfies efficiency, marginalism, symmetry and the null player out property. It does not satisfy union collusion neutrality.

Logical independence of the five axioms stated in Theorem 3.3.4 can be shown by similar alternative union values. To be precise, the first union value ($f_k(N, v, P) = 0$ for all $k \in M$), also satisfies union collusion neutrality.

Instead of applying the nucleolus to the game v^P, consider the union value $f_k(N, v, P) = \sum_{i \in P_k} Nuc_i(v)$, $k \in M$, that assigns to every union the sum of the nucleolus payoffs

over all its players. This union value satisfies efficiency, symmetry, and union collusion neutrality, but does not satisfy marginalism.

Considering exogenously given positive weights $\omega_i > 0$, assigned to all players $i \in N$, the union value given by $\overline{f}_k^{\omega}(N, v, P) = \sum_{i \in P_k} \sum_{\substack{H \subseteq N \\ i \in H}} \frac{\omega_i}{\sum_{j \in H} \omega_j} \Delta_v(H)$, $k \in M$, that assigns to every union the sum of the weighted Shapley values of all its players, satisfies efficiency, marginalism, and union collusion neutrality. It does not satisfy symmetry.

Obviously, the union-Shapley value satisfies efficiency, marginalism and symmetry, but does not satisfy union collusion neutrality.

Chapter 4

A local approach to games with a permission structure

It is known that the set of peer group games generated by the restriction of peer group situations is contained within the set of restricted games generated by both the conjunctive as wel as the disjunctive approach to games with a permission structure. However, the set of peer group games is also contained within the class of (weighted) digraph games. In this chapter we first argue that some known results on solutions for peer group games hold more generally for digraph games.

Second, we generalize both digraph games as well as games restricted by the conjunctive approach to permission structures by taking a local approach to games with a permission structure. This local approach reflects that every player needs permission from its predecessors in the digraph in order to generate worth, but that it does not need its predecessors in order to give permission to its own successors. We introduce and axiomatize a Shapley value type solution for these games, generalizing the conjunctive permission value for games with a permission structure and the β-measure for weighted digraphs.

This chapter is based on van den Brink and Dietz (2014b).

4.1 Introduction

The set of peer group games forms a subclass of the set of games obtained by the restriction of games with a permission structure. Besides being a special class of the TU-games generated by the restriction of a game with a permission structure, peer group games are also a special class of the (weighted) digraph games introduced in van den Brink and Borm (2002). Recall from Chapter 2 that for a given (weighted) digraph, where each node is assigned a weight, the associated digraph game assigns to every coalition of nodes the sum of the weights of all nodes in the coalition whose predecessors also belong to the coalition. For any peer group game we consider the weighted digraph game generated by the transitive closure of the rooted tree associated with that peer group situation and the same set of weights. It is seen that these games are equivalent. We show that results stated in the literature for peer group games can be extended straightforwardly to the more general class of weighted digraph games.

In this chapter we develop a new approach to permission structures. We have seen in Chapter 2 that for the conjunctive approach to permission structures a player within a coalition can only generate worth within the restricted game, if its complete set of superiors is also present within the coalition. With the new approach we weaken this requirement. For a player to be able to generate worth, only his direct predecessors within the digraph need to belong to the coalition. Compared to the set of superiors this constitutes a more local approach to permission structures, which is why the new approach will be referred to as the *local approach to permission structures*. Note that within this approach it is possible that a player grants permission to his successors, even though he himself does not have the permission to generate worth. This constitutes a certain degree of separation between authority and worth generation. A player might be needed within a coalition to give permission to other players, while he himself is unable to generate any worth. This is different from the conjunctive approach, where any player in a feasible coalition that is needed for permission, is also able to generate worth.

For hierarchically structured firms we can say that in a 'standard' permission structure a worker at the bottom can only be 'activated' by obtaining permission from its complete set of superiors. This is in line with Williamson (1967) where only these bottom level workers are able to generate worth, while the higher level managers organize and manage the production process, but do not actively take part in it. Often however, permission does not consist of a whole chain of predecessors being involved in every step of the production process. The new local approach to permission structures does not require this whole chain to be involved.

Similar to the conjunctive (and disjunctive) approach, we associate with every game with a permission structure a new restricted game, called the *locally restricted game*, where the worth of any coalition equals the worth of its worth generating set, that is, the subset of players in the coalition whose predecessors all belong to the coalition. We show that the set of locally restricted games developed in this chapter is a superset on both the set of conjunctively restricted games as well as the set of (weighted) digraph games. The set of locally restricted games generated by a game with a permission structure where the digraph is transitive is exactly the set of conjunctively restricted games. The set of locally restricted games generated by a game with a permission structure where the game is additive is exactly the set of weighted digraph games.

After studying several properties of the local approach to permission structures and locally restricted games, we introduce the local permission value as the solution that is obtained by applying the Shapley value to the locally restricted game, and compare it with the 'standard' conjunctive permission value by providing an axiomatization with axioms similar to those that characterize the conjunctive permission value.

4.2 Peer group games are digraph games

Brânzei, Fragnelli and Tijs (2002) already mention that peer group games are a subset of the set of conjunctively restricted games (see Chapter 2). However, peer group games are also a special class of weighted digraph games. For every peer group situation (a, T), the associated peer group game $v_{a,T}^P$ coincides with the weighted digraph game on the

transitive closure of the peer group tree $\overline{v}_{a,tr(T)}$ (see Chapter 2). (Note that in this chapter all results are obtained using a fixed player set N. Therefore we denote (N, v) by v, (N, a, T) by (a, T), etc.)

Proposition 4.2.1 *For every peer group situation (a, T) it holds that $v_{a,T}^P = \overline{v}_{a,tr(T)}$.*

PROOF
Let (a, T) be a peer group situation. By definition of the transitive closure it holds that $P_{tr(T)}(i) = \widehat{P}_T(i)$. Therefore, taking weighted digraph (a, T) it holds that $v_{a,T}^P(S) = \sum_{\substack{i \in S \\ \widehat{P}_T(i) \subseteq S}} a_i = \overline{v}_{\delta,tr(T)}(S)$, for all $S \subseteq N$. $\qquad\qquad\square$

On the other hand, not every weighted digraph game is a peer group game. This holds even if we restrict ourselves to rooted trees.

Example 4.2.2 *Consider the weighted digraph (δ, D) on $N = \{1, 2, 3\}$ given by $\delta = (1, 1, 1)$ and $D = \{(1, 2), (2, 3)\}$. The corresponding weighted digraph game is $\overline{v}_{\delta,D} = u_{\{1\}} + u_{\{1,2\}} + u_{\{2,3\}}$. There is no peer group situation (a, T) such that $v_{a,T}^P = \overline{v}_{\delta,D}$ since $\overline{v}_{\delta,D}(\{1\}) = 1$ implies that 1 is the root of T. But then $v_{a,T}^P(\{2, 3\})$ must be equal to zero, while $\overline{v}_{\delta,D}(\{2, 3\}) = 1$.*

We have the following corollary (recall from (2.2.6) that w^a is the additive game, where the worth of every coalition is given by the sum of the weights of the players in that coalition).

Corollary 4.2.3 *For every peer group situation (a, T) it holds that $v_{a,T}^P = r_{w^a,T}^c = r_{w^a,tr(T)}^c = \overline{v}_{a,tr(T)}$.*

So, in the literature we encounter two classes of games with a hierarchical structure on the player set which generalize peer group games. Whereas the results on solutions for peer group games as given in Brânzei, Fragnelli and Tijs (2002, Proposition 1) do not in general hold for conjunctively restricted games, they do hold for digraph games.[1]

Proposition 4.2.4 *Let (δ, D) be a weighted digraph and let $\overline{v}_{\delta,D}$ be the associated digraph game. Then*

(i) *The bargaining set of $\overline{v}_{\delta,D}$ coincides with the Core of $\overline{v}_{\delta,D}$;*

(ii) *The kernel of $\overline{v}_{\delta,D}$ coincides with the pre-kernel of $\overline{v}_{\delta,D}$ and consists of a unique point being the nucleolus of $\overline{v}_{\delta,D}$;*

(iii) *The nucleolus of $\overline{v}_{\delta,D}$ occupies a central position in its Core and is the unique point x satisfying $\{x \in \mathbb{R}^N : s_{ij}(x) = s_{ji}(x)$ for all i, j such that $i \neq j$, where $s_{ij}(x) = \max\{\overline{v}_{\delta,D}(S) - x(S) : i \in S \subseteq N \setminus \{j\}\}\}$;*

(iv) *The Core of $\overline{v}_{\delta,D}$ coincides with the Weber set of $\overline{v}_{\delta,D}$;*

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[1]We refer to the appendix at the end of this chapter for definitions of the solutions.

(v) *The Shapley value $Sh(\overline{v}_{\delta,D})$ is the center of gravity of the extreme points of the Core and is given by $Sh_i(\overline{v}_{\delta,D}) = \sum_{j \in F_D(i) \cup \{i\}} \frac{\delta_j}{(|P_D(j)|+1)}, \ i \in N$;*

(vi) *The τ-value $\tau(\overline{v}_{\delta,D})$ is given by $\tau_i(\overline{v}_{\delta,D}) = \begin{cases} \alpha\delta_i + (1-\alpha)M_i(\overline{v}_{\delta,D}) & \text{if } P_D(i) = \emptyset \\ (1-\alpha)M_i(\overline{v}_{\delta,D}) & \text{if } P_D(i) \neq \emptyset \end{cases}$, where $M_i(\overline{v}_{\delta,D}) = \sum_{j \in F_D(i) \cup \{i\}} \delta_j$ and $\alpha \in [0,1]$ is such that $\sum_{i \in N} \tau_i(\overline{v}_{\delta,D}) = \overline{v}_{\delta,D}(N)$.*

(vii) *The core of $\overline{v}_{\delta,D}$ coincides with the selectope of $\overline{v}_{\delta,D}$;*

Similar to Brânzei, Fragnelli and Tijs (2002), parts (i), (ii) and (iii) follow directly from convexity of $\overline{v}_{\delta,D}$, see Maschler, Peleg and Shapley (1972) and Maschler (1992), (vii) follows from the game being totally positive (meaning that all dividends are non-negative) as was mentioned by van den Brink and Borm (2002), see Derks, Haller and Peters (2000) and Vasil'ev and van der Laan (2002). This also implies (iv) since the Weber set is always a subset of the selectope and contains the Core. Computation of the τ-value in (vi) follows from its definition. Finally, (v) is shown in van den Brink and Borm (2002).

Brânzei, Fragnelli and Tijs (2002) conclude by mentioning some applications of peer group games. Since their above mentioned results on solutions also hold for more general digraph games, these results also can be applied to more situations, such as the ranking of teams in sports competitions, defining social choice correspondences or voting rules in social choice theory, and measuring relational power in social networks.

Theorem 4.4 of van den Brink and Borm (2002) shows that the set of marginal vectors equals the set of selectope vectors for those weighted digraph games $\overline{v}_{\delta,D}$ where the digraph D has no *anti-directed semi-circuit*. An anti-directed semi-circuit is a sequence $(i_1, i_2, \ldots, i_t)$, $t \geq 2$, of t even distinct nodes such that (i) $i_k \in N$ for all $k \in \{1, \ldots, t\}$, (ii) $i_k \in F_D(i_{k-1}) \cap F_D(i_{k+1})$ if $k \neq t$ is even, and (iii) $i_t \in F_D(i_{t-1}) \cap F_D(i_1)$. To discuss in what way this result holds for peer group games, we consider the existence of anti-directed semi-circuits within digraphs that are the transitive closure of some rooted tree (remember that the set of peer group games is exactly equal to the set of weighted digraph games on digraphs of this kind). It can easily be seen that for any rooted tree T with a path from the top node to another node containing at least four nodes, the transitive closure $tr(T)$ has an anti-directed semi-circuit.

Example 4.2.5 *Consider the peer group situation (a, T) on $N = \{1, 2, 3, 4\}$ given by $a = (1, 1, 1, 1)$ and $T = \{(1, 2), (2, 3), (3, 4)\}$. The corresponding peer group game is $v^P_{a,T} = u_{\{1\}} + u_{\{1,2\}} + u_{\{1,2,3\}} + u_{\{1,2,3,4\}}$. This game is equal to the digraph game $\overline{v}_{\delta,D}$ corresponding to $\delta = a$ and $D = tr(T) = T \cup \{(1,3),(1,4),(2,4)\}$. Digraph $tr(T)$ contains the anti-directed semi-circuit given by the sequence $(1, 3, 2, 4)$ and therefore the set of marginal vectors does not equal the set of selectope vectors.*

The set of marginal vectors equals the set of selectope vectors for the associated weighted digraph game of peer group game $(N, v^P_{a,T})$ if the rooted tree T has no path from the top node to any other node with more than 3 players.

4.3 Locally restricted games

Both the conjunctive approach to games with a permission structure as well as the weighted digraph games can be seen to use a digraph to restrict cooperation between the players in a classical TU-game. Two essential differences exist between these two models. In Chapter 2 we have seen that the conjunctively restricted game as well as the weighted digraph game can be obtained as the restriction from some underlying TU-game. For the conjunctive approach any characteristic function is allowed, while for the weighted digraph games only additive ones are considered. The second difference lies in the permission a player needs to generate worth. The conjunctive approach implies that a player needs permission from all its superiors in the digraph, while in the weighted digraph case a player only needs permission from its direct predecessors in the digraph. Note that for transitive digraphs it holds that for any player its set of direct predecessors is equal to its set of superiors. Therefore the weighted digraph game obtained from a set of weights and a transitive digraph, can also be obtained as the conjunctive restriction of the additive game on the weights and that same digraph.

Proposition 4.3.1 *For every weighted digraph* (δ, D) *it holds that* $\overline{v}_{\delta, tr(D)} = r^c_{w^\delta, D}$*. In particular, if* D *is transitive then* $\overline{v}_{\delta, D} = r^c_{w^\delta, D}$*.*

PROOF
Let (δ, D) be a weighted digraph. By definition of the transitive closure it holds that $P_{tr(D)}(i) = \widehat{P}_D(i)$. Therefore, $\overline{v}_{\delta, tr(D)}(S) = \sum_{\substack{i \in S \\ P_{tr(D)}(i) \subseteq S}} \delta_i = \sum_{\substack{i \in S \\ \widehat{P}_D(i) \subseteq S}} w^\delta(\{i\}) = w^\delta(\{i \in S : \widehat{P}_D(i) \subseteq S\}) = w^\delta(\sigma^c_D(S)) = r^c_{w^\delta, D}(S)$ for all $S \subseteq N$. $\square$

In what follows we consider a new approach to games with a permission structure. In this approach, called the *local approach to permission structures*, a player needs only his set of direct predecessors in the digraph to be present in order to generate worth. This allows for situations where a player himself is unable to generate worth, but (through his permission) his successors are. This is different from both the conjunctive as well as the disjunctive approach, where only players that are able to generate worth are able to give permission to other players, so that they may generate worth. Note that in a sense the local approach can still be considered a conjunctive approach, although on a local scale, since for a player *all* of its direct predecessors in the digraph need to be present in the coalition in order for that player to generate worth. A disjunctive local approach might also be defined, where a player only needs at least one of his direct predecessors in order to generate worth. Remarks on a disjunctive approach are made in the final section.

Definition 4.3.2 *For any coalition* $S \subseteq N$ *the worth generating set of* S *in* D *is given by*

$$\sigma^l_D(S) = \{i \in S : P_D(i) \subseteq S\}$$

So $\sigma^l_D(S)$ is the set of players in S for whom all predecessors also belong to S. A restricted game is obtained by assigning to any coalition S the worth of $\sigma^l_D(S)$.

Definition 4.3.3 *The locally restricted game* $r^l_{v,D} \in \mathcal{G}^N$ *of game with a permission structure* $(v, D) \in \mathcal{G}^N_D$ *is given by*

$$r^l_{v,D}(S) = v(\sigma^l_D(S)) \text{ for all } S \subseteq N.$$

An important difference with the conjunctive approach is the fact that $\sigma^l_D(\sigma^l_D(S))$ need not be equal to $\sigma^l_D(S)$, so the worth generating set of a coalition need not be equal to its own worth generating set.[2,3]

Example 4.3.4 *Consider the digraph* D *on* $N = \{1, 2, 3\}$ *given by* $D = \{(1, 2), (2, 3)\}$. *Then* $\sigma^l_D(\{2, 3\}) = \{3\}$ *but* $\sigma^l_D(\{3\}) = \emptyset$. *Now consider game* $v = u_{\{3\}} + u_{\{2\}}$. *For coalitions* $S = \{2, 3\}$ *and* $T = \{3\}$, *it holds that* $r^l_{v,D}(S) = v(\sigma^l_D(S)) = v(T) = 1$. *The worth of* S *in the unrestricted game however is given by* $v(S) = 2$. *Player 2 loses the ability to generate worth according to the local approach because* $P_D(2) = \{1\}$ *is not present in coalition* S. *Now it also holds that* $r^l_{v,D}(T) = v(\sigma^l_D(T)) = v(\emptyset) = 0$. *This shows that although player 2 is unable to generate any worth in coalition* $\{2, 3\}$, *he is still necessary for player 3 to generate worth.*

Because of the separation between authority (permission) and worth generation illustrated in the example, the cooperation structure cannot be described by a set of feasible coalitions as is the case for the conjunctive and disjunctive approach (see Chapter 2). Feasibility implies that if a coalition is the worth generating set or feasible part of some other coalition, then it is also able to generate worth on its own, i.e. a feasible set. In the example, coalition $\{3\}$ can not be considered feasible because it is not able to generate any worth, but there is a coalition, namely $\{2, 3\}$, such that $\{3\}$ is exactly the coalition that generates the worth of $\{3\}$. Therefore, we call $\{3\}$ a worth generating set in D.

Proposition 4.3.5 *Let* $(v, D) \in \mathcal{G}^N_D$ *and* $S \subseteq N$ *be given.*

(i) *For all* T *such that* $\sigma^c_D(S) \subseteq T \subseteq S$ *it holds that* $r^c_{v,D}(T) = v(\sigma^c_D(S))$

(ii) *For all* $T \subseteq S \setminus \sigma^c_D(S)$ *it holds that* $r^c_{v,D}(T) = 0$

(iii) *For all* T *such that* $\sigma^l_D(S) \cup P_D(\sigma^l_D(S)) \subseteq T \subseteq S$ *it holds that* $r^l_{v,D}(T) = v(\sigma^l_D(S))$

(iv) *For all* $T \subseteq S \setminus \sigma^l_D(S)$ *it holds that* $r^l_{v,D}(T) = 0$.

---

[2]In a similar way, for the games with restricted coalitions by Derks and Peters (1993) it holds that $\rho(\rho(S)) = \rho(S)$, where $\rho(S)$ can be considered the 'feasible' part of coalition S.

[3]In models restricted by feasible set systems, such as the communication graph games of Myerson (1977), and its generalization to games on union stable systems in Algaba, Bilbao, Borm and López (2000, 2001), it is the case that the worth of a feasible coalition in the restricted game equals the worth of this coalition in the original game.

PROOF

(i) For coalition S it holds that $\sigma_D^c(S)$ is the maximal subset $U \subseteq S$ such that for all players $i \in U$, $\widehat{P}_D(i) \subset S$. However by definition of $\sigma_D^c(S)$ it also holds for all players $i \in \sigma_D^c(S)$ that $\widehat{P}_D(i) \subset \sigma_D^c(S)$. Therefore $\sigma_D^c(T) = \sigma_D^c(S)$ for all T such that $\sigma_D^c(S) \subseteq T \subseteq S$.

(ii) For any player $i \in T \subseteq S \setminus \sigma_D^c(S)$ it holds that $\widehat{P}_D(i) \not\subseteq T$. Therefore $\sigma_D^c(T) = \emptyset$.

(iii) By definition of the local worth generating set it holds that $\sigma_D^l(S)$ is the maximal subset $U \subset S$ such that for all players $i \in U$, $P_D(i) \subset S$. However by definition of $\sigma_D^l(S)$ it does not have to hold for all players $i \in \sigma_D^l(S)$ that $P_D(i) \subset \sigma_D^l(S)$. However it does hold for all T such that $\sigma_D^l(S) \cup P_D(\sigma_D^l(S)) \subseteq T \subseteq S$, that $\sigma_D^l(T) \cup P_D(\sigma_D^l(T)) = \sigma_D^l(S) \cup P_D(\sigma_D^l(S))$.

(iv) For any player $i \in T \subseteq S \setminus \sigma_D^c(S)$ it holds that $P_D(i) \not\subseteq T$. Therefore $\sigma_D^l(T) = \emptyset$ and therefore $r_{v,D}^l(T) = 0$.

$\square$

Part (i) implies that for conjunctively restricted games, if a coalition of players S is able to generate its own worth, permission is not needed from the players outside S. Therefore worth generation implies permission. In part (iii) we find that the worth of a coalition S in the locally restricted game is that of its worth generating set $\sigma_D^l(S)$ in the original game v. Note that a coalition containing the worth generating set of S, but not all its predecessors might generate a different worth. Although it is true that $\sigma_D^l(T) = \sigma_D^l(S)$ for all T such that $\sigma_D^l(S) \cup P_D(\sigma_D^l(S)) \subseteq T \subseteq S$, this does not necessarily hold for all T such that $\sigma_D^l(S) \subseteq T \subseteq S$. This is an important difference with the 'standard' conjunctive approach to games with a permission structure. Next, we introduce some notions to describe the worth generation and permission in the local approach to games with a permission structure.

Definition 4.3.6 *For any coalition $S \subseteq N$ the active set of S is given by*

$$\overline{\alpha}_D^l(S) = \sigma_D^l(S) \cup P_D(\sigma_D^l(S))$$

The players in $\overline{\alpha}_D^l(S)$ are those that are necessary and sufficient to make the worth generating set $\sigma_D^l(S)$ of S active.

Call a set S *locally feasible* in D if $\overline{\alpha}_D^l(S) = S$.

Definition 4.3.7 *The set Ψ_D of locally feasible coalitions of permission structure D is given by*

$$\Psi_D = \{S \subseteq N : \overline{\alpha}_D^l(S) = S\}.$$

Note that not all the players belonging to a locally feasible set generate worth. Feasibility of a coalition, as it was discussed in Chapter 2, is understood to mean that every player in that coalition can generate worth in the coalition. This means that for feasible set system $\mathcal{F}$ if a set $S \in \mathcal{F}$ is feasible, it should hold that its worth in the game $v_{\mathcal{F}}$ restricted by $\mathcal{F}$ is equal to its worth in the original game v (so $v_{\mathcal{F}}(S) = v(S)$). Note that this holds for all of the set systems discussed in Chapter 2. The collection of locally feasible sets therefore is not a feasible set system in this sense. However for a coalition S to be locally feasible it holds that every player i belonging to that coalition is either giving permission to other players in S (meaning that $\sigma_D^l(S \setminus \{i\}) \neq S \setminus \{i\}$), generating worth (meaning that $i \in \sigma_D^l(S)$), or both.

The authorizing set of any coalition $S \subseteq N$ is the smallest coalition in Ψ_D that contains S.

Definition 4.3.8 *For any coalition $S \subseteq N$ the authorizing set of S is given by*

$$\alpha_D^l(S) = S \cup P_D(S).$$

This is the set of players in S together with all their predecessors. This is the set of players that is necessary and sufficient to make the players in S active. Compared to the authorizing set of coalitions S of the conjunctive approach (see Definition 2.2.16), only predecessors are needed instead of superiors.

Proposition 4.3.9

(i) *For all coalitions $S \subseteq N$, $\alpha_D^l(S)$ is locally feasible.*

(ii) *For all coalitions $S \subseteq N$, $\overline{\alpha}_D^l(S) = \alpha_D^l(\sigma_D^l(S))$*

PROOF

(i) We need to show that $\overline{\alpha}_D^l(\alpha_D^l(S)) = \alpha_D^l(S)$. By definition of $\overline{\alpha}_D^l$ we have that $\overline{\alpha}_D^l(\alpha_D^l(S)) \subseteq \alpha_D^l(S)$. Therefore suppose that $\overline{\alpha}_D^l(\alpha_D^l(S)) \neq \alpha_D^l(S)$ and therefore $\overline{\alpha}_D^l(\alpha_D^l(S)) \subset \alpha_D^l(S)$. This means that the set $T = \alpha_D^l(S) \setminus \overline{\alpha}_D^l(\alpha_D^l(S))$ is non-empty. By definition of $\overline{\alpha}_D^l$ the players in T are not part of the worth generating set $\sigma_D^l(\alpha_D^l(S))$ nor do they give permission to any of the players in $\sigma_D^l(\alpha_D^l(S))$. Since $S \subset \sigma_D^l(\alpha_D^l(S))$ by definition of the authorizing set, it must hold that T is a subset of $U = \alpha_D^l(S) \setminus S$. However by definition of the authorizing set $\alpha_D^l(S)$ it also holds that U contains only players i for which there exists some $j \in F_D(i)$ such that $j \in S$. Therefore $U \subseteq \overline{\alpha}_D^l(\alpha_D^l(S))$ and we have a contradiction.

(ii) By definition $\overline{\alpha}_D^l(S) = \sigma_D^l(S) \cup P_D(\sigma_D^l(S)) = \alpha_D^l(\sigma_D^l(S))$

$\square$

From the proposition it follows that the set of locally feasible coalitions can also be expressed as the set of those coalitions $S \subseteq N$ for which there exists some $T \subseteq N$ such that S is the authorizing set of T.

Example 4.3.10 *Consider the permission structure D of Example 4.3.4 and coalition $\{2,3\}$. We already saw that its worth generating set is given by $\{3\}$. Its active set is $\overline{\alpha}_D^l(\{2,3\}) = \{2,3\}$ since permission of 2 is necessary and sufficient to make its worth generating set $\{3\}$ active. Its authorizing set is $\alpha_D^l(\{2,3\}) = \{1,2,3\}$ since player 1 is necessary to make player 2 active who is not worth generating in $\{2,3\}$ but is still necessary to give permission to player 3. In this case the conjuctive feasible sets are given by $\Phi^c(N,D) = \{\{1\},\{1,2\},\{1,2,3\}\}$, while the locally feasible sets are given by $\Psi_D = \Phi^c(N,D) \cup \{\{2,3\}\}$.*

Consider permission structure $D \in \mathcal{D}^N$. The following properties follow naturally from the definition of locally feasible subsets.

(i) $\emptyset \in \Psi_D$,

(ii) $N \in \Psi_D$,

(iii) If $S, T \in \Psi_D$ then $S \cup T \in \Psi_D$.

Part (iii) shows that Ψ_D is union closed. The basic elements of Ψ_D are the sets $\{i\} \cup P_D(i)$, $i \in N$. The other elements of Ψ_D can be written as the union of two or more basic elements. However, unlike the conjuctive sovereign parts of two conjuctive feasible coalitions, for two coalitions $S, T \in \Psi_D$ it does not necessarily hold that $S \cap T \in \Psi_D$.

Example 4.3.11 *Consider the permission structure D of Example 4.3.4. Both coalitions $S = \{1,2\}$ and $T = \{2,3\}$ belong to Ψ_D. However, $S \cap T = \{2\}$ does not belong to Ψ_D.*

Next, we state some properties of worth generating sets and authorizing sets, similar to properties that hold for the sets of conjuctive feasible coalitions in Gilles, Owen and van den Brink (1992, Proposition 3.5). The proof is similar to their result, but using $F_D(i)$ and $P_D(i)$ instead of $\widehat{F}_D(i)$ and $\widehat{P}_D(i)$ respectively, and is therefore omitted.

Proposition 4.3.12 *Consider $D \in \mathcal{D}^N$ and $S, T \subseteq N$. Then*

(i) $\sigma_D^l(S) \cup \sigma_D^l(T) \subseteq \sigma_D^l(S \cup T)$;

(ii) $\sigma_D^l(S) \cap \sigma_D^l(T) = \sigma_D^l(S \cap T)$;

(iii) $\alpha_D^l(S) \cup \alpha_D^l(T) = \alpha_D^l(S \cup T)$;

(iv) $\alpha_D^l(S \cap T) \subseteq \alpha_D^l(S) \cap \alpha_D^l(T)$.

The following result is similar to another result in Gilles, Owen and van den Brink (1992, Theorem 4.2). While their proof however uses the fact that $\sigma_D^c(S) = S$ for every $S \in \Phi^c(N,D)$, we have seen that for the local approach to permission structures $\sigma_D^l(S)$ need not be equal to S for $S \in \Psi_D$.

Theorem 4.3.13 *For $S \subseteq N, S \neq \emptyset$ and $D \in \mathcal{D}^N$, it holds that $r_{u_S,D}^l = u_{\alpha_D^l(S)}$.*

PROOF
By definition of the locally restricted game, we have $r^l_{u_S,D}(T) = u_S(\sigma^l_D(T)) = 1$ if $S \subseteq \sigma^l_D(T)$, and $r^l_{u_S,D}(T) = 0$ otherwise. Since for $S \subseteq T$ we have $[S \subseteq \sigma^l_D(T)] \Leftrightarrow [S \cup P_D(S) \subseteq T] \Leftrightarrow [\alpha^l_D(S) \subseteq T]$, this implies that $r^l_{u_S,D} = u_{\alpha^l_D(S)}$. $\qquad\square$

Because of Proposition 4.3.4, as a corollary (similar to Corollary 4.3 in Gilles, Owen and van den Brink (1992)), any locally restricted game can be described as a linear combination of the unanimity games of its locally feasible sets.

Corollary 4.3.14 *Let $(v, D) \in \mathcal{G}^N_D$. Then $r^l_{v,D} = \sum_{S \in \Psi_D} \left[\sum_{\substack{T \subseteq N \\ \alpha^l_D(T)=S}} \Delta_v(T) \right] u_S$*

Inheritance properties similar to those of the conjunctively restricted game also hold for the locally restricted game.

Proposition 4.3.15 *Let $(v, D) \in \mathcal{G}^N_D$.*

(i) *If v is monotonic then $r^l_{v,D}$ is monotonic. Moreover, if v is also balanced, then $r^l_{v,D}$ is balanced as well.*

(ii) *If v is superadditive, then $r^l_{v,D}$ is superadditive.*

(iii) *If v is convex, then $r^l_{v,D}$ is convex.*

The proof is similar to the proof of Theorem 4.6 in Gilles, Owen and van den Brink (1992) and is therefore omitted. Part (iv) of their result states that the existence of a player i_0 such that $\widehat{S}(i_0) = N \setminus \{i_0\}$ is sufficient for the restriction of a monotonic game to be superadditive and balanced. That is not true for locally restricted games.

Example 4.3.16 *Consider the permission structure D of Example 4.3.4 and game $v \in \mathcal{G}^N$ given by $v(S) = 1$ for all $S \subseteq N$, $S \neq \emptyset$. Then $\widehat{F}_D(\{1\}) = N \setminus \{1\}$ and v is monotonic, but $r^l_{v,D}(\{2\}) = r^l_{v,D}(\{3\}) = 0$ and $r^l_{v,D}(S) = 1$ otherwise, and therefore $r^l_{v,D}$ is not superadditive, nor balanced (since $r^l_{v,D}(\{1\}) + r^l_{v,D}(\{2, 3\}) = 2 > 1 = r^l_{v,D}(\{1, 2, 3\})$).*

Next we argue that the locally restricted games generalize both the conjunctively restricted games as well as weighted digraph games.[4]
The conjunctively restricted game of a game with a permission structure equals the locally restricted game of that game on the transitive closure of the permission structure. A weighted digraph game equals the locally restricted game of the additive game determined by the weights and the digraph as the permission structure.

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[4]We want to remark that the generalization of games with a permission structure to games with a local permission structure is different from the generalization to games on antimatroids (see Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004a, 2004b)) which considers cooperative games with restricted coalition formation where the set of feasible coalitions is an antimatroid. The set of locally feasible coalitions is not an antimatroid.

Proposition 4.3.17

(i) *For every $(v, D) \in \mathcal{G}_D^N$, it holds that $r_{v,tr(D)}^l = r_{v,D}^c$. In particular, if $D \in \mathcal{D}^N$ is transitive then $r_{v,D}^l = r_{v,D}^c$.*

(ii) *For every weighted digraph (δ, D) it holds that $r_{w^\delta,D}^l = \overline{v}_{\delta,D}$.*

PROOF

(i) For every $D \in \mathcal{D}^N$ it holds that $\widehat{P}_D(i) = P_{tr(D)}(i)$ for all $i \in N$. So, $\sigma_D^c(S) = \{i \in S : \widehat{P}_D(i) \subseteq S\} = \{i \in S : P_{tr(D)}(i) \subseteq S\} = \sigma_{tr(D)}^l(S)$, and therefore $r_{v,D}^c(S) = v(\sigma_D^c(S)) = v(\sigma_{tr(D)}^l(S)) = r_{v,tr(D)}^l(S)$ for all $S \subseteq N$.

(ii) This follows straightforwardly since $\overline{v}_{\delta,D}(S) = \sum_{\substack{i \in S \\ P_D(i) \subseteq S}} \delta_i = \sum_{\substack{i \in S \\ P_D(i) \subseteq S}} w^\delta(\{i\}) = w^\delta(\{i \in S : P_D(i) \subseteq S\}) = w^\delta(\sigma_D^l(S)) = r_{w^\delta,D}^l(S)$ for all $S \subseteq N$.

$\square$

We end this section with an example illustrating some of the relations between the classes of games described above.

Example 4.3.18 *Consider the digraph $D = \{(1,3), (2,3), (3,5), (4,5)\}$ on $N = \{1, 2, 3, 4, 5\}$. If $v = u_{\{5\}}$ then v is additive and, clearly $r_{v,D}^l = u_{\{3,4,5\}} = \overline{v}_{\delta,D}$ with $\delta = (0, 0, 0, 0, 1)$. If $v = u_{\{1,5\}}$ then $r_{v,D}^l = u_{\{1,3,4,5\}}$. This cannot be the weighted digraph game of any $\delta \in \mathbb{R}_+^N$ on D since $v(S) = 0$ for all $S \subseteq N$ such that $S = P_D(i) \cup \{i\}$ for some $i \in N$, implying that all weights must be zero. In that case however $\overline{v}_{\delta,D}$ must be the zero game assigning worth zero to every coalition.*
For game $r_{v,D}^l = u_{\{1,3,4,5\}}$ it also holds that it cannot be the conjunctive restriction of some game $v \in \mathcal{G}^N$ on D since in that case $r_{v,D}^l(S)$ must be equal to $r_{v,D}^l(\sigma_D^c(S))$ for all $S \subseteq N$, but $u_{\{1,3,4,5\}}(\{1, 3, 4, 5\}) = 1$ while $u_{\{1,3,4,5\}}(\sigma_D^c(\{1, 3, 4, 5\}) = u_{\{1,3,4,5\}}(\{1, 4\}) = 0$.

4.4 The local permission value

The *local (conjunctive) permission value* φ^l is the solution that assigns to every game with a permission structure the Shapley value of the locally restricted game.

Definition 4.4.1 *The local permission value is the solution on $\mathcal{G}_D^N$ given by*

$$\varphi^l(v, D) = Sh(r_{v,D}^l).$$

We provide an axiomatization of the local permission value using axioms similar to those used by van den Brink and Gilles (1996) to axiomatize the conjunctive permission value. Recall from Theorem 2.2.18 that they axiomatized the latter value by efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.

The local permission value satisfies all properties used in the axiomatization of van den Brink and Gilles (1996) except for structural monotonicity. This is illustrated in the following example.

Example 4.4.2 *Consider the game with permission structure (v, D) on $N = \{1, 2, 3\}$ given by $D = \{(1, 2), (2, 3)\}$ and $v = u_{\{3\}}$. In that case $\varphi^l(v, D) = (0, \frac{1}{2}, \frac{1}{2})$, and player 2 earns more than player 1, even though player 2 is a successor of 1 and the game is monotonic.*

The local permission value satisfies a weaker version of structural monotonicity requiring the payoff of a player to at least be equal to the payoff of any of its successors in a monotonic game if at least one of its successors is a necessary player.

Local structural monotonicity For every $(v, D) \in \mathcal{G}_D^N$, if $i \in N$ and $j \in F_D(i)$ are such that there exists at least one $h \in F_D(i)$ who is a necessary player in v, then $f_i(v, D) \geq f_j(v, D)$.

As mentioned above, the local permission value does satisfy the inessential player property. It satisfies an even stronger version of the inessential player property, requiring the payoff of a null player to be zero as soon as all its successors, but not necessarily all its subordinates, are null players in the game. We say that player $i \in N$ is *weakly inessential* in game with permission structure (v, D) if i and all its successors are null players in v, i.e., if $v(S) = v(S \setminus \{j\})$ for all $S \subseteq N$ and $j \in F_D(i) \cup \{i\}$ (note that a player that is inessential is also weakly inessential).

Weakly inessential player property For every $(v, D) \in \mathcal{G}_D^N$, if $i \in N$ is a weakly inessential player in (v, D) then $f_i(v, D) = 0$.

It turns out that strengthening the inessential player property in this way and weakening structural monotonicity as done above characterizes the local permission value.

Theorem 4.4.3 *A solution on $\mathcal{G}_D^N$ is equal to the local permission value φ^l if and only if it satisfies efficiency, additivity, the necessary player property, the weakly inessential player property, and local structural monotonicity.*

PROOF
It is straightforward to verify that the local permission value satisfies the five axioms.

The proof of uniqueness follows steps similar to those of the uniqueness proof for the conjunctive permission value in van den Brink and Gilles (1996).

Let f be a solution satisfying the axioms. Let v_0 be the *null game* given by $v_0(S) = 0$ for all $S \subseteq N$. The weakly inessential player property then implies that $f_i(v_0, D) = 0$ for all $i \in N$.

Now, consider game (w_T, D) where $w_T = c_T u_T$ is a scaled unanimity game for some $c_T > 0$ and $\emptyset \neq T \subseteq N$. We distinguish the following three cases with respect to $i \in N$:

(i) If $i \in T$ then the necessary player property implies that there exists a $c^* \in \mathbb{R}$ such that $f_i(w_T, D) = c^*$ for all $i \in T$, and $f_i(w_T, D) \leq c^*$ for all $i \in N \setminus T$.

(ii) If $i \in N \setminus T$ and $T \cap F_D(i) \neq \emptyset$ then local structural monotonicity implies that $f_i(w_T, D) \geq f_j(w_T, D)$ for every $j \in T \cap F_D(i)$, and therefore with case 1 that $f_i(w_T, D) = c^*$.

(iii) If $i \in N \setminus T$ and $T \cap F_D(i) = \emptyset$ then the weakly inessential player property implies that $f_i(w_T, D) = 0$.

From (i) and (ii) follows that $f_i(w_T, D) = c^*$ for $i \in T \cup P_D(T)$. Efficiency and (iii) then imply that $\sum_{i \in N} f_i(w_T, D) = |T \cup P_D(T)|c^* = c_T$. Therefore $c^* = \frac{c_T}{|T \cup P_D(T)|}$ and $f(w_T, D)$ is uniquely determined.

Next, consider (w_T, D) with $w_T = c_T u_T$ for some $c_T < 0$ (and therefore we cannot apply the necessary player property and local structural monotonicity since w_T is not monotonic). Since $-w_T = -c_T u_T$ with $-c_T > 0$, and $v_0 = w_T + (-w_T)$, it follows from additivity of f that $f(w_T, D) = f(v_0, D) - f(-w_T, D) = -f(-w_T, D)$ is uniquely determined because $-w_T$ is monotonic.

Finally, since by (2.2.2) every characteristic function $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$ (with $\Delta_v(T)$ the *Harsanyi dividend* of coalition T, see Harsanyi (1959)), additivity uniquely determines $f(v, D) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} f(\Delta_v(T) u_T, D)$ for any $(v, D) \in \mathcal{G}_D^N$. $\qquad \square$

Logical independence of the five axioms stated in Theorem 4.4.3 is shown by the following alternative solutions for games with a permission structure.

1. The solution given by $f_i(N, v, D) = 0$ for all $i \in N$ satisfies additivity, the weakly inessential player property, the necessary player property and local structural monotonicity. It does not satisfy efficiency.

2. Let $Win(N, v, D)$ be the set of weakly inessential players in a game with permission structure (N, v, D). The solution $f(N, v, D)$ given by $f_i(N, v, D) = \frac{v(N)}{|N \setminus Win(N, v, D)|}$ for $i \notin Win(N, v, D)$ and $f_i(N, v, D) = 0$ for $i \in Win(N, v, D)$ satisfies efficiency, the weakly inessential player property, the necessary player property and local structural monotonicity. It does not satisfy additivity.

3. The solution $f(N, v, D)$ that assigns to any player $i \in N$ a payoff $f_i(N, v, D) = \frac{v(N)}{|N|}$, thereby equally dividing the worth of the grand coalition over all the players, satisfies efficiency, additivity, the necessary player property and local structural monotonicity. It does not satisfy the weakly inessential player property.

4. Consider the solution given by
   $f_i(N, v, D) = \sum_{\{S \subseteq N: i \in P_D(S)\}} \frac{\Delta_v(S)}{|P_D(S)|} + \sum_{\{S \subseteq N: i \in S, P_D(S) = \emptyset\}} \frac{\Delta_v(S)}{|S|}$ for all $i \in N$. This solution divides the dividend of a coalition S equally over the predecessors in S if these exist and otherwise divides it equally over the players in S. It satisfies efficiency, additivity, the weakly inessential player property and local structural monotonicity. It does not satisfy the necessary player property.

5. For any game (N, v, D) the Shapley value $Sh(N, v)$ satisfies efficiency, additivity, the weakly inessential player property and the necessary player property. It does not satisfy local structural monotonicity.

Instead of using local structural monotonicity, we can strengthen the necessary player property by saying that a player earns at least as much as any other player if this player is necessary or has at least one necessary successor in a monotonic game.

Strong necessary player property For every $(v, D) \in \mathcal{G}_D^N$, if at least one of the players in $F_D(i) \cup \{i\}$ is a necessary player in v then $f_i(v, D) \geq f_j(v, D)$ for all $j \in N$.

Theorem 4.4.4 *A solution on $\mathcal{G}_D^N$ is equal to the local permission value φ^l if and only if it satisfies efficiency, additivity, the strong necessary player property and the weakly inessential player property.*

The proof is similar to that of Theorem 4.4.3, except that in case (ii) the strong necessary player property is used instead of local structural monotonicity.

There exists another interesting difference between the conjunctive and the local permission value. Say that a player $i \in N$ is necessary for (or can veto) player $j \in N$ in game (N, v) if it holds that $v(S \cup \{j\}) - v(S) = 0$ for those coalitions $S \subset N$ such that $i \notin S$. Now define (N, v_j^i) that is derived from (N, v) by

$$v_j^i(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S) & \text{if } i \in S. \end{cases} \tag{4.4.1}$$

In game v_j^i player i has become necessary for player j. Compared to the collusion game $v_{ij}(S)$ used by Haller (see (1.0.1)) the only difference is that when $i \in S$ it holds that $v_{ij}(S) = v(S \cup \{j\})$, whereas $v_j^i(S) = v(S)$.

The property of veto monotonicity now states that a player earns at least as much in the game where it vetoes one of its successors, as in the game where it does not do so.

Veto monotonicity For every $(v, D) \in \mathcal{G}_D^N$ and $i, j \in N$ such that $j \in F_D(i)$, it holds that $f_i(v_j^i, D) \geq f_i(v, D)$.

Proposition 4.4.5 *The conjunctive permission value φ^c satisfies veto monotonicity.*

The proof is straightforward and therefore omitted.
As the following example shows the local permission value does not satisfy veto monotonicity.

Example 4.4.6 *Consider the game with permission structure (v, D) on $N = \{1, 2, 3\}$ given by $D = \{(1, 2), (2, 3)\}$ and $v = u_{\{3\}}$. Then $v_3^2 = u_{\{2,3\}}$, and $\varphi^l(v, D) = (0, \frac{1}{2}, \frac{1}{2})$ while $\varphi^l(v_3^2, D) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. So, in (v, D) player 2 earns more than in (v_3^2, D).*

The reason why the local permission value does not satisfy veto monotonicity is clear from the example. A player who is necessary but also has a necessary successor will share the payoff resulting from its own necessity with its predecessor. Therefore when at least one of the successors of a player is necessary, it is better for that player not to be necessary itself. In that case it does not have to share with its own predecessors, while

still receiving its share in the payoff from being needed to give permission to a necessary successor.

Applied to hierarchically structured firms this would imply that a manager is better off by having important (necessary) successors and not being necessary in the production process itself. In that way the manager will be more influential, since it does not have to ask for the permission from its own predecessor to perform a necessary task. It is better for the manager to delegate necessary tasks to its successors, creating a situation where the execution of this necessary task is 'invisible' for its own predecessors. Another example is given by criminals who entrust their property to others to make it appear as if they have no property in order to escape taxes or prosecution. Of course, this works only if the top criminal can be sure to have access to 'its' property when he or she wants.

4.5   Concluding remarks

At the start of section 4.3 we noted that the local approach to permission structures applied in this chapter can still be considered a conjunctive approach, only on a local scale. Instead of considering a conjunctive local approach, in a disjunctive local approach to games with permission structure, every coalition would earn the worth that can be generated by those players in the coalition having at least one predecessor present in the coalition. Applying this local disjunctive approach to games with a permission structure where the characteristic function is additive we would obtain a new class of digraph games, where the weight of a player is earned by any coalition containing this player and any of its direct predecessors. In that case applying solutions such as the Shapley value would yield new power measures for digraphs. These *disjunctive digraph games* have a similarity with apex power games where any coalition containing the apex player and at least one of its direct predecessors plus the coalition of all direct predecessors earns the weight of a player, see also van den Brink (2002) who defines power measures for digraphs using apex games. Although it might seem that a disjunctive local approach to games with a permission structure can be applied in a straightforward manner, it is less obvious how such an approach generalizes the (standard) disjunctive approach to games with a permission structure.

Appendix: Solution definitions

In this appendix we define the solutions that have been used in Proposition 4.2.4.

The *imputation set* of a game $v \in \mathcal{G}^N$ consists of those efficient payoff vectors $x \in \mathbb{R}^N$, where every player i obtains at least its singleton worth $v(\{i\})$.

$$I(v) = \{x \in \mathbb{R}^N : x_i \geq v(\{i\})) \text{ and } \sum_{i \in N} x_i = v(N) \text{ for all } i \in N\}. \qquad (4.5.2)$$

Let $x \in I(v)$ be an imputation vector of $v \in \mathcal{G}^N$. An *objection* of a player $i \in N$ against a different player $j \in N$ with respect to x is a pair (S, y), where $S \subset N$ is a coalition and $y \in \mathbb{R}^N$ a payoff vector, satisfying

   (i) $i \in S$, $j \notin S$

   (ii) $y_i > x_i$

   (iii) $y_k \geq x_k$, $k \in S$

   (iv) $\sum_{i \in S} y_i \leq v(S)$

A *counterobjection* of player j against (S, y) is a pair (T, z), where $T \subset N$ is a coalition and $z \in \mathbb{R}^N$ a payoff vector, satisfying

   (i) $j \in T$, $i \notin T$

   (ii) $z_j \geq x_j$

   (iii) $z_k \geq x_k$, $k \in T$

   (iv) $z_k \geq y_k$, $k \in S \cap T$

   (v) $\sum_{i \in S} z_i \leq v(T)$

Definition 4.5.1 *The bargaining set is the set-valued solution on $\mathcal{G}^N$ given by*

$$B(v) = \{x \in I(v) : \text{ for every objection against } x \text{ of a player } i \in N \text{ against a}$$

$$\text{different player } j \in N, \text{ there exists a counterobjection of player } j \text{ against player } i\}.$$

Given payoff vector $x \in \mathbb{R}^N$, the *excess* $e(S, x)$ of a coalition $S \subseteq N$ is given by

$$e(S, x) = v(S) - x(S). \tag{4.5.3}$$

Let $E(x) \in \mathbb{R}^{2^{|N|-2}}$ be the vector composed of the excesses of all coalition $S \subset N$, $S \neq \emptyset$, in a non-increasing order, so $E_1(x) \geq E_2(x) \geq \ldots \geq E_{2^{|N|-2}}(x)$. The nucleolus $Nuc(v)$ of $v \in \mathcal{G}^N$ is the unique payoff vector in $I(v)$, which lexicographically minimizes the vector of excesses E (let $\succeq_L$ denote the lexicographic order of vectors).

Definition 4.5.2 *The nucleolus is the solution on $\mathcal{G}^N$ given by*

$$Nuc(v) = \{x \in I(v) : E(x) \preceq_L E(y) \text{ for all } y \in I(v)\}.$$

Given payoff vector $x \in \mathbb{R}^N$, the *complaint* of a player $i \in N$ against a different player $j \in N$ is given by

$$s_{ij}(x) = max_{\substack{S \subseteq N \\ i \in S, j \notin S}} (v(S) - x(S)). \tag{4.5.4}$$

The prekernel of a game $v \in \mathcal{G}^N$ consists of those payoff vectors, where for every pair of players $i, j \in N$, $i \neq j$, the complaint i has against j is the same as the complaint j has against i.

Definition 4.5.3 *The prekernel is the set-valued solution on $\mathcal{G}^N$ given by*

$$PK(v) = \{x \in I(v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i, j \in N\}.$$

The kernel of a game $v \in \mathcal{G}^N$ consists of those payoff vectors in the imputation set, where for every pair of players $i, j \in N$, $i \neq j$, either the complaint i has against j is the same as the complaint j has against i or, if the complaint i has against j is bigger than the complaint j has against i, j obtains its singleton worth $v(\{j\})$.

Definition 4.5.4 *The kernel is the set-valued solution on $\mathcal{G}^N$ given by*

$$K(v) = \{x \in I(v) : [s_{ij}(x) = s_{ji}(x)] \text{ or } [s_{ij}(x) > s_{ji}(x) \text{ and } x_j = v(\{j\})], \ i, j \subset N\}.$$

For a game $v \in \mathcal{G}^N$ let $M(v) = \{x \in \mathbb{R}^N : \text{ there exists } \pi \in \Pi(N) \text{ such that } x = m^\pi(v)\}$ be the set of all marginal vectors (recall (2.2.8)). The Weber set of a game $v \in \mathcal{G}^N$ is given by the convex hull of all marginal vectors of v (let the convex hull of a set of vectors X be given by $\text{Con}(X)$).

Definition 4.5.5 *The Weber set is the set-valued solution on $\mathcal{G}^N$ given by*

$$W(v) = Con(M(v)).$$

A *sharing system* on N is defined to be a system $p = (p^S)_{S \subseteq N}$, where $p^S \in \mathbb{R}_+^S$ assigns a non-negative share $(p^S)_i$ to every player $i \in S$, with $\sum_{j \in S}(p^S)_j = 1$, $S \subseteq N$. The collection of sharing systems on N is given by

$$P^N = \{p = (p^S)_{S \subseteq N} : p^S \in \mathbb{R}_+^S \text{ with } \sum_{j \in S}(p^S)_j = 1 \text{ for each } S \subseteq N\}. \tag{4.5.5}$$

For a game $v \in \mathcal{G}^N$ and a sharing system $p \in P^N$ let the payoff vector $h^p(v) \in \mathbb{R}^N$ be given by

$$h_i^p(v) = \sum_{\substack{S \subseteq N \\ i \in S}}(p^S)_i \Delta_v(S) \text{ for all } i \in N, \tag{4.5.6}$$

i.e. the payoff $h_i^p(v)$ assigned to player $i \in N$ is given by the sum of player i's share $(p^S)_i$ in coalition S times the dividend of coalition S over all $S \subseteq N$ such that $i \in S$. The payoff vector $h^p(v)$ is therefore also called a *Harsanyi payoff vector*.

Definition 4.5.6 *The Selectope (or Harsanyi set) is the set-valued solution on $\mathcal{G}^N$ given by*

$$H(v) = \{x \in \mathbb{R}^N : x = h^p(v) \text{ for some } p \in P^N\}.$$

Given game $v \in \mathcal{G}^N$ let the *marginal contribution M* be given by

$$M_i(v) = v(N) - v(N \setminus \{i\}) \text{ for all } i \in N \tag{4.5.7}$$

Given game $v \in \mathcal{G}^N$ let the *minimal right m* be given by

$$m_i(v) = max_{E \ni i}[v(E) - \sum_{j \in E \setminus \{i\}} M_j(v)] \text{ for all } i \in N \tag{4.5.8}$$

A game $v \in \mathcal{G}^N$ is called *quasi-balanced* if the following two conditions are satisfied:

(i) $m(v) \le M(v)$

(ii) $\sum_{i \in N} m_i(v) \le v(N) \le \sum_{i \in N} M_i(v)$

The class of all quasi-balanced games on N is denoted by $\mathcal{Q}^N$. If $v \in \mathcal{Q}^N$, then the allocations $m(v)$ and $M(v)$ can be seen as lower and upper bounds for the distribution of the payoffs over the players in N respectively. The τ-value is the payoff vector that assigns to every quasi-balance game the unique efficient allocation on the line segment between $m(v)$ and $M(v)$.

Definition 4.5.7 *The τ-value is the solution on $\mathcal{Q}^N$ given by*

$$\tau(v) = m(v) + \alpha(v)(M(v) - m(v)),$$

where

$$\alpha(v) = \begin{cases} \frac{v(N) - \sum_{i \in N} m_i(v)}{\sum_{i \in N} M_i(v) - \sum_{i \in N} m_i(v)} & \text{if } m(v) \ne M(v) \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 5

Accessible union stable systems

In this chapter a new framework will be introduced that takes into account both communication as well as hierarchical aspects with respect to cooperation. The *accessible union stable systems* we introduce satisfy the properties of union stability and accessibility. Union stability reflects the communication aspects of cooperation, while accessibility reflects the hierarchical aspects. Through these properties they generalize both the feasible set systems resulting from communication graphs as well as those resulting from antimatroids (which themselves generalize the conjunctive as well as the disjunctive feasible sets given by a permission structure), see also Chapter 2. It is also shown that they generalize augmenting systems, which form another class of set systems with both communication as well as hierarchical aspects. We also consider cooperative games with restricted cooperation where the set of feasible coalitions is an accessible union stable system, and provide an axiomatization of an extension of the Shapley value for this class of games with restricted cooperation.

This chapter is based on Algaba, van den Brink and Dietz (2014).

5.1   Introduction

In a number of situations of cooperation between decision making agents it is possible that there exist some restrictions with respect to communication or that the agents are organized according to some hierarchical structure. Consider the example of a hierarchical organization, where different departments within the organization might have some form of autonomy within the organization. For these departments to communicate with each other members of the departments at a higher position in the hierarchy are needed. In the field of cooperative game theory these aspects have been addressed separately by a number of models.

Myerson (1977) models restrictions with respect to communication, by adding an undirected (communication) graph to a cooperative TU-game. A coalition is said to be feasible if it is connected in the graph. Recall from Chapter 2 that these communication feasible set systems are in fact set systems containing the empty set that can be characterized by the properties of *2-accessibility* (every feasible coalition with at least two players contains at least two players, such that if we remove either of these players from

the coalition, the remaining set of players is also a feasible coalition) and *union stability* (the union of two feasible coalitions with a non-empty intersection is itself also feasible).

Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996) consider situations where the players in a game are organized according to some hierachical structure. Their games with a permission structure model this hierarchical structure by a directed graph. Recall from Chapter 2 that the (hierarchical) permission structures are in fact special cases of antimatroids, well-known combinatorial structures that represent hierarchies. A set of feasible coalitions is an antimatroid if it contains the empty set and satisfies *accessibility* (every nonempty feasible coalition has at least one player that we can remove, such that the remaining set of players is also a feasible coalition) and is *union closed* (meaning that the union of two feasible coalitions is also feasible).

If we compare the communication feasible set systems with antimatroids, we find that antimatroids satisfy a stronger union property (since union closedness implies union stability) but a weaker accessibility property (since 2-accessibility implies accessibility).

In this chapter a framework is established that considers both communication- as well as hierarchical aspects with respect to cooperation. Our goal is to get a better understanding of organizations, where both these aspects play a role. We introduce the class of *accessible union stable systems*, which are set systems containing the empty set and satisfying accessibility and union stability. Note that these are exactly the weaker union and accessibility properties discussed before. Therefore, all sets of connected coalitions of some (undirected) communication graph as well as all antimatroids form a proper subclass of the accessible union stable systems. We also show that the augmenting systems of Bilbao (2003) form a proper subclass of these new structures. After discussing some results on these structures, we give special attention to the subclass of accessible union stable systems that are *cycle-free* and show that under accessibility and feasibility of the unitary coalitions, cycle-free union stable systems are exactly those that can be obtained as the set of connected coalitions in a cycle-free communication graph. Then, we consider cooperative TU-games with restricted cooperation where the set of feasible coalitions is an accessible union stable system. We consider the solution that assigns to every cooperative game on an accessible union stable system the Shapley value of a restricted game where the worths are generated only by feasible coalitions.

5.2 Accessible union stable systems

In this chapter our goal is to establish a framework that allows us to study situations of cooperation with both communication- as well as hierarchical aspects in more detail. For this reason we study set systems that generalize the already mentioned antimatroids as well as the communication feasible set systems. We formally introduce accessible union stable systems as those set systems that contain the empty set and satisfy union stability and accessibility. Although some of the properties used to characterize the systems discussed in this chapter have already been defined in Chapter 2, we recall those properties in defining accessible union stable systems for the convenience of the reader.

Definition 5.2.1 *A set system $\mathcal{F} \subseteq 2^N$ is an accessible union stable system if it satisfies*

the following properties:

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *implies that* $S \cup T \in \mathcal{F}$,

(accessibility) $S \in \mathcal{F}$ *with* $S \neq \emptyset$, *implies that there exists* $i \in S$ *such that* $S \setminus \{i\} \in \mathcal{F}$.

Union stability has a natural and intuitive interpretation in the context of partial cooperation. Common players will be vital for establishing communication in the union coalition. From union stability it is not clear in what way accessible union stable systems reflect hierarchy. For this we consider the axiom of accessibility. According to accessibility every feasible coalition contains at least one player that can be removed, such that the remaining coalition of players is also a feasible coalition. This implies the following augmentation property; every feasible coalition can be obtained by sequentially adding players to the empty set in at least one way, such that after each addition a feasible set is obtained. This way we can obtain an order on the set of players belonging to each feasible coalition, reflecting the hierarchical aspect of these systems.

Examples of accessible union stable systems on $N = \{1, \ldots, n\}$ are $\mathcal{F} = 2^N$, $\mathcal{F} = \{\emptyset, \{i\}\}$ where $i \in N$, and $\mathcal{F} = \{\emptyset, \{1\}, \ldots, \{n\}\}$. By definition the following holds.

Recall from Chapter 2 that a set system $\mathcal{F} \subseteq 2^N$ is called normal if $N = \bigcup_{S \in \mathcal{F}} S$, i.e. if every player belongs to at least one feasible coalition. This property is satisfied by all communication feasible set systems. It does not have to hold for antimatroids. For antimatroids $\mathcal{A} \subseteq 2^N$, that do satisfy normality, union closedness implies that $N \in \mathcal{A}$.

Proposition 5.2.2

(i) *An accessible union stable system is an antimatroid if and only if that system is union closed.*

(ii) *A normal accessible union stable system is a set system generated by the connected coalitions of some undirected graph if and only if that system satisfies 2-accessibility.*

An interesting relation between accessible union stable systems and antimatroids is that every set system, such that for every player the subcollection of feasible sets containing this player is an antimatroid, is an accessible union stable system.

Proposition 5.2.3 *Let* $\mathcal{F} \subseteq 2^N$ *be a set system containing the empty set. If* $\mathcal{F}_i = \{\emptyset\} \cup \{T \in \mathcal{F} : i \in T\}$ *is an antimatroid for all* $i \in N$, *then* $\mathcal{F}$ *is an accessible union stable system.*

PROOF
Consider $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$. If $j \in S \cap T$ then $S, T \in \mathcal{F}_j$. Since $\mathcal{F}_j$ is union closed it holds that $S \cup T \in \mathcal{F}_j$ and hence $S \cup T \in \mathcal{F}$, showing union stability of $\mathcal{F}$.
To show accessibility of $\mathcal{F}$, consider $S \in \mathcal{F}$, $S \neq \emptyset$. Then $S \in \mathcal{F}_j$ for some $j \in S$. By accessibility of $\mathcal{F}_j$ there exists $h \in S$ such that $S \setminus \{h\} \in \mathcal{F}_j$. But then also $S \setminus \{h\} \in \mathcal{F}$. This shows accessibility of $\mathcal{F}$.

$\square$

The reverse is not true as shown in the following example.

Example 5.2.4 *Consider set system $\mathcal{F}$ with $N = \{1, 2, 3\}$ given by $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2, 3\}, N\}$. This is an accessible union stable system but $\mathcal{F}_1 = \{\{1\}, N\}$ is not an antimatroid since it fails accessibility (no single player can be deleted from N).*

Now for any feasible coalition S define augmentation points as follows.

Definition 5.2.5 *A player $i \in N \setminus S$, such that $S \cup \{i\} \in \mathcal{F}$ is called an augmentation point of coalition S in $\mathcal{F}$.*

Denote the set of augmentation points of coalition $S \in \mathcal{F}$ by $au(S) = \{i \in N \setminus S : S \cup \{i\} \in \mathcal{F}\}$. Next, denote by $S^+ = S \cup au(S) = \{i \in N : S \cup \{i\} \in \mathcal{F}\}$ the set S together with all its augmentation points.[1]

Proposition 5.2.6 *Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system. Then the interval*

$$\left[T, T^+\right]_{\mathcal{F}} = \left\{S \in \mathcal{F} : T \subseteq S \subseteq T^+\right\}$$

is a Boolean algebra for every nonempty $T \in \mathcal{F}$.

PROOF
We need to prove that

$$\left\{S \in \mathcal{F} : T \subseteq S \subseteq T^+\right\} = \left\{S \subseteq N : T \subseteq S \subseteq T^+\right\},$$

i.e., for each $S \subseteq N$ such that $T \subseteq S \subseteq T^+$ it holds that $S \in \mathcal{F}$. We distinguish two cases.

(i) If the set $au(T) = \emptyset$ then $[T, T^+]_{\mathcal{F}} = \{T\}$.

(ii) Suppose that $au(T) \neq \emptyset$ and let $S \in [T, T^+]_{\mathcal{F}}, S \neq T$. Let $k(S) = |S| - |T|$. To prove that $S \in \mathcal{F}$ we perform induction on $k(S)$. If $k(S) = 1$, then for the unique player $i \in S \setminus T$ it holds that $i \in au(T)$. Since $S = T \cup \{i\}$, it therefore holds that $S \in \mathcal{F}$. Proceeding by induction, assume that $R \in [T, T^+]_{\mathcal{F}}$ is in $\mathcal{F}$, whenever $k(R) < k(S)$ and without loss of generality let $(s_1, \ldots, s_{k(S)})$ be any ordering of the players in $S \setminus T$. By the induction hypothesis it holds for $S' = T \cup \{s_1, \ldots, s_{k(S)-1}\}$ and $S'' = T \cup \{s_{k(S)}\}$ that $S', S'' \in \mathcal{F}$. By union stability of $\mathcal{F}$ and $S' \cap S'' = T \neq \emptyset$, we obtain $S \in \mathcal{F}$.

$\square$

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[1]From its proof it follows that the following proposition holds for every union stable system.

5.2.1 Accessible union stable systems vs. augmenting systems

Since accessible union stable systems generalize communication feasible set systems as well as antimatroids, we can use these systems to model organizations that have hierarchical as well as communication features. Next, we will show how accessible union stable systems are related to the class of augmenting systems introduced by Bilbao (2003), see also Chapter 2. These systems can also be said to include both communication as well as hierarchical aspects. Remember that besides containing the feasible set, these systems can be characterized by union stability plus an augmentation property. To prevent confusion with a different augmentation property introduced later in this chapter the property will be referred to as augmentation 1.

Definition 5.2.7 *Recall from Chapter 2 that a set system $\mathcal{F} \subseteq 2^N$ is an augmenting system if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, implies that $S \cup T \in \mathcal{F}$,

(augmentation 1) for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

Union stability expresses the communication aspect, where Augmentation 1 can be said to express the hierarchical aspect. It states that any feasible set can be sequentially 'built' from its subsets.

Recall that the set of extreme players of a coalition is defined as follows.

Definition 5.2.8 *A player $i \in S \in \mathcal{F} \subseteq 2^N$, such that $S \setminus \{i\} \in \mathcal{F}$, is an extreme player of coalition S in $\mathcal{F}$.*

The set of extreme points of coalition $S \in \mathcal{F}$ is denoted by
$ex(S) = \{i \in S : S \setminus \{i\} \in \mathcal{F}\}$.

A player i that is an augmentation point to feasible coalition S is by definition an extreme player to feasible coalition $S \cup \{i\}$. We now have the following proposition.

Proposition 5.2.9 *If $\mathcal{F} \subseteq 2^N$ is an augmenting system then $\mathcal{F}$ is an accessible union stable system.*

PROOF
Take any feasible coalition $T \neq \emptyset$. By augmentation 1 there exists a sequence of coalitions $T_0, T_1, \ldots, T_t$, with $T_h \in \mathcal{F}$, $|T_h| = h$, $0 \leq h \leq t$, such that $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_{t-1} \subset T_t = T$. The augmentation point of T_{t-1} is given by $i \in T$ such that $T \setminus \{i\} = T_{t-1} \in \mathcal{F}$. By definition this is an extreme player of $T_t = T$. This shows that augmenting systems also satisfy accessibility and therefore they are also accessible union stable systems. □

Accessibility and union stability on the other hand do not imply augmentation 1, and therefore the reverse is not true, i.e. there do exist accessible union stable systems that are not augmenting systems. We consider an example, where the fact that augmentation 1 does not have to be satisfied, is seen to provide more flexibility with respect to which players can initiate coalition formation.

Example 5.2.10 *Consider the set systems $\mathcal{F}_1$, $\mathcal{F}_2$ on $N = \{1, 2, 3, 4\}$ given by*

$$\mathcal{F}_1 = \{\emptyset, \{2\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}.$$
$$\mathcal{F}_2 = \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}.$$

Both of them are accessible union stable systems. Note that these sets only differ in the feasibility of coalitions $\{1\}$ and $\{2\}$. Where $\mathcal{F}_2$ is also an augmenting system, $\mathcal{F}_1$ is not. For $\mathcal{F}_1$ according to augmentation 1 there should be a player i in $\{2, 3, 4\} \setminus \{2\} = \{3, 4\}$ such that $\{2\} \cup \{i\}$ is feasible. However both $\{2, 3\}$ and $\{2, 4\}$ are not feasible coalitions and therefore $\mathcal{F}_1$ does not satisfy augmentation 1. In contrast to the augmenting systems player 2 is also able to initiate coalition formation.

5.2.2 Examples of accessible union stable systems

In this section we discuss applications of accessible union stable systems that are neither an augmenting system, nor an antimatroid nor a communication feasible set system.

Explorative and careful societies

Let N and M be two sets of players, where $N \cap M = \emptyset$. We interpret these as two different societies. Every subset of a society represents a feasible coalition. Furthermore, every subset of society N can form a coalition 'outside' of N, but there do not exist feasible coalitions consisting of proper subsets of M with players of N. Therefore, N can be considered to be a society of 'explorers', whereas M can be considered to be a 'careful' society. We can therefore describe the collection of feasible sets as follows:

$$\Phi_{N,M} = \{S \subseteq N \cup M : S \subseteq N \ \text{ or } \ S \subseteq M \ \text{ or } \ M \subset S\}. \tag{5.2.1}$$

The set of feasible coalitions $\Phi_{N,M}$ is an accessible union stable system. It is union stable since (i) any union of two non-disjoint feasible coalitions, that are either both a subset of N or both a subset of M, is feasible, and (ii) any union of two non-disjoint feasible coalitions where at least one of these coalitions contains players from both N and M, is feasible (since the coalition containing players from both N and M, must contain all players from M). The set system is accessible since (i) for any non-empty feasible coalition that is a subset of N or M it holds that every player in the coalition is an extreme player, and (ii) for every feasible coalition that contains players from both N and M each player from N is an extreme player (since the coalition of remaining players still contains all players from M, it is still feasible).

However, (i) it is not a communication feasible set system (since it does not satisfy 2-accessibility), (ii) it is not an antimatroid (since it is not union closed), and (iii) it is not an augmenting system (since it does not satisfy augmentation 1). We illustrate this by an example.

Example 5.2.11 *Let $N = \{1, 2\}$ represent the explorative society, with $M = \{3, 4, 5\}$ representing the careful society. The resulting set of feasible coalitions is given by*

$$\Phi_{N,M} \;=\; \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\},$$
$$\{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}$$

(The last three coalitions of $\Phi_{N,M}$ are the only coalitions that contain players from both societies)

To show that the set system $\Phi_{N,M}$ is not a communication feasible set system, consider feasible coalition $\{2,3,4,5\}$. This coalition contains one player from N, player 2, and all players from M. However since none of the players of M can be deleted, player 2 is the only extreme player, so the set system does not satisfy 2-accessibility.

To show that the set system $\Phi_{N,M}$ is not an antimatroid, consider coalitions $\{1\}$ and $\{3\}$. These are proper subsets of N, and M respectively, so their union contains a player from both N and M. It does however not contain all players from M, and therefore it is not feasible. This shows that $\mathcal{F}$ does not satisfy union closedness.

Finally, to show that the set system $\Phi_{N,M}$ is not an augmenting system, consider coalitions $N = \{1,2\}$ and $N \cup M = \{1,2,3,4,5\}$. Coalition N has no augmentation points, though it is a subset of $N \cup M$. Therefore augmentation 1 is not satisfied.

Network hierarchies

Define a network hierarchy (N, M, D, L) to be a tuple where N and M are two disjunct sets of players, $(N, D) \in \mathcal{D}_a$ is an acyclic digraph and $(N \cup M, L) \in \mathcal{L}$ is a communication graph. Furthermore we require that if $i, j \in N$, then $\{i, j\} \in L$, so any two players in N are by definition also connected in L. This means that subgraph $(N, L(N))$ is the complete graph on N. The idea is that (N, D) represents a hierarchical organization D on the set of agents N. These players are all able to communicate with each other. They also form part of a social network $(N \cup M, L)$ involving players outside of the organization (the players in M). With these systems we can model situations of interaction between a hierarchical organization and an outside environment.

A coalition S is said to be feasible within the network hierarchy, if two conditions are satisfied. First for every player $i \in S \cap N$ all superiors $\widehat{P}_D(i)$ in D are also present in S. Second it must hold that $(S, L(S))$ forms a connected subgraph of $(N \cup M, L)$. We can therefore describe the collection of feasible sets generated by a network hierarchy (N, M, D, L) as follows

$$\Phi_{N,M,D,L} = \{S \subseteq N \cup M : \widehat{P}_D(S \cap N) \subset S \text{ and } S \text{ is connected in } (N \cup M, L)\} \quad (5.2.2)$$

.

We will refer to the class of feasible set systems that can be generated from a network hierarchy as *network hierarchy systems*.

Example 5.2.12 *Consider network hierarchy (N, M, D, L), where $N = \{1,2,3\}$, $M = \{4,5,6\}$, $D = \{(1,2),(2,3)\}$ and $L = \{\{1,2\}, \{\{1,3\}, \{2,3\}, \{2,4\}, \{3,5\}, \{5,6\}\}$. The network hierarchy system $\Phi_{N,M,D,L}$ is given by*

$$\Phi_{N,M,D,L} \;=\; \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{4\}, \{5\}, \{6\}, \{5,6\},$$
$$\{1,2,4\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,4,5\}, \{1,2,3,5,6\}, \{1,2,3,4,5,6\}\}$$

In this example only subordinate agents 2 and 3 can communicate with agents outside of the hierarchical organization (N, D).

For feasible set system $\mathcal{F} \subseteq 2^N$ let $\mathcal{F}^S = \{T \in \mathcal{F} : T \subseteq S\}$. This is the set system obtained from $\mathcal{F}$ by considering only the feasible subsets of S in $\mathcal{F}$. We have the following proposition.

Proposition 5.2.13

(i) *For $\mathcal{F} = \Phi_{N,M,D,L}$ a network hierarchy system, $\mathcal{F}^M$ is a communication feasible set system.*

(ii) *For $\mathcal{F} = \Phi_{N,M,D,L}$ a network hierarchy system, $\mathcal{F}^N$ is the set system generated by the conjunctive feasible sets of some directed graph.*

The proof follows straightforwardly from the definition of $\Phi_{N,M,D,L}$.

It can be seen that the class of all network hierarchy systems is not part of either the class of augmenting systems, the class of antimatroids, or the communication feasible set systems. In fact these accessible union stable systems contain both the class of all communication feasible set systems as well as all conjunctive feasible sets generated by directed trees.

Proposition 5.2.14

(i) *Every communication feasible set systems belongs to the class of network hierarchy systems.*

(ii) *Every set system generated by the conjunctive feasible sets of some acyclic directed graph belongs to the class of network hierarchy systems.*

(iii) *The class of network hierarchy systems form a proper subset of the class of accessible union stable systems.*

(iv) *There exist feasible set systems in the class of network hierarchy systems that are not the communication feasible set systems of any graph.*

(v) *There exist feasible set systems in the class of network hierarchy systems that are not antimatroids.*

(vi) *There exist feasible set systems in the class of network hierarchy systems that are not are not augmenting systems.*

Proof

(i) For the communication feasible set system obtained from communication graph (M, L), consider network hierarchy (N, M, D, L), where $D = \emptyset$ and $N = \emptyset$. We obtain $\Phi_{N,M,D,L} = \{S \subseteq N \cup M : \widehat{P}_D(S \cap N) \subset S \text{ and } S \text{ is connected in } (N \cup M, L)\} = \{S \subseteq M : S \text{ is connected in } (M, L)\}$.

(ii) For the communication feasible set system obtained from directed graph (N, D), consider network hierarchy (N, M, D, L), where $M = \emptyset$ and L is the complete graph on N. We obtain $\Phi_{N,M,D,L} = \{S \subseteq N \cup M : \widehat{P}_D(S \cap N) \subset S$ and S is connected in $(N \cup M, L)\} = \{S \subseteq N : \widehat{P}_D(S) \subset S\} = \Phi^c(N, D)$ (recall from Chapter 2 that $\Phi^c(N, D)$ are the conjunctively feasible coalitions of D).

(iii) We begin by showing that any network hierarchy system is also an accessible union stable system.

First we show that network hierarchy systems satisfy union stability. For network hierarchy (N, M, D, L), consider any two sets $S, U \in \Phi_{N,M,D,L}$ that have a non empty intersection. For any player $i \in S \cup U$ such that $i \in N$ it must hold that, since $\widehat{P}_D(i) \subset S$ or $\widehat{P}_D(i) \subset U$, also $\widehat{P}_D(i) \subset S \cup U$. Since S and U are connected in L through their intersecting players, it holds that $S \cup U$ is also connected in L. Therefore the union of two feasible sets with a non-empty intersection is also feasible.

Second we show that network hierarchy systems satisfy accessibility. By communication feasible set systems satisfying 2-accessibility and Proposition 5.2.13.(i) it follows that any feasible set $U \in \Phi_{N,M,D,L}$ containing players only from M has an extreme player. The conjunctive feasible sets of some acyclic directed graph form a poset antimatroid (see Chapter 2). By antimatroids satisfying accessibility and Proposition 5.2.13.(ii) it follows that any feasible set $U \in \Phi_{N,M,D,L}$ containing players only from N has an extreme player. Now consider feasible sets U containing players from both N and M. Since D is an acyclic digraph and $U \in \Phi_{N,M,D,L}$, there exists at least one player $i \in U \cap N$ such that $\widehat{F}_D(i) \cap U = \emptyset$. If $U \setminus \{i\}$ is also connected in L, $U \setminus \{i\}$ is feasible and U has an extreme player. Now suppose there is no $i \in U \cap N$ such that $\widehat{F}_D(i) \cap U = \emptyset$ and $U \setminus \{i\}$ is connected in L. In that case there is no player $i \in U \cap N$ such that $U \setminus \{i\}$ is feasible. Therefore if U has an extreme player i it must hold that $i \in U \cap M$ and we only need to consider whether $U \setminus \{i\}$ is connected in L (since i does not give permission to players in $U \cap N$). Now select any player $i \in U \cap M$. If $U \setminus \{i\}$ is connected in L, $U \setminus \{i\}$ is feasible and U has an extreme player. Therefore suppose $U \setminus \{i\}$ is not connected in L. In that case $U \setminus \{i\}$ can be partitioned in a number of components $(K_1, \ldots, K_v)$ (recall from Chapter 2 that a coalition $K \subseteq N$ is a component in a graph (N, L) if and only if K is connected in L, and $K \cup \{k\}$ is not connected in L for every $k \in N \backslash K$). Let $V = \{1, \ldots, v\}$. For any $w \in V$ it holds that $\bigcup_{j \in V \backslash \{w\}} K_j \cup \{i\}$ is connected in L. All players from $U \cap N$ must be contained within one component, since by definition of network hierarchies there exists an edge between all players in N. Without loss of generality let K_1 be this component. Now consider component K_v. This component can only contain players from M. Suppose that this component consists of just one player k. We find that $U \backslash \{k\} = \bigcup_{j \in V \backslash \{v\}} K_j \cup \{i\}$ is a connected coalition in L, $U \backslash \{k\}$ is feasible and U has an extreme player. Therefore assume $|K_v| \geq 2$. From van den Brink (2012) we obtain that since K_v is a connected coalition in L, there exist at

least two players $l, m \in K_v$ such that $K_v \setminus \{m\}$ and $K_v \setminus \{l\}$ are also connected in L. There is at least one player $n \in K_v$ such that $\{i, n\} \in L$. Now select $k \in \{l, m\}$, such that $k \neq n$. We find that $U \setminus \{k\} = \bigcup\limits_{j \in V \setminus \{v\}} K_j \cup \{i\} \bigcup K_v \setminus \{k\}$ is a connected coalition in L, $U \setminus \{k\}$ is feasible and U has an extreme player. This shows that any network hierarchy satisfies both union stability and accessibility and is therefore an accessible union stable system.

Now consider the set system $\mathcal{F}$ given by $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$. This system satisfies accessibility and union stability. Suppose that this system is also a network hierarchy system generated by some network hierarchy (N, M, D, L). Since all singletons are feasible it must hold that $D = \emptyset$. Also since $\{1, 2, 3\}$ is feasible it must be a connected coalition in graph $(N \cup M, L)$. This implies that either $\{1, 2\} \in L$, $\{2, 3\} \in L$ or $\{1, 2\}, \{2, 3\} \in L$ and therefore at least one of the sets $\{1, 2\}, \{2, 3\} \in \mathcal{F}$. Since this is not the case, it must hold that $\mathcal{F}$ cannot be generated by a network hierarchy. We obtain that network hierarchy systems are a proper subset of accessible union stable systems.

(iv) Take example 5.2.12. The feasible coalition $\{1, 2, 3\}$ has only one extreme player and therefore 2-accessibility is not satisfied.

(v) Take example 5.2.12. The coalition $\{1, 4\}$ is not feasible, violating union closedness.

(vi) Take example 5.2.12. The feasible coalition $\{5, 6\}$ has no augmentation points.

$\square$

5.3 The dual of accessible union stable systems

The *dual structure* of a set system $\mathcal{F} \subseteq 2^N$ is the set system $\mathcal{F}^d$ given by

$$\mathcal{F}^d = \{S \subseteq N : N \setminus S \in \mathcal{F}\}. \tag{5.3.3}$$

It is known that convex geometries are the dual structures of normal antimatroids. Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison (1985).

Definition 5.3.1 *A set system $\mathcal{G} \subseteq 2^N$ is a convex geometry if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{G}$,

(intersection closed) $S, T \in \mathcal{G}$ *implies that* $S \cap T \in \mathcal{G}$,

(augmentation 2) $S \in \mathcal{G}$ *with* $S \neq N$, *implies that there exists* $i \in N \setminus S$ *such that* $S \cup \{i\} \in \mathcal{G}$.[2]

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[2]When $N \in \mathcal{G}$, this property is equal to augmentation 1 for N.

So, a convex geometry is a set system that contains the empty set, is intersection closed and satisfies an alternative augmentation property saying that for every feasible coalition that is a proper subset of the 'grand coalition' N there is a player that can be added resulting in a feasible coalition of higher cardinality.[3] Note that this implies that the 'grand coalition' is feasible, and thus convex geometries are normal set systems. Also, note that an augmenting system that is intersection closed and contains N is a convex geometry.

Accessibility and union stability do not imply augmentation 2, as is illustrated by the following example.

Example 5.3.2 *Let $N = \{1, 2, 3\}$ and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, N\}$. This is an accessible union stable system with $N \in \mathcal{F}$. But $au(\{3\}) = \emptyset$, and therefore it does not satisfy augmentation 2 and it is not a convex geometry.*

The dual structures of accessible union stable systems that satisfy normality form a class of set systems that contain all convex geometries. We will refer to these dual structures as intersecting stable systems with the augmentation property.

Definition 5.3.3 *A set system $\mathcal{F} \subseteq 2^N$ is an intersecting stable system with the augmentation property if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(weakly intersection closed) $S, T \in \mathcal{F}$ *with* $S \cup T \neq N$, *implies that* $S \cap T \in \mathcal{F}$

(augmentation 2) $S \in \mathcal{F}$ *with* $S \neq N$, *implies that there exists* $i \in N \setminus S$ *such that* $S \cup \{i\} \in \mathcal{F}$.

Proposition 5.3.4 *Let $\mathcal{F} \subseteq 2^N$ be a set system satisfying normality. Then $\mathcal{F}$ is an accessible union stable system if and only if $\mathcal{F}^d$ is an intersecting stable system with the augmentation property.*

PROOF
By normality N is contained in $\mathcal{F}$, and therefore $\emptyset \in \mathcal{F}^d$. We also have the following results

(i) $S, T \in \mathcal{F} \Leftrightarrow (N \backslash S), (N \backslash T) \in \mathcal{F}^d$

(ii) $S \cap T \neq \emptyset \Leftrightarrow (N \backslash S) \cup (N \backslash T) = N \backslash (S \cap T) \neq N$

(iii) $S \cup T \in \mathcal{F} \Leftrightarrow N \backslash (S \cup T) = (N \backslash S) \cap (N \backslash T) \in \mathcal{F}^d$

Therefore, union stability of $\mathcal{F}$ implies that $\mathcal{F}^d$ is weakly intersection closed. Finally, since for any $S \subseteq N$ with $i \in S$, $N \backslash S \neq N$, $i \in N \backslash (N \backslash S) = S$ and,

---

[3]Edelman and Jamison (1985) showed that L is a connected block graph if and only if the collection of subsets of N which induces connected subgraphs is a convex geometry.

(iv) $S, S \setminus \{i\} \in \mathcal{F} \Leftrightarrow N \setminus S \in \mathcal{F}^d$, $N \setminus (S \setminus \{i\}) = (N \setminus S) \cup \{i\} \in \mathcal{F}^d$,

accessibility of $\mathcal{F}$ implies that $\mathcal{F}^d$ satisfies augmentation 2. $\qquad\square$

Note that every convex geometry is an intersecting stable system with the augmentation property since it satisfies a stronger version of intersection closedness. However, not every intersecting stable system with the augmentation property is a convex geometry as the following example illustrates.

Example 5.3.5 *Consider the following set system on N*

$$\mathcal{F}^d = \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

This set system is weakly intersection closed and satisfies augmentation 2. However, since the intersection of the two feasible coalitions $\{1,3\}$ and $\{2,3\}$ (the singleton $\{3\}$) is not feasible, this set system is not intersection closed and is therefore not a convex geometry.

Given an intersecting stable system with the augmentation property $\mathcal{F} \subseteq 2^N$, by augmentation 2 the set $au(S) \neq \emptyset$, for all $S \in \mathcal{F}$, $S \neq N$. For some coalitions $S \in \mathcal{F}$, $S \neq N$ the set $ex(S)$ may be empty however. This shows that not all intersecting stable system with the augmentation property satisfy accessibility. We illustrate this by the following example.

Example 5.3.6 *Consider the set system $\mathcal{F}^d = \{\emptyset, \{1\}, \{1,3\}, \{2,3\}, N\}$ on $N = \{1,2,3\}$. This is an intersecting stable system with the augmentation point property and $ex(\{2,3\}) = \emptyset$.*

Notice that if an intersecting stable system with the augmentation property is an antimatroid then it is closed under union. By the following example we show that the reverse is not true.

Example 5.3.7 *Consider the set system $\mathcal{F} = \{\emptyset, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, N\}$ on $N = \{1,2,3\}$. This set system is closed under union, and it is an intersecting stable system with the augmentation property but, since $ex(\{1,2\}) = \emptyset$, it does not satisfy accessibility and therefore it is not an antimatroid.*

5.4 The supports of an accessible union stable system

Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system and $\mathcal{G} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is union stable, we can inductively define the feasible sets $\mathcal{G}^{(m)}$, $m = 0, 1, \ldots$, in the same way as Algaba, Bilbao, Borm and López (2000) did for arbitrary union stable systems, i.e.,

$$\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(m)} = \left\{ S \cup T : S, T \in \mathcal{G}^{(m-1)}, \ S \cap T \neq \emptyset \right\} \cup \{\emptyset\}, \quad m = 1, 2, \ldots \quad (5.4.4)$$

Notice that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(m-1)} \subseteq \mathcal{G}^{(m)} \subseteq \mathcal{F}$, since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F}$ is union stable. The inductive process is finite because $\mathcal{F}$ is finite.

Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system and let $\mathcal{G} \subseteq \mathcal{F}$. The closure $\overline{\mathcal{G}}$ is given by $\overline{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is defined as the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$.

Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system. We consider the set

$$R(\mathcal{F}) = \{R \in \mathcal{F} : R = S \cup T, \ S \neq R, \ T \neq R, \ S, T \in \mathcal{F}, \ S \cap T \neq \emptyset\} \tag{5.4.5}$$

consisting of those feasible coalitions which can be written as the union of two distinct feasible coalitions with a nonempty intersection. The basis of an accessible union stable system consists of those coalitions that cannot be written as the union of two feasible coalitions with a nonempty intersection.[4]

Definition 5.4.1 *Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system. The elements of the set $B(\mathcal{F}) = \mathcal{F} \setminus R(\mathcal{F})$ are called the supports of $\mathcal{F}$.*

By construction, the set $B(\mathcal{F})$ is unique, nonempty, and satisfies the following properties:

(i) $\emptyset \in B(\mathcal{F})$.

(ii) If $S \in \mathcal{F}$ and $|S| \leq 2$, then $S \in B(\mathcal{F})$.

Next, the following characterization of the set of supports can be obtained in the same way as for union stable systems (see Algaba, Bilbao, Borm and López (2001)) and follows directly from the definition of $B(\mathcal{F})$.

Proposition 5.4.2 *Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system and $B(\mathcal{F})$ the set of its supports. Then $B(\mathcal{F})$ is the unique minimal subset of $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$.*

The following example illustrates how to obtain an accessible union stable system from its supports in this way.

Example 5.4.3 *Consider the accessible union stable system $\mathcal{F}$ on $N = \{1, 2, 3, 4, 5\}$ given by*

$$\mathcal{F} = \left\{ \begin{array}{c} \emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{3, 4\}, \{4, 5\}, \{2, 3, 4\}, \\ \{3, 4, 5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, N \end{array} \right\}.$$

The supports of $\mathcal{F}$ are given by

$$B(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{3, 4\}, \{4, 5\}, \{2, 3, 4\}\}.$$

Note that

$$B^{(0)} = B(\mathcal{F}),$$

$$B^{(1)} = B^{(0)} \cup \{\{3, 4, 5\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}\},$$

$$B^{(2)} = B^{(1)} \cup \{\{1, 2, 3, 4, 5\}\},$$

$$B^{(3)} = B^{(2)} \ and \ B^{(2)} = \overline{B(\mathcal{F})} = \mathcal{F}$$

---

[4]In fact, such a basis can be defined for every union stable system.

We introduced accessible union stable systems as a model that generalizes communication feasible set systems as well as antimatroids in such a way that the two defining properties reflect communication (union stability) and hierarchy (accessibility). Next, we want to see how these features influence the basis of the system. It is known that the supports of a communication feasible set system are exactly those elements that have cardinality one or two, the first type being the singletons and the second type being the links or edges of the communication graph.

By accessibility, for any accessible union stable system it holds that $ex(S) \neq \emptyset$, for any nonempty $S \in \mathcal{F}$, where $ex(S)$ is the set of extreme players of S in $\mathcal{F}$. For accessible union stable systems it turns out that every support either has cardinality of at most two or is a path. (In the accessible union stable system of Example 5.4.3, the supports with cardinality of at most two are those in $B(\mathcal{F}) \setminus \{\{2, 3, 4\}\}$, while support $\{2, 3, 4\}$ is a path.)

Proposition 5.4.4 *Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system. If $B \in B(\mathcal{F})$ with $|B| > 2$ then $|ex(B)| = 1$, i.e., B is a path.*

PROOF
Suppose that $B \in B(\mathcal{F})$ is a support of $\mathcal{F}$ such that $|B| > 2$ and $|ex(B)| > 1$, i.e., B has at least two extreme points. Then there exists $i, j \in B$, with $i \neq j$, such that $B \setminus \{i\} \in \mathcal{F}$ and $B \setminus \{j\} \in \mathcal{F}$. Since $(B \setminus \{i\}) \cap (B \setminus \{j\}) = B \setminus \{i, j\} \neq \emptyset$ and $(B \setminus \{i\}) \cup (B \setminus \{j\}) = B$, B is the union of two feasible sets with a non-empty intersection. This contradicts the fact that B is a support of $\mathcal{F}$. $\qquad\square$

The reverse is not true, i.e., not every path with more than two players is a support.

Example 5.4.5 *Consider the set $N = \{1, 2, 3, 4\}$ and the accessible union stable system given by $\mathcal{F} = \{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, N\}$. Its basis is $B(\mathcal{F}) = \{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}\}$. Since the only extreme player of the grand coalition N is player 1, the grand coalition is a path but it is not a support.*

5.5 Cycle-free accessible union stable systems

Algaba, Bilbao, Borm and López (2001) considered the subclass USI^N of union stable systems on N that are intersection closed if the intersection contains at least two elements, and where every non-unitary feasible coalition can be written in a unique way as a union of non-unitary supports. A coalition S is non-unitary if $|S| \neq 1$ (i.e. S is not a singleton).

Definition 5.5.1 *The class of set systems USI^N consists of those union stable systems $\mathcal{F} \subseteq 2^N$ satisfying the following two conditions:*

(2-intersection closed) $S, T \in \mathcal{F}$ *with* $|S \cap T| \geq 2$ *implies that* $S \cap T \in \mathcal{F}$,

(cycle-free) *every non-unitary feasible coalition can be written in a unique way as a union of non-unitary supports.*

Note that for set systems $\mathcal{F}$ in the class USI^N, if $S, T \in B(\mathcal{F})$, $|S| \geq 2$, then $S \not\subseteq T$ (since otherwise T could be written as $T \cup \emptyset$ as well as $T \cup S$, and then the representation would not be unique.)

Algaba, Bilbao, Borm and López (2001) introduce this class as a generalization of cycle-free undirected graphs in the sense that any set system given by the connected coalitions of a cycle-free graph satisfies these properties.

Next we motivate why the property that every non-unitary feasible coalition can be written in a unique way as a union of non-unitary supports is called cycle-freeness. In van den Brink (2012) it is shown that the communication feasible set systems satisfying intersection-closedness are those set systems, where the feasible coalitions are the connected coalitions of a cycle-complete graph. A graph is *cycle-complete* if, whenever there is a cycle, the subgraph on that cycle is complete. Recall from (**??**) that the set of connected coalitions in a communication graph game (N, L) is given by $\mathcal{F}_L$.

Proposition 5.5.2 *[van den Brink (2012)] Let $\mathcal{F} \subseteq 2^N$ be a communication feasible set system. Then $\mathcal{F}$ satisfies normality, 2-accessibility, union stability and intersection closedness if and only if there is a cycle-complete graph L such that $\mathcal{F} = \mathcal{F}_L$.*

Note that every cycle-free communication graph is cycle-complete. Cycle-complete structures are often encountered in the economic literature, for example, cycle-free structures in, e.g. auction situations (see Graham, Marshall and Richard (1990)), airport games (see Littlechild and Owen (1973)), sequencing games (see Curiel, Potters, Rajendra Prasad, Tijs and Veltman (1993, 1994)), water distribution problems (see Ambec and Sprumont (2002)) or polluted river problems (see Ni and Wang (2007) and Dong, Ni and Wang (2012)), and other cycle-complete structures are encountered in, e.g. favor exchange networks (see Jackson, Rodríguez Barraquer and Tan (2012)).

Since in a communication feasible set system all singletons are feasible, Proposition 5.5.2 also holds if we use the weaker 2-intersection closedness instead of intersection closedness. It turns out that those communication feasible set systems satisfying the property of cycle-freeness from Definition 5.5.1 are those set systems, where the feasible coalitions are the connected coalitions of a cycle-free graph. Therefore we refer to this property of a union stable system as cycle-freeness.

Proposition 5.5.3 *Let $\mathcal{F} \subseteq 2^N$ be a communication feasible set system, i.e. there is an undirected graph L such that $\mathcal{F} = \mathcal{F}_L$. Then L is cycle-free if and only if every non-unitary feasible coalition in $\mathcal{F}$ can be written in a unique way as a union of non-unitary supports.*

PROOF
Let $\mathcal{F} \subseteq 2^N$ be such that there is an undirected graph L with $\mathcal{F} = \mathcal{F}_L$. **(Only if)** Suppose that L is cycle-free. Then for every connected coalition $S \in \mathcal{F}$ the unique way to write S as union of non-unitary supports is $S = \bigcup_{l \in L(S)} l$. **(If)** Suppose that L is not cycle-free. Let $(i_1, i_2, \ldots, i_k)$, $k \geq 3$, be a cycle in L. Every link on this cycle is a feasible coalition of 2 players and also a support. Therefore $S = \bigcup_{t \in \{1, \ldots, k-1\}} \{i_t, i_{t+1}\}$ and

$S = \{i_k, i_1\} \cup \left(\bigcup_{t \in \{2,\dots,k-1\}} \{i_t, i_{t+1}\}\right)$, and S can be written as a union of non-unitary supports in at least two ways. □

Since every cycle-free graph is cycle-complete, from Propositions 5.5.2 and 5.5.3 it follows that for a communication feasible set system $\mathcal{F} \subseteq 2^N$ cycle-freeness of $\mathcal{F}$ implies intersection closedness of $\mathcal{F}$ but not the other way around. For example, $\mathcal{F} = 2^N$ is intersection closed, but not cycle-free. For arbitrary union stable systems cycle-freeness does not imply 2-intersection closedness (and therefore also intersection closedness is not implied), as shown by the following example.

Example 5.5.4 *Consider a market with one buyer (player 1), two sellers (players 2 and 3) and one intermediary (player 4), so $N = \{1, 2, 3, 4\}$. Suppose that a nonempty coalition $S \subseteq 2^N$ is feasible if and only if it contains the buyer, the intermediary and at least one of the two sellers, i.e. $\mathcal{F} = \{\emptyset, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$. The basis is given by $B(\mathcal{F}) = \{\emptyset, \{1,2,4\}, \{1,3,4\}\}$. This is a cycle-free union stable set system. It does not satisfy 2-intersection closedness however, since $\{1,2,4\} \cap \{1,3,4\} = \{1,4\}$, which is not feasible.*

In Proposition 5.4.4 we saw that a support of an accessible union stable system either has a cardinality of at most 2 (so it is either a singleton or a link) or is a path. Now consider those accessible union stable systems that also satisfy the cycle-freeness from Definition 5.5.1. It turns out that for these accessible union stable systems every nonempty support has a cardinality of at most 2, and is therefore a link or singleton.

Proposition 5.5.5 *Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system that is cycle-free (i.e. all non-unitary feasible coalitions can be written in a unique way as a union of non-unitary supports). If $S \in B(\mathcal{F})$ then $|S| \leq 2$.*

PROOF
Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable system that is cycle-free. Suppose there is an $S \in B(\mathcal{F})$ such that $|S| > 2$. Let $T = S \setminus \{i\}$ for $i \in S$. We distinguish the following two cases:

(i) $|S| = 3$:

   We obtain $|T| = 2$ and therefore $T \in B(\mathcal{F})$ by definition. Since $T \subset S$, this contradicts $S \in B(\mathcal{F})$.

(ii) $|S| > 3$:

   If $T \in B(\mathcal{F})$, then $T \subset S$, contradicting $S \in B(\mathcal{F})$. Therefore suppose $T \notin B(\mathcal{F})$. In that case T can be written as a union of supports $T_1, T_2, \dots, T_k \in B(\mathcal{F})$, $k \geq 2$ such that $T_i \cap (\bigcup_{j=1}^{i-1} T_j) \neq \emptyset$ for $1 < i \leq k$. Since $T_l \subset S$, for $l \in \{1, \dots k\}$, this contradicts $S \in B(\mathcal{F})$.

□

Proposition 5.5.5 implies that a cycle-free accessible union stable system in which all singletons are feasible is the set of connected coalitions in some cycle-free graph, and

so for accessible union stable systems with all singletons feasible, cycle-freeness implies intersection closedness. Without the requirement that all singletons are feasible, the set system is still 2-intersection closed.

Corollary 5.5.6 *(i) Set system $\mathcal{F} \subseteq 2^N$ is a cycle-free accessible union stable system such that $\{i\} \in \mathcal{F}$ for all $i \in N$, if and only if there is a cycle-free graph L such that $\mathcal{F} = \mathcal{F}_L$.*
(ii) If set system $\mathcal{F} \subseteq 2^N$ is a cycle-free accessible union stable system such that $\{i\} \in \mathcal{F}$ for all $i \in N$, then $\mathcal{F}$ is intersection closed (and therefore 2-intersection closed).
(iii) A set system $\mathcal{F} \subseteq 2^N$ is a cycle-free accessible union stable system if and only if $\mathcal{F}$ is an accessible union stable system and $\mathcal{F} \in USI^N$.

It is interesting to point out however, that a cycle-free accessible union stable system does not necessarily satisfy intersection closedness as illustrated by the set system $\mathcal{F} = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ which is a cycle-free accessible union stable system but is not intersection closed since $\{1,2\} \cap \{2,3\} = \{2\} \notin \mathcal{F}$.

Example 5.2.11 provides an accessible union stable system where all singletons are feasible, but which is not cycle-free (since $\{1,2,3,4,5\} = \{1,3,4,5\} \cup \{1,2\} = \{2,3,4,5\} \cup \{1,2\}$).

It is known from Algaba, Bilbao, Slikker (2010) that within the class of augmenting systems requiring all singletons to be feasible characterizes the class of augmenting systems that can be the set of connected coalitions in some undirected graph.

Proposition 5.5.7 *[Algaba, Bilbao, Slikker (2010)] Let $\mathcal{F} \subseteq 2^N$ be an augmenting system. Then $\{i\} \in \mathcal{F}$ for all $i \in N$ if and only if there is an undirected graph L such that $\mathcal{F} = \mathcal{F}_L$.*

From Proposition 5.5.7 and 5.2.9 (respectively Theorem 2.2.45) it follows that we obtain connected coalitions from cycle-free graphs as augmenting systems that are cycle-free and contain all singletons.

Corollary 5.5.8 *(i) Set system $\mathcal{F} \subseteq 2^N$ is a cycle-free augmenting system with $\{i\} \in \mathcal{F}$ for all $i \in N$, if and only if $\mathcal{F} = \mathcal{F}_L$ for some cycle-free graph L.*
(ii) Let $\mathcal{F} \subseteq 2^N$ be a normal set system. Set system $\mathcal{F}$ is an augmenting system with $\{i\} \in \mathcal{F}$ for all $i \in N$, if and only if $\mathcal{F}$ is a 2-accessible union stable system.

5.6 Cooperative games on accessible union stable systems

Recall from Chapter 2 that a game on a union stable system is a triple $(N, v, \mathcal{F})$, with (N, v) a TU-game and $\mathcal{F} \subseteq 2^N$ a union stable system. Now let $\mathcal{G}_{AUS}$ be the subclass of games on union stable systems $(N, v, \mathcal{F})$, where $\mathcal{F}$ also satisfies accessibility. We will refer to the games in this class as *games on accessible union stable systems*. The set of all games on accessible union stable systems on player set N is denoted by $\mathcal{G}_{AUS}^N$. Since in

this chapter we take the player set to be fixed, we denote a game on an accessible union stable system by $(v, \mathcal{F})$. Recall from Definition 2.2.49 that the restricted game $v^{\mathcal{F}}$ is given by

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \text{ for all } S \subseteq N,$$

where $C_{\mathcal{F}}(S)$ denotes the set of all components of S.

5.6.1 A Shapley value for games on accessible union stable systems

A single-valued solution f on $\mathcal{G}_{AUS}^N$ assigns a unique payoff vector $f(v, \mathcal{F}) \in \mathbb{R}^N$ to every $(v, \mathcal{F}) \in \mathcal{G}_{AUS}^N$. We consider an adaptation of the Shapley value to games on accessible union stable systems. This solution is obtained by applying the Shapley value to restricted game $v^{\mathcal{F}}$.

Definition 5.6.1 *The restricted Shapley value is the solution φ on $\mathcal{G}_{AUS}^N$ given by*

$$\varphi(v, \mathcal{F}) = Sh(v^{\mathcal{F}}).$$

Note that the extension of the Shapley value defined on $\mathcal{G}_{US}^N$ by Algaba, Bilbao, Borm and López (2001) is a generalization of φ.

For games on accessible union stable systems, the value φ generalizes the Myerson value for games restricted by communication graphs and the (conjunctive and disjunctive) permission value for games with a permission structure.

We provide an axiomatization of φ on $\mathcal{G}_{AUS}^N$ using axioms similar to those used by Myerson (1980) to axiomatize the Myerson value on communication graph games. Recall from Theorem 2.2.9 that Myerson axiomatized it using component efficiency and balanced contributions for graph games.

Component efficiency of an allocation rule on a class of games $\mathcal{G}_{AUS}^N$ states that for every game with restricted cooperation in this class, the total payoff to every component equals its worth.

Component efficiency A solution f on $\mathcal{G}_{AUS}^N$ satisfies component efficiency if $\sum_{i \in M} f_i(v, \mathcal{F}) = v(M)$, for all $(v, \mathcal{F}) \in \mathcal{G}_{AUS}^N$.

A player $i \in N$ is called a *component dummy* in an accessible union stable system $\mathcal{F}$ if this player does not belong to any maximal component of the grand coalition, i.e. $i \notin \bigcup \{M \in \mathcal{F} : M \in C_{\mathcal{F}}(N)\}$. Note that a component dummy in $\mathcal{F}$ is a null player in any $v^{\mathcal{F}}$ that is derived from some $v \in \mathcal{G}^N$.

Component dummy property A solution f on $\mathcal{G}_{AUS}^N$ satisfies component dummy if for any component dummy i in $\mathcal{F}$, we have $f_i(v, \mathcal{F}) = 0$, for all $(v, \mathcal{F}) \in \mathcal{G}_{AUS}^N$..

Algaba, Bilbao, Borm and López (2001) show that their extension to the Shapley value on $\mathcal{G}_{US}^N$ satisfies component efficiency and the component dummy property, and therefore also on $\mathcal{G}_{AUS}^N$.

Given a set system $\mathcal{F} \subseteq 2^N$ and a player $i \in N$, the set system

$$\mathcal{F}_{-i} = \{S \in \mathcal{F} : i \notin S\} \tag{5.6.6}$$

is given by all those feasible coalitions in $\mathcal{F}$ which do not contain player i.

Proposition 5.6.2 *If $\mathcal{F} \subseteq 2^N$ is an accessible union stable system and $i \in N$, then $\mathcal{F}_{-i}$ is an accessible union stable system.*

PROOF
If $S, T \in \mathcal{F}_{-i}$ with $S \cap T \neq \emptyset$ then by union stability $S \cup T \in \mathcal{F}$. Since $i \notin S \cup T$, it holds that $S \cup T \in \mathcal{F}_{-i}$.

Moreover, if $S \in \mathcal{F}_{-i}$ then $S \in \mathcal{F}$, and by accessibility there exists a $j \in S, j \neq i$, such that $S \backslash \{j\} \in \mathcal{F}$. Since $i \notin S$, it holds that $S \backslash \{j\} \in \mathcal{F}_{-i}$. □

Observe that if $\mathcal{F} \subseteq 2^N$ is an accessible union stable system and $i \in N$, then i is a component dummy for the accessible union stable system $\mathcal{F}_{-i}$.

For a pair of players we consider the change in payoff of one player by deleting all feasible coalitions containing the other player, by using a balanced contributions axiom for games on accessible union stable systems.

Balanced contributions A solution f on $\mathcal{G}_{AUS}^N$ has balanced contributions if for every $(v, \mathcal{F}) \in \mathcal{G}_{AUS}^N$ and for any two players $i, j \in N$ with $i \neq j$, we have

$$f_i(v, \mathcal{F}) - f_i(v, \mathcal{F}_{-j}) = f_j(v, \mathcal{F}) - f_j(v, \mathcal{F}_{-i}).$$

Next we show that on the class of games on accessible union stable systems, the solution φ satisfies balanced contributions.

Proposition 5.6.3 . *The solution φ has balanced contributions $\mathcal{G}_{AUS}^N$.*

PROOF
Recall from Definition 1.0.16 that the Shapley value can be written, for every $v \in \mathcal{G}^N$, as

$$Sh_i(v) = \sum_{\{T \subseteq N : i \in T\}} \frac{\Delta_v(T)}{|T|} \text{ for all } i \in N,$$

where $\Delta_v(T)$ is the *Harsanyi dividend* of coalition $\emptyset \neq T \subseteq N$, see Harsanyi (1959).

If $T \in \mathcal{F}_{-i}$ then every $S \in \mathcal{F}$ such that $S \subseteq T$ satisfies $i \notin S$ and therefore, $S \in \mathcal{F}_{-i}$ and $S \setminus \{i\} = S$. Since $C_{\mathcal{F}_{-i}}(S) = C_{\mathcal{F}}(S \setminus \{i\})$ for all $S \subseteq N$, for $T \in \mathcal{F}_{-i}$, we have

$$\begin{aligned}
\Delta_{v^{\mathcal{F}_{-i}}}(T) &= \sum_{S \subseteq T} (-1)^{|T|-|S|} v^{\mathcal{F}_{-i}}(S) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v^{\mathcal{F}}(S \setminus \{i\}) \\
&= \sum_{S \subseteq T} (-1)^{|T|-|S|} v^{\mathcal{F}}(S) = \Delta_{v^{\mathcal{F}}}(T).
\end{aligned}$$

From Algaba, Bilbao, Borm and López (2000) we obtain that $\Delta_{v^{\mathcal{F}}}(T) = 0$ for those coalitions $T \notin \mathcal{F}$. Therefore, it holds that

$$\varphi_j\left(v, \mathcal{F}\right) - \varphi_j\left(v, \mathcal{F}_{-i}\right) = Sh_j\left(v^{\mathcal{F}}\right) - Sh_j\left(v^{\mathcal{F}_{-i}}\right)$$

$$= \sum_{\{T \subseteq N,\, :\, j \in T\}} \frac{\Delta_{v^{\mathcal{F}}(T)}}{|T|} - \sum_{\{T \subseteq N,\, :\, j \in T\}} \frac{\Delta_{v^{\mathcal{F}_{-i}}}(T)}{|T|}$$

$$= \sum_{\{T \in \mathcal{F}\, :\, j \in T\}} \frac{\Delta_{v^{\mathcal{F}}(T)}}{|T|} - \sum_{\{T \in \mathcal{F}\, :\, j \in T\}} \frac{\Delta_{v^{\mathcal{F}_{-i}}}(T)}{|T|}$$

$$= \sum_{\{T \in \mathcal{F}\, :\, i, j \in T\}} \frac{\Delta_{v^{\mathcal{F}}(T)}}{|T|},$$

where the third equality follows from the fact that $\Delta_{v^{\mathcal{F}}}(T) = \Delta_{v^{\mathcal{F}_{-j}}}(T)$ for all $T \in \mathcal{F}_{-j}$. Similarly we obtain that $\varphi_i\left(v, \mathcal{F}\right) - \varphi_i\left(v, \mathcal{F}_{-j}\right) = \sum_{\{T \in \mathcal{F}\, :\, i, j \in T\}} \frac{\Delta_{v^{\mathcal{F}}(T)}}{|T|}$. $\qquad\square$

The following theorem provides a characterization of φ on the class of games on accessible union stable systems.

Theorem 5.6.4 *A solution on $\mathcal{G}_{AUS}^N$ is equal to φ if and only if it satisfies component efficiency, component dummy, and has balanced contributions.*

PROOF
We have seen that φ satisfies the three axioms.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. We perform induction on the number of feasible coalitions in $\mathcal{F}$.[5]

If $|\mathcal{F}| = 1$ then $\mathcal{F} = \{\emptyset\}$ and hence every player $i \in N$ is a component dummy. By the component dummy property, $f_i(v, \mathcal{F}) = 0$ for all $i \in N$, so f is uniquely determined.

Proceeding by induction, suppose that f is uniquely determined whenever $|\mathcal{F}| < k$. Now consider $|\mathcal{F}| = k$. By the component dummy property, $f_i(v, \mathcal{F}) = 0$ for i a dummy player. To show uniqueness, it is sufficient to show for any component $M \in C_{\mathcal{F}}(N)$ that $f_i(v, \mathcal{F})$ is uniquely determined for all $i \in M$. If $|M| = 1$ then component efficiency uniquely determines the payoff for $i \in M$. If $|M| > 1$, take any $i \in M$. By applying balanced contributions we obtain

$$f_i\left(v, \mathcal{F}\right) - f_i\left(v, \mathcal{F}_{-j}\right) = f_j\left(v, \mathcal{F}\right) - f_j\left(v, \mathcal{F}_{-i}\right), \quad \text{for all } j \in M \setminus \{i\}. \tag{5.6.7}$$

This yields $|M| - 1$ linear independent equations, where the values $f(v, \mathcal{F}_{-i})$ and $f(v, \mathcal{F}_{-j})$ are known by the induction hypothesis.

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[5]Note that in fact uniqueness follows from a more general result in Katsev (2013). Here we provide the proof for completeness and to get a better insight into the structural properties.

Component efficiency requires that

$$\sum_{i \in M} f_i(v, \mathcal{F}) = v(M). \tag{5.6.8}$$

Together (5.6.7) and (5.6.8) yield $|M|$ linear independent equations in the $|M|$ unknown payoffs $f_h(v, \mathcal{F})$, $h \in M$, uniquely determining f. $\qquad\square$

5.7 Concluding remarks

When deleting coalitions from a set of feasible coalitions that satisfies certain properties we need to take care that the resulting set of feasible coalitions still satisfies these properties. For example, in a union stable system only supports can be deleted (since other coalitions are the union of two nondisjoint coalitions and deleting them violates union stability) while for antimatroids only paths (since otherwise union closedness is violated) that are not covered by a path. Note that when coalition $S \in \mathcal{F}$ is covered by $T \in \mathcal{F}$ then there is an $i \in T$ such that T is an i-path and $S = T \setminus \{i\}$. Algaba, Bilbao, Borm and López (2001) showed that Myerson (1977)'s axiomatization for communication graph games by component efficiency and fairness[6] also holds for games on union stable systems, i.e. the restricted Shapley value is the unique allocation rule for games on union stable systems that satisfies component efficiency, component dummy and fairness. The fact that not all supports can be deleted from an accessible union stable system creates problems when axiomatizing the restricted Shapley value for games on these systems. For example, van den Brink (1997) and Algaba, Bilbao, van den Brink and Jimenez-Lósada (2003) show that such axioms do not characterize the restricted Shapley value for games with a permission structure, respectively games on (normal) antimatroids. A main question then is whether the restricted Shapley value for accessible union stable systems is characterized by component efficiency, component dummy and a type of fairness. This is an important question, since our motivation to study accessible union stable systems is that they have a characterizing property from communication networks (union stability) as well as hierarchies (accessibility).

Finally, in the network formation and stability model of Jackson and Wolinsky (1996), worths are not assigned to coalitions of players, but to the network structures themselves. For example, in the two network structures $L = \{\{1, 2\}, \{1, 3\}\}$ respectively $L' = L \cup \{\{2, 3\}\}$ on $N = \{1, 2, 3\}$ (see Figure 5.1) the 'grand coalition' N can have a different value, although in a restricted game in the sense of Myerson (1977) coalition N should have the same worth in both restricted games. Since in this chapter we studied network structures richer than bilateral communication networks, it makes sense to consider network value functions similar to those in Jackson and Wolinsky (1996), only now for accessible union stable systems and other structures.

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[6]For communication graph games component dummy is not relevant since every singleton is feasible and therefore there are no dummies in the set of connected coalitions.

Figure 5.1: Two connected communication graphs on $N = \{1, 2, 3\}$

Chapter 6

Solutions for games with precedence constraints

We introduce the hierarchical solution for games with precedence constraints by Faigle and Kern (1992). Payoff assigned by this new solution to relevant players is unaffected by the presence of irrelevant players. We show that this is not the case for the precedence Shapley value by Faigle and Kern (1992). The hierarchical solution can be seen to belong to the class of precedence power solutions. The solutions in this class allocate the worth of a coalition relative to a power measure for acyclic digraphs. The hierarchical solution allocates proportionally to the hierarchical measure. We give an axiomatization of this power measure on the class of acyclic digraphs. In addition we extend the hierarchical measure to regular set systems. These feasible set systems contain the class of feasible set systems obtained from acyclic digraphs according to Faigle and Kern. Finally we consider a subclass of acyclic digraphs, given by forests and sink forests and consider the normalized version of the hierarchical measure on these subclasses as well as a number of other power measures. This chapter is based on Algaba, van den Brink and Dietz (2014).

6.1   Introduction

Faigle and Kern introduced games with precedence constraints as cooperative TU-games where the player set is endowed with a *precedence relation*. This precedence relation is represented by a partial order (i.e. reflexive, antisymmetric and transitive relation) on the player set. Equivalently it can be represented by an acyclic directed graph. Given an acyclic digraph, a permutation of the players is called *admissible*, if players are preceded in the permutation by their successors in the digraph. A coalition of players is considered *feasible*, if for any player in the coalition all of its successors in the digraph are also present in the coalition.[1] The absolute hierarchical strength of a player i, given a feasible coalition S, is now simply the number of admissible permutations in D where i is preceded in the permutation by the players in $S \setminus \{i\}$. The normalized hierarchical

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[1]Note that this is different from the conjunctive approach to permission structures by van den Brink, Gilles and Owen (1992). There a coalition is feasible if for any player in the coalition all of its predecessors in the digraph are present.

strength of a player i, given a feasible coalition S, is obtained by dividing the absolute hierarchical strength by the total number of permutations that are admissible in digraph D. Faigle and Kern (1992) use the hierarchical strength to axiomatically define the so-called *precedence Shapley value*. The axiomatization uses the axioms of efficiency, the precedence null player property (together presented by Faigle and Kern (1992) as the carrier axiom) and linearity, combined with an axiom called hierarchical strength, which replaces the 'standard' symmetry axiom. This axiom states that in a unanimity game with precedence constraints dividend is distributed among the players in the unanimity coalition proportionally to their normalized hierarchical strength applied to the unanimity coalition.

This chapter consists of three sections.

First we consider so-called *irrelevant players*. A player is called irrelevant if it is both a precedence null player in the game, and only precedes precedence null players in the digraph. Irrelevant player independence is subsequently defined as the property that states that the payoff to relevant players is not affected by the presence of irrelevant players. We consider this a desirable property for solutions for games with precedence constraints. We show that the precedence Shapley value does not satisfy this property. We consider this a negative aspect of the precedence Shapley value. Therefore we consider the hierarchical solution. Like the precedence Shapley value, this new solution for games with precedence constraints satisfies the axioms of efficiency, linearity and the precedence null player property. Payoffs to relevant players according to the hierarchical solution are not affected by the presence of irrelevant players. In addition it uses a weaker version of the hierarchical strength axiom used by Faigle and Kern to axiomatize the precedence Shapley value. Both solutions allocate the dividend in unanimity games with precedence constraints among the players in the unanimity coalition. The precedence Shapley value allocates proportionally to the hierarchical strength applied to the full digraph, while our new solution allocates proportionally to the hierarchical strength applied to the sub-graph on the unanimity coalition. We consider weight functions for digraphs. For every acyclic digraph and every feasible coalition within that digraph, a *weight function* assigns a weight to the players within that feasible coalition. Both the absolute as well as the normalized hierarchical strength are examples of weight functions. The class of so-called *weighted precedence solutions* consists of solutions that allocate the dividend of a coalition proportionally to some weight function. Both the precedence Shapley value as well as the hierarchical solution are weighted precedence solutions. The hierarchical solution is also contained in the subclass of weighted precedence solutions that allocate proportionally to so-called *subgraph-invariant* weight functions. The value of a subgraph-invariant weight function applied to a feasible coalition depends only on the subgraph on that coalition. We show that for solutions in this (sub)class payoff to relevant players is not affected by the presence of irrelevant players. We also show that this (sub)class of solutions can be obtained by allocating the dividend of a feasible coalition proportionally to some power measure for acyclic digraphs. We refer to such solutions as *precedence power solutions*. Power measures for acyclic digraphs assign values to the players in an acyclic digraph. These values can be interpreted as the 'strength' or 'influence' of these players in the digraph. Examples of power measures are the ones given by Gould (1987), White and

Borgatti (1994), the β-measure of van den Brink and Gilles (2000) and its reflexive version in van den Brink and Borm (2002), the λ-measure of Borm, van den Brink and Slikker (2002), the positional power measure of Herings, van der Laan and Talman (2005) or the centrality measures in del Pozo, Manuel, González-Arangüena and Owen (2011). Our approach of allocating the dividend of a feasible coalition proportionally to some power measure for acyclic digraphs is similar to that of van den Brink, van der Laan and Pruzhansky (2011) for (communication) graph games, where they compare solutions obtained from allocating dividend according to power measures for undirected graphs to solutions from the literature. The hierarchical solution is a precedence power solution. Therefore there exists a power measure, such that the hierarchical solution allocates dividend proportionally to this power measure. We refer to the power measure employed by the hierarchical strength as the *hierarchical measure*. For any digraph the hierarchical measure assigns to any player in the digraph the number of admissible permutations, where it is preceded in the permutation by all other players. For any digraph the hierarchical measure is given by the hierarchical strength applied to the grand coalition. We also give an axiomatization of the hierarchical measure.

In the literature the precedence Shapley value has been extended to games associated with a combinatorial structure more general than a digraph. For example, Bilbao and Edelman (2000) consider games on convex geometries and Bilbao and Ordoñez (2008) consider games on a class of augmenting systems. Convex geometries have been shown to be contained in the class of so-called regular set systems considered by Lange and Grabisch (2009), who also consider an extension of the precedence Shapley to games on regular set systems. A set of admissible permutations can be generated from these set systems similar to how this is done for digraphs. We consider an extension of the hierarchical measure on the class of regular set systems and proceed to give an axiomatization. The hierarchical measure can be seen to rank players based on a number of permutations of these players. An example where we also encounter ranking of players based on permutations comes from social choice theory. The permutations in this case are the preferences of the voters on a number of alternatives they can choose from and the players are the alternatives. It turns out that the hierarchical measure is similar to the plurality scoring rule for social choice situations. By considering different scoring rules, we can define new solutions for games with precedence constraints.

In the third section we consider the hierarchical measure on the classes of forests and sink forests, which are subclasses of acyclic digraphs. On these classes we consider the normalized version of the hierarchical measure. Finally we consider the application of a number of power measures to river games with multiple springs.

6.2 Solutions for games with precedence constraints and power measures for acyclic digraphs

6.2.1 Irrelevant player independence

Recall from (2.2.7) that the set of admissible permutations associated with an acyclic digraph (N, D), is given by

$$\Pi_D(N) = \{\pi \in \Pi(N) : \pi(i) > \pi(j) \text{ if } (i, j) \in D\},$$

and from (2.2.9) that

$$\Pi_D^i(N, S) = \{\pi \in \Pi_D(N) : \pi(i) > \pi(j) \text{ for all } j \in S \setminus \{i\}\}.$$

For convenience we will write $\Pi_D^i(N)$ instead of $\Pi_D^i(N, N)$ throughout this chapter.

Recall from Definition 2.2.30 that, for $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$, the absolute hierarchical strength of a player $i \in S$ is defined by

$$h_i(N, D, S) = |\Pi_D^i(N, S)|,$$

i.e. it is the number of those admissible permutations π in digraph D where i is predeced in the permutation by all the other players in coalition S. Recall from Definition 2.2.28, that the characteristic function v of a game with precedence constraints (N, v, D) is only defined on the set of coalitions $\Phi^p(N, D)$ and from (2.2.33) that the dividend of a coalition $S \in \Phi^p(N, D)$ in game with precedence constraints (N, v, D) is given by

$$\Delta_v^D(S) = v(S) - \sum_{T \subset S, T \in \Phi^p(N,D), T \neq \emptyset} \Delta_v^D(T).$$

Recall from Definition 2.2.35 that the precedence Shapley value allocates the dividend of a coalition $S \in \Phi^p(N, D)$ proportionally to the hierarchical strength $h(N, D, S)$ of the players in S, i.e.

$$H_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(N, D, S)}{\sum_{j \in S} h_j(N, D, S)} \Delta_v^D(S) \text{ for all } i \in N.$$

Furthermore, recall that a player $i \in N$ is a precedence null player in game with precedence constraints (N, v, D), if for every $\pi \in \Pi_D(N)$ it holds that $m_i^\pi(N, v, D) = 0$. For a classical TU-game (N, v), player $i \in N$ being a null player in v implies that $\Delta_v(S) = 0$ for all coalitions $S \subseteq N, i \in S$. However, for a game with precedence constraints (N, v, D), player $i \in N$ being a precedence null player does not imply that $\Delta_v(S) = 0$ for all coalitions $S \in \Phi^p(N, D), i \in S$. We illustrate this with an example.

Example 6.2.1 *Consider the game with precedence constraints (N, v, D), where $N = \{1, 2\}$, $v = u_{\{1,2\}}$ is the unanimity game on players 1,2 and D is given by $\{(1, 2)\}$. The set of feasible coalitions is given by $\Phi^p(N, D) = \{\emptyset, \{2\}, \{1, 2\}\}$. The set of admissible*

permutations is given by $\Pi_D(N) = \{(2,1)\}$. Therefore we only need to consider the precedence marginal vector $m^{(2,1)}(N, v, D)$, to decide which players are precedence null players. We obtain $m^{(2,1)}(N, v, D) = (1, 0)$ and therefore player 2 is a precedence null player. The dividends of (N, v, D) are given by $\Delta_v^D(\{2\}) = 0, \Delta_v^D(\{1, 2\}) = 1$. We find that even though player 2 is a precedence null player, not all feasible coalitions that it is contained in have 0 dividend.

The example shows that in games with precedence constraints, even though a player is a precedence null player, the Harsanyi dividend of a feasible coalition containing that player can still be non-zero. This is because that player precedes players in the digraph that are not precedence null players. Consider the example of a manufacturer of some good, where we can distinguish between agents that perform manual labor at an early stage in the production process, and management, that is in charge of distribution and sales of the good. The manual labor produces a good, but this in itself does not generate any worth. The real worth comes from selling the good.

The following proposition shows that coalitions containing a precedence null player, but not any of this player's predecessors in the digraph, always have zero dividend.

Proposition 6.2.2 *Consider game with precedence constraints (N, v, D). If coalition $S \in \Phi^p(N, D)$ contains a precedence null player i and $S \subseteq N \setminus P_D(i)$, then $\Delta_v^D(S) = 0$.*

PROOF

Let $H(S) = \{j : j \in S \text{ and } j \notin \widehat{F}_D(i)\}$ be the set of players in S that are not subordinates of player i in D. We perform induction on $|H(S)|$.

If $|H(S)| = 0$, then the only feasible subset of S containing player i is S itself. Therefore $v(S) - v(S \setminus \{i\}) = \Delta_v^D(S)$. Since i is a precedence null player, it holds that $v(S) - v(S \setminus \{i\}) = 0 = \Delta_v^D(S)$.

Proceeding by induction, assume that $\Delta_v^D(T) = 0$, when $0 \leq |H(T)| < |H(S)|$. Since $|H(S)| > 0$ it holds that S is no longer the only feasible subset of S containing player i. Let $K(S) = \{T \in \Phi^p(N, D) : i \in T \text{ and } T \subset S\}$. It now holds that $v(S) - v(S \setminus \{i\}) = \sum_{T \in K(S)} \Delta_v^D(T) + \Delta_v^D(S)$. Since $|H(T)| < |H(S)|$ for $T \in K(S)$, by induction we have $\Delta_v^D(T) = 0$ for all $T \in K(S)$. Since i is a precedence null player, it holds that $v(S) - v(S \setminus \{i\}) = 0 = \Delta_v^D(S)$.

$\square$

Player $i \in N$ an *irrelevant player* in game with precedence constraints (N, v, D) if i is a precedence null player, and any $j \in \widehat{P}_D(i)$ is also a precedence null player (this implies that any $j \in \widehat{P}_D(i)$ is also irrelevant). Call a player $i \in N$ *relevant* if it is not an irrelevant player. We have the following proposition.

Proposition 6.2.3 *Player $i \in N$ is an irrelevant player in game with precedence constraints (N, v, D) if and only if $\Delta_v^D(S) = 0$ for any coalition $S \in \Phi^p(N, D)$ that contains i.*

PROOF

Only if
For a coalition S, let $P_D^1(S) = P_D(S)$ and $P_D^k(S) = P_D(P_D^{k-1}(S))$. For an irrelevant player $i \in N$, let $\kappa(i)$ be the smallest integer such that $P_D^{\kappa(i)}(\{i\}) = \emptyset$. We show by induction on $\kappa(i)$ that $\Delta_v^D(S) = 0$ for all coalitions $S \in \Phi^p(N, D), i \in S$.

If $\kappa(i) = 1$, then player i has no predecessors in (N, D). In that case $S \subseteq N = N \setminus P_D(i)$ for any coalition $S \in \Phi^p(N, D)$ such that $i \in S$. Therefore by Proposition 6.2.2 $\Delta_v^D(S) = 0$ for $S \in \Phi^p(N, D)$ such that $i \in S$.

Proceeding by induction, assume that for any irrelevant player j such that $\kappa(j) < \kappa(i)$ it holds that $\Delta_v^D(S) = 0$ for all coalitions $S \in \Phi^p(N, D)$, $j \in S$. We already know that $\Delta_v^D(S) = 0$ for S such that $S \cap \widehat{P}_D(i) \neq \emptyset$, by the fact that predecessors of irrelevant players are themselves also irrelevant and $\kappa(j) < \kappa(i)$ for any $j \in \widehat{P}_D(i)$. Therefore we only need to consider those feasible sets $S \in \Phi^p(N, D)$ such that $S \cap \widehat{P}_D(i) = \emptyset$.
For these sets it holds that $S \subseteq N \setminus P_D(i)$. Therefore by Proposition 6.2.2 $\Delta_v^D(S) = 0$ for $S \in \Phi^p(N, D)$ such that $i \in S$.

If
If i is not a precedence null player, then there exists a coalition $S \in \Phi^p(N, D), i \in S$ such that $S \setminus \{i\} \in \Phi^p(N, D)$ and $v(S) - v(S \setminus \{i\}) \neq 0$. We also have $v(S) - v(S \setminus \{i\}) = \sum_{T \subseteq S, T \neq \emptyset, T \in \Phi^p(N, D)} \Delta_v^D(T)$. It follows that there exists at least one set $S \in \Phi^p(N, D), i \in S$ such that $\Delta_v^D(S) \neq 0$ and we obtain a contradiction.
If $j \in \widehat{P}_D(i)$ is not a precedence null player, we can reason in a similar way to obtain that there exists at least one set $S \in \Phi^p(N, D), j \in S$ such that $\Delta_v^D(S) \neq 0$. Since i is a subordinate of j in D, i must also be in S (since a coalition of players is considered feasible, if for any player in the coalition all of its successors in the digraph are also present in the coalition) and we obtain a contradiction. $\qquad\square$

Now let $Irr(N, v, D)$ be the set of irrelevant players in game with precedence constraints (N, v, D). Irrelevant player independence states that removal of irrelevant players from the game, does not affect the payoff to relevant players.

Irrelevant player independence Let $N' = N \setminus Irr(N, v, D)$. For every $(N, v, D) \in \mathcal{G}_{PC}$, it holds that $f_i(N, v, D) = f_i(N', v_{N'}, D(N'))$ for $i \in N'$

For a collection of sets $\mathcal{F} \subseteq 2^N$ let $\mathcal{F}_S = \{T \in \mathcal{F} : T \subseteq S\}$. This is the collection of sets obtained from $\mathcal{F}$ by considering only the subsets of S in $\mathcal{F}$. It can be seen that,

for $N' = N \setminus Irr(N, v, D)$, it holds that $\Phi_{N'}^p(N, D) = \Phi^p(N', D(N'))$: the collection of feasible subsets of coalition N' obtained from graph (N, D) is equal to the collection of feasible sets obtained from graph $(N', D(N'))$ (note that this does not have to be the case for all subsets of N). This means that removing irrelevant players from the game does not have an effect on the ability of relevant players to cooperate with each other.

We consider irrelevant player independence a desirable property for a solution for games with precedence constraints to satisfy. Since irrelevant players are precedence null players, they do not make any contribution to players that precede them in the digraph. Since they precede only players that are also precedence null players, they also do not make a contribution through players that they precede in any permutation. Therefore they should not be able to affect the payoffs of players that do make a contribution in the game. The precedence Shapley value does not satisfy irrelevant player independence. This is illustrated by the following example.

Example 6.2.4 *Consider the game (N, v, D), where $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$ is the unanimity game with precedence constraints on players 1,2 and D is given by $\{(3, 1)\}$. Player 3 is an irrelevant player in (N, v, D). The payoffs according to the precedence Shapley value are given by $H(N, u_{\{1,2\}}, D) = (\frac{1}{3}, \frac{2}{3}, 0)$.*

Next consider the game $(N', u_{\{1,2\}}, D')$, where $N' = N \setminus \{3\}$ and $D' = \emptyset$. The payoffs according to the precedence Shapley value are given by $H(N', u_{\{1,2\}}, D') = (\frac{1}{2}, \frac{1}{2})$.

The presence of irrelevant player 3 changes the payoffs of players 1 and 2 by the precedence Shapley value from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{3}, \frac{2}{3})$.

In fact it can be shown that for games $(N_m, u_{\{1,2\}}, D_m)$, where N_m is given by $\{1, \ldots, m\}$ and D_m by $\{(3, 1), (4, 3), \ldots, (m, m-1)\}$, the precedence Shapley value is given by $H_1(N_m, u_{\{1,2\}}, D_m) = \frac{1}{m+1}$, $H_2(N_m, u_{\{1,2\}}, D_m) = \frac{m}{m+1}$ and $H_i(N_m, u_{\{1,2\}}, D_m) = 0$ for $i \in N_m \setminus \{1, 2\}$ and so $\lim_{m \to \infty} H_1(N_m, u_{\{1,2\}}, D_m) = 0$ and $\lim_{m \to \infty} H_2(N_m, u_{\{1,2\}}, D_m) = 1$. We find that the fact that player 1 precedes many irrelevant players in the digraph, is detrimental to the payoff of player 1, even though, for different values of m, player 1 is present in exactly the same feasible coalitions that contain only relevant players.

6.2.2 The hierarchical solution for games with precedence constraints

Faigle and Kern axiomatize the precedence Shapley value by the axioms of efficiency, linearity, the precedence null player property and hierarchical strength. We consider the solution for games with precedence constraints that is obtained by replacing hierarchical strength by the axioms of irrelevant player independence and hierarchical strength 2. Hierarchical strength 2 is a weaker version of the hierarchical strength axiom in that it only holds for unanimity games of the grand coalition. This axiom can be interpreted as follows. If unanimity among all players must be reached before any non-zero worth can be generated, we might consider the players equals with respect to the game. Therefore worth allocation should depend only on the strength of players in the digraph. The strength of

each player in the digraph is measured by the hierarchical strength.[2]

Hierarchical strength 2 For every $N \subseteq \mathbb{N}, (N, D) \in \mathcal{D}_a$ and every $i, j \in N$, it holds that $\overline{h}_i(N, D, N) f_j(N, u_N, D) = \overline{h}_j(N, D, N) f_i(N, u_N, D)$.

We further note that the precedence null player property can be replaced by the following weaker property on irrelevant players.[3]

Irrelevant player property For each $(N, v, D) \in \mathcal{G}_{PC}$, if $i \in N$ is an irrelevant player in (N, v, D), then $f_i(N, v, D) = 0$.

We show that there is a unique solution for games with precedence constraints that satisfies efficiency, linearity, the irrelevant player property, independence of irrelevant players, and hierarchical strength 2.

Theorem 6.2.5 *There is a unique solution f on $\mathcal{G}_{PC}$ that satisfies efficiency, linearity, the irrelevant player property, independence of irrelevant players, and hierarchical strength 2.*

PROOF

Since f satisfies linearity it is sufficient to consider uniqueness of f on unanimity games with precedence constraints. For the unanimity game with precedence constraints (N, u_S, D) on some coalition $S \in \Phi^p(N, D)$, the set of irrelevant players is given by $N \setminus S$. By the irrelevant player property these players are assigned a 0 payoff by f. By irrelevant player independence for the players in S it holds that $f(N, u_S, D) = f(S, u_S, D)$.[4]
From efficiency it then follows that

$$\sum_{k \in S} f_k(S, u_S, D) = u_S(S) = 1. \tag{6.2.1}$$

Now consider any player $i \in S$. Since (S, u_S, D) is a unanimity game on the grand coalition S we can apply hierarchical strength 2 to player i and any player $k \in S$ to obtain that

$$\overline{h}_i(S, D, S) f_k(S, u_S, D) = \overline{h}_k(S, D, S) f_i(S, u_S, D). \tag{6.2.2}$$

We distinguish the following two cases:

(i) $\overline{h}_i(S, D, S) = 0$

Since $\sum_{j \in S} \overline{h}_j(S, D, S) = 1$, there exists at least one $l \in S \setminus \{i\}$ such that $h_l(S, D, S) \neq 0$. It follows from Equation (6.2.2) applied to players i and l that $f_i(S, u_S, D) = 0$.

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[2]We cannot use this same interpretation for the hierarchical strength by Faigle and Kern (1992), since in unanimity games with precedence constraints the worth of coalitions other than the grand coalition might be non-zero. Players in the unanimity coalition play a different role in the game than the irrelevant players outside the unanimity coalition.

[3]It is straightforward to show that the precedence null player property can also be replaced by the irrelevant player property in the axiomatization of the precedence Shapley value.

[4]For convenience we write the subgame on S by (S, u_S, D) instead of $(S, u_S|_S, D(S))$.

(ii) $\overline{h}_i(S, D, S) > 0$

For $k \in S$ it follows from Equation (6.2.2) that

$$f_k(S, u_S, D) = \frac{\overline{h}_k(S, D, S)}{\overline{h}_i(S, D, S)} f_i(S, u_S, D).$$

By substituting this expression in Equation (6.2.1) we obtain

$$\sum_{k \in S} \frac{\overline{h}_k(S, D, S)}{\overline{h}_i(S, D, S)} f_i(S, u_S, D) = 1$$

Since $\overline{h}_k(S, D, S)$ is known for $k \in S$ we find that $f_i(S, u_S, D)$ is uniquely determined.

$\square$

Next we define the hierarchical solution H^2 for games with precedence constraints.

Definition 6.2.6 *The hierarchical solution H^2 is the solution on $\mathcal{G}_{PC}$ given by*

$$H_i^2(N, v, D) = \sum_{\substack{S \in \Phi^P(N, D) \\ i \in S}} \frac{h_i(S, D(S), S)}{\sum_{j \in S} h_j(S, D(S), S)} \Delta_v^D(S), \; i \in N.$$

It is straightforward to show that the unique solution from Theorem 6.2.5 is the hierarchical solution.

Corollary 6.2.7 *A solution on $\mathcal{G}_{PC}$ is equal to the hierarchical solution H^2 if and only if it satisfies efficiency, linearity, the irrelevant player property, independence of irrelevant players, and hierarchical strength 2.*

We illustrate the hierarchical solution by an example.

Example 6.2.8 *Consider the game with precedence constraints (N, v, D), where $N = \{1, 2, 3\}$, v is given by $u_{\{2,3\}} + u_{\{1,2,3\}}$ and D is given by $\{(1, 2), (2, 3)\}$. The dividends of v are given by $\Delta_v^D(\{2,3\}) = 1$, $\Delta_v^D(\{1, 2, 3\}) = 1$ and $\Delta_v^D(S) = 0$ otherwise. The set of admissible permutations on subgraph $D(\{2, 3\})$ is given by $\{(3, 2)\}$. We obtain $h_2(\{2, 3\}, D(\{2, 3\}), \{2, 3\}) = 1, h_3(\{2, 3\}, D(\{2, 3\}), \{2, 3\}) = 0$. The set of admissible permutations on D is given by $\{(3, 2, 1)\}$. We obtain $h_1(N, D, N) = 1, h_2(N, D, N) = 0, h_3(N, D, N) = 0$. Finally the hierarchical solution is given by $H^2(N, v, D) = (1, 1, 0)$.*

6.2.3 Weighted precedence solutions for games with precedence constraints

Consider a game with precedence constraints (N, v, D). Both the precedence Shapley value H as well as the hierarchical solution H^2 allocate the dividend $\Delta_v^D(S)$ of a coalition $S \in \Phi^p(N, D)$ among the players in S. The precedence Shapley value allocates proportionally to the hierarchical strength $H(N, D, S)$, while the hierarchical solution allocates proportionally to the hierarchical strength $H(S, D(S), S)$.

A *weight function* assigns to every digraph $(N, D) \in \mathcal{D}_a$ and a coalition $S \in \Phi^p(N, D)$ a set of weights on the players in S.

Definition 6.2.9 *A weight function is a function w that assigns to every digraph $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$ a vector $w(N, D, S) \in \mathbb{R}^S$, where $\sum_{i \in S} w_i(N, D, S) > 0$.*

Let the collection of all weight functions be denoted by W. We note that both the absolute as well as the normalized hierarchical strength are weight functions. Let a weight function w be called *subgraph-invariant* if $w(N, D, S) = w(S, D(S), S)$ for all $(N, D) \in \mathcal{D}_a$, $S \in \Phi^p(N, D)$. Let the collection of all subgraph-invariant weight functions be denoted by W^I. The weights assigned to a digraph $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$ by a subgraph-invariant weight function depend only on the subgraph on S.

Next we consider the class of solutions that for a game (N, v, D) allocate the dividend of a coalition $S \in \Phi^p(N, D)$ according to $w(N, D, S)$ for some $w \in W$. We will refer to solutions in this class as *weighted precedence solutions*.

Definition 6.2.10 *For a game with precedence constraints (N, v, D) and weight function $w \in W$, the weighted precedence solution $f^w \in \mathbb{R}^N$ is given by*

$$f_i^w(N, v, D) = \sum_{\substack{S \in \Phi^p(N, D) \\ i \in S}} \frac{w_i(N, D, S)}{\sum_{j \in S} w_j(N, D, S)} \Delta_v^D(S) \text{ for all } i \in N.$$

Denote the set of all weighted precedence solutions by Ψ. It is clear that the precedence Shapley value H is the weighted precedence solution f^h, where h is the absolute (or normalized) hierarchical strength. It is straightforward to see that the hierarchical solution H^2 is the weighted precedence solution $f^{h'}$, where h' is the weight function given by

$$h'(N, D, S) = h(S, D(S), S) \text{ for } (N, D) \in \mathcal{D}_a \text{ and } S \in \Phi^p(N, D).$$

Let the set of weighted precedence solutions obtained from some subgraph-invariant weight function be denoted by Ψ^I. For these solutions dividend allocation of a feasible coalition depends only on the subgraph on that coalition. We have the following proposition.

Proposition 6.2.11 *A solution for games with precedence constraints in Ψ^I satisfies irrelevant player independence.*[5]

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[5] In the appendix to this chapter we consider the subclass of Ψ consisting of all precedence power solutions satisfying irrelevant player independence.

PROOF

Consider a game with precedence constraints (N, v, D) and $f^w \in \Psi^I$ obtained from weight function w. Let $N' = N \setminus Irr(N, v, D)$. By Proposition 6.2.3, it holds that $\Delta_v^D(S) = 0$ if $S \cap Irr(N, v, D) \neq \emptyset$ and from the expression of dividends given by Faigle and Kern (1992) it follows that $\Delta_v^D(S) = \Delta_{v_{N'}}^D(S)$ for $S \in \Phi_{N'}^p(N, D)$.

For a player $i \in N'$ the solution on (N, v, D) is given by

$f_i^w(N, v, D) = \sum_{\substack{S \in \Phi^p(N, D) \\ i \in S}} \frac{w_i(S, D(S), S)}{\sum_{j \in S} w_j(S, D(S), S)} \Delta_v^D(S) = \sum_{\substack{S \in \Phi_{N'}^p(N, D) \\ i \in S}} \frac{w_i(S, D(S), S)}{\sum_{j \in S} w_j(S, D(S), S)} \Delta_{v_{N'}}^D(S).$

For a player $i \in N'$ the solution on $(N', v_{N'}, D(N'))$ is given by $f_i^w(N', v_{N'}, D(N')) =$
$\sum_{\substack{S \in \Phi^p(N', D(N')) \\ i \in S}} \frac{w_i(S, D(N')(S), S)}{\sum_{j \in S} w_j(S, D(N')(S), S)} \Delta_{v_{N'}}^{D(N')}(S)$. For all coalitions $S \in \Phi^p(N', D(N')) = \Phi_{N'}^p(N, D)$

it holds that $D(S) = D(N')(S)$, and therefore $w_i(S, D(S), S) =$
$w_i(S, D(N')(S), S)$ for all $i \in S$. We obtain $f_i^w(N, v, D) = f_i^w(N', v_{N'}, D(N'))$.

$\square$

6.2.4 Precedence power solutions for games with precedence constraints

For a game with precedence constraints (N, v, D) the weighted precedence solution f^w obtained from $w \in W$ allocates the dividend of a coalition $S \in \Phi^p(N, D)$ proportionally to $w(N, D, S)$. Moreover if $f^w \in \Psi^I$ it holds that $w(N, D, S) = w(S, D(S), S)$ and dividend allocation depends only on the subgraph on S.

Next we consider power measures for acyclic digraphs. A *power measure* for acyclic digraphs is a function p, that for every acyclic digraph $(N, D) \in \mathcal{D}_a$ assigns a vector $p(N, D) \in \mathbb{R}^N$ to the players in N. For a player $i \in N$, $p_i(N, D)$ represents the 'power' or 'influence' of player i in (N, D). We call a power measure p positive if $\sum_{j \in N} p_j(N, D) > 0$ for all $(N, D) \in \mathcal{D}$. In this chapter we consider only positive power measures. Let the collection of all positive power measures be denoted by P.

Let $t : W^I \to P$ be the function that assigns to every subgraph-invariant weight function $w \in W^I$ the power measure $p \in P$ that for every acyclic digraph $(N, D) \in \mathcal{D}_a$ is given by $p(N, D) = w(N, D, N)$. We have the following proposition.

Proposition 6.2.12 *t is a bijection.*

PROOF

We show that t is both injective and surjective.

(i) t is injective

Suppose that t is not injective. In that case there exist $w_1, w_2 \in W^I$ such that for some digraph $(N, D) \in \mathcal{D}_a$ and coalition $S \in \Phi^p(N, D)$ it holds that $w_1(N, D, S) \neq w_2(N, D, S)$, but $t(w_1) = t(w_2)$.

Now let $p = t(w_1) = t(w_2)$. Since both w_1 and w_2 are subgraph-invariant, it holds that $w_1(N, D, S) = w_1(S, D(S), S)$ and $w_2(N, D, S) = w_2(S, D(S), S)$. By p we also have $p(S, D(S)) = w_1(S, D(S), S)$ and $p(S, D(S)) = w_2(S, D(S), S)$. Therefore

$w_1(N, D, S) = w_1(S, D(S), S) = p(S, D(S)) = w_2(S, D(S), S) = w_2(N, D, S)$ and we have a contradiction.

(ii) t is surjective

For any power measure $p \in P$ consider the weight function $w \in W^I$ given by $w(N, D, S) = p(S, D(S))$. Clearly it holds that $t(w) = p$.

$\square$

For $p \in P$ we define the *p-hierarchical solution* as the solution that allocates the dividend of a coalition $S \in \Phi^p(N, D)$ among the players in S proportionally to $p(S, D(S))$, the power of the players in S in digraph $D(S)$ according to power measure $p \in P$.

Definition 6.2.13 *For $p \in P$ the p-hierarchical solution is the solution on $\mathcal{G}_{PC}$ given by*

$$H_i^p(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{p_i(S, D(S))}{\sum_{j \in S} p_j(S, D(S))} \Delta_v^D(S) \text{ for all } i \in N.$$

Let the collection of all solutions f for games with precedence constraints such that $f = H^p$ for some $p \in P$, be denoted by Υ. We will refer to the solutions in this class as precedence power solutions. For these solutions dividend allocation of a feasible coalition depends only on the subgraph on that coalition. Allocation by weighted precedence solutions in Ψ^I also depends only on subgraphs. We have the following proposition.

Proposition 6.2.14 *The collection of weighted precedence solutions obtained from a subgraph-invariant weight function is equivalent to the collection of precedence power solutions: $\Psi^I = \Upsilon$.*

PROOF
Consider function t from Proposition 6.2.12. From Definition 6.2.10 and Definition 6.2.13 we have that $f^w = H^{t(w)}$. It follows from bijectivity of t that $\Psi^I = \Upsilon$.

$\square$

From Proposition 6.2.11 and 6.2.14 we obtain the following proposition.

Proposition 6.2.15 *A solution for games with precedence constraints in Υ satisfies irrelevant player independence.*

In order to axiomatize the p-hierarchical solution we introduce the p-strength axiom. This axiom has an interpretation similar to that of hierarchical strength 2 from Theorem 6.2.5. If unanimity among all players must be reached before any non-zero worth can be generated, we might consider the players equals with respect to the game. Therefore worth allocation should depend only on the strength of players in the digraph. The p-hierarchical solution uses the power measure p to measure the strength of each player in the digraph.

p-**strength** For every $N \subseteq \mathbb{N}, (N, D) \in \mathcal{D}_a$ and every $i, j \in N$, it holds that
$$p_i(N, D)f_j(N, u_N, D) = p_j(N, D)f_i(N, u_N, D).$$

The *p*-hierarchical solution is axiomatized by efficiency, linearity, the irrelevant player property, irrelevant player independence and *p*-strength.

Theorem 6.2.16 *A solution for games with precedence constraints is equal to the p-hierarchical solution H^p if and only if it satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence and p-strength.*

PROOF
It is straightforward to show that H^p satisfies efficiency, linearity, the irrelevant player property and *p*-strength. H^p satisfying irrelevant player independence follows from Proposition 6.2.15

The proof of uniqueness follows as in Theorem 6.2.5. □

6.2.5 The hierarchical measure for digraphs

Above we found that the hierarchical solution H^2 is equivalent to the weighted precedence solution $f^{h'}$, where h' is the subgraph-invariant weight function given by $h'(N, D, S) = h(S, D(S), S)$ for $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$. Proposition 6.2.14 implies that there exists a power measure p such that $H^p = f^{h'} = H^2$.[6] We will refer to this power measure as the hierarchical measure η. The following definition of the hierarchical measure follows straightforwardly.

Definition 6.2.17 *The hierarchical measure is the power measure on $\mathcal{D}_a$ given by*

$$\eta_i(N, D) = h'_i(N, D, N) = h_i(N, D, N) = |\Pi^i_D(N)|, i \in N.$$

From here on we will refer to the hierarchical solution H^2 as H^η.
Faigle and Kern (1992) only use the hierarchical strength as a tool to axiomatize the precedence Shapley value H. No motivation is given as to why the hierarchical strength is used, instead of any other measure. Here we motivate the use of power measure η in defining the hierarchical solution H^η by giving an axiomatization of the power measure η on the class of acyclic digraphs.
We first introduce a number of axioms that will be used in the axiomatization of the hierarchical measure for acyclic digraphs.

The first axiom, 1-normalization, states that if digraph (N, D) contains only one player then this player has power one. This property is satisfied by many power measures in the literature.

---

[6]For the hierarchical strength h itself, we cannot find such a corresponding power measure. This follows from the precedence Shapley value not satisfying irrelevant player independence.

1-Normalization For every $D \in \mathcal{D}_a^N$ with $|N| = 1$ and i the unique player in N, it holds that $p_i(N, D) = 1$.

The second axiom, the non-top property, states that players that are not top players in the graph cannot have non-zero power. The digraph is interpreted as a hierarchical structure, where the only players that are free to move on their own, and therefore have power, are players without predecessors. This property is also satisfied by, for example the λ-measure of Borm, van den Brink and Slikker (2002).

Non-top property For every $D \in \mathcal{D}_a^N$ and $i \in N$ such that $P_D(i) \neq \emptyset$, it holds that $p_i(N, D) = 0$.

The third axiom is independence of successors and states that the power of a player does not depend on its successors. This property reflects that the power of a player depends not so much on how many players it is able to dominate, but more on the players that it is dominated by.

For a player $i \in N$ let $out_D(i) := \{(k, l) \in D : k = i\}$ be the set of outgoing arcs from i in digraph D.

Independence of successors For every $D \in \mathcal{D}_a^N$ it holds that $f_i(N, D) = f_i(N, D \setminus out_D(i))$.

Finally, the isolated player property states that the power of an isolated player (i.e. having no successors nor predecessors) is equal to the sum of the powers of all other players in the subgraph without this isolated player. Since isolated players do not have any predecessors, these players might be considered to interact freely with any of the other players in the graph. Since isolated players also do not have any successors, it might be said that their power in the graph comes only from this interaction with other players. The isolated player property reflects that the power of an isolated player depends on the combined strength of the relations it is able to have with any of the other players, where the strength of each relation depends on the powers of the other players in the subgraph without the isolated player.

Isolated player property For every $D \in \mathcal{D}_a^N$ and $i \in N$ such that $F_D(i) \cup P_D(i) = \emptyset$, it holds that $p_i(N, D) = \sum_{j \in N \setminus \{i\}} p_j(N \setminus \{i\}, D_{-i})$.

The four previous axioms characterize the hierarchical measure on acyclic digraphs (N, D).

Now consider $(N, D) \in \mathcal{D}_a$. For any admissible permutation $\pi \in \Pi_D(N)$ and $S \subseteq N$ let $\pi_S \in \Pi(S)$ be such that $\pi_S(i) < \pi_S(j)$ if $\pi(i) < \pi(j), i, j \in S$, i.e. it is the permutation on players in coalition S obtained by considering the relative order of these players in π. Furthermore, for $1 \leq i \leq |N|$, let π_i be $j \in N$ such that $|\{k \in N : \pi(k) \leq \pi(j)\}| = i$, i.e. it is the player preceded by $i - 1$ players in permutation π.

In order to characterize the hierarchical measure by these four axioms, we need the following proposition.

Proposition 6.2.18 *Consider an acyclic digraph* $(N, D) \in \mathcal{D}$*. Let* $P = \{P_1, \ldots, P_m\}$ *be a partition of* N *such that for every pair* P_k, P_l *for* $k, l \in \{1, \ldots, m\}, k \neq l$ *there do not exist* $i \in P_k, j \in P_l$ *such that either* $(i, j) \in D$ *or* $(j, i) \in D$*. Then* $\pi \in \Pi_D(N)$*, if and only if for any* $k \in \{1, \ldots, m\}$ *it holds that* $\pi_{P_k} \in \Pi_{D(P_k)}(P_k)$*.*

The proof follows from the fact that the set of admissible permutations of acyclic digraph (N, D) is determined only by entry of successors before predecessors. Successors are guaranteed to enter before predecessors for those permutations π of N, where for any partition of (N, D) into subgraphs that have no arcs between them, the relative orders in π of players in those subgraphs are admissible permutations of those subgraphs.

Theorem 6.2.19 *A power measure for acyclic digraphs is equal to the hierarchical measure* η *if and only if it satisfies 1-normalization, the non-top property, independence of successors and the isolated player property.*

PROOF
Consider acyclic digraph (N, D). It is straightforward to show that the hierarchical measure η satisfies 1-normalization since the only permutation on $N = \{i\}$ is (i).

Since $j \in P_D(i)$ implies that $\pi(i) < \pi(j)$ for all $\pi \in \Pi_D(N)$, there is no $\pi \in \Pi_D(N)$ such that $\pi(i) = n$, and therefore $\eta_i(N, D) = 0$, showing that η satisfies the non-top property.

Suppose that $j \in F_D(i)$. Since $\Pi_D(N) \subset \Pi_{D\setminus\{(i,j)\}}(N)$ and $\pi \in \Pi_{D\setminus\{(i,j)\}}(N) \setminus \Pi_D(N)$ implies that $\pi(j) > \pi(i)$, it holds that $\eta_i(N, D) = \eta_i(N, D \setminus \{(i, j)\})$. Repeated application for all arcs in $out_D(i)$ shows that η satisfies the independence of successors property.

For an isolated player $i \in N$ there does not exist $j \in N \setminus \{i\}$ such that $(i, j) \in D$ or $(j, i) \in D$. By Proposition 6.2.18 it therefore holds that $\pi \in \Pi_D(N)$ if and only if $\pi_{N\setminus\{i\}} \in \Pi_{D_{-i}}(N \setminus \{i\})$. The number of admissible permutations in $\Pi_D^i(N)$ is therefore equal to the number of possible relative orders $\pi_{N\setminus\{i\}}$ of the players in $N \setminus \{i\}$. It follows that $|\Pi_D^i(N)| = |\Pi_{D_{-i}}(N \setminus \{i\})|$. Furthermore by definition of the hierarchical measure it holds that $|\Pi_{D_{-i}}(N \setminus \{i\})| = \sum_{j \in N\setminus\{i\}} \eta_j(N \setminus \{i\}, D_{-i})$. Therefore $\eta_i(N, D) = |\Pi_D^i(N)| = |\Pi_{D_{-i}}(N \setminus \{i\})| = \sum_{j \in N\setminus\{i\}} \eta_j(N \setminus \{i\}, D_{-i})$ showing that η satisfies the isolated player property.

The proof of uniqueness is given as follows.

Let p be a power measure satisfying the axioms. We perform induction on $|N|$. If $|N| = 1$ then $p_i(\{i\}, D) = 1$ by 1-normalization. Proceeding by induction, assume that $p(N', D')$ is uniquely determined whenever $|N'| < |N|$, and consider $(N, D) \in \mathcal{D}$. If $P_D(i) \neq \emptyset$ then $f_i(N, D) = 0$ follows by the non-top property. Therefore, suppose that $P_D(i) = \emptyset$. Then the independence of successor property implies that $p_i(N, D) = p_i(N, D \setminus out_D(i)) = p_i(N, D_{-i})$ and the isolated player property implies that $p_i(N, D_{-i}) = \sum_{j \in N\setminus\{i\}} p_j(N \setminus \{i\}, D_{-i})$ which is uniquely determined by the induction hypothesis. $\square$

We show logical independence by the following alternative power measures.

1. The power measure that is given by $p_i(N, D) = 0$ for all $(N, D) \in \mathcal{D}_a$ and $i \in N$ satisfies the non-top property, independence of successors and the isolated player property. It does not satisfy 1-normalization.

2. If $D = \emptyset$ then $|\Pi_D(N)| = |N|!$. In $\frac{1}{|N|}$ of these permutations player $i \in N$ is the last player. Ignoring the digraph D and assigning to every player the value equal to the number of permutations of N where it is last yields the power measure that is given by $p_i(N, D) = (|N| - 1)!$ for all $(N, D) \in \mathcal{D}$ and $i \in N$. This power measure satisfies 1-normalization, independence of successors and the isolated player property. It does not satisfy the non-top property.

3. Recall from chapter 2 that $TOP(N, D)$ denotes the set of top players in a directed graph. The power measure given by $p_i(N, D) = (|TOP(N, D)| - 1)!$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies 1-normalization, the non-top property and the isolated player property. It does not satisfy independence of successors.

4. The power measure given by $p_i(N, D) = 1$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies 1-normalization, the non-top property and independence of successors. It does not satisfy the isolated player property.

6.3 Regular set systems and the hierarchical measure

6.3.1 Chains and regular set systems

In the literature the precedence Shapley value has been extended to games associated with combinatorial structures more general than a digraph. Bilbao and Edelman (2000) consider games on convex geometries and Bilbao and Ordoñez (2008) consider games on augmenting systems, where the grand coalition is feasible. Convex geometries have been shown to be contained in the class of so-called regular set systems considered by Lange and Grabisch (2009), who also consider an extension of the precedence Shapley value to games on regular set systems. A set of admissible permutations is generated by considering the so-called chains of feasible sets.

Consider a feasible set system $\mathcal{F} \subseteq 2^N$ for which $\emptyset, N \in \mathcal{F}$. A *chain* in set system $\mathcal{F}$ from $S \in \mathcal{F}$ to $T \in \mathcal{F}$ is an ordered collection $C = (C_0, ..., C_k)$ of feasible sets, such that $C_0 = S$, $C_k = T$ and $C_i \subset C_{i+1}$ for $i \in \{0, \ldots, k-1\}$. A chain $C = (C_0, ..., C_k)$ is maximal when, there exists no set $S \in \mathcal{F}$ such that for some $i \in \{0, \ldots, k-1\}$ it holds that $C_i \subset S \subset C_{i+1}$. The length $l(C)$ of a chain C is defined as the number of sets that it contains. Let the collection of all maximal chains from $\emptyset$ to N be given by $C_{\mathcal{F}}^N$. For a chain $C = (C_0, ..., C_k)$, when $S = C_i$ for some $i \in \{0, \ldots, k\}$, we will say that set S is on chain C. From here on, when we refer to maximal chains, we will mean the maximal chains from $\emptyset$ to N.

Example 6.3.1 *Let $N = \{1, 2, 3, 4, 5\}$ and let $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3, 4\},$*
$\{2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4, 5\}\}$. The collection of maximal chains $C_{\mathcal{F}}^N$ is given by $\{c_1, c_2, c_3\}$,
where $c_1 = (\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}), c_2 = (\emptyset, \{2\}, \{2, 3\}, \{2, 3, 4\},$

$\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\})$ *and* $c_3 = (\emptyset, \{2\}, \{1, 2\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\})$. *Note that* $l(c_1) = 5$ *and* $l(c_3) = 5$, *while* $l(c_2) = 6$.

In this section we consider the hierarchical measure on regular set systems by Honda and Grabisch (2006).

Definition 6.3.2 *A set system* $\mathcal{F} \subseteq 2^N$ *is a regular set system if it satisfies the following axioms:*

*(**feasible empty set**)* $\emptyset \in \mathcal{F}$,

*(**feasible grand coalition**)* $N \in \mathcal{F}$,

*(**regularity property**)* *For every maximal chain* $C \in C_\mathcal{F}^N$ *it holds that* $l(C) = |N| + 1$

We obtain the following lemma with respect to regular set systems.

Lemma 6.3.3 *Let* $\mathcal{F} \subseteq 2^N$ *a regular set system and* $S, T \in \mathcal{F}$ *such that* $T \subset S$. *Then there exists at least one maximal chain* $C = (C_0, ..., C_k) \in C_\mathcal{F}^N$, *such that* S *and* T *are on* C.

The proof follows directly from the regularity property.

Honda and Grabisch (2006) show that all convex geometries are regular set systems. We show that the same holds for augmenting systems $\mathcal{F}$ satisfying $N \in \mathcal{F}$. Recall from Chapter 2 that a feasible set system is an augmenting system if it contains the empty set and satisfies union stability and augmentation 1.

Definition 6.3.4 *A set system* $\mathcal{F} \subseteq 2^N$ *is an augmenting system if it satisfies the following axioms:*

*(**feasible empty set**)* $\emptyset \in \mathcal{F}$,

*(**union stability**)* $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *implies that* $S \cup T \in \mathcal{F}$,

*(**augmentation 1**)* *for* $S, T \in \mathcal{F}$ *with* $S \subset T$, *there exists* $i \in T \backslash S$ *such that* $S \cup \{i\} \in \mathcal{F}$.

We obtain the following proposition, see also Grabisch (2013).

Proposition 6.3.5 *For* $\mathcal{F} \subseteq 2^N$ *an augmenting system satisfying* $N \in \mathcal{F}$, *it holds that* $\mathcal{F}$ *is a regular set system.*

PROOF
Since $\emptyset, N \in \mathcal{F}$ we only have to consider whether $\mathcal{F}$ satisfies the regularity property. Suppose that their exists a maximal chain $C \in C_\mathcal{F}^N$ such that $l(C) < |N| + 1$. Let $(C_0, ..., C_k)$ be this chain. Then for some $m \in \{0, \ldots, k-1\}$ it holds that $|C_{m+1} \backslash C_m| \geq 2$. By maximality of C there does not exist an $S \in \mathcal{F}$ such that $C_m \subset S \subset C_{m+1}$. However since $C_m \subset C_{m+1}$ by augmentation 1 we have that there exists $i \in C_{m+1} \backslash C_m$ such that $C_m \cup \{i\} \in \mathcal{F}$. But then $C_m \subset C_m \cup \{i\} \subset C_{m+1}$ and we obtain a contradiction.

□

Note that in general augmenting systems are not regular set systems, since N need not be feasible, see also Bilbao (2003).

For a feasible set system $\mathcal{F} \subseteq 2^N$ we can generate admissible permutations of the players in N by considering the maximal chains in $C_{\mathcal{F}}^N$. The set of *admissible permutations* $\Pi_{\mathcal{F}}(N)$ associated with a regular set system $\mathcal{F}$ is given by

$$
\Pi_{\mathcal{F}} = \left\{ \pi \in \Pi(N) : (\emptyset, \{\pi_1\}, \bigcup_{i=1}^{2}\{\pi_i\}, ..., \bigcup_{i=1}^{|N|-1}\{\pi_i\}, N) \in C_{\mathcal{F}}^N \right\}, \tag{6.3.3}
$$

where π_i is the player that is at position i in permutation π.

Note that for any feasible set system $\mathcal{F} \subseteq 2^N$ the set of admissible permutations $\Pi_{\mathcal{F}}$ maps $\mathcal{F}$ to the set of permutations on N. We find that this mapping is injective, but not surjective.

Proposition 6.3.6

(i) *For every two regular set systems $\mathcal{E}, \mathcal{F}$ if $\Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}$, then $\mathcal{E} = \mathcal{F}$.*

(ii) *There exist subsets $\Pi \subset \Pi(N)$ for which there is no regular set system $\mathcal{F}$ such that $\Pi_{\mathcal{F}} = \Pi$.*

PROOF

(i) Suppose $\Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}$, but $\mathcal{E} \neq \mathcal{F}$. In that case there exists a set $S \subset N$ such that $S \notin \mathcal{E} \cap \mathcal{F}$, but $S \in \mathcal{E} \cup S \in \mathcal{F}$. Without loss of generality suppose $S \in \mathcal{E}$. For every regular set system it holds that every feasible set is on a maximal chain. Therefore S is on some maximal chain C in $C_{\mathcal{E}}^N$. Let $\pi \in \Pi_{\mathcal{E}}$ be the permutation corresponding to this chain. Since also $\pi \in \Pi_{\mathcal{F}}$ the chain C must also be a chain in $\mathcal{F}$ and therefore $S \in \mathcal{F}$. This is a contradiction and therefore $\mathcal{E} = \mathcal{F}$.

(ii) Consider $N = \{1, 2, 3\}$ and $\Pi = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. If Π is the set of admissible permutations of some feasible set system $\mathcal{F}$ then it must hold that the set of maximal chains $C_{\mathcal{F}}^N$ from $\emptyset$ to N is given by $(\{1\}, \{1, 2\}, \{1, 2, 3\})$, $(\{2\}, \{2, 3\}, \{1, 2, 3\})$, $(\{3\}, \{1, 3\}, \{1, 2, 3\})$. The set system $\mathcal{F}$ is then given by the union of the sets on these chains and therefore $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. However in that case the ordered collection $(\{3\}, \{2, 3\}, \{1, 2, 3\})$ is also a chain in $\mathcal{F}$ and in that case it must hold that $(3, 2, 1) \in \Pi_{\mathcal{F}}$. Since $(3, 2, 1)$ is not in Π it holds that Π cannot be generated as the set of admissible permutations from the chains of any feasible set system.

□

6.3.2   The hierarchical measure for regular set systems

In this section we will study the hierarchical measure in the context of regular set systems. We will denote the class of all regular set systems by $\mathcal{R}$ and those on player set N by $\mathcal{R}^N$.

A *power measure p* for regular set systems is a function that assigns to every regular set system $(N, \mathcal{F}) \in \mathcal{R}$ a vector $p(N, \mathcal{F}) \in \mathbb{R}^N$, where $p_i(N, \mathcal{F})$ represents the 'power' or 'influence' of player i in N. We call a power measure p positive if $\sum_{j \in N} p_j(N, \mathcal{F}) > 0$ for all $(N, \mathcal{F}) \in \mathcal{R}$. In this chapter we consider only positive power measures.

For $i \in N$, the set of permutations $\Pi^i_{\mathcal{F}}(N)$ is defined by

$$\Pi^i_{\mathcal{F}}(N) = \{\pi \in \Pi_{\mathcal{F}}(N) : \pi(i) > \pi(j) \text{ for all } j \in N \setminus \{i\}\}. \tag{6.3.4}$$

This is the collection of those permutations in $\Pi_{\mathcal{F}}(N)$ where i is preceded by the players in $N \setminus \{i\}$.

The *hierarchical measure η* assigns to a player i the number of admissible permutations of regular set system $(N, \mathcal{F})$ such that player i is preceded by the players in $N \setminus \{i\}$.

Definition 6.3.7 *The hierarchical measure is the power measure on $\mathcal{R}$ given by*

$$\eta_i(N, \mathcal{F}) = \left| \Pi^i_{\mathcal{F}}(N) \right| \text{ for all } i \in N.$$

We use the following axioms to characterize the hierarchical measure on regular set systems. First, chain efficiency states that the sum of the power scores of the players is exactly equal to the number of maximal chains.

Chain Efficiency For every $(N, \mathcal{F}) \in \mathcal{R}$ it holds that $\sum_{i \in N} p_i(N, \mathcal{F}) = |C^N_{\mathcal{F}}|$.

The second axiom is similar to the non-top property for acyclic digraphs.

Non-tail property For $(N, \mathcal{F}) \in \mathcal{R}$, if for every maximal chain $C = (C_0, ..., C_{n-1}, C_{|N|}) \in C^N_{\mathcal{F}}$, it holds that $C_{|N|} \setminus C_{|N|-1} \neq \{i\}$, then $p_i(N, \mathcal{F}) = 0$.

The third axiom states that the only chains that affect the power of a player are the maximal chains where it is the last player.

Independence of irrelevant chains For $(N, \mathcal{F}), (N, \mathcal{F}') \in \mathcal{R}$, if $\Pi^i_{\mathcal{F}}(N) = \Pi^i_{\mathcal{F}'}(N)$ then $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}')$.

Now let $\mathcal{F}^i = \{S \in \mathcal{F} : \text{there is a chain } C = (C_0, ..., C_{|N|-1}, C_{|N|}) \text{ in } C^N_{\mathcal{F}}, \text{ such that } S = C_k \text{ for some } k \in \{1, \ldots, |N|\} \text{ and } C_{|N|} \setminus C_{|N|-1} = \{i\}\}$. This is the set system that is obtained from $(N, \mathcal{F})$ by removing exactly the sets that are not on any maximal chain where i only occurs in set N. We obtain the following propositions with respect to $\mathcal{F}^i$.

Proposition 6.3.8 *Let $\mathcal{F} \subseteq 2^N$ be a regular set system with $\Pi^i_{\mathcal{F}}(N) \neq \emptyset$. In that case $\mathcal{F}^i$ is also a regular set system.*

PROOF
Since $\emptyset, N \in \mathcal{F}^i$ we only have to consider whether $\mathcal{F}^i$ satisfies the regularity property. Suppose that their exists a maximal chain $C \in C^N_{\mathcal{F}^i}$ such that $l(C) < |N| + 1$. Let $(C_0, ..., C_k)$ be this chain. It holds that C is also a chain from $\emptyset$ to N in set system $\mathcal{F}$. In $\mathcal{F}$ every maximal chain has length $|N| + 1$. Therefore chain C is not maximal in $\mathcal{F}$. This means that for some $m \in \{0, \ldots, k-1\}$ there exists $S \in \mathcal{F}$ such that $C_m \subset S \subset C_{m+1}$. We distinguish two cases:

(i) $C_{m+1} = N$.
In that case it holds that $|C_m| = l$ for some $l < |N| - 1$. From the fact that $C_m \in \mathcal{F}^i$ it follows that there exists a maximal chain $C' \in C^N_{\mathcal{F}}$ such that $C_m = C'_l$ and $C'_{|N|} \setminus C'_{|N|-1} = \{i\}$. Now consider the set C'_{l+1}. It holds that $C'_{l+1} \in \mathcal{F}^i$ and $C'_{l+1} \neq N$ by $l < |N| - 1$. Since $C_m = C'_l \subset C'_{l+1} \subset N$ it must hold that $C \notin C^N_{\mathcal{F}^i}$ and we obtain a contradiction.

(ii) $C_{m+1} \neq N$.
Since $i \notin C_{m+1}$ we also have $i \notin S$. Now let $|C_{m+1}| = p$. By Lemma 6.3.3 there exists a chain $C' \in C^N_{\mathcal{F}}$ such that both S and C_{m+1} are on C'. From the fact that $C_{m+1} \in \mathcal{F}^i$ it follows that there exists a chain $C'' \in C^N_{\mathcal{F}}$ such that $C_{m+1} = C''_p$ and $C''_{|N|} \setminus C''_{|N|-1} = \{i\}$. From C' and C'' we construct a chain $C''' \in C^N_{\mathcal{F}}$ such that $C'''_{|N|} \setminus C'''_{|N|-1} = \{i\}$ and S is on C''' as follows. Let $C'''_j = C'_j$ for $j < p$, $C'''_p = C''_p = C'_p = C_{m+1}$ and $C'''_j = C''_j$ for $j > p$. It follows that $S \in \mathcal{F}^i$ and we obtain a contradiction.

$\square$

Let $C^N_{\mathcal{F}}|_i \subseteq C^N_{\mathcal{F}}$ be the collection of those maximal chains $C \in C^N_{\mathcal{F}}$ such that $i \notin C_k$ for $k \in \{1, \ldots, |N|-1\}$, i.e. it is the collection of those maximal chains generating admissible permutations where player i is preceded by the other players in N. We have the following proposition.

Proposition 6.3.9 $C^N_{\mathcal{F}}|_i = C^N_{\mathcal{F}^i}$

The proof follows straightforwardly from $\mathcal{F}^i$ containing exactly the sets in $\mathcal{F}$ that are on chains in $C^N_{\mathcal{F}}|_i$.

Theorem 6.3.10 *A power measure on $\mathcal{R}$ is equal to the hierarchical measure η if and only if it satisfies chain efficiency, the non-tail property and independence of irrelevant chains.*

PROOF
It is straightforward to show that the hierarchical measure satisfies these axioms.

The proof of uniqueness is given as follows.
Let p be a power measure satisfying the axioms. Let $(N, \mathcal{F}) \in \mathcal{R}$ and let $i \in N$. If for every maximal chain $C = (C_0, ..., C_{n-1}, C_{|N|}) \in C^N_{\mathcal{F}}$, it holds that $C_{|N|} \setminus C_{|N|-1} \neq \{i\}$,

then by the non-tail property $p_i(N, \mathcal{F}) = 0$. Now assume that $C_{|N|} \setminus C_{|N|-1} = \{i\}$ for at least one maximal chain in $C_{\mathcal{F}}^N$. We consider the regular set system $\mathcal{F}^i$. The non-tail property implies that $p_j(N, \mathcal{F}^i) = 0$ for all $j \in N \setminus \{i\}$. Then chain efficiency implies that $p_i(N, \mathcal{F}^i) = |\mathcal{F}^i|$. Since $C_{\mathcal{F}}^N|_i = C_{\mathcal{F}^i}^N$ we can apply independence of irrelevant chains to determine that $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}^i) = |C_{\mathcal{F}^i}^N|$ is uniquely determined. $\qquad\square$

We show logical independence by the following alternative power measures.

1. The power measure that is given by $p_i(N, \mathcal{F}) = 0$, $i \in N$ satisfies the the non-tail property and independence of irrelevant chains. It does not satisfy chain efficiency.

2. Consider the following power measure p. If $|N| = 1$, then $p(N, \mathcal{F}) = \eta(N, \mathcal{F})$. If $|N| > 1$ and players $1, 2 \in N$, then $p_1(N, \mathcal{F}) = \eta_1(N, \mathcal{F}) - 1$, $p_2(N, \mathcal{F}) = \eta_2(N, \mathcal{F}) + 1$ and $p_i(N, \mathcal{F}) = \eta_i(N, \mathcal{F})$ for $i \in N \setminus \{1, 2\}$. This power measure satisfies chain efficiency and independence of irrelevant chains. It does not satisfy the non-tail property.

3. Let $E = \{i \in N : C_{\mathcal{F}}^N|_i = \emptyset\}$ be the set of those players that are never the last player in a maximal chain. The power measure that is given by $p_i(N, \mathcal{F}) = 0$ for $i \in E$ and $p_i(N, \mathcal{F}) = \frac{|C_{\mathcal{F}}^N|}{|N \setminus E|}$ otherwise satisfies chain efficiency and the non-tail property. It does not satisfy independence of irrelevant chains.

A different axiomatization of the hierarchical measure for regular set systems is obtained by replacing the non-tail property by non-negativity.

Non-negativity For every $(N, \mathcal{F}) \in \mathcal{R}$ it holds that $p_i(N, \mathcal{F}) \geq 0$ for all $i \in N$.

A power measure on $\mathcal{R}$ that satisfies chain efficiency, non-negativity and independence of irrelevant chains also satisfies the non-tail property.

Proposition 6.3.11 *Let p be a power measure on $\mathcal{R}$ that satisfies chain efficiency, non-negativity and independence of irrelevant chains. Then p also satisfies the non-tail property.*

PROOF

For regular set system $(N, \mathcal{F})$ let $\mu_{\mathcal{F}} = |C_{\mathcal{F}}^N|$ be the total number of maximal chains. By independence of irrelevant chains, for $i \in N$ it holds that for every regular set system $(N, \mathcal{F})$ in $\mathcal{R}^N$ such that $C_{\mathcal{F}}^N|_i = \emptyset$, there exists a number $c_i \in \mathbb{R}$ such that $p_i(N, \mathcal{F}) = c_i$. For arbitrary $i \in N$ it holds for $j \in N \setminus \{i\}$ that $C_{\mathcal{F}^i}^N|_j = \emptyset$ and therefore $p_j(N, \mathcal{F}^i) = c_j$. By chain efficiency we therefore have $f_i(N, \mathcal{F}^i) = \mu_{\mathcal{F}^i} - \sum_{j \in N \setminus \{i\}} c_j$. Let $E = \{i \in N : C_{\mathcal{F}}^N|_i = \emptyset\}$ be the set of those players that are never the last player in a maximal chain. By independence of irrelevant chains, we also have $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}^i)$ if $i \in N \setminus E$. It follows that $p_i(N, \mathcal{F}) = \mu_{\mathcal{F}^i} - \sum_{j \in N \setminus \{i\}} c_j$. By chain efficiency we have $\mu_{\mathcal{F}} = \sum_{j \in N} p_j(N, \mathcal{F}) = \sum_{j \in E} c_j + \sum_{j \in N \setminus E}[\mu_{\mathcal{F}^j} - \sum_{k \in N \setminus \{j\}} c_k] = \sum_{j \in N \setminus E} \mu_{\mathcal{F}^j} - (|N \setminus E| - 1) \sum_{j \in N} c_j$. Since $\mu_{\mathcal{F}} = \sum_{j \in N \setminus E} \mu_{\mathcal{F}^j}$ it holds that $(|N \setminus E| - 1) \sum_{j \in N} c_j = 0$. By

non-negativity we therefore have that $c_j = 0$ for $j \in N$. It follows that p also satisfies the non-tail property.

$\square$

We obtain the following corollary.

Corollary 6.3.12 *A power measure on $\mathcal{R}$ is equal to the hierarchical measure η if and only if it satisfies chain efficiency, non-negativity and independence of irrelevant chains.*

Finally we note that Lange and Grabisch (2009) also considered games on regular set systems. New solutions for these so-called regular games, based on power measures, can be obtained similar to the p-hierarchical solutions for games with precedence constraints.

6.3.3 The Plurality and Borda measure

Note that in order to calculate the hierarchical measure for regular set systems, it is sufficient to know the set of admissible permutations. Instead of generating admissible permutations from some feasible set system, any set of permutations on the set of players can be used as the set of admissible permutations. The hierarchical measure can be seen to rank players based on these permutations. An example where we encounter the ranking of players from permutations comes from social choice theory. The 'permutations of the players' in this case are the preferences of the voters on the alternatives they can choose from if they all have a different preference ordering. Here the absolute hierarchical measure can be seen as the plurality scoring rule that assigns an alternative a score of 1 for any permutation that it leads (where it's the most preferred alternative).

The other way around, we can apply social choice theory to define new measures as alternatives for the hierarchical measure. A *social choice situation* is described by a triple (V, N, p) where V is the set of voters, N is the set of alternatives and $p = (p_k)_{k \in V}$ is a preference profile. A preference profile $p = (p_k)_{k \in V}$ consists of a preference relation p_k for every voter $k \in V$, being a weak order on the set of alternatives N. We denote the collection of all social choice situations by $\mathcal{S}$.[7] Two main questions social choice theory tries to answer for each social choice situation are (i) what can be considered the 'socially best' alternatives, and (ii) how to aggregate the individual preferences into one 'social preference relation'. The first question is dealt with by considering so-called social choice functions which assign to every social choice situation a subset of the alternatives that can be considered the 'social choice'. The second is dealt with by considering social welfare functions which assign a preference relation on the set of alternatives to every social choice situation. A specific class of social choice functions and social welfare functions are those based on a *scoring method* being a function $\sigma \colon \mathcal{S} \to \bigcup_{K \subseteq \mathbb{N}} \mathbb{R}^K$ such that $\sigma(V, N, p) \in \mathbb{R}^N$ for every $(V, N, p) \in \mathcal{S}$, which assigns a real number, its score, to every alternative in a social choice situation. As the social choice one can simply take the alternatives with the highest score, and one can define a social preference profile simply by ordering the alternatives in non-increasing order of their score. Given scoring method σ, in this chapter we consider the social welfare function that assigns to social choice situation (V, N, p) the

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[7]For the application we describe in this section the set of voters must be variable.

weak order $\succeq^\sigma$ given by $i \succeq^\sigma j$ if and only if $\sigma_i(V, N, p) \geq \sigma_j(V, N, P)$, where $i \succeq^\sigma j$ can be interpreted as 'i is at least as good as j'.[8]

Famous ranking methods are based on the plurality score and the Borda score. The *plurality score* of an alternative is simply the number of preference relations in the preference profile where it is ranked highest, i.e. $\sigma_i^{plur}(V, N, p) = \#\{k \in V : i \ p_k \ j \text{ for all } j \in N\}$. A main disadvantage of the plurality ranking method is that it only looks at the best alternative for every voter, but does not take into account the rest of the preference profile. For example, an alternative that is second best for every voter might be a 'good' social choice, but it will have the lowest plurality score (zero). Alternatively, one can use the *Borda score* which assigns for every voter $|N| - 1$ points to the best alternative, $n - 2$ points to the second best alternative, and so on to zero points for the worst alternative, i.e. $\sigma_i^{Borda}(V, N, p) = \sum_{k \in V}(\#\{j \in N : i \ p_k \ j\} - 1)$.[9]

Now, given a regular set system, we can consider the set of all admissible permutations (derived from the chains) as a preference profile with the nodes (players) as the alternatives. It does not matter what the set of voters is in this application, as long as there as many voters as maximal chains. Then we can define the *plurality measure* of player $i \in N$ in regular set system $\mathcal{F}$ as $\widehat{\sigma}_i^{plur}(N, \mathcal{F}) = \sigma_i^{plur}(V, N, p^\mathcal{F})$, where $(V, N, p^\mathcal{F})$ is the social choice situation derived from $\mathcal{F}$ with $V = C_\mathcal{F}^N$ and for every $k = (\emptyset, C_1, C_2, \ldots, C_n) \in C_\mathcal{F}^N$, the preference profile p_k is given by $i \ p_k \ j$ if and only if $i \in T_l \Rightarrow j \in T_l$. Obviously, since the plurality score just counts the number of preference profiles in which an alternative is ranked highest, applied in this way the plurality score of a player in a regular set system is the number of chains in which it enters last, i.e. its score according to the hierarchical measure.

Proposition 6.3.13 *If $\mathcal{F} \subseteq 2^N$ is a regular set system then $\eta_i(N, \mathcal{F}) = \widehat{\sigma}^{plur}(N, \mathcal{F})$ for all $i \in N$.*

In particular this gives an alternative definition of the hierarchical measure.

As a different measure we can use the Borda score for regular set systems; also taking into account positions in the chain other than the last one. We define the *Borda measure* of player $i \in N$ in regular set system $\mathcal{F} \subseteq 2^N$ as $\widehat{\sigma}^{Borda}(N, \mathcal{F}) = \sigma_i^{Borda}(V, N, p^\mathcal{F})$.

Other scoring methods from social choice theory (or multiple criteria decision making) can also be applied.

6.4 The normalized hierarchical measure for forests and sink forests

The *normalized hierarchical measure* $\overline{\eta}$ assigns to a player i the fraction of permutations in $\Pi_D(N)$ where i is preceded by the players in $N \setminus \{i\}$.

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[8]Although $\succeq^\sigma$ is a preference relation, just as p_k, $k \in V$, we use a different notation to stress that $\succeq^\sigma$ represents the social preference.

[9]Both the plurality score as well as the Borda score are special cases of the class of scoring methods where for a fixed set of r alternatives scoring numbers s_r, $r \in \{1, \ldots, n\}$ with $s_r \geq s_l$ if $r < l$, are given, and for every voter the best alternative gets s_1 points, the second best alternative gets s_2 points, and so on.

Definition 6.4.1 *The normalized hierarchical measure* $\overline{\eta}$ *is the power measure on* $\mathcal{D}_a$ *given by*

$$\overline{\eta}_i(N, D) = \frac{\eta_i(N, D)}{\sum_{j \in N} \eta_j(N, D)} \ \ \textit{for all } i \in N.$$

In this section we consider the normalized hierarchical measure $\overline{\eta}$ for two special classes of digraphs, namely rooted trees and sink trees. For the definition of a rooted tree see Chapter 2. A digraph (N, D) is a *sink tree* if the digraph (N, D^-) with $D^- = \{(i, j) \in N \times N : (j, i) \in D\}$ is a rooted tree. In other words, a digraph (N, D) is a sink tree if and only if there is an $i_s \in N$ such that (i) $F_D(i_s) = \emptyset$, (ii) $\widehat{P}_D(i_s) = N \setminus \{i_s\}$, and (iii) $|F_D(i)| = 1$ for all $i \in N \setminus \{i_s\}$. A digraph (N, D) is a line graph, when it is both a rooted tree and a sink tree. Rooted trees and sink trees are often encountered in the economic and OR literature. For example, in the literature on cooperative river water allocation games initiated by Ambec and Sprumont (2002).

Next, we consider the classes of digraphs where every component is a rooted tree, respectively the class of digraphs where every component is a sink tree. The first type of digraph is also known as a forest, the second as a sink forest. We denote the set of forest digraphs by $\mathcal{D}_R$ and the set of sink forest digraphs by $\mathcal{D}_S$.

6.4.1 An axiomatization of the normalized hierarchical measure for forests

From the axioms discussed in subsection 6.2.5, the normalized hierarchical measure satisfies the non-top property and 1-normalization. It satisfies an even stronger version of 1-normalization that normalizes the total power of all players to one for every digraph. Note that this property also holds on the class of all acyclic digraphs, although we will use it only on forests and sink forests.

Normalization For every $(N, D) \in \mathcal{D}_R$, it holds that $\sum_{i \in N} p_i(N, D) = 1$.

The normalized hierarchical measure satisfies the even stronger property that the cumulative power of the players in any one component is equal to the fraction of players in that component, i.e. when a component contains $|K|$ players this cumulative power is equal to $\frac{|K|}{|N|}$. Recall from Chapter 2 that the set of all components of a digraph D is given by $C_D(N)$.

Component normalization For every $(N, D) \in \mathcal{D}_R$, if $K \in C_D(N)$, we have

$$\sum_{i \in K} p_i(N, D) = \frac{|K|}{|N|}.$$

Note that component normalization implies normalization. In the axiomatization of the normalized hierarchical measure that follows, we will use component normalization. At the end of this subsection we remark on an axiomatization that uses normalization.

Before we continue with the axiomatization we need the following propositions.

Proposition 6.4.2 *Consider an digraph $(N, D) \in \mathcal{D}_a$ and let $K \in C_D(N)$. Now let permutation $\pi' \in \Pi_{D(K)}(K)$ and permutation $\pi'' \in \Pi_{D(N \setminus K)}(N \setminus K)$. The total number of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$ is given by $\binom{|N|}{|K|}$*

PROOF
Consider the vector $x = (x_1, \ldots, x_{|N \setminus K|+1}) \in \mathbb{N}^{|N \setminus K|+1}$, where x_1 represents the number of players in K that precede player π''_1 in permutation π, $x_{|N \setminus K|+1}$ represents the number of players in K that are preceded by player $\pi''_{|N \setminus K|}$ in permutation π and finally (if $|N \setminus K| > 1$) for $1 < i < |N \setminus K|+1$, x_i represents the number of players in K that are preceded by player π''_{i-1} in permutation π, but precede player π''_i. The number of permutations π satisfying $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$ is now equal to the number of solutions to $x_1 + \ldots + x_{|N \setminus K|+1} = |K|$, where $0 \leq x_i \leq |K|$ for $i \in \{1, \ldots, |N \setminus K| + 1\}$. From combinatorics we obtain that this number is equal to $\binom{|N|}{|K|}$.

$\square$

Next we consider the fraction of these permutations, where a player from K enters last.

Proposition 6.4.3 *Consider a digraph $(N, D) \in \mathcal{D}_a$ and let $K \in C_D(N)$. Now let permutation $\pi' \in \Pi_{D(K)}(K)$ and permutation $\pi'' \in \Pi_{D(N \setminus K)}(N \setminus K)$. The fraction of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$, and where $\pi_{|N|} \in K$ is given by $\frac{|K|}{|N|}$.*

PROOF
From Proposition 6.4.2 we obtain that the total number of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$ is given by $\binom{|N|}{|K|}$. To calculate the number of permutations with a player from K at the end, we can use a combinatorial argument similar to that from the proof of Proposition 6.4.2. Since the position of one of the players in K is now fixed (after any of the players in $N \setminus K$), we only have to know the number of ways to position the other $|K| - 1$ players in K relative to those in $N \setminus K$ to fully determine $\pi(i)$ for any player $i \in N$. Following the proof of Proposition 6.4.2, the total number of permutations such that a player from K is at the end is therefore given by the number of solutions to $x_1 + \ldots + x_{|N \setminus K|+1} = |K| - 1$, where $0 \leq x_i \leq |K| - 1$ for $i \in \{1, \ldots, |N \setminus K| + 1\}$. This number is equal to $\binom{|N|-1}{|K|-1}$. Therefore the fraction of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$, and where a player from K enters last is given by $\frac{\binom{|N|-1}{|K|-1}}{\binom{|N|}{|K|}} = \frac{|K|}{|N|}$

$\square$

For any specific pair of permutations $\pi' \in \Pi_{D(K)}(K)$ and $\pi'' \in \Pi_{D(N \setminus K)}(N \setminus K)$, by Proposition 6.4.3 we know that the fraction of permutations in $\Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$ and where a player from component K enters last is given by $\frac{|K|}{|N|}$. At the same time by Proposition 6.2.18, we know that for any permutation $\pi \in \Pi_D(N)$, there exist $\pi' \in \Pi_{D(K)}(K)$ and $\pi'' \in \Pi_{D(N \setminus K)}(N \setminus K)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$. The following proposition therefore follows immediately from these Propositions.

Proposition 6.4.4 *Consider a digraph $(N, D) \in \mathcal{D}_a$ and let $K \in C_D(N)$. The number of permutations $\pi \in \Pi_D(N)$, such that $\pi \in \Pi_D^i(N)$ for some $i \in K$, is given by $\frac{|K|}{|N|}$.*

Next we provide an axiomatization of the normalized hierarchical measure for forest digraphs. The non-top property used in this axiomatization is adapted in a straightforward way from the class of all acyclic digraphs to the class of forests $\mathcal{D}_R$, meaning that players with predecessors are assigned 0 power.

Theorem 6.4.5 *A power measure on $\mathcal{D}_R$ is equal to the normalized hierarchical measure $\overline{\eta}$ if and only if it satisfies component normalization and the non-top property.*

PROOF

The normalized hierarchical measure $\overline{\eta}$ satisfying component normalization follows straightforwardly from Proposition 6.4.4. The non-top property follows immediately from the absolute hierarchical measure η satisfying the non-top property.

The proof of uniqueness is given as follows.

Let p be a power measure satisfying the axioms. In a forest the top players in $TOP(N, D)$ are the roots of the tree components. The non-top property implies that $p_i(N, D) = 0$ for all $i \in N \setminus TOP(N, D)$. Since every component has a unique root, component normalization then uniquely determines the values for the players in $TOP(N, D)$.
□

We show logical independence by the following alternative power measures.

1. The absolute hierarchical measure η satisfies the non-top property on $\mathcal{D}_R$. It does not satisfy component normalization on $\mathcal{D}_R$.

2. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies component normalization on $\mathcal{D}_R$. It does not satisfy the non-top property on $\mathcal{D}_R$.

In the above axiomatization we used component normalization instead of the weaker normalization. Another weak version of component normalization requires only that the cumulative power of any one component is assigned proportionally to the number of players in that component.

Component comparability For every pair of components $K, K' \in C_D(N)$ of $(N, D) \in \mathcal{D}_R$, we have $\frac{\sum_{i \in K} p_i(N,D)}{\sum_{i \in K'} p_i(N,D)} = \frac{|K|}{|K'|}$.

It is straightforward to see that together normalization and component comparability are equivalent to component normalization.

Proposition 6.4.6 *A power measure satisfies component normalization if and only if it satisfies normalization and component comparability.*

Therefore, in the above axiomatization (and also in the one from the next subsection) it is possible to replace component normalization by normalization and component comparability.

6.4.2 An axiomatization of the normalized hierarchical measure for sink forests

Next, we consider the normalized hierarchical measure on the class of sink forests (i.e. digraphs whose components are sink trees).

The axioms of component normalization and the non-top property from Theorem 6.4.5 are adapted in a straightforward way to the class of sink forests $\mathcal{D}_S$, meaning that the cumulative power of the players in any one component is equal to the fraction of players in that component and that players with predecessors are assigned 0 power respectively. Together these axioms are not sufficient to axiomatize the normalized hierarchical measure on the class of sink forests. Therefore in addition we consider equal dependence on bottom players. This axiom states that if within some component a player is a subordinate to all the other players in that component (we refer to such a player as a bottom player), then deleting this player from the digraph does not change the power ratios of the top players in that component (who now belong to different components). This property reflects that removing a bottom player from a component should have the same effect on all top players in the same component, since these top players all dominate the bottom player.

Equal dependence on bottom players For every $(N, D) \in \mathcal{D}_S$, $K \in C_D(N)$ such that $|TOP(N, D) \cap K| \geq 2$, $i, j \in TOP(N, D) \cap K$ such that $i \neq j$ and $h \in K$ such that $F_D(h) = \emptyset$ (and therefore $\widehat{P}_D(h) = K \setminus \{h\}$), it holds that $\frac{p_i(N,D)}{p_j(N,D)} = \frac{p_i(N\setminus\{h\},D_{-h})}{p_j(N\setminus\{h\},D_{-h})}$.

Note that $(N \setminus \{h\}, D_{-h})$ is a sink forest if (N, D) is a sink forest and $\widehat{P}_D(h) = K \setminus \{h\}$ for some $K \in C_D(N)$.

Theorem 6.4.7 *A power measure on $\mathcal{D}_S$ is equal to the normalized hierarchical measure $\overline{\eta}$ if and only if it satisfies component normalization, the non-top property and equal dependence on bottom players.*

PROOF
The normalized hierarchical measure $\overline{\eta}$ satisfying component normalization follows straightforwardly from Proposition 6.4.4. The normalized hierarchical measure $\overline{\eta}$ satisfying the non-top property follows immediately from the absolute hierarchical measure η satisfying the non-top property.

To show that the normalized hierarchical measure $\overline{\eta}$ satisfies equal dependence on bottom players, we reason as follows.

Let $K \in C_D(N)$, $i, j \in TOP(N, D) \cap K$ and $h \in K$ such that $F_D(h) = \emptyset$ (and so $\widehat{P}_D(h) = K \setminus \{h\}$). Equal dependence on bottom players is satisfied if the ratio of the admissible permutations in $(N \setminus \{h\}, D_{-h})$ where i is at the end and those where j is at the end, is the same as in (N, D).

First we consider the digraph given by $(K, D(K))$. Since $F_D(h) = \emptyset$, none of the precedence relations between the players in $K \setminus \{h\}$ are changed if h is deleted from $(K, D(K))$. It follows that $\Pi_{D(K)_{-h}}(K\setminus\{h\}) = \{\pi' \in \Pi(K\setminus\{h\}) : \pi' = \pi_{K\setminus\{h\}}$ for some $\pi \in \Pi_{D(K)}(K)\}$. Since $\widehat{P}_D(h) = K\setminus\{h\}$, in any admissible permutation of $(K, D(K))$ the players in $K\setminus\{h\}$ are preceded by player h. Therefore any permutation $\pi' \in \Pi_{D(K)_{-h}}(K\setminus\{h\})$

corresponds to exactly one permutation $\pi \in \Pi_{D(K)}(K)$ such that $\pi' = \pi_{K \setminus \{h\}}$. This permutation is given by $\pi = (h, \pi'_1, \ldots, \pi'_{|K|})$. Since in both π and π' the same player is at the end, we obtain that the total number of permutations where a top player $i \in TOP(N, D) \cap K$ is at the end is the same for both $(K, D(K))$ and $(K \setminus \{h\}, D_{-h}(K))$. It follows for i, j and digraph $(K, D(K))$, that the ratio of the admissible permutations where i is at the end and those where j is at the end, stays the same if h is deleted from $D(K)$.

Next we consider the admissible permutations in (N, D) and $(N \setminus \{h\}, D_{-h})$. A partition of N into sets that are not connected through arcs in D is given by $\{N \setminus K, K\}$. A partition of $N \setminus \{h\}$ in (possibly) unconnected subsets in D_{-h} is given by $\{N \setminus K, K \setminus \{h\}\}$. From Proposition 6.2.18 we obtain that for $\pi \in \Pi_D(N)$, it holds that $\pi_K \in \Pi_{D(K)}(K)$. From the same Proposition we also obtain that for $\pi' \in \Pi_{D_{-h}}(N \setminus \{h\})$, it holds that $\pi_{K \setminus \{h\}} \in \Pi_{D_{-h}(K \setminus \{h\})}(K \setminus \{h\})$. From the result on $(K, D(K))$ it now follows straightforwardly that for a pair of top players $i, j \in TOP(N, D) \cap K$, the ratio of admissible permutations, where one player is at the end, does not change if h is deleted from D. This shows that $\overline{\eta}$ satisfies equal dependence on bottom players.

The proof of uniqueness is given as follows.

Let p be a power measure satisfying the axioms. Let K be a component in (N, D). We perform induction on $|K|$. If $|K| = 1$ then $p_i(\{i\}, D) = \frac{1}{|N|}$ by component normalization. Proceeding by induction, assume that $p_i(N, D')$ is uniquely determined for all $i \in K'$ whenever $|K'| < |K|$, with $p_i(N, D) > 0$ if $P_D(i) = \emptyset$. By component normalization, it holds that $\sum_{i \in K} p_i(N, D) = \frac{|K|}{|N|}$. If $P_D(i) \neq \emptyset$ then $p_i(N, D) = 0$ follows by the non-top property. Therefore, $\sum_{i \in TOP(K, D(K))} p_i(N, D) = \frac{|K|}{|N|}$. By equal dependence on bottom players, $\frac{p_i(N, D)}{p_j(N, D)} = \frac{p_i(N \setminus \{h\}, D_{-h})}{p_j(N \setminus \{h\}, D_{-h})}$, for all $i, j \in TOP(N, D) \cap K$ and $h \in K$ such that $F_D(h) = \emptyset$ (and therefore $\widehat{P}_D(h) = K \setminus \{h\}$). (Note that every component in a sink forest has exactly one such a player h.) Since $p_i(N \setminus \{h\}, D_{-h}) p_j(N \setminus \{h\}, D_{-h}) > 0$ by the induction hypothesis, this implies that the values $p_i(N, D)$, $i \in TOP(N, D) \cap K$, are uniquely determined. $\square$

We show logical independence by the following alternative power measures.

1. The absolute hierarchical measure η satisfies the non-top property and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy component normalization on $\mathcal{D}_S$.

2. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies component normalization and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy the non-top property on $\mathcal{D}_S$.

3. Let K_i be the component in (N, D) containing player i. Let ω assign to every digraph the exogenously given vector of weights $\omega(N, D) \in \mathbb{R}^N_{++}$, where $\omega_i(N, D) > 0$, $i \in N$. The weighted hierarchical measure h^ω given by
$$\eta_i^\omega(N, D) = \frac{\omega_i(N, D)}{\sum_{j \in TOP(K_i, D(K_i))} \omega_j(N, D)} \sum_{j \in TOP(K_i, D(K_i))} \overline{\eta}_j(N, D) \text{ if } P_D(i) = \emptyset, \text{ and}$$

$p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies component normalization and the non-top property on $\mathcal{D}_S$. It does not satisfy equal dependence on bottom players on $\mathcal{D}_S$.

Alternative normalizations and power measures for sink forests

In the literature on power measures it has been shown that just applying a different normalization can have an important impact.[10] In this subsection we consider two alternative versions of component normalization, and show that together with the non-top property and equal dependence on bottom players, these characterize other power measures for sink forests. The first alternative to component normalization requires that the cumulative power of the players in a component is the same for each component.

Component normalization 2 Let $(N, D) \in \mathcal{D}_S$. For $K \in C_D(N)$, it holds that $\sum_{i \in K} p_i(N, D) = \frac{1}{|C_D(N)|}$.

The second alternative requires that the cumulative power of the players in a component is equal to the share of top players in that component.

Component normalization 3 Let $(N, D) \in \mathcal{D}_S$. For $K \in C_D(N)$, it holds that $\sum_{i \in K} p_i(N, D) = \frac{|TOP(K, D(K))|}{|TOP(N, D)|}$.

Next, we show what power measures for sink forests are characterized by replacing component normalization in Theorem 6.4.7 by one of the above two alternatives.[11]

Theorem 6.4.8 *(i) A power measure for sink forests satisfies component normalization 2, the non-top property and equal dependence on bottom players if and only if it is the power measure p^2 given by*

$$p^2(N, D) = \begin{cases} \frac{1}{|C_D(N)| \cdot \prod_{j \in \widehat{F}_D(i)} |P_D(j)|} & \text{if } P_D(i) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

(ii) A power measure for sink forests satisfies component normalization 3, the non-top property and equal dependence on bottom players if and only if it is the power measure p^3 given by

$$p^3(N, D) = \begin{cases} \frac{1}{|TOP(N, D)|} & \text{if } P_D(i) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

---

[10]For example, van den Brink and Gilles (2000) provide axiomatizations of the outdegree (score) and β- measures that differ only in the normalization that is used, implying that a different normalization can lead to a different ranking of, for example, teams in a sports competition. In van den Brink, Rusinowska and Steffen (2012) axiomatizations of a power and satisfaction score in a (voting) model with opinion leaders are given differing only in the normalization that is applied.

[11]We can also define the axioms of component normalization 2 and component normalization 3 on the class of forest digraphs. It is straightforward to show that together with the non-top property both axioms characterize the solution that assigns a power of $p_i(N, D) = \frac{1}{|C_D(N)|} = \frac{|TOP(K, D(K))|}{|TOP(N, D)|} = \frac{1}{|TOP(N, D)|}$ if $i \in N$ is a top player and $p_i(N, D) = 0$ otherwise.

PROOF
It is straightforward to verify that the power measures p^2 and p^3 satisfy the corresponding axioms. Uniqueness follows as in the proof of Theorem 6.4.7, replacing component normalization however by component normalization 2 and component normalization 3 respectively. □

Together with the non-top property and equal dependence on bottom players, component normalization 2 yields the power measure p^2 where the power ratio between two top players equals the product of the number of predecessors of each of their subordinates. Considering a 'flow' argument, this power measure can be described as follows. Suppose you start a random walk at the sink and walk through the network along the edges to one of the top players. At every non-top player you select one of the arcs to its predecessors with equal probability and continue your walk along this arc. In that case p^2 describes the probability of ending up at each top player.[12]

Together with the non-top property and equal dependence on bottom players, component normalization 3 yields the power measure p^3 that distributes the power equally over the top players in the digraph.

We illustrate the three component normalizations and corresponding power measures discussed here with an example.

Example 6.4.9 *Consider the sink tree (N, D) with $N = \{1, 2, 3, 4, 5\}$ and $D = \{(1,3), (2,3), (3,5), (4,5)\}$. Then the normalized hierarchical measure $\overline{\eta}(N, D)$, $p^2(N, D)$ and $p^3(N, D)$, respectively, are given by*

$$\overline{\eta}(N, D) = (\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{4}, 0),$$

$$p^2(N, D) = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}, 0), and$$

$$p^3(N, D) = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0).$$

An application of power measures for sink forest to river games

We end this section by applying the hierarchical measure and the other power measures discussed in this section for sink forests to the river games mentioned before. These games are introduced by Ambec and Sprumont (2002) for rivers with a single spring and a single source. They consider river water allocation problems (N, e, b) where agents are located along a single-stream river. For agents $i, j \in N$, if $i < j$, then i is situated upstream of j along the river. There is a nonnegative water inflow $e_i \geq 0$ at the territory of every agent $i \in N$ and every agent is assumed to have quasi-linear preferences over river water and money. The benefit of consuming an amount of water is given by a differentiable, strictly increasing, and strictly concave benefit function $b_i \colon \mathbb{R}_+ \to \mathbb{R}$, where $b_i(x_i)$ is the benefit

---

[12]Alternatively, suppose that a unit of water flows from the sink to the top players in such a way that at each non-top player the water stream splits into multiple streams of equal amounts, for every arc to a predecessor. Then p^2 describes the expected amount of water that arrives at the top players.

of agent i of consuming an amount x_i of water. In addition, some other assumptions are considered. An allocation of the river water among the agents is efficient when it maximizes the total sum of benefits. Every water allocation and transfer schedule yields a welfare distribution, where the utility of an agent is equal to its benefit from water consumption plus its monetary transfer, which can be negative. Ambec and Sprumont (2002) derive a cooperative river game (N, v) where the worth of every coalition of agents that is connected along the river, equals the welfare to these agents when they optimally allocate (i.e. maximize the sum of their individual benefits) the water inflow in their own countries among each other (under the condition that water can be sent from upstream to downstream agents but not the other way around). The so-called *downstream incremental solution* they introduce is given by the marginal vector of the river game where upstream agents precede downstream agents. As noted by van den Brink, van der Laan and Vasil'ev (2007) this means that all the surplus of cooperation of a connected coalition is allocated to its most downstream agent. An alternative proposed by van den Brink, van der Laan and Vasil'ev (2007) and Ambec and Ehlers (2008) for single stream rivers is given by the so-called *upstream incremental solution*. This solution is given by the marginal vector where downstream agents precede upstream agents, and the surplus of cooperation is allocated to the upstream agents.

More general river structures than single stream rivers have also been considered in the literature. For these structures the river can no longer be inferred from the set of agents, so it is explicitly modeled by a digraph. Khmelnitskaya (2010) and van den Brink, van der Laan and Moes (2012) consider more general river structures that can have multiple springs, but still only a single sink.[13] Whereas it is straightforward to generalize the downstream incremental solution to multiple spring rivers (since all marginal vectors where upstream agents precede downstream agents are the same), it is less obvious how to generalize the upstream incremental solution to multiple spring rivers.

In van den Brink, van der Laan and Moes (2012) a class of solutions for multiple spring rivers, that contains the downstream incremental solution, is axiomatized. This is the class of weighted hierarchical solutions, which correspond to convex combinations of the hierarchical outcomes introduced by Demange (2004), see (2.2.10).

A multiple spring river problem can be modeled by a quadruple (N, e, b, D), where N represents a set of agents, $e_i \geq 0$ is the water inflow at agent i, b_i is the benefit function of agent i and $(N, D) \in \mathcal{D}_S$ represents a multiple spring river. If $(i, j) \in D$, then the river flows from agent i to agent j. Let $\mathcal{G}_{msr}^{(N,D)}$ be the class of cooperative TU-games that can be obtained from a multiple spring river problem on multiple spring river (N, D). Games from this class are obtained similar to the river games of Ambec and Sprumont (2002). The hierarchical outcome corresponding to player i for a river game $(N, v) \in \mathcal{G}_{msr}^{(N,D)}$ is given by the hierarchical outcome $t^i(N, v, L_D)$ of the communication graph game (N, v, L_D) (recall from Chapter 2 that $(N, L_D) \in \mathcal{L}$ is the undirected graph, where L_D is given by $\{\{i, j\} : (i, j) \in D$ or $(j, i) \in D\}$).

---

[13]Khmelnitskaya (2010) also considers rivers with a single spring but multiple sinks, whereas van den Brink, van der Laan and Moes (2012), like Ambec and Ehlers (2008), allow for more general benefit functions where the agents can be satiable.

Definition 6.4.10 *A solution* f *on* $\mathcal{G}_{msr}^{(N,D)}$ *is a weighted hierarchical solution if there exists an* $\alpha \in \mathbb{R}_{+}^{N}$ *with* $\sum_{i \in N} \alpha_i = 1$, *such that*

$$f_i(N,v) = \sum_{i \in N} \alpha_i t^i(N,v,L_D), \text{ for all } i \in N.$$

The downstream incremental solution is the weighted hierarchical solution obtained by assigning weight one to the hierarchical outcome corresponding to the (unique) most downstream agent and weight zero to the hierarchical outcome corresponding to all other agents. Next, for any power measure p on $\mathcal{D}_S$ that satisfies $\sum_{i \in N} p_i(N,D) = 1$, let $h^p(N,v)$ be the weighted hierarchical solution obtained by taking $\alpha_i = p_i, i \in N$. We consider the weighted hierarchical solutions $h^p(N,v)$ corresponding to the normalized hierarchical measure $\bar{\eta}$, power measure p^2 and power measure p^3. Since all three power measures satisfy the non-top property, only the weights corresponding to springs can be non-zero, and therefore the hierarchical solutions corresponding to these power measures can be considered generalizations of the upstream incremental solution. Although a theoretical analysis is beyond the scope of this chapter, we illustrate these three upstream incremental type solutions with the following example.

Example 6.4.11 *Consider the river problem* (N, e, b, D) *with* $N = \{1, 2, 3, 4, 5\}$ *digraph* D *as given in Example 6.4.9,* $e_1 = e_2 = e_4 = 1$, $e_3 = e_5 = 0$, $b_5(x_5) = \sqrt{x_5}$ *and* $b_i(x_i) = 0$ *for all* $i \in \{1, 2, 3, 4\}$.[14] *The associated river game is the game* (N, v) *given by* $v(\{4, 5\}) = v(\{1, 3, 5\}) = v(\{2, 3, 5\}) = v(\{1, 4, 5\}) = v(\{2, 4, 5\}) = v(\{3, 4, 5\}) = v(\{1, 2, 4, 5\}) = 1$, $v(\{1, 2, 3, 5\}) = v(\{1, 3, 4, 5\}) = v(\{2, 3, 4, 5\}) = \sqrt{2}$, $v(\{1, 2, 3, 4, 5\}) = \sqrt{3}$ *and* $v(S) = 0$ *otherwise. The three hierarchical outcomes* t^i, *where* i *is a top player, are given by*

$$t^1(N,v) = \left(\sqrt{3} - \sqrt{2}, 0, \sqrt{2} - 1, 0, 1 \right),$$

$$t^2(N,v) = \left(0, \sqrt{3} - \sqrt{2}, \sqrt{2} - 1, 0, 1 \right) \text{ and}$$

$$t^4(N,v) = \left(0, 0, 0, \sqrt{3} - \sqrt{2}, \sqrt{2} \right).$$

By Example 6.4.9, we have $\bar{\eta}(N,D) = (\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{4}, 0)$, $p^2(N,D) = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}, 0)$ *and* $p^3(N,D) = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$. *This yields the following weighted hierarchical outcomes according to* $\bar{\eta}, p^2$ *and* p^3.

$$h^{\bar{\eta}}(N,v) = \left(\frac{3\left(\sqrt{3} - \sqrt{2} \right)}{8}, \frac{3\left(\sqrt{3} - \sqrt{2} \right)}{8}, \frac{3\left(\sqrt{2} - 1 \right)}{4}, \frac{\left(\sqrt{3} - \sqrt{2} \right)}{4}, \frac{\left(3 + \sqrt{2} \right)}{4} \right),$$

$$h^{p^2}(N,v) = \left(\frac{\left(\sqrt{3} - \sqrt{2} \right)}{4}, \frac{\left(\sqrt{3} - \sqrt{2} \right)}{4}, \frac{\left(\sqrt{2} - 1 \right)}{2}, \frac{\left(\sqrt{3} - \sqrt{2} \right)}{2}, \frac{\left(1 + \sqrt{2} \right)}{2} \right) \text{ and}$$

---

[14]Although these benefit functions do not satisfy the assumptions of Ambec and Sprumont (2002), we use them for illustration.

$$h^{p^3}(N, v) = \left(\frac{(\sqrt{3} - \sqrt{2})}{3}, \frac{(\sqrt{3} - \sqrt{2})}{3}, \frac{2(\sqrt{2} - 1)}{3}, \frac{(\sqrt{3} - \sqrt{2})}{3}, \frac{(2 + \sqrt{2})}{3} \right).$$

Instead of using the power measures as weights to define hierarchical solutions, they can also be used as weights to define Harsanyi solutions (see Vasil'ev and van der Laan (2001)), also called sharing values by Derks, Haller and Peters (2000) or weighted Shapley values (see Shapley (1953) and Kalai and Samet (1987)).

Definition 6.4.12 *For any power measure p that satisfies $\sum_{i \in N} p_i(N, D) = 1$, the Harsanyi solution according to p on $\mathcal{G}_{msr}^{(N,D)}$ is given by*

$$H_i^p(N, v) = \sum_{\{S \in N : i \in S\}} p_i(S, D(S)) \Delta_v(S), \text{ for all } i \in N.$$

Example 6.4.13 *Consider the river problem from Example 6.4.11. The dividends of the river game are given by: $\Delta_v(\{4, 5\}) = \Delta_v(\{1, 3, 5\}) = \Delta_v(\{2, 3, 5\}) = 1, \Delta_v(\{1, 2, 3, 5\}) = \Delta_v(\{1, 3, 4, 5\}) = \Delta_v(\{2, 3, 4, 5\}) = \sqrt{2} - 2, \Delta_v(\{1, 2, 3, 4, 5\}) = \sqrt{3} - 3\sqrt{2} + 3$ and $\Delta_v(S) = 0$ otherwise. This yields the following welfare distribution according to the normalized hierarchical measure*

$$
\begin{aligned}
H_1^{\bar{\eta}}(v, D) = H_2^{\bar{\eta}}(v, D) &= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{2}{3}\left(\sqrt{2} - 2\right) + \frac{3}{8}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\
&= \frac{3}{8}\sqrt{3} + \frac{1}{24}\sqrt{2} - \frac{5}{24}
\end{aligned}
$$

and

$$
\begin{aligned}
H_4^{\bar{\eta}}(v, D) &= 1 + \frac{1}{3}\left(\sqrt{2} - 2\right) + \frac{1}{3}\left(\sqrt{2} - 2\right) + \frac{1}{4}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\
&= \frac{1}{4}\sqrt{3} - \frac{1}{12}\sqrt{2} + \frac{5}{12}.
\end{aligned}
$$

For those coalitions S, such that player 3 and/or player 5 is a top player of $(S, D(S))$, it holds that $\Delta_v(S) = 0$. For coalitions S such that $\Delta_v(S) \neq 0$, it holds that players 3 and 5 are not top players of $(S, D(S))$. Therefore their power is zero and these players obtain zero payoff. We have $H_3^{\bar{\eta}}(v, D) = H_5^{\bar{\eta}}(v, D) = 0$ (the same holds for power measures p^2 and p^3).

For power measure p^2 we obtain welfare distribution

$$
\begin{aligned}
H_1^{p^2}(v, D) = H_2^{p^2}(v, D) &= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{4}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\
&= \frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{2} - \frac{1}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
H_4^{p^2}(v, D) &= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\
&= \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} + \frac{1}{2}.
\end{aligned}
$$

with $H_3^{p^2}(v, D) = H_5^{p^2}(v, D) = 0$.

Finally, power measure p^3 always assigns equal power to the springs, so we obtain the following welfare distribution

$$
\begin{aligned}
H_1^{p^3}(v, D) = H_2^{p^3}(v, D) = H_4^{p^3}(v, D) \ &= \ 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{3}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\
&= \ \frac{1}{3}\sqrt{3}
\end{aligned}
$$

and $H_3^{p^3}(v, D) = H_5^{p^3}(v, D) = 0$.

6.5   Concluding remarks

In this chapter we considered the precedence Shapley value of Faigle and Kern (1992). This solution for games with precedence constraints does not satisfy irrelevant player independence. We introduced a class of solutions for games with precedence constraints that do satisfy irrelevant player independence. The solutions in this class allocate dividend according to power measures for acyclic digraphs. We introduced the hierarchical measure as a power measure for acyclic digraphs inspired by the hierarchical strength. We analyzed this measure by taking an axiomatic point of view. We also generalized the hierarchical measure to regular set systems. This wider class of structures allows for more applications. Score rankings may be considered in defining new solutions for games with precedence constraints. We also defined the normalized hierarchical measure and axiomatized it on the class of forests of rooted trees and forests of sink trees. On forests of sink trees we obtained different power measures, by using alternative versions of component normalization. Finally we considered the application of these measures on forests of sink trees to water allocation problems.

Appendix: Weighted precedence solutions satisfying irrelevant player independence

Let $W^{CI} \subset W$ be the set of weight functions w such that for every $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$ there exists $c_S^{(N,D)} > 0 \in \mathbb{R}^N$ in order that $w(N, D, S) = c_S^{(N,D)} w(S, D(S), S)$. Let the set of solutions for games with precedence constraints $\Psi^{CI} \subset \Psi$ be given by those weighted precedence solutions f^w, for which $w \in W^{CI}$.

Proposition 6.5.1 *A weighted precedence solution f^w is in Ψ^{CI} if and only if it satisfies irrelevant player independence.*

PROOF
First suppose that $f^w \in \Psi^{CI}$. Consider a game with precedence constraints (N, v, D). Let $N' = N \setminus Irr(N, v, D)$. It now holds for $i \in N'$ that

$$f_i^w(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{w_i(N, D, S)}{\sum_{j \in S} w_j(N, D, S)} \Delta_v^D(S)$$

$$= \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{c_S^{(N,D)} w_i(S, D(S), S)}{\sum_{j \in S} c_S^{(N,D)} w_j(S, D(S), S)} \Delta_v^D(S)$$

$$= \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{w_i(S, D(S), S)}{\sum_{j \in S} w_j(S, D(S), S)} \Delta_v^D(S)$$

$$= \sum_{\substack{S \in \Phi_{N'}^p(N,D) \\ i \in S}} \frac{w_i(S, D(S), S)}{\sum_{j \in S} w_j(S, D(S), S)} \Delta_{v_{N'}}^D(S)$$

$$= \sum_{\substack{S \in \Phi^p(N',D(N')) \\ i \in S}} \frac{w_i(S, D(N')(S), S)}{\sum_{j \in S} w_j(S, D(N')(S), S)} \Delta_{v_{N'}}^{D(N')}(S)$$

$$= \sum_{\substack{S \in \Phi^p(N',D(N')) \\ i \in S}} \frac{c_S^{(N',D(N'))} w_i(S, D(N')(S), S)}{\sum_{j \in S} c_S^{(N',D(N'))} w_j(S, D(N')(S), S)} \Delta_{v_{N'}}^{D(N')}(S)$$

$$= \sum_{\substack{S \in \Phi^p(N',D(N')) \\ i \in S}} \frac{w_i(N', D(N'), S)}{\sum_{j \in S} w_j(N', D(N'), S)} \Delta_{v_{N'}}^{D(N')}(S) = f_i^w(N', v_{N'}, D(N'))$$

and f^w satisfies irrelevant player independence.

Now suppose that f^w satisfies irrelevant player independence and suppose that $f^w \notin \Psi^{CI}$. We show that f^w cannot satisfy irrelevant player independence and we have a contradiction.

Since $f^w \notin \Psi^{CI}$, there exists a graph $(N, D) \in \mathcal{D}_a$ and $S \in \Phi^p(N, D)$ for which no $c \in \mathbb{R}$ exists such that $w(N, D, S) = cw(S, D(S), S)$. In that case for at least one player $i \in S$ it holds that

$$\frac{w_i(N, D, S)}{\sum_{j \in S} w_j(N, D, S)} \neq \frac{w_i(S, D(S), S)}{\sum_{j \in S} w_j(S, D(S), S)}, \tag{6.5.5}$$

otherwise we can take $c = \frac{w_i(N,D,S)}{w_i(S,D(S),S)}$ for any $i \in S$. Now consider unanimity game with precedence constraints (N, u_S, D). For this game it holds that $Irr(N, v, D)$ is $N \setminus S$ and it can be shown that for each player $i \in S$ such that Equation (6.5.5) holds, we have $f_i^w(N, u_S, D) \neq f_i^w(N', u_S|_{N'}, D(N'))$.

$\square$

Chapter 7

Comparable axiomatizations of four solutions for permission tree games

In this chapter we consider games with a permission structure where the permission structure is a rooted tree. The conjunctive permission value is considered for these games, as well as modifications of the Myerson value and the hierarchical outcomes for communication graph games (see Chapter 2). A fourth new solution for these games, the top value, is introduced. This solution assigns the worth of the grand coalition to the top player in the tree, and assigns zero payoff to the other players. These four solutions will be compared to each other by providing two comparable axiomatizations.

This chapter is based on van den Brink, Dietz and van der Laan (2014) and van den Brink, Dietz and van der Laan and Xu (2014).

7.1   Introduction

In this chapter we consider the class of games with a permission structure where the permission structure is a rooted tree. These games are also referred to as permission tree games. On this subclass of games with a permission structure, the conjunctive permission value coincides with the disjunctive permission value. Van den Brink, Herings, van der Laan and Talman (2013) introduced the *average tree permission value* for permission tree games and gave comparable axiomatizations for this solution as well as the conjunctive permission value. In this chapter we modify both the Myerson value and the hierarchical outcomes for communication graph games to define solutions for permission tree games. We modify the Myerson value by assigning to every permission tree game the Myerson value of the underlying undirected graph game. The hierarchical outcome of a permission tree game is simply the hierarchical outcome of the underlying communication graph game with the same root player as the root of the permission tree. In addition a new solution for these games, the top value, is introduced that assigns the worth of the grand coalition to the top player in the tree, and assigns zero payoff to the other players. These three solutions, along with the conjunctive permission value, are compared to each other by providing comparable axiomatizations. Comparing any two of these solutions, the axiomatizations we give differ in at most two axioms.

A number of criteria can be used to classify these four solutions. For the Myerson value and the hierarchical outcome, the permission structure is interpreted as more of a communication network. Players that are not connected in the graph, are restricted in cooperating with each other. As long as players are connected however, no permission from predecessors in the graph is needed and a coalition is feasible. This is different from both the conjunctive permission value, as well as the top value. For these solutions, in order for a coalition to be feasible it needs to contain all predecessors in the permission structure. A different classification can be made on the basis of how the solutions allocate dividend. In this case the Myerson value and the conjunctive permission value both distribute the dividend of a feasible coalition equally over the players needed to make that coalition feasible. This is different from the hierarchical outcome and the top value, where the dividend of a coalition is assigned to the top player among the players needed to make that coalition feasible.

The first set of comparable axiomatizations uses axioms that focus on predecessors in the permission structure. Next we provide axiomatizions were the predecessor axioms have been replaced by axioms that focus on superiors in the permission structure instead.

The chapter is organized as follows. Section 2 discusses the four solutions for permission tree games and gives comparable axiomatizations for these four solutions. In section 3 we compare the four solutions based on these axiomatizations. In section 4 we study a different set of comparable axiomatizations were the predecessor axioms have been replaced by (weaker) superior axioms. In section 5 we compare the four solutions based on this second set of axiomatizations. Finally section 6 ends with some concluding remarks.

7.2 Axiomatizations of four solutions for permission tree games using predecessor axioms

Recall from Chapter 2 that $\mathcal{D}_t$ is the class of directed rooted trees. Denote the collection of all games with a permission structure (N, v, T), where $(N, T) \in \mathcal{D}_t$ is a directed rooted tree, as $\mathcal{G}_{\mathcal{T}}$. We will refer to these games as *permission tree games*. Given a fixed player set N the set of all games with a permission structure (N, v, T), where $(N, T) \in \mathcal{D}_t$ is a directed rooted tree, is denoted by $\mathcal{G}_{\mathcal{T}}^N$. A single-valued solution f on $\mathcal{G}_{\mathcal{T}}$ assigns a unique payoff vector $f(N, v, T) \in \mathbb{R}^N$ to every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. The conjunctive permission value φ^c has been defined in Chapter 2 on the class of games with a permission structure (N, v, D), where D can be any digraph. The disjunctive permission value φ^d has been defined in Chapter 2 on the class of games with a permission structure (N, v, D), where D is a hierarchical digraph. When D is a directed rooted tree, the conjunctive permission value therefore coincides with the disjunctive permission value (see also Chapter 2). In this chapter we will therefore refer to the conjunctive permission value on permission tree games as the permission value φ. Next we define three other solutions for permission tree games. We compare these with each other and the permission value, by providing comparable axiomatizations.

Consider a directed rooted tree $(N, T) \in \mathcal{D}_t$. Recall from Chapter 2 that in that

case $(N, L_T) \in \mathcal{L}_t$ is the undirected tree, where L_T is given by $\{\{i, j\} : (i, j) \in T$ or $(j, i) \in T\}$. The Myerson value for permission tree games is the solution obtained by taking the Myerson value of the underlying communication graph game, i.e. by taking for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ the Myerson value of $(N, v, L_T) \in \mathcal{G}_G$ (see Chapter 2).

Definition 7.2.1 *The Myerson value μ is given by*

$$\mu(N, v, T) = \mu(N, v, L_T), \ (N, v, T) \in \mathcal{G}_\mathcal{T}.$$

The hierarchical outcome is the solution[1] that for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns to any player $i \in N$ its marginal contribution in the graph restricted game $(N, v^{L_T}) \in \mathcal{L}_t$ to the coalition of its successors in (N, T).

Definition 7.2.2 *The hierarchical outcome η is given by*

$$\eta_i(N, v, T) = v(\widehat{F}_T(i) \cup \{i\}) - \sum_{j \in F_T(i)} v(\widehat{F}_T(j) \cup \{j\}), \ (N, v, T) \in \mathcal{G}_\mathcal{T}, \ i \in N.$$

Now for $(N, T) \in \mathcal{D}_t$ and $R \subseteq N$ a connected coalition in (N, T), let $TOP_{(N,T)}(R) \in N$ be given by $i : (R \setminus \{i\}) \subseteq \widehat{F}_T(i)$; the unique top player of coalition R in (N, T). We introduce the top value as the solution that for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the worth $v(N)$ of the grand coalition N to the top player $TOP_{(N,T)}(N)$ of (N, T), while the other players get a payoff of zero.

Definition 7.2.3 *The top value τ is given by*

$$\tau_i(N, v, T) = \begin{cases} v(N) & i = TOP_{(N,T)}(N) \\ 0 & i \neq TOP_{(N,T)}(N) \end{cases}, \ (N, v, T) \in \mathcal{G}_\mathcal{T}.$$

As mentioned in Chapter 2, Demange (2004) showed that the hierarchical outcome is always in the Core of the restricted game of a communication graph game, when the game is superadditive and the graph is a tree. The top value satisfies a similar property with respect to the conjunctively restricted game of a permission tree game when the game is monotonic, since the top player is always a necessary player, i.e. $r^c_{N,v,T}(S) = 0$ for all $S \subseteq N \setminus \{TOP_{(N,T)}(N)\}$ and $(N, v, T) \in \mathcal{G}_\mathcal{T}$.

For a collection of sets $\mathcal{F} \subseteq 2^N$ and set $S \in 2^N$ let $\mathcal{F}_S = \{U \in \mathcal{F} : S \subseteq U\}$. This is the collection of sets obtained from $\mathcal{F}$ by considering only those sets in $\mathcal{F}$ containing S. Recall from (**??**) that the set of connected coalitions in a communication graph (N, L) is given by $\mathcal{F}_L$. For any coalition $R \subseteq N$ and $(N, T) \in \mathcal{D}_t$ we define the connected hull of R as the smallest connected coalition in (N, L_T) containing R.

Definition 7.2.4 *For any coalition $R \subseteq N$ the connected hull of R in (N, T) is given by*

$$\gamma_T(R) = S \in (\mathcal{F}_{L_T})_R : S \subseteq U \text{ for } U \in (\mathcal{F}_{L_T})_R.$$

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[1]When there is no confusion we will refer to this solution as well as the payoff vector it assigns to a permission tree game as hierarchical outcome.

For any coalition $R \subseteq N$ we define the permission hull of R as the smallest coalition with full permission in (N, T) containing R.

Definition 7.2.5 *For any coalition $R \subseteq N$ the permission hull of R in (N, T) is given by*

$$\alpha_T(R) = R \cup \widehat{P}_T(R).$$

Note that this is exactly the authorizing set from Definition 2.2.16 for the conjunctive approach to games with a permission structure.

Next we give expressions of the four solutions for permission tree games (N, v, T) considered in this chapter, in terms of the Harsanyi dividends of the underlying TU-game (N, v).

Proposition 7.2.6 *The Myerson value μ can be written in terms of dividends as follows:*

$$\mu_i(N, v, T) = \sum_{\{R \subseteq N : i \in \gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|}, \ (N, v, T) \in \mathcal{G}_\mathcal{T}, \ i \in N.$$

This result follows from Owen (1986).

Proposition 7.2.7 *The hierarchical outcome η can be written in terms of dividends as follows:*

$$\eta_i(N, v, T) = \sum_{\{R \subseteq N : i = TOP_{(N,T)}(\gamma_T(R))\}} \Delta_v(R), \ (N, v, T) \in \mathcal{G}_\mathcal{T}, \ i \in N.$$

The hierarchical outcome assigns the dividend of a coalition R exclusively to the top player $TOP_{(N,T)}(\gamma_T(R))$ of the connected hull of R. The proof follows from rewriting the expression of the hierarchical outcome in terms of dividends.

Proposition 7.2.8 *The top value τ can be written in terms of dividends as follows:*

$$\tau_i(N, v, T) = \begin{cases} \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) & i = TOP_{(N,T)}(N) \\ 0 & i \neq TOP_{(N,T)}(N) \end{cases}, \ (N, v, T) \in \mathcal{G}_\mathcal{T}.$$

The top value assigns the dividend of a coalition R exclusively to the top player of the permission hull of R (note that this must always be the top player $TOP_{(N,T)}(N)$ of (N, T)). The proof follows from the fact that $f_{TOP_{(N,T)}(N)}(N, v, T) = v(N)$, $f_i(N, v, T) = 0$ for $i \in N \setminus \{TOP_{(N,T)}(N)\}$ and $v(N) = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S)$ (see (2.2.2)).

Proposition 7.2.9 *The permission value φ can be written in terms of dividends as follows:*

$$\varphi_i(N, v, T) = \sum_{\{R \subseteq N : i \in \alpha_T(R)\}} \frac{\Delta_v(R)}{|\alpha_T(R)|}, \ (N, v, T) \in \mathcal{G}_\mathcal{T}, \ i \in N.$$

From Gilles, Owen and van den Brink (1992), we obtain that the dividend of a coalition R in the restricted game $r_{N,v,T}^c$ is given by $\Delta_{r^c}(R) = \sum\limits_{\{S \subseteq R: R = \alpha_T(S), S \neq \emptyset\}} \Delta_v(S)$. The expression of the permission value in terms of dividends now follows from the fact that the permission value of a game (N, v, T) is given by the Shapley value applied to $r_{N,v,T}^c$.

Next, we define the axioms that will be used in axiomatizing the four solutions. Efficiency states that the payoffs assigned to players must sum to $v(N)$, the worth of the grand coalition.

Efficiency For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$, it holds that $\sum_{i \in N} f_i(N, v, T) = v(N)$.

Additivity states that a solution should be additive on the class of permission tree games.

Additivity For every $(N, v, T), (N, w, T) \in \mathcal{G}_\mathcal{T}$, it holds that $f(N, v+w, T) = f(N, v, T) + f(N, w, T)$.

Call a player $i \in N$ a *pending null player* in $(N, v, T) \in \mathcal{G}_\mathcal{T}$ if it is both a null player in game (N, v) and a pending player in the undirected graph (N, L_T) (meaning there exists only one player $j \in N$ such that $\{i, j\} \in (N, L_T)$). The pending null player out property states that pending null players can be removed from the game without affecting the payoff distribution of the other players.

Pending null player out property For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$, when i is a pending null player in (N, v, T), it holds that $f_j(N, v, T) = f_j(N \setminus \{i\}, v_{N \setminus \{i\}}, T_{-i})$ for $j \in N \setminus \{i\}$.

Here $(N \setminus \{i\}, v_{N \setminus \{i\}}, T_{-i})$ denotes the subgame of (N, v, T) on the players in $N \setminus \{i\}$.

Now call player $i \in N$ a *strong pending null player* in $(N, v, T) \in \mathcal{G}_\mathcal{T}$ if player i is a pending null player in game (N, v, T) and $F_T(i) = \emptyset$. The weak pending null player out property weakens the pending null player out property by only considering strong pending null players.

Weak pending null player out property For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$, when i is a strong pending null player in (N, v, T), it holds that $f_j(N, v, T) = f_j(N \setminus \{i\}, v_{N \setminus \{i\}}, T_{-i})$ for $j \in N \setminus \{i\}$.

Recall from Chapter 2 that a player $i \in N$ is called *necessary* in game (N, v) if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$ and that the necessary player property states that necessary players should get at least as much as each of the other players, if the game is monotonic.

Necessary player property For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ with (N, v) monotonic, if $i \in N$ is a necessary player in v then $f_i(N, v, T) \geq f_j(N, v, T)$ for all $j \in N$.

The weak necessary player property weakens the necessary player property by stating that necessary players should only get at least as much as their subordinates in the permission structure, if the game is monotonic.

Weak necessary player property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ such that (N, v) is monotonic, if $i \in N$ is a necessary player in (N, v) then $f_i(N, v, T) \geq f_j(N, v, T)$ for all $j \in \widehat{F}_T(i)$.

Note that the weak necessary player property is in fact also a weak version of the structural monotonicity used by van den Brink and Gilles (1996) to axiomatize the conjunctive permission value (see also Chapter 2). It is weaker in the sense it that holds only for necessary players.

The one player property states that in a game where every player is necessary, only one of them can have a non-zero payoff.

One player property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ such that every $i \in N$ is a necessary player in (N, v) it holds that there is at most one player $j \in N$ such that $f_j(N, v, T) \neq 0$.

Say that a player $i \in N$ is necessary for (or can veto) player $j \in N$ in game (N, v) if it holds that $v(S \cup \{j\}) - v(S) = 0$ for those coalitions $S \subset N$ such that $i \notin S$. In (4.4.1) we defined (N, v_j^i) as the game derived from (N, v) by

$$v_j^i(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S) & \text{if } i \in S. \end{cases}$$

In game v_j^i player i has become necessary for player j.

It is seen that the dividend of a coalition R in game (N, v) such that $i \notin R$ and $j \in R$ is shifted to that of coalition $R \cup \{i\}$ in game (N, v_j^i). We therefore have the following expression:

$$v_j^i = v + \sum_{\{R \subseteq N : j \in R\}} \Delta_v(R)[u_{R \cup \{i\}} - u_R] \tag{7.2.1}$$

Predecessor necessity now states that the payoff distribution does not change if a predecessor i becomes necessary for a successor j. This expresses that players are able to veto their predecessors. Van den Brink, Herings, van der Laan and Talman (2013) use this property in axiomatizing the permission value for permission tree games.

Predecessor necessity For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ and $i, j \in N$ such that $(i, j) \in T$, it holds that $f(N, v, T) = f(N, v_j^i, T)$.

In the following weaker version of predecessor necessity it is no longer required that the payoff distribution does not change if a player i becomes necessary for its successor j. However if player j's marginal contribution to any coalition of his subordinates is zero, it does hold that the payoff distribution does not change if player i becomes necessary for player j. This expresses that a player is able to veto a successor, when it is needed to connect this successor to players in the graph necessary for this successor.

Weak predecessor necessity For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ such that $(i, j) \in T$ and $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq (\widehat{F}_T(j) \cup \{j\})$ with $j \in S$, it holds that $f(N, v, T) = f(N, v_j^i, T)$.

Before giving the axiomatizations we state some propositions that will be used in the proofs.

Proposition 7.2.10 *Consider* $(N, w_R, T) \in \mathcal{G}_\mathcal{T}$, *where* (N, w_R) *is a scaled unanimity game with* $w_R = cu_R, c \in \mathbb{R}, \emptyset \neq R \subseteq N$. *In that case*

(i) *For any solution f satisfying efficiency and the pending null player out property, it holds that $f_i(N, w_R, T) = 0$ if $i \in N \setminus \gamma_T(R)$ and $f_i(N, w_R, T) = f_i(\gamma_T(R), w_R|_{\gamma_T(R)}, T(\gamma_T(R)))$ for $i \in \gamma_T(R)$.*

(ii) *For any solution f satisfying efficiency and the weak pending null player out property, it holds that $f_i(N, w_R, T) = 0$ if $i \in N \setminus \alpha_T(R)$ and $f_i(N, w_R, T) = f_i(\alpha_T(R), w_R|_{\alpha_T(R)}, T(\alpha_T(R)))$ for $i \in \alpha_T(R)$.*

PROOF
Consider any permission tree game (N, v, T). For any null player $i \in N$ we have $v(N) = v(N \setminus \{i\}) = v_{N\setminus\{i\}}(N \setminus \{i\})$.

(i) Let $i \in N$ be a pending null player. Therefore i is a null player. By efficiency it holds that $\sum_{j \in N\setminus\{i\}} f_j(N \setminus \{i\}, v_{N\setminus\{i\}}, T_{-i}) = v_{N\setminus\{i\}}(N \setminus \{i\}) = v(N)$. By efficiency it also holds that $\sum_{j \in N} f_j(N, v, T) = v(N)$. By the pending null player out property it holds that $f_j(N, v, T) = f_j(N \setminus \{i\}, v_{N\setminus\{i\}}, T_{-i})$ for $j \in N \setminus \{i\}$. Therefore $\sum_{j \in N\setminus\{i\}} f_j(N, v, T) = v(N)$ and $f_i(N, v, T) = 0$. By repeated application of the pending null player out property and efficiency in this way it follows that $f_i(N, w_R, T) = 0$ if $i \in N \setminus \gamma_T(R)$ and $f_i(N, w_R, T) = f_i(\gamma_T(R), w_R|_{\gamma_T(R)}, T(\gamma_T(R)))$ for $i \in \gamma_T(R)$.

(ii) Similar to the proof of (i), but considering strong pending null players instead of pending null players. $\square$

Proposition 7.2.11 *Consider* $(N, w_R, T) \in \mathcal{G}_\mathcal{T}$, *where* (N, w_R) *is a scaled unanimity game with* $w_R = cu_R, c \in \mathbb{R}, \emptyset \neq R \subseteq N$. *In that case*

(i) *For any solution f satisfying predecessor necessity, it holds that $f(N, w_R, T) = f(N, w_{\alpha_T(R)}, T)$.*

(ii) *For any solution f satisfying weak predecessor necessity, it holds that $f(N, w_R, T) = f(N, w_{\gamma_T(R)}, T)$.*

The proofs follow straightforwardly from repeated application of (weak) predecessor necessity.

We first characterize the Myerson value for permission tree games by efficiency, additivity, the pending null player out property, weak predecessor necessity and the necessary player property.

Theorem 7.2.12 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the Myerson value μ if and only if it satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the necessary player property.*

PROOF
It is straightforward to verify that the Myerson value satisfies efficiency, additivity, the pending null player out property and the necessary player property.

To show that the Myerson value satisfies weak predecessor necessity we argue as follows. By Proposition 7.2.6, $\mu_k(N, v, T) = \sum\limits_{\{R \subseteq N : k \in \gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|}$ for all $k \in N$. Consider those coalitions R such that $j \in R$ and $i \notin R$. Denote by $\mathcal{V}$ the collection of these coalitions R such that $j \in R$ and $i \in \gamma_T(R)$. For $R \in \mathcal{V}$ it holds that $\gamma_T(R) = \gamma_T(R \cup \{i\})$. Denote by $\mathcal{W}$ the collection of these coalitions R such that $j \in R$ and $i \notin \gamma_T(R)$. By Proposition 7.2.6 and Equation (7.2.1), $\mu_k(N, v_j^i, T) =$

$$\sum\limits_{\{R \subseteq (N \setminus (\mathcal{V} \cup \mathcal{W})) : k \in \gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|} + \sum\limits_{\{R \in \mathcal{V} : k \in \gamma_T(R \cup \{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R \cup \{i\})|} + \sum\limits_{\{R \in \mathcal{W} : k \in \gamma_T(R \cup \{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R \cup \{i\})|}$$

$$= \sum\limits_{\{R \subseteq (N \setminus \mathcal{W} : k \in \gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|} + \sum\limits_{\{R \in \mathcal{W} : k \in \gamma_T(R \cup \{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R \cup \{i\})|}$$ for all $k \in N$. Weak predecessor

necessity can be applied when $(i, j) \in T$ and $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq (\widehat{F}_T(j) \cup \{j\})$ with $j \in S$. In that case $\Delta_v(R) = 0$ for coalitions R such that $j \in R$ and $i \notin \gamma_T(R)$. These are exactly the coalitions in $\mathcal{W}$. We obtain $\mu(N, v, T) = \mu(N, v_j^i, T)$, showing that the Myerson value satisfies weak predecessor necessity.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$, let there be one $c > 0$ such that $(N, w_U, T) \in \mathcal{G}_{\mathcal{T}}$ is the permission tree game, with $w_U = cu_U$ the unanimity game of U scaled by c. First, we consider $(N, w_R, T) \in \mathcal{G}_{\mathcal{T}}$, for R connected in the underlying undirected graph, so $R = \gamma_T(R)$. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0. Therefore by efficiency the players in R obtain $w_R(N) = c$. Since the players in R are all necessary players in (N, w_R, T) and w_R is monotonic, the necessary player property implies that $f_i(N, w_R, T) = \frac{c}{|R|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined.

Now consider those coalitions R not connected in (N, L_T), so $R \neq \gamma_T(R)$. By Proposition 7.2.11.(ii) and f satisfying weak predecessor necessity, it holds that $f(N, w_R, T) = f(N, w_{\gamma_T(R)}, T)$. Since $\gamma_T(R)$ is a connected coalition in (N, L_T), $f(N, w_{\gamma_T(R)}, T)$ has been uniquely determined above and therefore $f(N, w_R, T)$ is also uniquely determined.

Consider the game (N, v_0, T), in which all players are null players. By repeated application of the pending null player out property and efficiency it holds that $f_i(N, v_0, T) =$

0 for $i \in N$.

Next, consider (N, w_R, T) with $w_R = cu_R$ for some $c < 0$ (and therefore we cannot apply the necessary player property since w_R is not monotonic). Since $-w_R = -cu_R$ with $-c > 0$, and $v_0 = w_R + (-w_R)$, it follows from additivity of f that $f(N, w_R, T) = f(N, v_0, T) - f(N, -w_R, T) = -f(N, -w_R, T)$ is uniquely determined because $-w_R$ is monotonic.

Finally, since for every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ it holds that v can be written as $v = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) u_R$, where u_R is the unanimity game on coalition R, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. $\qquad\square$

We characterize the hierarchical outcome for permission tree games by replacing the necessary player property in Theorem 7.2.12 by the weak necessary player property and the one player property.

Theorem 7.2.13 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the hierarchical outcome η if and only if it satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property.*

PROOF
It is straightforward to verify that the hierarchical outcome satisfies efficiency, additivity, the pending null player out property, the weak necessary player property and the one player property.

By arguments similar to the proof that the Myerson value satisfies weak predecessor necessity (see Theorem 7.2.12) it can be shown that the hierarchical outcome satisfies weak predecessor necessity.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$ let there be one $c > 0$ such that $(N, w_U, T) \in \mathcal{G}_{\mathcal{T}}$ is the permission tree game, with $w_U = cu_U$ the unanimity game of U scaled by c. First, we consider $(N, w_R, T) \in \mathcal{G}_{\mathcal{T}}$, for R connected in the underlying undirected graph, so $R = \gamma_T(R)$. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0 and $f_i(N, w_R, T) = f_i(R, w_R, T(R))$ for $i \in R$.[2] Since in $(R, w_R, T(R))$ all players are necessary we can apply the one player property to obtain that there is only one player $j \in R$ such that $f_j(R, w_R, T(R)) \neq 0$. Let $r_0 = TOP_{(N,T)}(R)$. By the weak necessary player property and monotonicity it holds that $f_{r_0}(R, w_R, T(R)) \geq f_i(R, w_R, T(R))$ for all $i \in R \setminus \{r_0\}$. By efficiency it holds that $f_{r_0(R)}(R, w_R, T(R)) = c$ and $f_i(R, w_R, T(R)) = 0$ for all $i \in R \setminus \{r_0\}$. Therefore $f(N, w_R, T)$ is uniquely determined.

The cases (i) $R \neq \gamma_T(R)$ and $c > 0$, (ii) $c < 0$ and (iii) $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ follow along similar lines to those in the proof of Theorem 7.2.12 for the Myerson value. $\qquad\square$

Compared to Theorem 7.2.13 the top value is characterized by replacing the pending null player out property by the weak pending null player out property and by replacing weak predecessor necessity by predecessor necessity.

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[2]For convenience we write the subgame on R by $(R, w_R, T(R))$ instead of $(R, w_R|_R, T(R))$.

Theorem 7.2.14 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the top value τ if and only if it satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property.*

PROOF

It is straightforward to verify that the top value satisfies efficiency, additivity, the weak pending null player out property, the weak necessary player property and the one player property.

The top value satisfying predecessor necessity follows from $v_j^i(N) = v(N)$ and the fact that for any two games (N, v, T) and (N, v', T) such that $v(N) = v'(N)$ it holds that $\tau(N, v, T) = \tau(N, v', T)$.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$ let there be one $c > 0$ such that $(N, w_U, T) \in \mathcal{G}_{\mathcal{T}}$ is the permission tree game, with $w_U = cu_U$ the unanimity game of U scaled by c. First, we consider $(N, w_R, T) \in \mathcal{G}_{\mathcal{T}}$, with R having full permission in T, so $R = \alpha_T(R)$. By Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0 and $f_i(N, w_R, T) = f_i(R, w_R, T(R))$ for $i \in R$. Since all players in the game $(R, w_R, T(R))$ are necessary we can apply the one player property to obtain that there is only one player $j \in R$ such that $f_j(R, w_R, T(R)) \neq 0$. Let $i_0 = TOP_{(N,T)}(N)$. By the weak necessary player property and monotonicity it holds that $f_{i_0}(R, w_R, T(R)) \geq f_i(R, w_R, T(R))$ for all $i \in R \setminus \{i_0\}$. By efficiency it holds that $f_{i_0(R)}(R, w_R, T(R)) = c$ and $f_i(R, w_R, T(R)) = 0$ for all $i \in R \setminus \{i_0\}$. Therefore $f(N, w_R, T)$ is uniquely determined.

Now consider those coalitions R that do not have full permission in (N, T), so $R \neq \alpha_T(R)$. By Proposition 7.2.11.(ii) and f satisfying predecessor necessity, it holds that $f(N, w_R, T) = f(N, w_{\alpha_T(R)}, T)$. Since $\alpha_T(R)$ has full permission in (N, T), $f(N, w_{\alpha_T(R)}, T)$ has been uniquely determined above and therefore also $f(N, w_R, T)$ is uniquely determined.

The cases $c < 0$ and $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ follow along similar lines to those in the proof of Theorem 7.2.12 for the Myerson value. $\qquad\square$

Finally, we characterize the permission value using only axioms that were used in the three axiomatizations before. In particular, the permission value for permission tree games is characterized by efficiency, additivity, the weak pending null player out property, predecessor necessity and the necessary player property.

Theorem 7.2.15 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the permission value φ if and only if it satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the necessary player property.*

PROOF

It is straightforward to verify that the permission value satisfies efficiency, additivity, the weak pending null player out property and the necessary player property.

To show that the permission value satisfies predecessor necessity we argue as follows. By Proposition 7.2.9, $\varphi_k(N, v, T) = \sum_{\{R \subseteq N : k \in \alpha_T(R)\}} \frac{\Delta_v(R)}{|\alpha_T(R)|}$ for all $k \in N$. We consider $i, j \in N$ such that $j \in \widehat{F}_T(i)$. In that case, $i \in \alpha_T(R)$ for all R such that $j \in R$. Also $\alpha_T(R) = \alpha_T(R \cup \{i\})$. Denote by $\mathcal{W}$ the collection of these coalitions R such that $j \in R$ and $i \notin R$. By Proposition 7.2.9 and (7.2.1), $\varphi_k(N, v_j^i, T) = \sum_{\{R \subseteq N : R \notin \mathcal{W}, k \in \alpha_T(R)\}} \frac{\Delta_v(R)}{|\alpha_T(R)|} + \sum_{\{R \in \mathcal{W} : k \in \alpha_T(R \cup \{i\})\}} \frac{\Delta_v(R)}{|\alpha_T(R \cup \{i\})|} = \sum_{\{R \subseteq N : k \in \alpha_T(R)\}} \frac{\Delta_v(R)}{|\alpha_T(R)|}$ for all $k \in N$. Predecessor necessity can be applied when $(i, j) \in T$. In that case $j \in \widehat{F}_T(i)$. We obtain $\varphi(N, v, T) = \varphi(N, v_j^i, T)$, showing that the permission value satisfies predecessor necessity.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$ let there be one $c > 0$ such that $(N, w_U, T) \in \mathcal{G}_\mathcal{T}$ is the permission tree game, with $w_U = c u_U$ the unanimity game of U scaled by c. First, we consider $(N, w_R, T) \in \mathcal{G}_\mathcal{T}$, with R having full permission in T, so $R = \alpha_T(R)$. By Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0. Therefore by efficiency the players in R obtain $w_R(N) = c$. Since the players in R are all necessary players in (N, w_R, T) and w_R is monotonic, the necessary player property implies that $f_i(N, w_R, T) = \frac{c}{|R|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined.

Now consider those coalitions R that do not have full permission in (N, T), so $R \neq \alpha_T(R)$. By Proposition 7.2.11.(ii) and f satisfying predecessor necessity it holds that $f(N, w_R, T) = f(N, w_{\alpha_T(R)}, T)$. Since $\alpha_T(R)$ has full permission in (N, T), $f(N, w_{\alpha_T(R)}, T)$ has been uniquely determined above and therefore also $f(N, w_R, T)$ is uniquely determined.

The cases $c < 0$ and $(N, v, T) \in \mathcal{G}_\mathcal{T}$ follow along similar lines to those in the proof of Theorem 7.2.12 for the Myerson value. $\square$

Logical independence of the axioms used to characterize the above four solutions is shown in appendix A of this chapter.

7.3   Comparing the four solutions

Table 1:Axioms satisfied by the four solutions

Property	μ	η	φ	τ
Efficiency	++	++	++	++
Additivity	++	++	++	++
Weak pending null player out	+	+	++	++
Weak necessary player	+	+	++	++
Weak predecessor necessity	++	++	+	+
Pending null player out	++	++	-	-
Necessary player	++	-	++	-
Predecessor necessity	-	-	++	++
One player property	-	++	-	++

Table 1 gives an overview of the axioms used to characterize the four solutions for permission tree games (those marked with a ++). Moreover, it shows which other axioms are satisfied by the solutions (those marked with a +). In all four axiomatizations we use the first two axioms: efficiency and additivity. All four solutions also satisfy the next two axioms: the weak pending null player out property and weak predecessor necessity. Although not appearing explicitly in all axiomatizations, they do appear implicitly in all axiomatizations since the two axiomatizations that do not use the weak pending null player out property use the stronger pending null player out property, and the two axiomatizations that do not use weak predecessor necessity use the stronger predecessor necessity. Each of the four solutions satisfies exactly two of the final four axioms in the table, and together with the previous axioms these give an axiomatization of the corresponding solution. (Moreover, each of these four axioms appears in precisely two axiomatizations.) In appendix B of this chapter we show that these four combinations out of two of these four axioms are actually the only possible ones. This means that a solution satisfying efficiency, additivity, the weak pending null player out property and weak predecessor necessity cannot also satisfy both the pending null player out property and predecessor necessity nor satisfy both the necessary player property and the one player property. In this way we have comparable axiomatizations of the four solutions for permission tree games. Next, we describe this comparison in more detail.

The pending null player out property is satisfied by the Myerson value and the hierarchical outcome. It implies that players who do not contribute anything in the game nor connect any contributing players, obtain a zero payoff. This follows from the communication feature of these two solutions. The permission and top value may still grant such players a nonzero payoff, by domination of contributing successors. This follows from the hierarchical feature of these two solutions. This is also shown by these solutions satisfying the predecessor necessity property, whereas the Myerson value and hierarchical outcome do not. Therefore, the Myerson value and the hierarchical outcome may be thought of as 'communication solutions', whereas the permission value and the top value might be considered 'hierarchy solutions'. However this does not imply that 'bottom players' will always obtain a higher payoff from a 'communication value' than

from a 'hierarchy value'. For example, considering the unanimity game of a connected coalition R with $|R| \geq 2$ containing bottom player j, player j gets zero payoff according to the hierarchical outcome, but earns a positive payoff according to the permission value. The reason is that the hierarchical outcome satisfies the one player property, assigning the full unanimity payoff to the 'local' top player in the connected coalition, i.e. the player in the coalition who is closest to the root, while the permission value assigns equal payoffs to player j and all of its superiors because it satisfies the necessary player property. In this sense there is another distinction between the four solutions. On the one hand the Myerson value and the permission value satisfy the necessary player property, equally distributing the dividend of a coalition among the players needed to make that coalition feasible. The hierarchical outcome and the top value on the other hand satisfy the one player property, assigning the payoff of a coalition to the unique top player among the players needed to make that coalition feasible.

We summarize this in the following diagram.

Diagram 1: Solution classifications

$$\eta \quad \overline{\text{dividend assigned to top player}} \quad \tau$$

dividend distributed over connected coalitions $\diagup$ $\diagup$ dividend distributed over connected coalitions and superiors

$$\mu \quad \overline{\text{dividend distributed equally over 'essential' players}} \quad \varphi$$
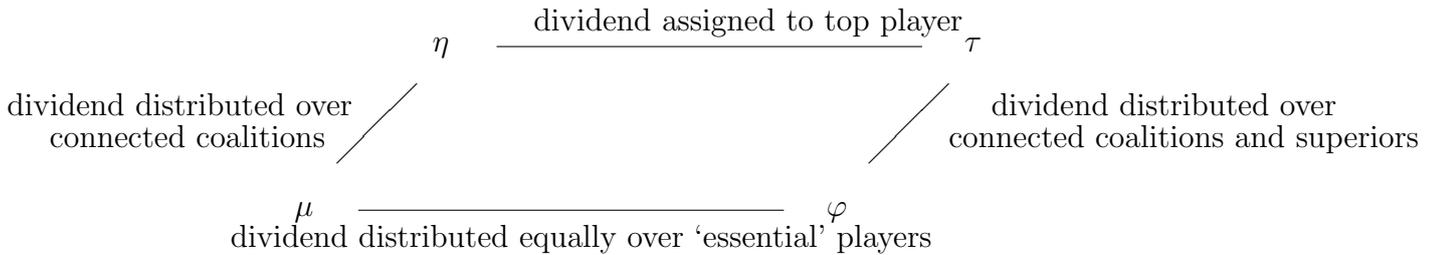
The interpretation of these solutions with respect to the hierarchy can be looked at in the following way. The Myerson value and the hierarchical outcome take a more local approach to hierarchies; coalitions within the hierarchy have some sense of autonomy in that they can operate without needing their predecessors to sign off on everything they do. The permission value and the top value take a global approach to hierarchies; everything a coalition does, needs to be approved by players located at a higher position in the hierarchy. The Myerson value and the permission value interpret the players of any feasible coalition in the hierarchy to play an equally important role. They should therefore be rewarded equally. The hierarchical outcome and the top value interpret the (local or global) top player of any feasible coalition to be the one that is in control, therefore this player should obtain all of the rewards.

7.4 Axiomatizations replacing the axioms on predecessors by axioms on superiors

In this section we consider different axiomatizations of the four solutions. We first introduce five new axioms that are used.

The first property is that of superior necessity, which is similar to predecessor necessity. The property of superior necessity is satisfied when the payoff to a player does not change when it becomes necessary for any of its subordinates. Superior necessity is weaker than predecessor necessity, in the sense that it only concerns the payoff of the player that becomes necessary, whereas predecessor necessity concerns the payoffs of all players. It is stronger than predecessor necessity, in the sense that it holds with respect to all the superiors of a player, instead of just the predecessors.

Superior necessity For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ and $i, j \in N$ such that $j \in \widehat{F}_T(i)$, it holds that $f_i(N, v, T) = f_i(N, v_j^i, T)$.

Consider a permission tree game (N, v, T). Let $C_j(i) \subset N$ be the set of those players $m \in N$, such that there exists a path $(i_0, \ldots, i_k)$ in (N, L_T), where $i_0 = j$, $i_k = m$ and $i_l \neq i$ for $l \in \{1, \ldots, k-1\}$. These are exactly the players in N that j is connected to in (N, T), such that player i is not needed to connect j to those players.

For the following weaker version of superior necessity it no longer holds that the payoff distribution to player i does not change if it becomes necessary for its subordinate j. However if j's marginal contribution to any coalition in the graph consisting only of players in $C_j(i)$ is zero, then it does hold that the payoff distribution does not change if player i becomes necessary for player j. Note that this property mirrors weak predecessor necessity, since if $(i, j) \in T$ then $C_j(i) = \widehat{F}_T(i)$. The difference is that it only concerns the payoff of the player that becomes necessary, whereas predecessor necessity concerns the payoffs of all players and that it holds with respect to all the superiors of a player, instead of just the predecessors.

Weak superior necessity For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ and $i, j \in N$ such that $j \in \widehat{F}_T(i)$, when $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq C_j(i) \cup \{j\}$ with $j \in S$, it holds that $f_i(N, v, T) = f_i(N, v_j^i, T)$.

A solution satisfies necessary player symmetry, when necessary players obtain the same payoff. Note that on the class of monotonic games, the necessary player property implies necessary player symmetry.

Necessary player symmetry For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ such that $i, j \in N$ are necessary players in (N, v) it holds that $f_i(N, v, T) = f_j(N, v, T)$.

A solution satisfies weak necessary player symmetry, when it assigns the same payoff to two necessary players i and j, where neither $i \in \widehat{F}_T(j)$, nor $j \in \widehat{F}_T(i)$. This expresses that neither player is able to dominate the other in the hierarchy.

Weak necessary player symmetry For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$, such that $i, j \in N$ are both necessary players in (N, v), $i \notin \widehat{F}_T(j)$ and $j \notin \widehat{F}_T(i)$, it holds that $f_i(N, v, T) = f_j(N, v, T)$.

Recall from (1.0.1) that if player i becomes a proxy for player j in game v, the game v_{ij} generated by the proxy agreement is given by

$$v_{ij}(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S \cup \{j\}) & \text{if } i \in S. \end{cases}$$

Compared to $v_j^i(S)$ the only difference is that when $i \in S$ it holds that $v_{ij}(S) = v(S \cup \{j\})$, whereas $v_j^i(S) = v(S)$.

It is seen that the dividend of a coalition R in game (N, v) such that $i \notin R$ and $j \in R$ is shifted to that of coalition $R \cup \{i\} \setminus \{j\}$ in game (N, v_{ij}). We therefore have the following expression:

$$v_{ij} = v + \sum_{\{R \subseteq N : j \in R\}} \Delta_v(R)[u_{R \cup \{i\} \setminus \{j\}} - u_R] \tag{7.4.2}$$

Next let players $i, j \in N$ be such that $j \in \widehat{F}_T(i)$ and i is necessary for j in game (N, v, T). The full domination property is satisfied, when the payoff distribution to player i does not change, if i acts on behalf of j in the sense of Haller. It expresses that when one player is able to 'dominate' another player in both the permission structure as well as in the game, it can act on behalf of that dominated player (with the latter player becoming a null player).

Full domination property For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ and $i, j \in N$ such that $j \in \widehat{F}_T(i)$ and i necessary for j in (N, v), it holds that $f_i(N, v, T) = f_i(N, v_{ij}, T)$.

We first characterize the Myerson value for permission tree games by efficiency, additivity, the pending null player out property, weak superior necessity and necessary player symmetry.

Theorem 7.4.1 *A solution on $\mathcal{G}_\mathcal{T}$ is equal to the Myerson value μ if and only if it satisfies efficiency, additivity, the pending null player out property, weak superior necessity and necessary player symmetry.*

PROOF
The Myerson value satisfying efficiency, additivity and the pending null player out property follows from Theorem 7.2.12. It is straightforward to verify that the Myerson value satisfies necessary player symmetry.

In the proof of Theorem 7.2.12 we have seen that the Myerson value satisfies weak predecessor necessity. In a similar way for $i, j \in N$ such that $j \in \widehat{F}_T(i)$ it can be shown that $\mu(N, v, T) = \mu(N, v_j^i, T)$ when $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq C_j(i) \cup \{j\}$ with $j \in S$. Therefore also $\mu_i(N, v, T) = \mu_i(N, v_j^i, T)$, showing that the Myerson value satisfies weak superior necessity.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$ let there be one $c \in \mathbb{R}$ such that $(N, w_U, T) \in \mathcal{G}_\mathcal{T}$ is the permission tree game, with $w_U = cu_U$ the unanimity

game of U scaled by c. For a set $\emptyset \neq R \subseteq N$ let $K(R) = \gamma_T(R) \setminus R$ be the set of players needed to connect players in R in undirected graph (N, L_T), but who are not in R themselves. Now consider $(N, w_R, T) \in \mathcal{G}_{\mathcal{T}}, \emptyset \neq R \subseteq N$. We show by induction on $k(R) = |K(R)|$ that

$$f_i(N, w_R, T) = \begin{cases} \frac{c}{|\gamma_T(R)|} & \text{if } i \in \gamma_T(R) \\ 0 & \text{otherwise,} \end{cases} \tag{7.4.3}$$

uniquely determining the payoff distribution.

If $k(R) = 0$, then R is a connected coalition in (N, T). So, $\gamma_T(R) = R$. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0. Therefore by efficiency the players in R obtain $w_R(N) = c$. Since the players in R are all necessary players in (N, w_R, T), necessary player symmetry implies that $f_i(N, w_R, T) = \frac{c}{|R|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.3).

Proceeding by induction, assume that $f_i(N, w_{R'}, T) = \frac{c}{|\gamma_T(R')|}$ for $i \in \gamma_T(R')$ and $f_i(N, w_{R'}, T) = 0$ otherwise, when $k(R') < k(R)$. Again by Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R)$ obtain a payoff of 0. Now consider the players in $K(R)$. This set is non-empty. Since any player $i \in K(R)$ is needed to connect players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply weak superior necessity to obtain $f_i(N, w_R, T) = f_i(N, w_{R \cup \{i\}}, T)$ (since $(w_R)_j^i = w_{R \cup \{i\}}$). Since $k(R \cup \{i\}) < k(R)$, $f_i(N, w_{R \cup \{i\}}, T) = \frac{c}{|\gamma_T(R \cup \{i\})|} = \frac{c}{|\gamma_T(R)|}$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = \frac{c}{|\gamma_T(R)|}$. By efficiency, the payoff that remains to be distributed over the players in R is given by $c - \frac{k(R)c}{|\gamma_T(R)|} = \frac{|\gamma_T(R)|c}{|\gamma_T(R)|} - \frac{k(R)c}{|\gamma_T(R)|} = \frac{|R|c}{|\gamma_T(R)|}$, since $|\gamma_T(R)| - k(R) = |R|$. Since the players in R are all necessary players in (N, w_R, T), necessary player symmetry implies that $f_i(N, w_R, T) = \frac{c}{|\gamma_T(R)|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.3).

Finally, since for every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ it holds that v can be written as $v = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) u_R$, where u_R is the unanimity game on coalition R, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. $\qquad \square$

We characterize the hierarchical outcome for permission tree games by replacing necessary player symmetry in Theorem 7.4.1 by weak necessary player symmetry and the full domination property.

Theorem 7.4.2 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the hierarchical outcome η if and only if it satisfies efficiency, additivity, the pending null player out property, weak superior necessity, weak necessary player symmetry and the full domination property.*

PROOF
The hierarchical outcome satisfying efficiency, additivity and the pending null player out property follows from Theorem 7.2.13. It is straightforward to verify that the hierarchical outcome satisfies weak necessary player symmetry.

In the proof of Theorem 7.2.13 we have seen that the hierarchical outcome satisfies weak predecessor necessity, i.e. $\eta(N, v, T) = \eta(N, v_j^i, T)$ when $(i, j) \in T$ and $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq (\widehat{F}_T(j) \cup \{j\})$ with $j \in S$. In a similar way it can be shown for $i, j \in N$ such that $j \in \widehat{F}_T(i)$ that $\eta(N, v, T) = \eta(N, v_j^i, T)$ when $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq C_j(i) \cup \{j\}$ with $j \in S$. Therefore also $\eta_i(N, v, T) = \eta_i(N, v_j^i, T)$, showing that the hierarchical outcome satisfies weak superior necessity.

To show that the hierarchical outcome satisfies the full domination property we argue as follows. For a coalition R, let $r_0(R) = TOP_{(N,T)}(\gamma_T(R))$. By Proposition 7.2.7, $\eta_k(N, v, T) = \sum\limits_{\{R \subseteq N : k = r_0(R)\}} \Delta_v(R)$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$ for all $k \in N$. Denote the collection of coalitions R such that $j \in R$ and $i \notin R$ by $\mathcal{V}$. Denote the collection of coalitions R such that $i, j \in R$ by $\mathcal{W}$. By Proposition 7.2.7 and (7.4.2), $\eta_k(N, v_{ij}, T) = \sum\limits_{\{R \subseteq N \setminus (\mathcal{W} \cup \mathcal{V}) : k = r_0(R)\}} \Delta_v(R) + \sum\limits_{\{R \in \mathcal{V} : k = r_0(R \setminus \{j\} \cup \{i\})\}} \Delta_v(R) + \sum\limits_{\{R \in \mathcal{W} : k = r_0(R \setminus \{j\})\}} \Delta_v(R)$ for all $k \in N$. The full domination property can be applied, when i is necessary for j and $j \in \widehat{F}_T(i)$. In that case $\Delta_v(R) = 0$ for $R \in \mathcal{V}$ and $r_0(R) = r_0(R \setminus \{j\})$ for $R \in \mathcal{W}$. We obtain $\eta(N, v, T) = \eta(N, v_{ij}, T)$, showing that the hierarchical outcome satisfies the full domination property.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For all $\emptyset \neq U \subseteq N$ let there be one $c \in \mathbb{R}$ such that $(N, w_U, T) \in \mathcal{G}_\mathcal{T}$ is the permission tree game, with $w_U = cu_U$ the unanimity game of U scaled by c. For a set $\emptyset \neq R \subseteq N$ as before let $K(R) = \gamma_T(R) \setminus R$ be the set of players needed to connect players in R in undirected graph (N, L_T), but who are not in R themselves. Now consider $(N, w_R, T) \in \mathcal{G}_\mathcal{T}, \emptyset \neq R \subseteq N$. Let $r_0 = TOP_{(N,T)}(\gamma_T(R))$ be the top player of the connected hull of R. We show by induction on $k(R) = |K(R)|$ that

$$f_i(N, w_R, T) = \begin{cases} c & \text{if } i = r_0 \\ 0 & \text{otherwise,} \end{cases} \tag{7.4.4}$$

uniquely determining the payoff distribution.

First consider those sets R such that $|R| = 1$. In that case the unique player in R must be the top player r_0 of the connected hull of R. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R = N \setminus \{r_0\}$ obtain a payoff of 0. It follows by efficiency that $f_{r_0}(N, w_R, T) = c$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.4).

Next consider any set R. If $k(R) = 0$, then R is a connected coalition in (N, T). So, $\gamma_T(R) = R$. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0. Since $(R \setminus \{r_0\}) \subseteq \widehat{F}_T(r_0)$ and r_0 is a veto player in w_R, by repeated application of the full domination property we obtain $f_{r_0}(N, w_R, T) = f_{r_0}(N, w_{r_0}, T)$. It follows from before that $f_{r_0}(N, w_R, T) = c$. By efficiency it now holds that $\sum_{i \in R \setminus \{r_0\}} f_i(N, w_R, T) = 0$. We perform a second induction on $l(R) = |R \setminus \{r_0\}|$. If $l(R) = 1$, then for the unique $i \in R \setminus \{r_0\}$ it holds that $f_i(N, w_R, T) = 0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.4). Proceeding by induction, assume that $f_{r_0}(N, w_{R'}, T) = c$ and $f_i(N, w_{R'}, T) = 0$ for $i \in R' \setminus$

$\{r_0\}$, when $l(R') < l(R)$. First consider those players $i \in R$ such that $\widehat{F}_T(i) \cap (R \setminus \{r_0\}) \neq \emptyset$. For any such an i we can select a player $j \in \widehat{F}_T(i)$ such that $\widehat{F}_T(j) \cap R = \emptyset$. By application of the full domination property to i and j we obtain $f_i(N, w_R, T) = f_i(N, w_{R \setminus \{j\}}, T)$. Since $l(R \setminus \{j\}) < l(R)$ (and $k(R \setminus \{j\}) = 0$), $f_i(N, w_{R \setminus \{j\}}, T) = 0$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = 0$. Next consider the set $S \subseteq (R \setminus \{r_0\})$ containing all players i such that $\widehat{F}_T(i) \cap (R \setminus \{r_0\}) = \emptyset$. If $|S| = 1$ then by efficiency it must hold for the unique player $i \in S$ that $f_i(N, w_R, T) = 0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.4). If $|S| > 1$ by weak necessary player symmetry and the fact that all players in S are necessary we have $f_i(N, w_R, T) = f_j(N, w_R, T)$ for any $i, j \in S$. It then follows from efficiency that $f_i(N, w_R, T) = 0$ for $i \in S$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.4).

Proceeding by induction, assume that $f_{r_0}(N, w_{R'}, T) = c$ and $f_i(N, w_{R'}, T) = 0$ for $i \in R' \setminus \{r_0\}$, when $k(R') < k(R)$. First we consider the payoff to r_0. If $r_0 \in R$, we obtain $f_{r_0}(N, w_R, T) = f_{r_0}(N, w_{r_0}, T(r_0))$ by repeated application of the full domination property. It follows that $f_{r_0}(N, w_R, T) = c$. If $r_0 \notin R$, then $r_0 \in K(R)$. Since then, r_0 is needed to connect players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply weak superior necessity to obtain $f_{r_0}(N, w_R, T) = f_{r_0}(N, w_{R \cup \{r_0\}}, T)$ (since $(w_R)_j^{r_0} = w_{R \cup \{r_0\}}$). Since $k(R \cup \{r_0\}) < k(R)$, $f_{r_0}(N, w_{R \cup \{r_0\}}, T) = c$ by the induction hypothesis. It follows that $f_{r_0}(N, w_R, T) = c$. Next we consider the payoff to players in $K(R) \setminus \{r_0\}$. Since any player in this set is needed to connect players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply weak superior necessity to obtain $f_i(N, w_R, T) = f_i(N, w_{R \cup \{i\}}, T)$ (since $(w_R)_j^i = w_{R \cup \{i\}}$). Since $k(R \cup \{i\}) < k(R)$, $f_i(N, w_{R \cup \{i\}}, T) = 0$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = 0$. Finally we consider the players in $R \setminus \{r_0\}$. By efficiency it holds that $\sum_{i \in R \setminus \{r_0\}} f_i(N, w_R, T) = 0$. Again we perform a second induction on $l(R) = |R \setminus \{r_0\}|$. In a similar way as for $k(R) = 0$, it follows that $f_i(N, w_R, T) = 0$ for $i \neq r_0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.4).

Finally, since for every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ it holds that v can be written as $v = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) u_R$, where u_R is the unanimity game on coalition R, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. $\qquad\square$

Compared to Theorem 7.4.2 the top value is characterized by replacing the pending null player out property by the weak pending null player out property and by replacing weak superior necessity by superior necessity.

Theorem 7.4.3 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the top value $\tau(N, v, T)$ if and only if it satisfies efficiency, additivity, the weak pending null player out property, superior necessity, weak necessary player symmetry and the full domination property.*

PROOF
The top value satisfying efficiency, additivity and the weak pending null player out property follows from Theorem 7.2.14. It is straightforward to verify that the top value satisfies weak necessary player symmetry.

The top value satisfying superior necessity follows from $v^i_j(N) = v(N)$ and the fact that for any two games (N, v, T) and (N, v', T) such that $v(N) = v'(N)$ it holds that $\tau(N, v, T) = \tau(N, v', T)$.

The top value satisfying the full domination property follows from the fact that $v_{ij}(N) = v(N)$ and the fact that for any two games (N, v, T) and (N, v', T) such that $v(N) = v'(N)$ it holds that $\tau(N, v, T) = \tau(N, v', T)$.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For a set $\emptyset \neq R \subseteq N$ let $K(R) = \{\widehat{P}_T(R) \setminus R\}$ be the set of superiors of players in R that is not included in R. Now consider $(N, w_R, T) \in \mathcal{G}_T$, where (N, w_R) is a scaled unanimity game with $w_R = cu_R, c \in \mathbb{R}, \emptyset \neq R \subseteq N$. Let $i_0 = TOP_{(N,T)}(N)$ be the top player of (N, T). We show by induction on $k(R) = |K(R)|$ that

$$f_i(N, w_R, T) = \begin{cases} c & \text{if } i = i_0 \\ 0 & \text{otherwise,} \end{cases} \tag{7.4.5}$$

uniquely determining the payoff distribution.

First consider $R = \{i_0\}$. By Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R = N \setminus \{i_0\}$ obtain a payoff of 0. It follows by efficiency that $f_{i_0}(N, w_R, T) = c$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.5).

Next consider any set R. If $k(R) = 0$, then R has full permission in (N, T). So, $\alpha_T(R) = R$ and $i_0 \in R$. By Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0. Since $(R \setminus \{i_0\}) \subseteq \widehat{F}_T(i_0)$ and i_0 is a veto player in w_R, by repeated application of the full domination property we obtain $f_{i_0}(N, w_R, T) = f_{i_0}(N, w_{i_0}, T)$. It follows from before that $f_{i_0}(N, w_R, T) = c$. By efficiency it now holds that $\sum_{i \in R \setminus \{i_0\}} f_i(N, w_R, T) = 0$. We perform a second induction on $l(R) = |R \setminus \{i_0\}|$. If $l(R) = 1$, then for the unique $i \in R \setminus \{i_0\}$ it holds that $f_i(N, w_R, T) = 0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.5). Proceeding by induction, assume that $f_{i_0}(N, w_{R'}, T) = c$ and $f_i(N, w_{R'}, T) = 0$ for $i \in R' \setminus \{i_0\}$, when $l(R') < l(R)$. First consider those players $i \in R$ such that $\widehat{F}_T(i) \cap (R \setminus \{i_0\}) \neq \emptyset$. For any such an i we can select a player $j \in \widehat{F}_T(i)$ such that $\widehat{F}_T(j) \cap R = \emptyset$. By application of the full domination property to i and j we obtain $f_i(N, w_R, T) = f_i(N, w_{R \setminus \{j\}}, T)$. Since $l(R \setminus \{j\}) < l(R)$ (and $k(R \setminus \{j\}) = 0$), $f_i(N, w_{R \setminus \{j\}}, T) = 0$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = 0$. Next consider the set $S \subseteq (R \setminus \{i_0\})$ containing all players i such that $\widehat{F}_T(i) \cap (R \setminus \{i_0\}) = \emptyset$. If $|S| = 1$ then by efficiency it must hold for the unique player $i \in S$ that $f_i(N, w_R, T) = 0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.5). If $|S| > 1$ by weak necessary player symmetry and the fact that all players in S are necessary we have $f_i(N, w_R, T) = f_j(N, w_R, T)$ for any $i, j \in S$. It then follows from efficiency that $f_i(N, w_R, T) = 0$ for $i \in S$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.5).

Proceeding by induction, assume that $f_{i_0}(N, w_{R'}, T) = c$ and $f_i(N, w_{R'}, T) = 0$ for $i \in R' \setminus \{i_0\}$, when $k(R') < k(R)$. First we consider the payoff to i_0. If $i_0 \in R$, we obtain $f_{i_0}(N, w_R, T) = f_{i_0}(N, w_{i_0}, T)$ by repeated application of the full domination

property. It follows that $f_{i_0}(N, w_R, T) = c$. If $i_0 \notin R$, then $i_0 \in K(R)$. Since then, i_0 is needed to give permission players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply superior necessity to obtain $f_{i_0}(N, w_R, T) = f_{i_0}(N, w_{R \cup \{i_0\}}, T)$ (since $(w_R)_j^{i_0} = w_{R \cup \{i_0\}}$). Since $k(R \cup \{i_0\}) < k(R)$, $f_{i_0}(N, w_{R \cup \{i_0\}}, T) = c$ by the induction hypothesis. It follows that $f_{i_0}(N, w_R, T) = c$. Next we consider the payoff to players in $K(R) \setminus \{i_0\}$. Since any player in this set is needed to give permission to players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply superior necessity to obtain $f_i(N, w_R, T) = f_i(N, w_{R \cup \{i\}}, T)$ (since $(w_R)_j^i = w_{R \cup \{i\}}$). Since $k(R \cup \{i\}) < k(R)$, $f_i(N, w_{R \cup \{i\}}, T) = 0$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = 0$. Finally we consider the players in $R \setminus \{i_0\}$. By efficiency it holds that $\sum_{i \in R \setminus \{i_0\}} f_i(N, w_R, T) = 0$. Again we perform a second induction on $l(R) = |R \setminus \{i_0\}|$. In a similar way as for $k(R) = 0$, it follows that $f_i(N, w_R, T) = 0$ for $i \neq i_0$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.5).

Finally, since for every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ it holds that v can be written as $v = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) u_R$, where u_R is the unanimity game on coalition R, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. $\square$

Finally, we characterize the permission value using only axioms that were used in the three axiomatizations before.

Theorem 7.4.4 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the permission value φ if and only if it satisfies efficiency, additivity, the weak pending null player out property, superior necessity and necessary player symmetry.*

PROOF
The permission value satisfying efficiency, additivity and the weak pending null player out property follows from Theorem 7.2.15. It is straightforward to verify that the permission value satisfies necessary player symmetry.

In the proof of Theorem 7.2.15 we have seen that the permission value satisfies predecessor necessity, i.e $\varphi(N, v, T) = \varphi(N, v_j^i, T)$ when $(i, j) \in T$. In a similar way it can be shown for $i, j \in N$ such that $j \in \widehat{F}_T(i)$ that $\varphi(N, v, T) = \varphi(N, v_j^i, T)$. Therefore also $\varphi_i(N, v, T) = \varphi_i(N, v_j^i, T)$, showing that the permission value satisfies superior necessity.

The proof of uniqueness is given as follows.

Let f be a solution satisfying the axioms. For a set $\emptyset \neq R \subseteq N$ let $K(R) = \widehat{P}_T(R) \setminus R$ be the set of superiors of players in R that is not included in R. Now consider $(N, w_R, T) \in \mathcal{G}_{\mathcal{T}}$, where (N, w_R) is a scaled unanimity game with $w_R = c u_R, c \in \mathbb{R}, \emptyset \neq R \subseteq N$. We show by induction on $k(R) = |K(R)|$ that

$$f_i(N, w_R, T) = \begin{cases} \frac{c}{|\alpha_T(R)|} & \text{if } i \in \alpha_T(R) \\ 0 & \text{otherwise,} \end{cases} \tag{7.4.6}$$

uniquely determining the payoff distribution.

If $k(R) = 0$, then R has full permission in (N, T). So, $\alpha_T(R) = R$. By Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the

players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0. Therefore by efficiency the players in R obtain $w_R(N) = c$. Since the players in R are all necessary players in (N, w_R, T), necessary player symmetry implies that $f_i(N, w_R, T) = \frac{c}{|R|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.6).

Proceeding by induction, assume that $f_i(N, w_{R'}, T) = \frac{c}{|\alpha_T(R')|}$ for $i \in \alpha_T(R')$ and $f_i(N, w_{R'}, T) = 0$ otherwise, when $k(R') < k(R)$. Again by Proposition 7.2.10.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R)$ obtain a payoff of 0. Now consider the players in $K(R)$. This set is non-empty. Since any player $i \in K(R)$ is needed to give permission to players in R, it must have at least one subordinate j in R (since (N, T) is a rooted tree). We can therefore apply superior necessity to obtain $f_i(N, w_R, T) = f_i(N, w_{R \cup \{i\}}, T)$ (since $(w_R)_j^i = w_{R \cup \{i\}}$). Since $k(R \cup \{i\}) < k(R)$, $f_i(N, w_{R \cup \{i\}}, T) = \frac{c}{|\alpha_T(R \cup \{i\})|} = \frac{c}{|\alpha_T(R)|}$ by the induction hypothesis. It follows that $f_i(N, w_R, T) = \frac{c}{|\alpha_T(R)|}$. By efficiency, the payoff that remains to be distributed over the players in R is given by $c - \frac{k(R)c}{|\alpha_T(R)|} = \frac{|\alpha_T(R)|c}{|\alpha_T(R)|} - \frac{k(R)c}{|\alpha_T(R)|} = \frac{|R|c}{|\alpha_T(R)|}$, since $|\alpha_T(R)| - k(R) = |R|$. Since the players in R are all necessary players in (N, w_R, T), necessary player symmetry implies that $f_i(N, w_R, T) = \frac{c}{|\alpha_T(R)|}$ for all $i \in R$ and thus $f(N, w_R, T)$ is uniquely determined as in (7.4.6).

Finally, since for every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ it holds that v can be written as $v = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) u_R$, where u_R is the unanimity game on coalition R, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. $\qquad \square$

Logical independence of the axioms used to characterize the above four solutions is shown in appendix C of this chapter.

As before we can compare the four solutions based on which axioms they satisfy. We obtain the following table.

Table 2:Axioms satisfied by the four solutions

Property	μ	η	φ	τ
Efficiency	++	++	++	++
Additivity	++	++	++	++
Weak pending null player out	+	+	++	++
Weak necessary player symmetry	+	+	++	++
Weak superior necessity	++	++	+	+
Pending null player out	++	++	-	-
Necessary player symmetry	++	-	++	-
Superior necessity	-	-	++	++
Full domination property	-	++	-	++

We find that the axioms of weak predecessor necessity and predecessor necessity from Table 1 can be substituted by weak superior necessity and superior necessity respectively. The one player property can be substituted for the full domination property. The

weak necessary player property and the necessary player property can be substituted for weak necessary player symmetry and necessary player symmetry respectively.

7.5   Concluding remarks

In this chapter we have given comparable axiomatizations of four solutions for permission tree games. We have seen that two distinctions can be made between these solutions. The first distinction is based on hierarchy and communication. The Myerson value and the hierarchical outcome can be considered communication solutions, whereas the permission value and the top value can be considered hierarchical solutions. The second distinction is based on distribution of payoff. The Myerson value and the permission value distribute payoff equally over players considered essential to a coalition, whereas the hierarchical outcome and the top value distribute all payoff to the (local) top player in the graph. One point of future research is to consider other solutions for permission tree games and compare them to the four solutions similar to the way done in this chapter. In their paper van den Brink, Herings, van der Laan and Talman (2013) considered both the permission value as well as the average tree permission value, which adapts the average tree solution for graph games to permission tree games. Other modifications of already existing solutions for graph games or games with a permission structure could also be considered for this, for example the local permission value introduced in Chapter 4 of this thesis.

A second comparable axiomatization has been given that replaces the predecessor axioms by superior axioms plus a weak version of necessary player symmetry. We might also consider even more different ways of axiomatizing the solutions from this chapter. Note that we can use Proposition 7.2.11 to replace the axioms of predecessor necessity and weak predecessor necessity in the axiomatizations of section 2 by the following consistency axioms. In theorems 7.2.12 and 7.2.13 we can replace weak predecessor necessity by communication associated consistency. This property is satisfied when payoff assigned by a solution remains unchanged, when the dividend of a coalition R is shifted to its connected hull $\gamma_T(R)$ in the graph.

Communication associated consistency For every permission tree game $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ and coalition $R \subseteq N$ it holds that $f(N, v, T) = f(N, v', T)$, where v' is given by $v' = v - \Delta_v(R)u_R + \Delta_v(R)u_{\gamma_T(R)}$.

In theorems 7.2.14 and 7.2.15 we can replace predecessor necessity by permission associated consistency. This property is satisfied when payoff assigned by a solution remains unchanged, when the dividend of a coalition R is shifted to its permission hull $\alpha_T(R)$ in the graph.

Permission associated consistency For every permission tree game $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ and coalition $R \subseteq N$ it holds that $f(N, v, T) = f(N, v', T)$, where v' is given by $v' = v - \Delta_v(R)u_R + \Delta_v(R)u_{\alpha_T(R)}$.

Appendix A: Logical independence of axiomatizations in section 2

Logical independence of the five axioms stated in Theorem 7.2.12 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N, v, T) = 0$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$, $i \in N$, satisfies additivity, the pending null player out property, weak predecessor necessity and the necessary player property. It does not satisfy efficiency.

2. For $(N, v, T) \in \mathcal{G}_\mathcal{T}$ let $Null(v)$ be the set of null players in v. Let f be the solution that for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ divides the worth $v(N)$ of the grand coalition equally over all non-null players and the players that connect these players in the graph and assigns a 0 payoff otherwise. So for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ solution f is given by $f_i(N, v, T) = \frac{v(N)}{|\gamma_T(N \setminus Null(v))|}$ for $i \in \gamma_T(N \setminus Null(v))$ and $f_i(N, v, T) = 0$ for $i \in N \setminus \gamma_T(N \setminus Null(v))$. This solution satisfies efficiency, the pending null player out property, weak predecessor necessity and the necessary player property. It does not satisfy additivity.

3. The permission value satisfies efficiency, additivity, weak predecessor necessity and the necessary player property. It does not satisfy the pending null player out property.

4. The solution $f(N, v, T) = Sh(N, v)$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$ satisfies efficiency, additivity, the pending null player out property and the necessary player property. It does not satisfy weak predecessor necessity.

5. The hierarchical outcome satisfies efficiency, additivity, the pending null player out property and weak predecessor necessity. It does not satisfy the necessary player property.

Logical independence of the six axioms stated in Theorem 7.2.13 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N, v, T) = 0$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$, $i \in N$, satisfies additivity, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. Consider the class of games (N, v, T), for which there is a coalition $R \subseteq N$ such that $\Delta_v(S) = 0$ if $\gamma_T(S) \neq R$. The solution that on this class of games assigns the hierarchical outcome and on games not in this class assigns the Myerson value satisfies efficiency, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy additivity.

3. The top value satisfies efficiency, additivity, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy the pending null player out property.

4. For a coalition S define $A(S) = (\gamma_T(S) \setminus (\widehat{F}_T(S) \cup S))$ to be those players in the connected hull $\gamma_T(S)$ that are not included in S nor are they subordinates of players in S. Now consider $f_i = \sum_{S \subseteq N, i \in S, i = TOP_{(N,T)}(\gamma_T(S))} \Delta_v(S) +$

    $\sum_{S \subseteq N, S \neq \emptyset, TOP_{(N,T)}(\gamma_T(S)) \notin S, i \in A(S)} \frac{\Delta_v(S)}{|\gamma_T(S) \setminus (\widehat{F}_T(S) \cup S)|}$ $i \in N$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$. This solution assigns the dividend of coalitions S, such that the top player of the connected hull $\gamma_T(S)$ is included in S uniquely to that top player, and equally distributes the dividend of coalitions S not containing the top player over those players in the connected hull $\gamma_T(S)$ that are not included in S nor are they followers of players in S. This solution satisfies efficiency, additivity, the pending null player out property, the weak necessary player property and the one player property. It does not satisfy weak predecessor necessity.

5. Consider the solution that for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the dividend $\Delta_v(S)$ of a coalition S to the player $i \in \gamma_T(S) \setminus \{TOP_{(N,T)}(\gamma_T(S))\}$, such that $i > j$ for $j \in (\gamma_T(S) \setminus \{TOP_{(N,T)}(\gamma_T(S))\}) \setminus \{i\}$ for games $(N, v, T) \in \mathcal{G}_\mathcal{T}$. This solution satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the one player property. It does not satisfy the weak necessary player property.

6. The Myerson value satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the weak necessary player property. It does not satisfy the one player property.

Logical independence of the six axioms stated in Theorem 7.2.14 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N, v, T) = 0$, $(N, v, T) \in \mathcal{G}_\mathcal{T}$, $i \in N$, satisfies additivity, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. Let V be the class of games (N, v, T), for which there is a coalition $R \subseteq N$ such that $\Delta_v(S) = 0$ if $\alpha_T(S) \neq R$. The solution that on this class of games assigns the top value and on games not in this class assigns the permission value satisfies efficiency, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy additivity.

3. Consider the solution that for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the dividend $\Delta_v(S)$ of coalitions S such that $\alpha_T(S) = N$ to the top player of (N, T) and distributes the dividend $\Delta_v(S)$ equally over N otherwise for games $(N, v, T) \in \mathcal{G}_\mathcal{T}$. This solution satisfies efficiency, additivity, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy the weak pending null player out property.

4. The hierarchical outcome satisfies efficiency, additivity, the weak pending null player out property, the weak necessary player property and the one player property. It does not satisfy predecessor necessity.

5. Consider the solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ assigns the dividend $\Delta_v(S)$ of a coalition S to the player $i \in \alpha_T(S) \setminus TOP_{(N,T)}(N)$ such that $i > j$ for $j \in (\alpha_T(S) \setminus TOP_{(N,T)}(N)) \setminus \{i\})$ for games $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. This solution satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the one player property. It does not satisfy the weak necessary player property.

6. The permission value satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the weak necessary player property. It does not satisfy the one player property.

Logical independence of the five axioms stated in Theorem 7.2.15 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N, v, T) = 0$, $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$, $i \in N$, satisfies additivity, the weak pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. The solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ equally divides the worth $v(N)$ of the grand coalition over all non-null players and their predecessors in the graph and assigns a 0 payoff otherwise satisfies efficiency, the weak pending null player out property, predecessor necessity and the necessary player property. It does not satisfy additivity.

3. The solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ equally divides the worth $v(N)$ of the grand coalition over the players in N satisfies efficiency, additivity, predecessor necessity and the necessary player property. It does not satisfy the weak pending null player out property.

4. The solution $f(N, v, T) = Sh(N, v)$ satisfies efficiency, additivity, the weak pending null player out property and the necessary player property. It does not satisfy predecessor necessity.

5. The top value satisfies efficiency, additivity, the weak pending null player out property and predecessor necessity. It does not satisfy the necessary player property.

Appendix B: Non-existence of solutions satisfying combinations discussed in Section 3

The Myerson value, the hierarchical outcome, the top value and the permission value all satisfy efficiency, additivity, the weak pending null player out property, the weak necessary player property and weak predecessor necessity. These four solutions also satisfy two out of the following four axioms: the pending null player out property, the necessary player property, predecessor necessity and the one player property. There are six combinations containing two out of these four axioms. This leaves two combinations that have not been

considered in this chapter. Here we show that a solution satisfying efficiency, additivity, the weak pending null player out property, the weak necessary player property and weak predecessor necessity and one of these two combinations of axioms leads to a contradiction and therefore cannot exist.

1. A solution f satisfying efficiency, additivity, the pending null player out property, the weak necessary player property and predecessor necessity.

   Consider a permission tree game $(N, u_{\{i\}}, T)$, where $|N| > 1$ and $(N, u_{\{i\}})$ is the unanimity game of a player $i \neq TOP_{(N,T)}(N)$. By Proposition 7.2.10.(i) and f satisfying efficiency and the pending null player out property, it holds that $f_i(N, u_{\{i\}}, T) = u_{\{i\}}(N) = 1$ and $f_j(N, u_{\{i\}}, T) = 0$ for $j \in N \setminus \{i\}$. However by repeatedly applying predecessor necessity, we also obtain that $f(N, u_{\{i\}}, T) = f(N, u_{\alpha_T(i)}, T)$. According to the weak necessary player property it holds that $1 = f_i(N, u_{\{i\}}, T) = f_i(N, u_{\alpha_T(i)}, T) \leq f_j(N, u_{\alpha_T(i)}, T) = f_j(N, u_{\{i\}}, T) = 0$ for $j \in \widehat{P}_T(i)$. Since $\widehat{P}_T(i) \neq \emptyset$ we obtain a contradiction and f cannot exist.

2. A solution f satisfying efficiency, additivity, the weak pending null player out property, the necessary player property and weak predecessor necessity and the one player property.

   Consider a permission tree game (N, u_N, T), where $|N| > 1$ and (N, u_N) is the unanimity game on the grand coalition. According to the one player property only one player can have a payoff that is non-zero. Efficiency implies that the payoff to this player must be $u_N(N) = 1$. However the necessary player property implies that $f_i(N, u_N, T) = f_j(N, u_N, T)$ for any two necessary players $i, j \in N$. Since in u_N all players in N are necessary and $|N| > 1$ we obtain a contradiction and f cannot exist.

Appendix C: Logical independence of axiomatizations in Section 4

Logical independence of the five axioms stated in Theorem 7.4.1 is shown by using the solutions from Appendix A, substituting weak predecessor necessity for weak superior necessity and the necessary player property for necessary player symmetry.

Logical independence of the axioms of efficiency, additivity, the pending null player out property, weak superior necessity and the full domination property stated in Theorem 7.4.2 is shown by using the solutions from Appendix A, substituting weak predecessor necessity for weak superior necessity, the weak necessary player property for weak necessary player symmetry and the one player property for the full domination property. Here we consider weak necessary player symmetry.

1. Define the collection D_{wns} as those coalitions R such that the top player of the connected hull of R is not included in R and this top player has two predecessors k, l, where $k < l$, and $R \subseteq (\widehat{F}_T(k) \cup \{k\} \cup \widehat{F}_T(l) \cup \{l\})$ and $R \cap (\widehat{F}_T(l) \cup \{l\}) \neq \emptyset$

and $R \cap (\widehat{F}_T(k) \cup \{k\}) \neq \emptyset$. Now consider the solution that assigns the dividend of any coalition R to the top player of the connected hull of R and for any coalition $R \in D_{wns}$ assigns to players l and k as before a payoff as follows: $-\Delta_v(R)$ to player k and $\Delta_v(R)$ to player l. This solution satisfies efficiency, additivity, the pending null player out property, weak superior necessity and the full domination property. It does not satisfy weak necessary player symmetry.

Logical independence of the axioms of efficiency, superior necessity and the full domination property stated in Theorem 7.4.3 is shown by using the solutions from Appendix A, substituting predecessor necessity for superior necessity, the weak necessary player property for weak necessary player symmetry and the one player property for the full domination property. Here we consider additivity, the weak pending null player out property and weak necessary player symmetry.

1. Consider the class of games (N, v, T) such that the following holds. The unique top player $i_0 = TOP_{(N,T)}(N)$ of directed rooted tree (N, T) has at least two successors, so $|F_T(i_0)| \geq 2$. Let the set of successors to top player i_0 be given by $\{1, 2, ..., m\}$ for $|F_T(i_0)| = m$. Also assume there exist two non-empty sets R, R' such that for R it holds that $\Delta_v(R) \neq 0$ and $R \subseteq (\widehat{F}_T(1) \cup \{1\})$ and for R' it holds that $\Delta_v(R') \neq 0$ and $R \subseteq (\widehat{F}_T(2) \cup \{2\})$. Define this class of games by $v \in \mathcal{G}^{add}$. Now consider the solution that for (N, v, T) in $\mathcal{G}^{add}$ assigns payoff $v(N)$ to player i_0, 1 to player 1, -1 to player 2 and 0 to the players in $N \setminus \{i_0, 1, 2\}$ and for (N, v, T) not in $\mathcal{G}^{add}$ assigns payoff according to the top value. This solution satisfies efficiency, the weak pending null player out property, superior necessity, weak necessary player symmetry and the full domination property. It does not satisfy additivity.

2. The solution that for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ equally divides $v(N)$ over the players in N satisfies efficiency, additivity, superior necessity, weak necessary player symmetry and the full domination property. It does not satisfy the weak pending null player out property.

3. Define the collection D_{wns2} as those coalitions R such that the top player of (N, T) is not included in R and this top player has two predecessors k, l, where $k < l$, and $R \subseteq (\widehat{F}_T(k) \cup \{k\} \cup \widehat{F}_T(l) \cup \{l\})$ and $R \cap (\widehat{F}_T(l) \cup \{l\}) \neq \emptyset$ and $R \cap (\widehat{F}_T(k) \cup \{k\}) \neq \emptyset$. Now consider the solution that for $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the dividend of any coalition R to the top player of (N, T) and for any coalition $R \in D_{wns2}$ assigns to players l and k as before a payoff as follows: $-\Delta_v(R)$ to player k and $\Delta_v(R)$ to player l. This solution satisfies efficiency, additivity, the weak pending null player out property, superior necessity and the full domination property. It does not satisfy weak necessary player symmetry.

Logical independence of the five axioms stated in Theorem 7.4.4 is shown by using the solutions from Appendix A.

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Samenvatting (summary in Dutch)

Restricties met betrekking tot coöperatie

Het vakgebied van speltheorie houdt zich bezig met de analyse van het gedrag van agenten die hun nut proberen te maximaliseren in situaties van sociale interactie tussen deze agenten. Een onderscheid kan worden gemaakt tussen twee takken van speltheorie: niet-coöperatieve speltheorie en coöperatieve speltheorie. Binnen de coöperatieve speltheorie wordt uitgegaan van de mogelijkheid van agenten om bindende contracten af te kunnen sluiten met elkaar. Dit is niet mogelijk binnen de niet-coöperatieve speltheorie. Het kunnen afsluiten van bindende contracten staat toe dat agenten met elkaar kunnen samenwerken en coalities kunnen vormen om daarmee hun afzonderlijke nut te maximaliseren. De coöperatieve speltheorie maakt gebruik van het model van *coöperatieve spelen (met overdraagbaar nut)* om situaties van samenwerking tussen agenten te bestuderen. Een coöperatief spel bestaat uit een verzameling spelers en een zogenaamde karakteristieke functie. De verzameling spelers representeert de agenten die met elkaar kunnen samenwerken en de karakteristieke functie wijst aan iedere coalitie van spelers een waarde toe die weergeeft wat deze spelers samen aan nut kunnen verkrijgen door samen te werken. Een oplossing voor coöperatieve spelen is een functie die aan ieder coöperatief spel een uitbetalingsvector toewijst. Deze uitbetalingsvector wijst aan iedere speler een waarde toe die de hoeveelheid nut, toegedeeld aan deze speler, weergeeft. Een van de bekendste oplossingen binnen de coöperatieve speltheorie is de Shapley-waarde. De twee hoofdvragen waar de coöperatieve speltheorie zich mee bezig houdt is welke coalities van spelers zich uiteindelijk vormen (coalitievorming) en hoe het nut verkregen door samenwerking over de afzonderlijke spelers verdeeld wordt (nutsverdeling). Beide vragen zijn sterk met elkaar verbonden; spelers zullen een voorkeur hebben voor die coalities, waarbij het nut op een voor hun zo gunstigst mogelijk manier verdeeld wordt. In plaats van te bestuderen hoe een bepaalde nutsverdeling zich verhoudt tot een bepaalde coalitievorming, wordt binnen de coöperatieve speltheorie ook wel uitgegaan van de zogenaamde axiomatische aanpak. Er wordt bij deze aanpak uitgegaan van een aantal wenselijke eigenschappen (axioma's), waar een oplossing aan zou moeten voldoen, waarna getracht wordt een oplossing te vinden die over deze eigenschappen beschikt.

In werkelijke situaties van sociale interactie tussen agenten, is het niet altijd mogelijk dat alle agenten met elkaar samen kunnen werken. Daarom worden binnen de coöperatieve speltheorie ook modellen met restricties op samenwerking bestudeerd. Myerson (1977) maakt gebruik van ongerichte grafen om spelen te bestuderen met *communicatierestricties*. Alleen de coalities die verbonden zijn in de graaf kunnen worden gevormd.

Gilles, Owen en vd Brink (1992) maken gebruik van een gerichte graaf om spelen te bestuderen, waarbij samenwerking plaatsvindt binnen een hiërarchische structuur, ook wel *permissiestructuur* genoemd. Spelers met een positie onderaan de hiërarchie kunnen alleen samenwerken in coalities met (een deel van) de spelers die boven hun staan in de hiërarchie. Gilles, Owen en vd Brink onderscheiden twee benaderingen van de permissiestructuren. Binnen de *conjunctieve benadering* van permissiestructuren zijn alleen die coalities van spelers mogelijk, waarbij voor iedere speler in de coalitie, ook alle voorgangers van deze speler in de gerichte graaf aanwezig zijn in de coalitie. Binnen de *disjunctieve benadering* van permissiestructuren zijn alleen die coalities van spelers mogelijk, waarbij voor iedere speler in de coalitie, ten minste één voorganger van deze speler in de gerichte graaf ook aanwezig is in de coalitie. Faigle en Kern (1992) maken gebruik van een vergelijkbaar model als Gilles, Owen en vd Brink. Faigle en Kern maken gebruik van een gerichte graaf om spelen met *opvolgingsrestricties* te bestuderen, waarbij samenwerking tussen spelers slechts volgens een bepaalde volgorde plaats kan vinden. Een coalitie van spelers is mogelijk, wanneer voor iedere speler in de coalitie, alle volgers van deze speler in de gerichte graaf ook aanwezig zijn in de coalitie. De hierboven genoemde modellen hebben met elkaar gemeen dat zij een verzameling van zogenaamde *realiseerbare coalities* toevoegen aan het klassieke model van coöperatieve spelen met overdraagbaar nut. Alleen de coalities die in de verzameling van realiseerbare coalities liggen, kunnen worden gevormd. In de literatuur is behalve aan ongerichte en gerichte grafen ook aandacht besteed aan andere combinatorische structuren om de verzameling realiseerbare coalities te genereren. Algaba, Bilbao, van den Brink and Jiménez-Losáda (2004) bestuderen spelen waarbij de realiseerbare coalities zijn gegenereerd door een antimatroïde. Ze laten zien hoe deze spelen een generalisatie vormen van de spelen met permissiestructuren en dat met deze spelen een grotere klasse van situaties met een hiërarchie op de spelers kan worden bestudeerd. Een andere belangrijke klasse van spelen met restricties op samenwerking zijn de spelen van Algaba, Bilbao, Borm and López (2001), waarbij de realiseerbare coalities gegenereerd zijn door een systeem dat stabiel is onder vereniging. Deze spelen vormen een generalisatie van zowel de spelen op antimatroïdes als de spelen met communicatie-restricties van Myerson (1977).

Deze dissertatie zal gaan over coöperatieve spelen met overdraagbaar nut, waarbij restricties aanwezig zijn op de samenwerking tussen spelers. Een aantal nieuwe modellen met restricties op samenwerking wordt geïntroduceerd en geanalyseerd. Met deze modellen is het mogelijk om situaties en toepassingen te bestuderen, die met eerdere modellen uit de literatuur niet of nauwelijks te bestuderen zijn. We laten ook zien hoe deze nieuwe modellen een uitbreiding zijn van al bestaande modellen uit de literatuur en hoe deze modellen zich met elkaar verhouden.

In Hoofdstuk 1 wordt een introductie gegeven van de coöperatieve speltheorie.

In Hoofdstuk 2 wordt een aantal modellen bestudeerd met restricties op samenwerking.

In Hoofdstuk 3 wordt gekeken naar de coöperatieve spelen met a-priori verenigingen van Aumann and Drèze (1974). De spelersverzameling is in deze spelen opgedeeld in een aantal losse verenigingen van de spelers. De oplossingen voor spelen met a-priori

verenigingen die in de literatuur zijn bekeken, wijzen aan iedere speler een bepaald nut toe. Wij introduceren een nieuwe klasse van oplossingen voor deze spelen, de *verenigingswaarden*. In plaats van nut toe te wijzen aan iedere speler wijzen deze verenigingswaarden nut toe aan iedere vereniging van spelers. Op deze manier is het mogelijk situaties van sociale interactie tussen agenten te bestuderen, waarbij de agenten bestaan uit een aantal sub-agenten. Denk hierbij bijvoorbeeld aan een bedrijf, dat uit meerdere afdelingen bestaat, of een sportteam, dat gevormd wordt door meerdere spelers. We bestuderen twee verenigingswaarden, die beiden een generalisatie zijn van de Shapley-waarde en geven van beide een axiomatisering.

In Hoofdstuk 4 wordt een nieuwe benadering van coöperatieve spelen met een permissiestructuur geïntroduceerd; de *lokale benadering* van permissiestructuren. Binnen deze benadering kan een speler alleen een bijdrage leveren aan het nut van een coalitie, als alle voorgangers van deze speler in de gerichte graaf ook in de coalitie aanwezig zijn. Met de lokale benadering bestuderen we de klasse van spelen met lokale restricties, die zowel een generalisatie vormt van de spelen met conjunctieve restricties van Gilles, Owen en vd Brink (1992) als de digraaf spelen van van de Brink en Borm (2002). Ook geven we in dit hoofdstuk een analyse van hoe een aantal deelklasses van de spelen met lokale restricties zich met elkaar verhouden. We sluiten af met de introductie en axiomatisering van een generalisatie van de Shapley-waarde voor spelen met lokale restricties.

In Hoofdstuk 5 bestuderen we systemen van realiseerbare coalities die zowel toegankelijk als stabiel onder vereniging zijn. Deze combinatorische structuren generaliseren zowel de communicatie-restricties gebruikt in Myerson (1977) als de antimatroïdes gebruikt in Algaba, Bilbao, van den Brink and Jiménez-Losáda (2004). We bestuderen hoe deze structuren zich verhouden tot de al bestaande structuren uit de literatuur en bekijken een aantal toepassingen dat exclusief door deze structuren gemodelleerd kan worden. Ook bekijken we een deelklasse van deze structuren, die overeenkomt met ongerichte grafen zonder cycles. We introduceren een klasse van coöperatieve spelen op deze structuren en sluiten af met de introductie en axiomatisering van een generalisatie van de Shapley-waarde voor deze spelen.

In Hoofdstuk 6 introduceren we de hiërarchische oplossing voor coöperatieve spelen met opvolgingsrestricties van Faigle en Kern (1992). We laten zien dat deze verkregen kan worden door nut in unanimiteitsspelen te verdelen via de zogenaamde *hiërarchische maat* toegepast op gerichte grafen. We geven een axiomatisering voor deze machtsmaat op gerichte grafen. Ook beschouwen we de toepassing en axiomatisering van deze maat op de reguliere systemen van realiseerbare coalities. We bestuderen ook een genormaliseerde versie van de hiërarchische maat en axiomatiseren deze en een aantal andere machtsmaten op twee deelklasses van de acyclische gerichte grafen.

In Hoofdstuk 7 bestuderen we coöperatieve spelen met conjunctieve restricties waar de permissiestructuur een gerichte boom is. We bestuderen generalisaties van de Myerson-waarde, de conjunctieve permissie-waarde en de hierarchische uitkomst voor spelen met communicatie-restricties. Ook introduceren we de zogenaamde top-waarde. Deze waarde wijst al het nut toe aan de speler die bovenaan de permissiestructuur staat en wijst geen nut toe aan de overige spelers. We vergelijken deze vier oplossingen met elkaar door vergelijkbare axiomatiseringen te geven.

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