Topics in Markov Chain Theory and Simulation Optimisation

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Appendices

Appendix A

IS A SMARTER ALLOCATION RULE FOR $\Pr(ACSR(K))$ POSSIBLE?

The goal of this appendix is to describe one of the attempts for a smarter allocation rule for $\Pr(ACSR(k))$. As we will explain with this note, we believe that the potential efficiency gain does not weight up against the computational effort of such an improved smarter allocation rule. The described attempt follows the same line of proof as in [53].

For notational easiness, we replace in the following $D_{\sigma_{j_a^k}(k)}$ by $D(k)$ and $\hat{z}_{n_\theta}(\theta)$ by $\hat{z}_{n_\theta}$. Recall that the objective for the budget allocation problem is to maximize $\Pr(CS(k))$ under the budget and the non-negative constraints. Define Δ as the computational budget, then the following approximated budget allocation problem is considered, where the objective uses the Bonferroni version of the lower bound for $\Pr(CSR(k))$ from Remark 2.1,

$$\begin{aligned} \underset{n_{\theta_1},\dots,n_{\theta_N}}{\text{Maximize}} \quad & \sum_{\theta\in D(k)} \left\{ 1 - \sum_{\theta'\in H(k)\setminus\{\theta\}} \Pr\left(\tilde{L}(\theta) > \tilde{L}(\theta')\right) \right\} \\ \text{Subject to:} \quad & \sum_{\theta\in H(k)} n_\theta = \Delta\,,\; n_\theta \in \{0,1,2,\dots\}\,. \end{aligned}$$

Using the posterior distribution based on the simulation effort, the second summation in the objective function can be rewritten as

$$\begin{aligned} \sum_{\theta'\in H(k)\setminus\{\theta\}} \Pr\left(\tilde{L}(\theta) > \tilde{L}(\theta')\right) &= \sum_{\theta'\in H(k)\setminus\{\theta\}} \int_{\frac{\hat{z}_{n_{\theta'}}-\hat{z}_{n_\theta}}{\sqrt{\frac{\sigma^2_{\theta'}}{n_{\theta'}}+\frac{\sigma^2_\theta}{n_\theta}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \\ &= \sum_{\theta'\in H(k)\setminus\{\theta\}} \int_{\frac{\delta_{\theta',\theta}}{\sigma_{\theta',\theta}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t\,, \end{aligned}$$

where in the last equality two new variables are introduced for notational easiness

$$\delta_{\theta',\theta} := \hat{z}_{n_{\theta'}} - \hat{z}_{n_\theta}\,,$$

$$\sigma_{\theta',\theta} := \sqrt{\frac{\sigma^2_{\theta'}}{n_{\theta'}} + \frac{\sigma^2_\theta}{n_\theta}}\,.$$

Furthermore, we relax the constraint that the variables $n_\theta, \theta \in H(k)$, should be non-negative. Also we allow the variables to be continuous in order to make the analysis applicable. Because of the relaxation the following problem is considered

$$\begin{aligned} &\underset{n_{\theta_1},\dots,n_{\theta_N}}{\text{Maximize}} && \sum_{\theta \in D(k)} \left\{ 1 - \sum_{\theta' \in H(k)\setminus\{\theta\}} \int_{\frac{\delta_{\theta',\theta}}{\sigma_{\theta',\theta}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \right\} \\ &\text{Subject to:} && \sum_{\theta \in H(k)} n_\theta = \Delta\,. \end{aligned} \tag{A.1}$$

The Lagrangian function Λ of problem (A.1) is

$$\Lambda = \sum_{\theta \in D(k)} \left\{ 1 - \sum_{\theta' \in H(k)\setminus\{\theta\}} \int_{\frac{\delta_{\theta',\theta}}{\sigma_{\theta',\theta}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \right\} - \lambda \left(\sum_{\theta \in H(k)} n_\theta - \Delta \right), \tag{A.2}$$

where λ is the Lagrange multiplier. In order to find the critical values of Λ we have to find the zeros of the gradient, i.e. $\nabla_{\{\forall \theta \in H(k), n_\theta\},\lambda} \Lambda = 0$. This comes down to the following first order conditions (where we make a distinction between designs from $H(k) \setminus D(k)$ and $D(k)$ respectively, which is needed for the derivation of the partial derivatives later on):

i) $\frac{\partial \Lambda}{\partial n_{\theta_i}} = 0,\ \theta_i \in D(k)$

ii) $\frac{\partial \Lambda}{\partial n_{\theta_j}} = 0,\ \theta_j \in H(k) \setminus D(k)$

iii) $\frac{\partial \Lambda}{\partial \lambda} = 0 \Leftrightarrow \sum_{\theta \in H(k)} n_\theta = \Delta\,.$

Observe that the last condition is just the budget constraint from Problem (A.1). Let us derive the partial derivatives of the first and the second condition, respectively. It holds for $\theta_i \in D(k)$ that

$$\begin{aligned} \frac{\partial \Lambda}{\partial n_{\theta_i}} = \; & \overbrace{-\frac{\partial}{\partial n_{\theta_i}} \left(\sum_{\theta' \in H(k)\setminus\{\theta_i\}} \int_{\frac{\delta_{\theta',\theta_i}}{\sigma_{\theta',\theta_i}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \right)}^{\text{Part I}} - \lambda \\ & \underbrace{-\frac{\partial}{\partial n_{\theta_i}} \left(\sum_{\theta' \in D(k)\setminus\{\theta_i\}} \int_{\frac{\delta_{\theta_i,\theta'}}{\sigma_{\theta_i,\theta'}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \right)}_{\text{Part II}}, \end{aligned} \tag{A.3}$$

where Part I is obtained by choosing θ_i in the first sum and Part II follows when choosing θ_i in the second sum in the Lagrangian function in (A.2), respectively. Part I of (A.3) is equal to

$$\begin{aligned} \text{Part I of (A.3)} &= -\frac{\partial}{\partial n_{\theta_i}} \sum_{\theta' \in H(k)\setminus\{\theta_i\}} \left\{ 1 - \Phi\left(\frac{\delta_{\theta',\theta_i}}{\sigma_{\theta',\theta_i}} \right) \right\} \\ &= \sum_{\theta' \in H(k)\setminus\{\theta_i\}} \phi\left(\frac{\delta_{\theta',\theta_i}}{\sigma_{\theta',\theta_i}} \right) \frac{\partial \left(\frac{\delta_{\theta',\theta_i}}{\sigma_{\theta',\theta_i}} \right)}{\partial \sigma_{\theta',\theta_i}} \frac{\partial \sigma_{\theta',\theta_i}}{\partial n_{\theta_i}} \\ &= \sum_{\theta' \in H(k)\setminus\{\theta_i\}} \phi\left(\frac{\delta_{\theta',\theta_i}}{\sigma_{\theta',\theta_i}} \right) \frac{\delta_{\theta',\theta_i} \sigma_{\theta_i}^2}{2 n_{\theta_i}^2 \sigma_{\theta',\theta_i}^3}\,, \end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density-and distribution function respectively. Similar, it holds for Part II in (A.3) that

$$\text{Part II of (A.3)} = \sum_{\theta' \in D(k)\setminus\{\theta_i\}} \phi\left(\frac{\delta_{\theta_i,\theta'}}{\sigma_{\theta_i,\theta'}}\right) \frac{\delta_{\theta_i,\theta'}\sigma^2_{\theta_i}}{2n^2_{\theta_i}\sigma^3_{\theta_i,\theta'}} .$$

For notational simplicity we define $q(\theta_1, \theta_2)$ as follows

$$q(\theta_1, \theta_2) := \phi\left(\frac{\delta_{\theta_1,\theta_2}}{\sigma_{\theta_1,\theta_2}}\right) \frac{1}{2\sigma^3_{\theta_1,\theta_2}} .$$

Note that it holds $q(\theta_1, \theta_2) = q(\theta_2, \theta_1)$. Using the above results and the definition of $q(\cdot, \cdot)$, the partial derivative in (A.3) becomes as follows,

$$\frac{\partial \Lambda}{\partial n_{\theta_i}} = \sum_{\theta' \in H(k)\setminus\{\theta_i\}} q(\theta', \theta_i) \frac{\delta_{\theta',\theta_i}\sigma^2_{\theta_i}}{n^2_{\theta_i}} + \sum_{\theta' \in D(k)\setminus\{\theta_i\}} q(\theta_i, \theta') \frac{\delta_{\theta_i,\theta'}\sigma^2_{\theta_i}}{n^2_{\theta_i}} - \lambda . \tag{A.4}$$

Splitting the first summation from (A.4) results in

$$\begin{aligned}\frac{\partial \Lambda}{\partial n_{\theta_i}} = & \sum_{\theta' \in H(k)\setminus D(k)} q(\theta', \theta_i) \frac{\delta_{\theta',\theta_i}\sigma^2_{\theta_i}}{n^2_{\theta_i}} + \sum_{\theta' \in D(k)\setminus\{\theta_i\}} q(\theta', \theta_i) \frac{\delta_{\theta',\theta_i}\sigma^2_{\theta_i}}{n^2_{\theta_i}} \\ & + \sum_{\theta' \in D(k)\setminus\{\theta_i\}} q(\theta_i, \theta') \frac{\delta_{\theta_i,\theta'}\sigma^2_{\theta_i}}{n^2_{\theta_i}} - \lambda ,\end{aligned}$$

and since it holds that $q(\theta_1, \theta_2) = q(\theta_2, \theta_1)$ and $\delta_{\theta_1,\theta_2} = -\delta_{\theta_2,\theta_1}$ this last result can be written as, $\theta_i \in D(k)$,

$$\frac{\partial \Lambda}{\partial n_{\theta_i}} = \sum_{\theta' \in H(k)\setminus D(k)} q(\theta', \theta_i) \frac{\delta_{\theta',\theta_i}\sigma^2_{\theta_i}}{n^2_{\theta_i}} - \lambda . \tag{A.5}$$

Following the same calculations as above, it holds that the partial derivative to n_{θ_j}, $\theta_j \in H(k) \setminus D(k)$, of the Lagrange function is

$$\frac{\partial \Lambda}{\partial n_{\theta_j}} = \sum_{\theta \in D(k)} q(\theta_j, \theta) \frac{\delta_{\theta_j,\theta}\sigma^2_{\theta_j}}{n^2_{\theta_j}} - \lambda . \tag{A.6}$$

Using (A.6) in the second condition of finding the critical values for Λ, we obtain $\forall \theta_j \in H(k) \setminus D(k)$,

$$\frac{\partial \Lambda}{\partial n_{\theta_j}} = 0 \Leftrightarrow \sum_{\theta \in D(k)} q(\theta_j, \theta)\delta_{\theta_j,\theta} = \lambda \frac{n^2_{\theta_j}}{\sigma^2_{\theta_j}} , \tag{A.7}$$

which becomes useful later on. The first condition of finding the critical values for Λ, when using (A.5), can be rewritten $\forall \theta_i \in D(k)$ as

$$\frac{\partial \Lambda}{\partial n_{\theta_i}} = 0 \Leftrightarrow \sum_{\theta' \in H(k)\setminus D(k)} q(\theta', \theta_i)\delta_{\theta',\theta_i} = \lambda \frac{n^2_{\theta_i}}{\sigma^2_{\theta_i}} ,$$

since this last result holds $\forall \theta_i \in D(k)$,

$$\Leftrightarrow \sum_{\theta_i \in D(k)} \sum_{\theta' \in H(k)\setminus D(k)} q(\theta', \theta_i)\delta_{\theta',\theta_i} = \lambda \sum_{\theta_i \in D(k)} \frac{n^2_{\theta_i}}{\sigma^2_{\theta_i}} ,$$

interchanging the order of summations and using (A.7),

$$\Leftrightarrow \sum_{\theta' \in H(k) \setminus D(k)} \lambda \frac{n_{\theta'}^2}{\sigma_{\theta'}^2} = \lambda \sum_{\theta_i \in D(k)} \frac{n_{\theta_i}^2}{\sigma_{\theta_i}^2} ,$$

rewriting gives, $\forall \theta_i \in D(k)$,

$$\Leftrightarrow n_{\theta_i} = \sigma_{\theta_i} \sqrt{\sum_{\theta' \in H(k) \setminus D(k)} \frac{n_{\theta'}^2}{\sigma_{\theta'}^2} - \sum_{\theta \in D(k) \setminus \{\theta_i\}} \frac{n_{\theta}^2}{\sigma_{\theta}^2}} . \tag{A.8}$$

Equation (A.8) shows how the number of simulations for a design in $D(k)$ relates to the number of simulations for other designs in $H(k)$. Furthermore, equation (A.8) corresponds with the original OCBA rule in [53] in case $|D(k)| = 1$, where $|D(k)|$ denotes the number of elements in $D(k)$. Let us now concentrate on the relationship between $n_{\theta_i}^2$ and $n_{\theta_j}^2$ for θ_i, $\theta_j \in D(k)$. It follows from (A.5) that $\forall \theta_i$, $\theta_j \in D(k)$

$$\frac{\partial \Lambda}{\partial n_{\theta_i}} = \frac{\partial \Lambda}{\partial n_{\theta_j}} \Leftrightarrow \sum_{\theta' \in H(k) \setminus D(k)} q(\theta', \theta_i) \frac{\delta_{\theta', \theta_i} \sigma_{\theta_i}^2}{n_{\theta_i}^2} = \sum_{\theta' \in H(k) \setminus D(k)} q(\theta', \theta_j) \frac{\delta_{\theta', \theta_j} \sigma_{\theta_j}^2}{n_{\theta_j}^2} ,$$

this can, e.g., be rewritten as follows

$$\Leftrightarrow \frac{n_{\theta_i}}{n_{\theta_j}} = \frac{\sigma_{\theta_i}}{\sigma_{\theta_j}} \sqrt{\frac{\sum_{\theta' \in H(k) \setminus D(k)} q(\theta', \theta_i) \delta_{\theta', \theta_i}}{\sum_{\theta' \in H(k) \setminus D(k)} q(\theta', \theta_j) \delta_{\theta', \theta_j}}} . \tag{A.9}$$

Lets focus on the relationship between n_{θ_i} and n_{θ_j} with θ_i, $\theta_j \in H(k) \setminus D(k)$. From (A.6) it follows $\forall \theta_i$, $\theta_j \in H(k) \setminus D(k)$,

$$\begin{aligned} \frac{\partial \Lambda}{\partial n_{\theta_i}} = \frac{\partial \Lambda}{\partial n_{\theta_i}} &\Leftrightarrow \sum_{\theta \in D(k)} q(\theta_i, \theta) \frac{\delta_{\theta_i, \theta} \sigma_{\theta_i}^2}{n_{\theta_i}^2} = \sum_{\theta \in D(k)} q(\theta_j, \theta) \frac{\delta_{\theta_j, \theta} \sigma_{\theta_j}^2}{n_{\theta_j}^2} \\ &\Leftrightarrow \sum_{\theta \in D(k)} \left\{ q(\theta_i, \theta) \frac{\delta_{\theta_i, \theta} \sigma_{\theta_i}^2}{n_{\theta_i}^2} - q(\theta_j, \theta) \frac{\delta_{\theta_j, \theta} \sigma_{\theta_j}^2}{n_{\theta_j}^2} \right\} = 0 , \end{aligned} \tag{A.10}$$

Lastly, for the relationship between n_{θ_i} and n_{θ_j} with $\theta_i \in D(k)$ and $\theta_j \in H(k) \setminus D(k)$ it holds

$$\begin{aligned} \frac{\partial \Lambda}{\partial n_{\theta_i}} = \frac{\partial \Lambda}{\partial n_{\theta_i}} &\Leftrightarrow \sum_{\theta' \in H(k) \setminus D(k)} q(\theta', \theta_i) \frac{\delta_{\theta', \theta_i} \sigma_{\theta_i}^2}{n_{\theta_i}^2} = \sum_{\theta \in D(k)} q(\theta_j, \theta) \frac{\delta_{\theta_j, \theta} \sigma_{\theta_j}^2}{n_{\theta_j}^2} \\ &\Leftrightarrow \sum_{\theta \in D(k)} \left\{ q(\theta_i, \theta) \frac{\delta_{\theta_i, \theta} \sigma_{\theta_i}^2}{n_{\theta_i}^2} - q(\theta_j, \theta) \frac{\delta_{\theta_j, \theta} \sigma_{\theta_j}^2}{n_{\theta_j}^2} \right\} = 0 , \end{aligned} \tag{A.11}$$

In conclusion, stationary points of the Lagrangian function Λ must satisfy equations (A.8), (A.9), (A.10) and (A.11). In [53] these conditions could be expressed as direct relations between n_{θ_i} and n_{θ_j} so that Problem (A.1) could be approximately solved. Many of our attempts to similarly express (A.8), (A.9), (A.10) and (A.11) in some direct relations between n_{θ_i} and n_{θ_j} failed because the extra summation makes life hard. For example, consider (A.10), in [53] they directly solved this expression by some extra simplifications, which were justified by earlier calculations. The extra summation in our case prevents us of following the same procedure and assumptions such as

$$\forall \theta \in D(k): \quad q(\theta_i, \theta)\frac{\delta_{\theta_i,\theta}\sigma^2_{\theta_i}}{n^2_{\theta_i}} = q(\theta_j, \theta)\frac{\delta_{\theta_j,\theta}\sigma^2_{\theta_j}}{n^2_{\theta_j}}$$

are needed to make the same analysis applicable. Unfortunately, these assumptions are highly unrealistic. As a second example, observe that the relation between n_{θ_i} and n_{θ_j} for θ_i, $\theta_j \in D(k)$ from equation (A.9) is not useful since the same values for n_{θ_i} and n_{θ_j} are trapped in the exponential of $q(\theta', \theta_i)$ and $q(\theta', \theta_j)$. In order to obtain a useful relation between n_{θ_i} and n_{θ_j} for θ_i, $\theta_j \in D(k)$ from equation (A.9) we must have, e.g., an assumption

$$\forall \theta' \in H(k) \setminus D(k): \ \delta_{\theta',\theta_i} \approx \delta_{\theta',\theta_j} \, ,$$

i.e., all design in $D(k)$ are nearly the same in performance. So that equation (A.9) reduces to $\frac{n_{\theta_i}}{n_{\theta_j}} = \frac{\sigma_{\theta_i}}{\sigma_{\theta_j}}$ by furthermore assuming that the number of simulations spend to a design in $D(k)$ is significant larger than for a design in $H(k) \setminus D(k)$. Obviously such an assumption is also very unrealistic and would require, e.g., an extra algorithmic procedure of updating $D(k)$ to ensure that the assumptions are realistic.

In conclusion, many of our attempts failed because of the extra summation in the accelerating stopping rule. Maybe there is some ingenious way of solving Problem (A.1), but we believe that the extra research effort and possible extra assumptions needed in finding such a smarter allocation rule is not worth the small amount of efficiency that it would possible realize. In light of our given arguments in Chapter 2, we believe that the combination of the OCBA and the accelerating stopping rule is already highly efficient.

HIDDEN SUDOKU PAGE

Congratulations, you found the hidden Sudoku page to kill some time! A Sudoku is a puzzle on a 9×9 grid. It is filled with nine numbers running from 1 to 9. Each number occurs only once in each row, each column and each of the nine 3×3 boxes. Initially some of the numbers are shown. The goal is to fill in all missing digits.

Level 1

	2						9	
3		1	9		6	5		2
			8		4			
	9						5	
5			2		3			6
	7						2	
			4		7			
8		2	5		1	7		3
	5						8	

Level PhD

6	7			1				4
		9	2					
	3							
7			8	9			4	
			3		2	7		
3	4		7					1
								3
9	5							
			1	8	6		5	

Appendix B

PROOFS OF JACKSON NETWORK RESULTS

B.1 Proof of Theorem 3.2

(i): Assume first that all nodes are in up status ($I = \emptyset$). We start the proof with evocation of the *Subnetwork Argument* from the proof of Theorem 7 in [172]. The Subnetwork Argument guarantees that the subnetwork W develops as a Jackson network where the source and sink represent $\{0\} \cup V$. The corresponding queueing process $\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$ is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \tilde{\eta}_j r(j,i), \quad i \in W, \quad \text{where } \tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i),$$

so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$. According to Jackson's theorem (see [107]), $\tilde{X}$ has the unique stationary and limiting distribution

$$\lim_{t \to \infty} P(X_i(t) = n_i : i \in W) = \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \quad \forall (n_i : i \in W) \in \mathbb{N}_0^{|W|}, \tag{B.1}$$

because $\eta_i < \mu_i$ for all $i \in W$ holds. Thus, even if the subnetwork V of nodes with infinite supply is not in equilibrium, the equilibrium on the subnetwork W of nodes without infinite supply is preserved, if the initial distribution has the joint marginal (B.1).

This joint queue length process $\tilde{X}$ is coupled with an availability process Y which only depends on the interaction of the nodes in $D \subseteq \tilde{J}$ but not on their load. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status. The network process $(Y, \tilde{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}_0^{|W|}$. The balance equations for the subnetwork W are for all $(\emptyset, n_k : k \in W) \in \{\emptyset\} \times \mathbb{N}_0^{|W|}$ given by

$$\pi(\emptyset, n_k : k \in W) \left(\sum_{i \in W} \Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i) \Big) + \sum_{i \in W} \mu_i (1 - r(i,i)) \mathbb{1}_{\{n_i > 0\}} + \sum_{\emptyset \neq I \subseteq D} \alpha(\emptyset, I) \right)$$
$$= \sum_{i \in W} \pi(\emptyset, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i) \Big) \cdot \mathbb{1}_{\{n_i > 0\}} +$$
$$+ \sum_{i \in W} \pi(\emptyset, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W} r(i,j) \Big) +$$

$$+\sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(\emptyset, n_k\in W\setminus\{i,j\}, n_i+1, n_j-1)\cdot\mu_i r(i,j)\cdot\mathbb{1}_{\{n_j>0\}}$$
$$+\sum_{\emptyset\neq I\subseteq D}\pi(I, n_k : k\in W)\cdot\beta(I,\emptyset)\,, \tag{B.2}$$

and for all $(I, n_k : k\in W)\in\mathcal{P}(D)\times\mathbb{N}_0^{|W|}$ with $I\neq\emptyset$

$$\pi(I, n_k : k\in W)\left(\sum_{I\subset H\subseteq D}\alpha(I,H)+\sum_{\emptyset\neq K\subset I}\beta(I,K)\right)$$
$$=\sum_{\emptyset\neq K\subset I}\pi(K, n_k : k\in W)\cdot\alpha(K,I)+\sum_{I\subset H\subseteq D}\pi(H, n_k : k\in W)\cdot\beta(H,I)\,. \tag{B.3}$$

We have to show, that (3.13) solves these equations. In the following we denote

$$\hat\pi(I, n_k : k\in W):=\frac{A(I)}{B(I)}\prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(I, n_k : k\in W)\in\mathcal{P}(D)\times\mathbb{N}_0^{|W|}$, which is (3.13) before normalization, and plug it into the above balance equations instead of $\pi(I, n_k : k\in W)$.

In the first equation (B.2) the term

$$\hat\pi(\emptyset, n_k : k\in W)\alpha(\emptyset, I)=\hat\pi(\emptyset, n_k : k\in W)A(I)=\hat\pi(I, n_k : k\in W)B(I)$$

on the left-hand side is equal to the term $\hat\pi(I, n_k : k\in W)\beta(I,\emptyset)=\hat\pi(I, n_k : k\in W)B(I)$ on the right-hand side for each $\emptyset\neq I\subseteq D$. The remainder of (B.2) is the global balance equation of a classical Jackson network which has the solution (see [107])

$$\hat\pi(\emptyset, n_k : k\in W):=\hat\pi(n_k : k\in W)=\prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i}\,.$$

Consider the second equation (B.3) for some fixed $I\neq\emptyset$. For any $K\subset I$, $K\neq\emptyset$, the term

$$\hat\pi(I, n_k : k\in W)\beta(I,K)=\hat\pi(I, n_k : k\in W)\frac{B(I)}{B(K)}=\hat\pi(\emptyset, n_k : k\in W)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat\pi(K, n_k : k\in W)\alpha(K,I)=\hat\pi(K, n_k : k\in W)\frac{A(I)}{A(K)}=\hat\pi(\emptyset, n_k : k\in W)\frac{A(I)}{B(K)}.$$

Moreover, for any $I\subset H\subseteq D$ the term

$$\hat\pi(I, n_k : k\in W)\alpha(I,H)=\hat\pi(I, n_k : k\in W)\frac{A(H)}{A(I)}=\hat\pi(\emptyset, n_k : k\in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat\pi(H, n_k : k\in W)\beta(H,I)=\hat\pi(H, n_k : k\in W)\frac{B(H)}{B(I)}=\hat\pi(\emptyset, n_k : k\in W)\frac{A(H)}{B(I)}$$

on the right-hand side. The proof of (i) is finished by normalization, which is possible because $\eta_i < \mu_i$ holds for all $i \in W$.

(ii): It is well known that ergodic Jackson networks have, in equilibrium, Poisson departure streams from node i to the sink with rate $\tilde{\eta}_i \tilde{r}(i,0)$, see [142, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network with unreliable nodes of its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i \tilde{r}(i,0) = \eta_i\Big(1 - \sum_{j\in W} r(i,j)\Big) = \eta_i\Big(r(i,0) + \sum_{j\in V} r(i,j)\Big).$$

Hence, if the subnetwork W is in equilibrium, as long as all nodes are in up status, departures to the sink from nodes $i \in W$ are Poisson streams with rate $\eta_i r(i,0)$ and departures from $i \in W$ to any node $j \in V$ are also Poisson streams with rate $\eta_i r(i,j)$, because a portion of $r(i,j)/(r(i,0) + \sum_{j\in V} r(i,j))$ of the departure stream from node $i \in W$ is directed to $j \in V$.

(iii): Under the condition that all nodes $j \in \tilde{J}$ are in up status, we start the proof with evocation of the *M/M/1 Argument* from the proof of theorem 13 in [172].

This argument leads to the conclusion, that if the subnetwork W is in equilibrium and if $r(i,i) = 0$ holds, node $i \in V$ behaves as a $M/M/1-$ system of its own. The corresponding queue length process $\hat{X}$ is a birth-death process on state space $\mathbb{N}_0$ with birth rates $\hat{\lambda}_i = \eta_i$ and death rates μ_i.

This queue length process $\hat{X}$ is here coupled with an availability process Y on $\mathcal{P}(D)$, $D \subseteq \tilde{J}$, where breakdown and repair of nodes only depend on the interaction of the nodes but not on their queue length. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status. The network process $(Y, \hat{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}_0$. The balance equations are

$$\begin{aligned} &\pi_i(\emptyset, n_i)\Big(\hat{\lambda}_i + \mu_i \mathbb{1}_{\{n_i>0\}} + \sum_{\emptyset\neq I\subseteq D} \alpha(\emptyset, I)\Big) \\ &\quad = \pi_i(\emptyset, n_i - 1)\cdot \hat{\lambda}_i \cdot \mathbb{1}_{\{n_i>0\}} + \pi_i(\emptyset, n_i+1)\cdot \mu_i + \sum_{\emptyset\neq I\subseteq D} \pi_i(I, n_i)\cdot \beta(I,\emptyset) \end{aligned} \tag{B.4}$$

for all $(\emptyset, n_i) \in \{\emptyset\} \times \mathbb{N}_0$ and

$$\begin{aligned} &\pi_i(I, n_i)\Big(\sum_{I\subset H\subseteq D} \alpha(I,H) + \sum_{\emptyset\neq K\subset I} \beta(I,K)\Big) \\ &= \sum_{\emptyset\neq K\subset I} \pi_i(K, n_i)\cdot \alpha(K,I) + \sum_{I\subset H\subseteq D} \pi_i(H, n_i)\cdot \beta(H,I) \end{aligned} \tag{B.5}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}_0$ with $I \neq \emptyset$.

We have to show, that (3.14) solves these equations. In the following we set

$$\hat{\pi}_i(I, n_i) := \frac{A(I)}{B(I)} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}_0$ as the non-normalized proposed solution density.

In the first equation (B.4) the term

$$\hat{\pi}_i(\emptyset, n_i)\alpha(\emptyset, I) = \hat{\pi}_i(\emptyset, n_i)A(I) = \hat{\pi}_i(I, n_i)B(I)$$

on the left-hand side is equal to the term $\hat{\pi}_i(I, n_i)\beta(I, \emptyset) = \hat{\pi}_i(I, n_i)B(I)$ on the right-hand side for each $\emptyset \neq I \subseteq D$. The remainder of (B.4) is the global balance equation of an $M/M/1-$system which has the solution

$$\hat{\pi}_i(\emptyset, n_i) := \hat{\pi}_i(n_i) = \left(\frac{\eta_i}{\mu_i}\right)^{n_i},$$

since $\hat{\lambda}_i = \eta_i$ holds.

Consider the second equation (B.5) for some fixed $I \neq \emptyset$. For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}_i(I, n_i)\beta(I, K) = \hat{\pi}_i(I, n_i)\frac{B(I)}{B(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}_i(K, n_i)\alpha(K, I) = \hat{\pi}_i(K, n_i)\frac{A(I)}{A(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}_i(I, n_i)\alpha(I, H) = \hat{\pi}_i(I, n_i)\frac{A(H)}{A(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}_i(H, n_i)\beta(H, I) = \hat{\pi}_i(H, n_i)\frac{B(H)}{B(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the right-hand side. The proof of (iii) is finished by normalization, which is possible from $\eta_i < \mu_i$.

Lastly, the limiting probability (3.15) for unstable nodes with infinite supply follows from the same arguments as in the proof of theorem 15 in [155].

B.2 Proof of Theorem 3.3

Consider the subset W of nodes without infinite supply. For any subset $I \subseteq D$ of broken down nodes, we have the following facts for the subset $W \setminus I$ which remain in force as long as I is unchanged:

- All service times of all up-nodes are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W \setminus I$ will either move to some node $j \in W \setminus I$ with probability $r^I(i, j)$ or leave the subnetwork with probability $1 - \sum_{j \in W \setminus I} r^I(i, j)$.

- At each node $i \in W \setminus I$, we have external arrivals from the source which are independent Poisson streams with rate $\lambda_i^I \geq 0$. Furthermore all arrivals from nodes $j \in V \setminus I$ with infinite supply into nodes $i \in W \setminus I$ are independent Poisson streams at rate $\mu_j r^I(j,i)$, see Theorem 3.1. The sum of independent Poisson streams is a Poisson stream, hence the arrival stream from the outside of the subset $W \setminus I$ into each node $i \in W \setminus I$ is a Poisson process with rate $\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i)$.

- All service times and all interarrival times are independent of each other.

Let $\tilde{X} := ((\tilde{X}_i(t) : i \in W \setminus I) : t \in \mathbb{R}_+)$ be the queueing process of this subnetwork. the process is supplemented with a Markov process $Y = (Y(t) : t \in \mathbb{R}_+)$ which describes the availability status of the nodes and therefore gives information on how long the network process on the subnet $W \setminus I$ lives until it jumps to the next Markov process on some randomly chosen subnet $W \setminus K$, $K \subseteq D$. Rerouting is according to the blocking rs-rd regime (skipping, resp.). The balance equations of the joint availability-queue length process $(Y, \tilde{X}_i : i \in W)$ are $\forall (I, n_i : i \in W) \in \mathcal{P}(D) \times \mathbb{N}_0^{|W|}$

$$\begin{aligned}
&\pi(I, n_k : k \in W) \cdot \Bigg(\sum_{i \in W \setminus I} \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \Big) + \\
&+ \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot \mathbb{1}_{\{n_i > 0\}} + \sum_{I \subset H \subseteq D} \alpha(I,H) + \sum_{K \subset I \subseteq D} \beta(I,K) \Bigg) \\
&= \sum_{i \in W \setminus I} \pi(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \Big) \cdot \mathbb{1}_{\{n_i > 0\}} \\
&+ \sum_{i \in W \setminus I} \pi(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j) \Big) + \\
&+ \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \pi(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot \mathbb{1}_{\{n_j > 0\}} \\
&+ \sum_{K \subset I \subseteq D} \pi(K, n_k : k \in W) \cdot \alpha(K,I) + \sum_{I \subset H \subseteq D} \pi(H, n_k : k \in W) \cdot \beta(H,I) \,. \qquad \text{(B.6)}
\end{aligned}$$

We have to show that the distribution given by (3.16) solves equation (B.6) for all $(n_i : i \in W) \in \mathbb{N}_0^{|W|}$ and all $I \subseteq D$. In the following we set

$$\hat{\pi}(I, n_k : k \in W) := \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i} \right)^{n_i}$$

for all $(n_i : i \in W) \in \mathbb{N}_0^{|W|}$ and all $I \subseteq D$, and consider equation (B.6) for some fixed $I \subseteq D$.

For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}(I, n_k : k \in W) \beta(I,K) = \hat{\pi}(I, n_k : k \in W) \frac{B(I)}{B(K)} = \hat{\pi}(\emptyset, n_k : k \in W) \frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}(K, n_k : k \in W) \alpha(K,I) = \hat{\pi}(K, n_k : k \in W) \frac{A(I)}{A(K)} = \hat{\pi}(\emptyset, n_k : k \in W) \frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}(I, n_k : k \in W)\alpha(I,H) = \hat{\pi}(I, n_k : k \in W)\frac{A(H)}{A(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}(H, n_k : k \in W)\beta(H,I) = \hat{\pi}(H, n_k : k \in W)\frac{B(H)}{B(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the right-hand side. The remainder of (B.6) is

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W\setminus I} \Big(\lambda_i^I + \sum_{j \in V\setminus I} \mu_j r^I(j,i) \Big) + \sum_{i \in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot \mathbb{1}_{\{n_i>0\}} \Big) \\
&= \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j \in V\setminus I} \mu_j r^I(j,i) \Big) \cdot \mathbb{1}_{\{n_i>0\}} + \\
&+ \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W\setminus I} r^I(i,j) \Big) + \\
&+ \sum_{i \in W\setminus I} \sum_{j \in W\setminus I, j\neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot \mathbb{1}_{\{n_j>0\}}.
\end{aligned}$$

With $\eta_i^I = \lambda_i^I + \sum_{j \in W\setminus I} \eta_j^I r^I(j,i) + \sum_{j\in V\setminus I} \mu_j r^I(j,i)$ (see (3.6)) this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W\setminus I} \Big(\eta_i^I - \sum_{j \in W\setminus I} \eta_j^I r^I(j,i) \Big) + \sum_{i \in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot \mathbb{1}_{\{n_i>0\}} \Big) \\
&= \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i^I - \sum_{j \in W\setminus I} \eta_j^I r^I(j,i) \Big) \cdot \mathbb{1}_{\{n_i>0\}} + \\
&+ \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W\setminus I} r^I(i,j) \Big) + \\
&+ \sum_{i \in W\setminus I} \sum_{j \in W\setminus I, j\neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot \mathbb{1}_{\{n_j>0\}}.
\end{aligned}$$

Under the required condition of either (3.7) and (3.8) in case of blocking rs-rd or (3.10) in case of skipping holds $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$ for the respective reduced traffic equations. Therefore from Lemma 3.1 or Lemma 3.2, respectively, this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W\setminus I} \Big(\eta_i - \sum_{j \in W\setminus I} \eta_j r^I(j,i) \Big) + \sum_{i \in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot \mathbb{1}_{\{n_i>0\}} \Big) \\
&= \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i - \sum_{j \in W\setminus I} \eta_j r^I(j,i) \Big) \cdot \mathbb{1}_{\{n_i>0\}} + \\
&+ \sum_{i \in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W\setminus I} r^I(i,j) \Big) + \\
&+ \sum_{i \in W\setminus I} \sum_{j \in W\setminus I, j\neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot \mathbb{1}_{\{n_j>0\}}.
\end{aligned}$$

Plugging in $\hat{\pi}(I, n_k : k \in W) = \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$ yields

$$\begin{aligned}
&\sum_{i \in W \setminus I} \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i)\Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot \mathbb{1}_{\{n_i > 0\}} \\
&= \sum_{i \in W \setminus I} \frac{\mu_i}{\eta_i} \cdot \Big(\eta_i - \sum_{j \in W \setminus I} \eta_j r^I(j,i)\Big) \cdot \mathbb{1}_{\{n_i > 0\}} + \sum_{i \in W \setminus I} \frac{\eta_i}{\mu_i} \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j)\Big) + \\
&+ \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \frac{\eta_i}{\mu_i} \frac{\mu_j}{\eta_j} \mu_i r^I(i,j) \cdot \mathbb{1}_{\{n_j > 0\}}
\end{aligned}$$

$$\Leftrightarrow 0 = - \sum_{i \in W \setminus I} \frac{\mu_i}{\eta_i} \cdot \sum_{j \in W \setminus I, j \neq i} \eta_j r^I(j,i) \cdot \mathbb{1}_{\{n_i > 0\}} + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \frac{\mu_j}{\eta_j} \eta_i r^I(i,j) \cdot \mathbb{1}_{\{n_j > 0\}}.$$

Thus $\hat{\pi}(I, n_k : k \in W) = \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$ solves the balance equations (B.6). The last step of proving (3.16) is by normalizing $\hat{\pi}$, which is possible because $\eta_i < \mu_i$ holds for all $i \in W$.

Appendix C

PROOF OF THE TABOO MATRIX REPRESENTATION OF THE DEVIATION MATRIX

The goal of this appendix is to proof under slightly more general conditions that the alternative deviation matrix expression from (4.9) from Chapter 4 holds true. Therefore we need Lemma C.1 which will be used in the proof of the alternative expression given in Theorem C.1. The proofs are based on Lemma 3.2 (p. 39) from [111].

Lemma C.1 (Based on [111]). *Consider a Markov uni-chain with transition matrix P with stationary distribution $\pi_P^\top$. Let $T = P - h\sigma^\top$ be a taboo matrix where h and $\sigma^\top$ are appropriate sized vectors. When there exists a matrix norm $\|\cdot\|$ such that $\|T\| < 1$ and $\pi_P^\top h \neq 0$, then it holds that*

$$\sigma^\top (I - T)^{-1} = \pi_P^\top / (\pi_P^\top h). \tag{C.1}$$

Proof. We may rewrite (C.1) as

$$(\text{C.1}) \Leftrightarrow \sigma^\top = \pi_P^\top (I - T)/(\pi_P^\top h)$$

inserting the expression $T = P - h\sigma^\top$ and using that $\pi_P^\top P = \pi_P^\top$, resp.,

$$\begin{aligned}
&\Leftrightarrow \sigma^\top = \pi_P^\top (I - P + h\sigma^\top)/(\pi_P^\top h) \\
&\Leftrightarrow \sigma^\top = \pi_P^\top h\sigma^\top/(\pi_P^\top h) \\
&\Leftrightarrow \sigma^\top = \sigma^\top
\end{aligned}$$

□

Theorem C.1 (Based on [111]). *Consider a Markov uni-chain with transition matrix P, stationary distribution $\pi_P^\top$, ergodic projector Π_P and for which its deviation matrix $D_P = \sum_{n=0}^{\infty} (P^n - \Pi_P)$ exists. Let $T = P - h\sigma^\top$ be a taboo matrix where h and $\sigma^\top$ are appropriate sized vectors and let $\bar{1}$ denote an appropriate sized vector of ones. When there exists a matrix norm $\|\cdot\|$ such that*

(i) $\|T\| < 1$,

(ii) $\sigma^\top (I - T)^{-1} \bar{1} \neq 0$ *(e.g., this holds true when T is non-negative and $\sigma^\top$ is a stochastic vector), and*

(iii) $\pi_P^\top h \neq 0$

then it holds that

$$D_P = (I - \Pi_P) \sum_{n=0}^{\infty} T^n (I - \Pi_P),$$

note that when $\|T\| < 1$ *the infinity sum is finite.*

Proof. From the definition of D_P it follows after some calculations that

$$D_P(I - P) = (I - \Pi_P)$$

since $P = T - h\sigma^\top$

$$\Leftrightarrow D_P(I - T) = I - \Pi_P - D_P h \sigma^\top$$

using that $\|T\| < 1$ gives

$$\Leftrightarrow D_P = (I - \Pi_P)(I - T)^{-1} - D_P h \sigma^\top (I - T)^{-1}. \tag{C.2}$$

If we multiply (C.2) from the right with $\bar{1}$ we get

$$D_P \bar{1} = (I - \Pi_P)(I - T)^{-1}\bar{1} - D_P h \sigma^\top (I - T)^{-1}\bar{1}$$

the definition of D_P shows that $D_P \bar{1} = 0$ so that rewriting gives

$$\Leftrightarrow D_P h \sigma^\top (I - T)^{-1}\bar{1} = (I - \Pi_P)(I - T)^{-1}\bar{1}$$

so when $\sigma^\top (I - T)^{-1}\bar{1} \neq 0$ (note that this is a scalar)

$$\Leftrightarrow D_P h = (I - \Pi_P)(I - T)^{-1}\bar{1}/(\sigma^\top (I - T)^{-1}\bar{1}). \tag{C.3}$$

Inserting the result from (C.3) into (C.2) gives

$$\begin{aligned}
\text{(C.2)} \Leftrightarrow D_P &= (I - \Pi_P)(I - T)^{-1} - \frac{(I - \Pi_P)(I - T)^{-1}\bar{1}\sigma^\top (I - T)^{-1}}{\sigma^\top (I - T)^{-1}\bar{1}} \\
\Leftrightarrow D_P &= (I - \Pi_P)(I - T)^{-1}(I - \bar{1}\sigma^\top (I - T)^{-1}/(\sigma^\top (I - T)^{-1}\bar{1}))
\end{aligned}$$

making use of Lemma C.1 which requires that $\pi_P^\top h \neq 0$

$$\Leftrightarrow D_P = (I - \Pi_P)(I - T)^{-1}[I - \bar{1}\pi_P^\top/(\pi_P^\top h)/(\pi_P^\top \bar{1}/(\pi_P^\top h))]$$

using that $\pi_P^\top \bar{1} = 1$

$$\Leftrightarrow D_P = (I - \Pi_P)(I - T)^{-1}(I - \bar{1}\pi_P^\top)$$

which ends the proof by noting that $\bar{1}\pi_P^\top = \Pi_P$ for Markov uni-chains. □

Remark C.1. *It can be shown that if* P *is a transition matrix of a Markov uni-chain which is geometric ergodic (for a definition see, e.g., Section 5.1) the deviation matrix expression from Theorem C.1 with taboo matrix* $T = P - h\sigma^\top$ *holds true when* $\sigma^\top \bar{1} = 1$ *and when there exists a matrix norm* $\|\cdot\|$ *such that* $\|T\| < 1$.