Measure-Valued Differentiation for Finite Products of Measures

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THEORY AND APPLICATIONS

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This thesis is devoted to the theory of weak differentiation of measures. The basic observation is that, formally, the weak derivative of a parameter-dependent probability distribution μ_θ is in general a finite signed measure which can be represented as the re-scaled difference between two probability distributions. This fact allows for a useful representations of the derivative $\frac{d}{d\theta}\mathbb{E}_\theta[g(X)]$ of the expected value $\mathbb{E}_\theta[g(X)]$, for some predefined class $\mathcal{D}$ of cost-functions g, where X is a random variable with distribution μ_θ.

Many mathematical models are described by a finite family of independent random variables and this is the reason why differentiability properties as well as representations for weak derivatives of product measures are studied in this thesis. To develop the theory, concepts and results from measure theory and functional analysis are required and the necessary prerequisites are presented in Chapter 1.

In Chapter 2 we develop the theory of first-order differentiation. Main results, such as the product rule of weak differentiation and a representation theorem for the weak derivatives of product measures, are established. A product rule for weak differentiation of probability measures was conjectured (without a proof) in [48]. At the end of the chapter two gradient estimation examples are provided.

In Chapter 3 we illustrate how the theory of measure-valued differentiation can be applied in order to establish bounds on perturbations for general stochastic models. Special attention is paid to the sequence of waiting times in the G/G/1 queue for which we show that the strong stability property holds true provided that the service-time distribution is weakly differentiable with respect to some class of sub-exponential cost-functions.

In Chapter 4 we extend our analysis to higher-order differentiation, which leads us to establish a measure-valued differential calculus. Analyticity issues are also treated and Taylor series approximation examples are provided.

Eventually, in Chapter 5 we apply the results established in Chapter 4 to the class of discrete event systems whose state dynamic can be formalized into a matrix-vector multiplication in some general, non-conventional algebra (e.g., max-plus or min-plus algebra). A key result shows that, for some class of polynomially bounded cost-functions, weak differentiability of two random matrices X_θ and Y_θ is inherited by their generalized product $X_\theta \odot Y_\theta$, which allows us to develop a weak differential calculus for random matrices.