Relaxed commutant lifting and Nehari interpolation

ter Horst, S.

2007

Relaxed commutant lifting

and

Nehari interpolation

Horst, Sanne ter, 1977–
Relaxed commutant lifting and Nehari interpolation
ISBN: 978-90-86591-58-9

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

Cover design by Gonneke H. Groenen with technical assistance of Bart de Graaff.
Cover photo by Savita Patil.

Printed by PrintPartners Ipskamp, The Netherlands.

VRIJE UNIVERSITEIT

Relaxed commutant lifting and Nehari interpolation

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. L.M. Bouter,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der Exacte Wetenschappen
op vrijdag 30 november 2007 om 15.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

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Chapter 1

Introduction

In this chapter we formulate the main problems treated in this thesis, and we present an overview of the main results. For our starting point we take the classical paper by Z. Nehari [72] from 1957, and we use the so-called Nehari problem as a role model for the type of problems considered in this thesis. As in Nehari's paper this thesis is concerned with a beautiful interplay between function theory and Hilbert space operator theory. The functions we shall deal with are not scalar-valued, as in Nehari's paper, but operator-valued. This transition, from scalar-valued to operator-valued, is natural and to a certain extent also motivated by applications.

The Nehari problem. The main problem treated by Nehari in [72] considered in this paper concerns the following question: given a sequence $\alpha_0, \alpha_1, \ldots$ of complex numbers, when is the bilinear form $\mathscr{A}$ defined by

$$\mathscr{A}(b,c) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{n+m} b_n c_m \quad (b = (b_0, b_1, \ldots), c = (c_0, c_1, \ldots)) \tag{1.1}$$

bounded on ℓ^2_+, the classical Hilbert space of square summable unilateral sequences of complex numbers.

Generalizing a classical result of I. Schur [83] Nehari proved that this happens if and only if there exists a function f in $L^\infty(\mathbb{T})$, the Banach space of all essentially bounded functions on the unit circle $\mathbb{T}$, such that for each nonnegative integer n the complex number α_n is equal to the n^{th} Fourier coefficient $\nu_n(f)$ of f, that is,

$$\alpha_n = \nu_n(f) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

In fact, it was shown in [72] that the supremum norm $\|f\|_\infty$ for any $f \in L^\infty(\mathbb{T})$ satisfying (1.2) serves as a bound for the bilinear form $\mathscr{A}$, and that the least bound for $\mathscr{A}$ is given by

$$\min\{\|f\|_\infty \mid f \in L^\infty(\mathbb{T}) \text{ such that (1.2) holds}\},$$

that is,

$$\sup_{\|b\|=\|c\|=1} \mathscr{A}(b,c) = \min\{\|f\|_\infty \mid f \in L^\infty(\mathbb{T}) \text{ such that (1.2) holds}\}.$$

Independent of [72], and coming from a different direction, V.M. Adamjan, D.Z. Arov and M.G. Kreĭn [3] showed that for any sequence $f_{-1}, f_{-2}, \ldots$ of complex numbers there exists a function f in $L^\infty(\mathbb{T})$ whose Fourier coefficients with negative index are prescribed by $\nu_n(f) = f_n$ for $n = -1, -2, \ldots$ if and only if the infinite matrix

$$C = \begin{bmatrix} f_{-1} & f_{-2} & f_{-3} & \cdots \\ f_{-2} & f_{-3} & f_{-4} & \cdots \\ f_{-3} & f_{-4} & f_{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.3}$$

induces a (bounded linear) operator on ℓ^2_+. The operator C in (1.3) is referred to as the *Hankel operator defined by* $f_{-1}, f_{-2}, \ldots$. When $\alpha_0 = f_{-1}, \alpha_1 = f_{-2}, \ldots$, the bilinear form $\mathscr{A}$ in (1.1) on ℓ^2_+ can be written as

$$\mathscr{A}(b, c) = \langle Cb, c \rangle \quad (b, c \in \ell^2_+),$$

where $\langle\ ,\ \rangle$ denotes the inner product of ℓ^2_+. It follows that the minimal possible supremum norm of a function f in $L^\infty(\mathbb{T})$ satisfying $\nu_n(f) = f_n$ for $n = -1, -2, \ldots$ is equal to the (operator) norm of the Hankel operator C defined by $f_{-1}, f_{-2}, \ldots$. In other words,

$$\|C\| = \min\{\|f\|_\infty \mid f \in L^\infty(\mathbb{T}) \text{ such that } \nu_n(f) = f \text{ for } n = -1, -2, \ldots\}.$$

Related to the problem in [3] is the following metric constrained interpolation problem, which by now is known as the *(scalar) Nehari problem*: given a sequence $f_{-1}, f_{-2}, \ldots$ of complex numbers, describe all functions f in $L^\infty(\mathbb{T})$ that satisfy

(i) $\nu_n(f) = f_n$ for $n = -1, -2, \ldots$,

(ii) $\|f\|_\infty \leq 1$.

The first condition is an interpolation condition and the second condition a metric constraint. From the results in [72] and [3] it follows that solutions to the Nehari problem exist if and only if the Hankel operator defined by $f_{-1}, f_{-2}, \ldots$ has norm at most one, that is, the Hankel operator is a contraction.

For this thesis it is important to know that a doubly infinite sequence of complex numbers $\ldots, f_{-2}, f_{-1}, f_0, f_1, \ldots$ is the sequence of Fourier coefficients of a function f in $L^\infty(\mathbb{T})$ if and only if the doubly infinite matrix

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & \cdots \\ \cdots & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & \cdots \\ \cdots & f_2 & f_1 & \boxed{f_0} & f_{-1} & f_{-2} & \cdots \\ \cdots & f_3 & f_2 & f_1 & f_0 & f_{-1} & \cdots \\ \cdots & f_4 & f_3 & f_2 & f_1 & f_0 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.4}$$

induces an operator on ℓ^2, the Hilbert space of square summable bilateral sequences of complex numbers. The box in (1.4) indicates the $(0, 0)$-position. In this case the norm of the operator on ℓ^2 induced by (1.4) is equal to the supremum norm $\|f\|_\infty$ of f. The operator on ℓ^2 induced by (1.4) is referred to as the *Laurent operator with symbol* f.

The facts mentioned in the preceding paragraph allow us to reformulate the Nehari problem as an extension problem. First note that the Laurent operator (1.4) has the same norm as its restriction to ℓ^2_+, that is, the operator from ℓ^2_+ into ℓ^2 induced by

$$\begin{bmatrix} \vdots & \vdots & \vdots & \iddots \\ f_{-2} & f_{-3} & f_{-4} & \cdots \\ f_{-1} & f_{-2} & f_{-3} & \cdots \\ \boxed{f_0} & f_{-1} & f_{-2} & \cdots \\ f_1 & f_0 & f_{-1} & \cdots \\ f_2 & f_1 & f_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.5}$$

Thus the Nehari problem admits the following reformulation: given a sequence $f_{-1}, f_{-2}, \ldots$ of complex numbers, describe all sequences $f_0, f_1, f_2, \ldots$ of complex numbers such that the infinite matrix (1.5) induces a contraction from ℓ^2_+ into ℓ^2.

In this form the Nehari problem has a natural truncated version. Namely: given a sequence $f_{-1}, f_{-2}, \ldots$ of complex numbers and a positive integer N, describe all sequences $f_0, f_1, f_2, \ldots$ of complex numbers such that the infinite matrix

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ f_{-2} & f_{-3} & \cdots & f_{-N-1} \\ f_{-1} & f_{-2} & \cdots & f_{-N} \\ \boxed{f_0} & f_{-1} & \cdots & f_{-N+1} \\ f_1 & f_0 & \cdots & f_{-N+2} \\ \vdots & \vdots & & \vdots \end{bmatrix} \tag{1.6}$$

induces a contractive operator mapping $\mathbb{C}^N$ into ℓ^2. This problem will be referred to as the *relaxed Nehari problem (with index N)*. Its operator-valued version (i.e., when the complex numbers f_j are replaced by operators acting between Hilbert spaces) is the main problem treated in the final chapter. In this relaxed version of the Nehari problem the role of the Hankel operator is taken over by the following truncated version:

$$\begin{bmatrix} f_{-1} & f_{-2} & \cdots & f_{-N} \\ f_{-2} & f_{-3} & \cdots & f_{-N-1} \\ f_{-3} & f_{-4} & \cdots & f_{-N-2} \\ \vdots & \vdots & & \vdots \end{bmatrix}. \tag{1.7}$$

From the lifting theory it follows that a solution to the relaxed Nehari problem exists if and only if the infinite matrix (1.7) induces a contractive operator from $\mathbb{C}^N$ into ℓ^2.

Intertwining relations. To explain what the Nehari problem and its relaxed version have to do with lifting theory it is important to know that the Hankel and Laurent operators and their truncated versions have a very special structure which can be expressed in terms of intertwining relations.

First we mention that a feature that distinguishes Laurent operators from arbitrary operators on ℓ^2 is the fact that a Laurent operator L commutes with the bilateral forward shift S on ℓ^2, that is, $SL = LS$, where

$$S = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \ddots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \ddots & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \boxed{0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{on } \ell^2.$$

The truncated Laurent operator in (1.5) and the Hankel operator (1.3) admit similar characterizations. To see this, write Π_+ for the orthogonal projection from ℓ^2 onto ℓ^2_+, and let S_+ denote the forward shift on ℓ^2_+, that is, $S_+ = \Pi_+ S|_{\ell^2_+}$. Then an operator B from ℓ^2_+ into ℓ^2 is of the form (1.5) if and only if

$$SB = BS_+, \tag{1.8}$$

while the Hankel operators C on ℓ^2_+ are characterized by the fact that they intertwine S_+ with its adjoint S_+^*, that is, $S_+^* C = C S_+$.

Now let ℓ^2_- be the orthogonal compliment of ℓ^2_+ in ℓ^2, and write Π_- for the orthogonal projection from ℓ^2 onto ℓ^2_-. We will denote S_- for the forward shift $\Pi_- S|_{\ell^2_-}$ on ℓ^2_-. Then S admits an operator decomposition of the form

$$S = \begin{bmatrix} S_- & 0 \\ X & S_+ \end{bmatrix} \text{ on } \begin{bmatrix} \ell^2_- \\ \ell^2_+ \end{bmatrix}, \quad \text{where} \quad X(\ldots, h_{-2}, h_{-1}) = (h_{-1}, 0, \ldots).$$

In the language of [69] this means that S is an isometric lifting of S_-.

For any sequence $f_0, f_1, f_2, \ldots$ of complex numbers the Hankel operator (1.3) is contractive if and only if the infinite matrix

$$\begin{bmatrix} \vdots & \vdots & \vdots & ⋰ \\ f_{-3} & f_{-4} & f_{-5} & \cdots \\ f_{-2} & f_{-3} & f_{-4} & \cdots \\ f_{-1} & f_{-2} & f_{-3} & \cdots \end{bmatrix} \tag{1.9}$$

induces a contraction from ℓ^2_+ into ℓ^2_-. Note that this operator appears in the upper half part of (1.5), and using an appropriate flip over operator the infinite matrix

(1.9) can be identified with the Hankel operator in (1.3). Furthermore, an operator A from ℓ^2_+ into ℓ^2_- is of the form (1.9) if and only if

$$S_- A = A S_+. \tag{1.10}$$

It is now clear that the Nehari problem admits the following alternative formulation: given an operator A from ℓ^2_+ into ℓ^2_- that satisfies the intertwining relation (1.10), describe all contractions B mapping ℓ^2_+ into ℓ^2 such that the intertwining relation (1.8) holds and $\Pi_- B = A$. In this way the Nehari problem fits into the commutant lifting framework. This fact, which was observed in [74], originates from the Sarason approach [81] to H^∞-interpolation; see also [38].

The relaxed Nehari problem admits a similar characterization with modified intertwining relations. To describe these relations we need operators R and Q both mapping $\mathbb{C}^{N-1}$ into $\mathbb{C}^N$ that are given by

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{1.11}$$

Then an operator B mapping $\mathbb{C}^N$ into ℓ^2 is of the form (1.6) if and only if

$$SBR = BQ.$$

Furthermore, for any sequence $f_{-1}, f_{-2}, \ldots$ of complex numbers the truncated Hankel matrix (1.7) is contractive if and only if the infinite matrix

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ f_{-3} & f_{-4} & \cdots & f_{-N-2} \\ f_{-2} & f_{-3} & \cdots & f_{-N-1} \\ f_{-1} & f_{-2} & \cdots & f_{-N} \end{bmatrix} \tag{1.12}$$

induces a contraction from $\mathbb{C}^N$ into ℓ^2_-. Operators A from $\mathbb{C}^N$ into ℓ^2_- of the form (1.12) are characterized by the fact that they satisfy

$$S_- A R = A Q. \tag{1.13}$$

Hence the relaxed Nehari problem also admits the following reformulation: given an operator A from $\mathbb{C}^N$ into ℓ^2_- that satisfies the intertwining relation (1.13), describe all contractions B mapping $\mathbb{C}^N$ into ℓ^2 such that the intertwining relation (1.8) holds and $\Pi_- B = A$. In this way the Nehari problem fits into the relaxed commutant lifting framework, which will be explained next.

Relaxed commutant lifting. One of the recent developments in lifting theory is the introduction in [42] of a relaxation of the commutant lifting setting. This relaxation is the main topic of the present thesis.

Let us explain this relaxation in precise terms. The starting point for the relaxed commutant lifting setting is a *lifting data set* $\{A, T', U', R, Q\}$, which is a set consisting of five Hilbert space operators. The operator A is a contraction mapping $\mathcal{H}$ into $\mathcal{H}'$, the operator U' on $\mathcal{K}'$ is a minimal isometric lifting of the contraction T' on $\mathcal{H}'$, and R and Q are operators from $\mathcal{H}_0$ to $\mathcal{H}$, satisfying the following constraints:

$$T'AR = AQ \quad \text{and} \quad R^*R \leq Q^*Q. \tag{1.14}$$

Given this data set the *relaxed commutant lifting problem* is to find all contractions B from $\mathcal{H}$ to $\mathcal{K}'$ such that

$$\Pi_{\mathcal{H}'}B = A \quad \text{and} \quad U'BR = BQ. \tag{1.15}$$

Here $\Pi_{\mathcal{H}'}$ is the orthogonal projection from $\mathcal{K}'$ onto $\mathcal{H}'$. A contraction B from $\mathcal{H}$ into $\mathcal{K}'$ that satisfies (1.15) will be called a *contractive interpolant for* $\{A, T', U', R, Q\}$.

The classical commutant lifting setting appears when R is the identity operator on $\mathcal{H}$, and thus $\mathcal{H}_0 = \mathcal{H}$, and Q is an isometry. In this case $R^*R = I_{\mathcal{H}} = Q^*Q$. Hence we have equality in the second condition of (1.14).

The relaxed Nehari problem can be seen as a special case of the relaxed commutant lifting problem. Indeed, for the relaxed Nehari problem $\mathcal{H}_0 = \mathbb{C}^{N-1}$, $\mathcal{H} = \mathbb{C}^N$, $\mathcal{H}' = \ell^2_+$, $\mathcal{K}' = \ell^2$, A is the operator given by (1.12), $T' = S_-$, $U' = S$, and R and Q are as in (1.11).

There are many other examples of interpolation and extension problems that fit into the relaxed commutant lifting framework (see [42]). Also the generalization the commutant lifting setting by Treil and Volberg [87] and the weighted commutant lifting result from Biswas, Foias and Frazho [27] fit into the relaxed setting; see [42].

It is known [42] that for any lifting data set a contractive interpolant exists. The description of all such contractive interpolants is one of the main themes of this thesis.

The underlying contraction. In the analysis of the relaxed commutant lifting problem a certain contraction plays an important role. To introduce this contraction we need some additional notation.

We denote by D_A and $D_{T'}$ the defect operators, and by $\mathcal{D}_A$ and $\mathcal{D}_{T'}$ the defect spaces of the contractions A and T', respectively. Put

$$D_\circ = (Q^*Q - R^*R)^{\frac{1}{2}} \quad \text{and} \quad \mathcal{D}_\circ = \overline{D_\circ \mathcal{H}_0}. \tag{1.16}$$

Because of (1.14), for each $h \in \mathcal{H}_0$ we have

$$\begin{aligned} \|D_AQh\|^2 &= \|Qh\|^2 - \|AQh\|^2 = \|D_\circ h\|^2 + \|Rh\|^2 - \|T'ARh\|^2 \\ &= \|D_\circ h\|^2 + \|ARh\|^2 - \|T'ARh\|^2 + \|Rh\|^2 - \|ARh\|^2 \\ &= \|D_\circ h\|^2 + \|D_{T'}ARh\|^2 + \|D_ARh\|^2. \end{aligned}$$

This computation shows that the following identity holds:

$$Q^*D_A^2Q = D_\circ^2 + R^*A^*D_{T'}^2AR + R^*D_A^2R. \tag{1.17}$$

Now let $\mathcal{F}$ be the subspace of $\mathcal{D}_A$ defined by $\mathcal{F} = \overline{D_A Q \mathcal{H}_0}$. Then (1.17) enables us to define a contraction ω by

$$\omega : \mathcal{F} = \overline{D_A Q \mathcal{H}_0} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix}, \quad \omega D_A Q = \begin{bmatrix} D_{T'} A R \\ D_A R \end{bmatrix}. \tag{1.18}$$

This contraction will be referred to as *the contraction underlying the lifting data set* $\{A, T', U', R, Q\}$.

It is interesting to note that the underlying contraction ω is an isometry if and only if the operator $D_\circ$ is the zero operator, and thus $D_\circ = \{0\}$. In terms of the operators appearing in the lifting data set the latter means that ω is an isometry if and only if $R^*R = Q^*Q$. In particular, in the classical commutant lifting setting the underlying contraction is an isometry. Moreover, in the commutant lifting setting the space $\Pi_{\mathcal{D}_A}\omega\mathcal{F}$ is dense in $\mathcal{D}_A$, where $\Pi_{\mathcal{D}_A}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{D}_A$. In general, in the setting of the relaxed commutant lifting problem these two properties are absent, which makes the relaxed commutant lifting problem more complicated than the classical one.

An equivalent problem. As for the classical commutant lifting problem, the fact that all minimal isometric liftings of T' are unitarily equivalent implies that without loss of generality we may assume U' to be equal to the Sz.-Nagy-Schäffer isometric lifting of T', that is,

$$U' = \begin{bmatrix} T' & 0 \\ E_{\mathcal{D}_{T'}} D_{T'} & S_{\mathcal{D}_{T'}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}. \tag{1.19}$$

Here $H^2(\mathcal{D}_{T'})$ is the Hardy space of $\mathcal{D}_{T'}$-valued functions on the unit disc $\mathbb{D}$, the operator $E_{\mathcal{D}_{T'}}$ is the canonical embedding of $\mathcal{D}_{T'}$ onto the subspace of constant functions in $H^2(\mathcal{D}_{T'})$, that is, $(E_{\mathcal{D}_{T'}} d)(\lambda) = d$ for all $\lambda \in \mathbb{D}$ and each $d \in \mathcal{D}_{T'}$, and $S_{\mathcal{D}_{T'}}$ denotes the unilateral shift on $H^2(\mathcal{D}_{T'})$.

Using some standard operator theory results (cf., Section IV.1 of [38] or Section XXVII.5 of [55]) it follows that an operator B from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is contractive and satisfies $\Pi_{\mathcal{H}'} B = A$ if and only if B is given by

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \quad \Gamma : \mathcal{D}_A \to H^2(\mathcal{D}_{T'}) \text{ a contraction.} \tag{1.20}$$

Moreover, the operator Γ in (1.20) is uniquely determined by B.

Assuming that B from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is of the form (1.20) and using the formula (1.19) for U' we obtain that the equation $U'BR = BQ$ can be expressed in terms of Γ in (1.20) as

$$E_{\mathcal{D}_{T'}} D_{T'} A R + S_{\mathcal{D}_{T'}} \Gamma D_A R = \Gamma D_A Q,$$

or equivalently, using the definition of the underlying contraction ω, as

$$E_{\mathcal{D}_{T'}} \omega_1 + S_{\mathcal{D}_{T'}} \Gamma \omega_2 = \Gamma|_{\mathcal{F}}.$$

Here ω_1 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_{T'}$ determined by the first component of ω and ω_2 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_A$ determined by the second component of ω.

This formulation suggests to consider the following general interpolation problem (introduced in Chapter 4): given Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, a subspace $\mathcal{F}$ of $\mathcal{U}$ and a contraction

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \tag{1.21}$$

find all contractions Γ mapping $\mathcal{U}$ into $H^2(\mathcal{Y})$ that satisfy

$$E_{\mathcal{Y}}\omega_1 + S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}. \tag{1.22}$$

Here $E_{\mathcal{Y}}$ and $S_{\mathcal{Y}}$ are defined in the same way as $E_{\mathcal{D}_{T'}}$ and $S_{\mathcal{D}_{T'}}$, with $\mathcal{Y}$ in place of $\mathcal{D}_{T'}$.

Clearly this problem covers the relaxed commutant lifting problem (see Proposition 4.3.1). We prove in Chapter 4 that conversely this problem can be reformulated as a relaxed commutant lifting problem.

It is straightforward that the problem (1.22) admits the following solution:

$$(\Gamma d)(\lambda) = \omega_1 \Pi_{\mathcal{F}} (I - \lambda \omega_2 \Pi_{\mathcal{F}})^{-1} d \quad (d \in \mathcal{U}, \lambda \in \mathbb{D}), \tag{1.23}$$

where $\Pi_{\mathcal{F}}$ denotes the orthogonal projection from $\mathcal{U}$ onto $\mathcal{F}$.

When ω is the contraction underlying a lifting data set $\{A, T', U', R, Q\}$ (see (1.18)), the operator B in (1.20) with Γ given by (1.23), where $\mathcal{U}$ is replaced by $\mathcal{D}_A$, is referred to as the *central contractive interpolant*. This contractive interpolant was already obtained in [42].

Main results concerning relaxed commutant lifting. The first main result of this thesis is the following representation of all contractive interpolants in terms of Schur class functions. By definition a *Schur class function* is a contractive analytic operator-valued function on the unit disc $\mathbb{D}$. For given Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ the symbol $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ stands for the set of Schur class functions whose values are operators from $\mathcal{U}$ into $\mathcal{Y}$.

Theorem 1.1. *Consider the lifting data set $\{A, T', U', R, Q\}$ with U' being the Sz.-Nagy-Schäffer isometric lifting of T', and let ω be the contraction underlying $\{A, T', U', R, Q\}$. Let Z be a Schur class function in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ such that $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$. Put*

$$(\Gamma d)(\lambda) = \Pi_{\mathcal{D}_{T'}} Z(\lambda)(I - \lambda \Pi_{\mathcal{D}_A} Z(\lambda))^{-1} d \quad (d \in \mathcal{D}_A, \lambda \in \mathbb{D}), \tag{1.24}$$

where $\Pi_{\mathcal{D}_{T'}}$ and $\Pi_{\mathcal{D}_A}$ are the orthogonal projections from $\mathcal{D}_{T'} \oplus \mathcal{D}_A$ onto $\mathcal{D}_{T'}$ and $\mathcal{D}_A$, respectively. Then Γ is a contraction from $\mathcal{D}_A$ into $H^2(\mathcal{D}_{T'})$ and the operator

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix} \tag{1.25}$$

is a contractive interpolant for $\{A, T', U', R, Q\}$. Moreover, in this way all contractive interpolants for $\{A, T', U', R, Q\}$ are obtained.

A Schur class function $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ satisfying the constraint $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$ always exists. For instance, one can take for Z the constant function with value $\omega\Pi_{\mathcal{F}}$. This choice for Z provides the central contractive interpolant.

In the commutant lifting setting, where $R = I_{\mathcal{H}}$ and Q is an isometry on $\mathcal{H}$, the formula for the contractive interpolants in (1.25) and (1.24) is known, cf., [37] and the earlier papers [36] and [17]. Moreover, due to the special properties of the underlying contraction in the commutant lifting setting, the representation via (1.25) and (1.24) provides a proper parameterization, that is, in the commutant lifting case the map $Z \mapsto B$ is one-to-one and onto. In general, the latter is not true. It can happen that different Z's yield the same B.

To give an example of the non-uniqueness in the representation (1.24), let $\mathcal{H}_0$, $\mathcal{H}$ and $\mathcal{H}'$ be equal to $\mathbb{C}$, let A, R and Q be the zero operator on $\mathbb{C}$, and take for T' the identity operator on $\mathbb{C}$. Since T' is an isometry, the Sz.-Nagy-Schäffer minimal isometric lifting of T' is equal to T'. The latter implies that there is only one contractive interpolant B for the data set $\{A, T', U', R, Q\}$, namely $B = A$. The fact that R and Q are the zero operators on $\mathbb{C}$ shows that $\mathcal{F} = \{0\}$. It follows that for this data set $\{A, T', U', R, Q\}$ the only contractive interpolant $B = A$ is given by (1.25) and (1.24) where for Z we can take any function in the Schur class $\mathbf{S}(\mathbb{C}, \mathbb{C})$.

To describe the non-uniqueness in the representation of all contractive interpolants in Theorem 1.1 we need some additional notation. Let B be a fixed contractive interpolant for the lifting data set $\{A, T', U', R, Q\}$. The fact that B is a contraction satisfying (1.15) implies that $\{B, U', U', R, Q\}$ is also a lifting data set. Using that the defect operator $D_{U'}$ of U' is the zero operator, the identity (1.17) for the lifting data set $\{B, U', U', R, Q\}$ yields

$$Q^* D_B^2 Q = D_\circ^2 + R^* D_B^2 R. \tag{1.26}$$

Since B is a contraction with $\Pi_{\mathcal{H}'} B = A$, we obtain that B admits an operator decomposition of the form (1.20). Therefore, the square of the defect operator D_B of B admits a factorization of the form

$$D_B^2 = I - B^*B = I - A^*A - D_A \Gamma^* \Gamma D_A = D_A(I - \Gamma^*\Gamma) D_A = D_A D_\Gamma^2 D_A,$$

where D_Γ denotes the defect operator of Γ. Hence with (1.26) we obtain

$$Q^* D_A D_\Gamma^2 D_A Q = D_\circ^2 + R^* D_A D_\Gamma^2 D_A R.$$

The latter identity implies that we can define a contraction ω_Γ by

$$\omega_\Gamma : \mathcal{F}_\Gamma = \overline{D_\Gamma D_A Q \mathcal{H}_0} \to \mathcal{D}_\Gamma, \quad \omega_\Gamma D_\Gamma D_A Q = D_\Gamma D_A R. \tag{1.27}$$

We are now ready to state the second main result.

Theorem 1.2. *Let B be a solution to the relaxed commutant lifting problem for the lifting data set $\{A, T', U', R, Q\}$ with U' being the Sz.-Nagy-Schäffer isometric lifting* *of T', and let Γ be the contraction determined by B via* (1.20). *Take $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$, and put*

$$J_\Gamma K = Z, \quad Z(\lambda) = \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda) - I) \end{bmatrix} F(\lambda)^{-1} \quad (\lambda \in \mathbb{D}),$$

where

$$\begin{array}{rcl} H(\lambda)d & = & (\Gamma d)(\lambda), \quad d \in \mathcal{D}_A, \\ F(\lambda) & = & \Gamma^*(I - \lambda S^*_{\mathcal{D}_{T'}})^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1} D_\Gamma. \end{array} \quad (\lambda \in \mathbb{D})$$

Then J_Γ is a one-to-one map from $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ into $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ that maps the set

$$\mathbf{K}_{\omega_\Gamma} := \{K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma) \mid K(\lambda)|_{\mathcal{F}_\Gamma} = \omega_\Gamma \text{ for each } \lambda \in \mathbb{D}\} \tag{1.28}$$

onto the set of all Schur class function Z in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ with the property that $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$ and such that B is given by (1.25) *and* (1.24).

Theorems 1.1 and 1.2 are proved in Chapter 4.

So far, the description of the contractive interpolants has been in terms of the contraction ω underlying the given lifting data set $\Omega = \{A, T', U', R, Q\}$. Often it is more convenient to have a description that is explicitly given in terms of the operators appearing in Ω.

This is done here under the additional assumptions that A is a strict contraction ($\|A\| < 1$) and R has a left inverse, which in many concrete examples is referred to as the sub-optimal case; cf., [53] and [22]. In this case the defect operator D_A of A is strictly positive (non-negative and invertible), and from the fact that $R^*R \leq Q^*Q$ it follows that Q also has a left inverse. In particular, D_AQ and D_AR are left invertible, or equivalently, $Q^*D_A^2Q$ and $R^*D_A^2R$ are strictly positive operators. Thus the contraction ω underlying $\{A, T', U', R, Q\}$ is given by

$$\omega = \begin{bmatrix} D_{T'}AR \\ D_AR \end{bmatrix} (Q^*D_AQ)^{-1}Q^*D_A : \mathcal{H} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix}.$$

The extra assumptions allow us to define analytic operator-valued functions K_1, K_2, K_3, K_4, K_5 and K_6 on $\mathbb{D}$, as follows:

$$\begin{aligned} K_1(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} D_A^{-2} \Pi^*_{\operatorname{Ker} Q^*}, \\ K_2(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} D_A^{-2} \Pi^*_{\operatorname{Ker} R^*}, \\ K_3(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} R(R^*D_A^2R)^{-1}J^*, \\ K_4(\lambda) &= D_{T'}AM(I - \lambda M)^{-1} D_A^{-2} \Pi^*_{\operatorname{Ker} Q^*}, \\ K_5(\lambda) &= \lambda D_{T'}AM(I - \lambda M)^{-1} D_A^{-2} \Pi^*_{\operatorname{Ker} R^*}, \\ K_6(\lambda) &= -\Pi_{\mathcal{D}'_T} + \lambda D_{T'}AM(I - \lambda M)^{-1} R(R^*D_A^2R)^{-1}J^*, \end{aligned} \quad (\lambda \in \mathbb{D}) \tag{1.29}$$

Here $\Pi_{\operatorname{Ker} Q^*}$ and $\Pi_{\operatorname{Ker} R^*}$ denote the orthogonal projections from $\mathcal{H}$ onto $\operatorname{Ker} Q^*$ and $\operatorname{Ker} R^*$, respectively, while

$$M = R(Q^*Q)^{-1}Q^* \text{ on } \mathcal{H} \quad \text{and} \quad J = \begin{bmatrix} D_\circ \\ D_{T'}AR \end{bmatrix} : \mathcal{H}_0 \to \begin{bmatrix} \mathcal{D}_\circ \\ \mathcal{D}_{T'} \end{bmatrix}, \qquad (1.30)$$

where $D_\circ$ and $\mathcal{D}_\circ$ are the operator and Hilbert space defined in (1.16). From the definition of M in (1.30) and the fact that $R^*R \le Q^*Q$ it follows that M is a contraction, and thus the functions K_1, K_2, K_3, K_4, K_5 and K_6 in (1.29) are all analytic on $\mathbb{D}$, indeed. Furthermore, let Δ_Q on $\operatorname{Ker} Q^*$, Δ_R on $\operatorname{Ker} R^*$ and Δ_Ω on $\mathcal{D}_\circ \oplus \mathcal{D}_{T'}$ be the strictly positive operators given by

$$\begin{aligned} \Delta_Q = \Pi_{\operatorname{Ker} Q^*} D_A^{-2} \Pi^*_{\operatorname{Ker} Q^*}, \qquad \Delta_R = \Pi_{\operatorname{Ker} R^*} D_A^{-2} \Pi^*_{\operatorname{Ker} R^*}, \\ \Delta_\Omega = I + J(R^* D_A^2 R)^{-1} J^*. \end{aligned} \qquad (1.31)$$

With these definitions we can state the following main result which epitomizes the results derived in Chapter 5; see Theorems 5.3.4 and 5.4.1 below.

Theorem 1.3. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that A is a strict contraction and R is left invertible. Define analytic functions $\Psi_{1,1}$, $\Psi_{1,2}$, $\Psi_{2,1}$ and $\Psi_{2,2}$ on $\mathbb{D}$ as follows*

$$\begin{aligned} \Psi_{1,1}(\lambda) &= -K_4(\lambda)\Delta_Q^{-\frac{1}{2}}, \\ \Psi_{1,2}(\lambda) &= -\begin{bmatrix} K_6(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_5(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix}, \\ \Psi_{2,1}(\lambda) &= K_1(\lambda)\Delta_Q^{-\frac{1}{2}}, \\ \Psi_{2,2}(\lambda) &= \lambda \begin{bmatrix} K_3(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_2(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix}. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Here K_1, K_2, K_3, K_4, K_5 and K_6 are the functions given by (1.29)*, and Δ_Q, Δ_R and Δ_Ω are as in* (1.31)*. Let W be a Schur class function in the Schur class $\mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*)$, and for each $h \in \mathcal{H}$ and $\lambda \in \mathbb{D}$ put*

$$(\Lambda h)(\lambda) = D_{T'}AM(I - \lambda M)^{-1} + G(\lambda)\Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1}, \qquad (1.32)$$

where M is as in (1.30)*, and G is the $\mathscr{L}(\operatorname{Ker} Q^*, \mathcal{D}_{T'})$-valued function on $\mathbb{D}$ given by*

$$G(\lambda) = (\Psi_{1,2}(\lambda)W(\lambda) + \Psi_{1,1}(\lambda))(\Psi_{2,2}(\lambda)W(\lambda) + \Psi_{2,1}(\lambda))^{-1} \quad (\lambda \in \mathbb{D}).$$

Then Λ is a contraction from $\mathcal{H}$ into $H^2(\mathcal{D}_{T'})$, and the operator

$$B = \begin{bmatrix} A \\ \Lambda \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}$$

is a contractive interpolant for $\{A, T', U', R, Q\}$. Moreover, in this way all contractive interpolants for $\{A, T', U', R, Q\}$ are obtained.

In many examples the operator M in (1.30) has a very simple structure. Indeed, for instance, in the classical Nehari problem M is the backward shift operator on ℓ^2_+, while for the relaxed Nehari problem M is the finite backward shift operator on $\mathbb{C}^N$.

Moreover, the operator M has an interesting property, namely, $M|_{\operatorname{Ker} Q^*} = 0$. From this property we obtain that Λ in (1.32) satisfies

$$(\Lambda e)(\lambda) = G(\lambda)e \quad (e \in \operatorname{Ker} Q^*, \lambda \in \mathbb{D}).$$

It follows that each contractive interpolant B is completely determined by its action on the kernel of Q^*. This observation holds true even without the additional assumptions $\|A\| < 1$ and R left invertible. The fact that a contractive interpolant B is completely determined by its action on $\operatorname{Ker} Q^*$ is in line with many of the examples. For instance, in the relaxed Nehari problem $\operatorname{Ker} Q^* = \mathbb{C} \oplus \{0\}^{N-1} \subset \mathbb{C}^N$, and thus $B|_{\operatorname{Ker} Q^*}$ corresponds to the first column in the truncated Laurent operator (1.5).

In the final chapter we return to the relaxed Nehari problem, not the scalar-valued, but the operator-valued version. We show that it fits into the relaxed commutant lifting framework in a way that parallels what was done for the scalar case earlier in this chapter. Theorem 1.3 is specified for the relaxed Nehari case. Finally, the classical (operator-valued) Nehari problem is studied as a limit case of the relaxed one.

The Banach space $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ and its unit ball. To prove the main results concerning relaxed commutant lifting it turns out to be useful to consider the relaxed commutant lifting problem from a more function theoretical perspective.

For this purpose it is interesting to specify Theorems 1.1 and 1.2 for a specific (trivial) lifting data set, namely, the lifting data set $\{A, T', U', R, Q\}$ given by

$$A = 0 : \mathcal{U} \to \mathcal{Y}, \quad T' = 0 \text{ on } \mathcal{Y}, \quad R = Q = 0 \text{ on } \mathcal{U},$$

and U' on $\mathcal{Y} \oplus H^2(\mathcal{Y})$ the Sz.-Nagy-Schäffer isometric lifting of T'. Here $\mathcal{U}$ and $\mathcal{Y}$ are arbitrary Hilbert spaces. In this case the underlying contraction ω is the zero operator from $\mathcal{F} = \{0\}$ into $\mathcal{Y} \oplus \mathcal{U}$, and for any operator B from $\mathcal{U}$ into $\mathcal{Y} \oplus H^2(\mathcal{Y})$ the condition $U'BR = BQ$ is satisfied trivially. Hence the contractive interpolants for this particular lifting data set are given by (1.20), that is, all contractive interpolants for $\{A, T', U', R, Q\}$ are given by

$$B = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \\ H^2(\mathcal{Y}) \end{bmatrix} \text{ with } \Gamma : \mathcal{U} \to H^2(\mathcal{Y}) \text{ a contraction.} \tag{1.33}$$

Theorem 1.1 for this case, and expressed in terms of the operator Γ in (1.33), reduces to the following result.

Theorem 1.4. *Let $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, and put*

$$(\Gamma u)(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} u \quad (u \in \mathcal{U}, \lambda \in \mathbb{D}), \tag{1.34}$$

where $\Pi_{\mathcal{Y}}$ and $\Pi_{\mathcal{U}}$ are the orthogonal projections from $\mathcal{Y} \oplus \mathcal{U}$ onto $\mathcal{Y}$ and $\mathcal{U}$, respectively. Then Γ is a contraction from $\mathcal{U}$ into $H^2(\mathcal{Y})$, and all contractions from $\mathcal{U}$ into $H^2(\mathcal{Y})$ are obtained in this way.

The non-uniqueness in the representation (1.34), as described in Theorem 1.2, also simplifies in this case. Indeed, let Γ be an arbitrary contraction from $\mathcal{U}$ into $H^2(\mathcal{Y})$, and let B be the contractive interpolant for $\{A, T', U', R, Q\}$ given by (1.33). Then the contraction ω_Γ in (1.27) is equal to the zero operator from $\mathcal{F}_\Gamma = \{0\}$ into $\mathcal{D}_\Gamma$, and the set $\mathbf{K}_{\omega_\Gamma}$ coincides with the Schur class $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$. Hence there is a unique Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying (1.34) if and only if Γ is an isometry. In fact, Theorem 1.2 and the above analysis yield the following result.

Theorem 1.5. *Let Γ be a contraction from $\mathcal{U}$ into $H^2(\mathcal{Y})$. Then the map J_Γ defined in Theorem* 1.2 *is a one-to-one map from $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ onto the set of all $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ that satisfy* (1.34).

The relation with function theory is obtained as follows. With an operator Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ we associate an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued analytic function H on $\mathbb{D}$ given by

$$H(\lambda)u = (\Gamma u)(\lambda) \quad (\lambda \in \mathbb{D}, u \in \mathcal{U}).$$

We write $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ for the set of all operator-valued functions obtained in this way. Then $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ is a Banach space under the norm $\|H\| = \|\Gamma\|$. The unit ball of $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, which is denoted by $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, is the main topic of study in Chapter 3.

Theorem 1.4 can now be interpreted as saying that the functions in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ are characterized by the fact that they admit a *Schur representation*, that is, an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function H on $\mathbb{D}$ is in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ if and only if there exists a $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ such that

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{1.35}$$

Then Theorem 1.5 provides a parameterization of the non-uniqueness in the Schur representation (1.35). See Theorems 3.2.1 and 3.2.2 below for the precise statements.

As it turns out, in order to prove Theorems 1.1 and 1.2, it is useful to first prove Theorems 1.4 and 1.5 (expressed in terms of the functions in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$). The general interpolation problem associated with the contraction ω in (1.21), which is equivalent to the relaxed commutant lifting problem, can also be formulated as follows: given a contraction ω as in (1.21), describe all functions $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ that satisfy

$$\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_{\mathcal{F}} \quad (\lambda \in \mathbb{D}).$$

This is precisely the problem we deal with in Chapter 4, and whose solution leads to the proof of Theorems 1.1 and 1.2.

Historical remarks and additional references. This thesis contributes to the theory of metric constrained interpolation. An area that started in the beginning of the previous century with the work of Carathéodory [28, 29], Schur [83, 84], Pick

[75, 76] and Nevanlinna [73]. The connection with operator theory was made at the end of the 1960's. In particular, the work of Adamjan, Arov and Kreĭn [3–5], using operator theory, presented new representations of the solutions to these interpolation problems. A further breakthrough came with a paper in 1967 by Sarason [81], in which a general framework was presented that includes most of the metric constrained interpolation problems. One year later this framework was vastly extended by B. Szökefalvi-Nagy and C. Foias in [68] using their work on dilation theory developed in the early 1960's. The latter method became known as the commutant lifting method. Throughout the years many of the results obtained for the individual metric constrained interpolation problems were translated to and proved for the commutant lifting setting; see [38] and the more recent [41]. Commutant lifting is one of the methods to solve metric constrained interpolation problems. There are several others, each with its own special features and advantages. For this and related references we refer to introductions and bibliographies in the monographs [30, 38, 41].

In the late 70's and early 80's of the previous century there was a renewed interest in interpolation theory, when it was observed that many control and electrical engineering problems, like optimal control, robust stabilization, sensitivity minimization, model matching and wideband disturbance attenuation, could be modeled as metric constrained interpolation problems; see the pioneering papers by Zames [88, 89] and Helton [57], and Francis' lecture notes [45]. This new impulse created a demand for non-scalar versions of the interpolation problems, as well as more explicit and better applicable formulas and representations for their solutions. As a form of cross-fertilization state space formulas, known from engineering and the theory of characteristic operator functions, gained a prominent role in operator theory, and not only for solving interpolation problems, cf., [24]. See also [41], where state space formulas for the solutions to the commutant lifting problem were obtained.

Francis [45] solved the model-matching problem in the matrix-valued case by reducing it to a Nehari problem. Moreover, the Nehari problem also proved to be useful for the H^∞ disc method due to Helton, which in turn could be used for certain control problems [59] and equalization in circuits [58]. One of the difficulties encountered in the Nehari problem is the infinite dimensionality of the Hankel operator. State space formulas for the solutions can be obtained, but unless the Hankel operator has finite rank, the state space will be infinite dimensional. Trefethen [86] proposed an approximation method for the scalar Nehari problem using N by N compressions of the Hankel operator, under the additional assumption that the Hankel operator attains norm one (the optimal case). This method, called the Carathéodory-Fejér method, was later modified by Helton and Young [61] to obtain better convergence; see also [60]. The relaxed Nehari method of approximating solutions to the Nehari problem, which we present in the final section of this thesis, can be seen as a variation on the Carathéodory-Fejér method. However, it also works for the operator-valued Nehari problem and does not require the Hankel operator to attain norm one.

One of the recent developments in lifting theory is the introduction of the relaxed commutant lifting framework [42]. It contains the classical commutant lifting setting as well as the Treil-Volberg version [87] and the weighted commutant lifting theorem from [27] as special cases. In [42] a central solution (the central contractive interpolant defined above) was obtained, and used to solve a number of relaxed versions of the classical metric constrained interpolation problems. Descriptions of all solutions were obtained in [48, 49, 63]. The ones in [48, 63] are based on the coupling method, which was introduced by Adamjan and Arov [1, 2] and was used to study commutant lifting for the first time by Arocena [15, 16]. The results of [48] were improved in [49], by generalizing certain results on harmonic majorants to operator-valued functions. As a by-product, a representation of certain matrix-valued H^2-functions given in [9] (see also [10, 11]) could be proved for to the operator-valued case as well. While the representations in [48, 49] are in terms of Schur class functions, the one in [63] uses a choice sequence approach. With Chapter VI of [41] as a source of inspiration, in [62] a Redheffer representation of all solutions is presented for a special case that corresponds to the so-called suboptimal case for many of the concrete examples. The latter result appears in the first part of Chapter 5 of the present thesis. In [50] the relaxed commutant lifting problem was proved to be equivalent to a general interpolation problem that involves the operator-valued H^2-functions encountered in [49]; this result and its corollaries appear in Chapter 4.

The appearance of more advanced interpolation problems spawned a large number of generalizations of the commutant lifting setting in various directions. We conclude this introduction by mentioning just a few of these more advanced interpolation problems and commutant lifting theorems. The Treil-Volberg version [87] enabled Nehari interpolation in a Bergman space setting; see also [85]. There exists a time-varying version of the commutant lifting theorem which is known as the three chain completion theorem [39, 40]. The analog for relaxed commutant lifting is not obvious. However, in the setting of the equivalent problem, as explained above, a generalization to the relaxed commutant lifting setting seems possible. This will be part of future work. There are various notions of multivariable interpolation, for instance, interpolation for function with non-commuting variables [79] (the Fock space setting), with commuting variables [8] (the Drury-Arveson space setting), and for functions in the Schur-Agler class for the polydisc [6, 7]. For all three cases there exists a commutant lifting theorem; see [46, 47] and the more recent [78] for the Fock space, [23] for the Drury-Arveson space and [21] for the polydisc. In the Fock space setting there also exists a relaxed commutant lifting theorem [80]. In fact, in [80] a generalization of the commutator version of the relaxed commutant lifting theorem from [43, 44] is obtained. A more abstract approach to multivariable interpolation and commutant lifting is presented in the work of Muhly and Solel [64–67], where so-called C^*-correspondences are studied. The C^*-correspondence approach contains the Fock space and Drury-Arveson space settings and also applies to quiver algebras, semi-crossed products, directed graphs and a number of other examples. However, the polydisc setting as well as the time-varying interpolation theory developed throughout the 1990's (see the papers [12, 13, 19], part B of the book [41] and other references given there) are not included. Moreover, there is no representation of all solutions to the C^*-correspondence commutant lifting theorem known. But the presentation of state space formulas for the C^*-correspondence version of the Schur class [18, 66, 67] might hint on a the possibility of a Schur type representation. Recently, Dritschel, Marcantognini and McCullough [32] (see also [33]) introduced a semigroupoid approach to interpolation, which, besides many of the examples covered by the C^*-correspondence framework, also contains the polydisc setting. A semigroupoid commutant lifting theorem has not yet been formulated. Much further work remains to be done.

Brief overview of the chapters. This thesis consists of six chapters. The first chapter is the present introduction. The second chapter has a preliminary character. Here we present a number of basic facts from operator theory that are used throughout this thesis. Some useful corollaries and lemmas are added too. The first section of this preliminary chapter contains an extended description of the notations and conventions used in the thesis. In Chapter 3 we introduce the space $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ and prove Theorems 1.4 and 1.5. For Theorem 1.5 two different proofs are given. The first uses a harmonic majorant argument and the second a state space approach. In Chapter 4 we prove Theorems 1.1 and 1.2, first in the setting of the general interpolation problem (1.22), and later as they are stated in the present introduction. In Chapter 5 we rewrite the Schur representation of Theorem 1.1 in the form of a linear fractional Redheffer representation, and later, under some additional conditions, in the form of a more classical linear fractional representation. In particular, Theorem 1.3 follows from the results derived in Chapter 5. The final chapter is devoted to the relaxed Nehari problem and its connection to the classical Nehari problem. Each chapter (except the first) ends with notes on the literature. In Chapter 4 some open problems are formulated.

Chapter 2

Preliminaries

This chapter has a preliminary character. We present a number of basic facts from operator theory that are used throughout this dissertation. Some useful corollaries and lemmas are added too. We also describe in some detail the standard notations and conventions that will appear in the sequel.

This chapter consists of five sections. We start with a section on notation and terminology. The remaining four sections deal with isometric liftings, systems and realizations, uniformly bounded functions and positive real functions; in that order.

2.1 Notation and terminology

Throughout capital calligraphic letters denote Hilbert spaces. The Hilbert space direct sum of two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ is denoted by

$$\mathcal{U} \oplus \mathcal{Y} \quad \text{or by} \quad \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}.$$

In case $\mathcal{U} = \mathcal{Y}$ we simply write $\mathcal{U}^2$, and analogously for each positive integer n the symbol $\mathcal{U}^n$ stands for the Hilbert space direct sum of n copies of $\mathcal{U}$. An *operator* is a bounded linear transformation acting between Hilbert spaces. With $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ we denote the set of all operators from $\mathcal{U}$ into $\mathcal{Y}$. The set $\mathscr{L}(\mathcal{U}, \mathcal{U})$ will be abbreviated by $\mathscr{L}(\mathcal{U})$. The *identity operator* on the space $\mathcal{U}$ is denoted by $I_{\mathcal{U}}$, or just by I when the underlying space is clear from the context. For a linear subset $\mathscr{E}$ of $\mathcal{U}$ we write $\overline{\mathscr{E}}$ for its closure in $\mathcal{U}$. By definition, a *subspace* is a closed linear manifold. Let $\mathcal{M}$ be a subspace of $\mathcal{U}$. Then $\mathcal{U} \ominus \mathcal{M}$ stands for the *orthogonal complement* of $\mathcal{M}$ in $\mathcal{U}$. We follow the convention that the symbol $\Pi_{\mathcal{M}}$ denotes the *orthogonal projection* from $\mathcal{U}$ onto $\mathcal{M}$ viewed as an operator from $\mathcal{U}$ to $\mathcal{M}$, whereas $P_{\mathcal{M}}$ stands for the *orthogonal projection* from $\mathcal{U}$ on $\mathcal{M}$ acting as an operator on $\mathcal{U}$. Note that with this notation $\Pi_{\mathcal{M}}^*$ is the *canonical embedding* of $\mathcal{M}$ into $\mathcal{U}$ and $P_{\mathcal{M}} = \Pi_{\mathcal{M}}^* \Pi_{\mathcal{M}}$.

Let C be an operator in $\mathscr{L}(\mathcal{U})$. Then C is called *nonnegative* (notation: $C \geq 0$) if $\langle Cu, u \rangle \geq 0$ for each $u \in \mathcal{U}$. If C is nonnegative as well as invertible, then we call C *strictly positive* (notation: $C \gg 0$). The operator C is referred to as *pointwise stable* if for each $u \in \mathcal{U}$ the sequence $C^n u$ converges to 0 in $\mathcal{U}$ as n goes to infinity. Moreover, we say that C is *pointwise $*$-stable* in case C^* is pointwise stable. In particular, when the *spectral radius* of C, which is denoted by $r_{\text{spec}}(C)$, is strictly less than one, then C is pointwise stable. The converse is in general not true. In case $r_{\text{spec}}(C) < 1$ we call C *exponentially stable*. Since $r_{\text{spec}}(C) = r_{\text{spec}}(C^*)$, we obtain that exponential stability implies both pointwise stability as well as pointwise $*$-stability. A subspace $\mathcal{M}$ of $\mathcal{U}$ is said to be *invariant under* C when C maps $\mathcal{M}$ into $\mathcal{M}$, and *reducing for* C if $\mathcal{M}$ is invariant under both C and C^*. The subspace $\mathcal{M}$ is called *cyclic for* C if

$$\mathcal{U} = \bigvee_{n=0}^{\infty} C^n \mathcal{M} := \overline{\{C^n m \mid n = 0, 1, 2, \ldots,\ m \in \mathcal{M}\}}.$$

Now let N be an operator in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$. If N is contractive, we write D_N for the nonnegative square root of $I - N^*N$, while $\mathcal{D}_N$ stands for the closure of the range of D_N. As usual, D_N and $\mathcal{D}_N$ are referred to as the *defect operator* and *defect space* of N, respectively. The operator N is called *left invertible* if there exists an operator M in $\mathscr{L}(\mathcal{Y},\mathcal{U})$ such that $MN = I_{\mathcal{U}}$, and in this case M is called a *left inverse of* N. Note that N is left invertible if and only if N^*N is strictly positive. The *adjoint of* N is denoted by N^*. If N is invertible, then we will sometimes write N^{-*} as a shorthand for the operator $(N^{-1})^*$, which is the same as $(N^*)^{-1}$.

We write $\ell^2(\mathcal{Y})$ for the Hilbert space of bilateral square summable sequences $(y_n)_{n\in\mathbb{Z}}$ with entries in $\mathcal{Y}$. As usual $\mathbb{Z}$ stands for the set of all integers. Let $\ell^2_+(\mathcal{Y})$ be the subspace of $\ell^2(\mathcal{Y})$ consisting of all sequences in $\ell^2(\mathcal{Y})$ that have the zero vector in entries with strictly negative index. Usually we omit the elements in the sequences in $\ell^2_+(\mathcal{Y})$ that have strictly negative index. So we identify vectors in $\ell^2_+(\mathcal{Y})$ with sequences of the form $(y_n)_{n\in\mathbb{N}}$, with $y_n \in \mathcal{Y}$, where $\mathbb{N}$ stands for the set of nonnegative integers (i.e., with zero included). We write $\ell^2_-(\mathcal{Y})$ for the orthogonal complement of $\ell^2_+(\mathcal{Y})$ in $\ell^2(\mathcal{Y})$, that is, $\ell^2_-(\mathcal{Y})$ is the Hilbert space of all square summable sequences $(\ldots, y_{-2}, y_{-1})$ with entries in $\mathcal{Y}$. The Hilbert space $\ell^2_+(\mathcal{Y})$ is unitarily equivalent with the Hardy space $H^2(\mathcal{Y})$ via the *Fourier transform* $\mathscr{F}_{\mathcal{Y}}$, which is the unitary operator mapping $\ell^2_+(\mathcal{Y})$ onto $H^2(\mathcal{Y})$ defined by

$$(\mathscr{F}_{\mathcal{Y}}(y_n)_{n\in\mathbb{N}})(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n \qquad ((y_n)_{n\in\mathbb{N}} \in \ell^2_+(\mathcal{Y}), \lambda \in \mathbb{D}).$$

The *unilateral forward shift* on $H^2(\mathcal{Y})$, that is, the operator of multiplication with the variable λ, is denoted by $S_{\mathcal{Y}}$, while we denote the *unilateral (forward) shift* on $\ell^2_+(\mathcal{Y})$ by $\tilde{S}_{\mathcal{Y}}$. Observe that $\mathscr{F}_{\mathcal{Y}}$ intertwines $\tilde{S}_{\mathcal{Y}}$ on $\ell^2_+(\mathcal{Y})$ with $S_{\mathcal{Y}}$ on $H^2(\mathcal{Y})$. We write $E_{\mathcal{Y}}$ for the canonical embedding of $\mathcal{Y}$ onto the subspace of constant functions in $H^2(\mathcal{Y})$, that is, $(E_{\mathcal{Y}}v)(\lambda) = v$ for all $\lambda \in \mathbb{D}$ and each $v \in \mathcal{Y}$. Finally, for two functions f and g on some set Ω we often write $f(\lambda) \equiv g(\lambda)$ whenever f and g coincide, that is, when $f(\lambda) = g(\lambda)$ for each $\lambda \in \Omega$.

2.2 Isometric liftings

In this section we review some facts concerning isometric liftings. For a more complete account we refer to the book [69]; see also Chapter VI in [38], and Section 11.3 in [42].

Let T' on $\mathcal{H}'$ be a contraction. Recall that an operator U on $\mathcal{K}$ is an *isometric lifting* of T' if $\mathcal{H}'$ is a subspace of $\mathcal{K}$ and U is an isometry satisfying $\Pi_{\mathcal{H}'}U = T'\Pi_{\mathcal{H}'}$. Isometric liftings exist. In fact, the *Sz.-Nagy-Schäffer isometric lifting* V of T' is given by

$$V = \begin{bmatrix} T' & 0 \\ E_{\mathcal{D}_{T'}} D_{T'} & S_{\mathcal{D}_{T'}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}. \tag{2.2.1}$$

To see that V in (2.2.1) is an isometric lifting of T' note that any operator U on $\mathcal{K} = \mathcal{H}' \oplus \mathcal{M}$ is an isometric lifting of T' if and only if U admits an operator matrix representation of the form

$$\begin{gathered} U = \begin{bmatrix} T' & 0 \\ Y_1 D_{T'} & Y_2 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix}, \\ \text{where } Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{M} \end{bmatrix} \to \mathcal{M} \text{ is an isometry.} \end{gathered} \tag{2.2.2}$$

An isometric lifting U of T' is called *minimal* when $\mathcal{H}'$ is cyclic for U, that is, when

$$\mathcal{K} = \bigvee_{n=0}^{\infty} U^n \mathcal{H}'. \tag{2.2.3}$$

If the isometric lifting U is given by (2.2.2), then the lifting is minimal if and only if the space $Y_1\mathcal{D}_{T'}$ is cyclic for Y_2. One can show that U in (2.2.2) is a minimal isometric lifting of T' in case Y is unitary and Y_2 is pointwise $*$-stable. In particular, the Sz.-Nagy-Schäffer isometric lifting of T' is minimal.

Two isometric liftings U_1 on $\mathcal{K}_1$ and U_2 on $\mathcal{K}_2$ of T' are said to be *unitarily equivalent* if there exists a unitary operator Φ from $\mathcal{K}_1$ onto $\mathcal{K}_2$ such that

$$\Phi U_1 = U_2 \Phi \quad \text{and} \quad \Phi h = h \text{ for all } h \in \mathcal{H}'.$$

In this case Φ is said to *unitarily intertwine the isometric liftings U_1 and U_2*. Minimality of an isometric lifting is preserved under unitary equivalence, and two minimal isometric liftings of T' are unitarily equivalent.

Finally, when U on $\mathcal{K}$ is a isometric lifting of T', then the subspace $\mathcal{K}'$ given by the right hand side of (2.2.3) is reducing for U. Furthermore, in that case the operator $U' = \Pi_{\mathcal{K}'}U|_{\mathcal{K}'}$ on $\mathcal{K}'$ is a minimal isometric lifting of T', and the operator U admits a operator matrix decomposition of the form

$$U = \begin{bmatrix} U' & 0 \\ 0 & \tilde{U} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{K}' \\ \tilde{\mathcal{K}} \end{bmatrix}, \tag{2.2.4}$$

where $\tilde{U}$ is an isometry on $\tilde{\mathcal{K}}$. We shall call U' in (2.2.4) the *minimal isometric lifting of T' associated with U*.

The following proposition summarizes the results referred to above in a form that will be convenient for us. For details we refer to Section 11.3 in [42].

Theorem 2.2.1. *Let T' be a contraction on $\mathcal{H}'$, let V on $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ be the Sz.-Nagy-Schäffer (minimal) isometric lifting of T', and let U on $\mathcal{H} \oplus \mathcal{M}$ be an arbitrary isometric lifting of T' given by (2.2.2). Then there exists a unique isometry Υ from $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ into $\mathcal{H}' \oplus \mathcal{M}$ such that $U\Upsilon = \Upsilon V$ and $\Upsilon h = h$ for all $h \in \mathcal{H}'$. In fact, Υ is given by*

$$\Upsilon = \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \Lambda \end{bmatrix} : \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}' \\ \mathcal{M} \end{bmatrix},$$

where Λ is defined by

$$\Lambda h = \sum_{n=0}^{\infty} Y_2^n Y_1 h_n \quad \textit{for} \quad h(\lambda) = \sum_{n=0}^{\infty} \lambda^n h_n, \ h \in H^2(\mathcal{D}_{T'}),$$

with Y_1 and Y_2 as in (2.2.2). Moreover, $(\Lambda^* m)(\lambda) = Y_1^*(I - \lambda Y_2^*)^{-1} m$ for all $m \in \mathcal{M}$ and each $\lambda \in \mathbb{D}$. Finally, Υ is unitary if and only if U is a minimal isometric lifting of T', and in that case the isometric liftings V and U of T' are unitarily equivalent.

2.3 Systems and realizations

In this section we review some of the classical results of system theory and realization theory for operator-valued functions, and we prove a few additional results that will be useful later on.

We call a quadruple $\{X, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}\}$ (or simply $\{X, B, C, D\}$) of operators a *system* in case X is an operator on $\mathcal{X}$ and B is an operator from $\mathcal{U}$ into $\mathcal{X}$, while C is an operator mapping $\mathcal{X}$ into $\mathcal{Y}$, and D is an operator from $\mathcal{U}$ into $\mathcal{Y}$. The Hilbert spaces $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are referred to as the *state*, *input space* and *output space*, respectively. To justify this terminology we refer to [52] and [90].

With a system $\{X, B, C, D\}$ we associate an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function G given by

$$G(\lambda) = D + \lambda C(I_{\mathcal{X}} - \lambda X)^{-1} B \tag{2.3.1}$$

for all λ in some open neighborhood of the origin in the complex plane. The function defined by the right hand side of (2.3.1) shall be referred to as the associated *transfer function.* On the other hand, given an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function G, a system $\{X, B, C, D\}$ that has G as its associated transfer function is called a *realization of* G.

In the remainder of this section we often omit the phrase 'for all λ in some open neighborhood of the origin'. Unless specified differently, we will always assume that λ is a complex number that is sufficiently close to zero. Note that the right hand side of (2.3.1) properly defines an analytic $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function on the disc around the origin with radius $\|X\|^{-1}$. All results obtained in this section will hold, in particular, for $\lambda \in \mathbb{C}$ with $|\lambda| < \|X\|^{-1}$.

Let $\{X, B, C, D\}$ be a system. The 2×2 operator matrix

$$M = \begin{bmatrix} D & C \\ B & X \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \tag{2.3.2}$$

is referred to as the *system matrix* associated with $\{X, B, C, D\}$. The transfer function G can also be expressed in terms of the associated system matrix. To be precise, we have

$$G(\lambda) = \Pi_{\mathcal{Y}} M (I_{\mathcal{U}\oplus\mathcal{X}} - \lambda J_{\mathcal{X}} M)^{-1} \Pi_{\mathcal{U}}^*,$$

where $J_{\mathcal{X}}$ is the partial isometry from $\mathcal{Y} \oplus \mathcal{X}$ to $\mathcal{U} \oplus \mathcal{X}$ given by

$$J_{\mathcal{X}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{X}} \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix}. \tag{2.3.3}$$

Indeed, for λ sufficiently close to zero we have

$$\begin{aligned} G(\lambda) &= D + \lambda C (I_{\mathcal{X}} - \lambda X)^{-1} B = \begin{bmatrix} D & C \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ \lambda (I_{\mathcal{X}} - \lambda X)^{-1} B \end{bmatrix} \\ &= \begin{bmatrix} D & C \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} & 0 \\ \lambda (I_{\mathcal{X}} - \lambda X)^{-1} B & (I_{\mathcal{X}} - \lambda X)^{-1} \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} D & C \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} & 0 \\ -\lambda B & I_{\mathcal{X}} - \lambda X \end{bmatrix}^{-1} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} D & C \end{bmatrix} \left(I_{\mathcal{U}\oplus\mathcal{X}} - \lambda \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} D & C \\ B & X \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \\ &= \Pi_{\mathcal{Y}} M (I_{\mathcal{U}\oplus\mathcal{X}} - \lambda J_{\mathcal{X}} M)^{-1} \Pi_{\mathcal{U}}^* . \end{aligned}$$

A system $\{X, B, C, D\}$, or just the pair $\{C, X\}$, is called *observable* if $CX^n x = 0$ for all integers $n \geq 0$ implies that the vector $x \in \mathcal{X}$ is equal to zero. Notice that $\{C, X\}$ is observable if and only if for each x in $\mathcal{X}$ we have that

$$C(I - \lambda X)^{-1} x = 0 \ \text{ for all } \ |\lambda| < \|X\|^{-1} \quad \Longrightarrow \quad x = 0.$$

A system $\{X, B, C, D\}$, or the pair $\{X, B\}$, is referred to as *controllable* if the space $\overline{B\mathcal{U}}$ is cyclic for X, that is, if

$$\mathcal{X} = \bigvee_{n=0}^{\infty} X^n B \mathcal{U}.$$

Note that the pair $\{X, B\}$ is controllable if and only if the pair $\{B^*, X^*\}$ is observable. In particular, $\{X, B, C, D\}$ is controllable if and only if the system $\{X^*, C^*, B^*, D^*\}$ is observable. The latter system is referred to as the *dual system* of $\{X, B, C, D\}$, and its transfer function $G^\sharp$ is given by

$$G^\sharp(\lambda) = G(\overline{\lambda})^*, \tag{2.3.4}$$

where G is the transfer function associated with $\{X, B, C, D\}$, and $\overline{\lambda}$ stands for the complex conjugate of λ. For any $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function G, we refer to the $\mathscr{L}(\mathcal{Y}, \mathcal{U})$-valued function $G^\sharp$ define by (2.3.4) as the *dual function* associated with G.

Observe that the system matrix associated with the dual system of $\{X, B, C, D\}$ is given by the adjoint of the system matrix associated with $\{X, B, C, D\}$ itself. In terms of the system matrix M in (2.3.2) we have that $\{X, B, C, D\}$ is controllable if and only if

$$\mathcal{X} = \Pi_{\mathcal{X}} \bigvee_{n=0}^{\infty} (J_{\mathcal{X}} M)^n \mathcal{U}, \tag{2.3.5}$$

where $J_{\mathcal{X}}$ is the operator in (2.3.3). Condition (2.3.5) is also equivalent to the requirement that $\{J_{\mathcal{X}} M, \Pi_{\mathcal{U}}^*\}$ is a controllable pair. So we have:

$$\begin{array}{lcl} \{X, B, C, D\} \text{ is controllable} & \Longleftrightarrow & \{J_{\mathcal{X}} M, \Pi_{\mathcal{U}}^*\} \text{ is controllable,} \\ \{X, B, C, D\} \text{ is observable} & \Longleftrightarrow & \{J_{\mathcal{X}}^* M^*, \Pi_{\mathcal{Y}}^*\} \text{ is controllable.} \end{array} \tag{2.3.6}$$

In the special case when $\mathcal{U} = \mathcal{Y}$ the conditions in the right hand side of (2.3.6) simplify, as explained in the following lemma.

Lemma 2.3.1. *Let $\{X, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{U}\}$ be a system with system matrix M. Then the pair $\{X, B\}$ is controllable if and only if $\mathcal{U}$ is cyclic for M, and the pair $\{C, X\}$ is observable if and only if $\mathcal{U}$ is cyclic for M^*.*

Proof. It suffices to prove the controllability part. The result for observability then follows by applying the controllability part to the dual system.

Let $\Pi_{\mathcal{U}}$ be the orthogonal projection of $\mathcal{U} \oplus \mathcal{X}$ onto $\mathcal{U}$, and define M_0 to be the operator

$$M_0 = \begin{bmatrix} 0 & 0 \\ B & X \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix}.$$

Then $M_0 = M - \Pi_{\mathcal{U}}^* \begin{bmatrix} C & D \end{bmatrix}$. This feedback relation implies that the pair $\{M_0, \Pi_{\mathcal{U}}^*\}$ is controllable if and only if the pair $\{M, \Pi_{\mathcal{U}}^*\}$ is controllable. In other words, $\mathcal{U}$ is cyclic for M if and only if $\mathcal{U}$ is cyclic for M_0. Now notice that for all integers $n \geq 1$, we have

$$M_0^n \Pi_{\mathcal{U}}^* = \begin{bmatrix} 0 & 0 \\ X^{n-1}B & X^n \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ X^{n-1}B \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix}.$$

It follows that

$$\bigvee_{n=0}^{\infty} M_0^n \Pi_{\mathcal{U}}^* \mathcal{U} = \begin{bmatrix} \mathcal{U} \\ \{0\} \end{bmatrix} \oplus \bigvee_{n=0}^{\infty} M_0^n \Pi_{\mathcal{U}}^* = \mathcal{U} \oplus \bigvee_{n=0}^{\infty} X^{n-1} B \mathcal{U}.$$

Hence $\mathcal{U}$ is cyclic for M_0 if and only if the pair $\{X, B\}$ is controllable. □

The following result is known as the Schur complement method.

Proposition 2.3.2. *Let N be an operator from $\mathcal{K} \oplus \mathcal{L}$ to $\mathcal{M} \oplus \mathcal{N}$ that admits the following decomposition:*

$$N = \begin{bmatrix} Q & R \\ S & T \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{L} \end{bmatrix} \to \begin{bmatrix} \mathcal{M} \\ \mathcal{N} \end{bmatrix}.$$

Assume that T is invertible. Then N is invertible if and only if $Q^\times := Q - RT^{-1}S$ is invertible. In that case

$$N^{-1} = \begin{bmatrix} (Q^\times)^{-1} & -(Q^\times)^{-1}RT^{-1} \\ -T^{-1}S(Q^\times)^{-1} & T^{-1} + T^{-1}S(Q^\times)^{-1}RT^{-1} \end{bmatrix}. \tag{2.3.7}$$

Assume that Q is invertible. Then N is invertible if and only if $T^\times := T - SQ^{-1}R$ is invertible. In that case

$$N^{-1} = \begin{bmatrix} Q^{-1} + Q^{-1}R(T^\times)^{-1}SQ^{-1} & -Q^{-1}R(T^\times)^{-1} \\ -(T^\times)^{-1}SQ^{-1} & (T^\times)^{-1} \end{bmatrix}. \tag{2.3.8}$$

The Schur complement method is well known; see Remark I.1.2 in [24], where also many applications are given. We refer to [25] for recent developments and [26] for an up-to-date exposition.

The next two results are corollaries of Proposition 2.3.2.

Corollary 2.3.3. *Let $\{X, B, C, D\}$ be a realization for an $\mathscr{L}(\mathcal{U})$-valued function G, and let M be the associated system matrix. Then for λ sufficiently close to zero*

$$(I - \lambda M)^{-1} = \begin{bmatrix} \Delta(\lambda)^{-1} & \lambda\Delta(\lambda)^{-1}C\Sigma(\lambda)^{-1} \\ \lambda\Sigma(\lambda)^{-1}B\Delta(\lambda)^{-1} & \Sigma(\lambda)^{-1} + \lambda^2\Sigma(\lambda)^{-1}B\Delta(\lambda)^{-1}C\Sigma(\lambda)^{-1} \end{bmatrix},$$

where

$$\Delta(\lambda) = I - \lambda G(\lambda) \quad \textit{and} \quad \Sigma(\lambda) = I - \lambda X.$$

Proof. Apply Proposition 2.3.2 with

$$N = I - \lambda M = \begin{bmatrix} I - \lambda D & -\lambda C \\ -\lambda B & I - \lambda X \end{bmatrix},$$

using that $I - \lambda X$ is invertible for λ sufficiently close to zero. □

Corollary 2.3.4. *Let $\{X, B, C, D\}$ be a realization for an $\mathscr{L}(\mathcal{U})$-valued function G. Assume that D is an invertible operator. Put $X^\times = X - BD^{-1}C$. Then $G(\lambda)$ is invertible if and only if $I - \lambda X^\times$ is invertible, for λ sufficiently small, and in that case*

$$G(\lambda)^{-1} = D^{-1} - \lambda D^{-1}C(I - \lambda X^\times)^{-1}BD^{-1}. \tag{2.3.9}$$

Proof. Apply Proposition 2.3.2 with

$$N = \begin{bmatrix} D & -\lambda C \\ B & I - \lambda X \end{bmatrix}. \tag{2.3.10}$$

By assumption D is invertible, and $I - \lambda X$ is invertible for λ sufficiently close to zero. Then (2.3.9) is obtained after comparing the left upper corners in the right hand sides of (2.3.7) and (2.3.8) specified for N in (2.3.10). □

2.4 Uniformly bounded analytic functions

With the symbol $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ we denote the class of uniformly bounded analytic functions on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ with values in $\mathscr{L}(\mathcal{U},\mathcal{Y})$. The set $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ forms a Banach space with respect to the *supremum norm* $\|\ \|_\infty$, which is defined by

$$\|F\|_\infty = \sup\{\|F(\lambda)\| \mid \lambda \in \mathbb{D}\}.$$

The closed unit ball in $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ is denoted by $\mathbf{S}(\mathcal{U},\mathcal{Y})$, and is referred to as the *Schur class* associated with $\mathcal{U}$ and $\mathcal{Y}$. Functions in $\mathbf{S}(\mathcal{U},\mathcal{Y})$ are called *Schur class functions*.

In this section we bring together some facts concerning functions in the Banach space $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$. The section is split in two parts. In the first part we associate two operators with a function in $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$, and recall some of their properties. In the second part we focus on realizations for Schur class functions.

Multiplication and Toeplitz operators. With a function $F \in \mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ we associate, in the usual way, a multiplication operator M_F mapping the Hardy space $H^2(\mathcal{U})$ into the Hardy space $H^2(\mathcal{Y})$, namely

$$(M_F g)(\lambda) = F(\lambda)g(\lambda) \quad (g \in H^2(\mathcal{U}), \lambda \in \mathbb{D}).$$

The operator M_F is called the *multiplication operator* defined by F. Moreover, the norm of M_F is equal to the supremum norm of F. In particular, M_F is a contraction if and only if F is in the Schur class $\mathbf{S}(\mathcal{U},\mathcal{Y})$. Multiplication operators for functions in $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ can be distinguished from arbitrary operators that map $H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$ by the fact that they are precisely those operators in $\mathscr{L}(H^2(\mathcal{U}), H^2(\mathcal{Y}))$ that intertwine the unilateral shift operators $S_\mathcal{U}$ on $H^2(\mathcal{U})$ and $S_\mathcal{Y}$ on $H^2(\mathcal{Y})$.

Next we associate with a function in $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ another operator. Let $F_0, F_1, \ldots$ be the sequence of Taylor coefficients of a function $F \in \mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$. Then the *Toeplitz operator* defined by F is the operator T_F given by the infinite operator matrix representation

$$T_F = \begin{bmatrix} F_0 & 0 & 0 & \cdots \\ F_1 & F_0 & 0 & \cdots \\ F_2 & F_1 & F_0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} : \ell^2_+(\mathcal{U}) \to \ell^2_+(\mathcal{Y}).$$

We have $\|T_F\| = \|M_F\| = \|F\|_\infty$. In fact, the Toeplitz operator T_F can be obtained from M_F via

$$T_F = \mathscr{F}_\mathcal{Y}^{-1} M_F \mathscr{F}_\mathcal{U},$$

where $\mathscr{F}_\mathcal{U}$ and $\mathscr{F}_\mathcal{Y}$ are the Fourier transforms defined in the final paragraph of Section 2.1. Since the Fourier transforms intertwine the unilateral shift operators, we obtain that the Toeplitz operators can be characterizes as those operators in $\mathscr{L}(\ell^2_+(\mathcal{U}), \ell^2_+(\mathcal{Y}))$ that intertwine the unilateral shift operators $\tilde{S}_\mathcal{U}$ on $\ell^2_+(\mathcal{U})$ and $\tilde{S}_\mathcal{Y}$ on $\ell^2_+(\mathcal{Y})$.

A function $F \in \mathbf{H}^\infty(\mathcal{U}, \mathcal{Y})$ is called an *inner function* whenever the multiplication operator M_F is an isometry, or equivalently, if the Toeplitz operator T_F is an isometry. In particular, inner functions are also Schur class functions.

From the infinite operator matrix representation of the Toeplitz operator and the fact that $F(0) = F_0$ we obtain the following lemma.

Lemma 2.4.1. *An $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function F on $\mathbb{D}$ is in $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ and satisfies $F(0) = 0$ if and only if there exists a $G \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ such that $F(\lambda) = \lambda G(\lambda)$ for each $\lambda \in \mathbb{D}$. In this case F is an inner function if and only G is inner.*

Proof. In case $G \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and $F(\lambda) \equiv \lambda G(\lambda)$ it is clear that $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and $F(0) = 0$. Conversely, assume that $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and $F(0) = 0$. Let $F_0, F_1, \ldots$ be the sequence of Taylor coefficients of F. Then we can define an analytic function G on $\mathbb{D}$ by

$$G(\lambda) = \sum_{n=0}^{\infty} \lambda^n F_{n+1} \quad (\lambda \in \mathbb{D}).$$

Since $F(0) = 0$, we have $F_0 = 0$. This implies that $G \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ and that the Toeplitz operator T_G defined by G satisfies $T_G = \tilde{S}^* T_F$. Moreover, we obtain that $\|T_F f\| = \|T_G f\|$ for each $f \in \ell^2_+(\mathcal{U})$. In particular, T_F is an isometry if and only if T_G is an isometry. In other words, F is inner if and only if G is inner. $\square$

Next observe that for any $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ valued function F on $\mathbb{D}$ the dual function $F^\sharp$ (see (2.3.4)) is properly defined on $\mathbb{D}$, and $\|F^\sharp(\lambda)\| = \|F(\overline{\lambda})^*\| = \|F(\overline{\lambda})\|$ for each $\lambda \in \mathbb{D}$. Therefore, $F \in \mathbf{H}^\infty(\mathcal{U}, \mathcal{Y})$ if and only if $F^\sharp \in \mathbf{H}^\infty(\mathcal{Y}, \mathcal{U})$. In fact, the computation shows that $\|F^\sharp\|_\infty = \|F\|_\infty$. Hence, in particular, $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ if and only if $F^\sharp \in \mathbf{S}(\mathcal{Y}, \mathcal{U})$.

Realizations for Schur class functions. A system $\{X, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}\}$ (as in Section 2.3) is called *contractive, co-isometric, isometric* or *unitary* whenever the system matrix

$$M = \begin{bmatrix} D & C \\ B & X \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix}$$

is contractive, co-isometric, isometric or unitary, respectively. Let F be the transfer function associated with $\{X, B, C, D\}$. Then we refer to $\{X, B, C, D\}$ as a *contractive, co-isometric, isometric* or *unitary realization of* F if $\{X, B, C, D\}$ is a contractive, co-isometric, isometric or unitary system, respectively. Co-isometric systems and realizations will be particularly useful in our setting, but contractive and unitary systems and realizations will also appear in some places.

The transfer function associated with a contractive (and so with a co-isometric, isometric or unitary) system is a Schur class function. Conversely, each Schur class function appears as the transfer function of a unitary (and so of a contractive, co-isometric or isometric) system.

The transfer function associated with a contractive, co-isometric, isometric or unitary system is a Schur class function. Conversely, each Schur class function appears as the transfer function of a contractive, co-isometric, isometric as well as a unitary system. In other words, each Schur class function admits a contractive, co-isometric, isometric as well as a unitary realization.

The relation between Schur class functions and co-isometric systems can be made more precise. In order to do this we need the notion of unitarily equivalent systems. Two systems $\{X \text{ on } \mathcal{X}, B, C, D\}$ and $\{X' \text{ on } \mathcal{X}', B', C', D'\}$ are said to be *unitarily equivalent* if $D = D'$ and if there exists a unitary operator Ψ mapping $\mathcal{X}$ onto $\mathcal{X}'$ such that

$$\Psi X = X'\Psi, \quad \Psi B = B' \quad \text{and} \quad C = C'\Psi.$$

One easily checks that the notions of observability and controllability are preserved under unitary equivalence, and that if two realizations are unitarily equivalent and one of them is co-isometric, then so is the other. Finally, two unitarily equivalent realizations have the same transfer function. In case we restrict to observable co-isometric realizations the converse also holds. In fact we have the following result.

Theorem 2.4.2. *Let F be an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function. Then F is in the Schur class $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ if and only if F admits a co-isometric realization. In this case, F admits an observable co-isometric realization, and all observable co-isometric realizations for F are unitarily equivalent.*

For a proof of Theorem 2.4.2 and additional references we refer to [14].

Let $\{X \text{ on } \mathcal{X}, B, C, D\}$ be a contractive realization. Then the equation

$$(W_o x)(\lambda) = C(I - \lambda X)^{-1} \quad (x \in \mathcal{X}, \lambda \in \mathbb{D}),$$

defines a contractive operator W_o from $\mathcal{X}$ into $H^2(\mathcal{Y})$; see Theorem 2.4.3 below. We refer to the operator W_o as the *observability operator* for $\{X, B, C, D\}$ (or for the pair $\{C, X\}$). Note that the system $\{X, B, C, D\}$ is observable if and only if $\operatorname{Ker} W_o = \{0\}$. Moreover, the Hilbert spaces $\mathcal{X}_{\bar{o}} = \operatorname{Ker} W_o$ and $\mathcal{X}_o = \mathcal{X} \ominus \mathcal{X}_{\bar{o}}$ are called the *unobservable subspace* and the *observable subspace* for $\{X, B, C, D\}$ (or for the pair $\{C, X\}$), respectively.

Theorem 2.4.3. *Let $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$, and let $\{X \text{ on } \mathcal{X}, B, C, D\}$ be a contractive realization for F. Let M_F be the multiplication operator defined by F, and let W_o be the observability operator for $\{X, B, C, D\}$. Then the operator*

$$M = \begin{bmatrix} M_F & W_o \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{U}) \\ \mathcal{X} \end{bmatrix} \to H^2(\mathcal{Y}) \tag{2.4.1}$$

is a contraction. Moreover, if $\{X, B, C, D\}$ is a unitary realization and X is pointwise stable, then M is unitary.

The statement for the case that $\{X, B, C, D\}$ is a unitary system and X pointwise stable is obtained from Theorem III.10.4 in [41], the statement that the operator in (2.4.1) is contractive in the general case can easily be derived from Theorem III.10.1 in [41]. It also follows by specifying the result from Proposition 1.7.2 in [77] concerning time-variant systems for the time-invariant case.

We conclude this section with two lemmas that will be of use in the sequel. The first lemma can be seen as a result on co-isometric systems $\{X, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}\}$ for which $\mathcal{U} = \{0\}$.

Lemma 2.4.4. *Assume that T is a co-isometry of the form:*

$$T = \begin{bmatrix} C \\ X \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix}. \tag{2.4.2}$$

Then the observability operator W_o for the pair $\{C, X\}$ is a co-isometry from $\mathcal{X}$ onto $H^2(\mathcal{Y})$. Moreover, the operators X and C admit matrix representations of the form

$$X = \begin{bmatrix} X_o & 0 \\ 0 & X_{\bar{o}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{X}_o \\ \mathcal{X}_{\bar{o}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_o & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_o \\ \mathcal{X}_{\bar{o}} \end{bmatrix} \to \mathcal{Y}, \tag{2.4.3}$$

where the pair $\{C_o, X_o\}$ is observable. Here $\mathcal{X}_o = \operatorname{Ker} W_o$ and $\mathcal{X}_{\bar{o}} = \mathcal{X} \ominus \mathcal{X}_o$.

Proof. Note that because T in (2.4.2) is a co-isometry, the operator

$$U = \begin{bmatrix} 0 & 0 \\ C^* & X^* \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix}$$

is an isometric lifting of the contraction $T' = 0$ on $\mathcal{Y}$. The fact that W_o is a co-isometry now follows with Theorem 2.2.1 and the observation that $W_o = \Lambda^*$, where Λ is the operator defined in Theorem 2.2.1. The decomposition of C and X in (2.4.3) is then obtained from the decomposition of U as in (2.2.4), where U' is the minimal isometric lifting associated with U and the observation that $\mathcal{K}'$ in (2.2.4) is equal to $\mathcal{Y} \oplus \mathcal{X}_o$. Moreover, from Theorem 2.2.1 we obtain that $W_o|_{\mathcal{X}_o}$ is unitary. In particular, this implies that the pair $\{C_o, X_o\}$ is observable. □

Lemma 2.4.5. *Let $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$, and let $C : \mathcal{C} \to \mathcal{Y}$ be a contraction, where $\mathcal{C} \subset \mathcal{U}$. Then $F(\lambda)|_{\mathcal{C}} = C$ for each $\lambda \in \mathbb{D}$ if and only if*

$$F(\lambda) = C\Pi_{\mathcal{C}} + D_{C^*} G(\lambda) \Pi_{\mathcal{U} \ominus \mathcal{C}} \quad (\lambda \in \mathbb{D}) \tag{2.4.4}$$

for some $G \in \mathbf{S}(\mathcal{U} \ominus \mathcal{C}, \mathcal{D}_{C^*})$. Moreover, each $G \in \mathbf{S}(\mathcal{U} \ominus \mathcal{C}, \mathcal{D}_{C^*})$ defines a function $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ by (2.4.4), and F and G in (2.4.4) determine each other uniquely.

Proof. Corollary XXVII.5.3 in [55] tells us that, in general, an operator T of the form

$$T = \begin{bmatrix} T_1 & T_1 \end{bmatrix} : \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix} \to \mathcal{K}$$

is contractive if and only if T_1 is contractive and $T_2 = D_{T_1^*}\Delta$ for a contraction Δ from $\mathcal{U}_2$ to $\mathcal{D}_{T_1^*}$. If T is contractive, then the contraction $\Delta \in \mathscr{L}(\mathcal{U}_1, \mathcal{D}_{T_1^*})$ such that $T_2 = D_{T_1^*}\Delta$ is unique.

Hence if $G \in \mathbf{S}(\mathcal{U} \ominus \mathcal{C}, \mathcal{D}_{C^*})$, and thus $G(\lambda)$ contractive for each $\lambda \in \mathbb{D}$, then we see that F defined by (2.4.4) is an analytic and contractive $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function on $\mathbb{D}$, that is, $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$. Moreover, F defined by (2.4.4) clearly has $F(\lambda)|_{\mathcal{C}} = C$ for each $\lambda \in \mathbb{D}$. Corollary XXVII.5.3 in [55] also implies that $F(\lambda)$ and $G(\lambda)$ in (2.4.4) determine each other uniquely for each $\lambda \in \mathbb{D}$, and thus F and G determine each other uniquely.

Conversely, assume that $F \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$ with $F(\lambda)|_{\mathcal{C}} \equiv C$. Again using Corollary XXVII.5.3 in [55] we see that there exists a function G on $\mathbb{D}$ whose values are contractions in $\mathscr{L}(\mathcal{C}, \mathcal{D}_{C^*})$ and such that (2.4.4) holds. By applying Lemma VI.4.3 in [41] twice we obtain that G is analytic on $\mathbb{D}$. Hence $G \in \mathbf{S}(\mathcal{U} \ominus \mathcal{C}, \mathcal{D}_{C^*})$. □

2.5 Positive real functions

In this section we recall a few properties of operator-valued positive real functions on the unit disc $\mathbb{D}$, and state the Naimark dilation theorem. The Cayley transform, relating Schur class functions to positive real functions, is also introduced.

Let W be an $\mathscr{L}(\mathcal{U})$-valued function on $\mathbb{D}$. The *real part of* W is the function $\operatorname{Re} W$ on $\mathbb{D}$ defined by

$$\operatorname{Re} W(\lambda) = \frac{1}{2}(W(\lambda) + W(\lambda)^*) \quad (\lambda \in \mathbb{D}).$$

We say that W is *positive real* if W is analytic on $\mathbb{D}$ and if the real part of W is nonnegative on $\mathbb{D}$, i.e., $\operatorname{Re} W(\lambda) \geq 0$ for each $\lambda \in \mathbb{D}$.

It is known (see, e.g., [51], Section 1.2) that an analytic $\mathscr{L}(\mathcal{U})$-valued function W on $\mathbb{D}$, $W(\lambda) = \sum_{n=0}^{\infty} \lambda^n W_n$ say, is positive real if and only if for each n the $n \times n$ Toeplitz operator matrix $T_{\operatorname{Re} W, n}$ given by

$$T_{\operatorname{Re} W, n} = \frac{1}{2} \begin{bmatrix} W_0 + W_0^* & W_1^* & \cdots & W_{n-1}^* \\ W_1 & W_0 + W_0^* & \cdots & W_{n-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & \cdots & W_0 + W_0^* \end{bmatrix}, \tag{2.5.1}$$

defines a nonnegative operator on $\mathcal{U}^n$.

The Naimark dilation theorem. Let $\{R_n\}_{n=0}^{\infty}$ be a sequence of operators on $\mathcal{U}$. Set $R_{-n} = R_n^*$ for all integers $n > 0$. The sequence $\{R_n\}_{n=0}^{\infty}$ is said to be *positive definite* if for each $n \in \mathbb{N}$ the $n \times n$ Toeplitz operator

$$T_{\{R_n\}_{n=0}^{\infty}} = \begin{bmatrix} R_0 & R_{-1} & \cdots & R_{-n+1} \\ R_1 & R_0 & \cdots & R_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-1} & R_{n-2} & \cdots & R_0 \end{bmatrix}$$

defines a non-negative operator on $\mathcal{U}^n$.

We see that an analytic $\mathscr{L}(\mathcal{U})$-valued function W on $\mathbb{D}$ with Taylor coefficients $W_0, W_1, \ldots$ at zero is positive real if and only if $W_0 + W_0^*, W_1, W_2, \ldots$ is a positive definite sequence. Furthermore, for any positive definite sequence $\{R_n\}_{n=0}^{\infty}$ and any operator W_0 on $\mathcal{U}$ with $W_0 + W_0^* = R_0$ the sequence $W_0, R_1, R_2, \ldots$ is the sequence of Taylor coefficients at zero for a positive real function.

A pair of operators $\{U, \Gamma\}$ is called a *Naimark pair for the sequence* $\{R_n\}_{n=0}^{\infty}$ if Γ maps $\mathcal{U}$ into a Hilbert space $\mathcal{K}$ and U is an isometry on $\mathcal{K}$ such that

$$R_k = \Gamma^* U^n \Gamma \text{ for each integer } n \geq 0.$$

In this case we also say that *the sequence* $\{R_n\}_{n=0}^{\infty}$ *admits a Naimark pair.*

A Naimark pair $\{U, \Gamma\}$ for $\{R_n\}_{n=0}^{\infty}$ is called *controllable* if $\{U, \Gamma\}$ is a controllable pair; see Section 2.3. Two Naimark pairs $\{U \text{ on } \mathcal{K}, \Gamma\}$ and $\{U' \text{ on } \mathcal{K}', \Gamma'\}$ are said to be *unitarily equivalent* when there exists a unitary operator Φ mapping $\mathcal{K}$ onto $\mathcal{K}'$ such that

$$U'\Phi = \Phi U \quad \text{and} \quad \Phi\Gamma = \Gamma'.$$

The following result is known as the Naimark dilation theorem, and goes back to [70] and [71]; see also [69] and [38].

Theorem 2.5.1. *Let* $\{R_n\}_{n=0}^{\infty}$ *be a sequence of operators on* $\mathcal{U}$. *Then* $\{R_n\}_{n=0}^{\infty}$ *is a positive definite sequence if and only if* $\{R_n\}_{n=0}^{\infty}$ *admits a Naimark pair. In that case* $\{R_n\}_{n=0}^{\infty}$ *admits a controllable Naimark pair and all controllable Naimark pairs for* $\{R_n\}_{n=0}^{\infty}$ *are unitarily equivalent.*

Positive real functions and the Cayley transform. For K in $\mathbf{S}(\mathcal{U},\mathcal{U})$ consider the map

$$K \mapsto W, \quad \text{where} \quad W(\lambda) = (I + \lambda K(\lambda))\,(I - \lambda K(\lambda))^{-1} \quad \text{for all } \lambda \in \mathbb{D}. \tag{2.5.2}$$

Since $K(\lambda)$ is a contraction for each $\lambda \in \mathbb{D}$, the function W in (2.5.2) is well defined. The map $K \mapsto W$ in (2.5.2) establishes a one-to-one correspondence between the Schur class $\mathbf{S}(\mathcal{U},\mathcal{U})$ and the set of all positive real functions W satisfying $W(0) = I$. Indeed, if W is defined by (2.5.2) for some $K \in \mathbf{S}(\mathcal{U},\mathcal{U})$, then W is analytic in $\mathbb{D}$ and $W(0) = I$ while an easy computation shows that

$$\operatorname{Re} W(\lambda) = (I - \lambda K(\lambda))^{-*}\left(I - |\lambda|^2 K(\lambda)^* K(\lambda)\right)(I - \lambda K(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{2.5.3}$$

Recall that for an invertible operator M, M^{-*} is short for $(M^{-1})^*$. From (2.5.3) it follows that $\operatorname{Re} W(\lambda) \gg 0$ for each $\lambda \in \mathbb{D}$, and hence W is positive real.

Conversely, for a positive real function W with $W(0) = I$, the (unique) function K in $\mathbf{S}(\mathcal{U},\mathcal{U})$ that is mapped onto W by (2.5.2) is recovered via

$$K(\lambda) = \frac{1}{\lambda}(W(\lambda) - I)(I + W(\lambda))^{-1} \quad (0 \neq \lambda \in \mathbb{D}). \tag{2.5.4}$$

For $K \in \mathbf{S}(\mathcal{U},\mathcal{U})$, the function W defined by (2.5.2) is called the *Cayley transform of* K, while for a positive real W with $W(0) = I$, the function K defined by (2.5.4) is called the *inverse Cayley transform of* W.

The next lemma shows how the positive real functions with a fixed nonnegative value at zero, rather than the identity operator, can be obtained.

Lemma 2.5.2. *Let W be a positive real function with values in $\mathscr{L}(\mathcal{U})$ such that $W(0) = X^*X$ for some operator $X \in \mathscr{L}(\mathcal{U}, \mathcal{V})$. Then there exists a positive real $\mathscr{L}(\mathcal{V})$-valued function V with $V(0) = I$ such that*

$$W(\lambda) = X^*V(\lambda)X \quad (\lambda \in \mathbb{D}). \tag{2.5.5}$$

Moreover, each positive real $\mathscr{L}(\mathcal{V})$-valued function V with $V(0) = I$ determines an $\mathscr{L}(\mathcal{U})$-valued positive real function W with $W(0) = X^*X$ via (2.5.5)*. Finally, if in addition $\overline{X\mathcal{U}} = \mathcal{V}$, then W and V in* (2.5.5) *determine each other uniquely.*

Proof. Let W be an $\mathscr{L}(\mathcal{U})$-valued positive real function such that $W(0) = X^*X$ for some operator $X \in \mathscr{L}(\mathcal{U}, \mathcal{V})$. Let $W_0, W_1, \ldots$ be the Taylor coefficients of W at the origin. Since $T_{\mathrm{Re}\,W,n}$ is a nonnegative Toeplitz matrix and $W(0) = X^*X$, we see that

$$\begin{bmatrix} 2X^*X & W_n^* \\ W_n & 2X^*X \end{bmatrix} \geq 0, \text{ for } n = 1, 2, \ldots. \tag{2.5.6}$$

Recall (see Theorem XVI.1.1. in [38]) that a 2×2 operator matrix

$$\begin{bmatrix} A & B^* \\ B & A \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{E} \\ \mathcal{E} \end{bmatrix}$$

induces a nonnegative operator on $\mathcal{E} \oplus \mathcal{E}$ if and only if A is nonnegative and $B = A^{1/2}CA^{1/2}$ for some contraction C on $\overline{A\mathcal{E}}$. In this case, B and C uniquely determine each other.

So from (2.5.6) it follows that for each integer $n \geq 0$ there exists a unique operator V_n on $\mathcal{V}$ such that $W_n = X^*V_nX$, the space $\overline{X\mathcal{U}}$ is reducing for V_n, we have $V_n|_{\mathrm{Ker}\,X^*} = 0$ in case $n > 0$, and $V_0 = I_{\mathcal{V}}$. In particular, if $\overline{X\mathcal{U}} = \mathcal{V}$, then there exists a unique operator V_n on $\mathcal{V}$ with $W_n = X^*V_nX$. Let $T_{\mathrm{Re}\,V,\,n}$ be the $n \times n$ block Toeplitz operator matrix obtained by replacing W_j by V_j in (2.5.1). Notice $\mathbf{D}_n^* T_{\mathrm{Re}\,V,\,n} \mathbf{D}_n = T_{\mathrm{Re}\,W,\,n}$, where $\mathbf{D}_n$ is the diagonal operator matrix $\mathrm{diag}\{X\}_1^n$ acting from $\mathcal{U}^n$ to $\mathcal{V}^n$. Since $T_{\mathrm{Re}\,W,\,n}$ is nonnegative, and $T_{\mathrm{Re}\,V,\,n}$ acts as the identity operator on $\mathrm{Ker}\,\mathbf{D}_n^*$, it follows that $T_{\mathrm{Re}\,R,\,n}$ is nonnegative for each integer $n \geq 0$. Hence $V(\lambda) = \sum_{n=0}^{\infty} \lambda^n V_n$ is a positive real function. Therefore $W(\lambda) = X^*V(\lambda)X$, where V is a positive real function satisfying $V(0) = I$, which proves our claim.

Since $\overline{X\mathcal{U}} = \mathcal{V}$ implies that the operator V_n on $\mathcal{V}$ such that $W_n = X^*V_nX$ is unique for each nonnegative integer n, we obtain that in this case V and W in (2.5.5) determine each other uniquely.

Finally, note that for each positive real $\mathscr{L}(\mathcal{V})$-valued function V with $V(0) = I$ we have that W given by (2.5.5) has $W(0) = X^*V(0)X = X^*X$ and

$$\mathrm{Re}\,W(\lambda) = X^*(\mathrm{Re}\,V(\lambda))X \geq 0 \quad (\lambda \in \mathbb{D}).$$

Hence W is positive real. □

Notes for Chapter 2

Most of the material of this chapter is standard, and can be found in textbooks. For the results in Section 2.2 see [69] and [38]. For the material about systems and realizations (Sections 2.3 and 2.4) we have used [24], [52], [90], [41] and [31] as the main sources. Section 2.5 is based on Chapters II and XV in [38] and Section 1 in [51].

Chapter 3

A Schur representation of functions in the unit ball of $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$

In this chapter we introduce a Banach space consisting of certain operator-valued H^2-functions, and we present a Schur class representation of the functions in the closed unit ball of this Banach space. The proofs are based on a detailed analysis of the relations between these operator-valued H^2-functions and corresponding positive real functions on the one hand, and Schur class functions on the other hand.

The chapter consists of four sections. In Section 1 the Banach space $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ and its unit ball are introduced. The main results are presented in the second section, and are proved there except for a theorem describing the set of positive real functions associated with a fixed operator-valued H^2-function; see Theorem 3.2.5 below. We present two different proofs of the latter theorem. The first, which uses the notion of a harmonic majorant, is given in the third section. In the final section a second proof is given using a state space approach.

3.1 The space $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ and its unit ball

We begin this section by introducing the space $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are given Hilbert spaces. By definition $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ is the space of all functions H on $\mathbb{D}$ with values in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ such that for each u in $\mathcal{U}$ the map $\lambda \mapsto H(\lambda)u$ defines a function in the Hardy space $H^2(\mathcal{Y})$. Such a function is automatically analytic on $\mathbb{D}$. In other words, $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ if and only if H is an analytic $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function on $\mathbb{D}$ whose Taylor coefficients $H_0, H_1, H_2, \ldots$ at zero satisfy the constraint $\sum_{n=0}^{\infty} \|H_n u\|^2 < \infty$ for each $u \in \mathcal{U}$. Obviously $\mathbf{H}^\infty(\mathcal{U}, \mathcal{Y})$ is properly contained in $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$.

The uniform boundedness principle guarantees that for a given $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ the formula

$$(\Gamma u)(\lambda) = H(\lambda)u \quad (u \in \mathcal{U}, \lambda \in \mathbb{D}) \tag{3.1.1}$$

defines an operator Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$, which we shall refer to as the *operator defined by* H. Note that $\|\Gamma u\|^2 = \sum_{n=0}^{\infty} \|H_n u\|^2$ for each $u \in \mathcal{U}$. Conversely, if Γ is an operator from $\mathcal{U}$ into $H^2(\mathcal{Y})$, then there is a unique $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ such that (3.1.1) holds, and in this case we call H the *defining function of* Γ.

The map which assigns to each function H in $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ the norm of the operator defined by H induces a norm on $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$, turning $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ into a Banach space. The closed unit ball in this Banach space is denoted by $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$. Thus H is in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$ if and only if $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ and the operator Γ defined by H is a contraction, or in other words

$$\sup_{u\in\mathcal{U},\|u\|\leq 1} \|H(\,\cdot\,)u\|_{H^2(\mathcal{Y})} \leq 1.$$

Note that $\mathbf{H}^2(\mathbb{C},\mathcal{Y})$ is equal to the Hardy space $H^2(\mathcal{Y})$, and thus is a Hilbert space. In general, the Banach space $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ will not be a Hilbert space.

Unlike for $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$, where the action of taking the dual function transforms $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$ into $\mathbf{H}^\infty(\mathcal{Y},\mathcal{U})$, it might happen that for a function $F \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ its dual function $F^\sharp$ is not in $\mathbf{H}^2(\mathcal{Y},\mathcal{U})$. Indeed, recall from Section 2.3 that for a function F on $\mathbb{D}$, its dual function $F^\sharp$ is defined by $F^\sharp(\lambda) \equiv F(\overline{\lambda})^*$. Now set $\mathcal{U} = \mathcal{Y} = \ell^2_+(\mathbb{C})$. Let $\tilde{S}$ denote the forward shift on $\ell^2_+(\mathbb{C})$ and P_0 the projection on the first coordinate space in $\ell^2_+(\mathbb{C})$. Put $F_n = P_0(\tilde{S}^*)^n$ for $n = 0, 1, \ldots$. Then

$$\sum_{n=0}^\infty \|F_n x\|^2 = \|x\|^2 \quad \text{and} \quad \sum_{n=0}^\infty \|F_n^* x\|^2 = \sum_{n=0}^\infty \|P_0 x\|^2 \quad (x \in \ell^2_+(\mathbb{C})).$$

The latter infinite sum is not finite unless $P_0 x = 0$. Thus $F_0, F_1, \ldots$ forms the sequence of Taylor coefficients at zero for a function $F \in \mathbf{H}^2(\ell^2_+(\mathbb{C}), \ell^2_+(\mathbb{C}))$, whose dual function $F^\sharp$ is not in $\mathbf{H}^2(\ell^2_+(\mathbb{C}), \ell^2_+(\mathbb{C}))$. In particular, this implies that F is not in $\mathbf{H}^\infty(\ell^2_+(\mathbb{C}), \ell^2_+(\mathbb{C}))$.

There are other ways, than the one used here, to extend the notion of a Hardy space to an operator-valued H^2-space; see for instance [9].

3.2 Schur representations

In this section we present a representation of the functions in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$ in terms of Schur class functions in the Schur class $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$. The following theorem is the first main result of the present chapter.

Theorem 3.2.1. *Let H be an $\mathscr{L}(\mathcal{U},\mathcal{Y})$-valued function on $\mathbb{D}$. Then H is a function in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$ if and only if H is given by*

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}) \tag{3.2.1}$$

for some Schur class function Z in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$.

The description in the right hand side of (3.2.1) with Z as in the theorem will be called a *Schur representation* of H. Hence Theorem 3.2.1 tells us that a function H is in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$ if and only if it admits a Schur representation.

In general, the map $Z \mapsto H$ given by (3.2.1) is not a one-to-one mapping from $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ to $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$. To see this, let H be the zero function in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$

and observe that the function Z defined by

$$Z(\lambda) = \begin{bmatrix} 0 \\ K(\lambda) \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \quad (\lambda \in \mathbb{D}), \text{ where } K \in \mathbf{S}(\mathcal{U},\mathcal{U}) \tag{3.2.2}$$

is in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and satisfies (3.2.1). In fact, as we will see in Example 3.2.3 below, (3.2.2) provides all $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ that satisfy (3.2.1) for H the zero function. So the map $Z \mapsto H$ given by (3.2.1) does not provide a proper parameterization.

The next theorem, which is the second main result of the present chapter, gives an explicit description of the non-uniqueness in the Schur representation of Theorem 3.2.1.

Theorem 3.2.2. *Let H in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, and let Γ be the operator defined by H. Then there is a one-to-one correspondence between $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ and the set of Schur class functions Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying* (3.2.1). *To be more explicit, let J_H be the map from $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ into $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ defined by the following rule: for $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ put $J_H K = Z$, where Z is the analytic function on $\mathbb{D}$ defined by*

$$\begin{aligned} Z(\lambda) &= \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda) - I) \end{bmatrix} F(\lambda)^{-1}, \\ F(\lambda) &= \Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1} D_\Gamma. \end{aligned} \qquad (\lambda \in \mathbb{D}) \tag{3.2.3}$$

Then J_H is a one-to-one map from $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ onto the set of all $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying (3.2.1). *In particular, there exists a unique $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying* (3.2.1) *if and only if Γ is an isometry.*

To illustrate the map J_H defined in Theorem 3.2.2 we give an example.

Example 3.2.3. Let $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ be constant, that is, $H(\lambda) \equiv C$ for some contraction $C \in \mathscr{L}(\mathcal{U}, \mathcal{Y})$. Then the operator Γ defined by H is given by $\Gamma = E_{\mathcal{Y}} C$. Hence $S_{\mathcal{Y}}^* \Gamma = 0$ and $D_\Gamma = D_C$. So for an arbitrary $K \in \mathbf{S}(\mathcal{D}_C, \mathcal{D}_C)$ the function F in (3.2.3) is given by

$$F(\lambda) = C^*C + D_C(I - \lambda K(\lambda))^{-1} D_C = I + \lambda D_C K(\lambda)(I - \lambda K(\lambda))^{-1} D_C \quad (\lambda \in \mathbb{D}).$$

Therefore, by Corollary 2.3.4,

$$F(\lambda)^{-1} = I - \lambda D_C K(\lambda)(I - \lambda C^* C K(\lambda))^{-1} D_C \quad (\lambda \in \mathbb{D}),$$

from which we obtain that

$$(J_H K)(\lambda) = \begin{bmatrix} C \\ 0 \end{bmatrix} + \begin{bmatrix} -\lambda C \\ I \end{bmatrix} D_C K(\lambda)(I - \lambda C^* C K(\lambda))^{-1} D_C \quad (\lambda \in \mathbb{D}).$$

When specified for the case that C is an isometry, i.e., $D_C = 0$, we obtain that $Z \equiv \begin{bmatrix} C^* & 0 \end{bmatrix}^*$ is the unique Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying (3.2.1). The case $C = 0$, i.e., $D_C = I_{\mathcal{U}}$, gives $J_H K = Z$, with Z as in (3.2.2). ◇

The following proposition provides a first step towards the proof of Theorems 3.2.1 and 3.2.2.

Proposition 3.2.4. *Let $Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$. Put*

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1} \text{ and } F(\lambda) = (I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{3.2.4}$$

Then $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$ and F is a positive real function such that

$$F(0) = I_{\mathcal{U}} \quad \text{and} \quad 2\,\mathrm{Re}\,F(\lambda) \geq H(\lambda)^* H(\lambda) + I \quad (\lambda \in \mathbb{D}). \tag{3.2.5}$$

Conversely, let $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$ and F a positive real function satisfying (3.2.5). *Then $F(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$ and the function Z defined by*

$$Z(\lambda) = \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda)-I) \end{bmatrix} F(\lambda)^{-1} \quad (\lambda \in \mathbb{D}) \tag{3.2.6}$$

is in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$. Moreover, the map $Z \mapsto (H,F)$ defined by (3.2.4) *is a one-to-one map from $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ onto the set*

$$\{(H,F) \mid H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y}), F \text{ positive real satisfying } (3.2.5)\}. \tag{3.2.7}$$

The inverse of this map is given by the map $(H,F) \mapsto Z$ defined by (3.2.6).

Proof. We split the proof into 3 parts.

Part 1. Let $Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$, and define H and F by (3.2.4). Since Z is a Schur class function, H and F are properly defined analytic functions on $\mathbb{D}$. Moreover, F is an analytic $\mathscr{L}(\mathcal{U})$-valued function which is invertible in each point of $\mathbb{D}$, and trivially $F(0) = (I-0)^{-1} = I_{\mathcal{U}}$. To see that F also satisfies the second requirement in (3.2.5), first note that

$$\begin{aligned} Z(\lambda)^* P_{\mathcal{Y}} Z(\lambda) &\leq I - Z(\lambda)^* P_{\mathcal{U}} Z(\lambda) \leq I - |\lambda|^2 Z(\lambda)^* P_{\mathcal{U}} Z(\lambda) \\ &= I - (I - I + \lambda\Pi_{\mathcal{U}}Z(\lambda))^*(I - I + \lambda\Pi_{\mathcal{U}}Z(\lambda)) \\ &= I - (I - F(\lambda)^{-*})(I - F(\lambda)^{-1}) \\ &= F(\lambda)^{-*} + F(\lambda)^{-1} - F(\lambda)^{-*}F(\lambda)^{-1}. \end{aligned} \quad (\lambda \in \mathbb{D})$$

Therefore

$$\begin{aligned} H(\lambda)^* H(\lambda) &= F(\lambda)^* Z(\lambda)^* P_{\mathcal{Y}} Z(\lambda) F(\lambda) \\ &\leq F(\lambda)^*(F(\lambda)^{-*} + F(\lambda)^{-1} - F(\lambda)^{-*}F(\lambda)^{-1})F(\lambda) \\ &= F(\lambda) + F(\lambda)^* - I = 2\,\mathrm{Re}\,F(\lambda) - I. \end{aligned} \quad (\lambda \in \mathbb{D})$$

So F satisfies (3.2.5), which immediately proves that F is positive real.

Next we show that H is in $\mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$. We already observed that H in analytic on $\mathbb{D}$; say $H(\lambda) \equiv \sum_{n=0}^{\infty} \lambda^n H_n$. Now define for r in the interval $(0,1)$ the function $H_{\{r\}}$ by $H_{\{r\}}(\lambda) = H(r\lambda)$ for $\lambda \in \mathbb{D}$. Then $H_{\{r\}}$ extends to an analytic function on an open neighborhood of $\overline{\mathbb{D}}$ for each $r \in (0,1)$. Moreover, we have

$$2\,\mathrm{Re}\,F(r\lambda) \geq H_{\{r\}}(\lambda)^* H_{\{r\}}(\lambda) + I \quad (\lambda \in \mathbb{D}, r \in (0,1)).$$

In particular, this implies that the zero-th Fourier coefficient of the function $\lambda \mapsto 2\mathrm{Re}\, F(r\lambda)$ is greater than or equal to the zero-th Fourier coefficient of the function $\lambda \mapsto H_{\{r\}}(\lambda)^* H_{\{r\}}(\lambda) + I$ for each $r \in (0,1)$, that is,

$$2I = F(0) + F(0)^* \geq \sum_{n=0}^{\infty} r^{2n} H_n^* H_n + I \quad \text{for each } r \in (0,1). \tag{3.2.8}$$

But then (3.2.8) must also hold for $r = 1$. Hence $\sum_{n=0}^{\infty} H_n^* H_n \leq 1$. In other words, $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$.

Part 2. Now let $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$ and F a positive real function on $\mathbb{D}$ satisfying (3.2.5). The second property of F in (3.2.5) shows that $\mathrm{Re}\, F(\lambda) \geq \frac{1}{2} I_{\mathcal{U}}$ for each $\lambda \in \mathbb{D}$. The latter inequality implies that the numerical range of $F(\lambda)$ is in the half plane $\{\sigma \in \mathbb{D} \mid \mathrm{Re}\,\sigma \geq \frac{1}{2}\}$, and hence $F(\lambda)$ in invertible for each $\lambda \in \mathbb{D}$; see Lemma X.3.2 in [54]. Since F is analytic on $\mathbb{D}$, we obtain that the map $\lambda \mapsto F(\lambda)^{-1}$ is also analytic on $\mathbb{D}$. Using, in addition, that $F(0) = I$, and that H is analytic on $\mathbb{D}$ we see that (3.2.6) defines an analytic function Z on $\mathbb{D}$. To prove that $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ it remains to show that $\|Z(\lambda)\| \leq 1$ for each $\lambda \in \mathbb{D}$. Note that for each $0 \neq \lambda \in \mathbb{D}$

$$\begin{aligned} |\lambda|^2 F(\lambda)^* Z(\lambda)^* Z(\lambda) F(\lambda) &= |\lambda|^2 H(\lambda)^* H(\lambda) + (F(\lambda)^* - I)(F(\lambda) - I) \\ &= |\lambda|^2 H(\lambda)^* H(\lambda) + F(\lambda)^* F(\lambda) + I - 2\,\mathrm{Re}\, F(\lambda) \\ &\leq F(\lambda)^* F(\lambda) - (1 - |\lambda|^2) H(\lambda)^* H(\lambda) \leq F(\lambda)^* F(\lambda). \end{aligned}$$

The invertibility of $F(\lambda)$ for each $\lambda \in \mathbb{D}$ proves that the function $\lambda \mapsto \lambda Z(\lambda)$ is a Schur class function. Using Lemma 2.4.1 we see that Z is in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$.

Part 3. In Parts 1 and 2 we obtained that the maps $Z \mapsto (H, F)$ and $(H, F) \mapsto Z$ defined by (3.2.6) and (3.2.4), respectively, are properly defined maps between $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and the set (3.2.7). We will now show that these maps are each others inverses. In particular, that they are one-to-one and onto.

First let $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and define H and F by (3.2.4). Then

$$\Pi_{\mathcal{Y}} Z(\lambda) = H(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda)) = H(\lambda) F(\lambda)^{-1} \quad (\lambda \in \mathbb{D}),$$

and

$$\lambda \Pi_{\mathcal{U}} Z(\lambda) = I - F(\lambda)^{-1} = (F(\lambda) - I) F(\lambda)^{-1} \quad (\lambda \in \mathbb{D}).$$

The continuity of Z proves that Z is given by (3.2.6).

Conversely, let $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$, and let F be a positive real function satisfying (3.2.5). Define Z by (3.2.6). Then $\lambda \Pi_{\mathcal{U}} Z(\lambda) = I - F(\lambda)^{-1}$ for each $\lambda \in \mathbb{D}$, which shows that F is given by the second identity in (3.2.4). Hence

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda) F(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}).$$

So H is given by the first identity in (3.2.4). □

Proof of Theorem 3.2.1. The first part of Proposition 3.2.4 tells us that for any $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ the function H defined by (3.2.1) is in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$.

Now let H be a function in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$. It remains to prove that there exists a positive real function F satisfying (3.2.5). Indeed, if F is a positive real function satisfying (3.2.5), then the second part of Proposition 3.2.4 shows that Z defined by (3.2.6) is in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ and satisfies (3.2.1).

Let Γ be the operator defined by H, and set $\Upsilon(\lambda)=I-\lambda S_{\mathcal{Y}}^*$ for each $\lambda\in\mathbb{D}$. Since $E_{\mathcal{Y}}E_{\mathcal{Y}}^*+S_{\mathcal{Y}}S_{\mathcal{Y}}^*=I$, we have

$$\begin{aligned} E_{\mathcal{Y}}E_{\mathcal{Y}}^* &= I-S_{\mathcal{Y}}S_{\mathcal{Y}}^* = I-|\lambda|^2S_{\mathcal{Y}}S_{\mathcal{Y}}^*-(1-|\lambda|^2)S_{\mathcal{Y}}S_{\mathcal{Y}}^*\\ &= I-(I-I+\overline{\lambda}S_{\mathcal{Y}})(I-I+\lambda S_{\mathcal{Y}}^*)-(1-|\lambda|^2)S_{\mathcal{Y}}S_{\mathcal{Y}}^*\\ &= I-(I-\Upsilon(\lambda)^*)(I-\Upsilon(\lambda))-(1-|\lambda|^2)S_{\mathcal{Y}}S_{\mathcal{Y}}^*\\ &= \Upsilon(\lambda)^*+\Upsilon(\lambda)-\Upsilon(\lambda)^*\Upsilon(\lambda)-(1-|\lambda|^2)S_{\mathcal{Y}}S_{\mathcal{Y}}^*. \end{aligned} \qquad (\lambda\in\mathbb{D})$$

Using that $H(\lambda)=E_{\mathcal{Y}}^*\Upsilon(\lambda)^{-1}\Gamma$ for each $\lambda\in\mathbb{D}$ we see that

$$\begin{aligned} H(\lambda)^*H(\lambda) &= \Gamma^*\Upsilon(\lambda)^{-*}E_{\mathcal{Y}}E_{\mathcal{Y}}^*\Upsilon(\lambda)^{-1}\Gamma\\ &= \Gamma^*\Upsilon(\lambda)^{-*}(\Upsilon(\lambda)^*+\Upsilon(\lambda)-\Upsilon(\lambda)^*\Upsilon(\lambda)+\\ &\qquad -(1-|\lambda|^2)S_{\mathcal{Y}}S_{\mathcal{Y}}^*)\Upsilon(\lambda)^{-1}\Gamma\\ &= \Gamma^*\Upsilon(\lambda)^{-1}\Gamma+\Gamma^*\Upsilon(\lambda)^{-*}\Gamma-\Gamma^*\Gamma+\\ &\qquad -(1-|\lambda|^2)\Gamma^*\Upsilon(\lambda)^{-*}S_{\mathcal{Y}}S_{\mathcal{Y}}^*\Upsilon(\lambda)^{-1}\Gamma\\ &= 2\,\mathrm{Re}\,(\Gamma^*(I-\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma)-\Gamma^*\Gamma+\\ &\qquad -(1-|\lambda|^2)\Gamma^*(I-\lambda S_{\mathcal{Y}}^*)^{-*}S_{\mathcal{Y}}S_{\mathcal{Y}}^*(I-\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma. \end{aligned} \qquad (\lambda\in\mathbb{D})$$

So for each $\lambda\in\mathbb{D}$ we have

$$\begin{aligned} 2\,\mathrm{Re}\,\Gamma^*(I-\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma &= (1-|\lambda|^2)\Gamma^*(I-\lambda S_{\mathcal{Y}}^*)^{-*}S_{\mathcal{Y}}S_{\mathcal{Y}}^*(I-\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma+\\ &\qquad +H(\lambda)^*H(\lambda)+\Gamma^*\Gamma \\ &\geq H(\lambda)^*H(\lambda)+\Gamma^*\Gamma. \end{aligned} \tag{3.2.9}$$

From (3.2.9) and the fact that $\Gamma^*\Gamma+D_\Gamma^2=I$ we obtain that F given by

$$F(\lambda)=\Gamma^*(I-\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma+D_\Gamma^2 \quad (\lambda\in\mathbb{D}) \tag{3.2.10}$$

satisfies the second requirement in (3.2.5), and thus is positive real. The identity $F(0)=I$ is trivial. So F satisfies (3.2.5). □

The proof of Theorem 3.2.1 shows that for an arbitrary $H\in\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ there exists a positive real function F that satisfies (3.2.5), namely, the function F given by (3.2.10). Note that the function F in (3.2.10) coincides with the F in (3.2.3) provided K is the zero function. In order to prove Theorem 3.2.2 it remains to show that the positive real functions F that satisfy (3.2.5) are precisely given by the formula for F in (3.2.3) for $K\in\mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$. This is what the next theorem says.

Theorem 3.2.5. *Let $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$, and let Γ be the operator defined by H. Given $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$, put*

$$F_K(\lambda) = \Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma \quad (\lambda \in \mathbb{D}). \tag{3.2.11}$$

Then the map $K \mapsto F_K$ is a one-to-one map from $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ onto the set of positive real functions F satisfying (3.2.5). *In particular, there exists a positive real function F satisfying* (3.2.5).

Proof of Theorem 3.2.2. To see that Theorem 3.2.2 holds fix a $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$. Proposition 3.2.4 shows that the map $F \mapsto Z$ given by (3.2.6) determines a bijection between the positive real functions F satisfying (3.2.5) and the Schur class functions Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ such that (3.2.1) holds. The characterization of the positive real functions F satisfying (3.2.5) given in Theorem 3.2.5 then proves Theorem 3.2.2. □

It is not difficult to prove that for $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ the function F_K given by (3.2.11) is positive real and satisfies (3.2.5). Indeed, it is immediately clear that F_K is analytic on $\mathbb{D}$, and that $F_K(0) = I$. Moreover, using (3.2.9) we obtain that it remains to show that

$$2\,\mathrm{Re}\, D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma \geq D^2_\Gamma \quad (\lambda \in \mathbb{D}).$$

To see that this is the case, let C denote the Cayley transform of K (see Section 2.5) and note that

$$\begin{aligned} D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma &= \tfrac{1}{2}D^2_\Gamma + \tfrac{1}{2}D_\Gamma(I + \lambda K(\lambda))(I - \lambda K(\lambda))^{-1}D_\Gamma \\ &= \tfrac{1}{2}D^2_\Gamma + \tfrac{1}{2}D_\Gamma C(\lambda)D_\Gamma. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Hence

$$2\,\mathrm{Re}\, D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma = D^2_\Gamma + D_\Gamma(\mathrm{Re}\, C(\lambda))D_\Gamma \geq D^2_\Gamma.$$

The proof of the converse statement requires more theory. We shall give two different proofs in Sections 3.3 and 3.4, respectively. In Section 3.3 we give a proof using harmonic majorants of functions of the form $\lambda \mapsto H(\lambda)^*H(\lambda)$ for H in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$. The proof in Section 3.4 uses a state space approach.

3.3 A harmonic majorant proof of Theorem 3.2.5

Let Θ be an $\mathscr{L}(\mathcal{U})$-valued function on $\mathbb{D}$. A *harmonic majorant of* Θ is the real part $\mathrm{Re}\, W$ of an analytic function W on $\mathbb{D}$ that satisfies $\mathrm{Re}\, W(\lambda) \geq \Theta(\lambda)$ for each $\lambda \in \mathbb{D}$. In this section we study analytic functions W on $\mathbb{D}$ whose real part is a harmonic majorant of a function Θ of the form

$$\Theta(\lambda) = H(\lambda)^*H(\lambda) \quad (\lambda \in \mathbb{D}), \quad \text{where} \quad H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y}). \tag{3.3.1}$$

In particular, such a function W is positive real. As a result of this we obtain a proof of Theorem 3.2.5.

For each function Θ as in (3.3.1) a harmonic majorant exists. Indeed, let Θ by as in (3.3.1). With Θ we associate the analytic $\mathscr{L}(\mathcal{U})$-valued function

$$V(\lambda) = \Gamma^*(I + \lambda S_{\mathcal{Y}}^*)(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma \quad (\lambda \in \mathbb{D}), \tag{3.3.2}$$

where Γ is the operator defined by H. An easy computation shows that V can also be written as

$$V(\lambda) = 2\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma - \Gamma^*\Gamma \quad (\lambda \in \mathbb{D}).$$

So with the identity in (3.2.9) we obtain for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned} \operatorname{Re} V(\lambda) &= 2\operatorname{Re}\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma - \Gamma^*\Gamma \\ &= H(\lambda)^*H(\lambda) + (1 - |\lambda|^2)\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-*}S_{\mathcal{Y}}S_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma \\ &\geq H(\lambda)^*H(\lambda). \end{aligned} \tag{3.3.3}$$

Hence $\operatorname{Re} V$ is a harmonic majorant of Θ.

The main result of this section is the following theorem which can be viewed as an operator-valued version of a classical result on harmonic majorants, cf., Section 2.6 in [34].

Theorem 3.3.1. *Let H be a function in $H^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$, and let Γ be the operator defined by H. An analytic $\mathscr{L}(\mathcal{U})$-valued function W on $\mathbb{D}$ satisfies*

$$W(0) = I \quad \text{and} \quad \operatorname{Re} W(\lambda) \geq H(\lambda)^*H(\lambda) \quad (\lambda \in \mathbb{D}) \tag{3.3.4}$$

if and only if there exists a $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ such that W is given by

$$W(\lambda) \equiv \Gamma^*(I + \lambda S_{\mathcal{Y}}^*)(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + D_\Gamma(I + \lambda K(\lambda))(I - \lambda K(\lambda))^{-1}D_\Gamma. \tag{3.3.5}$$

Moreover, W and K in (3.3.5) determine each other uniquely. Finally, there is only one analytic $\mathscr{L}(\mathcal{U})$-valued function W on $\mathbb{D}$ satisfying (3.3.4) if and only if Γ is an isometry.

Before we prove Theorem 3.3.1 it is convenient to first prove two lemmas. The first lemma concerns the harmonic majorant $\operatorname{Re} V$ of Θ in (3.3.1) with V as in (3.3.2).

Lemma 3.3.2. *Let $H \in H^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$, and let Γ be the operator defined by H. Let V be given by (3.3.2). Then $\operatorname{Re} V$ is a harmonic majorant of Θ in (3.3.1). Moreover, for any analytic $\mathscr{L}(\mathcal{U})$-valued function W on $\mathbb{D}$ such that $\operatorname{Re} W$ is a harmonic majorant of Θ we have $\operatorname{Re} V(\lambda) \leq \operatorname{Re} W(\lambda)$ for each $\lambda \in \mathbb{D}$.*

To give some further insight in (3.3.2), let us consider the scalar case, that is, $\mathcal{U}$ and $\mathcal{Y}$ are equal to $\mathbb{C}$. In that case V in (3.3.2) can be rewritten as

$$V(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\omega} + \lambda}{e^{i\omega} - \lambda}|H(e^{i\omega})|^2 d\omega \quad (\lambda \in \mathbb{D}),$$

and the above proposition is well known (see the proof of Theorem 2.12 in [34]).

Proof of Lemma 3.3.2. We already observed that V is a harmonic majorant of Θ in (3.3.1). The remainder of the proof is split into two parts.

Part 1. Fix $0 < r < 1$, and set $H_{\{r\}}(\lambda) = H(r\lambda)$ for each $\lambda \in \mathbb{D}$. Notice that $H_{\{r\}}$ is analytic in an open neighborhood of $\overline{\mathbb{D}}$, the closure of the open unit disc $\mathbb{D}$. Then $H_{\{r\}} \in \mathbf{H}^\infty(\mathcal{U}, \mathcal{Y})$, and thus, in particular, $H_{\{r\}} \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$.

Let $\Gamma_{\{r\}}$ denote the operator defined by $H_{\{r\}}$. Then $\Gamma_{\{r\}} = \Lambda_{\{r\}}\Gamma$, where $\Lambda_{\{r\}}$ is the operator on $H^2(\mathcal{Y})$ defined by

$$(\Lambda_{\{r\}}h)(\lambda) = h(r\lambda) \quad (h \in H^2(\mathcal{Y}), \lambda \in \mathbb{D}).$$

Note that $\Lambda_{\{r\}}$ is bounded, selfadjoint and $\lim_{r\uparrow 1} \Lambda_{\{r\}} = I$ with pointwise convergence. Moreover, the equality $\Lambda_{\{r\}}S_{\mathcal{Y}} = rS_{\mathcal{Y}}\Lambda_{\{r\}}$ holds. Therefore, we have $(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Lambda_{\{r\}} = \Lambda_{\{r\}}(I - r\lambda S_{\mathcal{Y}}^*)^{-1}$ for each $\lambda \in \mathbb{D}$. Let $V_{\{r\}}$ be the positive real function associated with $H_{\{r\}}$ via (3.3.2). Thus

$$\begin{aligned} V_{\{r\}}(\lambda) &= \Gamma^*\Lambda_{\{r\}}(I + \lambda S_{\mathcal{Y}}^*)(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Lambda_{\{r\}}\Gamma \\ &= \Gamma^*\Lambda_{\{r\}}^2(I + r\lambda S_{\mathcal{Y}}^*)(I - r\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Hence $V_{\{r\}}$ is analytic on an open neighborhood of $\overline{\mathbb{D}}$. From (3.3.3) we know for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned} &\operatorname{Re} V_{\{r\}}(\lambda) - H_{\{r\}}(\lambda)^* H_{\{r\}}(\lambda) = \\ &\quad = (1 - |\lambda|^2)\Gamma^*\Lambda_{\{r\}}(I - \overline{\lambda}S_{\mathcal{Y}})^{-1}S_{\mathcal{Y}}S_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Lambda_{\{r\}}\Gamma \\ &\quad = (1 - |\lambda|^2)\Gamma^*(I - r\overline{\lambda}S_{\mathcal{Y}})^{-1}\Lambda_{\{r\}}S_{\mathcal{Y}}S_{\mathcal{Y}}^*\Lambda_{\{r\}}(I - r\lambda S_{\mathcal{Y}}^*)^{-1}\Gamma. \end{aligned}$$

We conclude that

$$H_{\{r\}}(e^{\imath\omega})^* H_{\{r\}}(e^{\imath\omega}) = \operatorname{Re} V_{\{r\}}(e^{\imath\omega}) \quad (0 \le \omega \le 2\pi).$$

Let W be an $\mathscr{L}(\mathcal{U})$-valued positive real function with $H(\lambda)^*H(\lambda) \le \operatorname{Re} W(\lambda)$ for all $\lambda \in \mathbb{D}$. Set $W_{\{r\}}(\lambda) = W(r\lambda)$ for each $\lambda \in \mathbb{D}$. Then $H_{\{r\}}(\lambda)^*H_{\{r\}}(\lambda) \le \operatorname{Re} W_{\{r\}}(\lambda)$ for all $\lambda \in \mathbb{D}$. Again $W_{\{r\}}$ is analytic on an open neighborhood of $\overline{\mathbb{D}}$, and thus, by continuity, $H_{\{r\}}(e^{\imath\omega})^*H_{\{r\}}(e^{\imath\omega}) \le \operatorname{Re} W_{\{r\}}(e^{\imath\omega})$ for each $0 \le \omega \le 2\pi$. But then we can use the result of the previous paragraph to show that

$$\operatorname{Re} V_{\{r\}}(e^{\imath\omega}) \le \operatorname{Re} W_{\{r\}}(e^{\imath\omega}) \quad (0 \le \omega \le 2\pi). \tag{3.3.6}$$

Next we show that the latter inequality implies that $W_{\{r\}} - V_{\{r\}}$ is positive real. To accomplish this, let $L_{\operatorname{Re} V_{\{r\}}}$ and $L_{\operatorname{Re} W_{\{r\}}}$ be the block Laurent operators on $\ell^2(\mathcal{Y})$ defined by $\operatorname{Re} V_{\{r\}}$ and $\operatorname{Re} W_{\{r\}}$, respectively. Since $\operatorname{Re} V_{\{r\}}$ and $\operatorname{Re} W_{\{r\}}$ are both continuous on the unit circle $\mathbb{T}$, these operators are well defined and bounded. Furthermore, the inequality (3.3.6) implies that $L_{\operatorname{Re} V_{\{r\}}} \le L_{\operatorname{Re} W_{\{r\}}}$. Taking the compression to $\ell^2_+(\mathcal{Y})$ this implies that $T_{\operatorname{Re} V_{\{r\}}} \le T_{\operatorname{Re} W_{\{r\}}}$, where $T_{\operatorname{Re} V_{\{r\}}}$ and $T_{\operatorname{Re} W_{\{r\}}}$ are the block Toeplitz operators on $\ell^2_+(\mathcal{Y})$ defined by $\operatorname{Re} V_{\{r\}}$ and $\operatorname{Re} W_{\{r\}}$, respectively. Next, taking an n-th section of these block Toeplitz operators, we obtain for the $n \times n$ Toeplitz operator matrices defined by $\operatorname{Re} V_{\{r\}}$ and $\operatorname{Re} W_{\{r\}}$

(see (2.5.1)) that $T_{\mathrm{Re}\, V_{\{r\}},n} \leq T_{\mathrm{Re}\, W_{\{r\}},n}$ for all integers $n \geq 0$. This implies that $W_{\{r\}} - V_{\{r\}}$ is positive real; see Section 2.5.

Part 2. Put

$$\Delta = W - V \quad \text{and} \quad \Delta_{\{r\}} = W_{\{r\}} - V_{\{r\}} \text{ for each } 0 < r < 1.$$

Then Δ and $\Delta_{\{r\}}$ are analytic $\mathscr{L}(\mathcal{U})$-valued functions on $\mathbb{D}$ and $\Delta_{\{r\}}$ is positive real for each $0 < r < 1$. Furthermore, for $r \uparrow 1$ the n-th Taylor coefficient of $\Delta_{\{r\}}$ converges pointwise (i.e., in the strong operator topology) to the n-th Taylor coefficient of Δ for each integer $n \geq 0$. Hence for each $n = 0, 1, 2, \ldots$ we see that $T_{\mathrm{Re}\,\Delta_{\{r\}},n}x$ converges to $T_{\mathrm{Re}\,\Delta,n}x$ for each $x \in \mathcal{U}^n$ as $r \uparrow 1$. Since the operators $T_{\mathrm{Re}\,\Delta_{\{r\}},n}$ are all non-negative, the same holds true for $T_{\mathrm{Re}\,\Delta,n}$. This shows that $\Delta = W - V$ is positive real. □

We are now ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$, and define W by (3.3.5). Let C be the Cayley transform of K; see section 2.5. Then C is positive real, $C(0) = I$ and we have $W(\lambda) = V(\lambda) + D_\Gamma C(\lambda) D_\Gamma$ for each $\lambda \in \mathbb{D}$, where V is the harmonic majorant of Θ in (3.3.1) given by (3.3.2). We obtain that $W(0) = \Gamma^*\Gamma + D_\Gamma^2 = I$ and

$$\mathrm{Re}\, W(\lambda) = \mathrm{Re}\, V(\lambda) + D_\Gamma(\mathrm{Re}\, C(\lambda))D_\Gamma \geq \mathrm{Re}\, V(\lambda) \geq H(\lambda)^* H(\lambda) \quad (\lambda \in \mathbb{D}).$$

So for $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ the function W in (3.3.5) satisfies (3.3.4).

Conversely, let W be an $\mathscr{L}(\mathcal{U})$-valued function on $\mathbb{D}$ satisfying (3.3.4). In particular, $\mathrm{Re}\, W$ is a harmonic majorant of Θ in (3.3.1). According to Lemma 3.3.2 the function $\Delta = W - V$ is positive real, and satisfies $\Delta(0) = I - \Gamma^*\Gamma = D_\Gamma^2$. But then Lemma 2.5.2 says that there exists a unique $\mathscr{L}(\mathcal{D}_\Gamma)$-valued positive real function C with $C(0) = I$ such that $\Delta(\lambda) = D_\Gamma C(\lambda) D_\Gamma$ for each $\lambda \in \mathbb{D}$. Let $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ be the inverse Cayley transform of C. Then $W = V + \Delta$ is given by (3.3.5) for this choice of K. Moreover, since C and Δ determine each other uniquely, as well as K and C, we obtain that K and W in (3.3.5) determine each other uniquely. □

Proof of Theorem 3.2.5. Assume that W and F are $\mathscr{L}(\mathcal{U})$-valued functions on $\mathbb{D}$ that determine each other uniquely via

$$W(\lambda) = 2F(\lambda) - I \quad \text{and} \quad F(\lambda) = \frac{1}{2}(W(\lambda) + I) \quad (\lambda \in \mathbb{D}). \tag{3.3.7}$$

Then $W(0) = I$ implies that $F(0) = I$ and vice versa. Moreover,

$$2\mathrm{Re}\, F(\lambda) = \mathrm{Re}\, W(\lambda) + I \quad (\lambda \in \mathbb{D}).$$

Hence F satisfies (3.2.5) if and only if W satisfies (3.3.4).

Note that for $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ the function W in (3.3.5) can be written as

$$\begin{aligned} W(\lambda) &= 2\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + \Gamma^*\Gamma + 2D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma - D_\Gamma^2 \\ &= F_K(\lambda) - I, \end{aligned} \qquad (\lambda \in \mathbb{D})$$

where F_K is given by (3.2.11). Hence W in (3.3.5) and F_K in (3.2.11) are related via (3.3.7). So Theorem 3.2.5 is just a translation of Theorem 3.3.1 via the identities (3.3.7). □

3.4 A state space proof of Theorem 3.2.5

In this section we present a proof of Theorem 3.2.5 based on the notion of a state space triple. A triple of operators $\{\alpha, \beta, \gamma\}$ will be called a *state space triple* with *state space* $\mathcal{M}$ when α, β and γ are operators such that α acts on $\mathcal{M}$, β maps $\mathcal{U}$ into $\mathcal{M}$ and γ maps $\mathcal{M}$ into $\mathcal{Y}$. A state space triple $\{\alpha \text{ on } \mathcal{M}, \beta, \gamma\}$ for which the operator

$$\begin{bmatrix} \gamma \\ \alpha \end{bmatrix} : \mathcal{M} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{M} \end{bmatrix} \tag{3.4.1}$$

is a co-isometry will be called a *co-isometric state space triple.* Of particular interest will be co-isometric state space triples $\{\alpha, \beta, \gamma\}$ with β an isometry. For such state space triples we have the following result.

Lemma 3.4.1. *Let $\{\alpha, \beta, \gamma\}$ be a co-isometric state space triple such that β is an isometry. Put $\alpha^\times = (I - \beta\beta^*)\alpha$, and*

$$\begin{aligned} H(\lambda) &= \gamma(I - \lambda\alpha)^{-1}\beta, && \text{(3.4.2a)} \\ F(\lambda) &= \beta^*(I - \lambda\alpha)^{-1}\beta, \qquad (\lambda \in \mathbb{D}) && \text{(3.4.2b)} \\ Z(\lambda) &= \begin{bmatrix} \gamma \\ \beta^*\alpha \end{bmatrix} (I - \lambda\alpha^\times)^{-1}\beta. && \text{(3.4.2c)} \end{aligned}$$

Then $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, the function F is positive real and satisfies (3.2.5)*, and $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$. Moreover, the functions H, F and Z are related via* (3.2.4) *and* (3.2.6).

Proof. Let $\{\alpha, \beta, \gamma\}$ be a co-isometric state space triple with β an isometry. Then both α and $\alpha^\times = P_{\operatorname{Ker}\beta^*}\alpha$ are contractions. Hence the functions H, F and Z in (3.4.2) are properly defined analytic functions on $\mathbb{D}$. To see that Z is a Schur class function first observe that

$$\begin{aligned} \begin{bmatrix} \gamma^* & \alpha^*\beta \end{bmatrix} \begin{bmatrix} \gamma \\ \beta^*\alpha \end{bmatrix} &= \gamma^*\gamma + \alpha^*\beta\beta^*\alpha = \gamma^*\gamma + \alpha^*\alpha - \alpha^* P_{\operatorname{Ker}\beta^*}\alpha \\ &= \gamma^*\gamma + \alpha^*\alpha - (\alpha^\times)^*\alpha^\times \leq I - (\alpha^\times)^*\alpha^\times \\ &\leq I - |\lambda|^2(\alpha^\times)^*\alpha^\times \qquad (\lambda \in \mathbb{D}). \end{aligned}$$

Moreover, since $\beta^*\alpha^\times = \beta^* P_{\operatorname{Ker}\beta^*}\alpha = 0$, we have $\beta^*(I - \lambda\alpha^\times)^{-1} = \beta^*$ for each

$\lambda \in \mathbb{D}$. Therefore

$$\begin{aligned}
Z(\lambda)^* Z(\lambda) &\leq \beta^*(I-\lambda\alpha^\times)^{-*}(I-|\lambda|^2(\alpha^\times)^*\alpha^\times)(I-\lambda\alpha^\times)^{-1}\beta \\
&= \beta^*(I-\lambda\alpha^\times)^{-*}(I-\lambda\alpha^\times)^{-1}\beta \\
&\qquad -\beta^*((I-\lambda\alpha^\times)^{-*}-I)((I-\lambda\alpha^\times)^{-1}-I)\beta \qquad (\lambda \in \mathbb{D}) \\
&= \beta^*(I-\lambda\alpha^\times)^{-*}\beta + \beta^*(I-\lambda\alpha^\times)^{-1}\beta - \beta^*\beta \\
&= 2\beta^*\beta - \beta^*\beta = \beta^*\beta = I.
\end{aligned}$$

Hence $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y}\oplus\mathcal{U})$.

Next observe that for each $\lambda \in \mathbb{D}$ we have

$$\begin{aligned}
\Pi_{\mathcal{Y}} Z(\lambda)(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1} &= \gamma(I-\lambda\alpha^\times)^{-1}\beta(I-\lambda\beta^*\alpha(I-\lambda\alpha^\times)^{-1}\beta)^{-1} \\
&= \gamma(I-\lambda\alpha^\times)^{-1}(I-\lambda\beta\beta^*\alpha(I-\lambda\alpha^\times)^{-1})^{-1}\beta \\
&= \gamma(I-\lambda\alpha^\times-\lambda\beta\beta^*\alpha)^{-1}\beta = \gamma(I-\lambda\alpha)^{-1}\beta \\
&= H(\lambda).
\end{aligned}$$

So H is given by the first identity in (3.2.4).

Now apply Corollary 2.3.4 with $G = F$, where we use that $\beta^*\beta = I$ and

$$F(\lambda) = \beta^*(I-\lambda\alpha)^{-1}\beta = \beta^*\beta + \lambda\beta^*\alpha(I-\lambda\alpha)^{-1}\beta = I + \lambda\beta^*\alpha(I-\lambda\alpha)^{-1}\beta.$$

Then we obtain that the inverse of $F(\lambda)$ is given by

$$F(\lambda)^{-1} = I - \lambda\beta^*\alpha(I-\lambda\alpha^\times)^{-1}\beta = I - \lambda\Pi_{\mathcal{U}}Z(\lambda) \quad (\lambda \in \mathbb{D}).$$

Hence F is given by the second identity in (3.2.4).

From Proposition 3.2.4 we now obtain that $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$, the function F is a positive real function satisfying (3.2.5), and (3.2.6) holds. □

Remark 3.4.2. All arguments and computations in the proof of Lemma 3.4.1 also hold for state space triples $\{\alpha,\beta,\gamma\}$ for which β is an isometry and the operator in (3.4.1) is just contractive rather than co-isometric.

Let $\{\alpha,\beta,\gamma\}$ be a co-isometric state space triple and H an $\mathscr{L}(\mathcal{U},\mathcal{Y})$-valued function on $\mathbb{D}$. When (3.4.2a) holds, we call $\{\alpha,\beta,\gamma\}$ a *co-isometric representing triple* for H. Given $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$ it is easy to construct a co-isometric representing triple for H. Indeed, let Γ be the operator from $\mathcal{U}$ into $H^2(\mathcal{Y})$ defined by H, and take

$$\alpha = S^*_{\mathcal{Y}}, \quad \beta = \Gamma \quad \text{and} \quad \gamma = E^*_{\mathcal{Y}}.$$

For this state space triple $\{\alpha,\beta,\gamma\}$ the state space $\mathcal{M}$ is $H^2(\mathcal{Y})$, the operator defined by (3.4.1) is unitary, and H is given by (3.4.2a), but in general β is not an isometry. To obtain a co-isometric representing triple for H with β an isometry, we enlarge the state space and take

$$\alpha = \begin{bmatrix} S^*_{\mathcal{Y}} & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix}, \quad \beta = \begin{bmatrix} \Gamma \\ D_\Gamma \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} E^*_{\mathcal{Y}} & 0 \end{bmatrix}, \tag{3.4.3}$$

with the underlying state space $\mathcal{M}$ being given by $\mathcal{M} = H^2(\mathcal{Y}) \oplus \mathcal{D}_\Gamma$. Then $\{\alpha, \beta, \gamma\}$ is a co-isometric representing triple for H with β an isometry.

Making a more subtle choice allows us to prove that the map $K \mapsto F_K$ in Theorem 3.2.5 is properly defined and one-to-one.

Proof of Theorem 3.2.5 (first part). Fix a $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, and let Γ be the operator defined by H. Let K be a Schur class function in $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$. Choose a co-isometric realization $\{X \text{ on } \mathcal{X}, B, C, D\}$ of K, put $\mathcal{M} = H^2(\mathcal{Y}) \oplus \mathcal{D}_\Gamma \oplus \mathcal{X}$, and let $\{\alpha, \beta, \gamma\}$ be the state space triple with state space $\mathcal{M}$ defined by

$$\alpha = \begin{bmatrix} S^*_{\mathcal{Y}} & 0 & 0 \\ 0 & D & C \\ 0 & B & X \end{bmatrix}, \quad \beta = \begin{bmatrix} \Gamma \\ D_\Gamma \\ 0 \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} E^*_{\mathcal{Y}} & 0 & 0 \end{bmatrix}. \tag{3.4.4}$$

Clearly, $\{\alpha, \beta, \gamma\}$ is a co-isometric representing triple for H, and the operator β is an isometry. Let F be defined by formula (3.4.2b). Then F is positive real, and satisfies (3.2.5); by Lemma 3.4.1. Furthermore, for each $\lambda \in \mathbb{D}$

$$\begin{aligned} F(\lambda) &= \beta^*(I - \lambda\alpha)^{-1}\beta \\ &= \Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma + \begin{bmatrix} D_\Gamma & 0 \end{bmatrix} \begin{bmatrix} I - \lambda D & -\lambda C \\ -\lambda B & I - \lambda X \end{bmatrix}^{-1} \begin{bmatrix} D_\Gamma \\ 0 \end{bmatrix}. \end{aligned}$$

Now we apply Corollary 2.3.3 with $G = K$. It follows that

$$\begin{bmatrix} D_\Gamma & 0 \end{bmatrix} \begin{bmatrix} I - \lambda D & -\lambda C \\ -\lambda B & I - \lambda X \end{bmatrix}^{-1} \begin{bmatrix} D_\Gamma \\ 0 \end{bmatrix} = D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma \quad (\lambda \in \mathbb{D}).$$

We conclude that $F = F_K$, where F_K is the function given by (3.2.11). So the map $K \mapsto F_K$ is properly defined.

To see that the map $K \mapsto F_K$ is one-to-one, let $K' \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ and assume that $F_K = F_{K'}$. Then we have $D_\Gamma(I - \lambda K'(\lambda))^{-1}D_\Gamma = D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma$ for each $\lambda \in \mathbb{D}$. Since $\mathcal{D}_\Gamma$ is by definition the closure of the range of D_Γ, this implies that $(I - \lambda K'(\lambda))^{-1} = (I - \lambda K(\lambda))^{-1}$ for each $\lambda \in \mathbb{D}$. But then $K = K'$. $\square$

To prove that the map $K \mapsto F_K$ in Theorem 3.2.5 is onto, we need the notion of unitary equivalence for state space triples. Two state space triples $\{\alpha \text{ on } \mathcal{M}, \beta, \gamma\}$ and $\{\alpha' \text{ on } \mathcal{M}', \beta', \gamma'\}$ are said to be *unitarily equivalent* if there exists a unitary operator Φ mapping $\mathcal{M}$ onto $\mathcal{M}'$ satisfying

$$\Phi\alpha = \alpha'\Phi, \quad \Phi\beta = \beta' \quad \text{and} \quad \gamma'\Phi = \gamma.$$

In this case, we say that Φ *unitarily intertwines* $\{\alpha, \beta, \gamma\}$ *with* $\{\alpha', \beta', \gamma'\}$.

One easily checks that any state space triple that is unitarily equivalent to a co-isometric state space triple must also be co-isometric. Moreover, the functions H, F and Z associated with co-isometric state space triples $\{\alpha, \beta, \gamma\}$ with β an isometry via (3.4.2) are preserved under unitary equivalence. For state space triples that satisfy an additional minimality type of condition the converse of the latter statement also holds. We return to this at the end of the present section; see Proposition 3.4.10 below.

Lemma 3.4.3. *Let $H \in \mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ be represented by the co-isometric state space triple $\{\alpha,\beta,\gamma\}$ with β being an isometry, and let Γ be the operator defined by H. Then there exists a co-isometry*

$$M = \begin{bmatrix} D & C \\ B & X \end{bmatrix} \ on \ \begin{bmatrix} \mathcal{D}_\Gamma \\ \mathcal{X} \end{bmatrix} \tag{3.4.5}$$

such that $\{\alpha,\beta,\gamma\}$ is unitarily equivalent to the state space triple $\{\alpha',\beta',\gamma'\}$ with

$$\alpha' = \begin{bmatrix} S^*_\mathcal{Y} & 0 & 0 \\ 0 & D & C \\ 0 & B & X \end{bmatrix}, \quad \beta' = \begin{bmatrix} \Gamma \\ D_\Gamma \\ 0 \end{bmatrix}, \quad \gamma' = \begin{bmatrix} E^*_\mathcal{Y} & 0 & 0 \end{bmatrix}, \tag{3.4.6}$$

and with state space $\mathcal{M}' = H^2(\mathcal{Y}) \oplus \mathcal{D}_\Gamma \oplus \mathcal{X}$.

Proof. Let $\mathcal{M}_o$ and $\mathcal{M}_{\bar{o}}$ be the observable and unobservable subspaces for the pair $\{\gamma,\alpha\}$, that is,

$$\mathcal{M}_{\bar{o}} = \bigcap_{n=0}^{\infty} \operatorname{Ker} \gamma\alpha^n = \bigcap_{\lambda\in\mathbb{D}} \operatorname{Ker} \gamma(I-\lambda\alpha)^{-1} \quad \text{and} \quad \mathcal{M}_o = \mathcal{M} \ominus \mathcal{M}_{\bar{o}},$$

where $\mathcal{M}$ denotes the state space of $\{\alpha,\beta,\gamma\}$. Lemma 2.4.4 implies that α, β and γ decompose as

$$\begin{aligned} \alpha &= \begin{bmatrix} \alpha_o & 0 \\ 0 & \alpha_{\bar{o}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{M}_o \\ \mathcal{M}_{\bar{o}} \end{bmatrix}, \\ \beta &= \begin{bmatrix} \beta_o \\ \beta_{\bar{o}} \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{M}_o \\ \mathcal{M}_{\bar{o}} \end{bmatrix}, \\ \gamma &= \begin{bmatrix} \gamma_o & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M}_o \\ \mathcal{M}_{\bar{o}} \end{bmatrix} \to \mathcal{Y}, \end{aligned}$$

with the pair $\{\gamma_o,\alpha_o\}$ observable.

Now let W_o be the observability operator for the pair $\{\gamma,\alpha\}$. According to Lemma 2.4.4, W_o is a co-isometry and we have

$$S^*_\mathcal{Y} W_o = W_o\alpha, \quad E^*_\mathcal{Y} W_o = \gamma, \quad \Gamma = W_o\beta \quad \text{and} \quad \operatorname{Ker} W_o = \mathcal{M}_{\bar{o}}.$$

Hence $\Phi_o := W_o|_{\mathcal{M}_o}$ is a unitary operator mapping $\mathcal{M}_o$ onto $H^2(\mathcal{Y})$ satisfying

$$S^*_\mathcal{Y}\Phi_o = \Phi_o\alpha_o, \quad \Gamma = \Phi_o\beta_o \quad \text{and} \quad E^*_\mathcal{Y}\Phi_o = \gamma_o. \tag{3.4.7}$$

The second equality in (3.4.7) implies that $\beta_o^*\beta_o = \beta_o^*\Phi_o^*\Phi_o\beta_o = \Gamma^*\Gamma$. Since β is assumed to be isometric, we obtain $\beta_{\bar{o}}^*\beta_{\bar{o}} = I - \beta_o^*\beta_o = D_\Gamma^2$. Put $\mathcal{X} = \mathcal{M}_{\bar{o}} \ominus \overline{\beta_{\bar{o}}\mathcal{U}}$.

Then there exists a unitary operator $\Phi_{\bar{o}}$ mapping $\mathcal{M}_{\bar{o}} = \overline{\beta_{\bar{o}}\mathcal{U}} \oplus \mathcal{X}$ onto $D_\Gamma \oplus \mathcal{X}$ given by

$$\Phi_{\bar{o}}\beta_{\bar{o}}u = D_\Gamma u \quad \text{and} \quad \Phi_{\bar{o}}x = x \quad (u \in \mathcal{U}, x \in \mathcal{X}).$$

Put $M = \Phi_{\bar{o}}\alpha_{\bar{o}}\Phi_{\bar{o}}^*$. Then

$$M\Phi_{\bar{o}} = \Phi_{\bar{o}}\alpha_{\bar{o}} \quad \text{and} \quad \Pi_{\mathcal{D}_\Gamma}^* D_\Gamma = \Phi_{\bar{o}}\beta_{\bar{o}}.$$

Hence the map

$$\Phi := \begin{bmatrix} \Phi_o & 0 \\ 0 & \Phi_{\bar{o}} \end{bmatrix} : \begin{bmatrix} \mathcal{M}_o \\ \mathcal{M}_{\bar{o}} \end{bmatrix} \to \begin{bmatrix} H^2(\mathcal{Y}) \\ \mathcal{D}_\Gamma \oplus \mathcal{X} \end{bmatrix}$$

unitarily intertwines $\{\alpha, \beta, \gamma\}$ with the state space $\{\alpha', \beta', \gamma'\}$ in (3.4.6) via (3.4.5), where $M = \Phi_{\bar{o}}\alpha_{\bar{o}}\Phi_{\bar{o}}^*$. So $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ are unitarily equivalent. □

Proof of Theorem 3.2.5 (second part). Fix a $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, and let Γ be the operator defined by H. Let F be a positive real function satisfying (3.2.5). It remains to show that $F = F_K$ for some $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$.

Let Z be the function given by (3.2.6). Then $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and the equalities in (3.2.4) hold. Since $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, the function Z admits a co-isometric realization $\{X' \text{ on } \mathcal{X}', B', C', D'\}$. Decompose C' and D' as

$$C' = \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \quad \text{and} \quad D' = \begin{bmatrix} D_1' \\ D_2' \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.$$

Put

$$\alpha = \begin{bmatrix} D_2' & C_2' \\ B' & X' \end{bmatrix}, \quad \beta = \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix}, \quad \text{and} \quad \gamma = \begin{bmatrix} D_1' & C_1' \end{bmatrix}. \tag{3.4.8}$$

The triple $\{\alpha, \beta, \gamma\}$ is clearly a state space triple, and β is an isometry. Since $\{X', B', C', D'\}$ is a co-isometric system, we obtain that $\{\alpha, \beta, \gamma\}$ is a co-isometric state space triple. Note that $\alpha^\times = P_{\mathcal{X}'}\alpha$. The definition of $\{\alpha, \beta, \gamma\}$ shows that

$$\begin{aligned} \begin{bmatrix} \gamma \\ \beta^*\alpha \end{bmatrix} (I - \lambda\alpha^\times)^{-1}\beta &= \begin{bmatrix} D_1' & C_1' \\ D_2' & C_2' \end{bmatrix} \begin{bmatrix} I & 0 \\ -\lambda B' & I - \lambda X' \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} D' & C' \end{bmatrix} \begin{bmatrix} I \\ \lambda(I - \lambda X')^{-1}B' \end{bmatrix} \qquad (\lambda \in \mathbb{D}) \\ &= D' + \lambda C'(I - \lambda X')^{-1}B' = Z(\lambda). \end{aligned}$$

Hence Z is given by (3.4.2c). Since H and F are uniquely determined by Z via (3.2.4), Lemma 3.4.1 guarantees that H and F are given by (3.4.2a) and (3.4.2b), respectively.

Next we apply Lemma 3.4.3 to the state space triple $\{\alpha, \beta, \gamma\}$. We obtain a co-isometric state space triple $\{\alpha', \beta', \gamma'\}$ of the form (3.4.6) that is unitarily equivalent to $\{\alpha, \beta, \gamma\}$. In particular, $\{\alpha', \beta', \gamma'\}$ is a co-isometric representing triple for H, and F is the positive real function associated with $\{\alpha', \beta', \gamma'\}$ via (3.4.2b).

Now let K be the function defined by the system $\{X,B,C,D\}$ determined by $\{\alpha',\beta',\gamma'\}$ via (3.4.6). Since $\{\alpha',\beta',\gamma'\}$ is a co-isometric state space triple, the system $\{X,B,C,D\}$ is co-isometric. Hence K is in the Schur class $\mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$. Note that $\{\alpha',\beta',\gamma'\}$ is of the same form as the state space triple $\{\alpha,\beta,\gamma\}$ in (3.4.4). So with the same arguments as in the first part of the proof of Theorem 3.2.5 we obtain that the functions F and F_K coincide. □

Lemma 3.4.3 give us the following corollary.

Corollary 3.4.4. *Let $H \in \mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$, and let Γ be the operator defined by H. Then H has a co-isometric representing triple $\{\alpha',\beta',\gamma'\}$ with β' an isometry that is of the form given in* (3.4.5) *and* (3.4.6). *Moreover, any co-isometric state space triple with β an isometry is unitarily equivalent to a state space triple of the form given in* (3.4.5) *and* (3.4.6).

Proof. The second statement follows immediately from Lemma 3.4.3. To see that a function $H \in \mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ admits a co-isometric representing triple $\{\alpha,\beta,\gamma\}$ with β an isometry that is of the form given in (3.4.5) and (3.4.6) let M be an arbitrary co-isometry as in (3.4.5). Then $\{\alpha',\beta',\gamma'\}$ in (3.4.6) is a co-isometric state space triple and β' is an isometry. Since $H(\lambda) = E^*_{\mathcal{Y}}(I-\lambda S^*_{\mathcal{Y}})^{-1}\Gamma$, it is obviously the case that $\{\alpha',\beta',\gamma'\}$ is a representing triple for H. □

Let us give some examples to illustrate the constructions made above.

Example 3.4.5. Let H be an arbitrary function in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$, and let Γ be the operator defined by H. Let $\{\alpha,\beta,\gamma\}$ be the co-isometric representing triple for H with β isometric given by (3.4.3). The function F in (3.4.2b) is then given by

$$F(\lambda) = \Gamma^*(I-\lambda S^*_{\mathcal{Y}})^{-1}\Gamma + \frac{1}{1-\lambda} D^2_\Gamma \quad (\lambda \in \mathbb{D}). \tag{3.4.9}$$

We see that $F = F_K$, where $K \in \mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ is equal to the constant function with value $I_{\mathcal{D}_\Gamma}$.

Note that $\{\alpha,\beta,\gamma\}$ is a state space triple of the form given by (3.4.5) and (3.4.6), where $M = I_{\mathcal{D}_\Gamma}$. Moreover, $M = I_{\mathcal{D}_\Gamma}$ is the system matrix associated with the co-isometric system $\{0 \text{ on } \{0\}, 0, 0, I_{\mathcal{D}_\Gamma}\}$, which is a realization for the constant function with value $I_{\mathcal{D}_\Gamma}$, as could be expected from the second part of the proof of Theorem 3.2.5.

Next we compute the corresponding Schur class function Z (cf., formula (3.4.2c)). In this case

$$\alpha^\times := (I-\beta\beta^*)\alpha = \begin{bmatrix} D^2_{\Gamma^*}S^*_{\mathcal{Y}} & -D_{\Gamma^*}\Gamma \\ -\Gamma^* D_{\Gamma^*}S^*_{\mathcal{Y}} & \Gamma^*\Gamma|_{\mathcal{D}_\Gamma} \end{bmatrix} \text{ on } \begin{bmatrix} H^2(\mathcal{Y}) \\ \mathcal{D}_\Gamma \end{bmatrix}.$$

One can compute that

$$\begin{aligned}(I-\lambda\alpha^\times)^{-1}\beta \quad &\equiv \quad \begin{bmatrix} \Gamma \\ D_\Gamma \end{bmatrix} + \\ &+ \tfrac{\lambda}{1-\lambda} \begin{bmatrix} D_{\Gamma^*} \\ -\Gamma^* \end{bmatrix} D_{\Gamma^*}(S^*_{\mathcal{Y}}-I)\left(I - \tfrac{\lambda}{1-\lambda} D^2_{\Gamma^*}(S^*_{\mathcal{Y}}-I)\right)^{-1}\Gamma.\end{aligned}$$

The Schur class function $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ defined in (3.4.2c) can then be written as

$$Z(\lambda) = \begin{bmatrix} 0 \\ I \end{bmatrix} + \begin{bmatrix} E_{\mathcal{Y}}^* \\ \Gamma^*(S_{\mathcal{Y}}^* - I) \end{bmatrix} \left(I - \frac{\lambda}{1-\lambda} D_{\Gamma^*}^2 (S_{\mathcal{Y}}^* - I) \right)^{-1} \Gamma \quad (\lambda \in \mathbb{D}). \quad (3.4.10)$$

Example 3.4.6. Let H be an arbitrary function in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ and let Γ be the operator defined by H. Put

$$\alpha' = \begin{bmatrix} S_{\mathcal{Y}}^* & 0 \\ 0 & S_{\mathcal{D}_\Gamma}^* \end{bmatrix}, \quad \beta' = \begin{bmatrix} \Gamma \\ E_{\mathcal{D}_\Gamma} D_\Gamma \end{bmatrix}, \quad \gamma' = \begin{bmatrix} E_{\mathcal{Y}}^* & 0 \end{bmatrix},$$

with the underlying state space $\mathcal{M}'$ being given by $\mathcal{M}' = H^2(\mathcal{Y}) \oplus H^2(\mathcal{D}_\Gamma)$. Then $\{\alpha', \beta', \gamma'\}$ is a co-isometric representing triple for H with β' an isometry. Since $S_{\mathcal{D}_\Gamma}^* E_{\mathcal{D}_\Gamma} = 0$, the positive real function F' associated with $\{\alpha', \beta', \gamma'\}$ via (3.4.2b) is given by

$$F'(\lambda) = \Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + D_\Gamma^2 \quad (\lambda \in \mathbb{D}). \quad (3.4.11)$$

We see that $F' = F_K$ with K being the zero function in $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$.

On the other hand, $\{\alpha', \beta', \gamma'\}$ can be seen as a state space triple of the form (3.4.6) when we identify $\mathcal{D}_\Gamma$ with the subspace of constant functions in $H^2(\mathcal{D}_\Gamma)$ and $H^2(\mathcal{D}_\Gamma)$ with $\mathcal{D}_\Gamma \oplus H^2(\mathcal{D}_\Gamma)$. Under these identifications $S_{\mathcal{D}_\Gamma}^*$ can be viewed as the system matrix associated with the co-isometric system $\{S_{\mathcal{D}_\Gamma}^*, 0, E_{\mathcal{D}_\Gamma}^*, 0\}$, which is a realization for the zero function in $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$, as expected.

Next we compute the corresponding Schur class function Z' associated with $\{\alpha', \beta', \gamma'\}$ via (3.4.2c). We have

$$\alpha'^{\times} := (I - \beta'\beta'^*)\alpha' = \begin{bmatrix} D_{\Gamma^*}^2 S_{\mathcal{Y}}^* & -D_{\Gamma^*}\Gamma E_{\mathcal{D}_\Gamma}^* S_{\mathcal{D}_\Gamma}^* \\ -E_{\mathcal{D}_\Gamma}\Gamma^* D_{\Gamma^*} S_{\mathcal{Y}}^* & D_{D_\Gamma E_{\mathcal{D}_\Gamma}^*} S_{\mathcal{D}_\Gamma}^* \end{bmatrix} \text{ on } \begin{bmatrix} H^2(\mathcal{Y}) \\ H^2(\mathcal{D}_\Gamma) \end{bmatrix}.$$

This implies that for each $\lambda \in \mathbb{D}$ the operator $(I - \lambda\alpha'^{\times})^{-1}$ admits a decomposition of the form

$$(I - \lambda\alpha'^{\times})^{-1} = \begin{bmatrix} (I - \lambda D_{\Gamma^*}^2 S_{\mathcal{Y}}^*)^{-1} & *S_{\mathcal{D}_\Gamma}^* \\ E_{\mathcal{D}_\Gamma} * & I + *S_{\mathcal{D}_\Gamma}^* \end{bmatrix} \quad (\lambda \in \mathbb{D}),$$

where the $*$ stands for an unspecified operator. Using that $S_{\mathcal{D}_\Gamma}^* E_{\mathcal{D}_\Gamma} = 0$ we obtain that the Schur class function Z' in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ associated with $\{\alpha', \beta', \gamma'\}$ via (3.4.2c) can be written as

$$Z'(\lambda) = \begin{bmatrix} E_{\mathcal{Y}}^* \\ \Gamma S_{\mathcal{Y}}^* \end{bmatrix} (I - \lambda D_{\Gamma^*}^2 S_{\mathcal{Y}}^*)^{-1}\Gamma \quad (\lambda \in \mathbb{D}). \quad (3.4.12)$$

Example 3.4.7. Let L be a contraction from $\mathcal{U}$ into $\mathcal{Y} \oplus \mathcal{U}$, and partition L as

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.$$

Define

$$H(\lambda) = L_1(I-\lambda L_2)^{-1}, \quad F(\lambda) = (I-\lambda L_2)^{-1} \quad \text{and} \quad Z(\lambda) = L \quad (\lambda \in \mathbb{D}).$$

Then H, F and Z are related via (3.2.4) and (3.2.6). As a co-isometric realization for Z we take $\{X', B', C', D'\}$, where

$$\begin{bmatrix} D' & C' \\ B' & X' \end{bmatrix} := \begin{bmatrix} L & D_{L^*}E^*_{\mathcal{D}_{L^*}} \\ 0 & S^*_{\mathcal{D}_{L^*}} \end{bmatrix}.$$

The co-isometric state space triple $\{\alpha, \beta, \gamma\}$ with β an isometry associated with this co-isometric realization via (3.4.8) is defined by

$$\alpha = \begin{bmatrix} L_2 & \Pi_{\mathcal{U}} D_{L^*}E^*_{\mathcal{D}_{L^*}} \\ 0 & S^*_{\mathcal{D}_{L^*}} \end{bmatrix}, \quad \beta = \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} L_1 & \Pi_{\mathcal{Y}} D_{L^*}E^*_{\mathcal{D}_{L^*}} \end{bmatrix}.$$

We check:

$$\begin{aligned} \gamma(I-\lambda\alpha)^{-1}\beta &= \begin{bmatrix} L_1 & * \end{bmatrix} \begin{bmatrix} (I-\lambda L_2)^{-1} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} = L_1(I-\lambda L_2)^{-1}, \\ \beta^*(I-\lambda\alpha)^{-1}\beta &= \begin{bmatrix} I_{\mathcal{Y}} & 0 \end{bmatrix} \begin{bmatrix} (I-\lambda L_2)^{-1} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} = (I-\lambda L_2)^{-1}. \end{aligned}$$

Thus $\{\alpha, \beta, \gamma\}$ is a co-isometric representing triple for H, and F is given by (3.4.2b). Note that $\beta\beta^*$ is the orthogonal projection of $\mathcal{U} \oplus H^2(\mathcal{D}_{D_{L^*}})$ on $\mathcal{U}$. So that

$$\alpha^\times = (I - P_{\mathcal{U}})\alpha = \begin{bmatrix} 0 & 0 \\ 0 & S^*_{\mathcal{D}_{L^*}} \end{bmatrix}.$$

Hence for each $\lambda \in \mathbb{D}$

$$\begin{aligned} \begin{bmatrix} \gamma \\ \beta^*\alpha \end{bmatrix} (I-\lambda\alpha^\times)^{-1}\beta &= \begin{bmatrix} L_1 & \Pi_{\mathcal{U}} D_{L^*}E^*_{\mathcal{D}_{L^*}} \\ L_2 & \Pi_{\mathcal{Y}} D_{L^*}E^*_{\mathcal{D}_{L^*}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I-\lambda S^*_{\mathcal{D}_{L^*}})^{-1} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = Z(\lambda). \end{aligned}$$

So the identities in (3.4.2) hold. ◇

We conclude with a few results on co-isometric representing triples that give further insight in the results presented above and are interesting in their own right.

First we introduce an additional notion for state space triples. A state space triple $\{\alpha, \beta, \gamma\}$ is said to be *jointly observable* if the pair

$$\left\{ \begin{bmatrix} \gamma \\ \beta^* \end{bmatrix}, \alpha \right\}$$

is observable. It is not difficult to check that the notion of joint observability is invariant under unitary equivalence.

Proposition 3.4.8. *Let $H \in \mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$, and let F be a positive real function satisfying* (3.2.5). *Then there exists a jointly observable, co-isometric representing triple $\{\alpha,\beta,\gamma\}$ of H with β being an isometry, and such that F is given by* (3.4.2*b*).

Proof. According to Theorem 3.2.5 there exists a function $K \in \mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ such that $F = F_K$ in (3.2.11). Let $\{X \text{ on } \mathcal{X}, B, C, D\}$ be an observable co-isometric realization of K. Define α, β and γ by (3.4.4). Following the first part of the proof of Theorem 3.2.11, we obtain that $\{\alpha,\beta,\gamma\}$ is a co-isometric representing triple of H, and that F is given by (3.4.2b). So it remains to show that $\{\alpha,\beta,\gamma\}$ is jointly observable. Let $h = f \oplus k \in H^2(\mathcal{Y}) \oplus (\mathcal{D}_\Gamma \oplus \mathcal{X})$ such that

$$\begin{bmatrix} \gamma \\ \beta^* \end{bmatrix} (I - \lambda\alpha)^{-1} h = 0 \quad (\lambda \in \mathbb{D}). \tag{3.4.13}$$

The first row in 3.4.13 says that

$$E^*_{\mathcal{Y}}(I - \lambda S^*_{\mathcal{Y}})^{-1} f = 0 \quad (\lambda \in \mathbb{D}).$$

Since $\{E^*_{\mathcal{Y}}, S^*_{\mathcal{Y}}\}$ is an observable pair, we have $f = 0$. Then the second row of (3.4.13) reduces to

$$D_\Gamma \Pi_{\mathcal{D}_\Gamma}(I - \lambda M)^{-1} k = 0 \quad (\lambda \in \mathbb{D}),$$

where M is the system matrix associated with the realization $\{X, B, C, D\}$. In particular, we obtain that $\{\alpha',\beta',\gamma'\}$ is jointly observable if and only if the pair $\{D_\Gamma \Pi_{\mathcal{D}_\Gamma}, M\}$ is observable. Since by definition $\mathcal{D}_\Gamma = \overline{\operatorname{Im} D_\Gamma}$, the latter condition is equivalent to $\mathcal{D}_\Gamma$ being cyclic for M^*. According to Lemma 2.3.1 this condition is met because $\{X, B, C, D\}$ is an observable realization. □

From Lemma 2.3.1 and the proof of Proposition 3.4.8 we immediately obtain the next corollary.

Corollary 3.4.9. *Let $H \in \mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$, and let Γ be the operator defined by H. Let $\{\alpha',\beta',\gamma'\}$ be a co-isometric representing triple for H of the form* (3.4.6), *where M in* (3.4.5) *is a co-isometry. Then $\{\alpha',\beta',\gamma'\}$ is jointly observable if and only if $\mathcal{D}_\Gamma$ is cyclic for M, or equivalently, if and only if the co-isometric system $\{X, B, C, D\}$ given by* (3.4.5) *is observable.*

From Corollary 3.4.9 it is obvious that the co-isometric state space triples $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ given in Examples 3.4.5 and 3.4.6 are jointly observable because the systems $\{0 \text{ on } \{0\}, 0, 0, I_{\mathcal{D}_\Gamma}\}$ and $\{S^*_{\mathcal{D}_\Gamma}, 0, E^*_{\mathcal{D}_\Gamma}, 0\}$ both are observable.

Proposition 3.4.10. *Let $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ be jointly observable co-isometric state space triples with β and β' isometries. Let H, F and Z be the functions associated with $\{\alpha,\beta,\gamma\}$, and H', F' and Z' the functions associated with $\{\alpha',\beta',\gamma'\}$ via* (3.4.2). *Then the following are equivalent:*

(a) *$\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ are unitarily equivalent,*

(b) *$Z = Z'$,*

(c) $H = H'$ *and* $F = F'$.

Proof. The equivalence of (b) and (c) follows immediately from Lemma 3.4.1. The fact that (a) implies (b) and (c) follows because the functions H, F and Z in (3.4.2) are preserved under unitary equivalence, and does not require joint observability. To complete the proof we show that (c) implies (a). So assume that (c) holds. Since $\{\alpha,\beta,\gamma\}$ is a co-isometric state space triple, we have $\alpha\gamma^* = 0$ and $\gamma\gamma^* = I_{\mathcal{Y}}$. Similar equations hold for $\{\alpha',\beta',\gamma'\}$. Therefore

$$\gamma(I-\lambda\alpha)^{-1}\gamma^* = \gamma\gamma^* = I = \gamma'\gamma'^* = \gamma'(I-\lambda\alpha')^{-1}\gamma'^* \quad (\lambda\in\mathbb{D}).$$

Thus (c) is equivalent to

$$\begin{bmatrix}\gamma\\ \beta^*\end{bmatrix}(I-\lambda\alpha)^{-1}\begin{bmatrix}\gamma^* & \beta\end{bmatrix} = \begin{bmatrix}\gamma'\\ \beta'^*\end{bmatrix}(I-\lambda\alpha')^{-1}\begin{bmatrix}\gamma'^* & \beta'\end{bmatrix} \quad (\lambda\in\mathbb{D}).$$

In other words

$$\begin{bmatrix}\gamma\\ \beta^*\end{bmatrix}(\alpha^*)^n\begin{bmatrix}\gamma^* & \beta\end{bmatrix} = \begin{bmatrix}\gamma'\\ \beta'^*\end{bmatrix}(\alpha'^*)^n\begin{bmatrix}\gamma'^* & \beta'\end{bmatrix} \quad (n\geq 0). \tag{3.4.14}$$

Since $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ are co-isometric state space triples, α^* and α'^* are isometries. Thus from (3.4.14) we obtain that the pairs $\{\alpha^*, \begin{bmatrix}\gamma^* & \beta\end{bmatrix}\}$ and $\{\alpha'^*, \begin{bmatrix}\gamma'^* & \beta'\end{bmatrix}\}$ are both Naimark pairs, in the sense of Section 2.5, for the sequence $R_0, R_1, \ldots$ given by

$$R_n = \begin{bmatrix}\gamma\\ \beta^*\end{bmatrix}(\alpha^*)^n\begin{bmatrix}\gamma^* & \beta\end{bmatrix} \text{ for all integers } n\geq 0.$$

The joint observability of $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ yield that $\{\alpha^*, \begin{bmatrix}\gamma^* & \beta\end{bmatrix}\}$ and $\{\alpha'^*, \begin{bmatrix}\gamma'^* & \beta'\end{bmatrix}\}$ are controllable Naimark pairs. So with the Naimark dilation theorem (Theorem 2.5.1 above) we obtain that $\{\alpha^*, \begin{bmatrix}\gamma^* & \beta\end{bmatrix}\}$ and $\{\alpha'^*, \begin{bmatrix}\gamma'^* & \beta'\end{bmatrix}\}$ are unitarily equivalent Naimark pairs. That is, there exists a unitary map Φ from the state space $\mathcal{M}$ of $\{\alpha,\beta,\gamma\}$ onto the state space $\mathcal{M}'$ of $\{\alpha',\beta',\gamma'\}$ such that

$$\Phi\alpha^* = \alpha'^*\Phi \quad\text{and}\quad \Phi\begin{bmatrix}\gamma^* & \beta\end{bmatrix} = \begin{bmatrix}\gamma'^* & \beta'\end{bmatrix}.$$

In other words, $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ are unitarily equivalent state space triples. □

To illustrate Proposition 3.4.10 we compare the results on the state space triples $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ considered in Examples 3.4.5 and 3.4.6.

Example 3.4.11. Let $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ be the co-isometric representing triples for a fixed $H\in\mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$ given in Examples 3.4.5 and 3.4.6, respectively. Then both $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$ are jointly observable, co-isometric representing triples for H, and β and β' are isometries. Let F and Z, and F' and Z' be the functions associated with $\{\alpha,\beta,\gamma\}$ and $\{\alpha',\beta',\gamma'\}$, respectively, via (3.4.2b) and

(3.4.2c). Hence F, F', Z and Z' are the functions defined by (3.4.9), (3.4.11), (3.4.10) and (3.4.12), respectively. Observe that F and F' do not coincide. So according to Proposition 3.4.10 the state space triples $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ can not be unitarily equivalent. In line with this, we note that the Schur class functions Z and Z' also do not coincide. Moreover, we can see directly that the state space triples $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ can not be unitarily equivalent. Just observe that for $\{\alpha, \beta, \gamma\}$ the operator in (3.4.1) is unitary, while for $\{\alpha', \beta', \gamma'\}$ this in not the case. If $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$ would have been unitarily equivalent, this could not have happened. $\diamond$

Notes for Chapter 3

Schur representations, i.e., representations of the form (3.2.1) appear in [17] in the context of classical commutant lifting; see also page 386 in [38]. In general, the link of H^2-functions with Schur class functions as in (3.2.1), without commutant lifting restrictions, starts with Sarason [82]. In fact, in [82] a scalar version ($\mathcal{U} = \mathcal{Y} = \mathbb{C}$) of Theorem 3.2.1 is obtained for the special case that the function H in question has norm one as a vector in H^2. In this case, as was shown in [82], the Schur class function Z in the Schur representation is unique, which also follows from the final statement in Theorem 3.2.2. When $\mathcal{U} = \mathbb{C}^q$ and $\mathcal{Y} = \mathbb{C}^p$ Theorem 3.2.1 is Theorem 2.2 in [9]. The non-uniqueness in the Schur representation was observed in [9], and illustrated with an example. In full generality Theorem 3.2.1 was first obtained in [48] as a corollary of the description of all contractive interpolants to the relaxed commutant lifting problem. The description of the non-uniqueness as given in Theorem 3.2.2 appears in a less explicit form in [48], where it was obtained again as a consequence of the description of all contractive interpolants there. A direct proof of Theorems 3.2.1 and 3.2.2 is given in [49].

The proof of Theorem 3.2.1 and the part of the proof of Theorem 3.2.2 given in Section 3.2 are new. Section 3.3 coincides for a large part with Section 2 in [49]. The bijective map $Z \mapsto (H, F)$ described in Proposition 3.2.4 is implicitly present in [49]. The state space proof of Theorem 3.2.5 in Section 3.4 is roughly based on the coupling proof for the description of all contractive interpolants to the relaxed commutant lifting theorem given in [48].

Chapter 4

A general H^2 interpolation problem

In this chapter we consider an abstract interpolation problem for functions in the set $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$. The interpolation problem is introduced in Section 4.1, where also a Schur representation is given of the set of all solutions; see Theorem 4.1.1 below. In general this Schur representation does not provide a proper parameterization, that is, a given solution can be represented by many different Schur class functions. This non-uniqueness is the topic of the second section. In Sections 4.3 and 4.4 it is shown that the interpolation problem considered in Section 4.1 is equivalent to the relaxed commutant lifting problem. Moreover, the results of the first two sections are used to prove the first two main results concerning relaxed commutant lifting stated in the introduction (see Theorems 1.1 and 1.2). A particular choice of the parameter in the representation of all solutions provides us with the so-called central solution. The non-uniqueness in the Schur representation of this central solution is further specified in Section 4.5. In the final section we obtain, as an application of the results in Section 4.1, a generalization of the Schur representation in Theorem 3.2.1.

4.1 The main problem and its solution

We consider the following problem. Given a contraction

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \text{where } \mathcal{F} \subset \mathcal{U}, \tag{4.1.1}$$

find a (all) function(s) H in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ such that

$$\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_{\mathcal{F}} \quad (\lambda \in \mathbb{D}). \tag{4.1.2}$$

A function H in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ satisfying equation (4.1.2) will be called a *solution to the H^2 interpolation problem defined by (the contraction) ω*.

Recall (see Section 3.2) that $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ is the set all of functions H in $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ for which the operator Γ defined by H via (3.1.1) is contractive. Replacing H in (4.1.2) by the operator Γ defined by H we see that our problem has the following alternative formulation: find a (all) contraction(s) Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ satisfying the equation

$$E_{\mathcal{Y}}\omega_1 + S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}. \tag{4.1.3}$$

In this case we also say that Γ is a *solution to the H^2 interpolation problem defined by (the contraction) ω.*

Now let $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and put

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \text{ and } F(\lambda) = (I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{4.1.4}$$

We know from Section 3.2 that $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$, and that F is a positive real function satisfying

$$F(0) = I \quad \text{and} \quad 2 \operatorname{Re} F(\lambda) \geq H(\lambda)^* H(\lambda) + I \quad (\lambda \in \mathbb{D}). \tag{4.1.5}$$

Moreover, we have

$$\begin{aligned} H(\lambda) &= \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \\ &= \Pi_{\mathcal{Y}} Z(\lambda)(I + \lambda (I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \Pi_{\mathcal{U}} Z(\lambda)) \qquad (\lambda \in \mathbb{D}) \\ &= \Pi_{\mathcal{Y}} Z(\lambda) + \lambda H(\lambda) \Pi_{\mathcal{U}} Z(\lambda). \end{aligned}$$

Thus if $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfies $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$, then H in (4.1.4) satisfies (4.1.2), and hence H is a solution to the H^2 interpolation problem defined by ω.

We claim that the converse statement also holds, that is, we have the following result.

Theorem 4.1.1. *Let ω be a contraction as in* (4.1.1)*, and let H be an $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued function on $\mathbb{D}$. Then H is a solution to the H^2 interpolation problem defined by ω if and only if H is given by*

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}), \tag{4.1.6}$$

where Z is an arbitrary Schur class function in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying the constraint $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$.

As a first step towards a proof of Theorem 4.1.1 we prove the following result.

Proposition 4.1.2. *Let ω be a contraction as in* (4.1.1)*. Given $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, let H and F be the functions defined by* (4.1.4)*. Then we have*

$$\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_{\mathcal{F}} \quad \textit{and} \quad \Pi_{\mathcal{F}}^* + \lambda F(\lambda)\omega_2 = F(\lambda)|_{\mathcal{F}} \quad (\lambda \in \mathbb{D}) \tag{4.1.7}$$

if and only if $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$.

Proof. Assume $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. As we have seen this implies that the first identity in (4.1.7) holds. Next observe that

$$\begin{aligned} F(\lambda) &= (I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \\ &= I + \lambda (I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \Pi_{\mathcal{U}} Z(\lambda) \qquad (\lambda \in \mathbb{D}) \\ &= I + \lambda F(\lambda) \Pi_{\mathcal{U}} Z(\lambda). \end{aligned}$$

Since $Z(\lambda)|_{\mathcal{F}} \equiv \omega$, we obtain that F satisfies the second identity in (4.1.7).

It remains to prove the only if statement. Assume that the functions H and F satisfy (4.1.7). Take $f \in \mathcal{F}$ and fix $0 \neq \lambda \in \mathbb{D}$. Then it follows from the second identity in (4.1.7) that $F(\lambda)(f - \lambda\omega_2 f) = f$. We know that $F(\lambda)$ is invertible. Thus $F(\lambda)^{-1} f = f - \lambda\omega_2 f$. Proposition 3.2.4 shows that

$$Z(\lambda) = \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda) - I) \end{bmatrix} F(\lambda)^{-1}.$$

Using (4.1.7) we obtain

$$\begin{aligned} \Pi_{\mathcal{Y}} Z(\lambda) f &= H(\lambda)(f - \lambda\omega_2 f) \\ &= H(\lambda) f - \lambda H(\lambda)\omega_2 f = \omega_1 f, \\ \Pi_{\mathcal{U}} Z(\lambda) f &= \lambda^{-1} f - \lambda^{-1} F(\lambda)^{-1} f \\ &= \lambda^{-1} f - \lambda^{-1}(f - \lambda\omega_2 f) = \omega_2 f. \end{aligned}$$

Thus $Z(\lambda)|_{\mathcal{F}} = \omega$ holds for each $0 \neq \lambda \in \mathbb{D}$. By continuity we obtain that also $Z(0)|_{\mathcal{F}} = \omega$. □

The next lemma is a second step towards a proof of Theorem 4.1.1.

Lemma 4.1.3. *Let H be a solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1)*, and let Γ be the operator defined by H. Put $\mathcal{F}_H := \overline{D_\Gamma \mathcal{F}}$. Then there exists a unique contraction ω_H defined by*

$$\omega_H : \mathcal{F}_H \to \mathcal{D}_\Gamma, \quad \omega_H D_\Gamma|_{\mathcal{F}} = D_\Gamma \omega_2. \tag{4.1.8}$$

Moreover, ω_H is an isometry if and only if ω is an isometry.

Note that the final part of the above lemma tells us that the statement 'ω_H is isometric (or not)' is independent of the particular solution H.

Proof of Lemma 4.1.3. Since Γ is a contraction satisfying (4.1.3), we have for each $f \in \mathcal{F}$ that

$$\begin{aligned} \|D_\Gamma f\|^2 &= \|f\|^2 - \|\Gamma f\|^2 = \|f\|^2 - \|E_{\mathcal{Y}}\omega_1 f\|^2 - \|S_{\mathcal{Y}}\Gamma\omega_2 f\|^2 \\ &= \|f\|^2 - \|\omega_1 f\|^2 - \|\Gamma\omega_2 f\|^2 = \|f\|^2 - \|\omega f\|^2 + \|D_\Gamma\omega_2 f\|^2 \\ &= \|D_\omega f\|^2 + \|D_\Gamma\omega_2 f\|^2. \end{aligned}$$

This computation shows that the following identity holds:

$$\Pi_{\mathcal{F}} D_\Gamma^2 \Pi_{\mathcal{F}}^* = D_\omega^2 + \omega_2^* D_\Gamma^2 \omega_2. \tag{4.1.9}$$

In particular, we have that ω_H given by (4.1.8) is a contraction. Moreover, the identity (4.1.9) also shows that ω_H is an isometry if and only if ω is an isometry. □

Proof of Theorem 4.1.1. From Proposition 4.1.2 it is clear that for each Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ the function H given by (4.1.4) is a solution to the H^2 interpolation problem defined by ω.

Now assume that H is a solution to the H^2 interpolation problem defined by ω. In particular, $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, and thus the operator Γ defined by H is a contraction. According to Propositions 4.1.2 and 3.2.4 it suffices to construct a positive real function F satisfying (4.1.5) such that the second identity in (4.1.7) holds.

Let ω_H be the contraction defined by (4.1.8) and K the constant function in $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ with value $\omega_H \Pi_{\mathcal{F}_H}$. Put $F = F_K$ as in (3.2.11), that is,

$$F(\lambda) = \Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1} D_\Gamma \quad (\lambda \in \mathbb{D}).$$

Then F satisfies (4.1.5). We claim that F also satisfies the second identity in (4.1.7). Indeed, fix $\lambda \in \mathbb{D}$ and $f \in \mathcal{F}$. Since Γ satisfies (4.1.3), we have

$$\begin{aligned}
\Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma f &= \Gamma^*\Gamma f + \lambda\Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1} S^*_{\mathcal{Y}} \Gamma f \\
&= \Gamma^*\Gamma f + \lambda\Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1} S^*_{\mathcal{Y}}(E_{\mathcal{Y}}\omega_1 f + S_{\mathcal{Y}}\Gamma\omega_2 f) \\
&= \Gamma^*\Gamma f + \lambda\Gamma^*(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma\omega_2 f.
\end{aligned}$$

From the definition of ω_H it follows that

$$\begin{aligned}
D_\Gamma(I - \lambda\omega_H\Pi_{\mathcal{F}_H})^{-1} D_\Gamma f &= D^2_\Gamma f + \lambda D_\Gamma(I - \lambda\omega_H\Pi_{\mathcal{F}_H})^{-1}\omega_H D_\Gamma f \\
&= D^2_\Gamma f + \lambda D_\Gamma(I - \lambda\omega_H\Pi_{\mathcal{F}_H})^{-1} D_\Gamma\omega_2 f.
\end{aligned}$$

These two computations and the fact that $\Gamma^*\Gamma + D^2_\Gamma = I$ show that the second identity in (4.1.7) is satisfied. □

Note that Theorem 4.1.1 implies that a solution to the H^2 interpolation problem defined by ω always exists. Indeed, the constant function $Z(\lambda) \equiv \omega\Pi_{\mathcal{F}}$ is in the Schur class $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and obviously satisfies $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. In particular, a solution to the H^2 interpolation problem defined by ω is given by

$$H_c(\lambda) = \omega_1\Pi_{\mathcal{F}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} \quad (\lambda \in \mathbb{D}). \tag{4.1.10}$$

This solution H_c will be referred to as the *central solution to the H^2 interpolation problem defined by* ω, or just the *central solution* when no confusion concerning the contraction ω in question can arise.

In the following example we compute the central solution for a concretely given contraction ω of the form (4.1.1).

Example 4.1.4. We consider the general H^2 interpolation problem described above for the following data:

$$\mathcal{U} = \mathbb{C}^2, \quad \mathcal{Y} = \mathbb{C}, \quad \mathcal{F} = \mathbb{C} = \mathbb{C} \oplus \{0\} \quad \text{and} \quad \omega = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}. \tag{4.1.11}$$

Note that $\Pi_{\mathcal{F}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Hence the central solution H_c is given by

$$\begin{aligned} H_c(\lambda) &= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{2-\lambda}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \qquad (\lambda \in \mathbb{D}) \\ &= \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}. \end{aligned}$$

So the Taylor coefficients $H_0, H_1, \ldots$ of H_c at zero are given by $H_n = \begin{bmatrix} 2^{-n-1} & 0 \end{bmatrix}$ for $n = 0, 1, \ldots$. In particular, the norm of the operator Γ_c defined by H_c is $\|\Gamma_c\|^2 = \sum_{n=1}^{\infty} 4^{-n} = \frac{1}{3}$. $\diamond$

Next we consider the question when there exists a unique solution to the H^2 interpolation problem defined by ω. Necessary and sufficient conditions under which this happens are at this point not known. So we have the following open problem.

Open Problem 4.1.5. *Find necessary and sufficient conditions for the existence of a unique solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1).

Some sufficient conditions are simple to obtain. Indeed, each of the conditions

(a) $\mathcal{Y} = \{0\}$,

(b) $\mathcal{F} = \mathcal{U}$,

(c) ω is a co-isometry,

guarantees the existence of a unique solution to the H^2 interpolation problem defined by ω. Note that if $\mathcal{Y} = \{0\}$, then $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ consists of the zero function only, and hence in that case the existence of a unique solution is trivial. The fact that the same conclusion holds if (b) or (c) is satisfied follows directly from Theorem 4.1.1 and the following proposition.

Proposition 4.1.6. *Let ω be a contraction as in* (4.1.1). *Then there exists a unique $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if $\mathcal{F} = \mathcal{U}$ or ω is a co-isometry.*

Proof. If $\mathcal{F} = \mathcal{U}$, then trivially there can be only one $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. Next consider the case when ω is a co-isometry. Put $\mathcal{G} = \mathcal{U} \ominus \mathcal{F}$, and let $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ be such that $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. Since $D_{\omega^*} = 0$, we can use a variation on Douglas factorization lemma (see, e.g., Corollary XXVII.5.3 in [55]) to show that $Z(\lambda)|_{\mathcal{G}} \equiv 0$. It follows that $Z(\lambda) \equiv \omega \Pi_{\mathcal{F}}$, and hence Z is uniquely determined.

Conversely, assume that neither $\mathcal{F} = \mathcal{U}$ nor that ω is a co-isometry. Than $\mathcal{G} = \mathcal{U} \ominus \mathcal{F} \neq \{0\}$ and $\mathcal{D}_{\omega^*} \neq \{0\}$. Hence $\mathscr{L}(\mathcal{G}, \mathcal{D}_{\omega^*})$ consists of more than one element. Again using Corollary XXVII.5.3 in [55], we obtain that for any operator $X \in \mathscr{L}(\mathcal{G}, \mathcal{D}_{\omega^*})$ the constant function $Z(\lambda) \equiv \begin{bmatrix} \omega & D_{\omega^*} X \end{bmatrix}$ is in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ and satisfies $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. So there is more than one $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ satisfying $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. $\square$

The above proposition can also be derived as an immediate corollary of Lemma 2.4.5. In fact, by applying Lemma 2.4.5 with $C = \omega$ from $\mathcal{F}$ to $\mathcal{Y} \oplus \mathcal{U}$ we see that $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ and $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if

$$Z(\lambda) = \omega\Pi_{\mathcal{F}} + D_{\omega^*}V(\lambda)\Pi_{\mathcal{G}}, \quad (\lambda \in \mathbb{D}), \quad \text{where} \quad \mathcal{G} = \mathcal{U} \ominus \mathcal{F},$$

for some $V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$. Thus there is a unique $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{Y})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if $\mathcal{F} = \mathcal{U}$ or ω is a co-isometry.

If ω in (4.1.1) is a co-isometry, then ω^* is an isometry from $\mathcal{Y} \oplus \mathcal{U}$ into $\mathcal{F}$, and hence

$$\dim(\mathcal{Y} \oplus \mathcal{U}) \leq \dim \mathcal{F} \leq \dim \mathcal{U}.$$

Thus if $\mathcal{U}$ is finite dimensional, then condition (c) above implies that conditions (a) and (b) hold as well.

In the following example we return to the H^2 interpolation problem considered in Example 4.1.4, and we construct a solution different from the central solution.

Example 4.1.7. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). Take for $Z \in \mathbf{S}(\mathbb{C}^2, \mathbb{C}^3)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ the function

$$Z(\lambda) \equiv \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6}\sqrt{3}a(\lambda) \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad a \in \mathbf{S}(\mathbb{C}, \mathbb{C}).$$

Then the solution H of the H^2 interpolation problem defined by ω determined by Z via the Schur representation (4.1.6) is given by

$$\begin{aligned} H(\lambda) &= \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} \frac{1}{2} & \frac{1}{6}\sqrt{3}a(\lambda) \\ 0 & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & -\lambda\frac{1}{6}\sqrt{3}a(\lambda) \\ 0 & 1 \end{bmatrix}^{-1} \qquad (\lambda \in \mathbb{D}) \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{2-\lambda} & \frac{\lambda\sqrt{3}a(\lambda)}{3(2-\lambda)} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2-\lambda} & \frac{\lambda\sqrt{3}a(\lambda)}{6(2-\lambda)} \end{bmatrix}. \end{aligned}$$

In particular, we see that H is different from the central solution unless $a(\lambda) \equiv 0$.$\diamond$

The contraction ω_H defined in Lemma 4.1.3 will play an important role in the next section. Note that the identity (4.1.9) can be reformulated as:

$$\|D_{\omega_H} D_{\Gamma} f\| = \|D_{\omega} f\| \quad (f \in \mathcal{F}). \tag{4.1.12}$$

The latter identity gives us the following result.

Proposition 4.1.8. *Let H be a solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1)*, let Γ be the operator defined by H, and let ω_H be the contraction given by* (4.1.8)*. Then*

$$\|\Gamma|_{\mathcal{F}}\| \leq \|\omega\|, \quad \|\omega_H\| \leq \|\omega\|, \quad \|\omega\| \leq (1-(1-\|\omega_H\|^2)(1-\|\Gamma|_{\mathcal{F}}\|^2))^{\frac{1}{2}}. \quad (4.1.13)$$

Proof. Note that for an arbitrary contraction X from $\mathcal{V}$ to $\mathcal{W}$ the inequality $\|D_X v\|^2 \geq (1-\|X\|^2)\|v\|^2$ holds for each $v \in \mathcal{V}$. Conversely, if $\|D_X v\|^2 \geq (1-\delta^2)\|v\|$ holds for all $v \in \mathcal{V}$, then $\|X\| \leq \delta$. So from the identity (4.1.12) we can derive the following sequence of inequalities:

$$\|D_\Gamma f\|^2 \geq \|D_{\omega_H} D_\Gamma f\|^2 = \|D_\omega f\|^2 \geq (1-\|\omega\|^2)\|f\|^2 \geq (1-\|\omega\|^2)\|D_\Gamma f\|^2 \quad (f \in \mathcal{F}).$$

Comparing the first and last but one term, and the second and last term in this sequence we obtain the first two inequalities in (4.1.13). Next observe that

$$\|D_\omega f\|^2 = \|D_{\omega_H} D_\Gamma f\|^2 \geq (1-\|\omega_H\|^2)(1-\|\Gamma|_{\mathcal{F}}\|^2)\|f\|^2 \quad (f \in \mathcal{F}).$$

So the fact that $(1-\|\omega_H\|^2)(1-\|\Gamma|_{\mathcal{F}}\|^2) = (1-(1-(1-\|\omega_H\|^2)(1-\|\Gamma|_{\mathcal{F}}\|^2)))$ shows that also the last inequality in (4.1.13) holds. □

The next corollary follows immediately from Proposition 4.1.8.

Corollary 4.1.9. *Let ω in* (4.1.1) *be a strict contraction and H is an arbitrary solution to the H^2 interpolation problem defined by ω. Then $\Gamma|_{\mathcal{F}}$ and ω_H are strict contractions as well. Conversely, if for some solution $\tilde{H}$ to the H^2 interpolation problem defined by ω the operators $\tilde{\Gamma}|_{\mathcal{F}}$ and $\omega_{\tilde{H}}$ are strict contractions, then ω is a strict contraction, and hence $\Gamma|_{\mathcal{F}}$ and ω_H are strict contractions for any solution H to the H^2 interpolation problem defined by ω. Here $\tilde{\Gamma}$ and Γ are the operators defined by $\tilde{H}$ and H, respectively.*

The result of Proposition 4.1.8 is illustrated by the following example.

Example 4.1.10. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). Moreover, let H_c be the central solution to the H^2 interpolation problem defined by ω, that is, H_c is the function given by $H_c(\lambda) \equiv \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}$. From the results of Example 4.1.4 we see that $\|\omega\| = \frac{1}{2}\sqrt{2}$ and $\|\Gamma_c\| = \frac{1}{3}\sqrt{3}$, where Γ_c is the operator defined by H_c. Moreover, we have that $D_{\Gamma_c} = \operatorname{diag}(\frac{1}{3}\sqrt{6}, 1)$. Since $D_{\Gamma_c}\mathcal{F} = \mathcal{F}$ and $\omega_2\mathcal{F} = \mathcal{F}$, we have $\mathcal{F}_{H_c} = \mathcal{F}$ and $\omega_{H_c} = \omega_2$. So $\|\omega_{H_c}\| = \frac{1}{2}$. We check:

$$\|\Gamma_c|_{\mathcal{F}}\| = \tfrac{1}{3}\sqrt{3} < \tfrac{1}{2}\sqrt{2} = \|\omega\|, \quad \|\omega_{H_c}\| = \tfrac{1}{2} < \sqrt{2} = \|\omega\|,$$

and

$$\begin{aligned} \|\omega\| &= \tfrac{1}{2}\sqrt{2} = (1-\tfrac{3}{4}\tfrac{2}{3})^{\frac{1}{2}} = (1-(1-\tfrac{1}{4})(1-\tfrac{1}{3}))^{\frac{1}{2}} \\ &= (1-(1-\|\omega_{H_c}\|^2)(1-\|\Gamma_c|_{\mathcal{F}}\|^2))^{\frac{1}{2}}. \end{aligned}$$

So the inequalities in (4.1.13) hold. In fact, since we have equalities in each of the steps of the last computation, we obtain that the last inequality in (4.1.13) provides a sharp bound for $\|\omega\|$. ◇

4.2 Non-uniqueness in the Schur representation of a given solution

Let $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y})$ be a fixed solution to the H^2 interpolation problem defined by the contraction ω in (4.1.1). In this section we study the set of all $Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ such that (4.1.6) holds and $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$, that is, the set

$$\mathbf{Z}_{H,\omega} := \{Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U}) \mid H \text{ is given by (4.1.6) and } Z(\lambda)|_{\mathcal{F}} \equiv \omega\}. \qquad (4.2.1)$$

From Theorem 4.1.1 we know that the set $\mathbf{Z}_{H,\omega}$ is not empty. The next example shows that the set $\mathbf{Z}_{H,\omega}$ may consist of more than one element.

Example 4.2.1. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). Take for $Z \in \mathbf{S}(\mathbb{C}^2,\mathbb{C}^3)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ the constant function

$$Z(\lambda) \equiv \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & b(\lambda) \end{bmatrix}, \quad \text{where} \quad b \in \mathbf{S}(\mathbb{C},\mathbb{C}).$$

Then the solution H of the H^2 interpolation problem defined by ω determined by Z via the Schur representation (4.1.6) is given by

$$\begin{aligned} H(\lambda) &= \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda\Pi_{\mathcal{U}} Z(\lambda))^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & b(\lambda) \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & 0 \\ 0 & 1-\lambda b(\lambda) \end{bmatrix}^{-1} \qquad\qquad (\lambda \in \mathbb{D}) \\ &= \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{2-\lambda} & 0 \\ 0 & \frac{1}{1-\lambda b(\lambda)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}. \end{aligned}$$

In particular, we obtain that the solution H coincides with the central solution H_c, independently of the choice of $b \in \mathbf{S}(\mathbb{C},\mathbb{C})$. ◇

The following theorem provides a proper parameterization of the set $\mathbf{Z}_{H,\omega}$.

Theorem 4.2.2. *Assume that H is a solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1)*. Let Γ be the operator defined by H and ω_H the contraction given by* (4.1.8)*. Moreover, let J_H be the map from $\mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ into $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ defined in Theorem* 3.2.2*, that is, for each $K \in \mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ the function $J_H K \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ is given by*

$$\begin{aligned} (J_H K)(\lambda) &= \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda) - I) \end{bmatrix} F(\lambda)^{-1}, \\ F(\lambda) &= \Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1} D_\Gamma. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Put

$$\mathbf{K}_{\omega_H} := \big\{K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma) \mid K(\lambda)|_{\mathcal{F}_H} = \omega_H \text{ for each } \lambda \in \mathbb{D}\big\}. \tag{4.2.2}$$

Then J_H maps $\mathbf{K}_{\omega_H}$ in a one-to-one way onto the set $\mathbf{Z}_{H,\omega}$.

From Propositions 4.1.2 and 3.2.4 we know that there is a one-to-one correspondence between the set (4.2.1) and the set of all positive real functions F satisfying (4.1.5) and such that the second identity in (4.1.7) holds. Moreover, according to Proposition 3.2.4, this one-to-one correspondence is established by the map $F \mapsto Z$ in (3.2.6). Comparing (3.2.6) with the map J_H defined in Theorem 3.2.2, we see that Theorem 4.2.2 is an immediate consequence of the following result.

Theorem 4.2.3. *Let H be a solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1)*, and let Γ be the operator defined by H. Given K in the set $\mathbf{K}_{\omega_H}$ in* (4.2.2)*, put*

$$F_K(\lambda) = \Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma + D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma \quad (\lambda \in \mathbb{D}). \tag{4.2.3}$$

Then the map $K \mapsto F_K$ is a one-to-one map from the set $\mathbf{K}_{\omega_H}$ onto the set of positive real functions F satisfying (4.1.5) *and such that the second identity in* (4.1.7) *holds.*

Proof. We already observed in Theorem 3.2.5 that the map $K \mapsto F_K$ in (4.2.3) is a one-to-one map from the Schur class $\mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ onto the set of positive real functions F satisfying (4.1.5). So it remains to show that for a $K \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ we have that $K \in \mathbf{K}_{\omega_H}$ if and only if the function F_K satisfies the second identity in (4.1.7).

Fix $\lambda \in \mathbb{D}$. We have

$$\begin{aligned} \Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma &= \Gamma^*\Gamma + \lambda\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}S_{\mathcal{Y}}^*\Gamma, \\ D_\Gamma(I - \lambda K(\lambda))^{-1}D_\Gamma &= D_\Gamma^2 + \lambda D_\Gamma(I - \lambda K(\lambda))^{-1}K(\lambda)D_\Gamma. \end{aligned}$$

Hence

$$F_K(\lambda) = I + \lambda\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}S_{\mathcal{Y}}^*\Gamma + \lambda D_\Gamma(I - \lambda K(\lambda))^{-1}K(\lambda)D_\Gamma,$$

and for each $f \in \mathcal{F}$ we get

$$\begin{aligned} F_K(\lambda)f - \lambda F_K(\lambda)\omega_2 f &= f + \lambda\Gamma^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}(S_{\mathcal{Y}}^*\Gamma f - \Gamma\omega_2 f) \\ &\quad + \lambda D_\Gamma(I - \lambda K(\lambda))^{-1}(K(\lambda)D_\Gamma f - D_\Gamma\omega_2 f). \end{aligned}$$

The fact that condition (4.1.3) is fulfilled implies that $S_{\mathcal{Y}}^*\Gamma f = \Gamma\omega_2 f$ for each $f \in \mathcal{F}$. So in order that F_K satisfies the second identity in (4.1.7) for our fixed λ it is necessary and sufficient that $K(\lambda)D_\Gamma f = D_\Gamma\omega_2 f$ for each $f \in \mathcal{F}$, or equivalently, $K(\lambda)|_{\mathcal{F}_H} = \omega_H$. □

Corollary 4.2.4. *Given $\mathcal{F} \subset \mathcal{U}$ and $\tau : \mathcal{F} \to \mathcal{U}$ a contraction. Then all positive real functions F with $F(0) = I_{\mathcal{U}}$ and satisfying*

$$2\,\mathrm{Re}\,F(\lambda) \geq I \quad \text{and} \quad \Pi_{\mathcal{F}}^* + \lambda F(\lambda)\tau = F(\lambda)|_{\mathcal{F}} \quad (\lambda \in \mathbb{D})$$

are given by

$$F(\lambda) = (I - \lambda K(\lambda))^{-1} \quad (\lambda \in \mathbb{D}),$$

for some $K \in \mathbf{S}(\mathcal{U},\mathcal{U})$ *with* $K(\lambda)|_{\mathcal{F}} \equiv \tau$.

Proof. Apply Theorem 4.2.3 to the case $\mathcal{Y} = \{0\}$, $\omega = \tau$ and $H(\lambda) \equiv 0$. Note that in this case $H(\lambda) \equiv 0$ is the unique solution to the H^2 interpolation problem defined by ω. □

In the following example we revisit the H^2 interpolation problem considered in Example 4.1.4.

Example 4.2.5. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). Moreover, let H_c be the central solution to the H^2 interpolation problem defined by ω, that is, H_c is the funtion given by $H_c(\lambda) \equiv \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}$. In this example we use Theorem 4.2.2 to construct a Schur class function Z in $\mathbf{Z}_{H_c,\omega}$ other than the 'central choice' $Z(\lambda) \equiv \omega\Pi_{\mathcal{F}}$.

Let $K \in \mathbf{S}(\mathbb{C}^2, \mathbb{C}^2)$ be given by

$$K(\lambda) \equiv \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3}a(\lambda) \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad a \in \mathbf{S}(\mathbb{C}, \mathbb{C}).$$

Since the contraction ω_H defined in Lemma 4.1.3 is equal to ω_2 (see Example 4.1.10), we see that $K \in \mathbf{K}_{\omega_{H_c}}$. Now let Γ_c be the operator defined by H_c. Then $\Gamma_c^*\Gamma_c = \operatorname{diag}(\frac{1}{3}, 0)$ and $D_{\Gamma_c} = \operatorname{diag}(\frac{1}{3}\sqrt{6}, 1)$. So the positive real function F_K in (4.2.3) is given by

$$F_K(\lambda) = \begin{bmatrix} \frac{2}{2-\lambda} & \frac{\lambda a(\lambda)\sqrt{2}}{2-\lambda} \\ 0 & 1 \end{bmatrix}, \text{ and thus } F_K(\lambda)^{-1} = \begin{bmatrix} \frac{2-\lambda}{2} & \frac{-\lambda a(\lambda)\sqrt{2}}{2} \\ 0 & 1 \end{bmatrix} \quad (\lambda \in \mathbb{D}).$$

We obtain that the Schur class function $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ determined by H and F_K is given by

$$Z(\lambda) = \begin{bmatrix} H(\lambda)F_K(\lambda)^{-1} \\ \lambda^{-1}(I - F_K(\lambda)^{-1}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\lambda a(\lambda)\sqrt{2}}{2(2-\lambda)} \\ \frac{1}{2} & \frac{a(\lambda)\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} \quad (\lambda \in \mathbb{D}).$$

It is immediately clear that $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. Moreover

$$\begin{aligned} \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} &= \begin{bmatrix} \frac{1}{2} & \frac{-\lambda a(\lambda)\sqrt{2}}{2(2-\lambda)} \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & \frac{-\lambda a(\lambda)\sqrt{2}}{2} \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{-\lambda a(\lambda)\sqrt{2}}{2(2-\lambda)} \end{bmatrix} \begin{bmatrix} \frac{2}{2-\lambda} & \frac{\lambda a(\lambda)\sqrt{2}}{2-\lambda} \\ 0 & 1 \end{bmatrix} \quad (\lambda \in \mathbb{D}) \\ &= \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix} = H_c(\lambda). \end{aligned}$$

So indeed $Z \in \mathbf{Z}_{H_c,\omega}$. ◇

Fix a solution H to the H^2 interpolation problem defined by ω. Let Z be in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ be such that

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{Y}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}).$$

Then, in general, it is not the case that $Z(\lambda)|_{\mathcal{F}} \equiv \omega$. This fact is illustrated by the next example.

Example 4.2.6. Consider the general H^2 interpolation problem with data

$$\mathcal{Y} = \mathbb{C}, \quad \mathcal{U} = \mathbb{C}^2, \quad \mathcal{F} = \mathbb{C} = \mathbb{C} \oplus \{0\} \quad \text{and} \quad \omega = 0 : \mathcal{F} \to \mathbb{C}^3.$$

Then the central solution is the zero function $H_c(\lambda) \equiv 0$. In particular, we see that H_c is obtained by the Schur representations corresponding to the Schur class functions Z_1 and Z_2 given by

$$Z_1(\lambda) \equiv \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Z_2(\lambda) \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We immediately observe that $Z_1(\lambda)|_{\mathcal{F}} \not\equiv \omega$, while $Z_2(\lambda)|_{\mathcal{F}} \equiv \omega$. Now let F_1 and F_2 be the positive real functions defined by Z_1 and Z_2, respectively, via the second formula in (4.1.4). That is,

$$\begin{aligned} F_1(\lambda) &= (I - \lambda \Pi_{\mathcal{U}} Z_1(\lambda))^{-1} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{1-\lambda} & 0 \\ 0 & 1 \end{bmatrix}, \\ F_2(\lambda) &= (I - \lambda \Pi_{\mathcal{U}} Z_2(\lambda))^{-1} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (\lambda \in \mathbb{D})$$

So we have $F_1(\lambda)|_{\mathcal{F}} \not\equiv \Pi^*_{\mathcal{F}}$ and $F_2(\lambda)|_{\mathcal{F}} \equiv \Pi^*_{\mathcal{F}}$, as could be expected from Proposition 4.1.2. ◇

Let H_c be the central solution given by (4.1.10). Then the map J_{H_c} defined in Theorem 3.2.1 maps the 'central choice' $K(\lambda) \equiv \omega_{H_c} \Pi_{\mathcal{F}_{H_c}}$ in $\mathbf{K}_{\omega_H}$ to $Z(\lambda) \equiv \omega \Pi_{\mathcal{F}}$. To see this, let Γ_c be the operator defined by H_c. Observe that $S^*_{\mathcal{Y}} \Gamma_c = \Gamma_c \omega_2 \Pi_{\mathcal{F}}$, while $\omega_{H_c} \Pi_{\mathcal{F}_{H_c}} D_{\Gamma_c} = D_{\Gamma_c} \omega_2 \Pi_{\mathcal{F}}$. Hence for the positive real function F given by (4.2.3) we have

$$\begin{aligned} F(\lambda) &= \Gamma_c^*(I - \lambda S^*_{\mathcal{Y}})^{-1} \Gamma_c + D_{\Gamma_c}(I - \lambda \omega_{H_c} \Pi_{\mathcal{F}_{H_c}})^{-1} D_{\Gamma_c} \\ &= (\Gamma_c^* \Gamma_c + D^2_{\Gamma_c})(I - \lambda \omega_2 \Pi_{\mathcal{F}})^{-1} = (I - \lambda \omega_2 \Pi_{\mathcal{F}})^{-1}. \end{aligned} \quad (\lambda \in \mathbb{D})$$

Thus $Z = J_{H_c} K$ is given by

$$Z(\lambda) \equiv \begin{bmatrix} H_c(\lambda) F(\lambda)^{-1} \\ \lambda^{-1}(I - F(\lambda)^{-1}) \end{bmatrix} \equiv \begin{bmatrix} \omega_1 \Pi_{\mathcal{F}}(I - \lambda \omega_2 \Pi_{\mathcal{F}})^{-1}(I - \lambda \omega_2 \Pi_{\mathcal{F}}) \\ \lambda^{-1}(I - I + \lambda \omega_2 \Pi_{\mathcal{F}}) \end{bmatrix} \equiv \omega \Pi_{\mathcal{F}}.$$

This proves our claim.

Next we consider the question when the representation of all solutions to the H^2 interpolation problem defined by ω given in Theorem 4.1.1 provides a proper parameterization. In other words: When does the set $\mathbf{Z}_{H,\omega}$ consists of one element only for each solution H to the H^2 interpolation problem defined by ω? Necessary and sufficient conditions under which this happen are up to now not known. So we have the following open problem.

Open Problem 4.2.7. *Find necessary and sufficient conditions for the Schur representation of all solutions to the H^2 interpolation problem defined by the contraction ω in* (4.1.1) *given in Theorem* 4.1.1 *to provide a proper parameterization.*

The next proposition presents some conditions under which Theorem 4.1.1 does provide a proper parameterization.

Proposition 4.2.8. *Let ω be a contraction as in* (4.1.1). *Assume that one of the following conditions is satisfied:*

(a) $\mathcal{F} = \mathcal{U}$,

(b) *ω is a co-isometry,*

(c) *ω is an isometry and $\overline{\omega_2\mathcal{F}} = \mathcal{U}$.*

Then for each solution H to the H^2 interpolation problem defined by ω the set $\mathbf{Z}_{H,\omega}$ is a singleton.

Proof. We already observed in Proposition 4.1.6 that there exists a unique function Z in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if condition (a) or (b) holds. In particular, in that case $\mathbf{Z}_{H_c,\omega}$ is a singleton.

Now assume that condition (c) holds. Let H be a solution to the H^2 interpolation problem defined by ω. We claim that the contraction ω_H is unitary. Indeed, the fact that ω is an isometry implies that ω_H is an isometry; see Lemma 4.1.3. Furthermore, we have

$$\omega_H \mathcal{F}_H = \overline{D_\Gamma \omega_2 \mathcal{F}} = \overline{D_\Gamma \mathcal{U}} = \mathcal{D}_\Gamma,$$

where Γ is the operator defined by H. This proves our claim.

In particular, the operator ω_H is a co-isometry. Using a similar argument as in the proof of Proposition 4.1.6 we see that each Schur class function K in $\mathbf{K}_{\omega_H}$ has the property $K(\lambda)|_{\mathcal{G}_H} \equiv 0$, where $\mathcal{G}_H = \mathcal{D}_\Gamma \ominus \mathcal{F}_H$. So $\mathbf{K}_{\omega_H}$ consists of one element only. Then Theorem 4.2.2 implies that $\mathbf{Z}_{H,\omega}$ is a singleton. □

Note that in case $\overline{\omega_2\mathcal{F}} = \mathcal{U}$, we have $\dim \mathcal{U} \geq \dim \mathcal{F} \geq \dim \overline{\omega_2\mathcal{F}} = \dim \mathcal{U}$. So if $\mathcal{U}$ is finite dimensional, $\overline{\omega_2\mathcal{F}} = \mathcal{U}$ implies that $\mathcal{F} = \mathcal{U}$. In particular, if $\dim \mathcal{U} < \infty$, then condition (c) in Proposition 4.2.8 implies that condition (a) holds as well, and thus, that there exists a unique solution to the H^2 interpolation problem defined by ω; see Proposition 4.1.6.

Remark 4.2.9. Theorems 4.1.1 and 4.2.2 can be seen as further specifications of Theorems 3.2.1 and 3.2.2, respectively. On the other hand, by taking $\mathcal{F} = \{0\}$ and $\omega = 0$ from $\{0\}$ to $\mathcal{U} \oplus \mathcal{Y}$ Theorems 4.1.1 and 4.2.2 in turn reduce to Theorems 3.2.1 and 3.2.2, respectively.

4.3 The relaxed commutant lifting problem as a general H^2 interpolation problem

In this section we show how the relaxed commutant lifting problem can be viewed as a special case of the general H^2 interpolation problem considered in Section 4.1. Let $\{A, T', U', R, Q\}$ be a lifting data set, and recall that the contraction ω underlying $\{A, T', U', R, Q\}$ is defined by

$$\omega : \mathcal{F} = \overline{D_A Q \mathcal{H}_0} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix}, \quad \omega D_A Q = \begin{bmatrix} D_{T'} AR \\ D_A R \end{bmatrix}.$$

This ω proves to be precisely the contraction needed for the general H^2 interpolation problem of Section 4.1, as is made precise in the following proposition.

Proposition 4.3.1. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Let ω be the underlying contraction for $\{A, T', U', R, Q\}$. Then an operator B from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is a contractive interpolant for the lifting data set $\{A, T', U', R, Q\}$ if and only if B is given by*

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \tag{4.3.1}$$

where Γ is a solution to the H^2 interpolation problem defined by ω. Furthermore B and Γ in (4.3.1) *determine each other uniquely.*

Proof. Let B be an operator from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$. We already observed in the introduction that B is a contraction satisfying $\Pi_{\mathcal{H}'} B = A$ if and only if B is given by (1.20), that is,

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \quad \Gamma : \mathcal{D}_A \to H^2(\mathcal{D}_{T'}) \text{ a contraction.} \tag{4.3.2}$$

Moreover, B and Γ in (4.3.2) define each other uniquely. Now with B given by (4.3.2) and U' by (1.19), we obtain

$$U'BR = \begin{bmatrix} T' & 0 \\ E_{\mathcal{D}_{T'}} D_{T'} & S_{\mathcal{D}_{T'}} \end{bmatrix} \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} R = \begin{bmatrix} T'AR \\ E_{\mathcal{D}_{T'}} D_{T'} AR + S_{\mathcal{D}_{T'}} \Gamma D_A R \end{bmatrix},$$

while

$$BQ = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} Q = \begin{bmatrix} AQ \\ \Gamma D_A Q \end{bmatrix}.$$

Since $T'AR = AQ$ by assumption, the operator B is a contractive interpolant for $\{A, T', U', R, Q\}$ if and only if B is given by (4.3.2) with Γ satisfying the equation

$$E_{\mathcal{D}_{T'}} D_{T'} AR + S_{\mathcal{D}_{T'}} \Gamma D_A R = \Gamma D_A Q. \tag{4.3.3}$$

Using the definition of the underlying contraction ω we see that (4.3.3) is equivalent to

$$E_{\mathcal{D}_{T'}} \omega_1 + S_{\mathcal{D}_{T'}} \Gamma \omega_2 = \Gamma|_{\mathcal{F}},$$

where ω_1 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_{T'}$ determined by the first component of ω and ω_2 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_A$ determined by the second component of ω. Thus B is a contractive interpolant for $\{A, T', U', R, Q\}$ if and only if B is of the form (4.3.1) with Γ a solution to the H^2 interpolation problem defined by the contraction ω underlying the lifting data set $\{A, T', U', R, Q\}$. □

We conclude this section with a translation of some of the results of Sections 4.1 and 4.2 to the relaxed commutant lifting setting. We start with a proof of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. Theorem 1.1 follows immediately from Proposition 4.3.1 and Theorem 3.2.1. To see that Theorem 1.2 holds, it remains to show that the set $\mathbf{K}_{\omega_H}$ in (4.2.2) coincides with the set $\mathbf{K}_{\omega_\Gamma}$ in (1.28) when we take for ω the contraction underlying the lifting data set $\{A, T', U', R, Q\}$. To see that this is the case it suffices to show that the contraction ω_Γ in (1.27) is equal to the contraction ω_H in (4.1.8). But this follows immediately from their definitions and the fact that $\mathcal{F} = \overline{D_A Q \mathcal{H}_0}$ and $\omega_2 D_A Q = D_A R$. □

Note that the central solution, given in (4.1.10), to the H^2 interpolation problem defined by the contraction ω underlying the lifting data set $\{A, T', U', R, Q\}$ corresponds to the central contractive interpolant for $\{A, T', U', R, Q\}$.

From the remarks in the paragraph after Open Problem 4.1.5 we obtain that there exists a unique contractive interpolant if one of the following conditions is satisfied:

(a) T' is an isometry,

(b) $\overline{D_A Q \mathcal{H}_0} = \mathcal{D}_A$,

(c) ω is a co-isometry.

Moreover, there exists a unique $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if condition (b) or (c) is met; see Proposition 4.1.6. In particular, the conditions (b) and (c) imply that the representation of all contractive interpolants given in Theorem 1.1 provides a proper parameterization. Using the fact that ω is an isometry if and only if $R^*R = Q^*Q$, we obtain from Proposition 4.2.8 that Theorem 1.1 also provides a proper parameterization whenever

$$R^*R = Q^*Q \quad \text{and} \quad \overline{D_A R \mathcal{H}_0} = \mathcal{D}_A.$$

Classical commutant lifting. A case of special interest is the classical commutant lifting problem. As we know from [42] the classical commutant lifting problem appears when the lifting data set $\{A, T', U', R, Q\}$ has $\mathcal{H}_0 = \mathcal{H}$, $R = I_{\mathcal{H}}$ and Q an isometry. In this case, $R^*R = Q^*Q$ and $\overline{D_A R \mathcal{H}_0} = \overline{D_A \mathcal{H}} = \mathcal{D}_A$. So when specified for the classical commutant lifting problem the representation of all contractive interpolants in Theorem 1.1 does provide a proper parameterization. In particular, in the classical commutant lifting setting there exists a unique contractive interpolant if and only if there exists a unique $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$, where ω from $\mathcal{F} = \overline{D_A Q \mathcal{H}_0}$ to $\mathcal{D}_{T'} \oplus \mathcal{D}_A$ is the contraction underlying $\{A, T', U', R, Q\}$. The latter happens if and only if condition (b) or (c) above holds; see Proposition 4.1.6. Recall that $R^*R = Q^*Q$ implies that ω is an isometry. It then follows that for a $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ to have $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ it suffices to check whether $Z(0)|_{\mathcal{F}} = \omega$. In this way we recover the result of Theorem XIII.3.4 in [41].

Theorem 4.3.2. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' the Sz.-Nagy-Schäffer isometric lifting of T', $\mathcal{H}_0 = \mathcal{H}$, $R = I_{\mathcal{H}}$ and Q an isometry. Then all contractive interpolants for $\{A, T', U', R, Q\}$ are given by*

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \tag{4.3.4}$$

where Γ is the operator defined by the function $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{D}_A, \mathcal{D}_{T'})$ which is given by

$$H(\lambda) = \Pi_{\mathcal{D}_{T'}} Z(\lambda)(I - \lambda \Pi_{\mathcal{D}_A} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}) \tag{4.3.5}$$

with Z an arbitrary Schur class function in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ satisfying $Z(0)|_{\mathcal{F}} = \omega$. Here $\mathcal{F} = \overline{D_A Q \mathcal{H}_0}$ and ω is the contraction underlying the lifting data set $\{A, T', U', R, Q\}$. The solution B and the Schur class function Z in (4.3.4) *and* (4.3.5) *determine each other uniquely. Finally, there exists only one contractive interpolant for $\{A, T', U', R, Q\}$ if and only in $\mathcal{F} = \mathcal{D}_A$ or ω is unitary.*

4.4 The general H^2 interpolation problem in the relaxed commutant lifting setting

In the previous section we showed how the relaxed commutant lifting problem can be seen as a general H^2 interpolation problem. Conversely, each H^2 interpolation problem defined by a contraction ω as in (4.1.1) also fits into the relaxed commutant lifting setting. To be more precise, each contraction ω as in (4.1.1) appears as the underlying contraction of a lifting data set.

Proposition 4.4.1. *Let ω be a contraction as in* (4.1.1)*. Put*

$$\tilde{A} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{T}' = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

$$\tilde{R} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 \\ \Pi^*_{\mathcal{F}} \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

$$\tilde{U}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{U}} & 0 \\ E_{\mathcal{Y}} & 0 & S_{\mathcal{Y}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix}.$$

Then $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ is a lifting data set, and the underlying contraction is precisely the given contraction ω. Furthermore, $\tilde{U}'$ is the Sz-Nagy-Schäffer isometric lifting of $\tilde{T}'$.

Proof. The operators $\tilde{A}$ and $\tilde{T}'$ are orthogonal projections and hence contractions. Observe that $\tilde{T}'\tilde{A}$ and $\tilde{A}\tilde{Q}$ are both zero operators. Furthermore, note that $\tilde{R} = \omega$ is a contraction defined on $\mathcal{F}$ and $\tilde{Q}^*\tilde{Q}$ is the identity operator on $\mathcal{F}$. From these remarks we see that

$$\tilde{T}'\tilde{A}\tilde{R} = \tilde{A}\tilde{Q} \quad \text{and} \quad \tilde{R}^*\tilde{R} \leq \tilde{Q}^*\tilde{Q}.$$

Next, observe that

$$D_{\tilde{T}'} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.$$

Thus we can identify $\mathcal{D}_{\tilde{T}'}$ with the space $\mathcal{Y}$. With this identification in mind it is straightforward to check that $\tilde{U}'$ is the Sz-Nagy-Schäffer isometric lifting of $\tilde{T}'$. It follows that $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ is a lifting data set. Notice that in this case the space $\mathcal{H}_0$ appearing in the definition of a lifting data set is equal to the space $\mathcal{F}$. Using

$$D_{\tilde{A}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

we see that

$$D_{\tilde{A}}\tilde{Q} = \tilde{Q}, \quad D_{\tilde{T}'}\tilde{A}\tilde{R} = \omega_1, \quad D_{\tilde{A}}\tilde{R} = \omega_2.$$

So that the underlying contraction $\tilde{\omega}$ for the lifting data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ is given by

$$\tilde{\omega} : \tilde{\mathcal{F}} = \overline{\operatorname{Im} \tilde{Q}} = \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{\omega} = \tilde{\omega}\Pi_{\mathcal{F}^*} = \tilde{\omega} D_{\tilde{A}}\tilde{Q} = \begin{bmatrix} D_{\tilde{T}'}\tilde{A}\tilde{R} \\ D_{\tilde{A}}\tilde{R} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \omega.$$

Thus ω is the contraction underlying $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$. □

The next proposition now follows immediately from Propositions 4.4.1 and 4.3.1.

Proposition 4.4.2. *Let ω be a contraction as in* (4.1.1)*, and let $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ be the lifting data set constructed in Proposition* 4.4.1*. Then Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ is a solution to the H^2 interpolation problem defined by the contraction ω if and only if*

$$\tilde{B} = \begin{bmatrix} \tilde{A} \\ \Gamma D_{\tilde{A}} \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{Y} \oplus \mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix} \tag{4.4.1}$$

defines a contractive interpolant $\tilde{B}$ for the lifting data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$. Furthermore $\tilde{B}$ and Γ in (4.4.1) *determine each other uniquely.*

The construction in Proposition 4.4.1 is illustrated by the next example.

Example 4.4.3. Put

$$T' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{4.4.2}$$

Then T' and A are contractions,

$$T'AR = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} = AQ \quad \text{and} \quad R^*R = 1 < 2 = Q^*Q.$$

So $\{A, T', U', R, Q\}$ is a lifting data set, where U' is the Sz.-Nagy-Schäffer isometric lifting of T'. Note that A is a strict contraction, and thus $\mathcal{D}_A = \mathbb{C}^2$. Furthermore,

$$D_{T'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D_A = \begin{bmatrix} \frac{\sqrt{15}}{4} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Therefore,

$$D_A Q = \begin{bmatrix} \frac{\sqrt{15}}{2} \\ 0 \end{bmatrix}, \quad D_A R = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad \text{and} \quad D_{T'}AR = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now let us identify $\mathcal{D}_{T'}$ with $\mathbb{C}$. We obtain that the data for the general H^2 interpolation problem associated with the lifting data set $\{A, T', U', R, Q\}$ is given by

$$\mathcal{U} = \mathcal{D}_A = \mathbb{C}^2, \quad \mathcal{Y} = \mathcal{D}_{T'} = \mathbb{C}, \quad \mathcal{F} = \mathbb{C} = \mathbb{C} \oplus \{0\} \quad \text{and} \quad \omega = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5}\sqrt{5} \end{bmatrix}.$$

The lifting data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ associated with ω via Proposition 4.4.2 is then given by

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{T}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5}\sqrt{5} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and $\tilde{U}'$ the Sz.-Nagy-Schäffer isometric lifting of $\tilde{T}'$. ◇

In particular, Example 4.4.3 shows that different lifting data sets can have the same underlying contraction, and hence different lifting data sets can be related to the same H^2 interpolation problem via Proposition 4.3.1. Moreover, Proposition 4.4.1 shows that for each lifting data set $\{A, T', U', R, Q\}$ there exists a lifting data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ which has the same underlying contraction as $\{A, T', U', R, Q\}$ and is such that $\tilde{A}$ and $\tilde{T}'$ are complementary orthogonal projections. Apparently, it is not possible to determine from the contraction underlying a lifting data set $\{A, T', U', R, Q\}$ whether or not A is a strict contraction. The feature that A is a strict contraction in combination with R being left invertible is an important property a lifting data set can have; see Section 5.2. On the other hand it is not clear for which contractions ω of the form (4.1.1) there exists a lifting data set $\{A, T', U', R, Q\}$ with underlying contraction ω, and such that A is a strict contraction and R is left invertible.

Open Problem 4.4.4. *Given a contraction ω as in* (4.1.1) *does there exist a lifting data set $\{A, T', U', R, Q\}$, which has ω as its underlying contraction, and for which A is a strict contraction and R is left invertible?*

Some necessary conditions for Open Problem 4.4.4 will be obtained in the next chapter; see Corollary 5.2.3.

In Section 4.3 it was also noted that in the classical commutant lifting setting the underlying contraction ω is an isometry and $\overline{\omega_2\mathcal{F}} = \mathcal{U}$. It is not known whether this is also a necessary condition. That is, we have the following open problem.

Open Problem 4.4.5. *Find necessary and sufficient conditions for a contraction ω as in* (4.1.1) *to be the underlying contraction of a lifting data set $\{A, T', U', R, Q\}$ with $\mathcal{H} = \mathcal{H}'$, $R = I_\mathcal{H}$ and Q an isometry.*

4.5 The non-uniqueness in the description of the central solution

Given a solution H to the H^2 interpolation problem defined by the contraction ω in (4.1.1), Theorem 4.2.2 uses the map J_H defined in Theorem 3.2.2 to give a proper parameterization of the functions in the set $\mathbf{Z}_{H,\omega}$ in (4.2.1). In this section we consider the special case that H is the central solution H_c given by (4.1.10). Under the additional assumption that the operator Γ_{H_c} defined by H_c is a strict contraction we will derive a more transparent description of the functions in $\mathbf{Z}_{H_c,\omega}$. First we derive some properties of D_{Γ_c} and the contraction ω_{H_c} defined in Lemma 4.1.3.

Lemma 4.5.1. *Let H_c be the central solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1)*. Assume that the operator Γ_c defined by H_c is a strict contraction. Let ω_{H_c} and $\mathcal{F}_{H_c}$ be as defined in Lemma* 4.1.3*, and put $\mathcal{G}_{H_c} = \mathcal{U} \ominus \mathcal{F}_{H_c}$. Then*

(*a*) *D_{Γ_c} is invertible on $\mathcal{U}$,*

(*b*) *$D_{\Gamma_c} g = g$ for each $g \in \mathcal{G} := \mathcal{U} \ominus \mathcal{F}$,*

(*c*) *$\mathcal{F}_{H_c} = \mathcal{F}$ and $\mathcal{G}_{H_c} = \mathcal{G}$,*

(*d*) *$\omega_{H_c}\Pi_\mathcal{F} = D_{\Gamma_c}\omega_2\Pi_\mathcal{F} D_{\Gamma_c}^{-1}$.*

Proof. The assumption that Γ_c is a strict contraction is equivalent to the defect operator D_{Γ_c} being invertible. Moreover, Since $\mathcal{F} = \mathcal{U} \ominus \mathcal{G}$, the definition of the central solution H_c shows that $H_c(\lambda)|_\mathcal{G} \equiv 0$, and thus $\Gamma_c|_\mathcal{G} = 0$. In other words, $D_{\Gamma_c} g = g$ for each $g \in \mathcal{G}$. Since D_{Γ_c} is an invertible contraction and $D_{\Gamma_c}\mathcal{G} = \mathcal{G}$, we obtain that $D_{\Gamma_c}\mathcal{F} = \mathcal{F}$. Hence $\mathcal{F}_{H_c} = \mathcal{F}$ and $\mathcal{G}_{H_c} = \mathcal{G}$. Finally, note that

$$\omega_{H_c}\Pi_\mathcal{F} D_{\Gamma_c} = \omega_{H_c}\Pi_\mathcal{F} D_{\Gamma_c} P_\mathcal{F} = \omega_{H_c} D_{\Gamma_c} P_\mathcal{F} = D_{\Gamma_c}\omega_2\Pi_\mathcal{F}.$$

So the formula for $\omega_{H_c}\Pi_\mathcal{F}$ is obtained after multiplication from the right with $D_{\Gamma_c}^{-1}$ on both sides. □

Note that $\Gamma_{H_c}|_{\mathcal{G}} = 0$ also holds when Γ_{H_c} is not a strict contraction. So it suffices to assume that $\|\Gamma_c|_{\mathcal{F}}\| < 1$. According to Proposition 4.1.8 this happens for instance when ω is a strict contraction.

Moreover, in case $\|\Gamma_c\| < 1$, Lemmas 4.5.1 and 2.4.5 imply that a Schur class function $K \in \mathbf{S}(\mathcal{U},\mathcal{U})$ is in the set $\mathbf{K}_{\omega_{H_c}}$ if and only if

$$K(\lambda) = \omega_{H_c}\Pi_{\mathcal{F}} + D_{\omega^*_{H_c}}\tilde{K}(\lambda)\Pi_{\mathcal{G}} \quad (\lambda \in \mathbb{D}), \tag{4.5.1}$$

for some $\tilde{K} \in \mathbf{S}(\mathcal{D}_{\omega^*_H}, \mathcal{G})$, and K and $\tilde{K}$ in (4.5.1) determine each other uniquely.

We are now ready to prove the main result of this section.

Proposition 4.5.2. *Let H_c be the central solution to the H^2 interpolation problem defined by the contraction ω in* (4.1.1). *Assume that the operator Γ_c defined by H_c is a strict contraction. Let $\tilde{K} \in \mathbf{S}(\mathcal{D}_{\omega^*_H}, \mathcal{G})$, and put $Z = J_{H_c}K$, where K is the element of $\mathbf{K}_{\omega_{H_c}}$ given by* (4.5.1). *Then*

$$Z(\lambda) = \omega\Pi_{\mathcal{F}} + (\Pi^*_{\mathcal{U}} - \lambda\omega\Pi_{\mathcal{F}})D^2_{\Gamma_c}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}D^{-1}_{\Gamma_c}D_{\omega^*_{H_c}}\tilde{K}(\lambda)\Pi_{\mathcal{G}} \quad (\lambda \in \mathbb{D}), \tag{4.5.2}$$

and Z and $\tilde{K}$ in (4.5.2) *determine each other uniquely. Moreover,*

$$D^2_{\omega^*_{H_c}} = I - D_{\Gamma_c}\omega_2\Pi_{\mathcal{F}}D^{-2}_{\Gamma_c}\Pi^*_{\mathcal{F}}\omega^*_2 D_{\Gamma_c}.$$

Proof. For convenience put

$$L(\lambda) = D_{\Gamma_c}(I_{\mathcal{U}} - \lambda\omega_{H_c}\Pi_{\mathcal{F}})^{-1}D_{\omega^*_{H_c}}\tilde{K}(\lambda)\Pi_{\mathcal{G}} \quad (\lambda \in \mathbb{D}).$$

This formula properly defines a analytic function L on $\mathbb{D}$ because $\omega_{H_c}\Pi_{\mathcal{F}}$ is contractive on $\mathcal{U}$ and $\tilde{K}$ is a Schur class function.

Now fix $\lambda \in \mathbb{D}$. Since $D_{\Gamma_c}g = g$ for each $g \in \mathcal{G}$, we obtain that $L(\lambda) = L(\lambda)D_{\Gamma_c}$. This implies that

$$\begin{aligned} I - \lambda K(\lambda) &= I - \lambda\omega_{H_c}\Pi_{\mathcal{F}} - \lambda D_{\omega^*_{H_c}}\tilde{K}(\lambda)\Pi_{\mathcal{G}} \\ &= (I - \lambda\omega_{H_c}\Pi_{\mathcal{F}})D^{-1}_{\Gamma_c}\times \\ &\qquad (I - \lambda D_{\Gamma_c}(I_{\mathcal{U}} - \lambda\omega_{H_c}\Pi_{\mathcal{F}})^{-1}D_{\omega^*_{H_c}}\tilde{K}(\lambda)\Pi_{\mathcal{G}})D_{\Gamma_c} \\ &= (I - \lambda\omega_{H_c}\Pi_{\mathcal{F}})D^{-1}_{\Gamma_c}(I - \lambda L(\lambda))D_{\Gamma_c}. \end{aligned}$$

In particular, $I - \lambda L(\lambda)$ is invertible, and

$$(I - \lambda L(\lambda))^{-1} = D_{\Gamma_c}(I - \lambda K(\lambda))^{-1}(I - \lambda\omega_{H_c}\Pi_{\mathcal{F}})D^{-1}_{\Gamma_c}.$$

Using the formula for $\omega_{H_c}\Pi_{\mathcal{F}}$ obtained in Lemma 4.5.1 we get

$$\begin{aligned} D_{\Gamma_c}(I - \lambda K(\lambda))^{-1}D_{\Gamma_c} &= (I - \lambda L(\lambda))^{-1}D_{\Gamma_c}(I - \lambda\omega_{H_c}\Pi_{\mathcal{F}})^{-1}D_{\Gamma_c} \\ &= (I - \lambda L(\lambda))^{-1}D^2_{\Gamma_c}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}. \end{aligned} \tag{4.5.3}$$

Note that Γ_c is the observability operator for the pair $\{\omega_1\Pi_{\mathcal{F}}, \omega_2\Pi_{\mathcal{F}}\}$. In particular, $S^*_{\mathcal{Y}}\Gamma_c = \Gamma_c\omega_2\Pi_{\mathcal{F}}$. Moreover, $\Gamma_c|_{\mathcal{G}} = 0$ implies that $L(\lambda)\Gamma^*_c\Gamma_c = 0$. Therefore

$$\begin{aligned} \Gamma^*_c(I - \lambda S^*_{\mathcal{Y}})^{-1}\Gamma_c &= \Gamma^*_c\Gamma_c(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} \\ &= (I - \lambda L(\lambda))^{-1}\Gamma^*_c\Gamma_c(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}. \end{aligned} \tag{4.5.4}$$

From (4.5.4) and (4.5.3), and using $\Gamma_c^*\Gamma_c + D_{\Gamma_c}^2 = I$, it now follows that the positive real function $F = F_K$ in (3.2.11) is given by

$$\begin{aligned} F(\lambda) &= \Gamma_c^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma_c + D_{\Gamma_c}(I - \lambda K(\lambda))^{-1}D_{\Gamma_c} \\ &= (I - \lambda L(\lambda))^{-1}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}. \end{aligned}$$

The formula for $Z = J_{H_c}K$ in (3.2.3) can then be rewritten as

$$\begin{aligned} Z(\lambda) &= \begin{bmatrix} H(\lambda) \\ \lambda^{-1}(F(\lambda) - I) \end{bmatrix} F(\lambda)^{-1} = \begin{bmatrix} \omega_1\Pi_{\mathcal{F}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}F(\lambda)^{-1} \\ \lambda^{-1}(I - F(\lambda)^{-1}) \end{bmatrix} \\ &= \begin{bmatrix} \omega_1(I - \lambda L(\lambda)) \\ \lambda^{-1}(\lambda\omega_2\Pi_{\mathcal{F}} + \lambda(I - \lambda\omega_2\Pi_{\mathcal{F}})L(\lambda)) \end{bmatrix} \\ &= \begin{bmatrix} \omega_1\Pi_{\mathcal{F}} - \lambda\omega_1\Pi_{\mathcal{F}}L(\lambda) \\ \omega_2\Pi_{\mathcal{F}} + (I - \lambda\omega_2\Pi_{\mathcal{F}})L(\lambda) \end{bmatrix} \\ &= \omega\Pi_{\mathcal{F}} + (\Pi_{\mathcal{U}}^* - \lambda\omega\Pi_{\mathcal{F}})L(\lambda). \end{aligned}$$

To see that (4.5.2) holds just observe that the formula for $\omega_{H_c}\Pi_{\mathcal{F}}$ obtained in Lemma 4.5.1 implies that

$$L(\lambda) = D_{\Gamma_c}^2(I_{\mathcal{U}} - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}D_{\Gamma_c}^{-1}D_{\omega_{H_c}^*}\tilde{K}(\lambda)\Pi_{\mathcal{G}}.$$

Since K and $Z = J_{H_c}K$ determine each other uniquely, as well as K and $\tilde{K}$ in (4.5.1), we obtain that Z and $\tilde{K}$ in (4.5.2) determine each other uniquely. Finally, the formula for $D_{\omega_{H_c}^*}^2$ follows directly from the formula for $\omega_{H_c}\Pi_{\mathcal{F}}$. □

We conclude this section with an example that illustrates the result of Proposition 4.5.2 for the H^2 interpolation problem considered in Example 4.1.4.

Example 4.5.3. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). Recall that the central solution H_c to the H^2 interpolation defined by ω is given by $H_c(\lambda) \equiv \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}$, and $\|H_c\| = \frac{1}{3}\sqrt{3}$. Hence the operator Γ_c defined by H_c is a strict contraction. In fact, $D_{\Gamma_c} = \operatorname{diag}(\frac{1}{3}\sqrt{6}, 1)$. Moreover, the contraction ω_{H_c} defined in Lemma 4.1.3 is equal to ω_2. So $D_{\omega_{H_c}} = \operatorname{diag}(\frac{1}{2}\sqrt{3}, 1)$. Let $\tilde{K}$ in $\mathbf{S}(\mathbb{C}, \mathbb{C}^2)$, and define $K \in \mathbf{K}_{\omega_{H_c}}$ by (4.5.1). Since both $\omega_2\Pi_{\mathcal{F}}$ and D_{Γ_c} are diagonal matrices, we obtain via (4.5.2) that $Z = J_{H_c}K$ is given by

$$\begin{aligned} Z(\lambda) &= \omega\Pi_{\mathcal{F}} + (\Pi_{\mathcal{U}}^* - \lambda\omega\Pi_{\mathcal{F}})(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}D_{\Gamma_c}D_{\omega_2^*}\tilde{K}(\lambda)\Pi_{\mathcal{G}} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-\lambda}{2-\lambda} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \tilde{K}(\lambda) \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad (\lambda \in \mathbb{D}) \\ &= \begin{bmatrix} \frac{1}{2} & \frac{-\frac{1}{2}\sqrt{2}\lambda}{2-\lambda}\tilde{K}_1(\lambda) \\ \frac{1}{2} & \frac{1}{2}\sqrt{2}\tilde{K}_1(\lambda) \\ 0 & \tilde{K}_2(\lambda) \end{bmatrix}, \end{aligned}$$

where

$$\tilde{K}_1(\lambda) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{K}(\lambda) \quad \text{and} \quad \tilde{K}_2(\lambda) = \begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{K}(\lambda) \quad (\lambda \in \mathbb{D}).$$

We check that for each $\lambda \in \mathbb{D}$:

$$\begin{aligned}
\Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} &= \begin{bmatrix} \frac{1}{2} & \frac{-\frac{1}{2}\sqrt{2}\lambda}{2-\lambda} \tilde{K}_1(\lambda) \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & \frac{-\sqrt{2}\lambda}{2} \tilde{K}_1(\lambda) \\ 0 & 1 - \lambda \tilde{K}_2(\lambda) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{-\frac{1}{2}\sqrt{2}\lambda}{2-\lambda} \tilde{K}_1(\lambda) \end{bmatrix} \times \\
&\qquad \times \begin{bmatrix} \frac{2}{2-\lambda} & \frac{\sqrt{2}\lambda}{2-\lambda} \tilde{K}_1(\lambda)(1 - \lambda \tilde{K}_2(\lambda))^{-1} \\ 0 & (1 - \lambda \tilde{K}_2(\lambda))^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix} = H_c(\lambda).
\end{aligned}$$

So indeed $Z \in \mathbf{Z}_{H_c,\omega}$. ◇

4.6 A generalization of Theorem 3.2.1

Let $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ and Γ the operator defined by H. Thus Γ is an operator from $\mathcal{U}$ into $H^2(\mathcal{Y})$. The space $\mathcal{U}$ can be identified with the space of constant $\mathcal{U}$-valued functions via the canonical embedding $E_{\mathcal{U}}$ from $\mathcal{U}$ into $H^2(\mathcal{U})$. Moreover, the space $E_{\mathcal{U}}\mathcal{U}$ can also be written as $H^2(\mathcal{U}) \ominus S_{\mathcal{U}} H^2(\mathcal{U})$. Via this identification the operator Γ can then be interpreted as the restriction of the multiplication operator defined by H to the subspace $H^2(\mathcal{U}) \ominus S_{\mathcal{U}} H^2(\mathcal{U})$, viewed as an operator mapping into $H^2(\mathcal{Y})$. Hence the Banach space $H^2(\mathcal{U}, \mathcal{Y})$ can be characterized as all $\mathscr{L}(\mathcal{U}, \mathcal{Y})$-valued functions for which the multiplication operator induces an operator from $H^2(\mathcal{U}) \ominus S_{\mathcal{U}} H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$. We recall here that in our terminology an operator is by definition bounded.

In a similar way it follows that for each nonnegative integer n the multiplication operator defined by H also induces an operator from

$$H^2(\mathcal{U}) \ominus S_{\mathcal{U}}^{n+1} H^2(\mathcal{U}) = E_{\mathcal{U}}\mathcal{U} \oplus S_{\mathcal{U}} E_{\mathcal{U}}\mathcal{U} \oplus \cdots \oplus S_{\mathcal{U}}^n E_{\mathcal{U}}\mathcal{U} \tag{4.6.1}$$

into $H^2(\mathcal{Y})$. The elements in (4.6.1) are the polynomials of degree at most n. Note that the operator $S_{\mathcal{U}}^{n+1}$ is the multiplication operator defined by the inner function λ^{n+1}, and $H^2(\mathcal{U}) \ominus S_{\mathcal{U}}^{n+1} H^2(\mathcal{U})$ is the so-called model space corresponding to this inner function.

Now let $\Theta \in \mathbf{S}(\mathcal{E}, \mathcal{U})$ be an arbitrary inner function with $\Theta(0) = 0$. In general, it is not the case that the multiplication operator defined by H induces an operator from the model space $H^2(\mathcal{U}) \ominus M_\Theta H^2(\mathcal{E})$ into $H^2(\mathcal{Y})$. Indeed, take $\Theta \in \mathbf{S}(\{0\}, \mathcal{U})$ given by $\Theta(\lambda) \equiv 0$. Then $H^2(\mathcal{U}) \ominus M_\Theta H^2(\{0\}) = H^2(\mathcal{U})$. But the multiplication operator for any function in $\mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ that is not in $\mathbf{H}^\infty(\mathcal{U}, \mathcal{Y})$ can not induce an operator from $H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$.

The next theorem provides a Schur type representation of all functions in the space $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ for which the corresponding multiplication operator induces an operator from $H^2(\mathcal{U})\ominus M_\Theta H^2(\mathcal{E})$ into $H^2(\mathcal{Y})$, where $\Theta\in\mathbf{S}(\mathcal{E},\mathcal{U})$ is an arbitrary inner function with $\Theta(0)=0$.

Theorem 4.6.1. *Let $H\in\mathbf{H}^2(\mathcal{U},\mathcal{Y})$, and let $\Theta\in\mathbf{S}(\mathcal{E},\mathcal{U})$ be inner with $\Theta(0)=0$. Put $\mathcal{H}=H^2(\mathcal{U})\ominus M_\Theta H^2(\mathcal{E})$, where M_Θ is the multiplication operator defined by Θ. In order that the map $f\mapsto Hf$ defines a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ it is necessary and sufficient that H is given by*

$$H(\lambda)=\Pi_\mathcal{Y}Z(\lambda)(I-\Theta(\lambda)\Pi_\mathcal{E}Z(\lambda))^{-1}\quad(\lambda\in\mathbb{D}),\tag{4.6.2}$$

where Z is an arbitrary Schur class function in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{E})$.

Theorem 4.1.1 appears as the special case that $\Theta(\lambda)\equiv\lambda$ via the identification explained above. For $\Theta(\lambda)\equiv\lambda^n$, where n is a positive integer, the matrix-valued version of the above theorem can been found in [10] and [11] and for operator-valued functions in [62]. For the trivial choice $\Theta\in\mathbf{S}(\{0\},\mathcal{U})$ given by $\Theta(\lambda)\equiv 0$, Theorem 4.6.1 provides the well known result that a $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function induces a contractive multiplication operator from $H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$ if and only if it is a Schur class function in $\mathbf{S}(\mathcal{U},\mathcal{Y})$.

Before we give the proof of Theorem 4.6.1 it is useful to first prove two lemmas. Throughout let Θ be an inner function in $\in\mathbf{S}(\mathcal{E},\mathcal{U})$ such that $\Theta(0)=0$, and put $\mathcal{H}=H^2(\mathcal{U})\ominus M_\Theta H^2(\mathcal{E})$. Note that Lemma 2.4.1 implies that there exists an inner function $\Phi\in\mathbf{S}(\mathcal{E},\mathcal{U})$ such that $\Theta(\lambda)=\lambda\Phi(\lambda)$ for each $\lambda\in\mathbb{D}$.

Lemma 4.6.2. *Let $\Phi\in\mathbf{S}(\mathcal{E},\mathcal{U})$ be the inner function given by $\Phi(\lambda)\equiv\lambda^{-1}\Theta(\lambda)$, and put $\mathcal{H}_0=H^2(\mathcal{U})\ominus M_\Phi H^2(\mathcal{E})$. Then*

$$\mathcal{H}=E_\mathcal{U}\mathcal{U}\oplus S_\mathcal{U}\mathcal{H}_0,\quad \mathcal{H}=\mathcal{H}_0\oplus M_\Phi E_\mathcal{E}\,\mathcal{E}.\tag{4.6.3}$$

Proof. As usual, given any inner function $\alpha\in\mathbf{S}(\mathcal{X},\mathcal{Y})$, we shall denote the space $H^2(\mathcal{Y})\ominus M_\alpha H^2(\mathcal{X})$ by $\mathcal{H}(\alpha)$. The two identities in (4.6.3) then follow from the rule (see, e.g., Theorem X.1.9 in [38]) that for two inner functions $\alpha\in\mathbf{S}(\mathcal{X},\mathcal{Y})$ and $\beta\in\mathbf{S}(\mathcal{Y},\mathcal{Z})$ we have

$$\mathcal{H}(\beta\alpha)=\mathcal{H}(\beta)\oplus\beta\mathcal{H}(\alpha).$$

Indeed, we apply this rule twice. First with $\alpha(\lambda)=\Psi(\lambda)$ and $\beta(\lambda)=\lambda I_\mathcal{U}$, and next with $\alpha(\lambda)=\lambda I_\mathcal{E}$ and $\beta(\lambda)=\Psi(\lambda)$. Note that in both cases $\beta\alpha=\Theta$. With the first choice of α and β we get the first identity in (4.6.3), and the second choice yields the second identity in (4.6.3). □

The above lemma enables us to make the connection with the general H^2 interpolation problem considered in Section 4.1. Let $\mathcal{H}_0=H^2(\mathcal{U})\ominus M_\Phi H^2(\mathcal{E})$, where $\Phi\in\mathbf{S}(\mathcal{E},\mathcal{U})$ is the inner function given by $\Phi(\lambda)\equiv\lambda^{-1}\Theta(\lambda)$. Put $\mathcal{F}=S_\mathcal{U}\mathcal{H}_0$, and define

$$\omega=\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}:\mathcal{F}\to\begin{bmatrix}\mathcal{Y}\\ \mathcal{H}\end{bmatrix},\quad \omega_1=0,\quad \omega_2(S_\mathcal{U}h_0)=h_0\ (h_0\in\mathcal{H}_0).$$

Note that ω is an isometry. Moreover, since $\omega_1 = 0$, the equation (4.1.3) reduces to $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$.

Lemma 4.6.3. *Let Γ be an operator from $\mathcal{H}$ into $H^2(\mathcal{Y})$, and put*

$$G(\lambda) = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma E_{\mathcal{U}} \quad (\lambda \in \mathbb{D}). \tag{4.6.4}$$

Then $G \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and Γ satisfies $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$ if and only if for each $f \in \mathcal{H}$ we have

$$(\Gamma f)(\lambda) = G(\lambda)f(\lambda) \quad (\lambda \in \mathbb{D}). \tag{4.6.5}$$

Proof. According to the first identity in (4.6.3) the operator $E_{\mathcal{U}}$ maps $\mathcal{U}$ into $\mathcal{H}$. Thus $\Gamma E_{\mathcal{U}}$ is a well-defined operator from $\mathcal{U}$ into $H^2(\mathcal{Y})$. Hence G in (4.6.4) can also be defined by

$$G(\lambda)u = (\Gamma E_{\mathcal{U}}u)(\lambda) \quad (u \in \mathcal{U}, \lambda \in \mathbb{D}). \tag{4.6.6}$$

In particular, $G \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and G is the defining function for $\Gamma E_{\mathcal{U}}$.

Next assume that Γ satisfies the intertwining relation $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. From (4.6.6) we see that (4.6.5) holds for $f = E_{\mathcal{U}}u$ with $u \in \mathcal{U}$ arbitrary. Using the first equality in (4.6.3), we see that it suffices to prove (4.6.5) for $f = S_{\mathcal{U}}h_0$ with $h_0 \in \mathcal{H}_0$. However for such a function f we have

$$\Gamma f = S_{\mathcal{Y}}\Gamma\omega_2 f = S_{\mathcal{Y}}\Gamma h_0, \quad G(\lambda)f(\lambda) \equiv \lambda G(\lambda)h_0(\lambda).$$

Therefore, it suffices to prove (4.6.5) for $h_0 \in \mathcal{H}_0$.

Take $h_0 \in \mathcal{H}_0$. Note that the identity operator on $H^2(\mathcal{U})$ is equal to $S_{\mathcal{U}}S_{\mathcal{U}}^* + P_{\mathcal{U}}$, where $P_{\mathcal{U}}$ is the orthogonal projection of $H^2(\mathcal{U})$ onto $E_{\mathcal{U}}\mathcal{U}$. The first identity in (4.6.3) shows that $E_{\mathcal{U}}\mathcal{U} \subset \mathcal{H}$. So $\Gamma P_{\mathcal{U}}$ is well defined. Since $\mathcal{H}_0$ is invariant under $S_{\mathcal{U}}^*$, we see that $S_{\mathcal{U}}S_{\mathcal{U}}^*h_0$ belongs to $\mathcal{F} = S_{\mathcal{U}}\mathcal{H}_0$, and hence

$$\begin{aligned} \Gamma h_0 &= \Gamma S_{\mathcal{U}}S_{\mathcal{U}}^*h_0 + \Gamma P_{\mathcal{U}}h_0 = S_{\mathcal{Y}}\Gamma\omega_2 S_{\mathcal{U}}S_{\mathcal{U}}^*h_0 + \Gamma P_{\mathcal{U}}h_0 \\ &= S_{\mathcal{Y}}\Gamma S_{\mathcal{U}}^*h_0 + \Gamma P_{\mathcal{U}}h_0. \end{aligned}$$

Thus we get

$$\Gamma h_0 - S_{\mathcal{Y}}\Gamma S_{\mathcal{U}}^*h_0 = \Gamma P_{\mathcal{U}}h_0.$$

By induction, using that $\mathcal{H}_0$ is invariant under $S_{\mathcal{U}}^*$, the preceding identity yields

$$\Gamma h_0 = \sum_{k=0}^{n} S_{\mathcal{Y}}^k(\Gamma P_{\mathcal{U}})(S_{\mathcal{U}}^*)^k h_0 + S_{\mathcal{Y}}^{(n+1)}\Gamma(S_{\mathcal{U}}^*)^{(n+1)}h_0, \quad \text{for } n = 0, 1, 2, \ldots.$$

Now fix $\lambda \in \mathbb{D}$. From (4.6.4) we know that

$$(\Gamma P_{\mathcal{U}}f)(\lambda) = (\Gamma E_{\mathcal{U}}E_{\mathcal{U}}^*f)(\lambda) = G(\lambda)E_{\mathcal{U}}^*f \quad (f \in H^2(\mathcal{U})).$$

Hence for $h_0 \in \mathcal{H}_0$ we have

$$\begin{aligned} (\Gamma h_0)(\lambda) &= \sum_{k=0}^{n} \lambda^k G(\lambda)E_{\mathcal{U}}^*\,(S_{\mathcal{U}}^*)^k h_0 + \lambda^{n+1}\big(\Gamma(S_{\mathcal{U}}^*)^{(n+1)}h_0\big)(\lambda) \\ &= G(\lambda)\Big(\sum_{k=0}^{n} \lambda^k E_{\mathcal{U}}^*\,(S_{\mathcal{U}}^*)^k h_0\Big) + \lambda^{n+1}\big(\Gamma(S_{\mathcal{U}}^*)^{(n+1)}h_0\big)(\lambda). \quad (4.6.7) \end{aligned}$$

Note that for $n \to \infty$ the vector $\Gamma(S_{\mathcal{U}}^*)^{(n+1)}h_0$ converges to zero in the norm of $H^2(\mathcal{Y})$, and hence the same holds true for $S_{\mathcal{Y}}^{(n+1)}\Gamma(S_{\mathcal{U}}^*)^{(n+1)}h_0$. It follows that the second term in the right side of (4.6.7) converges to zero when $n \to \infty$. Furthermore, for $n \to \infty$ the vector $\sum_{k=0}^{n} \lambda^k E_{\mathcal{U}}^*(S_{\mathcal{U}}^*)^k h_0$ converges to $h_0(\lambda)$ in the norm of $\mathcal{U}$. Hence the first term in the right hand side of (4.6.7) converges to $G(\lambda)h_0(\lambda)$ when $n \to \infty$. Since h_0 was an arbitrary vector in $\mathcal{H}_0$, we have proved that (4.6.5) holds.

To prove the converse implication, assume that (4.6.5) holds. Let $f = S_{\mathcal{U}}h_0 \in \mathcal{F}$ with $h_0 \in \mathcal{H}_0$. Then for each $\lambda \in \mathbb{D}$ we have

$$(\Gamma f)(\lambda) = G(\lambda)\lambda h_0(\lambda) = \lambda G(\lambda)h_0(\lambda) = (S_{\mathcal{Y}}\Gamma h_0)(\lambda) = (S_{\mathcal{Y}}\Gamma\omega_2 f)(\lambda).$$

Since h_0 is an arbitrary element in $\mathcal{H}_0$, we see that $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. $\square$

We are now ready to prove the main result.

Proof of Theorem 4.6.1. Let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$, and let us assume that the map $f \mapsto Hf$ defines a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$. Denote this contraction by Γ. Moreover, let ω be the isometry in (4.6). Then (4.6.4) holds with H in place of G, and Lemma 4.6.3 shows that the contraction Γ satisfies the relation $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. Since $\omega_1 = 0$, we know from Theorem 4.1.1 that H is given by

$$H(\lambda) = \Pi_{\mathcal{Y}}\tilde{Z}(\lambda)(I_{\mathcal{H}} - \lambda\Pi_{\mathcal{H}}\tilde{Z}(\lambda))^{-1}E_{\mathcal{U}} \quad (\lambda \in \mathbb{D}), \tag{4.6.8}$$

where

$$\tilde{Z} \in \mathbf{S}(\mathcal{H},\mathcal{Y}\oplus\mathcal{H}), \quad \tilde{Z}(\lambda)|_{\mathcal{F}} = \omega \quad (\lambda \in \mathbb{D}). \tag{4.6.9}$$

Conversely, let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ be given by (4.6.8) with $\tilde{Z}$ as in (4.6.9). Then, again using Theorem 4.1.1, we know that there exists a contraction Γ from $\mathcal{H}$ into $H^2(\mathcal{Y})$ such that

$$(\Gamma h)(\lambda) = \Pi_{\mathcal{Y}}\tilde{Z}(\lambda)(I_{\mathcal{H}} - \lambda\Pi_{\mathcal{H}}\tilde{Z}(\lambda))^{-1}h \quad (h \in \mathcal{H}, \lambda \in \mathbb{D}).$$

Moreover, Γ satisfies $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. It follows that $H(\lambda) = (\Gamma E_{\mathcal{U}})(\lambda)$ for each $\lambda \in \mathbb{D}$. The fact that Γ satisfies the relation $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$ allows us to again apply Lemma 4.6.3. We conclude that $(\Gamma h)(\lambda) = H(\lambda)h(\lambda)$ for each $h \in \mathcal{H}$ and each $\lambda \in \mathbb{D}$. Thus the map $f \mapsto Hf$ induces a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ as desired.

From the previous results we see that it remains to show that the representations given by (4.6.8), (4.6.9) and by (4.6.2), with $Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{E})$, are equivalent. Consider the spaces

$$\begin{aligned} \mathcal{F} &= \lambda\mathcal{H}_0, \qquad & \mathcal{G} &= \mathcal{H} \ominus \mathcal{F} = E_{\mathcal{U}}\mathcal{U}, \\ \mathcal{F}' &= \mathcal{H}_0, \qquad & \mathcal{G}' &= \mathcal{H} \ominus \mathcal{F}' = M_{\Phi}E_{\mathcal{E}}\mathcal{E}, \end{aligned}$$

where $\mathcal{H}_0$ is the Hilbert space and Φ the inner function defined in Lemma 4.6.2.

Let $\tilde{Z}$ be as in (4.6.9), and define $F \in \mathbf{S}(\mathcal{H},\mathcal{Y})$ and $G \in \mathbf{S}(\mathcal{H},\mathcal{H})$ by

$$F(\lambda) = \Pi_{\mathcal{Y}}\tilde{Z}(\lambda) \quad \text{and} \quad G(\lambda) = \Pi_{\mathcal{H}}\tilde{Z}(\lambda) \quad (\lambda \in \mathbb{D}).$$

Fix $\lambda \in \mathbb{D}$. Since $G(\lambda)|_{\mathcal{F}} = \omega_2$ and ω_2 is an isometry which maps $\mathcal{F}$ onto $\mathcal{F}'$, we know that $G(\lambda)\mathcal{G} \subset \mathcal{G}' = M_\Phi E_\mathcal{E}\mathcal{E}$. Thus given $u \in \mathcal{U}$, we have $G(\lambda)E_\mathcal{U}u = M_\Phi E_\mathcal{E}e(\lambda)$ for some $e(\lambda) \in \mathcal{E}$. The fact that Φ is inner means, by definition, that M_Φ in an isometry. Put $C(\lambda) = E_\mathcal{E}^* M_\Phi^* G(\lambda)E_\mathcal{U}$. Then

$$C(\lambda)u = E_\mathcal{E}^* M_\Phi^* G(\lambda)E_\mathcal{U}u = E_\mathcal{E}^* M_\Phi^* M_\Phi E_\mathcal{E}e(\lambda) = E_\mathcal{E}^* E_\mathcal{E}e(\lambda) = e(\lambda).$$

We conclude that $G(\lambda)E_\mathcal{U} = M_\Phi E_\mathcal{E}C(\lambda)$. From the definition of $C(\lambda)$ it is clear that $C(\lambda)$ is an operator from $\mathcal{U}$ into $\mathcal{E}$. Moreover, $C(\lambda)$ depends analytically on $\lambda \in \mathbb{D}$.

From the result of the previous paragraph we know that

$$G(\lambda)(E_\mathcal{U}u \oplus S_\mathcal{U}h_0) = h_0 \oplus M_\Phi E_\mathcal{E}C(\lambda) = E_\mathcal{U}v \oplus S_\mathcal{U}k_0,$$

where

$$v = E_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda)u + E_\mathcal{U}^* h_0, \quad k_0 = S_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda)u + S_\mathcal{U}^* h_0.$$

Recall that $\mathcal{H} = E_\mathcal{U}\mathcal{U} \oplus S_\mathcal{U}\mathcal{H}_0$. Let J be the operator from $E_\mathcal{U}\mathcal{U} \oplus S_\mathcal{U}\mathcal{H}_0$ to $\mathcal{U} \oplus \mathcal{H}_0$ defined by $J(E_\mathcal{U}u \oplus S_\mathcal{U}h_0) = u \oplus h_0$. Obviously, J is unitary and its inverse is given by $J^{-1}(u \oplus h_0) = E_\mathcal{U}u \oplus S_\mathcal{U}h_0$.

It follows that relative to the direct sum decomposition $\mathcal{U} \oplus \mathcal{H}_0$ the operator $JG(\lambda)J^{-1}$ is given by the following 2×2 operator matrix:

$$JG(\lambda)J^{-1} = \begin{bmatrix} E_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) & E_\mathcal{U}^* \\ S_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) & S_\mathcal{U}^* \end{bmatrix}. \tag{4.6.10}$$

But then

$$J\big(I - \lambda G(\lambda)\big)J^{-1} = \begin{bmatrix} I - \lambda E_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) & -\lambda E_\mathcal{U}^* \\ -\lambda S_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) & I - \lambda S_\mathcal{U}^* \end{bmatrix},$$

and hence, using a Schur complement argument (see Proposition 2.3.2), we have

$$J\big(I - \lambda G(\lambda)\big)^{-1}J^{-1} = \begin{bmatrix} \Delta(\lambda)^{-1} & * \\ * & * \end{bmatrix},$$

where

$$\begin{aligned} \Delta(\lambda) &= I - \lambda E_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) - (-\lambda E_\mathcal{U}^*)(I - \lambda S_\mathcal{U}^*)^{-1}(-\lambda S_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda)) \\ &= I - \lambda E_\mathcal{U}^* M_\Phi E_\mathcal{E}C(\lambda) + \lambda E_\mathcal{U}^*(I - \lambda S_\mathcal{U}^*)^{-1}(I - \lambda S_\mathcal{U}^* - I)M_\Phi E_\mathcal{E}C(\lambda) \\ &= I - \lambda E_\mathcal{U}^*(I - \lambda S_\mathcal{U}^*)^{-1}M_\Phi E_\mathcal{E}C(\lambda) \\ &= I - \lambda\Phi(\lambda)C(\lambda) = I - \Theta(\lambda)C(\lambda). \end{aligned}$$

We also know that $F(\lambda)|_\mathcal{F} = \omega_1 = 0$. Thus $F(\lambda)J^{-1}$ admits the following representation:

$$F(\lambda)J^{-1} = \begin{bmatrix} F_1(\lambda) & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{H}_0 \end{bmatrix} \to \mathcal{Y}. \tag{4.6.11}$$

But then we have

$$\begin{aligned} H(\lambda) &= F(\lambda)\big(I-\lambda G(\lambda)\big)^{-1}E_{\mathcal{U}} \\ &= F(\lambda)J^{-1}J\big(I-\lambda G(\lambda)\big)^{-1}J^{-1}JE_{\mathcal{U}} \\ &= \begin{bmatrix} F_1(\lambda) & 0 \end{bmatrix}\begin{bmatrix} \Delta(\lambda)^{-1} & * \\ * & * \end{bmatrix}\begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \\ &= F_1(\lambda)\Delta(\lambda)^{-1} = F_1(\lambda)\big(I-\Theta(\lambda)C(\lambda)\big)^{-1}. \end{aligned}$$

Let τ be the canonical embedding of $\mathcal{H}_0$ into $\mathcal{H}$, that is, τ is defined by $\tau h_0 = h_0$ for all $h_0 \in \mathcal{H}_0$. From (4.6.10) and (4.6.11) it follows that

$$\begin{bmatrix} F(\lambda) \\ G(\lambda) \end{bmatrix} J^{-1} = \begin{bmatrix} F_1(\lambda) & 0 \\ M_\Phi E_{\mathcal{E}} C(\lambda) & \tau_{\mathcal{H}_0} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{H}_0 \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}.$$

Since $\operatorname{Im}\tau$ is perpendicular to $M_\Phi E_{\mathcal{E}}\mathcal{E}$, we see that for $h = E_{\mathcal{U}}u \oplus S_{\mathcal{U}}h_0$ we have

$$\|F(\lambda)h\|^2 + \|G(\lambda)h\|^2 = \|F_1(\lambda)u\|^2 + \|M_\Phi E_{\mathcal{E}} C(\lambda)u\|^2 + \|h_0\|^2$$

But $\|h\|^2 = \|u\|^2 + \|h_0\|^2$, and hence

$$\|h\|^2 - \big(\|F(\lambda)h\|^2 + \|G(\lambda)h\|^2\big) = \|u\|^2 - \big(\|F_1(\lambda)u\|^2 + \|M_\Phi E_{\mathcal{E}} C(\lambda)u\|^2\big).$$

Since the multiplication operator M_Φ and the map $E_{\mathcal{E}}$ are isometries, we conclude that

$$\tilde{Z} = \begin{bmatrix} F \\ G \end{bmatrix} \in \mathbf{S}(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}) \Longleftrightarrow Z = \begin{bmatrix} F_1 \\ C \end{bmatrix} \in \mathbf{S}(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{E} \end{bmatrix}).$$

We have now shown that the representations given by (4.6.8) and (4.6.9) imply those given by (4.6.2), with $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$. The reverse implication is obtained by reversing the arguments. □

Notes for Chapter 4

The general H^2 interpolation problem considered in this chapter was introduced in [50], where also the equivalence with the relaxed commutant lifting problem was proved; see Sections 4.3 and 4.4 above. In [50] this equivalence led to the proof of the main results, Theorems 4.1.1 and 4.2.2 above, via the description of all contractive interpolants for the relaxed commutant lifting problem given in [49]. The proof of the main results given in Sections 4.1 and 4.2 is based on the last section of [49]. The description of the non-uniqueness for the central solution given in the Section 4.5 is new. Finally, the material in the last section is taken from [50].

Chapter 5

Linear fractional representations

Throughout this chapter $\Omega = \{A, T', U', R, Q\}$ is a lifting data set with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Moreover ω is the contraction underlying $\{A, T', U', R, Q\}$, that is, ω is the contraction given by

$$\omega : \mathcal{F} = \overline{D_A Q \mathcal{H}_0} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix}, \quad \omega D_A Q = \begin{bmatrix} D_{T'} AR \\ D_A R \end{bmatrix}.$$

We present in this chapter various linear fractional representations of all contractive interpolants for Ω. In Section 5.1 a linear fractional Redheffer representation is derived, in which the corresponding Redheffer coefficients are expressed in terms of the underlying contraction ω. It is often more convenient to have a representation where the coefficients are given explicitly in terms of the operators appearing in the lifting data set. Such a Redheffer representation is derived in Section 5.2 under the additional assumptions that A is a strict contraction and that R is left invertible. The Redheffer coefficients in the representation obtained in Section 5.2 are further analyzed in Section 5.3. Alternative formulas for the Redheffer coefficients are obtained, and as a byproduct it is shown that each contractive interpolant is uniquely determined by its action on the kernel of Q^*. In fact, as it turns out this property does not depend on A being a strict contraction and R being left invertible. The final result of this chapter is given in Section 5.4, where under the conditions of the previous two sections a classical linear fractional representation of all contractive interpolants is obtained.

5.1 A Redheffer representation

In this section we use the Schur representation of all contractive interpolants in Theorem 1.1 to obtain the following Redheffer type representation.

Theorem 5.1.1. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T', and with underlying contraction ω. Put $\omega_1 =$*

$\Pi_{\mathcal{D}_{T'}}\omega$, $\omega_2 = \Pi_{\mathcal{D}_A}\omega$, $\mathcal{G} = \mathcal{D}_A \ominus \mathcal{F}$ *and*

$$\begin{aligned} \Upsilon_{1,1}(\lambda) &= \lambda\Pi_{\mathcal{G}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}\Pi_{\mathcal{D}_A}D_{\omega^*}, \\ \Upsilon_{1,2}(\lambda) &= \Pi_{\mathcal{G}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}, \\ \Upsilon_{2,1}(\lambda) &= \Pi_{\mathcal{D}_{T'}}D_{\omega^*} + \lambda\omega_1\Pi_{\mathcal{F}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}\Pi_{\mathcal{D}_A}D_{\omega^*}, \\ \Upsilon_{2,2}(\lambda) &= \omega_1\Pi_{\mathcal{F}}(I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1}. \end{aligned} \qquad (\lambda \in \mathbb{D}) \qquad (5.1.1)$$

Then

$$\Upsilon_{1,1} \in \mathbf{S}(\mathcal{D}_{\omega^*}, \mathcal{G}), \quad \Upsilon_{1,2} \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{D}_A, \mathcal{G}),$$
$$\Upsilon_{2,1} \in \mathbf{S}(\mathcal{D}_{\omega^*}, \mathcal{D}_{T'}), \quad \Upsilon_{2,2} \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{D}_A, \mathcal{D}_{T'}),$$

and for any $V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ *the function* H *given by*

$$H(\lambda) = \Upsilon_{2,2}(\lambda) + \Upsilon_{2,1}(\lambda)V(\lambda)(I - \Upsilon_{1,1}(\lambda)V(\lambda))^{-1}\Upsilon_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}), \qquad (5.1.2)$$

is in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{D}_A, \mathcal{D}_{T'})$, *and for* Γ *the operator defined by* H, *the operator*

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix} \qquad (5.1.3)$$

is a contractive interpolant for $\{A, T', U', R, Q\}$. *Moreover, all contractive interpolants are obtained in this way.*

Theorem 5.1.1 claims that the function $\Upsilon_{1,1}$ in (5.1.1) is a Schur class function. Since $\Upsilon_{1,1}(0) = 0$, this implies that $\Upsilon_{1,1}$ is of the form $\Upsilon_{1,1}(\lambda) \equiv \lambda\Xi(\lambda)$, where Ξ is also a Schur class function; see Lemma 2.4.1. In particular, this explains why the right hand side in (5.1.2) is properly defined for each $V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$.

To illustrate the result of Theorem 5.1.1 we give two examples.

Example 5.1.2. Let $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{F}$ and ω be as in (4.1.11). We shall use Theorem 5.1.1 to describe the set of all solutions to the H^2 interpolation problem defined by ω. In fact, we will show that all solutions H are of the form

$$H(\lambda) = \begin{bmatrix} \frac{1}{2-\lambda} & \frac{(\frac{1}{2}\sqrt{2}+1-\lambda)V_1(\lambda)+(\frac{1}{2}\sqrt{2}-1+\lambda)V_2(\lambda)}{(2-\lambda)(1-\lambda V_3(\lambda))} \end{bmatrix} \quad (\lambda \in \mathbb{D}), \qquad (5.1.4)$$

where the free parameters V_1, V_2 and V_3 are such that

$$V(\lambda) = \begin{bmatrix} V_1(\lambda) \\ V_2(\lambda) \\ V_3(\lambda) \end{bmatrix} \quad (\lambda \in \mathbb{D}) \qquad (5.1.5)$$

is an arbitrary Schur class function in $\mathbf{S}(\mathbb{C}, \mathbb{C}^3)$.

To prove our claim, note that Proposition 4.4.1 provides a procedure to construct a lifting data set $\{A, T', U', R, Q\}$ with $\mathcal{D}_A = \mathcal{U} = \mathbb{C}^2$, $\mathcal{D}_{T'} = \mathcal{Y} = \mathbb{C}$ and with the given ω being the contraction underlying $\{A, T', U', R, Q\}$. Then Proposition 4.4.2 and Theorem 5.1.1 show that all solutions H are given by (5.1.2), where

$V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ and $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ are as in (5.1.1). Note that in this case

$$\omega_1 = \frac{1}{2} \text{ on } \mathbb{C} \quad \text{and} \quad \omega_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} : \mathbb{C} \to \mathbb{C}^2.$$

Moreover, $\mathcal{G} = \mathcal{U} \ominus \mathcal{F} = \{0\} \oplus \mathbb{C} = \mathbb{C}$. So that $\Pi_{\mathcal{F}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\Pi_{\mathcal{G}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Furthermore,

$$D_{\omega^*} = \begin{bmatrix} \frac{1}{2} + \frac{1}{4}\sqrt{2} & -\frac{1}{2} + \frac{1}{4}\sqrt{2} & 0 \\ -\frac{1}{2} + \frac{1}{4}\sqrt{2} & \frac{1}{2} + \frac{1}{4}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and thus} \quad \mathcal{D}_{\omega^*} = \mathbb{C}^3 = \mathcal{D}_{T'} \oplus \mathcal{D}_A.$$

We obtain that the functions $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ in (5.1.1) are given by

$$\begin{aligned}
\Upsilon_{1,1}(\lambda) &= \lambda \Pi_{\mathcal{G}} (I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} \Pi_{\mathcal{D}_A} D_{\omega^*} \\
&= \lambda \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D_{\omega^*} = \begin{bmatrix} 0 & 0 & \lambda \end{bmatrix}, \\
\Upsilon_{1,2}(\lambda) &= \Pi_{\mathcal{G}} (I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad\qquad (\lambda \in \mathbb{D}) \\
\Upsilon_{2,1}(\lambda) &= \Pi_{\mathcal{D}_{T'}} D_{\omega^*} + \lambda\omega_1 \Pi_{\mathcal{F}} (I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} \Pi_{\mathcal{D}_A} D_{\omega^*} \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} D_{\omega^*} + \begin{bmatrix} \frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2-\lambda}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D_{\omega^*} \\
&= \begin{bmatrix} 1 & \frac{\lambda}{2-\lambda} & 0 \end{bmatrix} D_{\omega^*} = \begin{bmatrix} \frac{\frac{1}{2}\sqrt{2}+1-\lambda}{(2-\lambda)} & \frac{\frac{1}{2}\sqrt{2}-1+\lambda}{(2-\lambda)} & 0 \end{bmatrix}, \\
\Upsilon_{2,2} &= \omega_1 \Pi_{\mathcal{F}} (I - \lambda\omega_2\Pi_{\mathcal{F}})^{-1} = \begin{bmatrix} \frac{1}{2-\lambda} & 0 \end{bmatrix}.
\end{aligned}$$

An elementary computation then shows that H in (5.1.2) is indeed given by (5.1.4) when $V \in \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ is given by (5.1.5). ◇

Example 5.1.3. Let $\{A, T', U', R, Q\}$ be the lifting data set given by (4.4.2) with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Let V be an arbitrary Schur class function in $\mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ and $V_1, V_2, V_3 \in \mathbf{S}(\mathbb{C}, \mathbb{C})$ such that

$$V(\lambda) = \begin{bmatrix} V_1(\lambda) \\ V_2(\lambda) \\ V_3(\lambda) \end{bmatrix} \quad (\lambda \in \mathbb{D}). \tag{5.1.6}$$

Define H by

$$H(\lambda) = \frac{V_1(\lambda)}{5 - \lambda^2\sqrt{5}V_2(\lambda) - 2\lambda\sqrt{5}V_3(\lambda)} \begin{bmatrix} \lambda\sqrt{5} & 5 \end{bmatrix} \quad (\lambda \in \mathbb{D}). \tag{5.1.7}$$

We claim that $H \in \mathbf{H}^2_{\text{ball}}(\mathbb{C}^2, \mathbb{C})$, and that all contractive interpolants B for the lifting data set $\{A, T', U', R, Q\}$ are of the form (5.1.3), where Γ is the operator defined by H for some $V \in \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$.

Recall from Example 4.4.3 that $\mathcal{D}_A = \mathbb{C}^2$, $\mathcal{D}_{T'} = \mathbb{C}$, $\mathcal{F} = \mathbb{C} = \mathbb{C} \oplus \{0\}$ and that the contraction underlying $\{A, T', U', R, Q\}$ is given by

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5}\sqrt{5} \end{bmatrix}, \quad \text{so that} \quad D_{\omega^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{5}\sqrt{5} \end{bmatrix}.$$

We obtain that the functions $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ in (5.1.1) are given by

$$\begin{aligned} \Upsilon_{1,1}(\lambda) &= \begin{bmatrix} 0 & \frac{\lambda^2}{2}\sqrt{5} & \frac{2\lambda}{5}\sqrt{5} \end{bmatrix}, \quad \Upsilon_{1,2}(\lambda) = \begin{bmatrix} \frac{\lambda}{5}\sqrt{5} & 1 \end{bmatrix}, \\ \Upsilon_{2,1}(\lambda) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Upsilon_{2,2}(\lambda) = \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Our claim then follows from Theorem 5.1.1 and a straightforward computation that shows that H in (5.1.2) provided $V \in \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ as in (5.1.6) is given by (5.1.7). $\diamond$

Note that in Example 5.1.2 the function H (see (5.1.4)) has the property that $H(\lambda)|_{\mathcal{F}} \equiv H_c(\lambda)|_{\mathcal{F}}$, where H_c is the central solution to the H^2 interpolation problem defined by the contraction ω. In general, this does not happen, as can be seen in Example 5.1.3.

Recall from Theorem 1.1 that an operator B from $\mathcal{H}$ to $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is a contractive interpolant if and only if there exists a Schur class function Z in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_A \oplus \mathcal{D}_{T'})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ such that

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \tag{5.1.8}$$

where Γ is the operator defined by the function $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{D}_A, \mathcal{D}_{T'})$ which is given by the Schur representation

$$H(\lambda) = \Pi_{\mathcal{D}_{T'}} Z(\lambda)(I - \lambda \Pi_{\mathcal{D}_A} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{5.1.9}$$

Lemma 2.4.5 enables us to express the functions Z in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_A \oplus \mathcal{D}_{T'})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ in terms of functions in the Schur class $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ that are completely arbitrary. Indeed, according to Lemma 2.4.5 an $\mathscr{L}(\mathcal{D}_A, \mathcal{D}_A \oplus \mathcal{D}_{T'})$-valued function Z is in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_A \oplus \mathcal{D}_{T'})$ and satisfies $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if there exists a $V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ such that

$$Z(\lambda) = \omega \Pi_{\mathcal{F}} + D_{\omega^*} V(\lambda) \Pi_{\mathcal{G}} \quad (\lambda \in \mathbb{D}). \tag{5.1.10}$$

Moreover, for each $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_A \oplus \mathcal{D}_{T'})$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ there exists only one $V \in \mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ such that (5.1.10) holds.

The Schur representation of H in terms of Z in (5.1.9) and the representation of Z in terms of V in (5.1.10) combine into the representation of H directly in terms of V given by (5.1.2). Rather than proving this claim directly, we first present a general scheme which allows us to rewrite a Schur representation (5.1.9) as a representation of the form (5.1.2), assuming we have a relation as in (5.1.10). This general scheme will also be of use in the sequel.

Lemma 5.1.4. *Let*

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & 0 \\ Y_4 & Y_5 \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{W} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{Y} \end{bmatrix} \tag{5.1.11}$$

be a contraction, and define operator-valued functions $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ *and* $\Upsilon_{2,2}$ *by*

$$\begin{aligned} \Upsilon_{1,1}(\lambda) &= \lambda Y_3(I-\lambda Y_1)^{-1}Y_2, \\ \Upsilon_{1,2}(\lambda) &= Y_3(I-\lambda Y_1)^{-1}, \\ \Upsilon_{2,1}(\lambda) &= Y_5+\lambda Y_4(I-\lambda Y_1)^{-1}Y_2, \\ \Upsilon_{2,2}(\lambda) &= Y_4(I-\lambda Y_1)^{-1}. \end{aligned} \qquad (\lambda\in\mathbb{D}) \tag{5.1.12}$$

Then

$$\Upsilon_{1,1}\in\mathbf{S}(\mathcal{W},\mathcal{V}), \quad \Upsilon_{1,2}\in\mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{V}),$$
$$\Upsilon_{2,1}\in\mathbf{S}(\mathcal{W},\mathcal{Y}) \quad \textit{and} \quad \Upsilon_{2,2}\in\mathbf{H}^2_{\text{ball}}(\mathcal{U},\mathcal{Y}).$$

Moreover, let $M_{\Upsilon_{1,1}}$ *and* $M_{\Upsilon_{2,1}}$ *be the multiplication operators defined by* $\Upsilon_{1,1}$ *and* $\Upsilon_{2,1}$, *respectively, and* $\Gamma_{\Upsilon_{1,2}}$ *and* $\Gamma_{\Upsilon_{2,2}}$ *the operators defined by* $\Upsilon_{1,2}$ *and* $\Upsilon_{2,2}$, *respectively. Then the operator*

$$\tilde{M} = \begin{bmatrix} M_{\Upsilon_{1,1}} & \Gamma_{\Upsilon_{1,2}} \\ M_{\Upsilon_{2,1}} & \Gamma_{\Upsilon_{2,2}} \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{W}) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} H^2(\mathcal{V}) \\ H^2(\mathcal{Y}) \end{bmatrix} \tag{5.1.13}$$

is a contraction which is unitary whenever Y *in* (5.1.11) *is unitary and* Y_1 *is pointwise stable.*

Proof. Since Y is contractive, and thus Y_1 is contractive, the functions in (5.1.12) are properly defined, and analytic on $\mathbb{D}$. Now put

$$G(\lambda) = \begin{bmatrix} \Upsilon_{1,1}(\lambda) \\ \Upsilon_{2,1}(\lambda) \end{bmatrix} \quad \text{and} \quad F(\lambda) = \begin{bmatrix} \Upsilon_{1,2}(\lambda) \\ \Upsilon_{2,2}(\lambda) \end{bmatrix} \quad (\lambda\in\mathbb{D}).$$

Furthermore, put

$$\Xi = \left\{ Y_1, Y_2, \begin{bmatrix} Y_3 \\ Y_4 \end{bmatrix}, \begin{bmatrix} 0 \\ Y_5 \end{bmatrix} \right\}.$$

Then Ξ is a realization for G, and Y in (5.1.11) is the system matrix associated with Ξ. In particular, Ξ is a contractive realization, and hence G is a Schur class function; see Section 2.4. So $\Upsilon_{1,1}\in\mathbf{S}(\mathcal{W},\mathcal{V})$ and $\Upsilon_{2,1}\in\mathbf{S}(\mathcal{W},\mathcal{Y})$. Now let W_o be the observability operator for the contractive realization Ξ. Then W_o is a contraction from $\mathcal{U}$ into $H^2(\mathcal{V}\oplus\mathcal{Y})$; see Theorem 2.4.2. Moreover, F is the defining function of W_c. So we obtain $F \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{V} \oplus \mathcal{Y})$. Since $\Upsilon_{1,2}(\lambda) \equiv \Pi_{\mathcal{V}} F(\lambda)$ and $\Upsilon_{2,2}(\lambda) \equiv \Pi_{\mathcal{Y}} F(\lambda)$, this implies that $\Upsilon_{1,2} \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{V})$ and $\Upsilon_{2,2} \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$. The statement about the operator in (5.1.13) now follows from Theorem 2.4.3, when we make the identification $H^2(\mathcal{V} \oplus \mathcal{Y}) = H^2(\mathcal{V}) \oplus H^2(\mathcal{Y})$. □

Proposition 5.1.5. *Assume that Z is a Schur class function in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ given by*

$$Z(\lambda) = \begin{bmatrix} Y_4 \\ Y_1 \end{bmatrix} + \begin{bmatrix} Y_5 \\ Y_2 \end{bmatrix} V(\lambda) Y_3 : \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \qquad (\lambda \in \mathbb{D}), \tag{5.1.14}$$

where V is in the Schur class $\mathbf{S}(\mathcal{V}, \mathcal{W})$ and Y_1, Y_2, Y_3, Y_4 and Y_5 are operators such that the operator matrix Y in (5.1.11) *is a contraction. Then*

$$\begin{aligned} &\Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} \\ &\qquad = \Upsilon_{2,2}(\lambda) + \Upsilon_{2,1}(\lambda) V(\lambda)(I - \Upsilon_{1,1}(\lambda) V(\lambda))^{-1} \Upsilon_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}), \end{aligned} \tag{5.1.15}$$

where $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ are the functions defined in (5.1.12).

The formula on the right hand side of (5.1.15) is referred to as a *(linear fractional) Redheffer representation*, and the functions $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ in (5.1.15) as the corresponding *Redheffer coefficients.* The term Redheffer comes from scattering theory; see Chapter XIV in [38]. Indeed, let $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ be the functions in (5.1.12), where Y in (5.1.11) is a contraction. Consider the Redheffer scattering system

$$\begin{bmatrix} g \\ y \end{bmatrix} = \begin{bmatrix} M_{\Upsilon_{1,1}} & \Gamma_{\Upsilon_{1,2}} \\ M_{\Upsilon_{2,1}} & \Gamma_{\Upsilon_{2,2}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{subject to } x = M_V g, \tag{5.1.16}$$

where V is a Schur class function in $\mathbf{S}(\mathcal{V}, \mathcal{W})$. Here u is an element from $\mathcal{U}$ and the vectors y, g and x are functions from the Hardy spaces $H^2(\mathcal{Y})$, $H^2(\mathcal{V})$ and $H^2(\mathcal{W})$, respectively. Solving (5.1.16) we obtain that $y = \Gamma u$, where Γ is the operator defined by the function $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ given by the right hand side of (5.1.15), that is,

$$H(\lambda) = \Upsilon_{2,2}(\lambda) + \Upsilon_{2,1}(\lambda) V(\lambda)(I - \Upsilon_{1,1}(\lambda) V(\lambda))^{-1} \Upsilon_{1,2}(\lambda) \qquad (\lambda \in \mathbb{D}).$$

Proof of Proposition 5.1.5. Note that

$$\Pi_{\mathcal{Y}} Z(\lambda) = Y_4 + Y_5 V(\lambda) Y_3 \quad \text{and} \quad \Pi_{\mathcal{U}} Z(\lambda) = Y_1 + Y_2 V(\lambda) Y_3 \qquad (\lambda \in \mathbb{D}).$$

For each $\lambda \in \mathbb{D}$ we then obtain that

$$\begin{aligned} Y_3(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1} &= Y_3(I - \lambda Y_1 - \lambda Y_2 V(\lambda) Y_3)^{-1} \\ &= Y_3(I - \lambda (I - \lambda Y_1)^{-1} Y_2 V(\lambda) Y_3)^{-1} (I - \lambda Y_1)^{-1} \\ &= (I - \lambda Y_3 (I - \lambda Y_1)^{-1} Y_2 V(\lambda))^{-1} Y_3 (I - \lambda Y_1)^{-1} \\ &= (I - \Upsilon_{1,1}(\lambda) V(\lambda))^{-1} \Upsilon_{1,2}(\lambda), \end{aligned}$$

and

$$\begin{aligned}Y_4(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1} &= Y_4(I-\lambda(I-\lambda Y_1)^{-1}Y_2V(\lambda)Y_3)^{-1}(I-\lambda Y_1)^{-1}\\ &= Y_4(I-\lambda Y_1)^{-1}+\lambda Y_4(I-\lambda Y_1)^{-1}Y_2V(\lambda)\times\\ &\qquad \times Y_3(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1}\\ &= \Upsilon_{2,2}(\lambda)+(\Upsilon_{2,1}(\lambda)-Y_5)V(\lambda)\times\\ &\qquad \times(I-\Upsilon_{1,1}(\lambda)V(\lambda))^{-1}\Upsilon_{1,2}(\lambda).\end{aligned}$$

The combination of these two results gives

$$\begin{aligned}\Pi_{\mathcal{Y}}Z(\lambda)(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1} &= (Y_4+Y_5V(\lambda)Y_3)(I-\lambda\Pi_{\mathcal{U}}Z(\lambda))^{-1}\\ &= \Upsilon_{2,2}(\lambda)+(\Upsilon_{2,1}(\lambda)-Y_5+Y_5)V(\lambda)\times\\ &\qquad \times(I-\Upsilon_{1,1}(\lambda)V(\lambda))^{-1}\Upsilon_{1,2}(\lambda)\\ &= \Upsilon_{2,2}(\lambda)+\Upsilon_{2,1}(\lambda)V(\lambda)(I-\Upsilon_{1,1}(\lambda)V(\lambda))^{-1}\Upsilon_{1,2}(\lambda).\end{aligned}$$

So (5.1.15) holds. □

Proof of Theorem 5.1.1. Take $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ such that $Z(\lambda)|_{\mathcal{F}} \equiv \omega$, and let V be the Schur class function in $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ such that (5.1.10) holds. Then Z decomposes as in (5.1.14) with

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & 0 \\ Y_4 & Y_5 \end{bmatrix} = \begin{bmatrix} \Pi_{\mathcal{D}_A}\omega\Pi_{\mathcal{F}} & \Pi_{\mathcal{D}_A}D_{\omega^*} \\ \Pi_{\mathcal{G}} & 0 \\ \Pi_{\mathcal{D}_{T'}}\omega\Pi_{\mathcal{F}} & \Pi_{\mathcal{D}_{T'}}D_{\omega^*} \end{bmatrix} : \begin{bmatrix} \mathcal{D}_A \\ \mathcal{D}_{\omega^*} \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_A \\ \mathcal{G} \\ \mathcal{D}_{T'} \end{bmatrix}. \qquad (5.1.17)$$

After rearranging the rows we see that Y can be identified with

$$\begin{bmatrix} \omega\Pi_{\mathcal{F}} & D_{\omega^*} \\ \Pi_{\mathcal{G}} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_A \\ \mathcal{D}_{\omega^*} \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_{T'} \oplus \mathcal{D}_A \\ \mathcal{G} \end{bmatrix}.$$

Thus we obtain that Y is a contraction. Moreover, Y is unitary if and only if ω is an isometry, or equivalently, Y is unitary if and only if $R^*R = Q^*Q$. With Y_1, Y_2, Y_3, Y_4 and Y_5 as in (5.1.17), the formulas for the functions $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ in (5.1.12) correspond to those in (5.1.1). So Lemma 5.1.4 tells us that $\Upsilon_{1,1} \in \mathbf{S}(\mathcal{D}_{\omega^*}, \mathcal{G})$, $\Upsilon_{2,1} \in \mathbf{S}(\mathcal{D}_{\omega^*}, \mathcal{D}_{T'})$, $\Upsilon_{1,2} \in \mathbf{H}^2_{\text{ball}}(\mathcal{D}_A, \mathcal{G})$, $\Upsilon_{2,2} \in \mathbf{H}^2_{\text{ball}}(\mathcal{D}_A, \mathcal{D}_{T'})$. Moreover, Proposition 5.1.5 and Theorem 1.1 together provide the representation of all contractive interpolants via (5.1.3) and (5.1.2). □

Note that in the proof of Theorem 5.1.1 we did not use the full strength of Lemma 5.1.4. In fact, the next proposition follows immediately from the second part of Lemma 5.1.4 and the proof of Theorem 5.1.1.

Proposition 5.1.6. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' the Sz.-Nagy-Schäffer isometric lifting of T' and with underlying contraction ω. Let $M_{\Upsilon_{1,1}}$ and $M_{\Upsilon_{2,1}}$ be the multiplication operators defined by $\Upsilon_{1,1}$ and $\Upsilon_{2,1}$, respectively, and let*

$\Gamma_{\Upsilon_{1,2}}$ and $\Gamma_{\Upsilon_{2,2}}$ be the operators defined by $\Upsilon_{1,2}$ and $\Upsilon_{2,2}$, respectively, where $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ are the functions defined in (5.1.1). *Then*

$$\tilde{M} = \begin{bmatrix} M_{\Upsilon_{1,1}} & \Gamma_{\Upsilon_{1,2}} \\ M_{\Upsilon_{2,1}} & \Gamma_{\Upsilon_{2,2}} \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{D}_{\omega^*}) \\ \mathcal{D}_A \end{bmatrix} \to \begin{bmatrix} H^2(\mathcal{G}) \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}$$

is a contraction which is unitary whenever $R^*R = Q^*Q$ and $\omega_2\Pi_{\mathcal{F}}$ is pointwise stable.

We conclude this section with some comments on Theorem 5.1.1.

Note that $\Upsilon_{2,2}$ in (5.1.1) is the function in $\mathbf{H}^2(\mathcal{D}_A, \mathcal{D}_{T'})$ that provides the central contractive interpolant for $\{A, T', U', R, Q\}$. So we see that the central contractive interpolant is obtained in Theorem 5.1.1 by taking for V the zero function in $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$. This is in agreement with the remark made in the first paragraph after the proof of Theorem 4.1.1, because the Schur class function Z defined by $V(\lambda) \equiv 0$ via (5.1.10) is precisely $Z(\lambda) \equiv \omega\Pi_{\mathcal{F}}$.

When specified for the classical commutant lifting setting Theorem 5.1.1 reduces to the first part of Theorem VI.5.1 in [41], while Proposition 5.1.6 provides Proposition VI.5.3 in [41].

The remainder of Theorem VI.5.1 in [41] states that in the classical commutant lifting setting the Schur class function V in (5.1.2) is uniquely determined by H. Now recall that Theorem 4.3.2 implies that in the classical commutant lifting setting each contractive interpolant B uniquely determines a Schur class function Z in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ such that B is given by (5.1.8) and (5.1.9), while in (5.1.10) the Schur class function V is uniquely determined by Z. In this way we also obtain the remainder of Theorem VI.5.1 in [41].

Remark 5.1.7. The results of this section can also be translated to results for the H^2 interpolation problem considered in Chapter 4; see Example 5.1.2. This is not the case for the results in the remainder of the present chapter, because they rely heavily on the assumption that the operator A in the lifting data set $\{A, T', U', R, Q\}$ is a strict contraction. As we already observed in Section 4.4, the feature that $\|A\| < 1$ can not be determined from the contraction underlying $\{A, T', U', R, Q\}$. So this can not be expressed in the data for the H^2 interpolation problem of Chapter 4.

5.2 The sub-optimal case

The contraction ω underlying the lifting data set $\Omega = \{A, T', U', R, Q\}$ appears in a prominent way in the Redheffer representation of all contractive interpolants obtained in Theorem 5.1.1, as well as in the Schur representation in Theorem 1.1. However, often it is more convenient to have a description that is explicitly given in terms of the operators appearing in the lifting data set. To achieve this we need additional constraints on the operators in the lifting data set.

In this section we will assume that A is a strict contraction and R has a left inverse. In this case the defect operator D_A of A is strictly positive, and from the fact that $R^*R \leq Q^*Q$ we obtain that Q also has a left inverse. In particular, D_AQ and D_AR are left invertible, or equivalently, $Q^*D_A^2Q$ and $R^*D_A^2R$ are strictly positive operators. So the extra assumptions allow us to define operators X_1, X_2, X_3, X_4 and X_5 by

$$\begin{aligned}
X_1 &= R(Q^*D_A^2Q)^{-1}Q^*D_A^2 : \mathcal{H} \to \mathcal{H}, \\
X_2 &= -R(R^*D_A^2R)^{-1}J^*\Delta_\Omega^{-\frac{1}{2}}\Pi_{\mathcal{D}_\circ\oplus\mathcal{D}_{T'}} + \\
&\qquad -D_A^{-2}\Pi_{\mathrm{Ker}\,R^*}^*\Delta_R^{-\frac{1}{2}}\Pi_{\mathrm{Ker}\,R^*} : \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^* \to \mathcal{H}, \\
X_3 &= \Delta_Q^{-\frac{1}{2}}\Pi_{\mathrm{Ker}\,Q^*} : \mathcal{H} \to \mathrm{Ker}\,Q^*, \\
X_4 &= D_{T'}AR(Q^*D_A^2Q)^{-1}Q^*D_A^2 : \mathcal{H} \to \mathcal{D}_{T'}, \\
X_5 &= \Pi_{\mathcal{D}_{T'}}\Delta_\Omega^{-\frac{1}{2}}\Pi_{\mathcal{D}_\circ\oplus\mathcal{D}_{T'}} : \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^* \to \mathcal{D}_{T'}.
\end{aligned} \tag{5.2.1}$$

Here Δ_Q on $\mathrm{Ker}\,Q^*$, Δ_R on $\mathrm{Ker}\,R^*$ and Δ_Ω on $\mathcal{D}_\circ \oplus \mathcal{D}_{T'}$ are the strictly positive operators:

$$\begin{aligned}
&\Delta_Q = \Pi_{\mathrm{Ker}\,Q^*}D_A^{-2}\Pi_{\mathrm{Ker}\,Q^*}^*, \qquad \Delta_R = \Pi_{\mathrm{Ker}\,R^*}D_A^{-2}\Pi_{\mathrm{Ker}\,R^*}^*, \\
&\Delta_\Omega = I + J(R^*D_A^2R)^{-1}J^*, \quad \text{with} \quad J = \begin{bmatrix} D_\circ \\ D_{T'}AR \end{bmatrix} : \mathcal{H}_0 \to \begin{bmatrix} \mathcal{D}_\circ \\ \mathcal{D}_{T'} \end{bmatrix}.
\end{aligned} \tag{5.2.2}$$

Furthermore, $D_\circ$ and $\mathcal{D}_\circ$ are the operator and Hilbert space defined in (1.16).

In Proposition 5.2.2 below we show that the description of Z in (5.1.10) can be transformed into

$$Z(\lambda) = \begin{bmatrix} X_4 \\ D_AX_1 \end{bmatrix} D_A^{-1} + \begin{bmatrix} X_5 \\ D_AX_2 \end{bmatrix} W(\lambda)X_3D_A^{-1} \qquad (\lambda \in \mathbb{D}), \tag{5.2.3}$$

for some $W \in \mathbf{S}(\mathrm{Ker}\,Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^*)$. Here X_1, X_2, X_3, X_4 and X_5 are the operators defined in (5.2.1). The general scheme for rewriting a Schur representation to a Redheffer type representation presented in Section 5.1 can then be used to obtain the following result.

Theorem 5.2.1. *Let $\Omega = \{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that A is a strict contraction and R has a left inverse. Then all contractive interpolants for $\{A, T', U', R, Q\}$ are given by*

$$B = \begin{bmatrix} A \\ \Lambda \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \tag{5.2.4}$$

with Λ the operator defined by a function $L \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ of the form

$$L(\lambda) = \Phi_{2,2}(\lambda) + \Phi_{2,1}(\lambda)W(\lambda)(I - \Phi_{1,1}(\lambda)W(\lambda))^{-1}\Phi_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}), \tag{5.2.5}$$

where W is an arbitrary function in the Schur class $\mathbf{S}(\mathrm{Ker}\,Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^*)$. Moreover, each function $W \in \mathbf{S}(\mathrm{Ker}\,Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^*)$ defines a function $L \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ via (5.2.5). Here $\Phi_{1,1}$ and $\Phi_{2,1}$ are functions in the Schur classes

$\mathbf{S}(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*, \operatorname{Ker} Q^*)$ and $\mathbf{S}(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*, \mathcal{D}_{T'})$, respectively, and $\Phi_{1,2}$ and $\Phi_{2,2}$ are functions in $\mathbf{H}^2_{\text{ball}}(\mathcal{H}, \operatorname{Ker} Q^*)$ and $\mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$, respectively, and these functions are given by

$$\begin{aligned} \Phi_{1,1}(\lambda) &= \lambda X_3(I-\lambda X_1)^{-1}X_2, \\ \Phi_{1,2}(\lambda) &= X_3(I-\lambda X_1)^{-1}, \\ \Phi_{2,1}(\lambda) &= X_5 + \lambda X_4(I-\lambda X_1)^{-1}X_2, \\ \Phi_{2,2}(\lambda) &= X_4(I-\lambda X_1)^{-1}, \end{aligned} \qquad (\lambda \in \mathbb{D}) \tag{5.2.6}$$

where X_1, X_2, X_3, X_4 and X_5 are the operators defined in (5.2.1)*. Finally, let $M_{\Phi_{1,1}}$ and $M_{\Phi_{2,1}}$ be the multiplication operators defined by $\Phi_{1,1}$ and $\Phi_{2,1}$, respectively, and let $\Gamma_{\Phi_{1,2}}$ and $\Gamma_{\Phi_{2,2}}$ be the operators defined by $\Phi_{1,2}$ and $\Phi_{2,2}$, respectively. Put*

$$M = \begin{bmatrix} 0 & A \\ M_{\Phi_{1,1}} & \Gamma_{\Phi_{1,2}} \\ M_{\Phi_{1,1}} & \Gamma_{\Phi_{2,2}} \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*) \\ \mathcal{H} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\operatorname{Ker} Q^*) \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}.$$

Then M is a contraction, and M is an isometry whenever $R^*R = Q^*Q$ and X_1 is pointwise stable.

The proof of Theorem 5.2.1 is given after the proof of Proposition 5.2.2 below.

When taking the zero function for W in Theorem 5.2.1 we see that (5.2.5) reduces to $L(\lambda) \equiv \Phi_{2,2}(\lambda)$. The contractive interpolant obtained in this way is precisely the central contractive interpolant; see [42], where this formula for the central contractive interpolant was first obtained.

In [42], Proposition 5.3, it was shown that the spectral radius of X_1 in (5.2.1) is strictly less than one if R and Q are such that $R - \lambda Q$ is left invertible for each $\lambda \in \mathbb{D}$. In this case the functions $\Phi_{1,2}$ and $\Phi_{2,2}$ are uniformly bounded on $\mathbb{D}$, that is, $\Phi_{1,2}$ and $\Phi_{2,2}$ are functions in $\mathbf{H}^\infty(\mathcal{H}, \operatorname{Ker} Q^*)$ and $\mathbf{H}^\infty(\mathcal{H}, \mathcal{D}_{T'})$, respectively.

In the classical commutant lifting setting Theorem 5.1.1 reduces to the first part of Theorem VI.6.1 in [41] (see Corollary 5.2.5 below for further details). Moreover, in that case Theorem 5.2.1 provides a proper parameterization, that is, for each contractive interpolant B there exists a unique Schur class function W such that B is given by (5.2.4) and (5.2.5).

Proposition 5.2.2. *Let $\Omega = \{A, T', U', R, Q\}$ be a lifting data set. Assume that A is a strict contraction and R is left invertible. Let Z be an $\mathscr{L}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$-valued function on $\mathbb{D}$. Then $Z \in \mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ and $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if there exists a function W in the Schur class $\mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*)$ such that Z is given by* (5.2.3)*, where X_1, X_2, X_3, X_4 and X_5 are the operators defined in* (5.2.1)*. Finally, Z and W in* (5.2.3) *determine each other uniquely.*

Proof. Recall that $\|A\| < 1$ and R being left invertible imply that D_AQ and D_AR are left invertible, or equivalently, that $Q^*D_A^2Q$ and $R^*D_A^2R$ are strictly positive on $\mathcal{H}_0$. Then left inverses of D_AQ and D_AR are given by

$$L_{D_AQ} = (Q^*D_AQ)^{-1}Q^*D_A \quad \text{and} \quad L_{D_AR} = (R^*D_AR)^{-1}R^*D_A, \tag{5.2.7}$$

respectively. Notice $L_{D_AQ}|_{\operatorname{Ker} Q^*D_A} = 0$ and $L_{D_AR}|_{\operatorname{Ker} R^*D_A} = 0$.

The proof is split into four parts. The first three parts have a preparatory character. The actual proof of the proposition is given in Part 4.

Part 1. In this part we show that

$$\omega\Pi_{\mathcal{F}} = \begin{bmatrix} X_4 \\ D_A X_1 \end{bmatrix} D_A^{-1}.$$

Since $\mathcal{G} = \mathcal{D}_A \ominus \mathcal{F} = \operatorname{Ker} Q^*D_A$, we have $L_{D_AQ}|_{\mathcal{D}_A\ominus\mathcal{F}} = 0$, where L_{D_AQ} is the left inverse of D_AQ given by (5.2.7). From the definition of ω we then obtain that

$$\omega\Pi_{\mathcal{F}} = \begin{bmatrix} D_{T'}AR \\ D_AR \end{bmatrix} L_{D_AQ} = \begin{bmatrix} D_{T'}AR(Q^*D_A^2Q)^{-1}Q^*D_A \\ D_AR(Q^*D_A^2Q)^{-1}Q^*D_A \end{bmatrix} = \begin{bmatrix} X_4 \\ D_AX_1 \end{bmatrix} D_A^{-1}.$$

Part 2. Put

$$Y = \begin{bmatrix} \Delta_\Omega^{-\frac{1}{2}}\Pi^*_{\mathcal{D}_{T'}} & -\Delta_\Omega^{-\frac{1}{2}}JL_{D_AR} \\ 0 & -\Pi_{\operatorname{Ker} R^*D_A} \end{bmatrix} : \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \\ \operatorname{Ker} R^*D_A \end{bmatrix},$$

where Δ_Ω and J are the operators in (5.2.2). In this part we prove that $Y^*Y = D^2_{\omega^*}$ and $\operatorname{Ker} Y^* = \{0\}$.

We first show that J^* intertwines the operator Δ_Ω with $Q^*D_A^2Q(R^*D_A^2R)^{-1}$, that is,

$$J^*\Delta_\Omega = Q^*D_A^2Q(R^*D_A^2R)^{-1}J^*. \tag{5.2.8}$$

To see this, observe that the identity in (1.17) and the definition of J in (5.2.2) imply that $Q^*D_A^2Q = R^*D_A^2R + J^*J$. With this equality we obtain that

$$\begin{aligned} J^*\Delta_\Omega &= J^*(I + J(R^*D_A^2R)^{-1}J^*) = (I + J^*J(R^*D_A^2R)^{-1})J^* \\ &= (R^*D_A^2R + Q^*D_A^2Q - R^*D_A^2R)(R^*D_A^2R)^{-1}J^* \\ &= Q^*D_AQ(R^*D_AR)^{-1}J^*. \end{aligned}$$

So (5.2.8) holds.

From (5.2.8) it follows that $J^*\Delta_\Omega^{-1} = R^*D_A^2R(Q^*D_A^2Q)^{-1}J^*$ and

$$(I - J(Q^*D_A^2Q)^{-1}J^*)\Delta_\Omega = \Delta_\Omega - J(Q^*D_AQ)^{-1}J^*\Delta_\Omega = \Delta_\Omega - J(R^*D_A^2R)^{-1}J^* = I.$$

Since Δ_Ω is nonnegative, we see that Δ_Ω is strictly positive and that the inverse of Δ_Ω is given by

$$\Delta_\Omega^{-1} = I - J(Q^*D_A^2Q)^{-1}J^*. \tag{5.2.9}$$

The intertwining relation (5.2.8) also yields

$$\begin{aligned} \Delta_\Omega^{-1}JL_{D_AR} &= J(Q^*D_A^2Q)^{-1}R^*D_A^2R(R^*D_A^2R)^{-1}R^*D_A \\ &= J(Q^*D_A^2Q)^{-1}R^*D_A. \end{aligned} \tag{5.2.10}$$

Using $L_{D_AQ}L_{D_AQ}^* = (Q^*D_A^2Q)^{-1}$ and the formula for $\omega\Pi_{\mathcal{F}}$ obtained in Part 1 we see that

$$D_{\omega^*}^2 = \begin{bmatrix} I_{\mathcal{D}_{T'}} & 0 \\ 0 & I_{\mathcal{D}_A} \end{bmatrix} - \begin{bmatrix} D_{T'}AR \\ D_AR \end{bmatrix} (Q^*D_A^2Q)^{-1} \begin{bmatrix} R^*A^*D_{T'} & R^*D_A \end{bmatrix}.$$

From (5.2.9) we then obtain

$$\begin{aligned} \Pi_{\mathcal{D}_{T'}}Y^*Y\Pi_{\mathcal{D}_{T'}}^* &= \Pi_{D_{T'}}\Delta_\Omega^{-1}\Pi_{\mathcal{D}_{T'}}^* = I - D_{T'}AR(Q^*D_A^2Q)^{-1}R^*A^*D_{T'} \\ &= \Pi_{\mathcal{D}_{T'}}D_{\omega^*}^2\Pi_{\mathcal{D}_{T'}}^*. \end{aligned}$$

The formula for $\Delta_\Omega^{-1}JL_{D_AR}$ in (5.2.10) yields

$$\begin{aligned} \Pi_{\mathcal{D}_{T'}}Y^*Y\Pi_{\mathcal{D}_A} &= -\Pi_{\mathcal{D}_{T'}}\Delta_\Omega^{-1}JL_{D_AR} = -\Pi_{\mathcal{D}_{T'}}J(Q^*D_A^2Q)^{-1}R^*D_A \\ &= -D_{T'}AR(Q^*D_A^2Q)^{-1}R^*D_A = \Pi_{\mathcal{D}_{T'}}D_{\omega^*}^2\Pi_{\mathcal{D}_A}. \end{aligned}$$

Again using (5.2.10) and the observation that $J^*J = Q^*D_A^2Q - R^*D_A^2R$ we get

$$\begin{aligned} \Pi_{\mathcal{D}_A}Y^*Y\Pi_{\mathcal{D}_A} &= P_{\mathrm{Ker}\,R^*D_A} + L_{D_AR}^*J^*\Delta_\Omega^{-1}JL_{D_AR} \\ &= P_{\mathrm{Ker}\,R^*D_A} + D_AR(R^*D_A^2R)^{-1}J^*J(Q^*D_A^2Q)^{-1}R^*D_A \\ &= P_{\mathrm{Ker}\,R^*D_A} + P_{\mathrm{Im}\,D_AR} - D_AR(Q^*D_A^2Q)^{-1}R^*D_A \\ &= \Pi_{\mathcal{D}_A}D_{\omega^*}^2\Pi_{\mathcal{D}_A}. \end{aligned}$$

This proves that $Y^*Y = D_{\omega^*}^2$, as we claimed.

Next we show that $\mathrm{Ker}\,Y^* = \{0\}$. Since $Y^*|_{\mathrm{Ker}\,R^*D_A} = -\Pi_{\mathrm{Ker}\,R^*D_A}^*$ and

$$Y^*(\mathcal{D}_\circ \oplus \mathcal{D}_{T'}) \subset \mathcal{D}_\circ \oplus \mathrm{Im}\,D_AR = \mathcal{D}_\circ \oplus (\mathcal{D}_A \ominus \mathrm{Ker}\,R^*D_A),$$

it suffices to show that $\mathrm{Ker}\,Y^*|_{\mathcal{D}_\circ\oplus\mathcal{D}_{T'}} = \{0\}$. To prove the latter note that $R^*D_AL_{D_AR}^* = I$. So $\mathrm{Ker}\,L_{D_AR}^* = \{0\}$. Moreover, we have $\mathrm{Ker}\,D_\circ = \{0\}$, when $D_\circ$ is seen as an operator on $\mathcal{D}_\circ$. Therefore

$$\begin{aligned} \mathrm{Ker}\,Y^*|_{\mathcal{D}_\circ\oplus\mathcal{D}_{T'}} &= \mathrm{Ker}\begin{bmatrix} \Pi_{\mathcal{D}_{T'}}\Delta_\Omega^{-\frac{1}{2}} \\ L_{D_AR}^*J^*\Delta_\Omega^{-\frac{1}{2}} \end{bmatrix} \\ &= \mathrm{Ker}\begin{bmatrix} I_{\mathcal{D}_{T'}} & 0 \\ 0 & L_{D_AR}^* \end{bmatrix}\begin{bmatrix} \Pi_{\mathcal{D}_{T'}} \\ J^* \end{bmatrix}\Delta_\Omega^{-\frac{1}{2}} \\ &= \Delta_\Omega^{\frac{1}{2}}\mathrm{Ker}\begin{bmatrix} \Pi_{\mathcal{D}_{T'}} \\ J^* \end{bmatrix} = \Delta_\Omega^{\frac{1}{2}}\mathrm{Ker}\begin{bmatrix} 0 & I_{\mathcal{D}_{T'}} \\ D_\circ & R^*A^*D_{T'} \end{bmatrix} = \{0\}. \end{aligned}$$

Part 3. Next we show that the orthogonal projections on $\mathrm{Ker}\,Q^*D_A$ and $\mathrm{Ker}\,R^*D_A$ are given by

$$\begin{aligned} P_{\mathrm{Ker}\,Q^*D_A} &= D_A^{-1}\Pi_{\mathrm{Ker}\,Q^*}^*\Delta_Q^{-1}\Pi_{\mathrm{Ker}\,Q^*}D_A^{-1}, \\ P_{\mathrm{Ker}\,R^*D_A} &= D_A^{-1}\Pi_{\mathrm{Ker}\,R^*}^*\Delta_R^{-1}\Pi_{\mathrm{Ker}\,R^*}D_A^{-1}, \end{aligned} \tag{5.2.11}$$

where Δ_Q and Δ_R are the operators in (5.2.2).

Let N be any operator from $\mathcal{H}_0$ to $\mathcal{H}$. In particular, we can take R and Q for N. Then $D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*}$ has a left inverse, and thus $\Delta_N = \Pi_{\mathrm{Ker}\,N^*} D_A^{-2} \Pi^*_{\mathrm{Ker}\,N^*}$ is strictly positive on $\mathrm{Ker}\,N^*$. Since $D_A^{-1}(D_A N) = N$, we see that $D_A^{-1}\overline{\mathrm{Im}\,D_A N} = \overline{\mathrm{Im}\,N}$. This implies that $\mathrm{Im}\,D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*} = D_A^{-1}\mathrm{Ker}\,N^* = \mathrm{Ker}\,N^* D_A$. Since $D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*}$ is left invertible and $\Delta_N = (D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*})^* D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*}$, we obtain that orthogonal projections on $\mathrm{Ker}\,N^* D_A$ is given by

$$P_{\mathrm{Ker}\,N^* D_A} = P_{\mathrm{Im}\,D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*}} = D_A^{-1}\Pi^*_{\mathrm{Ker}\,N^*}\Delta_N^{-1}\Pi_{\mathrm{Ker}\,N^*} D_A^{-1}.$$

Inserting Q and R for N provides (5.2.11). Moreover, we obtain that the kernels of $\mathrm{Ker}\,Q^* D_A$ and $\mathrm{Ker}\,R^* D_A$ can be written as $\mathrm{Ker}\,Q^* D_A = \mathrm{Im}\,D_A^{-1}\Pi^*_{\mathrm{Ker}\,Q^*}$ and $\mathrm{Ker}\,R^* D_A = \mathrm{Im}\,D_A^{-1}\Pi^*_{\mathrm{Ker}\,R^*}$.

Part 4. In this part we combine the results from the Parts 1 to 3 to complete the proof of Proposition 5.2.2.

The identities $Y^*Y = D_{\omega^*}$ and $\mathrm{Ker}\,Y^* = \{0\}$ derived in Part 2 show that there exists a unitary operator ϕ_ω mapping $\mathcal{D}_{\omega^*}$ onto $\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^* D_A$ such that $Y^*\phi_\omega = D_{\omega^*}$. Furthermore, from (5.2.11) we obtain that there exist unitary operators ϕ_Q and ϕ_R mapping $\mathrm{Ker}\,Q^*$ and $\mathrm{Ker}\,R^*$ onto $\mathrm{Ker}\,Q^* D_A = \mathcal{G}$ and $\mathrm{Ker}\,R^* D_A$, respectively, such that

$$\phi_Q \Delta_Q^{-\frac{1}{2}} \Pi_{\mathrm{ker}\,Q^*} D_A^{-1} = \Pi_{\mathrm{Ker}\,Q^* D_A} \quad \text{and} \quad \phi_R \Delta_R^{-\frac{1}{2}} \Pi_{\mathrm{ker}\,R^*} D_A^{-1} = \Pi_{\mathrm{Ker}\,R^* D_A}.$$

Now put

$$\phi_* = \begin{bmatrix} I_{\mathcal{D}_\circ \oplus \mathcal{D}_{T'}} & 0 \\ 0 & \phi_R^* \end{bmatrix} \begin{bmatrix} \Pi_{\mathcal{D}_\circ \oplus \mathcal{D}_{T'}} \phi_\omega \\ \Pi_{\mathrm{Ker}\,R^* D_A} \phi_\omega \end{bmatrix} : \mathcal{D}_{\omega^*} \to \begin{bmatrix} \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \\ \mathrm{Ker}\,R^* \end{bmatrix}.$$

Then ϕ_* is unitary and we have

$$\begin{bmatrix} X_5 \\ D_A X_2 \end{bmatrix} \phi_* = \begin{bmatrix} \Pi_{\mathcal{D}_{T'}} \Delta_\Omega^{-\frac{1}{2}} & 0 \\ -L^*_{D_A R} J^* \Delta_\Omega^{-\frac{1}{2}} & -D_A^{-1}\Pi^*_{\mathrm{Ker}\,R^*}\Delta_R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I_{\mathcal{D}_\circ \oplus \mathcal{D}_{T'}} & 0 \\ 0 & \phi_R^* \end{bmatrix} \times$$

$$\times \begin{bmatrix} \Pi_{\mathcal{D}_\circ \oplus \mathcal{D}_{T'}} \phi_\omega \\ \Pi_{\mathrm{Ker}\,R^* D_A} \phi_\omega \end{bmatrix} = Y^* \phi_\omega = D_{\omega^*}.$$

From the definition of ϕ_Q we obtain that

$$\phi_Q X_3 D_A^{-1} = \phi_Q \Delta_Q^{-\frac{1}{2}} \Pi_{\mathrm{ker}\,Q^*} D_A^{-1} = \Pi_{\mathrm{Ker}\,Q^* D_A}.$$

In the second paragraph after Example 5.1.3 it was already observed that an $\mathscr{L}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$-valued function Z is in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ and satisfies the equality $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if there exists a Schur class function V in $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ such that Z is given by (5.1.10), and that in this case V in (5.1.10) is uniquely determined by Z.

The fact that ϕ_Q and ϕ_* are unitary implies that an operator-valued function W is in the Schur class $\mathbf{S}(\mathrm{Ker}\,Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \mathrm{Ker}\,R^* D_A)$ if and only if

$$W(\lambda) = \phi_* V(\lambda) \phi_Q \qquad (\lambda \in \mathbb{D})$$

for some Schur class function V in $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$, and W and V in (5.2) determine each other uniquely. Therefore we obtain that a function Z is in $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ and satisfies $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ if and only if there exists a Schur class function W in $\mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^* D_A)$ such that

$$\begin{aligned} Z(\lambda) &= \omega\Pi_{\mathcal{F}} + D_{\omega^*}\phi_*^* W(\lambda)\phi_Q^* \Pi_{\operatorname{Ker} Q^* D_A} \\ &= \begin{bmatrix} X_4 D_A^{-1} \\ D_A X_1 D_A^{-1} \end{bmatrix} + \begin{bmatrix} X_5 \\ D_A X_2 \end{bmatrix} W(\lambda) X_3 D_A^{-1} \qquad (\lambda \in \mathbb{D}), \end{aligned}$$

and W and Z determine each other uniquely. □

We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let B be a contractive interpolant. Let Z be in the Schur class $\mathbf{S}(\mathcal{D}_A, \mathcal{D}_{T'} \oplus \mathcal{D}_A)$ with $Z(\lambda)|_{\mathcal{F}} \equiv \omega$ such that B is given by (5.1.8), where Γ is the operator defined by $H \in \mathbf{H}^2(\mathcal{D}_A, \mathcal{D}_{T'})$ in (5.1.9). Let W be the Schur class function in $\mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^* D_A)$ such that Z is given by (5.2.3). Put

$$Y_1 = D_A X_1 D_A^{-1}, \quad Y_2 = D_A X_2, \quad Y_3 = X_3 D_A^{-1}, \quad Y_4 = X_4 D_A^{-1}, \quad Y_5 = X_5. \quad (5.2.12)$$

Then

$$\begin{aligned} \left[\begin{array}{c|c} Y_3 & 0 \\ \hline Y_4 & Y_5 \\ Y_1 & Y_2 \end{array}\right] &= \left[\begin{array}{c|c} X_3 D_A^{-1} & 0 \\ \hline X_4 D_A^{-1} & X_5 \\ D_A X_1 D_A^{-1} & D_A X_2 \end{array}\right] \\ &= \begin{bmatrix} \phi_Q^* & 0 \\ 0 & I_{\mathcal{D}_{T'} \oplus \mathcal{D}_A} \end{bmatrix} \begin{bmatrix} \Pi_{\mathcal{D}_A \ominus \mathcal{F}} & 0 \\ \omega \Pi_{\mathcal{F}} & D_{\omega^*} \end{bmatrix} \begin{bmatrix} I_{\mathcal{D}_A} & 0 \\ 0 & \phi_*^* \end{bmatrix}. \end{aligned} \qquad (5.2.13)$$

Here ϕ_Q and ϕ_* are the unitary operators constructed in Part 4 of the proof of Proposition 5.2.2. With the same argument as in the proof of Theorem 5.1.1, and after rearranging rows in (5.2.13), we obtain that Y in (5.1.11) is a contraction. Moreover, this contraction is unitary whenever $R^*R = Q^*Q$.

Let $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ be the functions defined in (5.1.12), and let $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ be the functions defined in (5.2.6). Using that the identity $(I - \lambda Y_1)^{-1} = D_A (I - \lambda X_1)^{-1} D_A^{-1}$ holds, we see from the relations in (5.2.12) that

$$\begin{aligned} &\Upsilon_{1,1}(\lambda) = \Phi_{1,1}(\lambda), \quad \Upsilon_{1,2}(\lambda) D_A = \Phi_{1,2}(\lambda), \\ &\Upsilon_{2,1}(\lambda) = \Phi_{2,1}(\lambda) \quad \text{and} \quad \Upsilon_{2,2}(\lambda) D_A = \Phi_{2,2}(\lambda). \end{aligned} \qquad (\lambda \in \mathbb{D}) \qquad (5.2.14)$$

Applying Lemma 5.1.4 we obtain that $\Phi_{1,1}$ and $\Phi_{2,1}$ are Schur class functions in $\mathbf{S}(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*, \operatorname{Ker} Q^*)$ and $\mathbf{S}(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*, \mathcal{D}_{T'})$, respectively, and that $\Phi_{1,2}$ and $\Phi_{2,2}$ are functions in $\mathbf{H}^2_{\text{ball}}(\mathcal{H}, \operatorname{Ker} Q^*)$ and $\mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$, respectively.

Moreover, Lemma 5.1.4 implies that the operator $\widetilde{M}$ in (5.1.13) is a contraction which is unitary in case Y in (5.1.11) is unitary and Y_1 is pointwise stable. Since X_1 and Y_1 are similar ($D_A X_1 = Y_1 D_A$ and $D_A \gg 0$), we see that Y_1 is pointwise stable if and only if X_1 is pointwise stable. We already observed that Y is unitary when

$R^*R = Q^*Q$. So we can conclude that the operator $\tilde{M}$ in (5.1.13) is a contraction which is unitary if X_1 is pointwise stable and $R^*R = Q^*Q$.

In terms of operators the identities in (5.2.14) become

$$\begin{aligned} M_{\Upsilon_{1,1}} &= M_{\Phi_{1,1}}, \quad \Gamma_{\Upsilon_{1,2}} = \Gamma_{\Phi_{1,1}} D_A^{-1}, \\ M_{\Upsilon_{2,1}} &= M_{\Phi_{2,1}} \quad \text{and} \quad \Gamma_{\Upsilon_{1,1}} = \Gamma_{\Phi_{1,1}} D_A^{-1}. \end{aligned} \tag{5.2.15}$$

Next define a contraction $\overline{A}$ by

$$\overline{A} = \begin{bmatrix} 0 & A \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{W}) \\ \mathcal{H} \end{bmatrix} \to \mathcal{H}',$$

where $\mathcal{W} = \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*$. The defect operator of $\overline{A}$ is given by

$$D_{\overline{A}} = \begin{bmatrix} I_{H^2(\mathcal{W})} & 0 \\ 0 & D_A \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{W}) \\ \mathcal{H} \end{bmatrix} \to \begin{bmatrix} H^2(\mathcal{W}) \\ \mathcal{D}_A \end{bmatrix}.$$

Then the identities in (5.2.15) show that

$$\begin{aligned} M &= \begin{bmatrix} 0 & A \\ M_{\Phi_{1,1}} & \Gamma_{\Phi_{1,2}} \\ M_{\Phi_{2,1}} & \Gamma_{\Phi_{2,2}} \end{bmatrix} = \begin{bmatrix} 0 & A \\ M_{\Upsilon_{1,1}} & \Gamma_{\Upsilon_{1,2}} D_A \\ M_{\Upsilon_{2,1}} & \Gamma_{\Upsilon_{2,2}} D_A \end{bmatrix} = \begin{bmatrix} \overline{A} \\ \tilde{M} D_{\overline{A}} \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \tilde{M} \end{bmatrix} \begin{bmatrix} \overline{A} \\ D_{\overline{A}} \end{bmatrix}. \end{aligned}$$

It is well known that the operator $\begin{bmatrix} \overline{A}^* & D_{\overline{A}} \end{bmatrix}^*$ is an isometry. Therefore we obtain that the operator M is a contraction which is isometric in case $R^*R = Q^*Q$ and X_1 is pointwise stable.

Finally, let $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{D}_A, \mathcal{D}_{T'})$ be given by (5.1.9). From Proposition 5.1.5 we obtain for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned} H(\lambda) D_A &= \Pi_{\mathcal{D}_{T'}} Z(\lambda)(I - \lambda \Pi_{\mathcal{D}_A} Z(\lambda))^{-1} D_A \\ &= \Upsilon_{2,2}(\lambda) D_A + \Upsilon_{2,1}(\lambda) W(\lambda)(I - \Upsilon_{1,1}(\lambda) W(\lambda))^{-1} \Upsilon_{1,2}(\lambda) D_A \\ &= \Phi_{2,2}(\lambda) + \Phi_{2,1}(\lambda) W(\lambda)(I - \Phi_{1,1}(\lambda) W(\lambda))^{-1} \Phi_{1,2}(\lambda) = L(\lambda), \end{aligned}$$

where L is the function defined in (5.2.5) with W the Schur class function determined by (5.2.3). □

The next corollary provides some necessary conditions for the problem posed in Open Problem 4.4.4.

Corollary 5.2.3. *Let $\{A, T', U', R, Q\}$ be a lifting data set with A a strict contraction and R left invertible, and let ω be the contraction underlying $\{A, T', U', R, Q\}$. Then $\omega_2 = \Pi_{\mathcal{D}_A} \omega$ is left invertible and $\omega_1 = \Pi_{\mathcal{D}_{T'}} \omega$ is a strict contraction.*

Proof. The formula for ω obtained in Part 1 of the proof of Proposition 5.2.2 shows that

$$\omega_2 = D_A R(Q^* D_A Q)^{-1} Q^* D_A|_{\mathcal{F}}.$$

Since $D_A Q$ is left invertible and $\mathcal{F} = \operatorname{Im} D_A Q$, the operator $Q^* D_A|_{\mathcal{F}}$ is invertible. The fact that $D_A R$ is left invertible then proves that ω_2 is left invertible as well. This implies that

$$I_{\mathcal{F}} - \omega_1^* \omega_1 \geq \omega_2^* \omega_2 \gg 0.$$

In other words, ω_1 is a strict contraction. □

Example 5.2.4. Let $\{A, T', U', R, Q\}$ be the lifting data set given by (4.4.2) with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Note that A is a strict contraction, or equivalently,

$$D_A = \begin{bmatrix} \frac{1}{4}\sqrt{15} & 0 \\ 0 & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

is invertible. Moreover, R is left invertible, and

$$D_\circ = \sqrt{3}, \quad Q^* D_A^2 Q = \frac{15}{4}, \quad R^* D_A^2 R = \frac{3}{4}, \quad \Delta_Q = \frac{4}{3}, \quad \Delta_R = \frac{16}{15},$$

$$J = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} \quad \text{and} \quad \Delta_\Omega = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

So we obtain that the functions $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ in (5.2.6) are given by

$$\begin{aligned} \Phi_{1,1}(\lambda) &= \begin{bmatrix} \frac{2\lambda}{5}\sqrt{5} & 0 & \frac{\lambda^2}{2}\sqrt{5} \end{bmatrix}, \\ \Phi_{1,2}(\lambda) &= \begin{bmatrix} \frac{\lambda}{4}\sqrt{3} & \frac{1}{2}\sqrt{3} \end{bmatrix}, \\ \Phi_{2,1}(\lambda) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ \Phi_{2,2}(\lambda) &= \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Let $W \in \mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*) = \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ be of the form

$$W(\lambda) = \begin{bmatrix} W_1(\lambda) \\ W_2(\lambda) \\ W_3(\lambda) \end{bmatrix} \quad (\lambda \in \mathbb{D}). \tag{5.2.16}$$

In this case the function L in (5.2.5) is given by

$$L(\lambda) = \frac{W_2(\lambda)}{20 + 4\lambda^2\sqrt{5}W_3(\lambda) + 8\lambda\sqrt{5}W_1(\lambda)} \begin{bmatrix} 5\lambda\sqrt{3} & 10\sqrt{3} \end{bmatrix} \quad (\lambda \in \mathbb{D}). \tag{5.2.17}$$

Next let H and V be as in Example 5.1.3. Then $L(\lambda) \equiv H(\lambda) D_A$ for the case when $V(\lambda) \equiv \phi_\omega W(\lambda)$, where

$$\phi_\omega = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

is the unitary operator constructed in Part 4 of the proof of Proposition 5.2.2. The identity $L(\lambda) \equiv H(\lambda)D_A$ also follows from the fact that the functions $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ obtained here can be written as

$$\Phi_{1,1}(\lambda) \equiv \Upsilon_{1,1}(\lambda)\phi_\omega^*, \quad \Phi_{1,2}(\lambda) \equiv \Upsilon_{1,2}(\lambda)D_A,$$
$$\Phi_{2,1}(\lambda) \equiv \Upsilon_{2,1}(\lambda)\phi_\omega^* \quad \text{and} \quad \Phi_{2,2}(\lambda) \equiv \Upsilon_{2,2}(\lambda)D_A,$$

where $\Upsilon_{1,1}$, $\Upsilon_{1,2}$, $\Upsilon_{2,1}$ and $\Upsilon_{2,2}$ are as in Example 5.1.3. $\diamond$

We conclude this section with a corollary that specifies Theorem 5.2.1 for the classical commutant lifting setting, cf., Theorem VI.6.1 in [41].

Corollary 5.2.5. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that Q is an isometry on $\mathcal{H}$, $R = I_\mathcal{H}$ and A is a strict contraction. Then $\operatorname{Ker} R^* = \{0\}$, $\mathcal{D}_\circ = \{0\}$ and the operator-valued functions $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ in (5.2.6) are functions in $\mathbf{S}(\mathcal{D}_{T'}, \operatorname{Ker} Q^*)$, $\mathbf{H}^\infty(\mathcal{H}, \operatorname{Ker} Q^*)$, $\mathbf{S}(\mathcal{D}_{T'}, \mathcal{D}_{T'})$ and $\mathbf{H}^\infty(\mathcal{H}, \mathcal{D}_{T'})$, respectively, and they are given by*

$$\begin{aligned}
\Phi_{1,1}(\lambda) &= -\lambda\Delta_Q^{-\frac{1}{2}}\Pi_{\operatorname{Ker} Q^*}(I - \lambda T_A)^{-1}D_A^{-2}A^*D_{T'}\Delta_\Omega^{-\frac{1}{2}},\\
\Phi_{1,2}(\lambda) &= \Delta_Q^{-\frac{1}{2}}\Pi_{\operatorname{Ker} Q^*}(I - \lambda T_A)^{-1},\\
\Phi_{2,1}(\lambda) &= \Delta_\Omega^{\frac{1}{2}} - D_{T'}A(I - \lambda T_A)^{-1}D_A^{-2}A^*D_{T'}\Delta_\Omega^{-\frac{1}{2}},\\
\Phi_{2,2}(\lambda) &= D_{T'}A(I - \lambda T_A)^{-1}T_A,
\end{aligned} \qquad (\lambda \in \mathbb{D}) \quad (5.2.18)$$

where T_A on $\mathcal{H}$, Δ_Q on $\operatorname{Ker} Q^*$ and Δ_Ω on $\mathcal{D}_{T'}$ are given by

$$\begin{gathered}
T_A = D_{AQ}^{-2}Q^*D_A^2,\\
\Delta_Q = \Pi_{\operatorname{Ker} Q^*}D_A^{-1}\Pi_{\operatorname{Ker} Q^*}^* \quad \textit{and} \quad \Delta_\Omega = I + D_{T'}AD_A^{-2}A^*D_{T'}.
\end{gathered} \qquad (5.2.19)$$

Proof. Since $R = I_\mathcal{H}$ and $R^*R = I_\mathcal{H} = Q^*Q$, we obtain that $\operatorname{Ker} R^* = \{0\}$ and $D_\circ = 0$ on $\mathcal{D}_\circ = \{0\}$. Clearly R is left invertible and, by assumption, A is a strict contraction. Moreover, we have that $R - \lambda Q = I_\mathcal{H} - \lambda Q$ is (left) invertible for each $\lambda \in \mathbb{D}$, because Q is contractive. Hence the result of Theorem 5.2.1, and the remark in the third paragraph underneath Theorem 5.2.1 hold for this lifting data set. In particular, $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ in (5.2.6) are functions in $\mathbf{S}(\mathcal{D}_{T'}, \operatorname{Ker} Q^*)$, $\mathbf{H}^\infty(\mathcal{H}, \operatorname{Ker} Q^*)$, $\mathbf{S}(\mathcal{D}_{T'}, \mathcal{D}_{T'})$ and $\mathbf{H}^\infty(\mathcal{H}, \mathcal{D}_{T'})$, respectively.

Observe that J in (5.2.2) is given by $J = D_{T'}A$ and we have

$$Q^*D_A^2Q = Q^*Q - Q^*A^*AQ = I - Q^*A^*AQ = D_{AQ}^2.$$

Therefore we have that $X_1 = T_A$ with X_1 from (5.2.1). Note that Δ_Q and Δ_Ω given in (5.2.2) reduce to the formulas in (5.2.19). So we obtain that the operators X_1, X_2, X_3, X_4 and X_5 in (5.2.1), under the present assumptions, are

$$X_1 = T_A, \quad X_2 = -D_A^{-2}A^*D_{T'}\Delta_\Omega^{-\frac{1}{2}}, \quad X_3 = \Delta_Q^{-\frac{1}{2}}\Pi_{\operatorname{Ker} Q^*},$$
$$X_4 = D_{T'}AT_A \quad \text{and} \quad X_5 = \Delta_\Omega^{-\frac{1}{2}}.$$

This immediately shows that the formulas for $\Phi_{1,1}$, $\Phi_{1,2}$ and $\Phi_{2,2}$ in (5.2.6) are given by (5.2.18) in the classical commutant lifting setting. Furthermore, we have for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned}
\Phi_{2,1}(\lambda) &= X_5 + \lambda X_4(I - \lambda X_1)^{-1} X_2 \\
&= \Delta_\Omega^{-\frac{1}{2}} - \lambda D_{T'} A T_A (I - \lambda T_A)^{-1} D_A^{-1} A^* D_{T'} \Delta_\Omega^{-\frac{1}{2}} \\
&= (I + D_{T'} A D_A^{-1} A^* D_{T'}) \Delta_\Omega^{-\frac{1}{2}} - D_{T'} A (I - \lambda T_A)^{-1} D_A^{-1} A^* D_{T'} \Delta_\Omega^{-\frac{1}{2}} \\
&= \Delta_\Omega \Delta_\Omega^{-\frac{1}{2}} - D_{T'} A (I - \lambda T_A)^{-1} D_A^{-1} A^* D_{T'} \Delta_\Omega^{-\frac{1}{2}} \\
&= \Delta_\Omega^{\frac{1}{2}} - D_{T'} A (I - \lambda T_A)^{-1} D_A^{-1} A^* D_{T'} \Delta_\Omega^{-\frac{1}{2}}.
\end{aligned}$$

So $\Phi_{2,1}$ in (5.2.6) also reduces to its formula in (5.2.18). □

5.3 New formulas for the Redheffer coefficients in the sub-optimal case

In this section we present new formulas for the Redheffer coefficients in Theorem 5.2.1. As before assume that A is a strict contraction and R is left invertible. In order to present the alternative formulas we first introduce analytic operator-valued functions K_1, K_2, K_3, K_4, K_5 and K_6 on $\mathbb{D}$, as follows

$$\begin{aligned}
K_1(\lambda) &= \Pi_{\mathrm{Ker}\, Q^*} (I - \lambda M)^{-1} D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^*, \\
K_2(\lambda) &= \Pi_{\mathrm{Ker}\, Q^*} (I - \lambda M)^{-1} D_A^{-2} \Pi_{\mathrm{Ker}\, R^*}^*, \\
K_3(\lambda) &= \Pi_{\mathrm{Ker}\, Q^*} (I - \lambda M)^{-1} R (R^* D_A^2 R)^{-1} J^*, \\
K_4(\lambda) &= D_{T'} A M (I - \lambda M)^{-1} D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^*, \\
K_5(\lambda) &= \lambda D_{T'} A M (I - \lambda M)^{-1} D_A^{-2} \Pi_{\mathrm{Ker}\, R^*}^*, \\
K_6(\lambda) &= -\Pi_{\mathcal{D}_T'} + \lambda D_{T'} A M (I - \lambda M)^{-1} R (R^* D_A^2 R)^{-1} J^*,
\end{aligned} \qquad (\lambda \in \mathbb{D}) \qquad (5.3.1)$$

where J is the operator in (5.2.2) and

$$M = R(Q^*Q)^{-1} Q^*. \qquad (5.3.2)$$

To see that the functions K_1, K_2, K_3, K_4, K_5 and K_6 are properly defined and analytic on $\mathbb{D}$, note that the inequality $R^*R \le Q^*Q$ implies that Q^*Q is invertible. In particular,

$$M^*M = Q(Q^*Q)^{-1} R^* R (Q^*Q)^{-1} Q^* \le Q(Q^*Q)^{-1} Q^* = P_{\mathrm{Im}\, Q} \le I,$$

which shows that M is contractive. This proves our claim.

In Proposition 5.3.1 below we present formulas for the functions $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ in (5.2.6), expressed in terms of the functions K_1, K_2, K_3, K_4, K_5 and

K_6 in (5.3.1). The reason for considering a representation in terms of the functions in (5.3.1) is justified by the applications. For many of the metric constrained interpolation problems that can be put into the (relaxed) commutant lifting setting, the operators R and Q appearing in the lifting data set are independent of the data of the interpolation problem in question. Thus the same holds true for M in (5.3.2). For instance, in the Nevanlinna-Pick interpolation problem R is the identity operator and Q the forward shift operator both on some Hardy space $H^2(\mathcal{U})$. In that case M is the backwards shift, and for each $\lambda \in \mathbb{D}$ the term $\Pi_{\mathrm{Ker}\, Q^*}(I - \lambda M)^{-1}$ appearing in the functions K_1, K_2 and K_3 is the operator of point evaluation in λ. For the relaxed interpolation problems considered in [42] and [62], the operator M is the finite backward shift on $\mathcal{U}^N$, for some Hilbert space $\mathcal{U}$; see also Section 6.1 below. In that case the functions in (5.3.1) are operator polynomials.

Proposition 5.3.1. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that A is a strict contraction and R is left invertible, and let $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ be as in* (5.2.6)*. Let K_1, K_2, K_3, K_4, K_5 and K_6 be the functions in* (5.3.1)*. Then $K_1(\lambda)$ is an invertible operator for each $\lambda \in \mathbb{D}$, and*

$$\begin{aligned}
\Phi_{1,1}(\lambda) &= -\lambda \Delta_Q^{\frac{1}{2}} K_1(\lambda)^{-1} \begin{bmatrix} K_3(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_2(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix},\\
\Phi_{1,2}(\lambda) &= \Delta_Q^{\frac{1}{2}} K_1(\lambda)^{-1} \Pi_{\mathrm{Ker}\, Q^*}(I - \lambda M)^{-1},\\
\Phi_{2,1}(\lambda) &= -\begin{bmatrix} K_6(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_5(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix} + \qquad\qquad (\lambda \in \mathbb{D}) \qquad (5.3.3)\\
&\qquad + \lambda K_4(\lambda) K_1(\lambda)^{-1} \begin{bmatrix} K_3(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_2(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix},\\
\Phi_{2,2}(\lambda) &= (D_{T'} A M - K_4(\lambda) K_1(\lambda)^{-1} \Pi_{\mathrm{Ker}\, Q^*})(I - \lambda M)^{-1}.
\end{aligned}$$

Here Δ_Q, Δ_R and Δ_Ω are the strictly positive operators defined in (5.2.2) *and $M = R(Q^*Q)^{-1}Q^*$.*

Proof. The proof is split in three parts. In the first part we show that

$$X_1 = M(I - D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^* \Delta_Q^{-1} \Pi_{\mathrm{Ker}\, Q^*}). \qquad (5.3.4)$$

Using this formula for X_1 we derive an expression for $(I - \lambda X_1)^{-1}$ that involves the inverse of $K_1(\lambda)$, where $\lambda \in \mathbb{D}$ is arbitrary. In particular, we obtain that $K_1(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$. The formula for the inverse of $I - \lambda X_1$ is used in Part 2 to derive the formulas in (5.3.3) for $\Phi_{1,2}$ and $\Phi_{1,1}$, and in Part 3 for $\Phi_{2,2}$ and $\Phi_{2,1}$.

Part 1. Since Q is left invertible, the projection on the range of Q can be written as $P_{\mathrm{Im}\, Q} = Q(Q^*Q)^{-1}Q^*$. Thus

$$X_1 P_{\mathrm{Im}\, Q} = R(Q^* D_A^2 Q)^{-1} Q^* D_A^2 Q (Q^*Q)^{-1} Q^* = R(Q^*Q)^{-1}Q^* = M.$$

Next observe that $X_1 D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^* = R(Q^* D_A^2 Q)^{-1} Q^* \Pi_{\mathrm{Ker}\, Q^*}^* = 0$. Hence

$$\begin{aligned}
X_1 \Pi_{\mathrm{Ker}\, Q^*}^* \Delta_Q &= X_1 P_{\mathrm{Ker}\, Q^*} D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^* = X_1(I - P_{\mathrm{Im}\, Q}) D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^*\\
&= -X_1 P_{\mathrm{Im}\, Q} D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^* = -M D_A^{-2} \Pi_{\mathrm{Ker}\, Q^*}^*.
\end{aligned}$$

This implies that

$$X_1 P_{\operatorname{Ker} Q^*} = -M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \Delta_Q^{-1} \Pi_{\operatorname{Ker} Q^*}.$$

The formula for X_1 in (5.3.4) then follows because $X_1 = X_1 P_{\operatorname{Im} Q} + X_1 P_{\operatorname{Ker} Q^*}$.

Now fix $\lambda \in \mathbb{D}$ and set $C = M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \Delta_Q^{-1}$. Then $X_1 = M - C \Pi_{\operatorname{Ker} Q^*}$ and

$$\begin{aligned} I + \lambda \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} C &= I + \lambda \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \Delta_Q^{-1} \\ &= \Pi_{\operatorname{Ker} Q^*}(I + \lambda (I - \lambda M)^{-1} M) D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \Delta_Q^{-1} \\ &= \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \Delta_Q^{-1} \\ &= K_1(\lambda) \Delta_Q^{-1}. \end{aligned}$$

Since M is contractive and $r_{\mathrm{spec}}(X_1) \leq 1$, we obtain that

$$(I - \lambda X_1)^{-1} = (I - M + C\Pi_{\operatorname{Ker} Q^*})^{-1} = (I + \lambda (I - \lambda M)^{-1} C \Pi_{\operatorname{Ker} Q^*})^{-1} (I - \lambda M)^{-1}.$$

In particular, $I + \lambda (I - \lambda M)^{-1} C \Pi_{\operatorname{Ker} Q^*}$ is invertible. Applying Corollary 2.3.4 we see that the operator $I - \lambda M + \lambda C \Pi_{\operatorname{Ker} Q^*}$ is invertible and

$$\begin{aligned} &(I + \lambda (I - \lambda M)^{-1} C \Pi_{\operatorname{Ker} Q^*})^{-1} = \\ &\quad = I - \lambda (I - \lambda M + \lambda C \Pi_{\operatorname{Ker} Q^*})^{-1} C \Pi_{\operatorname{Ker} Q^*} \\ &\quad = I - \lambda (I - \lambda M)^{-1} (I + \lambda C \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1})^{-1} C \Pi_{\operatorname{Ker} Q^*} \\ &\quad = I - \lambda (I - \lambda M)^{-1} C (I + \lambda \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} C)^{-1} \Pi_{\operatorname{Ker} Q^*} \\ &\quad = I - \lambda (I - \lambda M)^{-1} C (K_1(\lambda) \Delta_Q^{-1})^{-1} \Pi_{\operatorname{Ker} Q^*} \\ &\quad = I - \lambda (I - \lambda M)^{-1} C \Delta_Q K_1(\lambda)^{-1} \Pi_{\operatorname{Ker} Q^*} \\ &\quad = I - \lambda (I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* K_1(\lambda)^{-1} \Pi_{\operatorname{Ker} Q^*}. \end{aligned}$$

Since $\lambda \in \mathbb{D}$ was chosen arbitrarily, this implies that $K_1(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$ and

$$(I - \lambda X_1)^{-1} = (I - \lambda (I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* K_1(\lambda)^{-1} \Pi_{\operatorname{Ker} Q^*})(I - \lambda M)^{-1}. \quad (5.3.5)$$

Part 2. Using the identity in (5.3.5) and the fact that

$$K_1(\lambda) = \Delta_Q + \lambda \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* \quad (\lambda \in \mathbb{D}),$$

we obtain for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned} \Phi_{1,2}(\lambda) &= X_3 (I - \lambda X_1)^{-1} \\ &= \Delta_Q^{-\frac{1}{2}} \Pi_{\operatorname{Ker} Q^*}(I - \lambda (I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* K_1(\lambda)^{-1} \Pi_{\operatorname{Ker} Q^*}) \times \\ &\qquad \times (I - \lambda M)^{-1} \\ &= \Delta_Q^{-\frac{1}{2}} (I - \lambda \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} M D_A^{-2} \Pi_{\operatorname{Ker} Q^*}^* K_1(\lambda)^{-1}) \times \\ &\qquad \times \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} \\ &= \Delta_Q^{-\frac{1}{2}} (I - (K_1(\lambda) - \Delta_Q) K_1(\lambda)^{-1}) \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1} \\ &= \Delta_Q^{\frac{1}{2}} K_1(\lambda)^{-1} \Pi_{\operatorname{Ker} Q^*}(I - \lambda M)^{-1}. \end{aligned}$$

So $\Phi_{1,2}$ can be written as in (5.3.3). To see that the formula for $\Phi_{1,1}$ in (5.3.3) also holds, just observe that $\Phi_{1,1}(\lambda) \equiv \lambda\Phi_{1,2}(\lambda)X_2$ and

$$\Pi_{\mathrm{Ker}\,Q^*}(I-\lambda M)^{-1}X_2 = -\left[\begin{array}{cc} K_3(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_2(\lambda)\Delta_R^{-\frac{1}{2}} \end{array}\right] \quad (\lambda \in \mathbb{D}). \tag{5.3.6}$$

Part 3. Again using (5.3.5) we obtain for each $\lambda \in \mathbb{D}$ that

$$\begin{aligned}
\lambda\Phi_{2,2}(\lambda) &= \lambda X_4(I-\lambda X_1)^{-1} = \lambda D_{T'}AX_1(I-\lambda X_1)^{-1} \\
&= D_{T'}A((I-\lambda X_1)^{-1} - I) \\
&= D_{T'}A(I - \lambda(I-\lambda M)^{-1}MD_A^{-2}\Pi_{\mathrm{Ker}\,Q^*}^*K_1(\lambda)^{-1}\Pi_{\mathrm{Ker}\,Q^*} + \\
&\qquad -(I-\lambda M))(I-\lambda M)^{-1} \\
&= \lambda(D_{T'}AM - K_4(\lambda)K_1(\lambda)^{-1}\Pi_{\mathrm{Ker}\,Q^*})(I-\lambda M)^{-1}.
\end{aligned}$$

This proves that the formula for $\Phi_{2,2}$ in (5.3.3) holds for $0 \neq \lambda \in \mathbb{D}$, and hence, by continuity, also for $\lambda = 0$.

To see that $\Phi_{2,1}$ is given by the third identity in (5.3.3), first observe that

$$\begin{aligned}
\Phi_{2,1}(\lambda) &= X_5 + \lambda X_4(I-\lambda X_1)^{-1}X_2 = X_5 + \lambda\Phi_{2,2}(\lambda)X_2 \\
&= X_5 + \lambda D_{T'}AM(I-\lambda M)^{-1}X_2 + \qquad\qquad (\lambda \in \mathbb{D}) \\
&\qquad -\lambda K_4(\lambda)K_1(\lambda)^{-1}\Pi_{\mathrm{Ker}\,Q^*}(I-\lambda M)^{-1}X_2.
\end{aligned}$$

The formula for $\Phi_{2,1}$ then follows with (5.3.6) and the fact that

$$X_5 + \lambda D_{T'}AM(I-\lambda M)^{-1}X_2 = -\left[\begin{array}{cc} K_6(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_5(\lambda)\Delta_R^{-\frac{1}{2}} \end{array}\right].$$

So all identities in (5.3.3) hold. □

To illustrate the formulas in Proposition 5.3.1 we consider the following example.

Example 5.3.2. Let $\{A, T', U', R, Q\}$ be the lifting data set given by (4.4.2) with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Note that A is a strict contraction and R is left invertible. Furthermore,

$$M = R(Q^*Q)^{-1}Q^* = \left[\begin{array}{cc} 0 & 0 \\ \frac{1}{2} & 0 \end{array}\right], \quad \text{so} \quad (I-\lambda M)^{-1} = \left[\begin{array}{cc} 1 & 0 \\ \frac{\lambda}{2} & 1 \end{array}\right] \quad (\lambda \in \mathbb{D}). \tag{5.3.7}$$

With the formulas for D_A, $R^*D_A^2R$ and J derived in Example 5.2.4 we obtain that the functions $K_1, \ldots, K_6$ in (5.3.1) are given by

$$K_1(\lambda) \equiv \frac{4}{3}, \quad K_2(\lambda) \equiv \frac{8\lambda}{15}, \quad K_3(\lambda) \equiv \left[\begin{array}{cc} \frac{4}{3}\sqrt{3} & 0 \end{array}\right],$$

$$K_4(\lambda) \equiv 0, \quad K_5(\lambda) \equiv 0 \quad \text{and} \quad K_6(\lambda) \equiv \left[\begin{array}{cc} 0 & -1 \end{array}\right].$$

Using (5.3.3) we get

$$\begin{aligned}
\Phi_{1,1}(\lambda) &= -\lambda \tfrac{2}{\sqrt{3}}\tfrac{3}{4}\begin{bmatrix} \tfrac{4\sqrt{3}}{3}\tfrac{1}{\sqrt{5}} & 0 & \tfrac{8\lambda}{15}\tfrac{\sqrt{15}}{4}\end{bmatrix} = \begin{bmatrix} \tfrac{-2\lambda}{5}\sqrt{5} & 0 & \tfrac{\lambda^2}{5}\sqrt{5}\end{bmatrix},\\
\Phi_{1,2}(\lambda) &= \tfrac{2}{\sqrt{3}}\tfrac{3}{4}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \tfrac{\lambda}{2} & 1\end{bmatrix} = \begin{bmatrix} \tfrac{\lambda}{4}\sqrt{3} & \tfrac{1}{2}\sqrt{3}\end{bmatrix}, \qquad (\lambda \in \mathbb{D})\\
\Phi_{2,1}(\lambda) &= -\begin{bmatrix} 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\end{bmatrix},\\
\Phi_{2,2}(\lambda) &= \begin{bmatrix} 0 & 0 \end{bmatrix}.
\end{aligned}$$

These are precisely the same formulas as in Example 5.2.4. ◇

We now return to the classical commutant lifting setting in the following corollary.

Corollary 5.3.3. *Let $\{A, T', U', R, Q\}$ be a lifting data set, where U' is the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that Q is an isometry in $\mathscr{L}(\mathcal{H})$, $R = I_{\mathcal{H}}$ and A is a strict contraction. Define analytic operator-valued functions F_1, F_2, F_3 and F_4 on $\mathbb{D}$ by*

$$\begin{aligned}
F_1(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I-\lambda Q^*)^{-1} D_A^{-2}\Pi^*_{\operatorname{Ker} Q^*}\\
F_2(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I-\lambda Q^*)^{-1} D_A^{-2} A^* D_{T'} \qquad (\lambda \in \mathbb{D})\\
F_3(\lambda) &= D_{T'} A Q^*(I-\lambda Q^*)^{-1} D_A^{-2}\Pi^*_{\operatorname{Ker} Q^*}\\
F_4(\lambda) &= I_{\mathcal{D}_{T'}} - \lambda D_{T'} A Q^*(I-\lambda Q^*)^{-1} D_A^{-2} A^* D_{T'}.
\end{aligned}$$

Then $F_1(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$, and the functions $\Phi_{1,1}$, $\Phi_{1,1}$, $\Phi_{1,1}$ and $\Phi_{1,1}$ in (5.2.18) *are also given by*

$$\begin{aligned}
\Phi_{1,1} &= -\lambda \Delta_Q^{\frac{1}{2}} F_1(\lambda)^{-1} F_2(\lambda)\Delta_\Omega^{-\frac{1}{2}},\\
\Phi_{1,2} &= \Delta_Q^{\frac{1}{2}} F_1(\lambda)^{-1}\Pi_{\operatorname{Ker} Q^*}(I-\lambda Q^*)^{-1}, \qquad (\lambda \in \mathbb{D})\\
\Phi_{2,1} &= F_4(\lambda)\Delta_\Omega^{-\frac{1}{2}} + \lambda F_3(\lambda) F_1(\lambda)^{-1} F_2(\lambda)\Delta_\Omega^{-\frac{1}{2}},\\
\Phi_{2,2} &= (D_{T'} A Q^* - F_3(\lambda) F_1(\lambda)^{-1}\Pi_{\operatorname{Ker} Q^*})(I-\lambda Q^*)^{-1},
\end{aligned}$$

where Δ_Q and Δ_Ω are as in (5.2.19).

Proof. In the proof of Corollary 5.2.5 We already observed that in the classical commutant lifting setting the operators Δ_Q and Δ_Ω in (5.2.2) reduce to (5.2.19). So to complete the proof it suffices to show that the functions K_1, K_2, K_3, K_4, K_5 and K_6 in (5.3.1) are given by

$$\begin{gathered}
K_1(\lambda) \equiv F_1(\lambda), \quad K_2(\lambda) \equiv 0, \quad K_3(\lambda) \equiv F_2(\lambda),\\
K_4(\lambda) \equiv F_3(\lambda), \quad K_5(\lambda) \equiv 0 \quad \text{and} \quad K_6(\lambda) \equiv -F_4(\lambda).
\end{gathered}$$

Since $\operatorname{Ker} R^* = \{0\}$, it follows immediately that $K_2(\lambda) \equiv 0$ and $K_5(\lambda) \equiv 0$. Moreover, using that $R = I$ and Q is isometric we obtain that $M = Q^*$ and

$(R^*D_A^2R)^{-1} = D_A^{-2}$. Next observe that $R^*R = I = Q^*Q$ implies that $D_\circ = 0$ and $\mathcal{D}_\circ = \{0\}$. Hence the operator J in (5.2.2) is given by $J = D_{T'}A$ from $\mathcal{H}$ to $\mathcal{D}_{T'}$. Inserting these formulas for M, $(R^*D_A^2R)^{-1}$ and J into (5.3.1) we also obtain the claimed identities for the functions K_1, K_3, K_4 and K_6. □

Note that $MP_{\operatorname{Ker} Q^*} = 0$. Hence $(I - \lambda M)^{-1}P_{\operatorname{Ker} Q^*} = P_{\operatorname{Ker} Q^*}$ for each $\lambda \in \mathbb{D}$. Therefore the functions $\Phi_{1,2}$ and $\Phi_{2,2}$ in (5.3.3) satisfy

$$\begin{array}{lcl} \Phi_{1,2}(\lambda) & = & \Phi_{1,2}(\lambda)P_{\operatorname{Ker} Q^*}(I-\lambda M)^{-1}, \\ \Phi_{2,2}(\lambda) & = & D_{T'}AM(I-\lambda M)^{-1} + \Phi_{2,2}(\lambda)P_{\operatorname{Ker} Q^*}(I-\lambda M)^{-1}. \end{array} \quad (\lambda \in \mathbb{D}) \quad (5.3.8)$$

Now let $V \in \mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*)$, and define $L \in \mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ by (5.2.5). The identities for $\Phi_{1,2}$ and $\Phi_{2,2}$ in (5.3.8) then show that L satisfies

$$L(\lambda) = D_{T'}AM(I-\lambda M)^{-1} + L(\lambda)P_{\operatorname{Ker} Q^*}(I-\lambda M)^{-1} \quad (\lambda \in \mathbb{D}).$$

In particular, $L(\lambda)$ is uniquely determined by its action on $\operatorname{Ker} Q^*$. Using Theorem 5.2.1 this observation implies that each contractive interpolant for $\{A, T', U', R, Q\}$ is completely determined by its action on $\operatorname{Ker} Q^*$. It turns out that this result does not requires $\|A\| < 1$ and $R^*R \gg 0$, as the following theorem shows.

Theorem 5.3.4. *Let $\Omega = \{A, T', U', R, Q\}$ be a lifting data set with U' the Sz.-Nagy-Schäffer isometric lifting of T', and let M on $\mathcal{H}$ be the unique contraction such that $MQ = R$ and $M|_{\operatorname{Ker} Q^*} = 0$. Let B be a contractive interpolant for $\{A, T', U', R, Q\}$ given by*

$$B = \begin{bmatrix} A \\ \Lambda \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix},$$

and let $L \in \mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ be the defining function for Λ. Then

$$L(\lambda) = D_{T'}AM(I-\lambda M)^{-1} + L(\lambda)P_{\operatorname{Ker} Q^*}(I-\lambda M)^{-1} \quad (\lambda \in \mathbb{D}). \qquad (5.3.9)$$

Proof. From Theorem 5.1.1 and Proposition 4.3.1 we obtain that $L(\lambda) \equiv H(\lambda)D_A$, where H is a solution to the H^2 interpolation problem defined by the contraction ω underlying $\{A, T', U', R, Q\}$. Fix $\lambda \in \mathbb{D}$. The definition of ω and the fact that H is a solution to the H^2 interpolation problem defined by ω show that $D_{T'}AR + \lambda H(\lambda)D_AR = H(\lambda)D_AQ$. In terms of L and M the latter identity can be written as

$$D_{T'}AMQ + \lambda L(\lambda)MQ = L(\lambda)Q.$$

Since $M|_{\operatorname{Ker} Q^*} = 0$, this implies

$$D_{T'}AM + \lambda L(\lambda)M = L(\lambda)P_{\operatorname{Im} Q} = L(\lambda)(I - P_{\operatorname{Ker} Q^*}).$$

Hence $L(\lambda)(I - \lambda M) = D_{T'}AM + L(\lambda)P_{\operatorname{Ker} Q^*}$. Multiplying on both sides with $(I - \lambda M)^{-1}$ yields (5.3.9). □

Note that in case $Q^*Q \gg 0$, the operator M in Theorem 5.3.4 is given by (5.3.2).
The next corollary follows immediately from Theorem 5.3.4.

Corollary 5.3.5. *Each contractive interpolant for* $\{A, T', U', R, Q\}$ *is uniquely determined by its action on* $\operatorname{Ker} Q^*$.

The result of Corollary 5.3.5 is also in line with many of the classical examples, where the intertwining relation forces the contractive interpolants to be the multiplication operator of a Schur class function, while $\operatorname{Ker} Q^*$ corresponds to the subspace of constant functions.

Example 5.3.6. Let $\{A, T', U', R, Q\}$ be the lifting data set given by (4.4.2) with U' being the Sz.-Nagy-Schäffer isometric lifting of T'. Let W be a function in $\mathbf{S}(\ker Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*) = \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ and define L in $\mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'}) = \mathbf{H}^2_{\text{ball}}(\mathbb{C}^2, \mathbb{C})$ by (5.2.5). In Example 5.2.4 we obtained that in case W is given by (5.2.16) the function L is as in (5.2.17). Since $\operatorname{Ker} Q^* = \{0\} \oplus \mathbb{C} \subset \mathbb{C}^2$, we have

$$L(\lambda)|_{\operatorname{Ker} Q^*} = \frac{5\sqrt{3}W_2(\lambda)}{10 + 2\lambda^2\sqrt{5}W_3(\lambda) + 4\lambda\sqrt{5}W_1(\lambda)} \quad (\lambda \in \mathbb{D}).$$

Note that $D_{T'}AM = 0$ with M as in (5.3.7). Therefore, we obtain from Theorem 5.3.4 that

$$\begin{aligned} L(\lambda) &= (D_{T'}AM + L(\lambda)P_{\operatorname{Ker} Q^*})(I - \lambda M)^{-1} \\ &= \tfrac{5\sqrt{3}W_2(\lambda)}{10+2\lambda^2\sqrt{5}W_3(\lambda)+4\lambda\sqrt{5}W_1(\lambda)} \begin{bmatrix} \frac{\lambda}{2} & 1 \end{bmatrix} \qquad (\lambda \in \mathbb{D}) \\ &= \tfrac{W_2(\lambda)}{20+4\lambda^2\sqrt{5}W_3(\lambda)+8\lambda\sqrt{5}W_1(\lambda)} \begin{bmatrix} 5\lambda\sqrt{3} & 10\sqrt{3} \end{bmatrix}, \end{aligned}$$

which is in agreement with (5.2.17). ◇

5.4 An alternative linear fractional representation

As before assume that the operators A and R in the lifting data set $\{A, T', U', R, Q\}$ are such that $\|A\| < 1$ and $R^*R \gg 0$. Let B be a contractive interpolant for $\{A, T', U', R, Q\}$, let Λ be the contraction from $\mathcal{H}$ into $H^2(\mathcal{D}_{T'})$ such that

$$B = \begin{bmatrix} A \\ \Lambda \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix},$$

and let $L \in \mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ be the defining function for Λ. According to Theorem 5.2.1 there exists a function $W \in \mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*)$ such that L is given by the Redheffer representation (5.2.5), where the coefficients $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ are as in (5.2.6). From the results of the previous section (Theorem 5.3.4 and Corollary 5.3.5) we know that for each $\lambda \in \mathbb{D}$ the operator $L(\lambda)$ is uniquely determined by $L(\lambda)|_{\operatorname{Ker} Q^*}$. Now let us restrict both sides of (5.2.5) to $\operatorname{Ker} Q^*$. This yields

$$L(\lambda)|_{\operatorname{Ker} Q^*} = \tilde{\Phi}_{2,2}(\lambda)+\Phi_{2,1}(\lambda)W(\lambda)(I-\Phi_{1,1}(\lambda)W(\lambda))^{-1}\tilde{\Phi}_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}). \tag{5.4.1}$$

Here

$$\tilde{\Phi}_{1,2}(\lambda) = \Phi_{1,2}(\lambda)|_{\operatorname{Ker} Q^*} \quad \text{and} \quad \tilde{\Phi}_{2,2}(\lambda) = \Phi_{2,2}(\lambda)|_{\operatorname{Ker} Q^*} \quad (\lambda \in \mathbb{D}).$$

Next we use that $\Phi_{1,2}$ and $\Phi_{2,2}$ are also given by (5.3.3), and the fact that $M|_{\operatorname{Ker} Q^*} = 0$. The latter implies that $(I - \lambda M)^{-1}\Pi^*_{\operatorname{Ker} Q^*} \equiv \Pi^*_{\operatorname{Ker} Q^*}$. We obtain

$$\tilde{\Phi}_{1,2}(\lambda) = \Delta_Q^{\frac{1}{2}} K_1(\lambda)^{-1} \quad \text{and} \quad \tilde{\Phi}_{2,2}(\lambda) = -K_4(\lambda)K_1(\lambda)^{-1} \quad (\lambda \in \mathbb{D}). \tag{5.4.2}$$

Here K_1 and K_4 are as in (5.3.1). It follows that in the Redheffer representation (5.4.1) the coefficient $\tilde{\Phi}_{1,2}$ has the property that its values are invertible operators in $\mathscr{L}(\operatorname{Ker} Q^*)$. This fact enables us to rewrite (5.4.1) in the form of a more classical linear fractional representation.

Theorem 5.4.1. *Let $\{A, T', U', R, Q\}$ be a lifting data set with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that A is a strict contraction and R is left invertible. Let $W \in \mathbf{S}(\operatorname{Ker} Q^*, \mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*)$, and define $L \in \mathbf{H}^2_{\text{ball}}(\mathcal{H}, \mathcal{D}_{T'})$ by* (5.2.5)*. Then $L(\lambda)|_{\operatorname{Ker} Q^*}$ can be written as*

$$L(\lambda)|_{\operatorname{Ker} Q^*} = (\Psi_{1,2}(\lambda)W(\lambda) + \Psi_{1,1}(\lambda))(\Psi_{2,2}(\lambda)W(\lambda) + \Psi_{2,1}(\lambda))^{-1} \ (\lambda \in \mathbb{D}), \tag{5.4.3}$$

where

$$\begin{aligned} \Psi_{1,1}(\lambda) &= -K_4(\lambda)\Delta_Q^{-\frac{1}{2}}, \\ \Psi_{1,2}(\lambda) &= -\begin{bmatrix} K_6(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_5(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix}, \\ \Psi_{2,1}(\lambda) &= K_1(\lambda)\Delta_Q^{-\frac{1}{2}}, \\ \Psi_{2,2}(\lambda) &= \lambda \begin{bmatrix} K_3(\lambda)\Delta_\Omega^{-\frac{1}{2}} & K_2(\lambda)\Delta_R^{-\frac{1}{2}} \end{bmatrix}. \end{aligned} \qquad (\lambda \in \mathbb{D}) \tag{5.4.4}$$

Here K_1, K_2, K_3, K_4, K_5 and K_6 are the functions given by (5.3.1)*, and Δ_Q, Δ_R and Δ_Ω are the strictly positive operators in* (5.2.2)*.*

Proof. In order to prove Theorem 5.4.1, we first recall some general facts concerning the Redheffer transform; cf., Section 13.1 in [20]. Let $\sigma_{1,1}$, $\sigma_{1,2}$, $\sigma_{2,1}$ and $\sigma_{2,2}$ be operators such that

$$\begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{bmatrix} : \begin{bmatrix} \mathcal{V} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{W} \end{bmatrix} \quad \text{with } \sigma_{2,2} \text{ invertible.}$$

Put

$$\delta_{1,1} = \sigma_{1,1} - \sigma_{1,2}\sigma_{2,2}^{-1}\sigma_{2,1}, \quad \delta_{1,2} = \sigma_{1,2}\sigma_{2,2}^{-1}, \quad \delta_{2,1} = -\sigma_{2,2}^{-1}\sigma_{2,1}, \quad \delta_{2,2} = \sigma_{2,2}^{-1}. \tag{5.4.5}$$

Then

$$\begin{bmatrix} \delta_{1,1} & \delta_{1,2} \\ \delta_{2,1} & \delta_{2,2} \end{bmatrix} : \begin{bmatrix} \mathcal{V} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{W} \end{bmatrix} \quad \text{with } \delta_{2,2} \text{ invertible.} \tag{5.4.6}$$

The transformation (5.4.5) is referred to as the *Redheffer transform.* To retrieve the operators $\sigma_{1,1}$, $\sigma_{1,2}$, $\sigma_{2,1}$ and $\sigma_{2,2}$ from (5.4.6) one applies the Redheffer transform once more, that is,

$$\sigma_{1,1} = \delta_{1,1} - \delta_{1,2}\delta_{2,2}^{-1}\delta_{2,1}, \quad \sigma_{1,2} = \delta_{1,2}\delta_{2,2}^{-1}, \quad \sigma_{2,1} = -\delta_{2,2}^{-1}\delta_{2,1} \quad \text{and} \quad \sigma_{2,2} = \delta_{2,2}^{-1}.$$

Moreover, for any operator U from $\mathcal{W}$ to $\mathcal{V}$ we have:

$$I_{\mathcal{W}} - \sigma_{2,1}U \text{ is invertible} \iff \delta_{2,1}U + \delta_{2,2} \text{ is invertible,}$$

and in that case

$$\sigma_{1,2} + \sigma_{1,1}U(I - \sigma_{2,1}U)^{-1}\sigma_{2,2} = (\delta_{1,1}U + \delta_{1,2})(\delta_{2,1}U + \delta_{2,2})^{-1}. \tag{5.4.7}$$

Now fix $\lambda \in \mathbb{D}$ and put

$$\sigma_{1,1} = \Phi_{2,1}(\lambda), \quad \sigma_{1,2} = \tilde{\Phi}_{2,2}(\lambda), \quad \sigma_{2,1} = \Phi_{1,1}(\lambda) \quad \text{and} \quad \sigma_{2,2} = \tilde{\Phi}_{2,1}(\lambda),$$

where $\Phi_{1,1}$ and $\Phi_{2,1}$ are as in (5.3.3), and $\tilde{\Phi}_{1,2}$ and $\tilde{\Phi}_{2,2}$ are the functions defined in (5.4.2). According to Proposition 5.3.1 the operator $K_1(\lambda)$ is invertible, and thus also $\sigma_{2,2}$ is invertible. Moreover, since W and $\Phi_{1,1}$ are Schur class functions and $\Phi_{1,1}(0) = 0$, we have that $I - \sigma_{2,1}W(\lambda)$ is invertible; see Lemma 2.4.1. So from the Redheffer transform facts referred to in the previous paragraph we obtain equality (5.4.7), where $\delta_{1,1}$, $\delta_{1,2}$, $\delta_{2,1}$ and $\delta_{2,2}$ are given by (5.4.5) and $U = W(\lambda)$. One easily computes that

$$\delta_{1,1} = \Psi_{1,2}(\lambda), \quad \delta_{1,2} = \Psi_{1,1}(\lambda), \quad \delta_{2,1} = \Psi_{2,2}(\lambda) \quad \text{and} \quad \delta_{2,2} = \Psi_{2,1}(\lambda).$$

Hence the right hand side in (5.4.1) coincides with the right hand side in (5.4.3). $\square$

Next we give one more example where we consider the lifting data set constructed in Example 4.4.3.

Example 5.4.2. Let $\{A, T', U', R, Q\}$ be the lifting data set given by (4.4.2) with U' the Sz.-Nagy-Schäffer isometric lifting of T'. Note that A is a strict contraction and R is left invertible. Using the formulas for $K_1, \ldots, K_6$ found in Example 5.3.2 we obtain that the functions in (5.4.4) are given by

$$\Psi_{1,1}(\lambda) \equiv 0, \quad \Psi_{1,2}(\lambda) \equiv \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$$\Psi_{2,1}(\lambda) \equiv \tfrac{2}{3}\sqrt{3} \quad \text{and} \quad \Psi_{2,2}(\lambda) \equiv \begin{bmatrix} \tfrac{4\lambda}{15}\sqrt{15} & 0 & \tfrac{2\lambda^2}{15}\sqrt{15} \end{bmatrix}.$$

So for $W \in \mathbf{S}(\mathbb{C}, \mathbb{C}^3)$ of the form (5.2.16) Theorem 5.4.1 shows that the function L in (5.2.5) satisfies

$$\begin{aligned} L(\lambda)|_{\operatorname{Ker} Q^*} &= (\Psi_{1,2}(\lambda)W(\lambda) + \Psi_{1,1}(\lambda))(\Psi_{2,2}(\lambda)W(\lambda) + \Psi_{2,1}(\lambda))^{-1} \\ &= W_2(\lambda)\left(\tfrac{4\lambda}{15}\sqrt{15}W_1(\lambda) + \tfrac{2\lambda^2}{15}\sqrt{15}W_3(\lambda) + \tfrac{2}{3}\sqrt{3}\right)^{-1} \qquad (\lambda \in \mathbb{D}) \\ &= \frac{5\sqrt{3}W_2(\lambda)}{10+4\lambda\sqrt{5}W_1(\lambda)+2\lambda^2\sqrt{5}W_3(\lambda)}. \end{aligned}$$

Note that $\operatorname{Ker} Q^* = \{0\} \oplus \mathbb{C} \subset \mathbb{C}^2$. It follows that the above formula for $L(\lambda)|_{\operatorname{Ker} Q^*}$ is in accordance with the formula for L in (5.2.17). ◇

We conclude this chapter by specifying Theorem 5.4.1 for the classical commutant lifting setting. The result follows immediately from the formulas for the functions K_1, K_2, K_3, K_4, K_5 and K_6 in (5.3.1) derived in the proof of Corollary 5.3.3.

Corollary 5.4.3. *Let $\{A, T', U', R, Q\}$ be a lifting data set, where U' is the Sz.-Nagy-Schäffer isometric lifting of T'. Assume that Q is an isometry in $\mathscr{L}(\mathcal{H})$, $R = I_{\mathcal{H}}$ and A is a strict contraction. Then* $\operatorname{Ker} R^* = \{0\}$, $\mathcal{D}_\circ = \{0\}$ *and the operator-valued functions $\Psi_{1,1}$, $\Psi_{1,2}$, $\Psi_{2,1}$ and $\Psi_{2,2}$ in* (5.4.4) *are given by*

$$\begin{aligned} \Psi_{1,1}(\lambda) &= -D_{T'}AQ^*(I-\lambda Q^*)^{-1}D_A^{-2}\Pi^*_{\operatorname{Ker} Q^*}\Delta_Q^{-\frac{1}{2}} \\ \Psi_{1,2}(\lambda) &= I_{\mathcal{D}_{T'}} - \lambda D_{T'}AQ^*(I-\lambda Q^*)^{-1}D_A^{-2}A^*D_{T'}\Delta_\Omega^{-\frac{1}{2}} \\ \Psi_{2,1}(\lambda) &= \Pi_{\operatorname{Ker} Q^*}(I-\lambda Q^*)^{-1}D_A^{-2}\Pi^*_{\operatorname{Ker} Q^*}\Delta_Q^{-\frac{1}{2}} \\ \Psi_{2,2}(\lambda) &= \lambda\Pi_{\operatorname{Ker} Q^*}(I-\lambda Q^*)^{-1}D_A^{-2}A^*D_{T'}\Delta_\Omega^{-\frac{1}{2}}. \end{aligned} \qquad (\lambda \in \mathbb{D})$$

Here Δ_Q and Δ_Ω are the strictly positive operators defined in (5.2.19).

Notes for Chapter 5

The general scheme for rewriting a Schur representation to a Redheffer representation presented in Section 5.1 appeared in [62]. However, it is a well known technique, cf., Chapter XIV in [38], where Redheffer products are used to obtain a linear fractional Redheffer representation of the solutions to the classical commutant lifting problem. Theorem 5.1.1 did not appear before. Section 5.2 coincides for a large part with [62]. The formula for the central contractive interpolant $\Phi_{2,2}$ in (5.2.6) is the same as the formula for the central contractive interpolant given in Corollary 5.2 of [42]. In their full generality the results in the last two sections are new. On the other hand, it is well known that a Redheffer representation can be rewritten to a linear fractional representation as appears in Theorem 5.4.1 when an appropriate invertibility condition is satisfied ; see Section 13.1 in [20] and the references therein. Representations as the one appearing in Theorem 5.4.1 are also very common in the classical literature, cf., Chapter 3 in [35] and the references given there.

Chapter 6

The relaxed Nehari problem

In this chapter we employ the theory developed in the previous chapters to solve a relaxed version of the classical (operator-valued) Nehari problem. This relaxed Nehari problem is introduced in the first section, where it is also shown that the problem can be put into a relaxed commutant lifting setting. The latter allows us to show that a solution exists if and only if a so-called truncated Hankel operator is contractive. In the second section, assuming this truncated Hankel operator is strictly contractive, the results derived in the previous chapter are used to obtain a classical, as well as a Redheffer, linear fractional representation of all solutions. The data for the relaxed Nehari problem consists of a sequence of operators and a positive integer N. By fixing the sequence of operators and letting the integer N go to infinity, the classical Nehari problem appears as the limit case. In the third section we recall the classical Nehari problem and review some well known facts concerning its solutions. In the final section we make the convergence behavior referred to above more precise, and we show how the solutions to the classical Nehari problem can be obtained as the limits of the solutions to the relaxed Nehari problem.

6.1 The relaxed Nehari problem and relaxed commutant lifting

Throughout this chapter $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces. We consider the following relaxation of the operator-valued Nehari problem. Given a sequence $F_{-1}, F_{-2}, \ldots$ of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ and a positive integer N, find a (all) sequence(s) $H_0, H_1, \ldots$ of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ such that the operator from $\mathcal{U}^N$ into $\ell^2(\mathcal{Y})$ given by the operator matrix representation

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ F_{-2} & F_{-3} & \cdots & F_{-(N+1)} \\ F_{-1} & F_{-2} & \cdots & F_{-N} \\ \boxed{H_0} & F_{-1} & \cdots & F_{-(N-1)} \\ H_1 & H_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_{-1} \\ \vdots & & \ddots & H_0 \\ \vdots & & & \vdots \end{bmatrix} : \mathcal{U}^N \to \ell^2(\mathcal{Y}) \tag{6.1.1}$$

has operator norm at most one.

The box in (6.1.1) indicates the first entree in the 0^{th} row which maps into the zero position in $\ell^2(\mathcal{Y})$. We shall refer to the sequence $F_{-1}, F_{-2}, \ldots$ as a *Nehari data sequence.* The positive integer N is called the *relaxation index of the relaxed Nehari problem defined by* $F_{-1}, F_{-2}, \ldots$, or just the relaxation index when the Nehari data sequence in question is clear from the context. Moreover, a sequence of operators $H_0, H_1, \ldots$ in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ such that the operator in (6.1.1) is contractive is said to be an *N-complementary sequence associated with* $F_{-1}, F_{-2}, \ldots$, or just an N-complementary sequence if no confusion concerning the sequence $F_{-1}, F_{-2}, \ldots$ can arise.

For this relaxed Nehari problem to be solvable it is clearly necessary that the operator given by

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ F_{-3} & F_{-4} & \cdots & F_{-(N+2)} \\ F_{-2} & F_{-3} & \cdots & F_{-(N+1)} \\ F_{-1} & F_{-2} & \cdots & F_{-N} \end{bmatrix} : \mathcal{U}^N \to \ell^2_-(\mathcal{Y}) \tag{6.1.2}$$

is a contraction. The operator in (6.1.2) will be referred to as the *N-truncated Hankel operator defined by the sequence* $F_{-1}, F_{-2}, \ldots$. We claim that for the existence of an N-complementary sequence it is also sufficient that the N-truncated Hankel operator is contractive. One way to see this is by repeatedly applying Parrott's lemma (see Corollary IV.3.6 in [38]). It also follows from the fact that the relaxed Nehari problem can be put into a relaxed commutant lifting setting. The latter is the topic of the remaining part of this section.

Our first remark is that an operator A from $\mathcal{U}^N$ into $\ell^2_-(\mathcal{Y})$ is an N-truncated Hankel operator if and only if A satisfies the intertwining relation $T'AR = AQ$, where T' is the forward shift operator on $\ell_-^2(\mathcal{Y})$ given by

$$T' = \begin{bmatrix} \ddots & & \vdots & \vdots & \vdots \\ \ddots & \ddots & \vdots & \vdots & \vdots \\ \ddots & \ddots & 0 & 0 & 0 \\ \cdots & 0 & I_{\mathcal{Y}} & 0 & 0 \\ \cdots & 0 & 0 & I_{\mathcal{Y}} & 0 \end{bmatrix}, \tag{6.1.3}$$

and R and Q are the operators from $\mathcal{U}^{N-1}$ to $\mathcal{U}^N$ defined by

$$R = \begin{bmatrix} I_{\mathcal{U}} & 0 & \cdots & 0 \\ 0 & I_{\mathcal{U}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_{\mathcal{U}} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_{\mathcal{U}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & I_{\mathcal{U}} & 0 \\ 0 & \cdots & 0 & I_{\mathcal{U}} \end{bmatrix}. \tag{6.1.4}$$

Indeed, if A is an operator from $\mathcal{U}^N$ into $\ell_-^2(\mathcal{Y})$, then A admits an operator matrix decomposition of the form

$$A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A_{31} & A_{32} & \cdots & A_{3N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ A_{11} & A_{12} & \cdots & A_{1N} \end{bmatrix},$$

and hence

$$AR = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A_{31} & A_{32} & \cdots & A_{3\,N-1} \\ A_{21} & A_{22} & \cdots & A_{2\,N-1} \\ A_{11} & A_{12} & \cdots & A_{1\,N-1} \end{bmatrix} \quad \text{and} \quad AQ = \begin{bmatrix} \vdots & \vdots & & \vdots \\ A_{32} & A_{33} & \cdots & A_{3N} \\ A_{22} & A_{23} & \cdots & A_{2N} \\ A_{12} & A_{13} & \cdots & A_{1N} \end{bmatrix}.$$

It follows that $T'AR = AQ$ is equivalent to $A_{kj} = A_{k-1\,j+1}$ for appropriate indices k and j, that is, $T'AR = AQ$ is equivalent to A being an N-truncated Hankel operator.

We shall also need the bilateral forward shift V' on $\ell^2(\mathcal{Y})$ which is given by

$$V' = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & 0 & I_{\mathcal{Y}} & 0 & 0 & 0 & & & \\ & & 0 & I_{\mathcal{Y}} & \boxed{0} & 0 & 0 & & \\ & & & 0 & I_{\mathcal{Y}} & 0 & 0 & 0 & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6.1.5}$$

Relative to the decomposition $\ell^2(\mathcal{Y}) = \ell^2_-(\mathcal{Y}) \oplus \ell^2_+(\mathcal{Y})$ the bilateral forward shift V' partitions as

$$V' = \begin{bmatrix} T' & 0 \\ X' & \tilde{S}_{\mathcal{Y}} \end{bmatrix} : \begin{bmatrix} \ell^2_-(\mathcal{Y}) \\ \ell^2_+(\mathcal{Y}) \end{bmatrix} \to \begin{bmatrix} \ell^2_-(\mathcal{Y}) \\ \ell^2_+(\mathcal{Y}) \end{bmatrix}. \tag{6.1.6}$$

Here T' is given by (6.1.3), the operator $\tilde{S}_{\mathcal{Y}}$ is the unilateral forward shift on $\ell^2_+(\mathcal{Y})$, and X' is the operator from $\ell^2_-(\mathcal{Y})$ into $\ell^2_+(\mathcal{Y})$ given by

$$X'(\ldots, y_{-3}, y_{-2}, y_{-1}) = (y_{-1}, 0, 0, \ldots) \qquad ((\ldots, y_{-3}, y_{-2}, y_{-1}) \in \ell^2_-(\mathcal{Y})). \tag{6.1.7}$$

Since V' is unitary, the partitioning in (6.1.6) shows that V' is an isometric lifting of T'. As is easily seen this lifting is also minimal. In fact, as we shall show later (see the proof of Proposition 6.1.2 below) up to some simple identification the operator V' is equal to the Sz.-Nagy-Schäffer isometric lifting of T'.

We are now ready to state the main result of this section.

Proposition 6.1.1. *Let $F_{-1}, F_{-2}, \ldots$ be a sequence of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$, and let N be a positive integer. Assume that the N-truncated Hankel operator A defined by $F_{-1}, F_{-2}, \ldots$ is a contraction. Put $\tilde{\Omega} = \{A, T', V', R, Q\}$, where T', V', R and Q are defined by* (6.1.3), (6.1.5) *and* (6.1.4)*. Then $\tilde{\Omega}$ is a lifting data set, and there exists a contractive interpolant for $\tilde{\Omega}$ if and only if there exists an N-complementary sequence associated with $F_{-1}, F_{-2}, \ldots$. More precisely, an operator $\tilde{B}$ from $\mathcal{U}^N$ to $\ell^2(\mathcal{Y})$ is a contractive interpolant for $\tilde{\Omega}$ if and only if $\tilde{B}$ is of the form*

$$\tilde{B} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ F_{-2} & F_{-3} & \cdots & F_{-(N+1)} \\ F_{-1} & F_{-2} & \cdots & F_{-N} \\ \boxed{H_0} & F_{-1} & \cdots & F_{-(N-1)} \\ H_1 & H_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & F_{-1} \\ \vdots & & \ddots & H_0 \\ \vdots & & & \vdots \end{bmatrix} : \mathcal{U}^N \to \ell^2(\mathcal{Y}) \tag{6.1.8}$$

with $H_0, H_1, \ldots$ an N-complementary sequence associated with $F_{-1}, F_{-2}, \ldots$.

Proof. We already know that $T'AR = AQ$ and that V' is a minimal isometric lifting of T'. Since R and Q are both isometries, we have $R^*R = Q^*Q$. This proves that $\tilde{\Omega}$ is a lifting data set.

Now let $\tilde{B}$ be an operator from $\mathcal{U}^N$ into $\ell^2(\mathcal{Y})$. Using similar arguments as in the third paragraph of this section we obtain that $\tilde{B}$ satisfies $V'\tilde{B}R = \tilde{B}Q$ if and only if

$$\tilde{B} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ H_{-2} & H_{-3} & \cdots & H_{-(N+1)} \\ H_{-1} & H_{-2} & \cdots & H_{-N} \\ \boxed{H_0} & H_{-1} & \cdots & H_{-(N-1)} \\ H_1 & H_0 & \cdots & H_{-(N-2)} \\ H_2 & H_1 & \cdots & H_{-(N-3)} \\ \vdots & \vdots & & \vdots \end{bmatrix} : \mathcal{U}^N \to \ell^2(\mathcal{Y}) \tag{6.1.9}$$

for some bi-infinite sequence of operators $\ldots, H_{-2}, H_{-1}, H_0, H_1, H_2, \ldots$ in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$. Moreover, if $\tilde{B}$ is given by (6.1.9), then $\Pi_{\ell^2_-(\mathcal{Y})}\tilde{B} = A$ holds if and only if $H_k = F_k$ for $k = -1, -2, -3, \ldots$. Hence $\tilde{B}$ is a contractive interpolant for $\tilde{\Omega}$ if and only if $\tilde{B}$ is given by (6.1.8) with $H_0, H_1, \ldots$ an N-complementary sequence associated with $F_{-1}, F_{-2}, \ldots$. □

Note that for any sequence $H_0, H_1, \ldots$ of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ to be an N-complementary sequence it is necessary that $\sum_{n=0}^{\infty} \|H_n u\|^2 \le \|u\|^2$ for each $u \in \mathcal{U}$. The latter condition implies that $H_0, H_1, \ldots$ is the sequence of Taylor coefficients at zero of a function $H \in \mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$. So, alternatively, we seek functions H in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ such that $\tilde{B}$ in (6.1.8) is a contraction, where H_n is the n^{th} Taylor coefficient of H at zero.

Proposition 6.1.2. *Let $F_{-1}, F_{-2}, \ldots$ be a sequence of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$, and let N be a positive integer. Assume that the N-truncated Hankel operator A defined by $F_{-1}, F_{-2}, \ldots$ is a contraction. Put $\Omega = \{A, T', U', R, Q\}$ with T' as in* (6.1.3), *R and Q defined by* (6.1.4) *and U' the Sz.-Nagy-Schäffer isometric lifting of T'. Then Ω is a lifting data set, and* $\operatorname{Ker} Q^*$ *and $\mathcal{D}_{T'}$ can be identified with $\mathcal{U}$ and $\mathcal{Y}$, respectively. Let B be a contractive interpolant for $\{A, T', U', R, Q\}$, and let H in $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$ be the defining function of the contraction $\Gamma = \Pi_{H^2(\mathcal{D}_{T'})} B|_{\operatorname{Ker} Q^*}$ from $\mathcal{U}$ into $H^2(\mathcal{Y})$. Then the Taylor coefficients of H at zero form a N-complementary sequence associated with $F_{-1}, F_{-2}, \ldots$, and all N-complementary sequences associated with $F_{-1}, F_{-2}, \ldots$ are obtained in this way.*

Proof. Since $\tilde{\Omega}$ defined in Proposition 6.1.1 is a lifting data set, it follows immediately that Ω is a lifting data set as well. Moreover,

$$\operatorname{Ker} Q^* = \mathcal{U} \oplus \{0\} \oplus \cdots \oplus \{0\} \subset \mathcal{U}^N \quad \text{and} \quad \mathcal{D}_{T'} = \cdots \oplus \{0\} \oplus \{0\} \oplus \mathcal{Y} \subset \ell^2_-(\mathcal{Y}).$$

Thus the spaces $\operatorname{Ker} Q^*$ and $\mathcal{D}_{T'}$ can be identified with $\mathcal{U}$ and $\mathcal{Y}$, respectively. With these identifications

$$\Pi_{\operatorname{Ker} Q^*} = \begin{bmatrix} I_{\mathcal{U}} & 0 \cdots & 0 \end{bmatrix} \quad \text{and} \quad D_{T'} = \Pi_{\mathcal{D}_{T'}} = \begin{bmatrix} \cdots & 0 & 0 & I \end{bmatrix}. \tag{6.1.10}$$

Let Φ be the unique unitary operator from $\ell^2(\mathcal{Y})$ onto $\ell^2_+(\mathcal{Y}) \oplus H^2(\mathcal{Y})$ that unitarily intertwines the minimal isometric liftings V' in (6.1.5) and U'. Since for an operator $\tilde{B}$ from $\mathcal{U}^N$ into $\ell^2(\mathcal{Y})$

$$\Pi_{\ell^2_-(\mathcal{Y})}\tilde{B} = \Pi_{\ell^2_-(\mathcal{Y})}\Phi\tilde{B} \quad \text{and} \quad \Phi V'\tilde{B}R = U'\Phi\tilde{B}R,$$

the operator $\tilde{B}$ is a contractive interpolant for $\tilde{\Omega}$ if and only if $B = \Phi\tilde{B}$ is a contractive interpolant for Ω.

Next we compute Φ. For that purpose recall that the Fourier transform $\mathscr{F}_{\mathcal{Y}}$ from $\ell^2_+(\mathcal{Y})$ onto $H^2(\mathcal{Y})$ intertwines the unilateral forward shifts $\tilde{S}_{\mathcal{Y}}$ on $\ell^2_+(\mathcal{Y})$ and $S_{\mathcal{Y}}$ on $H^2(\mathcal{Y})$. Moreover, from (6.1.10) we obtain $\mathscr{F}_{\mathcal{Y}}X' = E_{\mathcal{Y}}D_{T'}$, where X' is given by (6.1.7). This implies that Φ is given by

$$\Phi = \begin{bmatrix} I_{\ell^2_-(\mathcal{Y})} & 0 \\ 0 & \mathscr{F}_{\mathcal{Y}} \end{bmatrix} : \begin{bmatrix} \ell^2_-(\mathcal{Y}) \\ \ell^2_+(\mathcal{Y}) \end{bmatrix} \to \begin{bmatrix} \ell^2_-(\mathcal{Y}) \\ H^2(\mathcal{Y}) \end{bmatrix}.$$

To complete the proof, let $\tilde{B}$ be an operator from $\mathcal{U}^N$ into $\ell^2(\mathcal{Y})$ given by (6.1.9), and let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ be the defining function of $\Gamma = \Pi_{H^2(\mathcal{Y})}\Phi\tilde{B}|_{\operatorname{Ker} Q^*}$. It remains to show that the operators $H_0, H_1, \ldots$ in (6.1.9) form the sequence of Taylor coefficients of H at zero. But this follows immediately because

$$\Gamma = \Pi_{H^2(\mathcal{Y})}\Phi\tilde{B}|_{\operatorname{Ker} Q^*} = \mathscr{F}_{\mathcal{Y}}\Pi_{\ell^2_+(\mathcal{Y})}\tilde{B}|_{\operatorname{Ker} Q^*}$$

and

$$\Pi_{\ell^2_+(\mathcal{Y})}\tilde{B}|_{\operatorname{Ker} Q^*} = \begin{bmatrix} H_0^* & H_1^* & \cdots \end{bmatrix}^*.$$

This proves our claim. □

Notice that Q in (6.1.4) is an isometry, and thus left invertible. We obtain that the operator M in (5.3.2) is given by

$$M = R(Q^*Q)^{-1}Q^* = RQ^* = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & I \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \text{ on } \mathcal{U}^N. \qquad (6.1.11)$$

That is, M is the finite backward shift operator on $\mathcal{U}^N$.

Remark 6.1.3. Let us see what Theorem 5.3.4 means in the setting of the relaxed Nehari problem. For this purpose let $H_0, H_1, \ldots$ be an N-complementary sequence, and let $\tilde{B}$ be the contractive interpolant given by (6.1.8) for the lifting data set

$\tilde{\Omega} = \{A, T', V', R, Q\}$ defined in Proposition 6.1.1. Note that

$$\tilde{B} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \boxed{H_0} & 0 & \cdots & 0 \\ H_1 & H_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & H_0 \\ \vdots & & & \vdots \end{bmatrix} + \begin{bmatrix} \vdots & \vdots & & \vdots \\ F_{-2} & F_{-3} & \cdots & F_{-(N+1)} \\ F_{-1} & F_{-2} & \cdots & F_{-N} \\ \boxed{0} & F_{-1} & \cdots & F_{-(N-1)} \\ 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & F_{-1} \\ \vdots & & \ddots & 0 \\ \vdots & & & \vdots \end{bmatrix}.$$

We shall show that the two summands in the right hand side of the above identity are in one-to-one correspondence with the two summands in the right hand side of (5.3.9); see also (6.1.12) below. To prove this, let H be the function in $\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}, \mathcal{Y})$ such that $H_0, H_1, \ldots$ is its sequence of Taylor coefficients at zero. Furthermore, put $\Lambda = \mathscr{F}_{\mathcal{Y}} \Pi_{\ell^2_+(\mathcal{U})} \tilde{B}$ from $\mathcal{U}^N$ into $H^2(\mathcal{Y})$, and let $L \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U}^N, \mathcal{Y})$ be the defining function for Λ. According to Theorem 5.3.4 we have

$$L(\lambda) = D_{T'} A M (I - \lambda M)^{-1} + H(\lambda) \Pi_{\mathrm{Ker}\, Q^*} (I - \lambda M)^{-1} \quad (\lambda \in \mathbb{D}). \qquad (6.1.12)$$

Here M is the operator defined in (6.1.11); see the remark in the first paragraph after the proof of Theorem 5.3.4. Let L_1 and L_2, both in $\mathbf{H}^2(\mathcal{U}^N, \mathcal{Y})$, be given by the first and second summand in the right hand side of (6.1.12), and let Γ_{L_1} and Γ_{L_2} be the operators defined by L_1 and L_2, respectively. Using (6.1.10) we see that

$$D_{T'} A M = \begin{bmatrix} 0 & F_{-1} & \cdots & F_{-N+1} \end{bmatrix} : \mathcal{U}^N \to \mathcal{Y}.$$

Then one easily checks that

$$\mathscr{F}^*_{\mathcal{Y}} \Gamma_{L_1} = \begin{bmatrix} 0 & F_{-1} & \cdots & F_{-N+1} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_{-1} \\ \vdots & & \ddots & 0 \\ \vdots & & & 0 \\ \vdots & & & \vdots \end{bmatrix}, \quad \mathscr{F}^*_{\mathcal{Y}} \Gamma_{L_2} = \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ H_1 & H_0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ H_{N-1} & \cdots & \cdots & H_0 \\ H_N & \cdots & \cdots & H_1 \\ \vdots & & & \vdots \end{bmatrix}.$$

Hence our claim follows.

6.2 All solutions to the relaxed Nehari problem

In this section we use the connection with the relaxed commutant lifting problem obtained in the previous section and the results of Chapter 5 to derive a representation of all solutions to the relaxed Nehari problem.

As before, $F_{-1}, F_{-2}, \ldots$ is a sequence of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$, and N is a positive integer. We assume additionally that the N-truncated Hankel operator A defined by $F_{-1}, F_{-2}, \ldots$ is a strict contraction. This additional condition corresponds to the strictly contractive condition used in Sections 5.2 and 5.4.

To state the main results of this section we need some preliminaries. We already observed that

$$\mathcal{D}_{T'} = \mathcal{Y} \quad \text{and} \quad D_{T'} = \begin{bmatrix} \cdots & 0 & 0 & I_{\mathcal{Y}} \end{bmatrix} : \ell^2_-(\mathcal{Y}) \to \mathcal{Y}.$$

The defect operator of the N-truncated Hankel operator A is the positive square root of

$$D_A^2 = \begin{bmatrix} \Lambda_{1,1} & \cdots & \Lambda_{1,N} \\ \vdots & \ddots & \vdots \\ \Lambda_{N,1} & \cdots & \Lambda_{N,N} \end{bmatrix} \text{ on } \mathcal{U}^N, \tag{6.2.1}$$

where

$$\Lambda_{i,j} = \begin{cases} I - \sum_{n=i}^{\infty} F_{-n}^* F_{-n} & \text{if } i = j, \\ -\sum_{n=i}^{\infty} F_{-n}^* F_{-n+i-j} & \text{if } i \neq j. \end{cases}$$

Since the N-truncated Hankel operator A is assumed to be a strict contraction, the defect operator D_A is strictly positive, hence, in particular, invertible. We shall also use the entries in the $N \times N$ operator matrix representation of D_A^{-2},

$$D_A^{-2} = \begin{bmatrix} \Lambda^\times_{1,1} & \cdots & \Lambda^\times_{1,N} \\ \vdots & \ddots & \vdots \\ \Lambda^\times_{N,1} & \cdots & \Lambda^\times_{N,N} \end{bmatrix} \text{ on } \mathcal{U}^N. \tag{6.2.2}$$

The fact that D_A^2 is a strictly positive operator implies that D_A^{-2} as well as $\Lambda^\times_{j,j}$ for $j = 1, \ldots, N$ in (6.2.2) are also strictly positive. In particular, this is true for $n = 1$ and $n = N$. Moreover, the $N-1$ by $N-1$ left upper block matrix operator of D_A^2 in (6.2.1) is strictly positive. Therefore there exists a unique solution $\begin{bmatrix} G_1 & \cdots & G_{N-1} \end{bmatrix}$ to the equation

$$\begin{bmatrix} \Lambda_{1,1} & \cdots & \Lambda_{1,N-1} \\ \vdots & \ddots & \vdots \\ \Lambda_{N-1,1} & \cdots & \Lambda_{N-1,N-1} \end{bmatrix} \begin{bmatrix} G_1^* \\ \vdots \\ G_{N-1}^* \end{bmatrix} = \begin{bmatrix} F_{-1}^* \\ \vdots \\ F_{-N+1}^* \end{bmatrix}. \tag{6.2.3}$$

With the Nehari data sequence $F_{-1}, F_{-2}, \ldots$, the relaxation index N and the operators defined in (6.2.2) and (6.2.3) we associate the following operator-valued polynomials:

$$\begin{aligned} K_1(\lambda) &= \textstyle\sum_{k=0}^{N-1} \lambda^k \Lambda^\times_{k+1,1}, & K_4(\lambda) &= \textstyle\sum_{k=0}^{N-2} \lambda^k \sum_{i=1}^{N-1-k} F_{-i} \Lambda^\times_{i+1+k,1} \\ K_2(\lambda) &= \textstyle\sum_{k=0}^{N-1} \lambda^k \Lambda^\times_{k+1,N}, & K_5(\lambda) &= \textstyle\sum_{k=1}^{N-1} \lambda^k \sum_{i=1}^{N-k} F_{-i} \Lambda^\times_{i+k,N} \\ K_3(\lambda) &= \textstyle\sum_{k=0}^{N-2} \lambda^k G_{k+1}^*, & K_6(\lambda) &= -I + \textstyle\sum_{k=1}^{N-2} \lambda^k \sum_{i=1}^{N-1-k} F_{-i} G_{i+k}^*. \end{aligned} \tag{6.2.4}$$

Note that the operator R in (6.1.4) is isometric, and thus, is left invertible. The fact that the N-truncated Hankel matrix A is a strict contraction, then enables us to derive the following result.

Theorem 6.2.1. *Let $F_{-1}, F_{-2}, \dots$ be a sequence of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$, and let N be a positive integer. Assume that the N-truncated Hankel operator A defined by $F_{-1}, F_{-2}, \dots$ is a strict contraction. Let $\Psi_{1,1}$, $\Psi_{1,2}$, $\Psi_{2,1}$ and $\Psi_{2,2}$ be the operator-valued polynomials given by*

$$\begin{aligned}
\Psi_{1,1}(\lambda) &= -K_4(\lambda)(\Lambda^{\times}_{1,1})^{-\frac{1}{2}} : \mathcal{U} \to \mathcal{Y}, \\
\Psi_{1,2}(\lambda) &= -\begin{bmatrix} K_6(\lambda)(I+FG^*)^{-\frac{1}{2}} & K_5(\lambda)(\Lambda^{\times}_{N,N})^{-\frac{1}{2}} \end{bmatrix} : \mathcal{Y} \oplus \mathcal{U} \to \mathcal{Y}, \\
\Psi_{2,1}(\lambda) &= K_1(\lambda)(\Lambda^{\times}_{1,1})^{-\frac{1}{2}} \text{ on } \mathcal{U}, \qquad\qquad (6.2.5)\\
\Psi_{2,2}(\lambda) &= \lambda \begin{bmatrix} K_3(\lambda)(I+FG^*)^{-\frac{1}{2}} & K_2(\lambda)(\Lambda^{\times}_{N,N})^{-\frac{1}{2}} \end{bmatrix} : \mathcal{Y} \oplus \mathcal{U} \to \mathcal{U},
\end{aligned}$$

with $K_1, \dots, K_6$ the polynomials in (6.2.4), and F and G the operators from $\mathcal{U}^N$ into $\mathcal{Y}$ given by

$$F = \begin{bmatrix} F_{-1} & \cdots & F_{-N+1} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} G_1 & \cdots & G_{N-1} \end{bmatrix}. \qquad (6.2.6)$$

Then, given a Schur class function W in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, the function H defined by

$$H(\lambda) = (\Psi_{1,2}(\lambda)W(\lambda) + \Psi_{1,1}(\lambda))(\Psi_{2,2}(\lambda)W(\lambda) + \Psi_{2,1}(\lambda))^{-1} \quad (\lambda \in \mathbb{D}) \qquad (6.2.7)$$

belongs to $\mathbf{H}^2_{\text{ball}}(\mathcal{U}, \mathcal{Y})$, and the Taylor coefficients $H_0, H_1, \dots$ of H at zero form an N-complementary sequence associated with $F_{-1}, F_{-2}, \dots$. Moreover, all N-complementary sequences associated with $F_{-1}, F_{-2}, \dots$ are obtained in this way.

From the definition of G in (6.2.6) and (6.2.3) it follows that FG^* is a nonnegative operator. Hence $I+FG^*$ is strictly positive, and thus the operator $(I+FG^*)^{-\frac{1}{2}}$ appearing in (6.2.5) is properly defined.

As a computational remark, note that to obtain the solution $\begin{bmatrix} G_1 & \cdots & G_{N-1} \end{bmatrix}$ to the equation (6.2.3) it suffices to compute the inverse of the operator $\Lambda^{\times}_{N,N}$ in (6.2.2). Indeed, with a standard Schur complement type of argument (see Proposition 2.3.2) we see that the inverse of the $N-1$ by $N-1$ left upper corner of D_A^2 in (6.2.1) is given by

$$\begin{bmatrix} \Lambda^{\times}_{1,1} & \cdots & \Lambda^{\times}_{1,N-1} \\ \vdots & \ddots & \vdots \\ \Lambda^{\times}_{N-1,1} & \cdots & \Lambda^{\times}_{N-1,N-1} \end{bmatrix} - \begin{bmatrix} \Lambda^{\times}_{1,N} \\ \vdots \\ \Lambda^{\times}_{N-1,N} \end{bmatrix} (\Lambda^{\times}_{N,N})^{-1} \begin{bmatrix} (\Lambda^{\times}_{N,1})^* \\ \vdots \\ (\Lambda^{\times}_{N,N-1})^* \end{bmatrix}^* .$$

Furthermore, if $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional, and the Schur class function W is a rational matrix function, then the function H given by (6.2.7) is also a rational matrix function, independent of the Nehari data sequence in question. Thus in this case all computations only involve finite matrix operations. In this sense the relaxed Nehari problem is very different from the classical Nehari problem.

Proof of Theorem 6.2.1. We already observed that the operator R in (6.1.4) is left invertible. Moreover, the N-truncated Hankel operator A is, by assumption, a strict contraction. Hence we can apply Theorem 5.4.1 to the lifting data set $\Omega = \{A, T', U', R, Q\}$ defined in Proposition 6.1.2. In fact, from the results of Proposition 6.1.2 and Theorem 5.4.1 we obtain that to prove Theorem 6.2.1 it remains to show that the functions K_1, K_2, K_3, K_4, K_5 and K_6 in (5.3.1) specified for the lifting data set Ω, coincide with the polynomials K_1, K_2, K_3, K_4, K_5 and K_6 defined by (6.2.4), and that in this case the operators Δ_Q, Δ_R and Δ_Ω in (5.2.2) are given by

$$\Delta_Q = \Lambda^\times_{1,1}, \quad \Delta_R = \Lambda^\times_{N,N} \quad \text{and} \quad \Delta_\Omega = I + FG^*.$$

The formulas for Δ_Q and Δ_R follow immediately from (6.2.2). Since $\operatorname{Ker} Q^* = \mathcal{U} \oplus \{0\}^{N-1}$ and $\operatorname{Ker} R^* = \{0\}^{N-1} \oplus \mathcal{U}$, these two spaces both can be identified with $\mathcal{U}$, so that

$$\Pi_{\operatorname{Ker} Q^*} = \begin{bmatrix} I_\mathcal{U} & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \Pi_{\operatorname{Ker} R^*} = \begin{bmatrix} 0 & \cdots & 0 & I_\mathcal{U} \end{bmatrix}.$$

Note that R and Q are both isometries. In particular, we have $R^*R = Q^*Q$. Hence $D_\circ = 0$ and $\mathcal{D}_\circ = \{0\}$, where $D_\circ$ and $\mathcal{D}_\circ$ are the operator and Hilbert space defined in (1.16). It follows that J in (5.2.2) is given by $J = D_{T'}AR = F$. The $N-1$ by $N-1$ left upper block matrix operator of D_A^2 can be written as $R^*D_A^2R$. Thus $G^* = (R^*D_A^2R)^{-1}J^*$. This implies that

$$\Delta_\Omega = I + J(R^*D_A^2R)^{-1}J^* = I + FG^*.$$

So the identities in (6.2) hold.

Recall from the previous section that the operator M in (5.3.2) is the backward shift operator on $\mathcal{U}^N$ given by (6.1.11). Hence for arbitrary operators $X = \begin{bmatrix} X_1 & \cdots & X_N \end{bmatrix}$ from $\mathcal{U}^N$ into $\mathcal{X}$ and $Y = \begin{bmatrix} Y_1 & \cdots & Y_N \end{bmatrix}$ from $\mathcal{U}^N$ into $\mathcal{Y}$ we have

$$X(I - \lambda M)^{-1}Y^* = \sum_{j=0}^{N-1} \lambda^j \sum_{k=1}^{N-j} X_k Y^*_{k+j} \quad (\lambda \in \mathbb{D}).$$

Applying this identity with

$$X = \Pi_{\operatorname{Ker} Q^*} \quad \text{and} \quad X = D_{T'}AM = \begin{bmatrix} 0 & F \end{bmatrix},$$

while taking

$$Y^* = D_A^{-2}\Pi^*_{\operatorname{Ker} Q^*}, \quad Y^* = D_A^{-2}\Pi^*_{\operatorname{Ker} R^*}, \quad Y^* = R(R^*D_A^2R)^{-1}J^* = \begin{bmatrix} G & 0 \end{bmatrix}^*,$$

we obtain that the functions K_1, K_2, K_3, K_4, K_5 and K_6 in (5.3.1) are given by (6.2.4). □

The results of Chapter 5 also provide the following linear fractional Redheffer representation.

Proposition 6.2.2. *Let $F_{-1}, F_{-2}, \ldots$ be a sequence of operators in $\mathscr{L}(\mathcal{U},\mathcal{Y})$, and let N be a positive integer. Assume that the N-truncated Hankel operator A defined by $F_{-1}, F_{-2}, \ldots$ is a strict contraction. Then the operator-valued polynomial K_1 in* (6.2.4) *has the property that $K_1(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$. Moreover, let $W \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, and define $H \in \mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$ by* (6.2.7). *Then the function H can alternatively be written as*

$$H(\lambda) = \Phi_{2,2}(\lambda) + \Phi_{2,1}(\lambda)W(\lambda)(I - \Phi_{1,1}(\lambda)W(\lambda))^{-1}\Phi_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}), \tag{6.2.8}$$

where $\Phi_{1,1}$ and $\Phi_{2,1}$ are Schur class functions in $\mathbf{S}(\mathcal{Y} \oplus \mathcal{U}, \mathcal{U})$ and $\mathbf{S}(\mathcal{Y} \oplus \mathcal{U}, \mathcal{Y})$, respectively, and $\Phi_{1,2}$ and $\Phi_{2,2}$ are functions in $\mathbf{H}^\infty(\mathcal{U},\mathcal{U})$ and $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$, respectively, and these functions are given by

$$\begin{aligned}
\Phi_{1,1}(\lambda) &\equiv -\lambda(\Lambda^\times_{1,1})^{\frac{1}{2}} K_1(\lambda)^{-1} \begin{bmatrix} K_3(\lambda)(I+FG^*)^{-\frac{1}{2}} & K_2(\lambda)(\Lambda^\times_{N,N})^{-\frac{1}{2}} \end{bmatrix}, \\
\Phi_{1,2}(\lambda) &\equiv (\Lambda^\times_{1,1})^{\frac{1}{2}} K_1(\lambda)^{-1}, \\
\Phi_{2,1}(\lambda) &\equiv -\begin{bmatrix} K_6(\lambda)(I+FG^*)^{-\frac{1}{2}} & K_5(\lambda)(\Lambda^\times_{N,N})^{-\frac{1}{2}} \end{bmatrix} + \\
&\qquad + \lambda K_4(\lambda)K_1(\lambda)^{-1} \begin{bmatrix} K_3(\lambda)(I+FG^*)^{-\frac{1}{2}} & K_2(\lambda)(\Lambda^\times_{N,N})^{-\frac{1}{2}} \end{bmatrix}, \\
\Phi_{2,2}(\lambda) &\equiv -K_4(\lambda)K_1(\lambda)^{-1}.
\end{aligned} \tag{6.2.9}$$

Here K_1, K_2, K_3, K_4, K_5 and K_6 are the operator-valued polynomials given by (6.2.4), *and F and G are the operators in* (6.2.6). *Moreover, let $M_{\Phi_{1,1}}$ and $M_{\Phi_{2,1}}$ be the multiplication operators defined by $\Phi_{1,1}$ and $\Phi_{2,1}$, respectively, an let $\Gamma_{\Phi_{1,2}}$ and $\Gamma_{\Phi_{2,2}}$ be the operators defined by $\Phi_{1,2}$ and $\Phi_{2,2}$, respectively. Then the operator*

$$\begin{bmatrix} 0 & A\tilde{E}_{\mathcal{U}} \\ M_{\Phi_{11}} & \Gamma_{\Phi_{12}} \\ M_{\Phi_{21}} & \Gamma_{\Phi_{22}} \end{bmatrix} : \begin{bmatrix} H^2(\mathcal{Y} \oplus \mathcal{U}) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \ell^2_-(\mathcal{Y}) \\ H^2(\mathcal{U}) \\ H^2(\mathcal{Y}) \end{bmatrix}, \tag{6.2.10}$$

is isometric, where $\tilde{E}_{\mathcal{U}}$ is the operator that is embedding $\mathcal{U}$ into $\ell^2_+(\mathcal{U})$ given by $\tilde{E}^*_{\mathcal{U}} = \begin{bmatrix} I_{\mathcal{U}} & 0 & 0 & \cdots \end{bmatrix}$.

Proof. The identity (6.2.8) follows immediately from Proposition 5.3.1, Theorems 5.4.1 and 6.2.1, and the discussion in Section 5.4 preceding Theorem 5.4.1. Here $\{A, T', U', R, Q\}$ is the lifting data set defined in Proposition 6.1.1, but with the Sz.-Nagy-Schäffer isometric lifting U' of T' rather than with V' in (6.1.5). Note that $R - \lambda Q$ is left invertible for each $\lambda \in \mathbb{D}$. Hence, using the remark in the third paragraph after Theorem 5.2.1, we obtain that the spectral radius of the operator X_1 in (5.2.1) is strictly less than one. In particular, it follows that $\Phi_{1,2} \in \mathbf{H}^\infty(\mathcal{U},\mathcal{U})$ and $\Phi_{2,2} \in \mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$. Finally, the observation concerning the operator in (6.2.10) now follows because $R^*R = Q^*Q$, and because the operator (6.2.10) is the restriction of the operator M in Theorem 5.2.1 to the subspace

$$H^2(\mathcal{Y} \oplus \mathcal{U}) \oplus \mathcal{U} = H^2(\mathcal{D}_\circ \oplus \mathcal{D}_{T'} \oplus \operatorname{Ker} R^*) \oplus \operatorname{Ker} Q^*;$$

see Theorem 5.2.1. □

6.3 The classical Nehari problem

The purpose of this section is to recall the classical (operator-valued) Nehari problem and review some well known facts about its solutions. We do not present an extensive treatment of the problem, but merely provide the results that are required in the sequel. For a more detailed account we refer to Sections IX.8 and XIV.9 in [38] and the references given there.

Let $F_{-1}, F_{-2}, \ldots \in \mathscr{L}(\mathcal{U}, \mathcal{Y})$ be a Nehari data sequence; see Section 6.1. The classical Nehari problem is to describe all sequences $H_0, H_1, \ldots$ of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ such that the operator from $\ell^2_+(\mathcal{U})$ into $\ell^2(\mathcal{Y})$ given by

$$\begin{bmatrix} \vdots & \vdots & \vdots & \\ F_{-2} & F_{-3} & F_{-4} & \cdots \\ F_{-1} & F_{-2} & F_{-3} & \ddots \\ \boxed{H_0} & F_{-1} & F_{-2} & \ddots \\ H_1 & H_0 & F_{-1} & \ddots \\ H_2 & H_1 & H_2 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathcal{U}) \to \ell^2(\mathcal{Y}) \tag{6.3.1}$$

has operator norm at most one. A sequence $H_0, H_1, \ldots$ of operators in $\mathscr{L}(\mathcal{U}, \mathcal{Y})$ such that the operator in (6.3.1) is contractive will be referred to as an *∞-complementary sequence associated with* $F_{-1}, F_{-2}, \ldots$, or just an ∞-complementary sequence when the Nehari data sequence in question is clear from the context.

As is well known, an ∞-complementary sequence exists if and only if the *Hankel operator defined by* $F_{-1}, F_{-2}, \ldots$

$$A_\infty = \begin{bmatrix} \vdots & \vdots & \vdots & \\ F_{-3} & F_{-4} & F_{-5} & \cdots \\ F_{-2} & F_{-3} & F_{-4} & \cdots \\ F_{-1} & F_{-2} & F_{-3} & \cdots \end{bmatrix} : \ell^2_+(\mathcal{U}) \to \ell^2_-(\mathcal{Y}) \tag{6.3.2}$$

is a contraction. Notice that (6.3.2) is not the standard form of the Hankel operator. The more common form is obtained after applying the so-called flip over operator from $\ell^2_-(\mathcal{Y})$ onto $\ell^2_+(\mathcal{Y})$, which is given by

$$(\ldots, y_{-3}, y_{-2}, y_{-1}) \mapsto (y_{-1}, y_{-2}, y_{-3}, \ldots),$$

on the left side of A_∞ in (6.3.2).

The classical Nehari problem can be put into a classical commutant lifting setting as follows. Consider the lifting data set $\Omega_\infty = \{A_\infty, T', V', I_{\ell^2_+(\mathcal{U})}, \tilde{S}_{\mathcal{U}}\}$, where A_∞ is the Hankel operator defined by $F_{-1}, F_{-2}, \ldots$, the operators T' and V' are given by (6.1.3) and (6.1.5), respectively, and $\tilde{S}_{\mathcal{U}}$ is the forward shift operator on $\ell^2_+(\mathcal{U})$.

An operator B from $\ell^2_+(\mathcal{U})$ into $\ell^2(\mathcal{Y})$ is a contractive interpolant for Ω_∞ if and only if there exists an ∞-complementary sequence $H_0, H_1, \ldots$ such that B is equal to the contraction in (6.3.1). Thus the ∞-complementary sequences can be obtained from the contractive interpolants for Ω_∞, and even from their first columns, in other words, from their restrictions to Ker $\tilde{S}^*_{\mathcal{U}}$.

Next we assume that the Hankel operator A_∞ is a strict contraction. Following the line of thought in Section 6.2 one sees that the ∞-complementary sequences can be obtained as the Taylor coefficients at zero of functions in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ that are characterized by Schur class functions in $\mathbf{S}(\mathcal{U},\mathcal{Y})$. In this Schur class $\mathbf{S}(\mathcal{U},\mathcal{Y})$ the spaces $\mathcal{U}$ and $\mathcal{Y}$ correspond to Ker $\tilde{S}^*_{\mathcal{U}}$ and $\mathcal{D}_{T'}$, respectively. Unlike in the relaxed Nehari setting there is no second occurrence of $\mathcal{U}$ because Ker $I_{\ell^2_+(\mathcal{U})} = \{0\}$. Moreover, it is well known that in the classical Nehari case the functions in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ that provide the ∞-complementary sequences via their Taylor coefficients at zero are also elements of $\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$.

In fact, let $V \in \mathbf{S}(\mathcal{U},\mathcal{Y})$ and define the function $H^{(\infty)}_V$ on $\mathbb{D}$ by

$$H^{(\infty)}_V(\lambda) = (\Psi^{(\infty)}_{1,2}(\lambda)V(\lambda)+\Psi^{(\infty)}_{1,1}(\lambda))(\Psi^{(\infty)}_{2,2}(\lambda)V(\lambda)+\Psi^{(\infty)}_{2,1}(\lambda))^{-1}\ (\lambda\in\mathbb{D}),\quad (6.3.3)$$

or alternatively by

$$H^{(\infty)}_V(\lambda) = \Phi^{(\infty)}_{2,2}(\lambda)+\Phi^{(\infty)}_{2,1}(\lambda)V(\lambda)(I-\Phi^{(\infty)}_{1,1}(\lambda)V(\lambda))^{-1}\Phi^{(\infty)}_{1,2}(\lambda),\ (\lambda\in\mathbb{D})\quad (6.3.4)$$

where $\Psi^{(\infty)}_{1,1}$, $\Psi^{(\infty)}_{1,2}$, $\Psi^{(\infty)}_{2,1}$ and $\Psi^{(\infty)}_{2,2}$ are the coefficients in the linear fractional representation (5.4.3) in Theorem 5.4.1, while $\Phi^{(\infty)}_{1,1}$, $\Phi^{(\infty)}_{1,2}$, $\Phi^{(\infty)}_{2,1}$ and $\Phi^{(\infty)}_{2,2}$ denote the Redheffer coefficients in (5.4.1), all specified for the lifting data set Ω_∞. Then the Taylor coefficients of $H^{(\infty)}_V$ at zero form an ∞-complementary sequence, and all ∞-complementary sequences are obtained in this way. Furthermore, in (6.3.3) (and (6.3.4)) the Schur class function V is uniquely determined by the function $H^{(\infty)}_V$. We will not specify the coefficients appearing in (6.3.3) and (6.3.4) any further. This is not required for the results we are after in the sequel, and there are many formulas available in the literature; we refer to Section XIV.9 in [38] and the references mentioned there.

6.4 The classical Nehari problem as a limit case

Let $F_{-1}, F_{-2}, \ldots \in \mathscr{L}(\mathcal{U},\mathcal{Y})$ be a Nehari data sequence, and assume that the Hankel operator A_∞ defined by $F_{-1}, F_{-2}, \ldots$ is contractive. For each positive integer N we will view $\mathcal{U}^N$ as the subspace of $\ell^2_+(\mathcal{U})$ consisting of all $\mathcal{U}$-valued sequences that have non-zero vectors only in the first N positions. In that case the N-truncated Hankel operator A_N is equal to $A_\infty|_{\mathcal{U}^N}$, and hence is also a contraction. Moreover, each solution to the classical Nehari problem, that is, each ∞-complementary sequence, is an N-complementary sequence as well, for any positive integer N. The converse statement is also true, that is, a sequence $H_0, H_1, \ldots \in \mathscr{L}(\mathcal{U},\mathcal{Y})$ is an

∞-complementary sequence if and only if it is an N-complementary sequence for each positive integer N.

In this way the classical Nehari problem can be seen as the limit case of the relaxed Nehari problem as the Nehari index N goes to infinity. Under the additional assumption that the Hankel operator is a strict contraction there is also convergence of the solutions. The following theorem presents the precise result.

Theorem 6.4.1. *Let $F_{-1}, F_{-2}, \ldots \in \mathscr{L}(\mathcal{U}, \mathcal{Y})$ be a Nehari data sequence such that the Hankel operator A_∞ defined by $F_{-1}, F_{-2}, \ldots$ is a strict contraction. Let W be in $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$, and define V in $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ by $V(\lambda) = \Pi_{\mathcal{Y}} W(\lambda)$ for each $\lambda \in \mathbb{D}$. Define $H_V^{(\infty)}$ by (6.3.3), and for each positive integer N let $H_W^{(N)}$ be the function given by the right hand side of (6.2.7). Then $H_W^{(N)}$ converges to $H_V^{(\infty)}$ in the strong operator topology and uniformly on compact subsets of $\mathbb{D}$ as N goes to infinity.*

Before we give the proof of Theorem 6.4.1 we first recall some facts concerning convergence in the strong operator topology; cf., Chapter 13 in [56].

Let $X_0, X_1, \ldots$ be a sequence of operators in $\mathscr{L}(\mathcal{V}, \mathcal{W})$. Then $X_0, X_1, \ldots$ is said to converge to $X \in \mathscr{L}(\mathcal{V}, \mathcal{W})$ in the strong operator topology if

$$\lim_{n\to\infty} \|Xv - X_n v\| = 0 \text{ for each } v \in \mathcal{V}.$$

The property that $X_0, X_1, \ldots$ converges to X in the strong operator will denoted by $X_n \overset{\mathbf{SOT}}{\longrightarrow} X$.

Now let $Y, Y_0, Y_1, \ldots \in \mathscr{L}(\mathcal{V}, \mathcal{W})$ and $Z, Z_0, Z_1, \ldots \in \mathscr{L}(\mathcal{W}, \mathcal{X})$, and assume that $X_n \overset{\mathbf{SOT}}{\longrightarrow} X \in \mathscr{L}(\mathcal{V}, \mathcal{W})$ and $Y_n \overset{\mathbf{SOT}}{\longrightarrow} Y \in \mathscr{L}(\mathcal{V}, \mathcal{W})$, while $Z_n \overset{\mathbf{SOT}}{\longrightarrow} Z \in \mathscr{L}(\mathcal{W}, \mathcal{X})$. then the following statements hold:

(i) $X_n + Y_n \overset{\mathbf{SOT}}{\longrightarrow} X + Y$,

(ii) $Z_n X_n \overset{\mathbf{SOT}}{\longrightarrow} ZX$,

(iii) if $X, X_n \geq 0$ for all $n \in \mathbb{N}$, then $X_n^{\frac{1}{2}} \overset{\mathbf{SOT}}{\longrightarrow} X^{\frac{1}{2}}$,

(iv) if X and X_n are invertible such that $\|X_n^{-1}\| \leq \delta < \infty$ for all $n \in \mathbb{N}$, then $X_n^{-1} \overset{\mathbf{SOT}}{\longrightarrow} X^{-1}$.

Statement (i) follows trivially from the definitions. For (ii), see Problem 113 in [56]. Claim (iii) can be proved with (i) and (ii), and a standard approximation argument based on the functional calculus for self adjoint operators involving sequences of polynomials in X and X_n that converge to $X^{\frac{1}{2}}$ and $X_n^{\frac{1}{2}}$. To see that (iv) holds let $h \in \mathcal{W}$ and put $k = X^{-1}h \in \mathcal{V}$. Then

$$\|X^{-1}h - X_n^{-1}h\| = \|X_n^{-1}(X_N X^{-1}h - XX^{-1}h)\| \leq \delta\|X_n k - Xk\| \to 0 \quad \text{as } n \to \infty,$$

which proves (iv).

The following lemma will also be useful in the sequel.

Lemma 6.4.2. *Let $X, X_0, X_1, \ldots$ be operators on $\mathcal{X}$ such that $X_n \overset{\textbf{SOT}}{\longrightarrow} X$. Assume that $r_{\mathrm{spec}}(X) < 1$, and assume that there exists an positive number $\delta < 1$ such that $r_{\mathrm{spec}}(X_n) < \delta$ for each integer $n \geq 0$. Then*

$$(I - X_n)^{-1} \overset{\textbf{SOT}}{\longrightarrow} (I - X)^{-1}.$$

Proof. Let n be an arbitrary nonnegative integer. Since $r_{\mathrm{spec}}(X_n) < \delta < 1$, we have that the spectrum of $I - X_n$ is outside the open ball $\mathrm{B}_{1-\delta}(0)$ with radius $1 - \delta$ and center 0. Therefore the spectral mapping theorem, Theorem I.3.3 in [54], shows that

$$\|(I - X_n)^{-1}\| \leq r_{\mathrm{spec}}(I - X_n)^{-1} \leq (1 - \delta)^{-1} \quad (n \in \mathbb{N}).$$

Our claim now follows using property (iv) in the list given above. □

Finally, before we start with the proof of Theorem 6.4.1 we introduce the following notation. For $\mathscr{L}(\mathcal{V}, \mathcal{W})$-valued functions $Z, Z_0, Z_1, \ldots$ on $\mathbb{D}$ we write $Z_n(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} Z(\,\cdot\,)$ (or $Z_n(\lambda) \overset{\textbf{SOT/UC}}{\longrightarrow} Z(\lambda)$ when this causes no confusion) in case $Z_n(\lambda) \overset{\textbf{SOT}}{\longrightarrow} Z(\lambda)$ for each $\lambda \in \mathbb{D}$, where the convergence in λ is uniform on compact subsets of $\mathbb{D}$.

Proof of Theorem 6.4.1. The proof is split into five parts.

Part 1. We will first set the notation used throughout the proof. Let Ω_∞ and Ω_N be the lifting data sets corresponding to the classical Nehari problem and the relaxed Nehari problem with relaxation index N, respectively, both for the Nehari data sequence $F_{-1}, F_{-2}, \ldots$. Hence $\Omega_\infty = \{A_\infty, T', V', I_{\ell^2_+(\mathcal{U})}, \tilde{S}_{\mathcal{U}}\}$ and $\Omega_N = \{A_N, T', V', R_N, Q_N\}$, where A_∞ denotes the Hankel operator (6.3.2) while A_N stands for the N-truncated Hankel operator (6.1.2), the operators T' and V' are given by (6.1.3) and (6.1.5), respectively, $\tilde{S}_{\mathcal{U}}$ is the forward shift operator on $\ell^2_+(\mathcal{U})$, and R_N and Q_N denote the operators R and Q in (6.1.4), respectively.

As we mentioned earlier in this section, the space $\mathcal{U}^N$ is seen as the subspace of $\ell^2_+(\mathcal{U})$ consisting of all $\mathcal{U}$-valued sequences that have non-zero vectors only in the first N positions. We write Π_N for the orthogonal projection from $\ell^2_+(\mathcal{U})$ onto $\mathcal{U}^N$ viewed as an operator from $\ell^2_+(\mathcal{U})$ to $\mathcal{U}^N$, while P_N will denote the orthogonal projection from $\ell^2_+(\mathcal{U})$ on $\mathcal{U}^N$ acting as an operator on $\ell^2_+(\mathcal{U})$.

The functions in (5.3.1) associated with the lifting data sets Ω_∞ and Ω_N will be written as $K_1^{(\infty)}, \ldots, K_6^{(\infty)}$ and $K_1^{(N)}, \ldots, K_6^{(N)}$, respectively. For each $i, j = 1, 2$ the coefficients for the linear fractional representation (5.4.3) given in Theorem 5.4.1 and the Redheffer coefficients in (5.4.1) specified for the lifting data sets Ω_∞ and Ω_N will be written as $\Psi_{i,j}^{(\infty)}$, $\Phi_{i,j}^{(\infty)}$, and $\Psi_{i,j}^{(N)}$, $\Phi_{i,j}^{(N)}$, respectively.

Part 2. Let N be an arbitrary positive integer. In this part we express the functions $K_1^{(N)}$, $K_2^{(N)}$, $K_3^{(N)}$, $K_4^{(N)}$, $K_5^{(N)}$ and $K_6^{(N)}$, as well as the strictly positive operators Δ_{Q_N}, Δ_{R_N} and Δ_{Ω_N}, in terms of the operators appearing in Ω_∞ and the projections Π_n and P_n for $n = N, N-1$. In fact, we will show that $K_1^{(N)}$, $K_2^{(N)}$, $K_3^{(N)}$, $K_4^{(N)}$,

$K_5^{(N)}$ and $K_6^{(N)}$ can be written as

$$\begin{aligned}
K_1^{(N)}(\lambda) &= \Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\Pi_1^*,\\
K_2^{(N)}(\lambda) &= \Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^*,\\
K_3^{(N)}(\lambda) &= \Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_{N-1}}^{-2}P_{N-1}A_\infty^*D_{T'},\\
K_4^{(N)}(\lambda) &= D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\Pi_1^*,\\
K_5^{(N)}(\lambda) &= \lambda D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^*,\\
K_6^{(N)}(\lambda) &= -I_{\mathcal{Y}}+\lambda D_{T'}A_\infty P_{N-1}\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_{N-1}}^{-2}P_{N-1}A_\infty^*D_{T'},
\end{aligned}\quad (\lambda\in\mathbb{D}) \tag{6.4.1}$$

while Δ_{Q_N}, Δ_{R_N} and Δ_{Ω_N} are given by

$$\Delta_{Q_N}=\Pi_1D_{A_\infty P_N}^{-2}\Pi_1^*,\quad \Delta_{R_N}=\Pi_1(\tilde{S}_{\mathcal{U}}^*)^{N-1}D_{A_\infty P_N}^{-2}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* \tag{6.4.2}$$

and

$$\Delta_{\Omega_N}=I+D_{T'}A_\infty P_{N-1}D_{A_\infty P_{N-1}}^{-2}P_{N-1}A_\infty^*D_{T'}. \tag{6.4.3}$$

First note that from the definitions of R_N and Q_N, and since $A_N=A_\infty|_{\mathcal{U}^N}$, we immediately get the following identities:

$$A_N=A_\infty\Pi_N^*,\quad R_N=\Pi_N\Pi_{N-1}^*,\quad Q_N=\Pi_N\tilde{S}_{\mathcal{U}}\Pi_{N-1}^*,$$

$$\Pi_{\operatorname{Ker}R_N^*}=\Pi_1(\tilde{S}_{\mathcal{U}}^*)^{N-1}\Pi_N^*\quad\text{and}\quad \Pi_{\operatorname{Ker}Q_N^*}=\Pi_1\Pi_N^*.$$

Thus we obtain

$$\begin{aligned}
M_N &:= R_NQ_N^*=\Pi_NP_{N-1}\tilde{S}_{\mathcal{U}}^*\Pi_N^*=\Pi_N\tilde{S}_{\mathcal{U}}^*\Pi_N^*,\\
D_{A_N}^{-2} &= D_{A_\infty\Pi_N^*}=\Pi_ND_{A_\infty P_N}^{-2}\Pi_N^*,\\
(R_N^*D_{A_N}R_N)^{-2} &= D_{A_{N-1}}^{-1}=\Pi_{N-1}D_{A_\infty P_{N-1}}^{-2}\Pi_{N-1}^*,\\
J_N &:= D_{T'}A_NR_N=D_{T'}A_\infty P_N\Pi_{N-1}^*=D_{T'}A_\infty\Pi_{N-1}^*.
\end{aligned} \tag{6.4.4}$$

From these identities and the definitions in (5.2.2) it follows that (6.4.2) and (6.4.3) hold.

The formula for M_N in (6.4.4) and the fact that $\tilde{S}_{\mathcal{U}}^*\mathcal{U}^N=\mathcal{U}^{N-1}\subset\mathcal{U}^N$ imply that $(I-\lambda M_N)^{-1}=\Pi_N(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}\Pi_N^*$ for each $\lambda\in\mathbb{D}$. So, using that $\mathcal{U}^N$ is an invariant subspace for both $D_{A_\infty P_N}^{-2}$ and $\tilde{S}_{\mathcal{U}}^*$, we see that

$$\begin{aligned}
K_1^{(N)}(\lambda) &= \Pi_{\operatorname{Ker}Q^*}(I-\lambda M_N^*)^{-1}D_{A_N}^{-2}\Pi_{\operatorname{Ker}Q^*}^*\\
&= \Pi_1P_N(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}P_ND_{A_\infty P_N}^{-2}P_N\Pi_1^*\\
&= \Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\Pi_1^*,\\
K_4^{(N)}(\lambda) &= D_{T'}A_NM_N(I-\lambda M_N^*)^{-1}D_{A_N}^{-2}\Pi_{\operatorname{Ker}Q^*}^*\\
&= D_{T'}A_\infty P_{N-1}\tilde{S}_{\mathcal{U}}^*P_N(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}P_ND_{A_\infty P_N}^{-2}P_N\Pi_1^*\\
&= D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}D_{A_\infty P_N}^{-2}\Pi_1^*.
\end{aligned}\quad (\lambda\in\mathbb{D})$$

Hence the formulas for $K_1^{(N)}$ and $K_4^{(N)}$ in (6.4.1) hold. The formulas for $K_2^{(N)}$, $K_3^{(N)}$, $K_5^{(N)}$ and $K_6^{(N)}$ in (6.4.1) follow with similar computations using the identities derived in (6.4.4).

Part 3. In this part we show that for $i = 1, 3, 4$ and $j = 2, 5$ we have

$$K_i^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} K_i^{(\infty)}(\,\cdot\,) \quad \text{and} \quad K_j^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} 0, \tag{6.4.5}$$

while for the operators Δ_{Q_N}, Δ_{R_N} and Δ_{Ω_N} we obtain

$$\Delta_{Q_N}^{\pm 1} \overset{\text{SOT}}{\longrightarrow} \Delta_{\tilde{S}_{\mathcal{U}}}^{\pm 1}, \quad \Delta_{R_N}^{\pm 1} \overset{\text{SOT}}{\longrightarrow} I_{\mathcal{U}} \quad \text{and} \quad \Delta_{\Omega_N}^{\pm 1} \overset{\text{SOT}}{\longrightarrow} \Delta_{\Omega_\infty}^{\pm 1}. \tag{6.4.6}$$

Here $\Delta_{\tilde{S}_{\mathcal{U}}}$ and Δ_{Ω_∞} denote the strictly positive operators Δ_Q and Δ_Ω in (5.2.2), respectively, specified for Ω_∞. As a result of these limits and Theorem 5.4.1 it follows trivially that

$$\Psi_{1,1}^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} \Psi_{1,1}^{(\infty)}(\,\cdot\,), \quad \Psi_{1,2}^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} \begin{bmatrix} \Psi_{1,2}^{(\infty)}(\,\cdot\,) & 0 \end{bmatrix},$$

$$\Psi_{2,1}^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} \Psi_{2,1}^{(\infty)}(\,\cdot\,), \quad \Psi_{2,2}^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} \begin{bmatrix} \Psi_{2,2}^{(\infty)}(\,\cdot\,) & 0 \end{bmatrix}.$$

We will first show that the limits in (6.4.6) hold. To see that this is the case first notice that $P_N \overset{\text{SOT}}{\longrightarrow} I_{\ell_+^2(\mathcal{U})}$. Hence $P_N A_\infty^* A_\infty P_N \overset{\text{SOT}}{\longrightarrow} A_\infty^* A_\infty$. Since

$$r_{\text{spec}}(P_N A_\infty^* A_\infty P_N) = \|A_\infty P_N\|^2 \leq \|A_\infty\| < 1 \text{ for each integer } N > 0,$$

we obtain from Lemma 6.4.2 that

$$D_{A_\infty P_N}^{-2} = (I - P_N A_\infty^* A_\infty P_N)^{-1} \overset{\text{SOT}}{\longrightarrow} (I - A_\infty^* A_\infty)^{-1} = D_{A_\infty}^{-2}.$$

This implies that $\Delta_{Q_N} \overset{\text{SOT}}{\longrightarrow} \Delta_{\tilde{S}_{\mathcal{U}}}$ and $\Delta_{\Omega_N} \overset{\text{SOT}}{\longrightarrow} \Delta_{\Omega_\infty}$. Next observe that

$$\begin{aligned} \Delta_{R_N} &= \Pi_1 (\tilde{S}_{\mathcal{U}}^*)^{N-1} D_{A_\infty P_N}^{-2} \tilde{S}_{\mathcal{U}}^{N-1} \Pi_1^* \\ &= I_{\mathcal{U}} + \Pi_1 (\tilde{S}_{\mathcal{U}}^*)^{N-1} D_{A_\infty P_N}^{-2} P_N A_\infty^* A_\infty P_N \tilde{S}_{\mathcal{U}}^{N-1} \Pi_1^* \\ &= I_{\mathcal{U}} + \Pi_1 (\tilde{S}_{\mathcal{U}}^*)^{N-1} D_{A_\infty P_N}^{-2} P_N A_\infty^* T'^{N-1} A_\infty \Pi_1^*. \end{aligned}$$

Since $\tilde{S}_{\mathcal{U}}^*$ and T' are pointwise stable operators, we see that $\Delta_{R_N} \overset{\text{SOT}}{\longrightarrow} I_{\mathcal{U}}$.

The fact that $D_{A_\infty P_N}$ is a strict contraction, implies that $D_{A_\infty P_N}^{-2} \gg I$. In particular, using that Π_1^* and $\tilde{S}_{\mathcal{U}}^{N-1} \Pi_1^*$ are isometries, we obtain that $\Delta_{Q_N} \gg I$ and $\Delta_{R_N} \gg I$. Moreover, from the formula for Δ_{Ω_N} in (6.4.3) it follows that $\Delta_{\Omega_N} \geq I$. Thus the operators $\Delta_{Q_N}^{-1}$, $\Delta_{R_N}^{-1}$ and $\Delta_{\Omega_N}^{-1}$ are all contractions. This implies that the limits for the inverses of Δ_{Q_N}, Δ_{R_N} and Δ_{Ω_N} in (6.4.6) also hold.

Notice that $\Pi_{\text{Ker}\, \tilde{S}_{\mathcal{U}}^*}$ can be identified with Π_1. Hence, using the formulas in (6.4.1) and $D_{A_\infty P_N} \overset{\text{SOT}}{\longrightarrow} D_{A_\infty}$, we see that $K_i^{(N)}(\,\cdot\,) \overset{\text{SOT/UC}}{\longrightarrow} K_i^{(\infty)}(\,\cdot\,)$ holds for $i = 1, 3, 4, 6$.

Next observe that

$$\begin{aligned} D_{A_\infty P_N}^{-2}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* &= \tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* + D_{A_\infty P_N}^{-2}P_N A_\infty^* A_\infty P_N \tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* \\ &= \tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* + D_{A_\infty P_N}^{-2}P_N A_\infty^* T'^{N-1}A_\infty \Pi_1. \end{aligned}$$

Again using that T' is pointwise stable, we obtain that the second summand in the right hand side of last identity converges to zero in the strong operator topology. Hence to prove (6.4.5) it remains to show that

$$\begin{aligned} &\Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* \overset{\text{SOT/UC}}{\longrightarrow} 0, \\ &D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* \overset{\text{SOT/UC}}{\longrightarrow} 0. \end{aligned} \tag{6.4.7}$$

Since $\tilde{S}_{\mathcal{U}}^*\tilde{S}_{\mathcal{U}} = I$ and $\Pi_1\tilde{S}_{\mathcal{U}} = 0$, we see that $\Pi_1(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* = \lambda^{N-1}I_{\mathcal{U}}$ and

$$\begin{aligned} D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^*(I-\lambda\tilde{S}_{\mathcal{U}}^*)^{-1}\tilde{S}_{\mathcal{U}}^{N-1}\Pi_1^* &= \textstyle\sum_{n=0}^{N-2}\lambda^n D_{T'}A_\infty\tilde{S}_{\mathcal{U}}^{N-2-n} \\ &= D_{T'}\textstyle\sum_{n=0}^{N-2}\lambda^n T'^{N-2-n}A_\infty. \end{aligned}$$

Thus the first limit in (6.4.7) clearly holds. To see that the second limit in (6.4.7) also holds we show that $\sum_{n=0}^{N}\lambda^n T'^{N-n} \overset{\text{SOT/UC}}{\longrightarrow} 0$. Fix a $h \in \ell_-^2(\mathcal{U})$, $\varepsilon > 0$ and a σ in the open interval $(0,1)$. Let $\lambda \in \mathbb{D}$ be such that $|\lambda| < \sigma$. Since T' is pointwise stable, we have that $\|T'^N h\| \to 0$ as $N \to \infty$. In particular, there is an $M_0 > 0$ such that $\|T'^N h\| < M_0$ for each $N \in \mathbb{N}$. Note that for each $N_0 \in \mathbb{N}$ we have

$$\sum_{n=N_0}^{\infty}|\lambda|^n < \frac{\sigma^{N_0}}{1-\sigma}.$$

Now let $N_0 \in \mathbb{N}$ be such that $\sigma^{N_0} < \varepsilon(1-\sigma)(2M_0)^{-1}$. From the fact that $\|T'^N h\| \to 0$ as $N \to \infty$ we obtain that there exists a $N_1 \in \mathbb{N}$ such that $N_1 > N_0$ and for each $\tilde{N} > N_1$ we have $\|T'^{\tilde{N}-N_0}h\| < \varepsilon(2N_0)^{-1}$. Then we have for each $\tilde{N} > N_1$ that

$$\begin{aligned} \|\sum_{n=0}^{\tilde{N}}\lambda^k T'^{\tilde{N}-n}h\| &\leq \sum_{n=0}^{\tilde{N}}|\lambda|^n\|T'^{\tilde{N}-n}h\| \\ &= \sum_{n=0}^{N_0-1}|\lambda|^n\|T'^{\tilde{N}-n}h\| + \sum_{n=N_0}^{\tilde{N}}|\lambda|^n\|T'^{\tilde{N}-1-n}h\| \\ &< \sum_{n=0}^{N_0-1}\|T'^{\tilde{N}-n}h\| + M_0\sum_{n=N_0}^{\tilde{N}}|\lambda|^n \\ &< N_0\varepsilon(2N_0)^{-1} + M_0\varepsilon(2M_0)^{-1} = \varepsilon. \end{aligned}$$

This proves our claim.

Part 4. Next we show that $K_1^{(N)}(\,\cdot\,)^{-1} \overset{\textbf{SOT/UC}}{\longrightarrow} K_1^{(\infty)}(\,\cdot\,)^{-1}$. From Proposition 5.3.1 and the convergence results in the previous part it then follows that

$$\Phi_{1,1}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \begin{bmatrix} \Phi_{1,1}^{(\infty)}(\,\cdot\,) & 0 \end{bmatrix}, \quad \Phi_{1,2}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \Phi_{1,2}^{(\infty)}(\,\cdot\,),$$

$$\Phi_{2,1}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \begin{bmatrix} \Phi_{2,1}^{(\infty)}(\,\cdot\,) & 0 \end{bmatrix}, \quad \Phi_{2,2}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \Phi_{2,2}^{(\infty)}(\,\cdot\,).$$

Using Proposition 5.3.1 and Theorem 5.2.1 we see that for each positive integer N

$$K_1^{(N)}(\lambda)^{-1} = \Delta_{Q_N}^{-\frac{1}{2}}\Phi_{1,2}^{(N)}(\lambda) = \Delta_{Q_N}^{-1}\Pi_1\Pi_N^*(I_{\mathcal{U}^N} - \lambda X_1^{(N)})^{-1}\Pi_N\Pi_1^* \quad (\lambda \in \mathbb{D}),$$

where (using (5.3.4))

$$\begin{aligned} X_1^{(N)} &= R_N(Q_N^*D_{A_N}^2Q_N)^{-1}Q_N^*D_{A_N}^2 = M_N(I - D_{A_N}^{-2}\Pi_{\mathrm{Ker}\,Q_N^*}^*\Delta_{Q_N}^{-1}\Pi_{\mathrm{Ker}\,Q_N^*}) \\ &= \Pi_N\tilde{S}_{\mathcal{U}}^*\Pi_N^*(I - \Pi_N D_{A_\infty P_N}P_N\Pi_1^*\Delta_{Q_N}^{-1}\Pi_1\Pi_N^*) \\ &= \Pi_N\tilde{S}_{\mathcal{U}}^*(I - D_{A_\infty P_N}\Pi_1^*\Delta_{Q_N}^{-1}\Pi_1)\Pi_N^*. \end{aligned}$$

With similar arguments we obtain that

$$K_1^{(\infty)}(\lambda)^{-1} = \Delta_{\tilde{S}_{\mathcal{U}}}^{-1}\Pi_1(I_{\ell_+^2(\mathcal{U})} - \lambda X_1^{(\infty)})^{-1}\Pi_1^* \quad (\lambda \in \mathbb{D}),$$

where

$$X_1^{(\infty)} = \tilde{S}_{\mathcal{U}}^*(I - D_{A_\infty}\Pi_1^*\Delta_{\tilde{S}_{\mathcal{U}}}^{-1}\Pi_1).$$

With the results of Part 3 we see that $\Pi_N^*X_1^{(N)}\Pi_N \overset{\textbf{SOT}}{\longrightarrow} X_1^{(\infty)}$. Moreover, since $R_N - \lambda Q_N$ as well as $I - \lambda\tilde{S}_{\mathcal{U}}$ is left invertible for each $\lambda \in \mathbb{D}$ and each positive integer N, we obtain from the remark in the third paragraph after Theorem 5.2.1 that $r_{\mathrm{spec}}(X_1^{(\infty)}) < 1$ and $r_{\mathrm{spec}}(\Pi_N^*X_1^{(N)}\Pi_N) = r_{\mathrm{spec}}(X_1^{(N)}) < 1$. Thus for each $\sigma \in (0,1)$ we have $r_{\mathrm{spec}}(\lambda\Pi_N^*X_1^{(N)}\Pi_N) < \sigma$ for $\lambda \in \mathbb{D}$ with $|\lambda| < \sigma$. According to Lemma 6.4.2 we then have

$$\Pi_N^*(I - \lambda X_1^{(N)})^{-1}\Pi_N = P_N(I - \lambda\Pi_N^*X_1^{(N)}\Pi_N)^{-1}P_N \overset{\textbf{SOT/UC}}{\longrightarrow} (I - \lambda X_1^{(\infty)})^{-1}.$$

Hence $K_1^{(N)}(\,\cdot\,)^{-1} \overset{\textbf{SOT/UC}}{\longrightarrow} K_1^{(\infty)}(\,\cdot\,)^{-1}$, as we claimed.

Part 5. In this final part we complete the proof of Theorem 6.4.1. Notice that $H_W^{(N)}$ and $H_V^{(\infty)}$ are given by

$$\begin{aligned} H_W^{(N)}(\lambda) &= \Phi_{2,2}^{(N)}(\lambda) + \Phi_{2,1}^{(N)}(\lambda)W(\lambda)(I - \Phi_{1,1}^{(N)}(\lambda)W(\lambda))^{-1}\Phi_{1,2}^{(N)}(\lambda), \\ H_V^{(\infty)}(\lambda) &= \Phi_{2,2}^{(\infty)}(\lambda) + \Phi_{2,1}^{(\infty)}(\lambda)V(\lambda)(I - \Phi_{1,1}^{(\infty)}(\lambda)V(\lambda))^{-1}\Phi_{1,2}^{(\infty)}(\lambda). \end{aligned} \quad (\lambda \in \mathbb{D})$$

We know from Part 4 that

$$\Phi_{1,2}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \Phi_{1,2}^{(\infty)}(\,\cdot\,) \quad \text{and} \quad \Phi_{2,2}^{(N)}(\,\cdot\,) \overset{\textbf{SOT/UC}}{\longrightarrow} \Phi_{2,2}^{(\infty)}(\,\cdot\,),$$

and from the definition of V we obtain

$$\Phi_{1,1}^{(N)}(\,\cdot\,)W(\,\cdot\,) \overset{\mathbf{SOT/UC}}{\longrightarrow} \Phi_{1,1}^{(\infty)}(\,\cdot\,)V(\,\cdot\,) \quad \text{and} \quad \Phi_{2,1}^{(N)}(\,\cdot\,)W(\,\cdot\,) \overset{\mathbf{SOT/UC}}{\longrightarrow} \Phi_{2,1}^{(\infty)}(\,\cdot\,)V(\,\cdot\,).$$

So in order to prove that $H_W^{(N)}(\,\cdot\,) \overset{\mathbf{SOT/UC}}{\longrightarrow} H_V^{(\infty)}(\,\cdot\,)$ it suffices to show that

$$((I - \Phi_{1,1}^{(N)}(\,\cdot\,)W(\,\cdot\,))^{-1}) \overset{\mathbf{SOT/UC}}{\longrightarrow} (I - \Phi_{1,1}^{(\infty)}(\,\cdot\,)V(\,\cdot\,))^{-1}.$$

The latter limit holds because the functions $\Phi_{1,1}^{(\infty)}(\,\cdot\,)V(\,\cdot\,)$ and $\Phi_{1,1}^{(N)}(\,\cdot\,)W(\,\cdot\,)$, with N a positive integer, are Schur class functions whose value at zero is the zero operator. In particular, they are all of the form $\lambda \mapsto \lambda Z(\lambda)$, where Z is also a Schur class function; see Lemma 2.4.1. Thus for each $\delta \in (0,1)$ we have

$$r_{\text{spec}}(\Phi_{1,1}^{(N)}(\lambda)W(\lambda)) \le \|\Phi_{1,1}^{(N)}(\lambda)W(\lambda)\| < \delta \quad (|\lambda| < \delta, N \in \mathbb{Z}, N > 0).$$

Our claim then follows directly from Lemma 6.4.2. □

From Theorem 6.4.1 it follows that the part of the parameter $W \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ that maps into $\mathcal{U}$, that is, the function $\lambda \mapsto \Pi_{\mathcal{U}} W(\lambda)$, does not play a role in the limit of the corresponding N-complementary sequence when we let the Nehari index N goes to infinity.

Notes for Chapter 6

The relaxed Nehari problem was introduced in [62], where it served as an example to illustrate the Redheffer representation of the solutions to the relaxed commutant lifting problem obtained there; see Theorem 5.2.1 above. Notice that its formulation is similar to that of the relaxed versions of the Nevanlinna-Pick, Schur and Sarason interpolation problems introduced in [42]. The linear fractional representations in Section 6.2 are new, and, in particular, different from the one obtained in [62]. In Section 6.3 standard material is presented concerning the classical Nehari problem; see the references mentioned there and in the introduction. The convergence result of Theorem 6.4.1 is new. For a slightly different, but related, approximation method for the scalar case ($\mathcal{U} = \mathcal{Y} = \mathbb{C}$) assuming that the Hankel operator has norm one we refer to [86] and [61].

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List of symbols

$\mathbb{Z}$	the integers $\dots, -2, -1, 0, 1, 2, \dots$
$\mathbb{N}$	the nonnegative numbers $0, 1, 2, 3, \dots$
$\mathbb{C}$	the complex numbers
$\mathbb{D}$	the open unit disc in $\mathbb{C}$
$\mathbb{T}$	the unit circle in $\mathbb{C}$
A^*	the adjoint of the operator A
A^{-1}	the inverse of the invertible operator A
A^{-*}	the adjoint of the inverse of the invertible operator A
$A^{\frac{1}{2}}$	the positive square root of the positive operator A
$A\|_{\mathcal{M}}$	the restriction of A to the subspace $\mathcal{M}$
$A \geq 0$	the operator A is nonnegative
$A \gg 0$	the operator A is nonnegative and invertible
$r_{\text{spec}}(A)$	the spectral radius of the operator A
D_A	the defect operator of the contraction A
$\mathcal{D}_A$	the defect space of the contraction A
$\\|A\\|$	the norm of the operator A
$I_{\mathcal{U}}$	the identity operator on $\mathcal{U}$, also denoted by I
$\Pi_{\mathcal{M}}$	the orthogonal projection onto the subspace $\mathcal{M}$
$P_{\mathcal{M}}$	the orthogonal projection in a Hilbert space onto the subspace $\mathcal{M}$, viewed as an operator on the larger space
$\mathscr{L}(\mathcal{U}, \mathcal{Y})$	the set of all operator from $\mathcal{U}$ into $\mathcal{Y}$
$\mathscr{L}(\mathcal{U})$	shorthand notation for $\mathscr{L}(\mathcal{U},\mathcal{U})$
$\mathcal{U} \oplus \mathcal{Y}$	the Hilbert space direct sum of $\mathcal{U}$ and $\mathcal{Y}$
$\mathcal{U} \ominus \mathcal{M}$	the orthogonal complement of the subspace $\mathcal{M}$ in $\mathcal{U}$
$\mathcal{U}^n$	the Hilbert space direct sum of n copies of $\mathcal{U}$

ℓ^2	the set of all square summable bilateral sequences with values in $\mathbb{C}$
ℓ^2_+	the set of all square summable unilateral sequences with values in $\mathbb{C}$
ℓ^2_-	the subspace $\ell^2 \ominus \ell^2_+$
$\ell^2(\mathcal{U})$	the set of all square summable bilateral sequences with values in $\mathcal{U}$
$\ell^2_+(\mathcal{U})$	the set of all square summable unilateral sequences with values in $\mathcal{U}$
$\ell^2_-(\mathcal{U})$	the subspace $\ell^2(\mathcal{U}) \ominus \ell^2_+(\mathcal{U})$
$L^\infty(\mathbb{T})$	the set of Lebesgue measurable functions uniformly bounded a.e. on $\mathbb{T}$
$H^2(\mathcal{U})$	the Hardy space of analytic functions on $\mathbb{D}$ with values in $\mathcal{U}$ and square summable Taylor coefficients
$\mathbf{H}^2(\mathcal{U},\mathcal{Y})$	the Banach space of $\mathscr{L}(\mathcal{U},\mathcal{Y})$-valued functions H on $\mathbb{D}$ such that for each $u \in \mathcal{U}$ the function $\lambda \mapsto H(\lambda)u$ is in $H^2(\mathcal{Y})$
$\mathbf{H}^2_{\mathrm{ball}}(\mathcal{U},\mathcal{Y})$	the unit ball of the Banach space $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$
$\mathbf{H}^\infty(\mathcal{U},\mathcal{Y})$	the space of uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathscr{L}(\mathcal{U},\mathcal{Y})$
$\mathbf{S}(\mathcal{U},\mathcal{Y})$	the set of contractive analytic functions on $\mathbb{D}$ with values in $\mathscr{L}(\mathcal{U},\mathcal{Y})$
$\mathscr{F}_\mathcal{U}$	the Fourier transform from $\ell^2_+(\mathcal{U})$ onto $H^2(\mathcal{U})$
$\tilde{S}_\mathcal{U}$	the unilateral forward shift on $\ell^2_+(\mathcal{U})$
$S_\mathcal{U}$	the unilateral forward shift on $H^2(\mathcal{U})$
$E_\mathcal{U}$	the canonical embedding of $\mathcal{U}$ into $H^2(\mathcal{U})$, $E_\mathcal{U} u \equiv u$
T_F	the Toeplitz operator defined by the function F
M_F	the multiplication operator defined by the function F
$F^\sharp$	the dual function of the function F
$\mathrm{Re}\, F$	the real part of the function F

Acknowledgements

It is my pleasure to thank the following people who contributed to this thesis in one way or another.

I am deeply grateful to prof.dr. M.A. Kaashoek and prof.dr. A.C.M. Ran for their professional guidance and stimulating advice. They where always ready to answer my questions and provide me with new ideas. Prof.dr. A.E. Frazho was involved in my research from the beginning. I am truly indebted to him for his vast knowledge of the subject and his perseverance in explaining it to me. I thank the members of the reading committee for their careful reading of this thesis and their useful comments.

During the lunches at the Vrije Universiteit I had many interesting discussions with the members of the operator theory group. Thank you for that. I thank my former roommates Frank van der Meulen, Pawel Zareba and Sara Lombardo for their companionship. Guit-Jan Ridderbos was a great help in getting the layout of this thesis right. Thanks.

The Vrije Universiteit provided me with the financial support to visit a large number of conferences and meeting. The Promovendifonds of the Vrije Universiteit made it possible for me to visit the operator theory group at Virginia Tech for three months. I truly appreciate this.

I want to thank the members of the operator theory group at Virginia Tech, in particular, prof.dr. J.A. Ball and his Ph.D. student Quanlei Fang, for the pleasant time I had there.

Further, I am grateful to the staff of the department of mathematics at the Radboud Universiteit Nijmegen, where I received my M.Sc. degree. They introduced me to mathematics and showed me how beautiful it can be. In particular, I want to thank prof.dr. A.C.M. van Rooij, dr. R.A. Kortram and dr. W. Veldman.

My family and friends where always there when I needed support. Thank you. I thank my girlfriend and partner Gonneke for her interest in me and my work, and for reminding me that there are other things in this world than mathematics. Our daughter Lian brought new inspiration to my life.

Samenvatting

Een “verlichte” versie van de methode van het optillen van de commutant met toepassingen voor het interpolatieprobleem van Nehari

De methode van het optillen van de commutant (in het Engels: de commutant lifting method en daarom afgekort als de CL-methode) is een algemene methode voor het construeren van oplossingen voor metrisch begrensde interpolatieproblemen zoals die van Schur, Nevanlinna-Pick en Nehari. Een “verlichte” versie van deze methode, die in het Engels bekend is onder de naam relaxed commutant lifting method en die we daarom de RCL-methode noemen, werd in het begin van deze eeuw geïntroduceerd. Behalve voor de eerder genoemde interpolatieproblemen kan de RCL-methode ook gebruikt worden om oplossingen voor getrunkeerde versies van deze interpolatieproblemen te construeren. Een dataset voor de RCL-methode bestaat uit vijf operatoren. Wanneer zo een dataset gegeven is, dan beschrijft de RCL-methode zogenoemde contractieve interpolanten. Op het niveau van de interpolatieproblemen komen deze contractieve interpolanten overeen met de oplossingen (de interpolanten). In het artikel waar de RCL-methode geïntroduceerd werd, werd een specifieke contractieve interpolant geconstrueerd, die bekend staat als de centrale contractieve interpolant. De vraag naar een beschrijving van alle contractieve interpolanten bleef open. Deze vraag was het beginpunt van het onderzoek dat heeft geleid tot deze dissertatie.

Het beschrijven van alle contractieve interpolanten voor de RCL-methode blijkt overeen te komen met het beschrijven van alle oplossingen van een abstract interpolatieprobleem. De oplossingen voor dit interpolatieprobleem zijn analytische operatorwaardige functies op de eenheidsschijf $\mathbb{D}$ van het complexe vlak $\mathbb{C}$. Om precies te zijn, veronderstel dat twee Hilbertruimten $\mathcal{U}$ en $\mathcal{Y}$ gegeven zijn. We schrijven $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ voor de verzameling van analytische functies H op $\mathbb{D}$ waarvan de waarden operatoren van $\mathcal{U}$ naar $\mathcal{Y}$ zijn, zo dat de formule

$$(\Gamma u)(\lambda) = H(\lambda)u \quad (u \in \mathcal{U}, \lambda \in \mathbb{D}) \tag{1}$$

een operator van $\mathcal{U}$ naar de Hardyruimte $H^2(\mathcal{Y})$ definieert. De verzameling $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ is een Banachruimte onder de norm $\|H\| := \|\Gamma\|$, waar $\|\Gamma\|$ de norm van de operator Γ in (1) is. De eenheidsbol in de Banachruimte $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ wordt aangeduid met $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$. Het abstracte interpolatieprobleem kan nu als volgt worden geformuleerd: Gegeven twee Hilbertruimten $\mathcal{U}$ en $\mathcal{Y}$, een deelruimte $\mathcal{F}$ van $\mathcal{U}$ en een contractie

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \tag{2}$$

beschrijf alle functies H in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ zo dat

$$\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_{\mathcal{F}} \quad (\lambda \in \mathbb{D}). \tag{3}$$

Het is nuttig om eerst de functies in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ te beschrijven. In deze dissertatie geven we een beschrijving van deze functies uitgedrukt in de functies van de Schurklasse $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$. Een functie Z op $\mathbb{D}$ is een Schurklasse-functie waneer Z analytisch is op $\mathbb{D}$ en de waarden van Z contractieve operatoren zijn. Het symbool $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ staat voor de verzameling van alle Schurklasse-functies waarvan de waarden operatoren van $\mathcal{U}$ naar $\mathcal{Y}\oplus\mathcal{U}$ zijn. De beschrijving van de functies in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ is als volgt: Voor elke functie H in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ bestaat er een functie Z in de Schurklasse $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ zo dat

$$H(\lambda) = \Pi_{\mathcal{U}} Z(\lambda)(I - \lambda\Pi_{\mathcal{Y}} Z(\lambda))^{-1} \quad (\lambda \in \mathbb{D}). \tag{4}$$

Hier staan de symbolen $\Pi_{\mathcal{Y}}$ en $\Pi_{\mathcal{U}}$ voor de orthogonale projecties van de Hilbertruimte directe som $\mathcal{Y}\oplus\mathcal{U}$ op $\mathcal{Y}$, respectievelijk op $\mathcal{U}$. Omgekeerd is voor elke Z in de Schurklasse $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$, de functie H gegeven door (4) een element van $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$. Het is echter niet het geval dat er voor elke H in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ een unieke Z in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ bestaat zodat H door (4) gegeven wordt. Dit gebrek aan uniciteit is daarentegen wel precies te beschrijven.

De oplossingen voor het abstracte interpolatieprobleem kunnen nu als volgt worden verkregen: Neem een functie Z in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ zodat $Z(\lambda)|_{\mathcal{F}} = \omega$ voor elke $\lambda \in \mathbb{D}$, de functie H in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ gegeven door (4) voldoet dan aan (3) en op deze manier krijgen we alle functies H in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ die aan (3) voldoen. Het gebrek aan uniciteit bij de beschrijving van alle functies in $\mathbf{H}^2_{\rm ball}(\mathcal{U},\mathcal{Y})$ speelt ook hier een rol. Over het algemeen is het niet zo, dat er voor elke oplossing H een unieke Z in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ bestaat met $Z(\lambda)|_{\mathcal{F}} = \omega$ voor elke $\lambda \in \mathbb{D}$, zo dat de oplossing H gegeven wordt door (4). Echter, in sommige gevallen geeft de bovenstaande procedure wel een één-op-één beschrijving van alle oplossingen. Een van de voorbeelden waarbij we wel een één-op-één beschrijving krijgen is de CL-methode.

Via de beschrijving van alle oplossingen van het abstracte interpolatieprobleem krijgen we vervolgens een beschrijving van alle contractieve interpolanten voor de RCL-methode. Deze beschrijving heeft twee minpunten met betrekking tot zijn toepasbaarheid. Ten eerste kunnen we niet zomaar elke Schurklasse-functie Z in de Schurklasse $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ nemen. De functie Z moet voldoen aan de extra eis dat $Z(\lambda)| = \omega$ voor elke $\lambda \in \mathbb{D}$. Ten tweede wordt er bij de vertaling van de RCL-methode naar het bijbehorende abstracte interpolatieprobleem een onderliggende contractie ω van de vorm (2) afgeleid. Deze onderliggende contractie speelt een belangrijke rol in de beschrijving van alle contractieve interpolanten en is over het algemeen niet expliciet uit te drukken in de operatoren in de dataset voor de RCL-methode. Om het eerste minpunt te ondervangen merken we op dat de functies Z in $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ die voldoen aan $Z(\lambda) = \omega$ voor elke $\lambda \in \mathbb{D}$ beschreven kunnen worden via de formule

$$Z(\lambda) = \omega\Pi_{\mathcal{F}} + D_{\omega^*}V(\lambda)\Pi_{\mathcal{G}} \quad (\lambda \in \mathbb{D}), \tag{5}$$

waar V een functie van de Schurklasse $\mathbf{S}(\mathcal{G}, \mathcal{D}_{\omega^*})$ is zonder verdere beperkingen. Hier staat $\mathcal{G}$ voor het orthogonaal complement van $\mathcal{F}$ in $\mathcal{U}$ en staan D_{ω^*} en $\mathcal{D}_{\omega^*}$ voor de defectoperator en defectruimte van de contractie ω^*. Met enkele algebraïsche manipulaties is de formule voor H in (4), gegeven dat Z van de vorm (5) is, om te schrijven tot een gebroken lineaire vorm van het Redheffer-type:

$$H(\lambda) = \Phi_{2,2}(\lambda) + \Phi_{2,1}(\lambda)V(\lambda)(I - \Phi_{1,1}(\lambda)V(\lambda))^{-1}\Phi_{1,2}(\lambda) \quad (\lambda \in \mathbb{D}),$$

waar de Redheffercoefficïenten $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ en $\Phi_{2,2}$ operatorwaardige analytische functies op $\mathbb{D}$ zijn die uitgedrukt kunnen worden in de onderliggende contractie ω. Door extra eisen te stellen aan de dataset voor de RCL-methode is het mogelijk om een gebroken lineaire vorm van het Redheffer-type voor de oplossingen te krijgen waarbij de Redheffercoefficienten expliciet in de operatoren in dataset uit te drukken zijn. Naast de gebroken lineaire vorm van het Redheffer-type leiden we een beschrijving van alle oplossingen af in de vorm van een klassieke gebroken lineaire vorm.

Aan het eind van de dissertatie passen we de ontwikkelde theorie toe op het Nehari-interpolatieprobleem. We formuleren een getrunkeerde versie van het Nehari-interpolatieprobleem en beschrijven alle oplossingen voor dit probleem. Het Nehari-interpolatieprobleem kan gezien worden als een limiet geval van de getrunkeerd versie. We maken dit in het laatste deel van de dissertatie precies door te laten zien dat er ook in een wel bepaalde zin convergentie van de oplossingen is.