

# VU Research Portal

## Measure-Valued Differentiation for Finite Products of Measures

Leahu, H.

2008

### **document version**

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

### **citation for published version (APA)**

Leahu, H. (2008). *Measure-Valued Differentiation for Finite Products of Measures: Theory and Applications*. [PhD-Thesis - Research and graduation internal, Vrije Universiteit Amsterdam].

### **General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

### **E-mail address:**

[vuresearchportal.ub@vu.nl](mailto:vuresearchportal.ub@vu.nl)

*MEASURE-VALUED DIFFERENTIATION FOR FINITE PRODUCTS OF  
MEASURES:  
THEORY AND APPLICATIONS*

*MEASURE-VALUED DIFFERENTIATION FOR FINITE PRODUCTS OF  
MEASURES:  
THEORY AND APPLICATIONS*

This thesis is devoted to the theory of weak differentiation of measures. The basic observation is that, formally, the weak derivative of a parameter-dependent probability distribution  $\mu_\theta$  is in general a finite signed measure which can be represented as the re-scaled difference between two probability distributions. This fact allows for a useful representations of the derivative  $\frac{d}{d\theta}\mathbb{E}_\theta[g(X)]$  of the expected value  $\mathbb{E}_\theta[g(X)]$ , for some predefined class  $\mathcal{D}$  of cost-functions  $g$ , where  $X$  is a random variable with distribution  $\mu_\theta$ .

Many mathematical models are described by a finite family of independent random variables and this is the reason why differentiability properties as well as representations for weak derivatives of product measures are studied in this thesis. To develop the theory, concepts and results from measure theory and functional analysis are required and the necessary prerequisites are presented in Chapter 1.

In Chapter 2 we develop the theory of first-order differentiation. Main results, such as the product rule of weak differentiation and a representation theorem for the weak derivatives of product measures, are established. A product rule for weak differentiation of probability measures was conjectured (without a proof) in [48]. At the end of the chapter two gradient estimation examples are provided.

In Chapter 3 we illustrate how the theory of measure-valued differentiation can be applied in order to establish bounds on perturbations for general stochastic models. Special attention is paid to the sequence of waiting times in the G/G/1 queue for which we show that the strong stability property holds true provided that the service-time distribution is weakly differentiable with respect to some class of sub-exponential cost-functions.

In Chapter 4 we extend our analysis to higher-order differentiation, which leads us to establish a measure-valued differential calculus. Analyticity issues are also treated and Taylor series approximation examples are provided.

Eventually, in Chapter 5 we apply the results established in Chapter 4 to the class of discrete event systems whose state dynamic can be formalized into a matrix-vector multiplication in some general, non-conventional algebra (e.g., max-plus or min-plus algebra). A key result shows that, for some class of polynomially bounded cost-functions, weak differentiability of two random matrices  $X_\theta$  and  $Y_\theta$  is inherited by their generalized product  $X_\theta \odot Y_\theta$ , which allows us to develop a weak differential calculus for random matrices.