Power homogeneity in Topology

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Power homogeneity in Topology

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door

Guit-Jan Frederik Ridderbos

geboren te Haarlem

promotor: prof.dr. J. van Mill

In loving memory of my grandfathers.

Preface

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Guit-Jan
Amstelveen, April 18, 2007

Contents

Guide to the reader

The first section of Chapter 1 is intended for a general audience. Considering the fact that my family and friends will probably like to know what I have been doing during the last few years, I have tried to give a simple introduction to power homogeneous spaces in this section. If you have no trouble reading the next sections, then you are probably a mathematician with some background in topology. For those people who decide for themselves that they belong to the class of family and friends, a Dutch summary is provided at the very end of this thesis.

The two main chapters are Chapters 3 and 4. Chapter 2 is largely introductory in nature and the reader can skip this chapter and refer back to it as needed. Chapter 5 contains examples and counterexamples of topological spaces with additional properties. Some open problems related to the topics discussed in this thesis are listed in a special section after Chapter 5. Our standard references for all undefined notions are ENGELKING [22] and KUNEN [39]. *In this thesis we only consider spaces that are at least Hausdorff.*

The reader can find indices at the end of this book. The index of special symbols contains short descriptions of the symbols listed. In the subject index, an italicized page number refers to the page on which a term is defined. A special section containing notes to the text is also included. This section contains detailed references and additional remarks.

Chapter 1

Introduction

1.1 Homogeneity in Topology

This thesis is all about power homogeneous spaces. In this section we introduce this class of spaces and give some simple examples. In the following sections we give an outline of the two main chapters of this thesis.

So what is a power homogeneous space? We explain this in three steps; first we explain what a '*space*' is, next we look at '*homogeneous* spaces' and finally we shall arrive at '*power* homogeneous spaces'.

When we say *space* we really mean *topological space*. A topological space can be almost anything you like, in its most general form it is some mathematical object endowed with a certain structure. The mathematical object is called the *space* and the structure is called the *topology* of the space. Members of our spaces will be called points and the topology determines some relationship between points. Examples of topological spaces are given in Figure 1.1 on page 2. Some of these spaces can also be viewed as geometric objects and this is no coincidence, since Geometry and Topology are two closely related subjects. In general one can obtain topological spaces from geometric objects by throwing away some but not all parts of the geometric structure. To get an idea of what a topology really is, one can think of it in terms of *closeness*. In particular, the closer points are to each other, the stronger is their relationship determined by the topology.

Viewed in this way, a topology determines properties that points can have in a space. These properties are referred to as topological properties. Points in spaces can share the same properties, but more often different points have different topological properties. In the special case where all points share exactly the same topological properties, we say that a space is

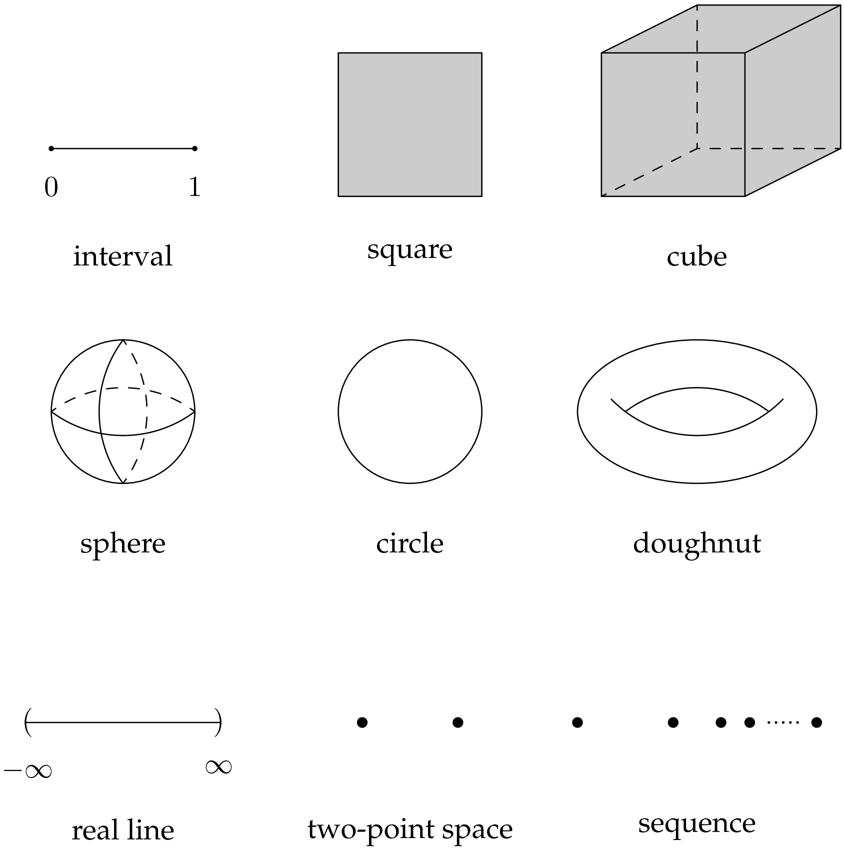

Figure 1.1: Some topological spaces

homogeneous. One can also say that a space is homogeneous if it looks the same everywhere from a topological point of view. Homogeneous spaces are very nice for lazy topologists, because if one knows the topological behaviour of only one point in a homogeneous space, then one also knows the behaviour of all the other points in that space! Of course, we can say a lot more about homogeneous spaces than we can about ordinary spaces and this is what really keeps us busy. Let's have a look at Figure 1.1 again. The two-point space is homogeneous but this is a very simple space. Another example of a homogeneous space is the real line ranging from minus infinity to plus infinity. One might think that the real line is not homogeneous, because for example 0 is exactly in the middle but the point 100 is not. However 'being in the middle of the real line' is not a topological property. To see an example of a typical property which is shared by every point of the real line, consider what happens if you remove a point. If you do this, then the line falls apart into two separate pieces. We can express this fact by saying that every point has the following property: 'If you throw me away, then the real line falls apart into two separate pieces'. This provides only one topological property that is shared by all points of the real line. Of course, to verify that the real line is homogeneous, one has to check for *all* topological properties. Fortunately there are easier ways to prove that a space is homogeneous.

One can also find spaces in Figure 1.1 that are not homogeneous. An example is the unit interval. To see that this space is not homogeneous, consider again what happens if you remove a point. Suppose that the point 0 is removed. In this case we are left with only one part, namely an interval from 0 to 1 (not including 0). However, if the point $1/2$ is removed, then the interval falls apart into two pieces. So the topological behaviour of the point 0 differs from that of $1/2$ in the unit interval and therefore this space is not homogeneous. Of the nine spaces depicted in Figure 1.1 only the following are homogeneous: sphere, circle, doughnut, real line and the two-point space.

We now come to power homogeneous spaces. In general, homogeneous spaces are very nice, but unfortunately not every space is homogeneous. Power homogeneous spaces generalize homogeneous spaces and they are still nice enough to be interesting for topologists. So what is a power homogeneous space? Take a look at the unit interval, the square and the cube in Figure 1.1. In the square, one can find two natural copies of the interval, namely a horizontal and a vertical one. We say that the square is the *product* of these two copies of the interval and we write (interval $\times$ in-

is homogeneous: 6, +1

terval) = square. Similarly (interval × interval × interval) = cube. So given a space X we can form many new spaces; $X \times X$, $X \times X \times X$, etc. This notation is not very appealing and we prefer to use the notation X^2 and X^3 for these spaces. Topologists have borrowed this from Number Theory, where people are used to writing 5^3 instead of $5 \times 5 \times 5$. So given a space X and a number κ, we make a new space X^κ which corresponds to taking the product of κ-many copies of X. The number κ can also be infinite which corresponds to taking an infinite product. The space X^κ is called a power space of X. Even if a space X is not homogeneous, it can happen that some power space of it is homogeneous. If this is the case then X is called power homogeneous. More formally; a topological space X is called *power homogeneous* if some power space of X is homogeneous.

Note that X^1 is just the space X, so every homogeneous space is also power homogeneous. Of course, not every power homogeneous space is homogeneous and an example is provided by the convergent sequence in Figure 1.1. This space is not homogeneous because the limit point on the right is different from all the other points. However if we take the infinite product of the convergent sequence, we obtain a compact zero-dimensional metrizable space without isolated points. By a well-known result of BROUWER [16] this infinite product is homogeneous because it is homeomorphic to the Cantor set. So the convergent sequence is power homogeneous but not homogeneous. The interval, the square and the cube are also power homogeneous but not homogeneous. In fact, their infinite products are all homeomorphic to the Hilbert cube which is homogeneous by a result of KELLER [38]. So we have seen that all spaces in Figure 1.1 are power homogeneous. In Chapter 5 the reader can also find simple examples of spaces that are not power homogeneous.

1.2 History and Background

An important question in the study of homogeneity in topology dates back to 1955 and it concerns the Čech-Stone compactification of ω which is denoted by $\beta\omega$. This compact space contains the countably infinite discrete space ω as a dense subspace. We denote $\beta\omega \setminus \omega$ by ω^* and this is called the *remainder* of the compactification. Clearly, $\beta\omega$ is not homogeneous; simply note that no point of ω^* can be an isolated point of $\beta\omega$ whereas every point of ω is isolated in $\beta\omega$. This settles the question whether $\beta\omega$ is homogeneous. A much more difficult problem concerns the homogeneity of ω^*. This problem was raised in 1955 and has received considerable attention from many topologists. We now know that ω^* is not homogeneous and there are many ways to prove this. The first partial answer was provided by RUDIN [62] in 1956 who showed that assuming the Continuum Hypothesis some but not all elements of ω^* are P-points. The first full answer was provided by FROLÍK [26] in 1967. Frolík showed that ω^* is not homogeneous by showing that certain sequences of subsets do not cluster in the same way at all points of ω^*. The argument of Frolík is essentially a cardinality argument and it does not produce a simple topological property which is possessed by some but not all points of ω^*. However, some years later such a property was provided by KUNEN [40] in 1980 who showed that some but not all points of ω^* are weak P-points.

Instead of asking whether $\beta\omega$ and ω^* are homogeneous, we can also ask whether these spaces are *power homogeneous*. These questions were considered and answered by VAN DOUWEN [19] in 1978. Van Douwen realized that the ideas of Frolík may also be applied to power homogeneous spaces and this led to the first restrictions on the size of such spaces. Using these restrictions, Van Douwen proved that neither $\beta\omega$ nor ω^* is power homogeneous. The results of Van Douwen are surprising, but to explain *why* his results are surprising, we must first get acquainted with the general structure of arguments in the field of cardinal functions.

A cardinal function assigns to every topological space a cardinal number in such a way that the same cardinal number is assigned to homeomorphic spaces. A cardinal function ϕ on a space X is often defined in terms of collections of open subsets of X. The value of ϕ may then be determined by the minimum size of such a collection. For example, the *weight* of X is defined to be the minimum size of a basis for X. In this setting, a cardinal restriction provides us with some upper bound on the size of X in terms of the sizes of certain collections of open subsets of X. There are many cardinal restrictions of this type in the literature and the arguments of almost all of these results consist of separating points of X by suitable families of open subsets. The idea is to choose a family of open subsets of X in such a way that we can retrieve the space X from it. We illustrate this by considering a family $\mathcal{B}$ which forms a basis for the topology on X. Assuming enough separation axioms, all points of X are separated by members of $\mathcal{B}$. Since we only consider Hausdorff spaces, if $x \neq y$ then there are disjoint sets B_x and B_y in $\mathcal{B}$ such that B_x is a neighbourhood of x and B_y is a neighbourhood of y. So if we let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$, then we have just shown that if $x \neq y$, then $\mathcal{B}_x \neq \mathcal{B}_y$. In other words, the map which sends x to $\mathcal{B}_x$ is an injection from X into $\mathcal{P}(\mathcal{B})$. If $\mathcal{B}$ is a countable family, then $\mathcal{P}(\mathcal{B})$ has the cardinality of the continuum, so if X has a countable basis (i.e., X has countable weight) then the size of X is at most $\mathfrak{c}$.

We have just proved a very simple upper bound on the size of X. By considering different families that separate points, we can obtain stronger results. In addition, the more we know about a space, the better we can choose our separating families. We can roughly say that nice spaces satisfy stronger bounds than spaces which are not so nice. For example, the size of separable *metrizable* spaces is always bounded by $\mathfrak{c}$, since such spaces have a countable basis. Without the assumption of metrizability this may fail; there are examples of separable spaces whose size exceeds $\mathfrak{c}$. The same phenomenon occurs for homogeneous spaces; results that are valid for arbitrary spaces, can often be improved for homogeneous spaces. The key fact here, is that using homogeneity, one can choose separating families in some uniform way and these are often of smaller size.

Now let us return to Eric Van Douwen. In [19] he proved an upper bound on the size of power homogeneous spaces. We will refer to this result as Van Douwen's Theorem. This theorem has a very short proof for homogeneous spaces. In this case, points of X are separated in some clever way using families of open subsets. Of course, Van Douwen did more because he proved his theorem for power homogeneous spaces. So what is so special about this? Well, suppose that X is power homogeneous. Then there is some cardinal number κ such that X^κ is homogeneous. So in X^κ we can construct good separating families of open subsets of X^κ. However, if we want to prove something about the size of X, then we have to be able to construct separating families using only the open subsets of X. The problem here is that we do not know the size of κ for which X^κ is homogeneous and therefore the space X^κ can have considerably more open subsets than the space X. Van Douwen found some clever way to use the homogeneity of some (possibly very large) power of X to construct separating families in some smaller power of X. The strength of the argument lies in the fact that this smaller power of X is small enough to prove useful bounds on the size of X.

In Chapter 3 of this thesis we prove more cardinality bounds for the size of power homogeneous spaces. All of these bounds are known in the case of homogeneous spaces. Our arguments are similar in spirit to the proof of Van Douwen's Theorem because we have some way to get separating families in small powers of spaces given that some large power is homogeneous. The key result in this context is proved in the first section of Chapter 3. This result has proven itself to be very useful in the study of power homogeneous spaces and it seems to be more powerful than the clustering technique used by Van Douwen. However, it should be pointed out that our approach resembles Van Douwen's techniques. The resemblance lies in the fact that both techniques make good use of the simple fact that basic open subsets of power spaces only depend on finitely many coordinates.

I think that one of the most intuitively appealing results of Chapter 3 is a reflection theorem for power homogeneous spaces. Recall the problem with power homogeneous spaces; if X is power homogeneous, then the least power κ for which X^κ is homogeneous might be very large. It would be a very strong result if we can provide a relatively small upper bound on the size of this power in some way. The reflection theorem almost does this; it provides a small power in which X possesses enough homogeneity properties to construct good families of separating collections. In particular, this leads to a quick and transparent proof of Van Douwen's Theorem.

1.3 Honest proofs

Although it is not our main interest, the results of Chapter 3 can be used to prove that certain spaces are not power homogeneous. For example, if a space does not satisfy the bound provided by Van Douwen's Theorem, then it is not power homogeneous. Such an argument is based on cardinality restrictions, and this is in general not a very powerful way of proving non-power homogeneity. In particular, based on the results in Chapter 3 it is impossible to determine whether separable metric spaces are power homogeneous or not. To prove non-power homogeneity of such spaces, we need topological properties that behave well with respect to products. We study such properties in Chapter 4.

We have already come across two different arguments to show that ω^* is not homogeneous. The first argument is really a cardinality argument and the second argument is based on the existence and non-existence of weak P-points. The second argument provides a topological property which is shared by some but not all points of ω^*. Van Douwen called arguments of the second kind 'honest' proofs of non-homogeneity. In Chapter 4 we will provide tools to give honest proofs of non-power homogeneity; i.e., we will provide topological properties that are shared by either all or no points in power homogeneous spaces. Suppose that we have some topological property which is either true or false in points of a topological space. If a space is homogeneous, then all points are topologically equivalent, and therefore it is clear that if this property is true in some point then it is true in all points of the space. Such reasoning can fail in power homogeneous spaces, because not all points in power homogeneous spaces are necessarily topologically equivalent. However, we will provide properties that *can* be used in honest proofs because they behave well with respect to taking products. The idea is to start with some property $\mathcal{P}$ and a power homogeneous space X. If $\mathcal{P}$ is preserved by taking products and projections, then power homogeneity of X implies that if some point in X satisfies $\mathcal{P}$, then all points in X satisfy $\mathcal{P}$. This will give new proofs of non-power homogeneity since it now suffices to determine whether the behaviour with respect to $\mathcal{P}$ is the same in all points of a space X. If this is not the case then X is not power homogeneous.

Arguments of this kind are not new. In fact, VAN DOUWEN [19] already provided such a property to prove that the one-point compactification of a discrete space of uncountable size is not power homogeneous. In recent years, ARHANGEL'SKIĬ [11] has also used such arguments. One of his results states that if X is compact and power homogeneous and the set of points at which X is first-countable is dense in X, then X is first-countable at all points. In Chapter 3 we will prove similar statements for π-character and tightness.

The study of 'nice' topological properties in spaces also extends to different classes than just the class of power homogeneous spaces. These classes are considered in Chapter 4. In particular we study properties of topological groups and coset spaces. In addition to being homogeneous, such spaces can be endowed with an algebraic structure which interacts well with the topology. As a consequence, this often means that the group of homeomorphisms of such spaces is very rich. This leads to statements about the existence of many homeomorphisms on topological groups and coset spaces. If such homeomorphisms or even continuous mappings are lacking on a space, then one concludes that such a space cannot be endowed with certain algebraic structures.

Chapter 2

Preliminaries

2.1 Set theory

Whenever X is a set, we use $|X|$ to denote its cardinality. Every ordinal number is the set of all of its predecessors and cardinal numbers are initial ordinal numbers. We use Greek letters to denote ordinal numbers. Usually α, β and γ denote ordinals and κ, λ and μ denote cardinals. Whenever κ is a cardinal number, then by κ^+ we denote its successor cardinal, i.e., κ^+ is the first cardinal number which is strictly larger than κ. We use ω to denote the first infinite ordinal number, it is the set of natural numbers, including 0. If we want to emphasize that we are dealing with cardinal numbers, we use $\aleph_0$ to denote the cardinality of ω. The successor of $\aleph_0$ is denoted by $\aleph_1$ and $\mathfrak{c}$ is the size of the continuum, $2^{\aleph_0}$.

The Continuum Hypothesis, abbreviated CH, is the statement '$\aleph_1 = \mathfrak{c}$'. The Generalized Continuum Hypothesis (abbreviated GCH) is the assertion that $\kappa^+ = 2^\kappa$ for all infinite cardinals κ.

If X is any set and κ a cardinal number, then by $[X]^\kappa$ we denote the set of all subsets of X of cardinality κ. The collections $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined similarly. Whenever $f : [X]^\kappa \to Y$ is a function, we call a set $Z \subseteq X$ homogeneous for f if for every $A, B \in [Z]^\kappa$ we have $f(A) = f(B)$. If κ, η, μ and λ are cardinal numbers then we say that the partition relation

$$\kappa \to (\eta)^\mu_\lambda$$

holds if for every function $f : [X]^\mu \to \lambda$ such that $|X| \geq \kappa$, there is a homogeneous set Z for f with $|Z| \geq \eta$. The following theorem is known as the Erdös-Rado Theorem, a proof can be found in MARKER [42].

2.1.1. ERDÖS-RADO THEOREM. *If κ is an infinite cardinal number, then the following partition relation holds:*

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa,$$

i.e., whenever $|X| > 2^\kappa$ and $f : [X]^2 \to \kappa$ is a function, then there is a homogeneous set for f of cardinality $\geq \kappa^+$.

2.2 Basic topology

By $\mathbb{R}$ we denote the set of real numbers, $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of natural numbers (not including 0). If $n \in \omega$, then by $\mathbb{S}^n$ we denote the n-dimensional sphere given by $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. The set B^n is the n-dimensional ball which is given by $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$. So $\mathbb{S}^n$ is the boundary of B^{n+1}. We use C to denote the Cantor set.

A Polish space is a separable completely metrizable space. A space X is called normal if disjoint closed subsets of X can be separated by disjoint open neighbourhoods. A space X is called hereditarily normal provided that every subspace of X is normal. We abbreviate 'hereditarily normal' with T_5.

The topology in product spaces is always the standard Tychonoff product topology. Whenever $\{X_i : i \in I\}$ is a collection of topological spaces and Y is the product space $\prod\{X_i : i \in I\}$, then for $A \subseteq I$ by Y_A we denote the product $\prod\{X_i : i \in A\}$. By π_A we denote the natural projection of Y onto Y_A. If $i \in I$, then we write π_i instead of $\pi_{\{i\}}$. If $y \in Y$ then by y_A we denote the point $\pi_A(y)$ and for $i \in I$, y_i is the point $\pi_i(y)$. By π we denote π_0, the projection onto the first coordinate. Finally, if $Z \subseteq Y$, then $Z_A = \pi_A[Z]$.

A topological space X is said to be homogeneous if for every $x, y \in X$, there is a homeomorphism h of X mapping x onto y. The space X is called power homogeneous if X^μ is homogeneous for some cardinal μ. If A is any set, then by $\Delta(X, A)$ we denote the diagonal in X^A which is given by

$$\Delta(X, A) = \{x \in X^A : \forall\, \alpha, \beta \in A\, (x_\alpha = x_\beta)\}.$$

By $\Delta(X)$ we denote the diagonal in X^2. The product space X^A is called Δ-homogeneous if for every $x, y \in \Delta(X, A)$ there is a homeomorphism h of X^A mapping x onto y. A space X is called Δ-power homogeneous if X^κ is Δ-homogeneous for some cardinal κ. If X^κ is homogeneous, then it is clearly Δ-homogeneous, so every power homogeneous space is Δ-power homogeneous. Conversely, every Δ-power homogeneous space is power homogeneous, we shall prove this in §3.2.

A subset G of X is called a G_δ-subset if it is the intersection of countably many open subsets of X. In general, if κ is a cardinal number, then G is called a G_κ-subset of X if it is the intersection of κ many open subsets of X. By $\mathcal{G}_\kappa(X)$ we denote the collection of all closed G_κ-subsets of a space X. We say that the G_κ-density at a point x in X does not exceed κ if there exists a closed G_κ-subset H of X and a set $S \in [X]^{\leq\kappa}$ such that $x \in H \subseteq \overline{S}$. We say that the G_κ-density of X does not exceed κ, if the G_κ-density does not exceed κ at all points x in X.

A collection $\mathcal{F}$ of subsets of a space X is called discrete provided that for every $x \in X$, there is an open neighbourhood U of x such that $|\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1$. A subset D of X is called a discrete subspace if it is a discrete space in the relative topology. It is not hard to verify that if D is a subset of X and $\mathcal{F} = \{\{d\} : d \in D\}$, then $\mathcal{F}$ is a discrete collection of closed subsets of X if and only if D is a closed and discrete subspace of X. If $\mathcal{F}$ is a collection of pairwise disjoint closed subsets of X, then we say that a collection $\mathcal{U}$ of open subsets of X separates $\mathcal{F}$ if $\mathcal{U}$ consists of pairwise disjoint open subsets of X and for every $F \in \mathcal{F}$ there is some $U(F) \in \mathcal{U}$, containing F, such that the map which sends F to $U(F)$ is an injection. We say that $\mathcal{U}$ separates a subset D of X if it separates the collection $\{\{d\} : d \in D\}$.

A space X is called weakly κ-collectionwise Hausdorff provided that every closed discrete subspace D of X of cardinality κ contains a subset D_0 of cardinality κ which is separated by a collection of open subsets of X. A space satisfies property $wD(\kappa)$ if every closed discrete subspace D of cardinality κ contains a subset D_0 of cardinality κ which can be separated by a discrete collection of open subsets of X.

2.3 Cardinal functions

In this section we introduce cardinal functions. For a good introduction to some elementary inequalities, we refer the reader to JUHÁSZ [34]. Whenever X is a space, then by $\tau(X)$ we denote its topology, i.e., $\tau(X)$ is the collection of all open subsets of X. By $\tau^*(X)$ we denote the collection $\tau(X) \setminus \{\emptyset\}$. We fix a topological space X. Whenever $\mathcal{B} \subseteq \tau(X)$ and $x \in X$, then by $\mathcal{B}_x$ we denote the collection $\{B \in \mathcal{B} : x \in B\}$.

2.3.1. DEFINITION. If F is a subset of X, then $\mathcal{B} \subseteq \tau(X)$ is called a local neighbourhood base at F in X if every member of $\mathcal{B}$ contains F and for every open neighbourhood U of F there is some $B \in \mathcal{B}$ such that $F \subseteq B \subseteq U$. If $x \in X$, then $\mathcal{B}$ is called a local neighbourhood base at x in X if it is a local neighbourhood base at $\{x\}$ in X. A collection $\mathcal{B} \subseteq \tau(X)$ is called a basis for X if for each $x \in X$, the collection $\mathcal{B}_x$ is a local neighbourhood base at x in X. We define the weight and character as follows (note that we write $\chi(x, X)$ instead of $\chi(\{x\}, X)$);

$$\begin{aligned} w(X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a basis for } X\} + \aleph_0, \\ \chi(F, X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local base at } F \text{ in } X\} + \aleph_0, \\ \chi(X) &= \sup\{\chi(x, X) : x \in X\}. \end{aligned}$$

A collection $\mathcal{N}$ of subsets of a space X is called a network in X if for every $x \in X$ and every open neighbourhood U of x, there is some $N \in \mathcal{N}$ such that $x \in N \subseteq U$. So a basis is a network consisting of open subsets. The network-weight of a space X is defined by

$$nw(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a network in } X\} + \aleph_0.$$

2.3.2. Definition. If F is a subset of X, then $\mathcal{B} \subseteq \tau(X)$ is called a local ψ-base at F in X if $F = \bigcap \mathcal{B}$. If $x \in X$, then $\mathcal{B} \subseteq \tau(X)$ is called a local ψ-base at x in X if $\{x\} = \bigcap \mathcal{B}$. A collection $\mathcal{B} \subseteq \tau(X)$ is called a ψ-base for X if for every $x \in X$, the set $\mathcal{B}_x$ is a local ψ-base at x in X. We define the ψ-weight and pseudo character as follows (note that we write $\psi(x, X)$ instead of $\psi(\{x\}, X)$);

$$\begin{aligned} \psi w(X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \psi\text{-base for } X\} + \aleph_0, \\ \psi(F, X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local } \psi\text{-base at } F \text{ in } X\} + \aleph_0, \\ \psi(X) &= \sup\{\psi(x, X) : x \in X\}. \end{aligned}$$

2.3.3. Definition. If $x \in X$, then $\mathcal{B} \subseteq \tau^*(X)$ is called a local π-base at x in X if for every open neighbourhood U of x, there is some $B \in \mathcal{B}$ such that $B \subseteq U$. A collection $\mathcal{B} \subseteq \tau^*(X)$ is called a π-base for X if $\mathcal{B}$ is a local π-base at x in X for all $x \in X$. We define the π-weight and π-character as follows;

$$\begin{aligned} \pi w(X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base for } X\} + \aleph_0, \\ \pi\chi(x, X) &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local } \pi\text{-base at } x \text{ in } X\} + \aleph_0, \\ \pi\chi(X) &= \sup\{\pi\chi(x, X) : x \in X\}. \end{aligned}$$

A family $\mathcal{B}$ of non-empty subsets of a space X is called a local π-network of X at a point x if every open neighbourhood of x contains some member of

$\mathcal{B}$. For a family $\mathcal{E}$ of non-empty subsets of X we define the $\pi_{\mathcal{E}}$-character of X at x by

$$\pi\chi_{\mathcal{E}}(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{E} \text{ is a local } \pi\text{-network of } X \text{ at } x\}.$$

This cardinal function is only defined if $\mathcal{E}$ contains some local π-network of X at x. Examples of possible families are $\tau(X)$ and $\mathcal{G}_\kappa(X)$. If $\mathcal{E} = \mathcal{G}_\kappa(X)$ we write $\pi\kappa\chi(x, X)$ instead of $\pi\chi_{\mathcal{E}}(x, X)$. By $\pi\kappa\chi(X)$ we denote the supremum of $\pi\kappa\chi(x, X)$ for all $x \in X$. Note that $\pi\kappa\chi(X) \leq \pi\chi(X)$.

2.3.4. DEFINITION. If $x \in X$, then the tightness at x in X, $t(x, X)$, is the least infinite cardinal number κ with the property that if $A \subseteq X$ and $x \in \overline{A}$, then there is some set $B \in [A]^{\leq\kappa}$ such that $x \in \overline{B}$. The tightness of X is defined by,

$$t(X) = \sup\{t(x, X) : x \in X\}.$$

2.3.5. DEFINITION. A collection $\mathcal{C} \subseteq \tau^*(X)$ is called a cellular family in X, if $\mathcal{C}$ consists of pairwise disjoint open subsets of X. The cellularity and density are defined as follows;

$$\begin{aligned} c(X) &= \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a cellular family in } X\}, \\ d(X) &= \min\{|D| : D \subseteq X \text{ and } \overline{D} = X\}. \end{aligned}$$

We say that X satisfies the countable chain condition (abbreviated c.c.c.) if $c(X) \leq \aleph_0$.

2.3.6. DEFINITION. The set of all autohomeomorphisms of a space X is denoted by $\mathcal{H}(X)$ and we let $\operatorname{tpe}(x, X) = \{h(x) : h \in \mathcal{H}(X)\}$ be the type of x in X. The homogeneity index of a space X is defined as the number of different types in X, so

$$\operatorname{hind}(X) = |\{\operatorname{tpe}(x, X) : x \in X\}|.$$

2.3.7. DEFINITION. The pointwise compactness type of a space X, $\operatorname{pct}(X)$, is defined as the least cardinal κ such that for every $x \in X$, there is a compact subset G of X containing x such that $\chi(G, X) \leq \kappa$. A space X is said to be of point-countable type if $\operatorname{pct}(X) \leq \aleph_0$. A space X is said to be of countable type if for every compact subset F of X there is a compact subset G of X containing F with $\chi(G, X) \leq \aleph_0$.

Some of the cardinal functions that we have encountered in this section may be considered to be local cardinal functions. We say that a cardinal function ϕ is a local cardinal function if $\phi(X)$ is defined as either the supremum or the infimum over the values of $\phi(x, X)$ for all $x \in X$, where $\phi(x, X)$ has the property that if $h : X \to X$ is a homeomorphism, then $\phi(x, X) = \phi(h(x), X)$. Examples of local cardinal functions are character, pseudo-character, π-character and tightness.

If ϕ is a local cardinal function and X is a topological space, then we say that X is homogeneous with respect to ϕ if for all $x, y \in X$, $\phi(x, X) = \phi(y, X)$. So if X is homogeneous with respect to a local cardinal function ϕ, then $\phi(X) = \phi(x, X)$ for all $x \in X$. Of course, every homogeneous space is homogeneous with respect to local cardinal functions. We will show in Chapter 3 that certain power homogeneous spaces are also homogeneous with respect to certain local cardinal functions.

The following non-trivial result is due to Šapirovskiĭ. We omit the proof, which may be found in JUHÁSZ [34, 2.37].

2.3.8. PROPOSITION. *If X is regular, then $d(X) \leq \pi\chi(X)^{c(X)}$.*

Some cardinal functions behave nicely with respect to products, this is summarized in the following results. More information on cardinal functions and products can be found in JUHÁSZ [34, Chapter 5].

2.3.9. THEOREM. *Suppose $\{X_i : i \in I\}$ is a collection of topological spaces and let Y be the product space. Let $\phi \in \{w, \pi w, \chi, \pi\chi, d\}$ and suppose that for all $i \in I$, $\phi(X_i) \leq \kappa$. If $|I| \leq \kappa$, then $\phi(Y) \leq \kappa$. Furthermore, if $|I| \leq 2^\kappa$, then $d(Y) \leq \kappa$.*

2.3.10. THEOREM. *Let $Y = \prod\{X_i : i \in I\}$ and suppose that $d(X_i) \leq \kappa$ for all $i \in I$. If U is an open subset of Y, then its closure depends on not more than κ many coordinates. This means that*

$$\overline{U} = \pi_A^{-1}[\pi_A[\overline{U}]]$$

for some $A \in [I]^{\leq\kappa}$.

2.4 Cardinal functions and compact spaces

In this section we study the behaviour of some cardinal functions on compact spaces. The tightness of a compact space X may be characterized as the supremum over all the lengths of free sequences in X. This was proved by ARHANGEL′SKIĬ [3]. We shall not prove this characterization here, but we shall need the following result which is one ingredient in the proof. Recall that a sequence $\{x_\alpha : \alpha < \kappa\}$ is called a free sequence if for all $\beta < \kappa$ the sets $\{x_\alpha : \alpha < \beta\}$ and $\{x_\alpha : \alpha \geq \beta\}$ have disjoint closures.

2.4.1. PROPOSITION. *If X is compact and $t(X) \leq \kappa$, then X contains no free sequences of length κ^+.*

PROOF. Suppose $\{x_\alpha : \alpha < \kappa^+\}$ is a sequence in X of length κ^+. For $\beta < \kappa$ we let $G_\beta = \mathrm{Cl}\,\{x_\alpha : \alpha < \beta\}$ and $F_\beta = \mathrm{Cl}\,\{x_\alpha : \alpha \geq \beta\}$. The collection $\{F_\beta : \beta < \kappa^+\}$ is a decreasing collection of non-empty closed subsets of the compact space X and therefore its intersection is non-empty. Let $y \in \bigcap_{\beta<\kappa^+} F_\beta$. Then y belongs to the closure of the set $\{x_\alpha : \alpha < \kappa^+\}$. Since the tightness of X does not exceed κ and κ^+ is regular, it follows that there is some $\beta < \kappa^+$ with $y \in G_\beta$. But then $y \in G_\beta \cap F_\beta$ and this means that the sequence $\{x_\alpha : \alpha < \kappa^+\}$ is not a free sequence. □

2.4.2. THEOREM. *If X is a compact space, then $\pi\chi(X) \leq t(X)$.*

PROOF. Let $t(X) = \kappa$ and suppose to the contrary that $\pi\chi(x, X) \geq \kappa^+$ for some $x \in X$. We will show how to construct collections $\{A_\alpha : \alpha < \kappa^+\}$ and $\{B_\alpha : \alpha < \kappa^+\}$ of open subsets of X satisfying;

(1) $\overline{A}_\alpha \cap \overline{B}_\alpha = \emptyset$,

(2) $x \in B_\alpha$,

(3) ξ_β has the finite intersection property, where

$$\xi_\beta = \{\overline{B}_\alpha : \alpha \leq \beta\} \cup \{\overline{A}_\alpha : \beta < \alpha\}.$$

Suppose we have constructed such collections. Then by compactness, we may pick $x_\beta \in \bigcap \xi_\beta$ for all $\beta < \kappa^+$. But then for all $\gamma < \kappa^+$ we have

$$\begin{aligned} \{x_\beta : \beta < \gamma\} &\subseteq \bigcap_{\alpha \geq \gamma} \overline{A}_\alpha \subseteq \overline{A}_\gamma \\ \{x_\beta : \beta \geq \gamma\} &\subseteq \bigcap_{\alpha \leq \gamma} \overline{B}_\alpha \subseteq \overline{B}_\gamma, \end{aligned}$$

and this means that $\{x_\alpha : \alpha < \kappa^+\}$ is a free sequence of length κ^+ in X which is impossible by Proposition 2.4.1.

We will now show how to carry out the construction. We may just pick A_0 and B_0 as required. Suppose $\beta < \kappa^+$ and for all $\alpha < \beta$ the sets A_α

and B_α have been constructed. Let $\mathcal{E}_\beta = \{A_\alpha, B_\alpha : \alpha < \beta\}$ and let $\mathcal{P}_\beta$ be the closure of $\mathcal{E}_\beta$ under finite intersections, excluding the empty set. Then $|\mathcal{P}_\alpha| \leq \kappa$, so this is not a local π-base at x in X. Therefore we may find an open neighbourhood U_β of x such that no member of $\mathcal{P}_\beta$ is contained in U_β. We now pick A_β and B_β such that $x \in B_\beta$ and $X \setminus U_\beta \subseteq A_\beta$ and $\overline{A}_\beta \cap \overline{B}_\beta = \emptyset$. This completes the construction. It remains to verify that ξ_β has the f.i.p.

We will actually prove that the collection $\eta_\beta = \{B_\alpha : \alpha \leq \beta\} \cup \{A_\alpha : \beta < \alpha\}$ has the f.i.p. Suppose that η_β does not have the f.i.p. for some $\beta < \kappa^+$ and let

$$B_{\alpha_1}, \ldots, B_{\alpha_N}, A_{\gamma_1}, \ldots, A_{\gamma_M}$$

be some sequence of minimal length in η_β with empty intersection. Since $x \in \bigcap_{n=1}^N B_{\alpha_n}$, it must be the case that $M \geq 1$. Since the sequence is of minimal length, $V = \bigcap_{n=1}^N B_{\alpha_n} \cap \bigcap_{n=1}^{M-1} A_{\gamma_n} \neq \emptyset$ and thus $V \in \mathcal{P}_{\gamma_M}$. But then V is not contained in U_{γ_M} and thus $V \cap A_{\gamma_M} \neq \emptyset$ by construction. This is impossible since this means that the chosen sequence does not have empty intersection. It follows that for all $\beta < \kappa^+$, η_β has the f.i.p. and therefore ξ_β has the f.i.p. This completes the proof. □

2.4.3. LEMMA. *Suppose F is a compact subset of Y and Y is a compact subset of X and $x \in F$, then $\pi\chi(x, X) \leq \pi\chi(x, F)\chi(F, X)$ and $\chi(F, Y) \leq \chi(F, Y)\chi(Y, X)$.*

PROOF. We will show that $\chi(F, X) \leq \chi(F, Y)\chi(Y, X)$, the other statement has a similar proof. Let $\kappa = \chi(F, Y)\chi(Y, X)$. We fix collections $\mathcal{U}$ and $\mathcal{V}$ of open subsets of X such that $|\mathcal{U}| \leq \kappa$ and $|\mathcal{V}| \leq \kappa$ and $\mathcal{V}$ is a local neighbourhood base at Y in X and $\{U \cap Y : U \in \mathcal{U}\}$ is a local neighbourhood base at F in Y. The collection $\mathcal{W}$ is given by

$$\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

We have $|\mathcal{W}| \leq \kappa$ and we will show that $\mathcal{W}$ is a local neighbourhood base at F in X. So let G be an arbitrary open neighbourhood of F in X. By normality of Y we may find some $U \in \mathcal{U}$ such that $\overline{U} \cap Y \subseteq Y \cap G$. Then $(X \setminus \overline{U}) \cup G$ is an open neighbourhood of Y in X, so we may also find some $V \in \mathcal{V}$ such that $V \subseteq (X \setminus \overline{U}) \cup G$. Then $U \cap V \in \mathcal{W}$ and $U \cap V \subseteq G$ by construction. This completes the proof. □

2.4.4. COROLLARY. *If X is any space, then $\pi\chi(X) \leq t(X)\mathrm{pct}(X)$.*

PROOF. Let $x \in X$ be arbitrary and suppose $\kappa = t(X)\mathrm{pct}(X)$. Then there is a compact set $F \subseteq X$ such that $x \in F$ and $\chi(F, X) \leq \kappa$. Since F is a closed subset of X we have $t(F) \leq t(X) \leq \kappa$ and therefore $\pi\chi(x, F) \leq \kappa$ by Theorem 2.4.2. It follows from Lemma 2.4.3 that $\pi\chi(x, X) \leq \kappa$. □

2.4.5. PROPOSITION. *Suppose X is a compact space and F is a closed subset of X. Then $\psi(F, X) = \chi(F, X)$.*

PROOF. We clearly have $\psi(F, X) \leq \chi(F, X)$, so we will show that $\chi(F, X) \leq \psi(F, X)$. Let $\psi(F, X) = \kappa$ and fix a ψ-base $\mathcal{U}$ at F in X with $|\mathcal{U}| \leq \kappa$. Since X is normal, we may assume without loss of generality that $F = \bigcap\{\overline{U} : U \in \mathcal{U}\}$.

Let V be an arbitrary open neighbourhood of F in X. The set $X \setminus V$ is compact and since $F = \bigcap\{\overline{U} : U \in \mathcal{U}\}$, the collection $\{X \setminus \overline{U} : U \in \mathcal{U}\}$ covers $X \setminus V$. By compactness, we may find a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{X \setminus \overline{U} : U \in \mathcal{V}\}$ covers $X \setminus V$. But then $\bigcap \mathcal{V}$ is an open neighbourhood of F which is contained in V. Since V was arbitrary, we have shown that the collection

$$\left\{\bigcap \mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{<\omega}\right\}$$

is a neighbourhood base at F in X. Since $|[\mathcal{U}]^{<\omega}| \leq \kappa$, this completes the proof. □

2.4.6. COROLLARY. *If X is any space then $\psi(X)\mathrm{pct}(X) = \chi(X)$ and if X is of point-countable type, then $\psi(x, X) = \chi(x, X)$ for all $x \in X$. If X is a space of countable type and F is a compact subset of X, then $\psi(F, X) = \chi(F, X)$.*

PROOF. This follows from Lemma 2.4.3 and Proposition 2.4.5. □

2.4.7. PROPOSITION. *Let X be a compact space and suppose that G is a closed G_κ-subset of X. If $t(x, X) \geq \kappa^+$ for some $x \in G$ then $t(G) \geq \kappa^+$.*

PROOF. Let $D \subseteq X$ be such that $x \in \overline{D}$ but $x \notin \overline{E}$ whenever $E \in [D]^{\leq\kappa}$. Let $F = \bigcup\{\overline{E} : E \in [D]^{\leq\kappa}\}$. Since $D \subseteq F$, we have $x \in \overline{F}$, however we also have that $x \notin \overline{E}$ whenever $E \in [F]^{\leq\kappa}$. Since $\chi(G, X) = \psi(G, X) \leq \kappa$ by Proposition 2.4.5, we may fix a neighbourhood base $\mathcal{U}$ at G in X such that $|\mathcal{U}| \leq \kappa$.

Let $F' = F \cap G$. Then $x \notin \overline{E}$ whenever $E \in [F']^{\leq\kappa}$. We will show that $x \in \overline{F'}$ and this shows that $t(G) \geq \kappa^+$. So let V be an arbitrary open neighbourhood of x in X and let W be an open neighbourhood of x such that $\overline{W} \subseteq V$. For every $U \in \mathcal{U}$, the set $U \cap W$ is an open neighbourhood of x, so as $x \in \overline{D}$, we may find $x(U) \in D$ such that $x(U) \in U \cap W$. Let $E = \{x(U) : U \in \mathcal{U}\}$. Then $|E| \leq \kappa$, and since $\mathcal{U}$ is a neighbourhood base at G in X it follows that $\overline{E} \cap G \neq \emptyset$. Let $e \in \overline{E} \cap G$. Since $\overline{E} \subseteq F$ we have $e \in F$ and since $\overline{E} \subseteq \overline{W} \subseteq V$, we have $e \in V$. So we have $e \in V \cap F \cap G = V \cap F'$. Since V was arbitrary, this shows that $x \in \overline{F'}$ and this completes the proof. □

2.4.8. LEMMA. *Let X be a normal space and let U be an open neighbourhood of the closed subset F of X. Then there is a closed G_δ-set G in X such that $F \subseteq G \subseteq U$.*

PROOF. Since X is normal, we may find a continuous function $f : X \to \mathbb{I}$ such that $F \subseteq f^{-1}(0)$ and $X \setminus U \subseteq f^{-1}(1)$. Then $f^{-1}(0)$ is a closed G_δ-set which contains F and is contained in U. □

2.4.9. LEMMA. *Every locally compact space is of countable type.*

PROOF. For an arbitrary compact subset F of X, let U be an open neighbourhood of F with compact closure. Since $\overline{U}$ is compact it is normal, so by the previous lemma there is a closed G_δ-set G in $\overline{U}$ such that $F \subseteq G \subseteq U$. Since the closure of U is compact, G is also compact and it follows from Proposition 2.4.5 that $\chi(G, \overline{U}) \leq \omega$. Since $G \subseteq U \subseteq \overline{U}$, it follows that $\chi(G, U) \leq \omega$ and hence that $\chi(G, X) \leq \omega$ since U is an open subset of X. Since F was arbitrary, this completes the proof. □

2.4.10. PROPOSITION. *If X is a compact space with $t(X) \leq \kappa$, then the G_κ-density does not exceed κ at some point $e \in X$.*

PROOF. There is no sequence of points $\{x_\alpha : \alpha < \kappa^+\}$ and a sequence of closed G_κ-subsets $\{G_\alpha : \alpha < \kappa^+\}$ of X satisfying the following conditions;

(1) $\forall\, \alpha < \kappa^+ \, (x_\alpha \in G_\alpha)$,

(2) $\forall\, \alpha < \beta < \kappa^+ \, (G_\beta \subseteq G_\alpha)$,

(3) $\forall\, \beta < \kappa^+ \, \big(\mathrm{Cl}_X(\{x_\alpha : \alpha < \beta\}) \cap \overline{G}_\beta = \emptyset\big)$.

To see this simply apply Proposition 2.4.1; since $t(X) \leq \kappa$, there is no free sequence in X of length κ^+. Condition (3) implies that the sequence $\{x_\alpha : \alpha < \kappa^+\}$ is a free sequence of length κ^+ which is impossible.

Now consider the following recursive construction. We pick $x_0 \in X$ and $G_0 = X$. If $\{x_\alpha : \alpha < \beta\}$ and $\{G_\alpha : \alpha < \beta\}$ have been defined as above (where $\beta < \kappa^+$), and $\bigcap_{\alpha<\beta} G_\alpha \not\subseteq \mathrm{Cl}_X(\{x_\alpha : \alpha < \beta\})$, then we define x_β and G_β as follows.

We may pick

$$x_\beta \in \bigcap_{\alpha<\beta} G_\alpha \setminus \mathrm{Cl}_X(\{x_\alpha : \alpha < \beta\}).$$

We may find an open neighbourhood U of x_β such that

$$\mathrm{Cl}_X(\{x_\alpha : \alpha < \beta\}) \cap \overline{U} = \emptyset.$$

By compactness, the space X is normal, so by Lemma 2.4.8 we may find a closed G_δ-set G such that $x_\beta \in G \subseteq U$. Note that the set $\bigcap_{\alpha<\beta} G_\alpha$ is a G_κ-set in X, so we may set $G_\beta = G \cap \bigcap_{\alpha<\beta} G_\alpha$.

This completes the recursion. By what we have shown, we cannot continue this process for all $\beta < \kappa^+$. So there must be some $\beta < \kappa^+$ such that

$$\bigcap_{\alpha<\beta} G_\alpha \subseteq \mathrm{Cl}_X(\{x_\alpha : \alpha < \beta\}).$$

But then we have a closed G_κ-set which is contained in the closure of a set of size $\leq \kappa$. Note that by compactness, $\bigcap_{\alpha<\beta} G_\alpha \neq \emptyset$, so for e we may take any member of this set. □

2.4.11. COROLLARY. *Suppose X is any space with $t(X)\mathrm{pct}(X) \leq \kappa$. Then the G_κ-density does not exceed κ at some point $e \in X$.*

PROOF. Since $\mathrm{pct}(X) \leq \kappa$, we may find a compact G_κ-subset F of X. Since F is a closed subset of X, we have $t(F) \leq \kappa$. By Proposition 2.4.10 there is some point $e \in F$ such that $e \in H \subseteq \overline{S}$ for some G_κ-subset H of F and $S \in [F]^{\leq\kappa}$. Since F is a G_κ-subset of X, it follows that H is also a G_κ-subset of X. So the G_κ-density at e does not exceed κ in X. □

The remaining results of this section are consistency results. These results will be used in §3.5 to show that consistently every power homogeneous T_5 compactum is first-countable. We first prove that if $\mathfrak{c} < 2^{\aleph_1}$, then locally compact T_5 spaces are hereditarily $\aleph_1$-collectionwise Hausdorff.

2.4.12. LEMMA ($\mathfrak{c} < 2^{\aleph_1}$). *Let X be a normal space. Suppose further that $\mathcal{F}$ is a discrete collection of closed subsets of X such that $\chi(F, X) \leq \mathfrak{c}$ for all $F \in \mathcal{F}$. If $|\mathcal{F}| = \aleph_1$, then there is a subcollection $\mathcal{G}$ of $\mathcal{F}$ such that $|\mathcal{G}| = \aleph_1$ and $\mathcal{G}$ is separated by a collection of open subsets of X.*

PROOF. Suppose to the contrary that whenever $\mathcal{G} \subseteq \mathcal{F}$ is separated by a collection of open subsets of X, then $|\mathcal{G}| \leq \omega$. For every $F \in \mathcal{F}$, we may find a neighbourhood base $\mathcal{B}_F$ at F in X such that $|\mathcal{B}_F| \leq \mathfrak{c}$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_F : F \in \mathcal{F}\}$. Since $|\mathcal{F}| = \aleph_1$, we have $|\mathcal{B}| \leq \mathfrak{c}$.

Fix $\mathcal{G} \subseteq \mathcal{F}$ and let $G = \bigcup \mathcal{G}$ and $H = \left(\bigcup \mathcal{F}\right) \setminus G$. Since $\mathcal{F}$ is a discrete collection of closed subsets, the sets G and H are disjoint closed subsets of X. By normality there is an open neighbourhood U_G of G in X such that $\overline{U}_G \cap H = \emptyset$. Let $\mathcal{U}_\mathcal{G}$ be a maximally disjoint collection of members of the set $\{B \in \mathcal{B} : B \subseteq U_G\}$. Then $\mathcal{U}_\mathcal{G}$ separates some subset of $\mathcal{G}$, so by assumption we have that $|\mathcal{U}_\mathcal{G}| \leq \omega$. If we set $V = \bigcup \mathcal{U}_\mathcal{G}$, then $F \cap \overline{V} \neq \emptyset$ whenever $F \in \mathcal{G}$. Furthermore, $\overline{V} \cap H = \emptyset$ and therefore $F \cap \overline{V} = \emptyset$ if $F \in \mathcal{F} \setminus \mathcal{G}$.

We have shown that if $\mathcal{G} \neq \mathcal{G}'$, then $\mathcal{U}_\mathcal{G} \neq \mathcal{U}_{\mathcal{G}'}$. So the map which sends $\mathcal{G}$ to $\mathcal{U}_\mathcal{G}$ is an injection from $\mathcal{P}(\mathcal{F})$ into $[\mathcal{B}]^{\leq\omega}$. But then

$$2^{\aleph_1} = |\mathcal{P}(\mathcal{F})| \leq |[\mathcal{B}]^{\leq\omega}| = \mathfrak{c}^\omega = \mathfrak{c}.$$

This contradicts the assumption $\mathfrak{c} < 2^{\aleph_1}$. □

2.4.13. COROLLARY ($\mathfrak{c} < 2^{\aleph_1}$). *Suppose X is normal and locally compact. Then X is weakly $\aleph_1$-collectionwise Hausdorff.*

PROOF. Let D be a closed discrete subset of X with $|D| = \aleph_1$. For every $d \in D$, we will find a closed set F_d containing d such that $\chi(F_d, X) \leq \aleph_1$ and the collection $\mathcal{F} = \{F_d : d \in D\}$ is discrete in X. The result then follows from Lemma 2.4.12.

Fix $d \in D$. Since X is Tychonoff, we may find an open F_σ-subset G_d such that $d \in G_d$ and $\overline{G}_d \cap (D \setminus \{d\}) = \emptyset$. By normality, there is an open set U in X such that

$$D \subseteq U \subseteq \overline{U} \subseteq \bigcup\{G_d : d \in D\}.$$

Since X is locally compact and normal, by Lemma 2.4.8 we may find a compact G_δ-set F'_d in X such that $d \in F'_d$ and $F'_d \subseteq G_d \cap U$. We now define the set F_d as follows;

$$F_d = F'_d \setminus \bigcup\{G_e : e \in D, e \neq d\}.$$

Then F_d is a closed subset of F'_d and therefore it is compact. Recall that each set G_e is an F_σ and the set F'_d is a G_δ. Since $|D| = \aleph_1$, it follows that the set F_d is the intersection of $\aleph_1$-many open subsets of X. Since X is locally compact and normal it is of countable type by Lemma 2.4.9 and therefore it follows from Corollary 2.4.6 that $\chi(F_d, X) \leq \aleph_1$.

So it remains to verify that the collection $\mathcal{F} = \{F_d : d \in D\}$ is discrete. Let $x \in X$ be arbitrary. If $x \notin \overline{U}$, then $X \setminus \overline{U}$ is a neighbourhood of x which misses F_d for all $d \in D$ since $F_d \subseteq U$. If on the other hand $x \in \overline{U}$, then by construction $x \in G_d$ for some $d \in D$. Since $F_e \cap G_d \neq \emptyset$ if any only if $d = e$, the neighbourhood G_d of x witnesses the fact that $\mathcal{F}$ is discrete in x. This shows that $\mathcal{F}$ is discrete and this completes the proof. □

2.4.14. COROLLARY ($\mathfrak{c} < 2^{\aleph_1}$). *If X is a locally compact T_5 space, then X is hereditarily weakly $\aleph_1$-collectionwise Hausdorff.*

PROOF. Suppose Y is a subspace of X and let D be a closed discrete subset of Y of cardinality $\aleph_1$. Then D is a discrete subset of X and therefore the set $F = \overline{D} \setminus D$ is a closed subset of X. Let $U = X \setminus F$. Then U is an open subset of X and therefore U is locally compact. Since X is T_5, the subspace U is normal. It follows from Corollary 2.4.13 that U is weakly $\aleph_1$-collectionwise Hausdorff. By construction of U, the set D is a closed discrete subset of U, and therefore we may find a subset D_0 of D such that $|D_0| = \aleph_1$ and D_0 is separated by a collection $\mathcal{V}$ of open subsets of U. Then D_0 is also separated by the collection $\{V \cap Y : V \in \mathcal{V}\}$ which consists of open subsets of Y. Since D was arbitrary, this shows that Y is weakly $\aleph_1$-collectionwise Hausdorff. □

2.4.15. PROPOSITION. *If X is normal and weakly $\aleph_1$-collectionwise Hausdorff, then X satisfies property $wD(\aleph_1)$.*

PROOF. Suppose D is a closed discrete subset of X of cardinality $\aleph_1$. Since X is weakly $\aleph_1$-collectionwise Hausdorff, we may find a subset D_0 of D of size $\aleph_1$, such that D_0 is separated by a collection $\mathcal{U}$ of open subsets of X. Let $U = \bigcup \mathcal{U}$ and consider the set $F = \overline{U} \setminus U$. Since U is open, the set F is closed and since $D_0 \subseteq U$, the sets D_0 and F are disjoint. Note that D_0 is also a closed subset of X. Since X is normal, we may find an open neighbourhood V of D_0 such that $F \cap \overline{V} = \emptyset$. Let $\mathcal{W}$ be given by $\{W \cap V : W \in \mathcal{U}\}$. Then $\mathcal{W}$ is a collection of open subsets of X and it separates D_0. By construction, the collection $\mathcal{W}$ is also discrete. Since D was arbitrary, this shows that X satisfies property $wD(\aleph_1)$. □

2.4.16. THEOREM ($\mathfrak{c} < 2^{\aleph_1}$). *If X is a locally compact T_5 space, then X satisfies property $wD(\aleph_1)$ hereditarily.*

PROOF. This follows from Corollary 2.4.14 and Proposition 2.4.15. □

2.5 Further results

In this section we collect some further results from the literature that will be needed in the sequel. The following theorem is known as the Čech-Pospišil Theorem, for a proof see JUHÁSZ [34, 3.16].

2.5.1. ČECH-POSPIŠIL THEOREM. *If X is a compact space without isolated points and κ is an infinite cardinal number such that $\chi(x, X) \geq \kappa$ for all $x \in X$, then $|X| \geq 2^\kappa$.*

This theorem is particularly interesting for compact spaces that are homogeneous with respect to character. By Arhangel′skiĭ's Theorem (see §3.1) the size of compact spaces X is always bounded by $2^{\chi(X)}$. It follows from the Čech-Pospišil Theorem that whenever X is a compact space without isolated points which is homogeneous with respect to character, then $|X| = 2^{\chi(X)}$; this was noted by HART and KUNEN [30]. We will apply similar reasoning in §3.4. The following theorem is due to Šapirovskiĭ, for a proof we refer the reader to JUHÁSZ [34, 3.18, 3.20 & 3.21].

2.5.2. THEOREM. *If X is a compact T_5 space, then the set $\{x \in X : \pi\chi(x, X) \leq \omega\}$ is dense in X.*

The following theorem can be found in JUHÁSZ and SZENTMIKLÓSSY [37].

2.5.3. THEOREM. *Let κ be an uncountable regular cardinal. If a compact space X contains a free sequence of length κ, then it also contains a converging free sequence of length κ.*

If X is a topological space, then $y \in X$ is called a pseudo P-point if every G_δ-set containing y has non-empty interior. A set $D \subseteq X$ is called G_δ-dense, if D intersects every non-empty G_δ-subset of X. Note that if the set of all pseudo P-points of X is G_δ-dense in X, then every point of X is a pseudo P-point. The following lemma can be found in JUHÁSZ, NYIKOS and SZENTMIKLÓSSY [36].

2.5.4. LEMMA. *Let Y be a locally compact space without isolated points. The set of points y which fail to satisfy at least one of the following conditions is dense in Y:*

(1) $\pi\chi(y, Y) \leq \omega$,

(2) *y is a pseudo P-point.*

In particular, not all points of Y can satisfy both (1) and (2).

The following theorem is an extension of a result of MATVEEV [43], it was proved by VAN MILL [47, Theorem 4.2]. Recall that the cardinal $\mathfrak{p}$ is the least size of a family of subsets of ω which has the strong finite intersection property but no infinite pseudo-intersection. We always have $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{c}$ and under $\mathsf{MA} + \neg\mathsf{CH}$ it is even the case that $\aleph_1 < \mathfrak{p}$. For more information on the cardinal $\mathfrak{p}$ and other small uncountable cardinals, see VAN DOUWEN [20].

2.5.5. THEOREM. *Let $a\omega$ and $b\omega$ be compactifications of ω. Assume that*

(1) *there is a retraction $r : a\omega \to a\omega \setminus \omega$,*

(2) *there is a retraction $s : b\omega \to b\omega \setminus \omega$,*

(3) *$f : a\omega \setminus \omega \to b\omega \setminus \omega$ is a homeomorphism*

If the weight of $a\omega \setminus \omega$ is less than $\mathfrak{p}$, then f can be extended to a homeomorphism $\bar{f} : a\omega \to b\omega$.

So the theorem says that there is a permutation $\pi : \omega \to \omega$ such that $\bar{f} = f \cup \pi$ is a homeomorphism. Finally we shall need the following result, a proof may be found in VAN MILL [47].

2.5.6. PROPOSITION. *Let $\gamma\omega$ be a compactification of ω such that $\gamma\omega \setminus \omega$ is a product of second countable compacta. Then $\gamma\omega \setminus \omega$ is a retract of $\gamma\omega$.*

Chapter 3

Cardinal restrictions

3.1 Introduction

In this chapter we study cardinal restrictions for spaces on which some form of homogeneity is assumed. Except for Theorem 3.4.24, we are exclusively concerned with power homogeneous spaces. A cardinal restriction for some space X is an upper bound on the cardinality of that space in terms of its cardinal functions. One can find many such restrictions in the literature. For example, ARHANGEL′SKIǏ [2] proved that if X is an arbitrary Hausdorff space, then

(A) $$|X| \leq 2^{\chi(X)L(X)}.$$

In particular, the size of first-countable compact spaces is bounded by $\mathfrak{c}$. Another restriction on the size of Hausdorff spaces was proved by HAJNAL and JUHÁSZ [29]; they proved that if X is a Hausdorff space, then

(HJ) $$|X| \leq 2^{\chi(X)c(X)}.$$

So the size of first-countable c.c.c. spaces is also bounded by $\mathfrak{c}$. Applying results of ŠAPIROVSKIǏ [64], it was noted by ARHANGEL′SKIǏ [7], that for regular homogeneous spaces the character in the Hajnal-Juhász bound (HJ) may be replaced with the π-character; if X is a homogeneous regular space, then

(AS) $$|X| \leq 2^{\pi\chi(X)c(X)}.$$

So in the presence of homogeneity, one obtains sharper cardinality restrictions. This phenomenon also occurs for power homogeneous spaces. This was first noted by VAN DOUWEN [19], who proved that if X is a power homogeneous Hausdorff space, then

$$|X| \leq 2^{\pi w(X)}. \tag{vD}$$

This bound is not valid for arbitrary Hausdorff spaces, and it can be used to show that certain spaces are not power homogeneous. This is in itself surprising, since if a space X is power homogeneous, then the least power μ for which X^μ is homogeneous might be very large. The techniques developed by Van Douwen are very powerful since one can prove cardinality results about a space X by looking at some power X^μ, regardless of the size of the cardinal number μ. Applying his results, Van Douwen proved that the Čech-Stone compactification $\beta\omega$ of ω and its remainder $\beta\omega \setminus \omega$ are not power homogeneous. The method which he used is known as a 'clustering-method' and it was later applied by VAN MILL [49] to prove the inequality (AS) also for power homogeneous compact spaces.

In recent years, cardinal restrictions for homogeneous compact spaces have been studied by DE LA VEGA [74] and JUHÁSZ, NYIKOS and SZENTMIKLÓSSY [36]. De la Vega proved that if X is a homogeneous compact space, then

$$|X| \leq 2^{t(X)}. \tag{dlV}$$

This strengthens Arhangel′skiĭ's Theorem (A) for homogeneous compact spaces and it follows in particular that the size of countably tight homogeneous compacta is bounded by $\mathfrak{c}$. It was proved by Juhász, Nyikos and Szentmiklóssy that consistently every homogeneous T_5 compactum is of cardinality $\leq \mathfrak{c}$. The results in [36] and [74] even imply that consistently every homogeneous T_5 compactum is first-countable.

The cardinality restrictions mentioned in this introduction for homogeneous spaces, may also be proved for power homogeneous spaces. We will provide the proofs in this chapter. We will in turn study Hausdorff spaces, compact spaces and compact T_5 spaces. In the first section we prove two cardinality restrictions for homogeneous spaces. In particular, we will prove the inequality (AS) for homogeneous Hausdorff spaces. The argument is a simple modification of a proof of the Hajnal-Juhász bound (HJ) which can be found in JUHÁSZ [34, 2.15]. In the first section we will also prove a technical tool which is used in all subsequent sections to prove cardinal restrictions for power homogeneous spaces. Contrary to the clustering-technique used by Van Douwen, we will use this result to prove cardinal restrictions for power homogeneous spaces by only looking at relatively small powers of spaces. This leads in particular to a new proof of Van Douwen's Theorem in §3.2.

The proof of the following result is a slight modification of techniques used by ISMAIL [33].

3.1.1. PROPOSITION. *Suppose A is some type in X. Then $|A| \leq d(X)^{\pi\chi(X)}$.*

PROOF. Let $A = \operatorname{tpe}(x, X)$. For every $y \in A$ we may fix a homeomorphism $h_y : X \to X$ such that $h_y(x) = y$. We fix a dense set D in X with $|D| = d(X)$ and a local π-base $\mathcal{U}$ at x in X with $|\mathcal{U}| \leq \pi\chi(X)$.

We define a map $H : A \to D^{\mathcal{U}}$ as follows. Fix some well-ordering on D and for $y \in A$ and $U \in \mathcal{U}$, let

$$H(y)(U) = \min\bigl\{d \in D : d \in h_y[U]\bigr\}.$$

Since D is dense this is well-defined. It remains to verify that H is injective. So let $w, z \in A$ with $w \neq z$. Since X is Hausdorff, we may find two disjoint open subsets V_w and V_z of X such that $w \in V_w$ and $z \in V_z$. Now let

$$V = h_w^{-1}[V_w] \cap h_z^{-1}[V_z].$$

Then V is an open neighbourhood of x and since $\mathcal{U}$ is a local π-base at x, there is some $U \in \mathcal{U}$ with $U \subseteq V$. But then $H(w)(U) \in V_w$ and $H(z)(U) \in V_z$. Since V_w and V_z are disjoint, it follows that $H(w)(U) \neq H(z)(U)$ and thus $H(w) \neq H(z)$. This shows that H is injective and therefore $|A| \leq |D|^{|\mathcal{U}|}$. □

Since $d(X)^{\pi\chi(X)} \leq 2^{\pi w(X)}$ for any space X, it follows that the cardinality of homogeneous Hausdorff spaces X is bounded by $2^{\pi w(X)}$. This is Van Douwen's Theorem (vD) for homogeneous spaces. We will use the previous proposition to prove this theorem also for power homogeneous spaces, see Theorem 3.2.6. It was asked by CARLSON [17] whether power homogeneous Hausdorff spaces satisfy the inequality (AS). The following proposition provides a positive solution for homogeneous spaces. The proof of this result is an obvious generalization of the proof of (HJ) which appears in JUHÁSZ [34, 2.15].

3.1.2. PROPOSITION. *Suppose $f : X \to Y$ is an open and continuous map. If A is some type in X, then*

$$|f[A]| \leq 2^{\pi\chi(X)c(Y)}.$$

PROOF. Let $\kappa = \pi\chi(X)c(Y)$. We may fix $B \subseteq A$, such that $f{\restriction}B : B \to f[A]$ is a bijection. Fix $p \in B$ and choose for every $x \in B$ a homeomorphism $h_x : X \to X$ such that $h_x(p) = x$. Let $\mathcal{U}$ be a local π-base at p in X with $|\mathcal{U}| \leq \kappa$ and fix some well-ordering on $\mathcal{U}$.

Suppose $x, y \in B$ with $x \neq y$. Then there are open neighbourhoods U_x and U_y of $f(x)$ and $f(y)$ respectively (in Y), such that $U_x \cap U_y = \emptyset$. The set $U = h_x^{-1} f^{-1}[U_x] \cap h_y^{-1} f^{-1}[U_y]$ is an open neighbourhood of p in X. Since $\mathcal{U}$ is a local π-base at p, we may find $V \in \mathcal{U}$ such that $V \subseteq U$. Clearly, we have $fh_x[V] \cap fh_y[V] = \emptyset$.

We now define a map $G : [B]^2 \to \mathcal{U}$ as follows:

$$G(\{x, y\}) = \min\{V \in \mathcal{U} : fh_x[V] \cap fh_y[V] = \emptyset\}.$$

We have just shown that G is well-defined. Next suppose that $|B| \geq (2^\kappa)^+$. By Theorem 2.1.1, we may find a subset B' of B such that $|B'| \geq \kappa^+$ and $G(b) = V$ for all $b \in [B']^2$. But this means that the collection $\mathcal{C}$ given by

$$\mathcal{C} = \{fh_x[V] : x \in B'\},$$

is a cellular family in Y. Since $|\mathcal{C}| = |B'| = \kappa^+$ this contradicts the assumption that $c(Y) \leq \kappa$. It follows that $|f[A]| = |B| \leq 2^\kappa$. □

We shall prove in Theorem 3.3.8 that the size of a power homogeneous space X is also bounded by $2^{c(X)\pi\chi(X)}$. It already follows from the previous proposition that this is true if X^κ is Δ-homogeneous where $\kappa = c(X)\pi\chi(X)$.

The following theorem was proved by ARHANGEL'SKIĬ, VAN MILL and RIDDERBOS [12]. It is the starting point for all subsequent results in this chapter. The result was inspired by the notion of a κ-twister, studied by ARHANGEL'SKIĬ [10, 11].

3.1.3. THEOREM. *Let $Y = \prod\{X_i : i \in I\}$ and suppose $h : Y \to Y$ is a homeomorphism. Suppose that $h(z) = x$ and for some $i \in I$, $\pi\kappa\chi(x_i, X_i) \leq \kappa$. Then there is some set $A \in [I]^{\leq\kappa}$ such that for all $y \in Y$;*

$$y_A = z_A \Rightarrow h(y)_i = x_i.$$

PROOF. Fix a local $\pi\kappa$-network $\mathcal{U}$ at x_i in X_i with $|\mathcal{U}| \leq \kappa$. For every $U \in \mathcal{U}$, we pick $p_U \in \pi_i^{-1}[U]$ such that $(p_U)_B = x_B$, where $B = I \setminus \{i\}$. Note that for every $U \in \mathcal{U}$, the set $\pi_i^{-1}[U]$ is a G_κ-subset of Y and therefore we may fix some basic G_κ-set G_U such that

$$h^{-1}(p_U) \in G_U \subseteq h^{-1}[\pi_i^{-1}[U]].$$

By A_U we denote the set of coordinates on which G_U is determined. Thus $G_U = \pi_{A_U}^{-1}[\pi_{A_U}[G_U]]$ and $A_U \in [I]^{\leq\kappa}$. We let $A = \bigcup\{A_U : U \in \mathcal{U}\}$ so that $A \in [I]^{\leq\kappa}$. We will prove that if $y \in Y$ and $y_A = z_A$, then $h(y)_i = x_i$.

Suppose to the contrary that $h(y)_i \neq x_i$ for some $y \in Y$ with $y_A = z_A$. We fix a neighbourhood W of x_i such that $h(y)_i \notin \overline{W}$. Let $\mathcal{V} = \{U \in \mathcal{U} : U \subseteq W\}$. Then $\mathcal{V}$ is a local $\pi\kappa$-network at x_i. So we have

$$x_i \in \mathrm{Cl}\,\{\pi_i(p_V) : V \in \mathcal{V}\}.$$

By construction we have

$$x \in \mathrm{Cl}\,\{p_V : V \in \mathcal{V}\}.$$

But then we also have $y_A = z_A \in \mathrm{Cl}\,\{h^{-1}(p_V)_A : V \in \mathcal{V}\}$.

For $V \in \mathcal{V}$ we now define the point $q_V \in Y$ as follows,

$$(q_V)_j = \begin{cases} h^{-1}(p_V)_j & \text{if } j \in A, \\ y_j & \text{if } j \notin A. \end{cases}$$

Note that $q_V \in G_V$ and that $y \in \mathrm{Cl}\,\{q_V : V \in \mathcal{V}\}$. But then $h(y)_i \in \mathrm{Cl}\,\{h(q_V)_i : V \in \mathcal{V}\}$. However, since $q_V \in G_V$ we also have that $h(q_V)_i \in V$ for all $V \in \mathcal{V}$, and therefore $\mathrm{Cl}\,\{h(q_V)_i : V \in \mathcal{V}\} \subseteq \overline{W}$. Since $h(y)_i \notin \overline{W}$, this is impossible. □

3.1.4. COROLLARY. *Let $Y = \prod\{X_i : i \in I\}$ and suppose $h : Y \to Y$ is a homeomorphism. Suppose that $h(z) = x$ and for some $i \in I$, $\pi\chi(x_i, X_i) \leq \kappa$. Then there is some set $A \in [I]^{\leq\kappa}$ such that for all $y \in Y$,*

$$y_A = z_A \Rightarrow h(y)_i = x_i.$$

3.2 Delta homogeneity

In our study of cardinal restrictions for power homogeneous spaces we attempt to look at relatively small powers to prove our results. A useful tool in this line of reasoning is the notion of Δ-homogeneity. A careful examination of the results in the literature reveals that when reasoning about power homogeneous spaces, it is often sufficient to know that certain 'large' subsets of power spaces are types. Since we want to obtain restrictions on the size of some space X, 'large' here means 'at least as large as X'. The diagonal in any power space of X is of the same cardinality as X, so it seems natural to consider power spaces in which the diagonal is (contained in) a type. This led to the introduction of Δ-power homogeneous spaces in RIDDERBOS [59]. It turns out that if a space X is Δ-power homogeneous, then $X^{\pi w(X)}$ is Δ-homogeneous. This result allows a quick proof of the Van Douwen bound, see Theorem 3.2.6 below.

The introduction of Δ-power homogeneity leads to the question of its relation to power homogeneity. We will prove that Δ-power homogeneity and power homogeneity are equivalent notions. However, we do not know whether $X^{\pi w(X)}$ is homogeneous provided that X is power homogeneous. We therefore introduce the following cardinal function for power homogeneous spaces X; if X is power homogeneous, then its homogeneity degree, $\mathrm{hdeg}(X)$, is defined as the least cardinal number μ for which X^μ is homogeneous. We will prove that if X is power homogeneous, then

$$\mathrm{hdeg}(X) \le \big(\pi w(X)\mathrm{hind}(X)\big)^+.$$

We do not know whether this bound is sharp.

We first turn towards proving that $X^{\pi w(X)}$ is Δ-homogeneous provided that X is Δ-power homogeneous. Recall from Theorem 2.3.10 that if $Y = \prod\{X_i : i \in I\}$ and $d(X_i) \le \kappa$ for all $i \in I$, then the closure of an open set U in Y depends on not more than κ many coordinates, which means that $\overline{U} = \pi_A^{-1}\big[\pi_A[\overline{U}]\big]$ for some $A \in [I]^{\le\kappa}$.

3.2.1. THEOREM. *Let $Y = \prod\{X_i : i \in I\}$ and suppose that $h : Y \to Y$ is a homeomorphism. Suppose further that for all $i \in I$, $d(X_i) \le \kappa$ and $\pi w(X_j) \le \kappa$ for some $j \in I$. Then there is a set $A \in [I]^{\le\kappa}$ such that for all $w, z \in Y$,*

$$w_A = z_A \Rightarrow h(w)_j = h(z)_j.$$

PROOF. Fix a π-base $\mathcal{U}$ for X_j of size $\le \kappa$. Then we may fix a set of coordinates $A \in [I]^{\le\kappa}$ such that for every $U \in \mathcal{U}$, the closure of $h^{-1}\pi_j^{-1}[U]$ depends on the coordinates in A. We will show that A is as required.

So let $w, z \in Y$ with $w_A = z_A$ and suppose $p = h(w)_j \ne h(z)_j = q$. Then we may fix a neighbourhood V of p in X_j with $q \notin \overline{V}$. Let $\mathcal{V} = \{U \in \mathcal{U} : U \subseteq V\}$. Since $\mathcal{U}$ is a π-base in X_j we have $p \in \mathrm{Cl} \bigcup \mathcal{V}$. But then $w \in \overline{F}$ where

$$F = \bigcup\big\{\mathrm{Cl}\, h^{-1}\pi_j^{-1}[U] : U \in \mathcal{V}\big\}.$$

By construction we have $F = \pi_A^{-1}\big[\pi_A[F]\big]$, so also $\overline{F} = \pi_A^{-1}\big[\pi_A[\overline{F}]\big]$. Since $z_A = w_A \in \pi_A[\overline{F}]$, it follows that $z \in \pi_A^{-1}\big[\pi_A[\overline{F}]\big] = \overline{F}$. But we also have $\overline{F} \subseteq h^{-1}\pi_j^{-1}[\overline{V}]$, and therefore it follows that $q = h(z)_j \in \overline{V}$, which is impossible. $\square$

3.2.2. COROLLARY. *Let $h : X^\mu \to X^\mu$ be a homeomorphism and suppose $\pi w(X) \leq \kappa$. If $B \in [\mu]^{\leq\kappa}$ then there is a set $A \in [\mu]^{\leq\kappa}$ such that for all $w, z \in X^\mu$,*

$$w_A = z_A \Rightarrow h(w)_B = h(z)_B.$$

PROOF. We may view X^μ as the product space of X^B and X_α for $\alpha \in \mu \setminus B$. Since $\pi w(X^B) = \pi w(X) \cdot |B| \leq \kappa$ by Theorem 2.3.9, the statement follows from Theorem 3.2.1. □

Note that in the previous corollary, any set containing the set A also satisfies the conclusion. In particular, the set A may be taken so that it contains B.

3.2.3. THEOREM. *Let $h : X^\mu \to X^\mu$ be a homeomorphism and let $\pi w(X) \leq \kappa$. Suppose $B \in [\mu]^{\leq\kappa}$, then there is a set $A \in [\mu]^{\leq\kappa}$ such that $B \subseteq A$ and for all $w, z \in X^\mu$*

$$w_A = z_A \Longleftrightarrow h(w)_A = h(z)_A.$$

PROOF. By Corollary 3.2.2 we may construct a sequence $(A_n)_n$ which satisfies the following conditions for all $n < \omega$,

(1) $A_0 = B$, $A_n \in [\mu]^{\leq\kappa}$ and $A_n \subseteq A_{n+1}$.

(2) For all $w, z \in X^\mu$, $w_{A_{2n+1}} = z_{A_{2n+1}} \Longrightarrow h(w)_{A_{2n}} = h(z)_{A_{2n}}$,

(3) For all $w, z \in X^\mu$, $h(w)_{A_{2n+2}} = h(z)_{A_{2n+2}} \Longrightarrow w_{A_{2n+1}} = z_{A_{2n+1}}$.

Now set $A = \bigcup_{n<\omega} A_n$ and the theorem follows. □

Note that if A is as in the previous theorem, then it is also the case that for all $w, z \in X^\mu$,

$$w_A = z_A \Longleftrightarrow h^{-1}(w)_A = h^{-1}(z)_A.$$

3.2.4. THEOREM. *Let X be a topological space and suppose that $\pi w(X) \leq \kappa \leq \mu$. Suppose further that $h : X^\mu \to X^\mu$ is a homeomorphism. If $B \in [\mu]^{\leq\kappa}$ then there are an $A \in [\mu]^{\leq\kappa}$ and a homeomorphism $h_A : X^A \to X^A$ such that $B \subseteq A$ and $h_A \circ \pi_A = \pi_A \circ h$.*

PROOF. By Theorem 3.2.3 we may choose $A \subseteq \mu$ such that $B \subseteq A$, $|A| \leq \kappa$ and A has the following property: for all $w, z \in X^\mu$,

$$(*) \quad w_A = z_A \Leftrightarrow h(w)_A = h(z)_A \text{ and } w_A = z_A \Leftrightarrow h^{-1}(w)_A = h^{-1}(z)_A.$$

Let $i : X^A \to X^\mu$ be any continuous injection such that $\pi_A \circ i = \mathrm{id}_{X^A}$ and let $f = \pi_A \circ h \circ i$ and $g = \pi_A \circ h^{-1} \circ i$. Then f and g are continuous and it follows from (∗) that $f \circ \pi_A = \pi_A \circ h$ and $g \circ \pi_A = \pi_A \circ h^{-1}$. So we have that

$$\begin{aligned} f \circ g &= f \circ \pi_A \circ h^{-1} \circ i = \pi_A \circ h \circ h^{-1} \circ i \\ &= \pi_A \circ i = \mathrm{id}_{X^A}. \end{aligned}$$

Similarly $g \circ f = \mathrm{id}_{X^A}$ and this shows that f is a homeomorphism. So for h_A we may take f and this proves the theorem. □

3.2.5. THEOREM. *If X is Δ-power homogeneous, then $X^{\pi w(X)}$ is Δ-homogeneous.*

PROOF. Let $\kappa = \pi w(X)$ and choose $\mu \geq \kappa$ such that X^μ is Δ-homogeneous. Instead of choosing arbitrary elements of $\Delta(X, \kappa)$, we choose $x, y \in \Delta(X, \mu)$ arbitrarily and show that for some homeomorphism g of X^κ, $g(x_\kappa) = y_\kappa$.

Since X^μ is Δ-homogeneous, we may fix a homeomorphism $h : X^\mu \to X^\mu$ with $h(x) = y$. Let $A \in [\mu]^\kappa$ and $h_A : X^A \to X^A$ be as in the previous theorem. Then h_A is a homeomorphism with $h_A(x_A) = h(x)_A = y_A$. By a suitable change of coordinates we obtain a homeomorphism g of X^κ which maps x_κ onto y_κ. It is essential here that both x and y are constant as functions from μ into X. □

This is what is known as a *reflection theorem*; the Δ-homogeneity of some (large) power X^μ implies that $X^{\pi w(X)}$ has the same property: one says that Δ-homogeneity *reflects down* to $\pi w(X)$. As a corollary we give a proof of Van Douwen's Theorem from [19].

3.2.6. VAN DOUWEN'S THEOREM. *If X is power homogeneous, then*

$$|X| \leq 2^{\pi w(X)}.$$

PROOF. Let $\kappa = \pi w(X)$. By Theorem 3.2.5 it follows that X^κ is Δ-homogeneous. Therefore the diagonal $\Delta(X, \kappa)$ is contained in some type of X^κ. Since $\pi w(X^\kappa) \leq \kappa$ by Theorem 2.3.9, we have $d(X^\kappa) \leq \kappa$ and $\pi\chi(X^\kappa) \leq \kappa$. It follows from Proposition 3.1.1 that

$$|X| = |\Delta(X, \kappa)| \leq \kappa^\kappa = 2^\kappa. \qquad \square$$

We now turn to proving that a space is Δ-power homogeneous if and only if it is power homogeneous. We start with two combinatorial lemmas.

3.2.7. LEMMA. *Suppose X is a topological space and suppose further that κ is a (possibly finite) cardinal number such that X^κ is Δ-homogeneous. If μ is a cardinal number such that $\mu \geq \kappa$, then X^μ is Δ-homogeneous in each of the following cases:*

(1) *μ is infinite,*

(2) *μ is finite and μ is a multiple of κ.*

Whenever X is a topological space, we say that a subset Q of X is a set of representatives for the types in X if for every $x \in X$, there is a *unique* member q of Q such that $x \in \mathrm{tpe}(q, X)$. Note that in this case $|Q| = \mathrm{hind}(X)$.

3.2.8. LEMMA. *Suppose X^κ is Δ-homogeneous, where κ is infinite. Let $\mu \geq \kappa$ and $x \in X^\mu$. If for every $\alpha < \mu$ we have:*

$$|\{\beta \in \mu : x_\beta \in \mathrm{tpe}(x_\alpha, X)\}| \geq \kappa,$$

then there is a homeomorphism h of X^μ such that $h(x) \in \Delta(X, \mu)$.

PROOF. Let $p \in \Delta(X, \mu)$ be arbitrary. We will find a homeomorphism h of X^μ such that $h(x) = p$. Choose a set Q which is a set of representatives for the types in X. Without loss of generality we may assume that $x \in Q^\mu$. For every $q \in Q$, we let

$$A(q) = \{\alpha < \mu : x_\alpha = q\}.$$

By assumption we have that for every $q \in Q$, either $A(q)$ is empty or $|A(q)| \geq \kappa$. We also have that $\mu = \bigcup_{q \in Q} A(q)$ and if $q, q' \in Q$ then $A(q) \cap A(q') = \emptyset$ provided $q \neq q'$.

If $q \in Q$ and $A(q) \neq \emptyset$, then $|A(q)| \geq \kappa$ and therefore $X^{A(q)}$ is Δ-homogeneous by Lemma 3.2.7. Since $x_{A(q)} \in \Delta(X, A(q))$, we may find a homeomorphism h_q of $X^{A(q)}$ such that $h(x_{A(q)}) = p_{A(q)}$.

If we let h be the product homeomorphism of all h_q's where $q \in Q$ and $A(q) \neq \emptyset$, then h is a homeomorphism of X^μ which maps x onto p. □

3.2.9. THEOREM. *Let X be a topological space and μ an infinite cardinal number. Suppose that either $\mu \geq \pi w(X)$ and $\mathrm{hind}(X) < \mathrm{cf}(\mu)$ or $\mu > \mathrm{hind}(X)\pi w(X)$. The following statements are equivalent:*

(1) *X^μ is homogeneous,*

(2) *X is power homogeneous,*

(3) *X is Δ-power homogeneous,*

(4) $X^{\pi w(X)}$ *is* Δ*-homogeneous.*

PROOF. It is obvious that $(1) \to (2)$ and $(2) \to (3)$. The fact that $(3) \to (4)$ follows from Theorem 3.2.5. It remains to verify that $(4) \to (1)$.

Choose a set Q which is a set of representatives for the types in X and let $x \in X^\mu$ be arbitrary. Since X^μ is Δ-homogeneous by Lemma 3.2.7, it suffices to show that x can be mapped into the diagonal of X^μ by a homeomorphism of X^μ. By Lemma 3.2.8, it suffices to show that there is a homeomorphism h of X^μ such that $h(x) = y$ where $y \in Q^\mu$ and for every $q \in Q$ we have:

$$(*) \qquad |\{\alpha < \mu : y_\alpha = q\}| \geq \kappa,$$

where $\kappa = \pi w(X)$. Without loss of generality we may assume that $x \in Q^\mu$. Again, for $q \in Q$, we let $A(q) = \{\alpha < \mu : x_\alpha = q\}$ and as before we have:

$$(**) \qquad \mu = \bigcup_{q \in Q} A(q).$$

We first show that there is some $q \in Q$ with $|A(q)| \geq \kappa \cdot |Q|$. We consider the two cases we have for the value of μ. If $|Q| < \mathrm{cf}(\mu)$, then it follows from $(**)$ that $|A(r)| = \mu$ for some $r \in Q$. If on the other hand $\mu > \kappa \cdot |Q|$, then it follows from $(**)$ that there is some $r \in Q$ with $|A(r)| > \kappa \cdot |Q|$. In either case we find an $r \in Q$ such that $|A(r)| \geq \kappa \cdot |Q|$.

We fix such an r and a partition $\{B(q) : q \in Q\}$ of $A(r)$ such that for all $q \in Q$, $|B(q)| \geq \kappa$. We let the point $y \in X^\mu$ be defined as follows;

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \notin A(r), \\ q & \text{if } \alpha \in A(r) \text{ and } \alpha \in B(q). \end{cases}$$

Note that for every $q \in Q$, we have $B(q) \subseteq \{\alpha < \mu : y_\alpha = q\}$. Since $|B(q)| \geq \kappa$, it follows that the point y satisfies $(*)$. Also, for every $q \in Q$, $y_{B(q)} \in \Delta(X, B(q))$.

If $q \in Q$, then $X^{B(q)}$ is Δ-homogeneous and $x_{B(q)} \in \Delta(X, B(q))$ since $B(q) \subseteq A(r)$. So for $q \in Q$, we may fix a homeomorphism h_q of $X^{B(q)}$ such that $h_q(x_{B(q)}) = y_{B(q)}$. Let h be the product homeomorphism consisting of the product of the h_q's for $q \in Q$ and the identity map on the space $X^{(\mu \setminus A(r))}$. Then h is a homeomorphism of X^μ which maps x onto y and this completes the proof. □

This characterization reduces the study of power homogeneity of a space X to the study of Δ-homogeneity of relatively small powers of the space X. For example, if X is a separable metrizable space, then X is power homogeneous if and only if X^ω is Δ-homogeneous. Also note that if X is power homogeneous and $\mathrm{hind}(X)$ is finite, then X^κ is homogeneous if and only if it is Δ-homogeneous. In particular if $\pi w(X) = \kappa$, it follows that in this case X^κ is homogeneous. As a corollary to the previous theorem, we obtain the following result mentioned at the beginning of this section;

3.2.10. COROLLARY. *If X is power homogeneous, then*

$$\mathrm{hdeg}(X) \leq \big(\pi w(X)\mathrm{hind}(X)\big)^+.$$

3.3 Hausdorff spaces

The argument which we used to prove Proposition 3.1.1 may also be applied to power homogeneous spaces. This leads to the proof of Theorem 3.3.4 which generalizes Van Douwen's Theorem. Of course, we will only use 'small' powers of spaces, where in this case the π-character is small. We first develop the necessary tools.

We fix topological spaces X and Y and an open subset V of Y. Furthermore f is an open and continuous mapping from X onto Y. We let μ be an infinite cardinal number and π is the projection of X^μ onto X. We have the following situation:

$$X^\mu \xrightarrow{\pi} X \xrightarrow{f} Y \supseteq V.$$

We define the π-character of Y with respect to f as follows;

$$\begin{aligned} \pi\chi(y,f) &= \min\{\pi\chi(x,X) : x \in f^{-1}(y)\}, \\ \pi\chi(Y,f) &= \sup\{\pi\chi(y,f) : y \in Y\}. \end{aligned}$$

Fix $p \in \Delta(X,\mu)$ and assume that $\mathcal{U}$ is a local π-base at $\pi(p)$ in X. If $B \subseteq A \subseteq \mu$, then $\pi_{A\to B}$ denotes the projection from X^A onto X^B. Whenever $A \subseteq \mu$, then by $\mathcal{U}(A)$ we denote the collection

$$\Big\{\pi^{-1}_{A\to B}\Big[\prod\nolimits_{b\in B} U_b\Big] : B \in [A]^{<\omega}, \forall\, b \in B\,\big(U_b \in \mathcal{U}\big)\Big\}.$$

The set $\mathcal{U}(A)$ is a local π-base at p_A in X^A and $|\mathcal{U}(A)| \leq |A|\cdot|\mathcal{U}|$. Furthermore if $A = \bigcup_{n<\omega} A_n$ where $\{A_n : n < \omega\}$ is some increasing sequence of subsets of μ, then

$$\mathcal{U}(A) = \bigcup_{n<\omega} \pi^{-1}_{A\to A_n}\mathcal{U}(A_n),$$

where $\pi^{-1}_{A\to B}\,\mathcal{U}(B)$ denotes the collection $\{\pi^{-1}_{A\to B}[U] : U \in \mathcal{U}(B)\}$. The following theorem follows from Corollary 3.1.4 and a recursive construction.

3.3.1. LEMMA. *Let D be a (dense) subset of V and suppose that $\pi\chi(d,f) \leq \kappa$ for all $d \in D$. If $h : X^\mu \to X^\mu$ is a homeomorphism, $|\mathcal{U}| \leq \kappa$ and $B \in [\mu]^{\leq\kappa}$, then there is a set $A \in [\mu]^{\leq\kappa}$ such that $B \subseteq A$ and for all $U \in \mathcal{U}(A)$ with $f\pi h\pi_A^{-1}[U] \cap D \neq \emptyset$ there is some $e \in X^\mu$ satisfying:*

(1) *$f\pi h(e) = d \in D$ and $e \in \pi_A^{-1}[U]$,*

(2) *$h\pi_A^{-1}(e_A)$ is contained in $\pi^{-1}(x)$ for some $x \in X$ with $f(x) = d$.*

PROOF. We may construct an increasing sequence $\{A_n : n < \omega\} \subseteq [\mu]^{\leq\kappa}$ where $A_0 = B$, such that for all $U \in \mathcal{U}(A_n)$ with $f\pi h\pi_{A_n}^{-1}[U] \cap D \neq \emptyset$ there is some $e \in X^\mu$ satisfying:

(1) $f\pi h(e) = d \in D$ and $e \in \pi_{A_n}^{-1}[U]$,

(2) $h\pi_{A_{n+1}}^{-1}(e_{A_{n+1}})$ is contained in $\pi^{-1}(x)$ for some $x \in X$ with $f(x) = d$.

For (1), this follows from the fact that $f\pi h\pi_{A_n}^{-1}[U] \cap D \neq \emptyset$. For (2), we just apply Corollary 3.1.4; given A_n we may find A_{n+1} such that (2) is satisfied since $|\mathcal{U}(A_n)| \leq \kappa$ and $\pi\chi(d,f) \leq \kappa$ for all $d \in D$.

We set $A = \bigcup_{n<\omega} A_n$. Then the conditions in the theorem are satisfied since $\mathcal{U}(A) = \bigcup_{n<\omega} \pi^{-1}_{A\to A_{n+1}}\mathcal{U}(A_{n+1})$. □

Whenever A and B are subsets of μ such that $|A| = |B|$ and $|\mu \setminus A| = |\mu \setminus B|$, then there is a natural homeomorphism of X^μ that realizes a coordinate change from A to B. We denote this homeomorphism by $g_{A\to B}$. It is defined as follows. Let $g : \mu \to \mu$ be a bijection such that $g[A] = B$. Then $g_{A\to B} : X^\mu \to X^\mu$ is defined coordinate wise for $\beta < \mu$:

$$g_{A\to B}(x)_\beta = x_\alpha \overset{\text{def}}{\iff} g(\alpha) = \beta.$$

The following is a simple lemma concerning the homeomorphism $g_{A\to B}$.

3.3.2. LEMMA. *Suppose A and B are subsets of μ with $|A| = |B|$ and $|\mu \setminus A| = |\mu \setminus B|$. Then*

(1) *$\left(g_{A\to B}[\pi_A^{-1}[U]]\right)_B \in \mathcal{U}(B)$ if and only if $U \in \mathcal{U}(A)$,*

(2) *For all* $Z \subseteq X^\mu$,

$$g_{A\to B}\big[\pi_A^{-1}[Z_A]\big] = \pi_B^{-1}\big[\big(g_{A\to B}[Z]\big)_B\big].$$

Using a coordinate change, we can control the set A which is provided by Lemma 3.3.1. We make this precise in the following corollary.

3.3.3. COROLLARY. *Let D be a (dense) subset of V and suppose that $\pi\chi(d, f) \leq \kappa$ for all $d \in D$ and $|\mathcal{U}| \leq \kappa$. Let $\mu \geq \kappa$ be such that X^μ is homogeneous. Then for every $q \in \Delta(X, \mu)$ there is a homeomorphism $h_q : X^\mu \to X^\mu$ satisfying the following conditions.*

(1) $h_q(p) = q$,

(2) *For all $U \in \mathcal{U}(\kappa)$ such that $f\pi h\pi_\kappa^{-1}[U] \cap D \neq \emptyset$ there is some $e \in X^\mu$ satisfying:*

  (a) $f\pi h_q(e) = d \in D$ *and* $e \in \pi_\kappa^{-1}[U]$,

  (b) $h_q\pi_\kappa^{-1}(e_\kappa)$ *is contained in* $\pi^{-1}(x)$ *for some* $x \in X$ *with* $f(x) = d$.

PROOF. Since X^μ is Δ-homogeneous, there is a homeomorphism h of X^μ with $h(p) = q$. We apply Lemma 3.3.1 with $B = \kappa$ to obtain $A \in [\mu]^{\leq\kappa}$ with the given properties. Since $B \subseteq A$, we have $|A| = \kappa$. There are two cases to consider. First of all, if $\kappa = \mu$, then $A = \kappa$ and the conditions (2a) and (2b) are valid for h.

Next assume that $\kappa < \mu$. Then $|A| = \kappa$ and $|\mu \setminus A| = \mu = |\mu \setminus \kappa|$. Therefore we may apply the coordinate change $g_{\kappa\to A}$. We let $h_q = h \circ g_{\kappa\to A}$. Since $g_{\kappa\to A}(p) = p$, we have $h_q(p) = q$. By Lemma 3.3.2 the conditions (2a) and (2b) are valid for h_q since they are valid for h when κ is replaced by A. □

3.3.4. THEOREM. *Let $f : X \to Y$ be an open and continuous onto mapping and suppose that X is power homogeneous. If V is an open subset of Y, then*

$$|V| \leq d(V)^{\pi\chi(V,f)}.$$

PROOF. Let $\kappa = \pi\chi(V, f)$. We may assume that X^μ is homogeneous where $\mu \geq \kappa$. Let D be some dense subset of V with $|D| = d(V)$. For every $q \in \Delta(X, \mu)$ we fix a homeomorphism h_q as in the previous corollary. Since $\pi\chi(V, f) = \kappa$, we may assume that the size of the local π-base $\mathcal{U}$ is equal to κ. Then $\mathcal{U}(\kappa)$ is a local π-base at p_κ in X^κ of size equal to κ.

We define a map $H : V \to (D \cup \emptyset)^{\mathcal{U}(\kappa)}$ as follows. Whenever $y \in V$, we pick some $q \in \Delta(X, \mu)$ such that $y = f\pi(q)$. For $U \in \mathcal{U}(\kappa)$ we let

$$H(y)(U) = \begin{cases} f\pi h_q(e) & \text{if } f\pi h\pi_\kappa^{-1}[U] \cap D \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

The notation refers to that of 3.3.3. To make this definition precise for the first case, we may fix a well-ordering of X^μ and we let

$$e = \min\{e' \in X^\mu : e' \text{ satisfies conditions (2a) and (2b) of Corollary 3.3.3}\}.$$

Note that in particular $H(y)(U) \in D \cup \{\emptyset\}$ so H is well-defined. We will show that H is injective, which will complete the proof. So suppose $y, z \in V$ where $y \neq z$. We let q and r be the corresponding elements of $\Delta(X, \mu)$ chosen for the definition of H such that $y = f\pi(q)$ and $z = f\pi(r)$. In Y we may fix disjoint open neighbourhoods V_y and V_z of y and z respectively, such that $V_y \cup V_z \subseteq V$. The set W given by

$$\pi_\kappa h_q^{-1}\pi^{-1}[f^{-1}[V_y]] \cap \pi_\kappa h_r^{-1}\pi^{-1}[f^{-1}[V_z]],$$

is an open neighbourhood of p_κ in X^κ. Since $\mathcal{U}(\kappa)$ is a local π-base at p_κ, there is some $U \in \mathcal{U}(\kappa)$ which is contained in the neighbourhood W. We will prove the following claim,

CLAIM 1. $H(y)(U) \in V_y$ and $H(z)(U) \in V_z$.

PROOF OF CLAIM. We prove the statement only for y, the case for z is identical.

Since $U \subseteq W \subseteq \pi_\kappa h_q^{-1}\pi^{-1}[f^{-1}[V_y]]$ and $V_y \subseteq V$, it follows that $f\pi h\pi_\kappa^{-1}[U] \cap V \neq \emptyset$. Since f is an open map, this is an open subset of V. It follows that $f\pi h_q\pi_\kappa^{-1}[U] \cap D \neq \emptyset$ since D is dense in V. Therefore we may assume that $H(y)(U) = f\pi h_q(e) = d \in D$.

Then $e \in \pi_\kappa^{-1}[U]$, so $e_\kappa \in U$. Since $U \subseteq W \subseteq \pi_\kappa h_q^{-1}\pi^{-1}[f^{-1}[V_y]]$ it follows that

$$\pi_\kappa^{-1}(e_\kappa) \cap h_q^{-1}\pi^{-1}[f^{-1}[V_y]] \neq \emptyset,$$

and by applying h_q we have

$$h_q\pi_\kappa^{-1}(e_\kappa) \cap \pi^{-1}[f^{-1}[V_y]] \neq \emptyset.$$

Since $h_q\pi_\kappa^{-1}(e_\kappa) \subseteq \pi^{-1}(x)$ for some $x \in X$ with $f(x) = d$, it follows that $x \in f^{-1}[V_y]$ and thus $f(x) = d \in V_y$ and this proves the claim. ◀

Since $V_y \cap V_z = \emptyset$ it follows from the claim that $H(y)(U) \neq H(z)(U)$, and thus $H(y) \neq H(z)$. We have shown that H is injective and therefore $|V| \leq |D|^{|\mathcal{U}(\kappa)|} = d(V)^\kappa$. $\square$

Although we refer to Theorem 3.2.6 as Van Douwen's Theorem, he proved a slightly more general result. Van Douwen proved that if $f : X \to Y$ is an open and continuous onto mapping and X is power homogeneous with $d(X) \leq \pi w(Y)$, then the size of Y is bounded by $2^{\pi w(Y)}$. The following corollary is a variation on this result.

3.3.5. COROLLARY. *Suppose $f : X \to Y$ is an open, continuous and onto mapping. If X is power homogeneous and $\pi\chi(X) \leq \pi\chi(Y)$, then*

$$|Y| \leq d(Y)^{\pi\chi(Y)}.$$

PROOF. This follows from Theorem 3.3.4 and the fact that $\pi\chi(Y, f) \leq \pi\chi(X)$. $\square$

The following generalizes Van Douwen's Theorem and also Proposition 3.1.1. Note that an open subspace of a (power) homogeneous space need not be power homogeneous; let X be the disjoint sum of two circles in the plane. Then X is homogeneous, but if we remove any point from X then it is not even power homogeneous. To see that 'X minus a point' is not power homogeneous, simply note that every power of this space contains a compact and a non-compact component.

3.3.6. COROLLARY. *If X is an open subset of a power homogeneous space, then*

$$|X| \leq d(X)^{\pi\chi(X)}.$$

PROOF. Suppose Z is power homogeneous where X is an open subset of Z. We let $f : Z \to Z$ be the identity mapping. Since X is an open subset of Z, we have that $\pi\chi(x, Z) = \pi\chi(x, X)$ for all $x \in X$ and therefore $\pi\chi(X, f) = \pi\chi(X)$. Now apply Theorem 3.3.4. $\square$

Recall that the density of a regular space X is bounded by $\pi\chi(X)^{c(X)}$ (see Proposition 2.3.8). So it follows from the previous corollary, that if X is regular and power homogeneous, then its size is bounded by $2^{c(X)\pi\chi(X)}$. We shall now show that this statement remains true if we drop the assumption of regularity on X. Since always $c(X)\pi\chi(X) \leq \pi w(X)$, this provides us with yet another proof of Van Douwen's Theorem. We need the following lemma:

3.3.7. LEMMA. *Suppose X^μ is homogeneous where $\pi\chi(X) \leq \kappa$. Fix a point $p \in \Delta(X,\mu)$ and a local π-base $\mathcal{U}$ at $\pi(p)$ in X of size $\leq \kappa$. Then for all $x \in \Delta(X,\mu)$ there is a homeomorphism $h_x : X^\mu \to X^\mu$ such that $h_x(p) = x$ and the following conditions are satisfied;*

(1) *For all $z \in X^\mu$, if $z_\kappa = p_\kappa$ then $\pi(h_x(z)) = \pi(x)$,*

(2) *For all $U \in \mathcal{U}(\kappa)$, there is a point $q(U) \in \pi_\kappa^{-1}[U]$ and a basic open neighbourhood U_x of $h_x(q(U))_\kappa$ in X^κ such that;*

  (a) *$q(U)_\alpha = p_\alpha$ for all $\alpha \in \mu \setminus \kappa$,*
  (b) *$\pi_\kappa^{-1}[U_x] \subseteq h_x[\pi_\kappa^{-1}[U]]$.*

PROOF. Since X^μ is homogeneous we pick $h : X^\mu \to X^\mu$ such that $h(p) = x$. Applying Corollary 3.1.4 we find $A \in [\mu]^{\leq\kappa}$ such that (1) is satisfied for A instead of κ. Next for all $U \in \mathcal{U}(A)$, we pick $q(U) \in \pi_A^{-1}[U]$ as in (2a), where κ is replaced by A. For (2b), we may just pick a basic open neighbourhood of $h(q(U))$ in X^μ which is contained in $h[\pi_A^{-1}[U]]$. Since $|\mathcal{U}(A)| \leq \kappa$, we obtain a set B of at most κ many coordinates such that all the basic open sets obtained in this way depend only on the coordinates in B.

By applying suitable coordinate changes, we obtain h_x as required. □

We point out that for a fixed p, the points $q(U)$ from the previous lemma depend on x. In the proof of the following theorem we will not write $q(x,U)$ to express this dependence because we only consider points of the form $h_x(q(U))$. We will implicitly assume that in this notation, the point $q(U)$ is the point $q(x,U)$.

3.3.8. THEOREM. *If X is power homogeneous, then $|X| \leq 2^{\pi\chi(X)c(X)}$.*

PROOF. We let $\kappa = \pi\chi(X)c(X)$ and we fix $\mu \geq \kappa$ such that X^μ is homogeneous. We fix a point $p \in \Delta(X,\mu)$ and a local π-base $\mathcal{U}$ at $\pi(p)$ in X of size $\leq \kappa$. For every $x \in \Delta(X,\mu)$ we pick a homeomorphism $h_x : X^\mu \to X^\mu$ as in the previous lemma. For $x \in \Delta(X,\mu)$ and $U \in \mathcal{U}(\kappa)$, the open set U_x is a basic open subset of X^κ, so we may fix a collection $\{U_{x,\alpha} : \alpha \in \kappa\}$ of open subsets of X such that

$$U_x = \bigcap_{\alpha<\kappa} \pi_\alpha^{-1}[U_{x,\alpha}].$$

For every $\alpha \in \kappa$ we also fix a local π-base $\{V(x,U,\alpha,\beta) : \beta < \kappa\}$ at the point $h_x(q(U))_\alpha$ in X. We first observe the following;

CLAIM 1. Whenever $x, y \in \Delta(X, \mu)$ are different, there are some $U \in \mathcal{U}(\kappa)$ and $\alpha, \beta < \kappa$ such that

$$V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}.$$

PROOF OF CLAIM. Since $\pi(x) \neq \pi(y)$ we have that $h_y^{-1}(x)_\kappa \neq p_\kappa$. Fix an open neighbourhood W of p_κ in X^κ such that $h_y^{-1}(x)_\kappa \notin \overline{W}$ and let

$$\mathcal{W} = \{U \in \mathcal{U}(\kappa) : U \subseteq W\}.$$

Note that $\mathcal{W}$ is a local π-base at p_κ in X^κ and $h_y^{-1}(x)_\kappa \notin \operatorname{Cl} \bigcup \mathcal{W}$. So we have that;

$$x \in \operatorname{Cl} \{h_x(q(U)) : U \in \mathcal{W}\},$$

and since $\pi_\kappa^{-1}[U_y] \subseteq h_y \pi_\kappa^{-1}[U]$ for all $U \in \mathcal{W}$, we have

$$x \notin \operatorname{Cl} \bigcup \{\pi_\kappa^{-1}[U_y] : U \in \mathcal{W}\}.$$

But this means that there is some $U \in \mathcal{W}$ such that

$$h_x(q(U))_\kappa \notin \overline{U}_y.$$

Since U_y is a basic open subset of X^κ, it follows that there is some $\alpha < \kappa$ such that

$$h_x(q(U))_\alpha \notin \overline{U}_{y,\alpha}.$$

Since $\{V(x, U, \alpha, \beta) : \beta < \kappa\}$ is a local π-base at $h_x(q(U))_\alpha$ in X and $h_x(q(U))_\alpha \in U_{x,\alpha}$, we may pick $\beta < \kappa$ such that $V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}$ and this completes the proof. ◀

We now prove the desired inequality. So assume to the contrary that $|X| > 2^\kappa$. We fix a well-ordering $\prec$ on X and define a map $G : [X]^2 \to \mathcal{U}(\kappa) \times \kappa \times \kappa$ as follows; let $\{x, y\} \in [X]^2$ and assume that $x \prec y$. Applying the previous claim, we may let $G(\{x, y\}) = \langle U, \alpha, \beta \rangle$ be such that

$$V(x, U, \alpha, \beta) \subseteq U_{x,\alpha} \setminus \overline{U}_{y,\alpha}.$$

Here we have identified $\Delta(X, \mu)$ with X. Note that $|\mathcal{U}(\kappa) \times \kappa \times \kappa| = \kappa$. Since $|X| > 2^\kappa$, we apply the Erdös-Rado Theorem (Theorem 2.1.1) to find $Y \subseteq X$ and $\langle U, \alpha, \beta \rangle \in \mathcal{U}(\kappa) \times \kappa \times \kappa$ such that $|Y| = \kappa^+$ and for all $\{x, y\} \in [Y]^2$, $G(\{x, y\}) = \langle U, \alpha, \beta \rangle$. By possibly removing the $\prec$-largest element from Y, we may assume that for all $y \in Y$, $V(y, U, \alpha, \beta) \subseteq U_{y,\alpha}$.

Consider the collection $\mathcal{C} = \{V(x, U, \alpha, \beta) : x \in Y\}$ of open subsets of X^κ. If $x, y \in Y$ are different with $x \prec y$, then we have

$$V(x, U, \alpha, \beta) \cap \overline{U}_{y,\alpha} = \emptyset \qquad \text{and} \qquad V(y, U, \alpha, \beta) \subseteq U_{y,\alpha},$$

and therefore $V(x, U, \alpha, \beta)$ and $V(y, U, \alpha, \beta)$ are disjoint. But this means that the collection $\mathcal{C}$ consists of pairwise disjoint open subsets of X. Since $|\mathcal{C}| = |Y| = \kappa^+$ and $c(X) \leq \kappa$, this is impossible. □

If X is a homogeneous space, then it is obviously homogeneous with respect to π-character. In particular, if $\pi\chi(x, X) \leq \kappa$ for some $x \in X$, then the π-character of all points of X does not exceed κ. If we assume instead that X is power homogeneous, then this result is no longer true. However, if the set D consisting of all points x such that $\pi\chi(x, X) \leq \kappa$ is dense in X, then the π-character of all points of X does not exceed κ.

3.3.9. PROPOSITION. *Suppose X is power homogeneous and let D be a dense subset of X. If $\pi\chi(d, X) \leq \kappa$ for all $d \in D$, then for all $x \in X$ there is some $E \subseteq D$ such that $|E| \leq \kappa$ and $x \in \overline{E}$.*

PROOF. We may assume without loss of generality that X^μ is homogeneous where $\mu \geq \kappa$. Let $x \in \Delta(X, \mu)$ be arbitrary. We may find a homeomorphism $h : X^\mu \to X^\mu$ such that $h(p) = x$ where we assume that $p \in \Delta(D, \mu)$. Since $\pi(p) \in D$, we have $\pi\chi(\pi(p), X) \leq \kappa$, so we may find a local π-base $\mathcal{U}$ at $\pi(p)$ in X such that $|\mathcal{U}| \leq \kappa$. Since D is dense in X, we apply Lemma 3.3.1 to find a non-empty set $A \in [\mu]^{\leq\kappa}$ such that whenever $U \in \mathcal{U}(A)$ there is some $e \in X^\mu$ satisfying:

(1) $\pi h(e) = d \in D$ and $e \in \pi_A^{-1}[U]$,

(2) $h\pi_A^{-1}(e_A)$ is contained in $\pi^{-1}(d)$.

For every $U \in \mathcal{U}(A)$ we pick an element e satisfying these conditions. We let the set E consist of all elements of the form $\pi h(e)$ obtained in this way. By construction we have $E \subseteq D$ and since $|A| \leq \kappa$, we also have that $|E| \leq \kappa$. It remains to show that $\pi(x) \in \overline{E}$.

So let V be an arbitrary open neighbourhood of $\pi(x)$ in X. Then $p \in h^{-1}\pi^{-1}[V]$ and therefore the set W which is given by $\pi_A h^{-1}\pi^{-1}[V]$ is an open neighbourhood of p_A in X^A. Since $\mathcal{U}(A)$ is a local π-base at p_A, we may find $U \in \mathcal{U}(A)$ such that $U \subseteq W$. By construction, there is an element $e \in \pi_A^{-1}[U]$ satisfying conditions (1) and (2) such that $\pi h(e) \in E$.

Since $e_A \in U \subseteq W = \pi_A h^{-1}\pi^{-1}[V]$, it follows that

$$\pi_A^{-1}(e_A) \cap h^{-1}\pi^{-1}[V] \neq \emptyset.$$

and therefore

$$h\pi_A^{-1}(e_A) \cap \pi^{-1}[V] \neq \emptyset.$$

But $h\pi_A^{-1}(e_A) \subseteq \pi^{-1}(\pi h(e))$, so it follows that $\pi h(e) \in V$ and therefore $E \cap V \neq \emptyset$. Since V was an arbitrary open neighbourhood of $\pi(x)$, we have shown that $\pi(x) \in \overline{E}$ and this completes the proof. □

3.3.10. COROLLARY. *Suppose X is power homogeneous and let κ be some infinite cardinal. If the set $D = \{x \in X : \pi\chi(x, X) \leq \kappa\}$ is dense in X, then $\pi\chi(X) \leq \kappa$.*

PROOF. For every $d \in D$, we may fix a local π-base $\mathcal{U}_d$ at d in X such that $|\mathcal{U}_d| \leq \kappa$. For $E \subseteq D$, we let $\mathcal{U}_E = \bigcup\{\mathcal{U}_d : d \in E\}$. So if $|E| \leq \kappa$, then $|\mathcal{U}_E| \leq \kappa$. Now let $x \in X$ be arbitrary. By Proposition 3.3.9 we may find $E \subseteq D$ such that $|E| \leq \kappa$ and $x \in \overline{E}$. We will show that $\mathcal{U}_E$ is a local π-base at x in X. For let V be an arbitrary open neighbourhood of x. Since $x \in \overline{E}$, there is some $d \in E$ such that $d \in V$. Since $\mathcal{U}_d$ is a local π-base at d, we may find $U \in \mathcal{U}_d \subseteq \mathcal{U}_E$ such that $U \subseteq V$. Since V was arbitrary, this completes the proof. □

The final result of this section may be summarized as follows; if we consider the π-character to be small, then for power homogeneous spaces X it is either the case that the pseudo character of X is small, or otherwise X is homogeneous with respect to pseudo character.

3.3.11. PROPOSITION. *Suppose $X = \prod\{X_i : i \in I\}$ is homogeneous. Suppose further that for all $i \in I$, X_i contains some G_κ-point. If for some $j \in I$, $\pi\kappa\chi(X_j) \leq \kappa$ then $\psi(X_j) \leq \kappa$.*

PROOF. For all $i \in I$, let $e_i \in X_i$ be such that $\psi(e_i, X_i) \leq \kappa$. Let $z \in X_j$ be arbitrary. Choose $x \in X$ with $x_j = z$ and let $h : X \to X$ be some homeomorphism with $h(x) = e$, where of course e is the point of X whose i^{th} coordinate is e_i. By Theorem 3.1.3 there is some set $A \in [I]^{\leq\kappa}$ such that

$$\pi_j h^{-1}\pi_A^{-1}(e_A) = \{z\}.$$

Let $B = I \setminus \{j\}$ and $Y = \{y \in X : y_B = x_B\}$. Then $\pi_j{\restriction}Y : Y \to X_j$ is a homeomorphism. Since $G = h^{-1}\pi_A^{-1}(e_A)$ is a G_κ-subset of X, the set $G \cap Y$ is a G_κ-subset of Y. Since $G \cap Y = \{x\}$, it follows that $\{z\}$ is a G_κ-subset of X_j. □

3.3.12. COROLLARY. *If X is power homogeneous and $\pi\kappa\chi(X) \le \kappa$, then either $\psi(X) \le \kappa$ or X is homogeneous with respect to ψ-character.*

PROOF. Let $\mu = \min\{\psi(x, X) : x \in X\}$. If $\mu \le \kappa$, then $\psi(X) \le \kappa$ by Proposition 3.3.11. If $\mu > \kappa$, then it follows from Proposition 3.3.11 that $\psi(X) \le \mu$. In this case it follows from the choice of μ that $\psi(x, X) = \mu$ for all $x \in X$. □

3.4 Compact spaces

It was asked by Arhangel′skiĭ whether the cardinality of homogeneous countably tight compact spaces is bounded by $\mathfrak{c}$. This question, which has remained open for over 30 years, was answered positively by DE LA VEGA [74]. In fact, De la Vega proved that if the G_κ-density of a compact space X does not exceed κ, then its density is bounded by 2^κ. It follows from this result that if X is compact and homogeneous, then the cardinality of X is bounded by $2^{t(X)}$. To see this, recall that if X is a compact homogeneous space with $t(X) = \kappa$, then it contains some point e at which the G_κ-density does not exceed κ (see Proposition 2.4.10). By homogeneity it follows that the G_κ-density of X does not exceed κ at all points. Since $\pi\chi(X) \le t(X)$ for compact spaces, an application of Corollary 3.3.6 and De la Vega's result give that the cardinality of X is bounded by 2^κ.

In this section we will prove this cardinality bound also for power homogeneous compact spaces. By De la Vega's results, it suffices to prove that in a power homogeneous compact space X with $t(X) = \kappa$, the G_κ-density of X does not exceed κ at all points. We will show this below. Next we will present a proof of De la Vega's result.

We fix a product space $X = \prod\{X_i : i \in I\}$ and some collection $\{S_i : i \in I\}$ where $S_i \subseteq X_i$ for all $i \in I$. We set $S = \prod\{S_i : i \in I\} \subseteq X$ and for each $i \in I$, we pick some $s_{0,i} \in S_i$. Whenever $A \in [I]^{\le\kappa}$ we let

$$\mathcal{S}(A) = \big\{x \in S : \{i : x_i \ne s_{0,i}\} \text{ is finite}\big\}.$$

This is also called the σ-product of the S_i's with base point $(s_{0,i})_i$. We will need the following lemma, its proof is straightforward.

3.4.1. LEMMA. *If $A \in [I]^{\le\kappa}$ and for all $i \in A$, $|S_i| \le \kappa$ then $|\mathcal{S}(A)| \le \kappa$ and $\mathcal{S}(A)_A$ is dense in S_A. Furthermore if $A \subseteq B$, then $\mathcal{S}(A) \subseteq \mathcal{S}(B)$. If (A_n) is an increasing sequence of infinite subsets of I and $A = \bigcup_n A_n$ then*

$$\mathcal{S}(A) = \textstyle\bigcup_n \mathcal{S}(A_n). \qquad \square$$

3.4.2. THEOREM. *Let* $X = \prod\{X_i : i \in I\}$ *and* $S = \prod\{S_i : i \in I\} \subseteq X$ *where* $|S_i| \leq \kappa$ *for all* $i \in I$. *Suppose that for some* $j \in I$, $\pi\kappa\chi(X_j) \leq \kappa$. *If* $h : X \to X$ *is a homeomorphism and* $B \in [I]^{\leq\kappa}$ *then there is a set* $A \in [I]^{\leq\kappa}$ *such that* $B \subseteq A$ *and for all* $s \in \mathcal{S}(A)$ *and all* $x \in X$;

$$(*) \qquad x_A = s_A \Rightarrow h(s)_j = h(x)_j.$$

PROOF. Set $A_0 = B$. By applying Theorem 3.1.3 to all members of the set $h[\mathcal{S}(A_n)]$, we get a set $A_{n+1} \in [I]^{\leq\kappa}$ such that for all $s \in \mathcal{S}(A_n)$ and all $x \in X$;

$$x_{A_{n+1}} = s_{A_{n+1}} \Rightarrow h(s)_j = h(x)_j.$$

We may assume that $A_n \subseteq A_{n+1}$. If we let $A = \bigcup_n A_n$, then $(*)$ follows from Lemma 3.4.1. □

3.4.3. COROLLARY. *Let* $X = \prod\{X_i : i \in I\}$. *Suppose that* X *is homogeneous and for all* $i \in I$, *the* G_κ*-density of some point of* X_i *does not exceed* κ. *If for some* $j \in I$, $\pi\kappa\chi(X_j) \leq \kappa$ *then the* G_κ*-density of all points of* X_j *does not exceed* κ.

PROOF. For each $i \in I$, fix some point $e_i \in X_i$, a closed G_κ-subset $H_i \subseteq X_i$ and $S_i \in [X_i]^{\leq\kappa}$ such that $e_i \in H_i \subseteq \overline{S}_i$. By H we denote the set $\prod\{H_i : i \in I\}$.

Let $z \in X_j$ be arbitrary and pick $w \in X$ with $w_j = z$. Since X is homogeneous, there is a homeomorphism h of X with $h(e) = w$, where of course e is the point of X whose i^{th} coordinate is e_i. By the previous Theorem and Lemma 3.4.1 we get a set $A \in [I]^{\leq\kappa}$ and a subset $\mathcal{S}$ of X with $|\mathcal{S}| \leq \kappa$, such that

(1) $e_A \in H_A \subseteq \overline{\mathcal{S}_A}$,

(2) For all $s \in \mathcal{S}$ and all $x \in X$,

$$x_A = s_A \Rightarrow h(x)_j = h(s)_j.$$

For every $s \in \mathcal{S}$, the set $\pi_j h \pi_A^{-1}(s_A)$ consists of the single point $h(s)_j$. We let $T = \{h(s)_j : s \in \mathcal{S}\}$. Since $H_A \subseteq \overline{\mathcal{S}_A}$ and π_A is open, we have

$$\pi_A^{-1}[H_A] \subseteq \overline{\pi_A^{-1}[\mathcal{S}_A]}.$$

It follows that

$$\pi_j h[\pi_A^{-1}[H_A]] \subseteq \operatorname{Cl} \pi_j h \pi_A^{-1}[\mathcal{S}_A] = \overline{T}.$$

Let $B = I \setminus \{j\}$ and $Y = \{y \in X : y_B = w_B\}$. Then $\pi_j{\restriction}Y : Y \to X_j$ is a homeomorphism. Now let $K = Y \cap h[\pi_A^{-1}[H_A]]$. Then K is a closed G_κ-subset of Y which contains w. We let $F = \pi_j[K]$. Then F is a closed G_κ-subset of X_j and we have

$$z = w_j \in F = \pi_j[K] \subseteq \pi_j h[\pi_A^{-1}[H_A]] \subseteq \overline{T}.$$

Then $z \in F \subseteq \overline{T}$ is the desired statement. □

3.4.4. COROLLARY. *Suppose $X = \prod\{X_i : i \in I\}$ is homogeneous and suppose that $t(X_i)\mathrm{pct}(X_i) \leq \kappa$ for each $i \in I$. Then for all $i \in I$, the G_κ-density of X_i does not exceed κ.*

PROOF. It follows from Corollary 2.4.11 that for every $i \in I$, the G_κ-density does not exceed κ at some point in X_i. By Corollary 2.4.4 we have that $\pi\kappa\chi(X_i) \leq \kappa$ for all $i \in I$. Since X is homogeneous, it follows from Corollary 3.4.3 that for all $i \in I$, the G_κ-density of X_i does not exceed κ. □

For power homogeneous spaces we obtain the following.

3.4.5. COROLLARY. *Suppose X is power homogeneous with $t(X)\mathrm{pct}(X) \leq \kappa$. Then the G_κ-density of X does not exceed κ.* □

It follows from the previous corollary that if X is a discrete space of size $\geq \omega_1$, then its one-point compactification, αX, is not power homogeneous. Using a different argument, this was also noted by VAN DOUWEN [19]. We will now present a proof of De la Vega's Theorem from [74]. We shall prove this theorem by using a classical closing-off argument. We first prove some preliminary results.

3.4.6. LEMMA. *Suppose G is a closed G_κ-subset of X and $S \in [X]^{\leq\kappa}$ is such that $G \subseteq \overline{S}$. Then there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$, such that $\mathcal{B}$ is a local ψ-base for every point $x \in G$.*

PROOF. Let $G = \bigcap_{\alpha<\kappa} G_\alpha$, where $G_\alpha \subseteq X$ is open. The collection $\mathcal{B}$ is given by

$$\mathcal{B} = \{G_\alpha : \alpha < \kappa\} \cup \{X \setminus \overline{C} : C \subseteq S\}.$$

Then $|\mathcal{B}| \leq \kappa + 2^\kappa = 2^\kappa$ and it is easily verified that $\mathcal{B}$ is as desired. □

3.4.7. LEMMA. *Suppose G is a compact subset of X with $\chi(G, X) \leq 2^\kappa$ and $w(G) \leq 2^\kappa$. Then there is a collection $\mathcal{B}$ such that $\mathcal{B}$ is a local neighbourhood basis in X for all points of G and $|\mathcal{B}| \leq 2^\kappa$.*

PROOF. We define a collection $\mathcal{V}$ of open subsets of X as follows. Let $\mathcal{E} \subseteq \mathcal{P}(G)$ be a basis for G with $|\mathcal{E}| \leq 2^\kappa$. We assume that $\mathcal{E}$ is closed under finite unions. For all $E, F \in \mathcal{E}$ with $\overline{E} \cap \overline{F} = \emptyset$ we may fix disjoint open sets U_{EF} and V_{EF} in X such that $\overline{E} \subseteq U_{EF}$ and $\overline{F} \subseteq V_{EF}$. This does not require normality of X, it suffices to know that X is Hausdorff and G is compact. We let

$$\mathcal{V} = \{V_{EF} : E, F \in \mathcal{E},\ \overline{E} \cap \overline{F} = \emptyset\}.$$

Let $\mathcal{W}$ be a collection of open subsets of X such that $\mathcal{W}$ is a neighbourhood basis for G in X and $|\mathcal{W}| \leq 2^\kappa$. We define

$$\mathcal{U} = \{W \cap V : W \in \mathcal{W},\ V \in \mathcal{V}\}.$$

We will show that $\mathcal{U}$ is the required collection. So let U be an arbitrary open neighbourhood of some element $x \in G$. Since $G \setminus U$ is compact and the fact that $\mathcal{E}$ is closed under finite unions, we may find members E and F of $\mathcal{E}$ with disjoint closures such that $G \setminus U \subseteq E$ and $x \in F$.

Since $G \subseteq U_{EF} \cup U$ we may find $W \in \mathcal{W}$ such that $W \subseteq U_{EF} \cup U$. Clearly, $W \cap V_{EF}$ is an open neighbourhood of x. Furthermore, since $W \subseteq U_{EF} \cup U$ and $U_{EF} \cap V_{EF} = \emptyset$, it follows that $W \cap V_{EF} \subseteq U$. This completes the proof. □

3.4.8. LEMMA. *Suppose* $\operatorname{pct}(X) \leq \kappa$ *and* $nw(X) \leq 2^\kappa$*, then* $w(X) \leq 2^\kappa$*.*

PROOF. If $\mathcal{N}$ is a network in X with $|\mathcal{N}| \leq 2^\kappa$, we may assume without loss of generality that $\mathcal{N}$ is closed under κ-intersections. Thus if G is a G_κ-subset of X and $x \in G$, then there is some $N \in \mathcal{N}$ such that $x \in N \subseteq G$.

Therefore since $\operatorname{pct}(X) \leq \kappa$ and $nw(X) \leq 2^\kappa$, we may fix a cover $\mathcal{G}$ of compact subsets G of X with $\chi(G, X) \leq 2^\kappa$ such that $|\mathcal{G}| \leq 2^\kappa$. Note that $w(G) = nw(G)$ for $G \in \mathcal{G}$, cf. JUHÁSZ [34, 2.8 & 3.11]. The conclusion now follows from Lemma 3.4.7 since if $G \in \mathcal{G}$, then $w(G) = nw(G) \leq nw(X) \leq 2^\kappa$. □

3.4.9. LEMMA. *Suppose X is a regular space with* $t(X)\operatorname{pct}(X) \leq \kappa$*. Then for any closed subspace Y of X with* $d(Y) \leq 2^\kappa$*, we have* $w(Y) \leq 2^\kappa$*.*

PROOF. Let D be dense in Y with $|D| \leq 2^\kappa$. Note that Y is regular and since it is a closed subset of X, we have $t(Y) \leq \kappa$. Let $\mathcal{F}$ be the collection given by

$$\mathcal{F} = \{\overline{A} : A \in [D]^{\leq\kappa}\}.$$

Then $\mathcal{F}$ forms a network for Y. To see this, suppose that U is an open subset of Y and let $x \in U$ be arbitrary. Since Y is regular and $t(Y) \leq \kappa$, we may find

$A \in [D]^{\leq\kappa}$ such that $x \in \overline{A} \subseteq U$. This shows that $\mathcal{F}$ forms a network for Y. Since $|\mathcal{F}| \leq 2^\kappa$, it follows that $nw(Y) \leq 2^\kappa$. Since Y is closed in X, it is also the case that $\mathrm{pct}(Y) \leq \kappa$. The conclusion follows from Lemma 3.4.8. □

3.4.10. LEMMA. *Let X be a topological space with $w(X) \leq 2^\kappa$ and $L(X) \leq \kappa$. Then the cardinality of the family of all closed G_κ-subsets of X does not exceed 2^κ.*

PROOF. Let $\mathcal{B}$ be a base for X of size $\leq 2^\kappa$. We may assume without loss of generality that $\mathcal{B}$ is closed under unions of length κ. If H is a closed G_κ-subset of X, then there is a subfamily $\mathcal{U}$ of $\mathcal{B}$ with $|\mathcal{U}| \leq \kappa$, such that $H = \bigcap \mathcal{U}$. This follows from the fact that $L(X) \leq \kappa$. Thus we may define an injection from the family of all closed G_κ-subsets of X into $\mathcal{B}^\kappa$ and this completes the proof. □

3.4.11. LEMMA. *Suppose X is a topological space with $t(X)\mathrm{pct}(X)\,L(X) \leq \kappa$. Suppose further that the G_κ-density of X does not exceed κ. If Y is a closed subspace of X with $d(Y) \leq 2^\kappa$ then there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \leq 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points of Y.*

PROOF. Since the G_κ-density of X does not exceed κ, we may find a cover $\mathcal{G}$ of Y which consists of closed G_κ-subsets of X such that every member of $\mathcal{G}$ is contained in the closure of some member of $[X]^{\leq\kappa}$. By Lemma 3.4.10 and Lemma 3.4.9 we may assume that $|\mathcal{G}| \leq 2^\kappa$. By Lemma 3.4.6 we find for every $G \in \mathcal{G}$ a ψ-base $\mathcal{B}_G$ for all points of G such that $|\mathcal{B}_G| \leq 2^\kappa$. Then $\mathcal{B} = \bigcup\{\mathcal{B}_G : G \in \mathcal{G}\}$ is a ψ-base in X for all points of Y and clearly $|\mathcal{B}| \leq 2^\kappa$. □

We now come to the closing-off argument. This kind of argument was first used by POL [54] to give a proof of Arhangel′skiĭ's Theorem from [2]. The following proof is due to DE LA VEGA [73, 74] although he used a different terminology. In the proof of the next theorem, we recursively construct a subspace Y of X. The term 'closing-off' argument refers to the fact that at every stage of the recursion we add sufficiently many points to Y to make sure that in the end Y satisfies certain closure conditions. By choosing these conditions appropriately, we will be able to show that in fact Y equals X.

In the original proof of De la Vega, the subspace Y is constructed by taking an elementary submodel. This construction will guarantee that Y is closed under 'all possible closure conditions'. Below we only make sure that Y is closed under certain conditions that are necessary to carry out the proof. Although the use of elementary submodels provides a general framework for carrying out closing-off arguments, I have chosen to include the following proof because it does not require any additional knowledge of logic.

3.4.12. THEOREM. *Suppose X is a regular space with $t(X)\mathrm{pct}(X)\,L(X) \le \kappa$. Suppose further that the G_κ-density of X does not exceed κ. Then $w(X) \le 2^\kappa$.*

PROOF. By transfinite recursion we construct an increasing sequence $\{Y_\alpha : \alpha < \kappa^+\}$ of closed subspaces of X and an increasing sequence $\{\mathcal{P}_\alpha : \alpha < \kappa^+\}$ of families of open subsets of X such that the following conditions are satisfied for all $\alpha < \kappa^+$.

(1) $\mathcal{P}_\alpha$ is a local ψ-base for all points of Y_α,

(2) $|\mathcal{P}_\alpha| \le 2^\kappa$,

(3) $d(Y_\alpha) \le 2^\kappa$,

(4) Whenever $\mathcal{U}$ is a subfamily of $\mathcal{P}_\alpha$ of size $\le \kappa$ which covers Y_α such that $X \setminus \bigcup \mathcal{U}$ is non-empty, then $Y_{\alpha+1} \setminus \bigcup \mathcal{U}$ is non-empty.

We put $Y_0 = \emptyset$ and $\mathcal{P}_0 = \emptyset$. Take any $\beta < \kappa^+$ and suppose that Y_α and $\mathcal{P}_\alpha$ have been defined for all $\alpha < \beta$. Then we proceed as follows.

Case 1: β is a limit ordinal. Put $Y_\beta = \mathrm{Cl}_X \bigcup\{Y_\alpha : \alpha < \beta\}$. Then by (3), the density of Y_β does not exceed 2^κ. It follows from 3.4.11 that there is a collection $\mathcal{B}$ of open subsets of X with $|\mathcal{B}| \le 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points of Y_β. Now let $\mathcal{P}_\beta = \mathcal{B} \cup \bigcup\{\mathcal{P}_\alpha : \alpha < \beta\}$.

Case 2: $\beta = \alpha+1$ for some $\alpha < \omega_1$. Let $\mathcal{E}_\alpha$ be the collection of all members $\mathcal{U}$ of $[\mathcal{P}_\alpha]^{\le\kappa}$ such that $\mathcal{U}$ covers Y_α and $X \setminus \bigcup \mathcal{U}$ is non-empty. Since $|\mathcal{P}_\alpha| \le 2^\kappa$, the cardinality of the collection $\mathcal{E}_\alpha$ does not exceed 2^κ. For each $\mathcal{U} \in \mathcal{E}_\alpha$, pick $c(\mathcal{U}) \in X \setminus \bigcup \mathcal{U}$, and put

$$Y_\beta = \mathrm{Cl}_X\{c(\mathcal{U}) : \mathcal{U} \in \mathcal{E}_\alpha\} \cup Y_\alpha.$$

The density of Y_β does not exceed 2^κ, so we may find a collection $\mathcal{B}$ with $|\mathcal{B}| \le 2^\kappa$ such that $\mathcal{B}$ is a local ψ-base for all points $y \in Y_\beta$. We put $\mathcal{P}_\beta = \mathcal{B} \cup \mathcal{P}_\alpha$.

This completes the recursion. We put $Y = \bigcup\{Y_\alpha : \alpha < \kappa^+\}$. Then clearly $d(Y) \le \kappa^+ \cdot 2^\kappa = 2^\kappa$. Furthermore, since the sequence $\{Y_\alpha : \alpha < \kappa^+\}$ is increasing and $t(X) \le \kappa$, it follows that Y is closed in X. We will show that $X = Y$.

Assume to contrary that $x \in X \setminus Y$. Put $\mathcal{P} = \bigcup\{\mathcal{P}_\alpha : \alpha < \kappa^+\}$. Clearly, $\mathcal{P}$ is a ψ-base for all points $y \in Y$. Therefore, there exists a cover $\mathcal{U}$ of Y

which consists of members of $\mathcal{P}$ such that $x \notin \bigcup \mathcal{U}$. Since $L(Y) \leq \kappa$ we may assume that $|\mathcal{U}| \leq \kappa$. It follows that $\mathcal{U} \subseteq \mathcal{P}_\alpha$ for some $\alpha < \kappa^+$. But then by condition (4), it follows that $\mathcal{U}$ does not cover $Y_{\alpha+1}$, which contradicts the fact that $\mathcal{U}$ covers Y. It follows that $X = Y$.

Since $d(Y) \leq 2^\kappa$, it follows from Lemma 3.4.9 that $w(X) = w(Y) \leq 2^\kappa$. □

3.4.13. COROLLARY. *If $X = \prod\{X_i : i \in I\}$ is homogeneous and $t(X_i)\mathrm{pct}(X_i) \leq \kappa$ for all $i \in I$, then whenever X_j is regular,*

$$w(X_j) \leq 2^{\kappa \cdot L(X_j)}.$$

PROOF. It follows from Corollary 3.4.4 that for all $i \in I$, the G_κ-density of X_i does not exceed κ. If in addition X_j is regular, then an application of Theorem 3.4.12 yields the inequality. □

3.4.14. COROLLARY. *If X is a power homogeneous regular space then*

$$|X| \leq 2^{t(X)\mathrm{pct}(X)L(X)}.$$

PROOF. Since X is a power homogeneous regular space, it follows from Corollary 3.4.13 that

$$d(X) \leq w(X) \leq 2^{t(X)\mathrm{pct}(X)L(X)}.$$

Furthermore, $\pi\chi(X) \leq t(X)\mathrm{pct}(X)$ by Corollary 2.4.4. So from Corollary 3.3.6 it follows that

$$|X| \leq d(X)^{\pi\chi(X)} \leq 2^{t(X)\mathrm{pct}(X)L(X)}.$$ □

3.4.15. COROLLARY. *If X is a power homogeneous compact space, then*

$$|X| \leq 2^{t(X)}.$$

PROOF. Since X is compact, $\mathrm{pct}(X)L(X) = \aleph_0$. Now apply Corollary 3.4.14. □

We now present some more homogeneity results for local cardinal functions on compact power homogeneous spaces. This is a continuation of such results for power homogeneous spaces from the final part of the previous section.

3.4.16. PROPOSITION. *Suppose* $X = \prod\{X_i : i \in I\}$ *is homogeneous. Suppose further that for all* $i \in I$*,* X_i *contains some point* e_i *with* $\chi(e_i, X_i) \leq \kappa$*. If for some* $j \in I$*,* $\pi\kappa\chi(X_j) \leq \kappa$ *and* X_j *is of pointwise countable type then* $\chi(X_j) \leq \kappa$*.*

PROOF. It follows from Proposition 3.3.11 that $\psi(X_j) \leq \kappa$. Since X_j is of pointwise countable type, it follows from Corollary 2.4.6 that $\psi(X_j) = \chi(X_j)$. ☐

3.4.17. PROPOSITION. *Suppose* X *is a power homogeneous space of pointwise countable type with* $\pi\kappa\chi(X) \leq \kappa$*. Then either* $\chi(X) \leq \kappa$ *or* X *is homogeneous with respect to character.*

PROOF. By Corollary 2.4.6, $\psi(x, X) = \chi(x, X)$ for all $x \in X$. So X is homogeneous with respect to ψ-character if and only if it is homogeneous with respect to character. Now apply Corollary 3.3.12. ☐

3.4.18. COROLLARY (GCH). *If* X *is a power homogeneous compactum then* $\chi(X) = t(X)$*.*

PROOF. Since $|X| \leq 2^{t(X)}$, it follows from the Čech-Pospišil Theorem (Theorem 2.5.1) and GCH that for some $e \in X$, $\chi(e, X) \leq t(X)$. Since $\pi\chi(X) \leq t(X)$, it follows from Proposition 3.4.17 that $\chi(X) \leq t(X)$. ☐

3.4.19. COROLLARY ($\mathfrak{c} < 2^{\aleph_1}$). *A power homogeneous compactum has countable tightness if and only if it is first-countable.* ☐

3.4.20. COROLLARY. *If* X *is a power homogeneous compactum, then assuming* GCH, $\chi(X) \leq c(X)\pi\chi(X)$*. In particular, assuming* $\mathfrak{c} < 2^{\aleph_1}$*, every power homogeneous compactum of countable* π*-weight is first-countable.*

PROOF. By Theorem 3.3.8 we have $|X| \leq 2^{c(X)\pi\chi(X)}$. We may repeat the proof of Corollary 3.4.18 to conclude that $\chi(X) \leq c(X)\pi\chi(X)$. ☐

3.4.21. LEMMA. *Suppose* X *is a power homogeneous compactum and* $t(p, X) \leq \kappa$ *for some* $p \in X$*. Then for all* $x \in X$*, if* $\pi\chi(x, X) \leq \kappa$*, then* $t(x, X) \leq \kappa$*.*

PROOF. Let μ be large enough so that X^μ is homogeneous. We will abuse notation and assume that $p, x \in \Delta(X, \mu)$ and $t(\pi(p), X) \leq \kappa$ and $\pi\chi(\pi(x), X) \leq \kappa$. Since $\Delta(X, \mu)$ is homeomorphic to X, it suffices to show that if $Y \subseteq \Delta(X, \mu)$ and $x \in \overline{Y}$, then $x \in \overline{Z}$ for some $Z \in [Y]^{\leq\kappa}$.

Fix a homeomorphism h of X^μ such that $h(p) = x$. By Corollary 3.1.4 and the fact that $\pi\chi(\pi(x), X) \leq \kappa$, we may find a set of coordinates $A \subseteq \mu$ such that $|A| \leq \kappa$ and

$$h\pi_A^{-1}(p_A) \subseteq \pi^{-1}(\pi(x)).$$

Since X is compact and $|A| \leq \kappa$, it follows that $t(p_A, X^A) \leq \kappa$ (see for example JUHÁSZ [34, 5.9]). But π_A is continuous and therefore p_A is contained in the closure of $\pi_A h^{-1}(Y)$ in X^A. Therefore we may find a set $W \subseteq h^{-1}(Y)$ with $|W| \leq \kappa$ and $p_A \in \overline{\pi_A[W]}$. Let $Z = h[W] \subseteq Y$. By compactness, it follows that

$$\pi_A^{-1}(p_A) \cap \overline{W} \neq \emptyset.$$

and therefore

$$h\pi_A^{-1}(p_A) \cap \overline{Z} \neq \emptyset.$$

Since $\left(h\pi_A^{-1}(p_A)\right) \cap \Delta(X, \mu) = \{x\}$ and $Z \subseteq Y \subseteq \Delta(X, \mu)$ it must be the case that $x \in \overline{Z}$. Since $|Z| = |W| \leq \kappa$ and $Z \subseteq Y$ this completes the proof. □

3.4.22. COROLLARY. *Suppose X is a power homogeneous compactum. Then either $t(X) = \pi\chi(X)$ or X is homogeneous with respect to tightness.*

PROOF. We have $\pi\chi(X) \leq t(X)$ by Theorem 2.4.2. Let $\mu = \min\{t(x, X) : x \in X\}$. If $\mu \leq \pi\chi(X)$, then $t(X) \leq \pi\chi(X)$ by Lemma 3.4.21. If $\mu > \kappa$, then it follows from Lemma 3.4.21 that $t(x, X) = \mu$ for all $x \in X$. □

Unlike all the other results in this chapter, the final theorem of this section is not about power homogeneous spaces. A continuous function $\mu : X^3 \to X$ which satisfies the equalities $\mu(x, y, y) = x = \mu(y, y, x)$ for all $x, y \in X$ is called a Mal'tsev function. If there is a Mal'tsev function on a space X, then X is called a Mal'tsev space. For more information on Mal'tsev spaces we refer the reader to the introduction of Chapter 4. For now we note that every compact Mal'tsev space is a retract of a topological group. This was proved by SIPACHEVA [66]. We shall call a space retral if it is a retract of a topological group.

It was shown by ISMAIL [33] that if G is a topological group, then $\chi(G) = \pi\chi(G)$. We shall prove below that this equality is also valid for Mal'tsev spaces of point-countable type. For compact Mal'tsev spaces, this follows from known results because USPENSKIĬ [72] proved that every compact retral space is dyadic and GERLITS [28] proved that in dyadic spaces weight and π-character coincide. We first prove the following lemma.

3.4.23. LEMMA. *Suppose that X is a compact subset of a Mal'tsev space Z. Then for all $x \in X$, $\chi(x, X) \leq \pi\chi(x, Z)$.*

PROOF. Let Z be a Mal'tsev space such that $X \subseteq Z$ and fix a Mal'tsev function μ on Z. All closures are taken in Z. Fix a point $x \in X$ and a local π-base $\mathcal{B}$ at x in Z. For every $B \in \mathcal{B}$, we pick $y_B \in B$. For $y \in Z$ we define $f_y : Z \to Z$ by $f_y(z) = \mu(y, x, z)$. Note that $f_y(x) = y$ for every $y \in Z$.

Let $\mathcal{U} = \{X \cap f_{y_B}^{-1}[B] : B \in \mathcal{B}\}$. We will show that this is a local basis at x in X. Since $|\mathcal{U}| \leq |\mathcal{B}|$, this suffices to prove the theorem. Note in particular that $\mathcal{U}$ is a collection of open neighbourhoods of x in X.

So let U be some open neighbourhood of x in X. Then $X \setminus U$ is a compact subset of Z which misses x and therefore we may find an open neighbourhood V of x in Z such that

$$X \setminus U \subseteq Z \setminus \overline{V}.$$

It follows that

$$\{x\} \times \{x\} \times (X \setminus U) \subseteq \mu^{-1}[Z \setminus \overline{V}].$$

By a standard compactness argument, we may find an open neighbourhood W of x in Z such that

$$W \times W \times (X \setminus U) \subseteq \mu^{-1}[Z \setminus \overline{V}].$$

Since $\mathcal{B}$ is a local π-base at x in Z, we may find $B \in \mathcal{B}$ such that $B \subseteq W \cap V$. We prove the following claim;

CLAIM 1. If $y \in B$, then $X \cap f_y^{-1}[B] \subseteq U$.

PROOF OF CLAIM. Let $z \in X \cap f_y^{-1}[B]$ and suppose that $z \notin U$. Then $z \in X \setminus U$ and since $x, y \in W$, it follows that $(y, x, z) \in W \times W \times (X \setminus U)$. This implies that $f_y(z) = \mu(y, x, z) \in Z \setminus \overline{V}$. So we have $f_y(z) \in B \cap (Z \setminus \overline{V})$. But $B \subseteq V$ and therefore $B \cap (X \setminus \overline{V}) = \emptyset$. This is a contradiction. ◀

From the claim it follows that $X \cap f_{y_B}^{-1}[B]$ is a neighbourhood of x in X which is contained in U. Since U was arbitrary, we have proved that $\mathcal{U}$ is a local basis at x in X and this completes the proof. □

3.4.24. THEOREM. *If X is a Mal'tsev space, then $\chi(X) = \pi\chi(X)\mathrm{pct}(X)$. In particular, if X is of point-countable type, then $\chi(X) = \pi\chi(X)$.*

PROOF. Fix $x \in X$ and a compact subset F of X such that $x \in F$ and $\chi(F,X) \leq \mathrm{pct}(X)$. We have just proved that $\chi(x,F) \leq \pi\chi(x,X)$. Applying Lemma 2.4.3, it follows that:

$$\chi(x,X) \leq \chi(x,F)\chi(F,X) \leq \pi\chi(x,X)\mathrm{pct}(X).$$

This shows that $\chi(X) \leq \pi\chi(X)\mathrm{pct}(X)$. The reverse inequality is always valid, so this completes the proof. □

3.5 Hereditarily normal compact spaces

In this section we study power homogeneous T_5 compacta. It was asked by VAN MILL [49] whether the size of homogeneous T_5 compacta is bounded by $\mathfrak{c}$. A consistent positive answer was provided by JUHÁSZ, NYIKOS and SZENTMIKLÓSSY [36]. This section contains generalizations from both [49] and [36] to the power homogeneous case. The main observation is the following result;

3.5.1. PROPOSITION. *Suppose X is a power homogeneous T_5 compactum. Then $\pi\chi(X) \leq \omega$.*

PROOF. Since X is a T_5 compactum, it follows from Theorem 2.5.2 that the set $D = \{x \in X : \pi\chi(x,X) \leq \omega\}$ is dense in X. By Corollary 3.3.10 it follows that $\pi\chi(X) \leq \omega$. □

3.5.2. COROLLARY. *If X is a power homogeneous T_5 compactum, then X is homogeneous with respect to tightness and character.*

PROOF. This follows from Proposition 3.5.1, Corollary 3.4.22 and Proposition 3.4.17. □

3.5.3. THEOREM. *If X is a power homogeneous T_5 compactum, then*

$$|X| \leq 2^{c(X)}.$$

PROOF. This follows from Proposition 3.5.1 and the fact that $|X| \leq 2^{\pi\chi(X)c(X)}$ by Theorem 3.3.8. □

3.5.4. THEOREM. *If X is a compact space satisfying $wD(\aleph_1)$ hereditarily and the set of points of uncountable tightness in X is G_δ-dense, then every point of X is a pseudo P-point.*

PROOF. By the remarks made after Lemma 2.5.4, it suffices to show that the set of pseudo P-points of X is G_δ-dense in X. It is proved in [36, Theorem 2.7], that if some free ω_1-sequence converges to y, then y is a pseudo P-point of X. Therefore it suffices to show that every non-empty G_δ-subset of X contains a limit point of some converging free ω_1-sequence.

So let G be a non-empty G_δ-subset of X. Without loss of generality we may assume that G is closed. Then by assumption, G contains points of uncountable tightness (in X) and therefore the tightness of G is uncountable. This follows from Proposition 2.4.7. Since G is compact we apply Theorem 2.5.3 to conclude that G contains a limit point of some converging free ω_1-sequence. Since G is closed in X, this sequence is also a converging free ω_1-sequence in X and this completes the proof. □

3.5.5. COROLLARY. *If X is a power homogeneous T_5 compactum satisfying $wD(\aleph_1)$ hereditarily, then X is countably tight and hence of cardinality $\leq \mathfrak{c}$.*

PROOF. If X contains an isolated point, then it contains a point of countable tightness and therefore by Corollary 3.5.2 the tightness of X is countable. So suppose that X does not contain isolated points. Since the π-character of X is countable by Proposition 3.5.1, it follows from Lemma 2.5.4 that not every point of X is a pseudo P-point. Since X is homogeneous with respect to tightness by Corollary 3.5.2, if follows from Theorem 3.5.4 that X is countably tight.

In both cases it follows from Corollary 3.4.15 that $|X| \leq \mathfrak{c}$. □

3.5.6. COROLLARY ($\mathfrak{c} < 2^{\aleph_1}$). *Every power homogeneous T_5 compactum is first-countable.*

PROOF. The space X is homogeneous with respect to character by Corollary 3.5.2. So if X contains isolated points, then it follows that X is first-countable. If X contains no isolated points, then $|X| = 2^{\chi(X)}$ by the Čech-Pospišil Theorem (Theorem 2.5.1). It follows from Theorem 2.4.16 that X satisfies $wD(\aleph_1)$ hereditarily and therefore $|X| \leq \mathfrak{c}$ by Corollary 3.5.5. So we have $2^{\chi(X)} \leq \mathfrak{c}$, and therefore X is first-countable, since $\mathfrak{c} < 2^{\aleph_1}$. □

A space X is called monotonically normal provided that for every $x \in X$ and an open neighbourhood U of x, there is an open set $H(x, U)$ such that $x \in H(x, U) \subseteq U$ and whenever $H(x, U) \cap H(y, V) \neq \emptyset$, then either $x \in V$ or $y \in U$. Every monotonically normal power homogeneous compact space is first-countable, see ARHANGEL'SKIĬ [9, Theorem 8]. An elementary proof of this fact for the homogeneous case is given in [36]; with the results from this section this proof also works for the power homogeneous case.

3.5.7. PROPOSITION. *Every monotonically normal power homogeneous compact space is first-countable.*

PROOF. Let X be a monotonically normal power homogeneous compact space. Recall that monotone normality is hereditary and every monotonically normal space is normal. It follows that X is hereditarily normal and therefore $\pi\chi(X) \leq \omega$ and X is homogeneous with respect to character (see Proposition 3.5.1 and Corollary 3.5.2). If X has isolated points, then it follows that X is first-countable, so we assume that X has no isolated points.

Let $P(p, X)$ denote $\min\{\kappa : p$ is not a P_{κ^+}-point of $X\}$, i.e., $P(p, X)$ is the smallest size of a family of neighbourhoods of p in X whose intersection is not a neighbourhood of p. In a monotonically normal compact space X, the set of all points p such that $\chi(p, X) = P(p, X)$ is dense in X (this is due to WILLIAMS and ZHOU [76], see also JUHÁSZ [35, 3.12]).

We will show that $P(p, X)$ is countable for all $p \in X$. For suppose not and let x be a P-point in X. Let $\mathcal{U}$ be a countable local π-base at x in X. Since x is non-isolated, we may assume without loss of generality that $x \notin \overline{U}$ for all $U \in \mathcal{U}$. But then $F = \bigcup\{\overline{U} : U \in \mathcal{U}\}$ is an F_σ-set which misses x. Since x is a P-point, there is an open neighbourhood V of X such that $V \cap F = \emptyset$. This contradicts the fact that $\mathcal{U}$ is a local π-base at x.

We have shown that $P(p, X)$ is countable for all $p \in X$ and therefore X contains a dense set of points of countable character. Since X is homogeneous with respect to character, it follows that X is first-countable. $\square$

Chapter 4

Structural properties

4.1 Introduction

In this chapter we study spaces which are retracts of certain homogeneous spaces. Examples are retracts of coset spaces, retracts of topological groups and of course power homogeneous spaces. Such spaces are usually not homogeneous, but they do satisfy certain 'nice' topological properties. A natural question in this context is the following; suppose we are given some class $\mathcal{A}$ of homogeneous spaces. Which spaces are retracts of members of $\mathcal{A}$?

Several authors have considered this question for various classes of homogeneous spaces. It was shown by USPENSKIĬ [69] that for every space X there is a space Y such that $X \times Y$ is homogeneous. So every space is a retract of a homogeneous space. This positive results leads to the question whether every compact space is a retract of a compact homogeneous space. The answer to this question is NO; Motorov showed that the $\sin(1/x)$-continuum is not a retract of a compact homogeneous space, by showing that such retracts are weakly $1^1/_2$-homogeneous (see Definition 4.1.1). The $\sin(1/x)$-continuum does not have this property.

We study similar properties for retracts of some class of homogeneous spaces. One of the properties that we study is the weak form of Ungar's Theorem which is defined as follows. If $\mathcal{U}$ is an open cover of the space X and $A \subseteq X$, then a function $f : A \to X$ is said to be limited by $\mathcal{U}$ provided that for every $x \in A$ there is an element $U \in \mathcal{U}$ containing both x and $f(x)$. If $p \in X$, then we say that X satisfies Ungar's Theorem locally at p if for every open cover $\mathcal{U}$ of X and compact subset $K \subseteq X$, we may find an open neighbourhood V of p with the following property; whenever $x, y \in V$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$ and $f \restriction K$ is limited by $\mathcal{U}$. If a space X satisfies these conditions where we replace 'homeomorphism' by 'continuous map', then we say that X satisfies the weak form of Ungar's Theorem locally at p. We say that a space X satisfies (the weak form of) Ungar's Theorem if it satisfies (the weak form of) Ungar's Theorem locally at p for all $p \in X$.

We will prove that coset spaces satisfy Ungar's Theorem and that the weak form of this property is preserved by taking retractions. So in particular, retracts of coset spaces and retral spaces satisfy the weak form of Ungar's Theorem. Recall from §3.4 that a continuous function $\mu : X^3 \to X$ which satisfies the equalities $\mu(x, y, y) = x = \mu(y, y, x)$ for all $x, y \in X$ is called a Mal'tsev function. If there is a Mal'tsev function on a space X, then X is called a Mal'tsev space.

If G is a topological group, then the formula $xy^{-1}z$ defines a Mal'tsev function on G. Furthermore, if μ is a Mal'tsev function on X and $r : X \to Y$ is a retraction, then the restriction to Y^3 of the function $r \circ \mu$ defines a Mal'tsev function on Y. So the class of Mal'tsev spaces is closed under taking retractions. Since we have just shown that every topological group is a Mal'tsev space, it follows that retral spaces are also Mal'tsev spaces.

This raises the question whether conversely every Mal'tsev space is retral. It was shown by SIPACHEVA [66] that every compact Mal'tsev space is retral, but GARTSIDE, REZNICHENKO and SIPACHEVA [27] provide an example of a Mal'tsev space which is not retral. Since every retral space satisfies the weak form of Ungar's Theorem, this leads to the question whether Mal'tsev spaces also have this property. We will show that this is indeed the case for regular Mal'tsev spaces.

In §4.3 we study some consequences of the weak form of Ungar's Theorem. The study of this property was motivated by a result of AARTS and OVERSTEEGEN [1], who proved that if a homogeneous space X is locally compact and separable metrizable, then it is homeomorphic to a product space $Z \times Y$, where Z is zero-dimensional and Y is connected. One ingredient in the proof of this representation theorem is to show that a certain quotient map is an open mapping. We will prove that for spaces satisfying the weak form of Ungar's Theorem, this quotient map is also open. It is not the case that such spaces have a similar product structure, but we will show that these spaces still have certain 'nice' topological properties.

Next we consider power homogeneous spaces. A power homogeneous space X is a retract of a homogeneous space which has a very strong relation with X, namely it is a power space of X. In Chapter 3 we studied cardinal restrictions for power homogeneous spaces. Such cardinal restrictions allow us to show that certain spaces are not power homogeneous. In this chapter we provide more tools for proving non-power homogeneity of certain spaces. An example of such a tool can be found in ARHANGEL'SKIĬ [8]. It is noted there that if a power homogeneous space X contains a point which has a basis consisting of clopen subsets of X, then X is zero-dimensional. For example, the subspace $[0,1] \cup \{2\}$ of the real line is not power homogeneous because it contains an isolated point but it is not zero-dimensional.

One could express this by saying that power homogeneous spaces are homogeneous with respect to 'having a local basis of clopen sets': either all or none of the points in the space have such a local base. We prove similar statements for other local properties. We show that compact power homogeneous spaces are homogeneous with respect to the weak form of Ungar's Theorem. Next we show that connected power homogeneous spaces are homogeneous with respect to local connectedness and we also prove a generalization of this result. This provides us with tools for showing that certain spaces are not power homogeneous. We will apply these results in the next chapter to provide examples of separable metrizable spaces that are not power homogeneous. This task can usually not be achieved by way of cardinal functions, since such spaces are first-countable and their size is bounded by $\mathfrak{c}$.

To give a flavour of the arguments used in this chapter, we first demonstrate a classical result from the literature. This result, which is due to Motorov, states that a retract of a compact homogeneous space is weakly $1^1\!/_2$-homogeneous. The argument can be divided into two steps. The first step is to prove that every compact homogeneous space is $1^1\!/_2$-homogeneous. To conclude that retracts of compact homogeneous spaces are weakly $1^1\!/_2$-homogeneous, it suffices to note that weak $1^1\!/_2$-homogeneity is preserved under taking retractions.

4.1.1. DEFINITION. We call a space X (weakly) $1^1\!/_2$-homogeneous if for every closed subset A of X, $x \in A$ and $y \notin A$, there is a homeomorphism (continuous function) $f : X \to X$ such that $f(x) \notin A$ and $f(y) \in A$.

4.1.2. THEOREM. *If X is a compact homogeneous space, then it is $1^1\!/_2$-homogeneous.*

PROOF. Let A be a closed subset of X and let $\mathcal{A} = \{h[A] : h \in \mathcal{H}(X)\}$. Furthermore, let $\mathcal{B}$ be the family of all non-empty intersections of subfamilies of $\mathcal{A}$. If $\mathcal{C}$ is a decreasing chain in $\mathcal{B}$ then by compactness of X, the intersection $\bigcap \mathcal{C}$ is non-empty. So if we order $\mathcal{B}$ by inclusion, then every decreasing chain in $\mathcal{B}$ has a lower bound. It follows from Zorn's lemma that the collection $\mathcal{E}$ which consists of minimal elements of $\mathcal{B}$ is non-empty. Note that if h is a homeomorphism of X and $E \in \mathcal{E}$, then $h[E] \in \mathcal{B}$ and this set is also minimal in $\mathcal{B}$ and therefore $h[E] \in \mathcal{E}$. Since X is homogeneous, it follows that $\mathcal{E}$ forms a cover of X. Furthermore, since every member of $\mathcal{E}$ is minimal and $\mathcal{B}$ is closed under taking intersections, it follows that $\mathcal{E}$ forms a partition of X.

Now let $x \in A$ and $y \notin A$ be arbitrary. For $z \in X$, the set E_z is the unique element of $\mathcal{E}$ containing z. Note that if $a \in A$, then $E_a \subseteq A$ since $A \in \mathcal{B}$. Since $y \notin A$, it follows that $E_y \cap E_a = \emptyset$ whenever $a \in A$ and therefore $E_y \cap A = \emptyset$. Since E_y is the intersection of members of $\mathcal{A}$, it follows from compactness of X that there is some homeomorphism h of X such that $h[A] \cap A = \emptyset$ and $y \in E_y \subseteq h[A]$.

Note that $A \cap h^{-1}[A] = \emptyset$ and since $y \in h[A]$, we have $h^{-1}(y) \in A$. So if we let $f = h^{-1}$, then $f(y) \in A$ and $f(x) \notin A$. This completes the proof. □

4.1.3. PROPOSITION. *If X is weakly $1\frac{1}{2}$-homogeneous and $r : X \to Y$ is a retraction, then Y is weakly $1\frac{1}{2}$-homogeneous.*

PROOF. Let A be a closed subset of Y, $x \in A$ and $y \in Y \setminus A$. Then $r^{-1}[A]$ is a closed subset of X, so there is a continuous function $f : X \to X$ such that $f(x) \notin r^{-1}[A]$ and $f(y) \in r^{-1}[A]$. Let $g : Y \to Y$ be defined by the formula $g(z) = r(f(z))$, where $z \in Y$. Then $g(x) \notin A$ and $g(y) \in A$ and this completes the proof. □

Since every $1\frac{1}{2}$-homogeneous space is also weakly $1\frac{1}{2}$-homogeneous, the following corollary follows from the previous two results.

4.1.4. COROLLARY. *If X is a retract of a compact homogeneous space, then it is weakly $1\frac{1}{2}$-homogeneous.*

4.1.5. PROPOSITION. *If X is a Mal'tsev space and $x, y \in X$, then there is a continuous function $f : X \to X$ such that $f(x) = y$ and $f(y) = x$. In particular every Mal'tsev space is weakly $1\frac{1}{2}$-homogeneous.*

PROOF. If $\mu : X^3 \to X$ is a Mal'tsev function and $x, y \in X$, then define $f : X \to X$ by the formula $f(z) = \mu(x, z, y)$ for $z \in X$. Then f is continuous and since $\mu(x, x, y) = y$ and $\mu(x, y, y) = x$, it follows that $f(x) = y$ and $f(y) = x$. This clearly implies that X is $1\frac{1}{2}$-homogeneous. □

4.2 Retracts of coset spaces and Mal'tsev spaces

In this section we prove that retracts of coset spaces and regular Mal'tsev spaces satisfy the weak form of Ungar's Theorem. Note that a space X satisfies (the weak form of) Ungar's Theorem if and only if for every open cover $\mathcal{U}$ of X and compact subset $K \subseteq X$ there is an open cover $\mathcal{V}$ of X with the following property; whenever $x, y \in V \in \mathcal{V}$ then there is a homeomorphism (continuous map) $f : X \to X$ such that $f(x) = y$ and $f \restriction K$ is limited by $\mathcal{U}$. We will always use this characterization to prove that a space satisfies some form of Ungar's Theorem.

Recall that a space X is called a coset space provided that there is a topological group G with closed subgroup H such that X and $G/H = \{gH : g \in G\}$ are homeomorphic. Every coset space is homogeneous, see for example HEWITT and ROSS [32, Theorem 5.20]. If G is a topological group acting transitively on a space X, then for every $x \in X$ we let $\gamma_x : G \to X$ be defined by $\gamma_x(g) = gx$. It is easy to show that Z is a coset space if and only if there is a topological group G acting transitively on Z such that for some $z \in Z$ (equivalently: for all $z \in Z$) the function $\gamma_z : G \to Z$ is open, see VAN MILL [48].

4.2.1. THEOREM. *Coset spaces satisfy Ungar's Theorem.*

PROOF. Let Z be a coset space, $\mathcal{U}$ an open cover of Z and $K \subseteq Z$ a compact subset. Let G be a topological group acting transitively on Z such that for every $z \in Z$ we have that the function $\gamma_z : G \to Z$ is open. For $z \in K$ let V_z be an open neighbourhood of e in G such that $\gamma_z[V_z^2]$ is contained in an element of $\mathcal{U}$. There is a finite $F \subseteq K$ such that

$$K \subseteq \bigcup_{z \in F} \gamma_z[V_z].$$

Let $V = \bigcap_{z \in F} V_z$, and let W be a symmetric open neighbourhood of e in G such that $W^2 \subseteq V$. Put $\mathcal{V} = \{\gamma_z[W] : z \in Z\}$. Then $\mathcal{V}$ is an open cover of Z, and we claim that it is as desired. To this end pick arbitrary $z, p, q \in Z$ such that $p, q \in \gamma_z[W]$. There are $h, g \in W$ such that $hz = p$ and $gz = q$. Then $\xi = gh^{-1} \in W^2$ and $\xi p = q$. So it suffices to prove that if $\alpha \in W^2$ and $y \in K$ are arbitrary then there exists $U \in \mathcal{U}$ containing both y and αy. Pick $z \in F$ such that $y \in \gamma_z[V_z] \subseteq \gamma_z[V_z^2]$. Then there is an element $f \in V_z$ such that $fz = y$. Since $\alpha y = (\alpha f)z \in \gamma_z[V_z^2]$ and $\gamma_z[V_z^2]$ is contained in an element of $\mathcal{U}$, this completes the proof. □

It was shown by UNGAR [68] that every homogeneous, separable, metrizable and locally compact space is a coset space. This is a consequence of the Effros theorem on transitive actions of Polish groups on Polish spaces, see EFFROS [21] and also VAN MILL [48]. It follows that such spaces satisfy Ungar's Theorem by Theorem 4.2.1, so we have the following;

4.2.2. COROLLARY. *A locally compact, homogeneous and separable metrizable space satisfies Ungar's Theorem.*

Suppose X is a compact power homogeneous metric space such that X^ω is homogeneous. Then by the previous result X is a retract of a space satisfying Ungar's Theorem. The following proposition shows that in this case X satisfies the weak form of Ungar's Theorem.

4.2.3. PROPOSITION. *Let $r : X \to Y$ be a retraction and suppose that X satisfies the weak form of Ungar's Theorem. Then Y also satisfies the weak form of Ungar's Theorem.*

PROOF. Let $\mathcal{U}$ be an open cover of Y and $K \subseteq Y$ a compact subset. We apply the fact that X satisfies the weak for of Ungar's Theorem to the cover $\{r^{-1}[U] : U \in \mathcal{U}\}$ and the compact set $K \subseteq X$ to find a cover $\mathcal{W}$ of X with the desired properties. We let $\mathcal{V} = \{W \cap Y : W \in \mathcal{W}\}$. Clearly, $\mathcal{V}$ is an open cover of Y. If $x, y \in V$ for some $V \in \mathcal{V}$, then $x, y \in W$ for some $W \in \mathcal{W}$, so there is a continuous map $f : X \to X$ such that $f(x) = y$ and $f \restriction K$ is limited by $\{r^{-1}[U] : U \in \mathcal{U}\}$. If we define $f : Y \to Y$ by $f(z) = r(h(z))$ for $z \in Y$, then it is clear that $f(x) = y$ and it is easily verified that $f \restriction K$ is limited by $\mathcal{U}$. □

It follows that retracts of coset spaces satisfy the weak form of Ungar's Theorem. So in particular retral spaces also have this property. Although a Mal'tsev space need not be retral, we will now prove that regular Mal'tsev spaces also satisfy the weak form of Ungar's Theorem. Recall that $\Delta(X)$ denotes the diagonal in X^2.

4.2.4. THEOREM. *Regular Mal'tsev spaces satisfy the weak form of Ungar's Theorem.*

PROOF. Let X be a regular Mal'tsev space with Mal'tsev function μ. Fix an open cover $\mathcal{U}$ of X and a compact subset $K \subseteq X$. By regularity we may refine the open cover $\mathcal{U}$ by an open cover $\mathcal{W}$ of X, such that the cover

$\{\overline{W} : W \in \mathcal{W}\}$ is also a refinement of $\mathcal{U}$. For every $W \in \mathcal{W}$ we choose $U_W \in \mathcal{U}$ such that $\overline{W} \subseteq U_W$. We have that;

$$\Delta(X) \times \overline{W} \subseteq \mu^{-1}[U_W].$$

CLAIM. For every $W \in \mathcal{W}$ there is an open cover $\mathcal{V}_W$ of X such that

$$\bigcup\{V \times V \times \big(K \cap \overline{W}\big) : V \in \mathcal{V}_W\} \subseteq \mu^{-1}[U_W].$$

PROOF OF CLAIM. Let $W \in \mathcal{W}$ be given. If $K \cap \overline{W} = \emptyset$ then there is nothing to prove. So assume $K \cap \overline{W} \neq \emptyset$. We fix $x \in X$. For every $w \in K \cap \overline{W}$, there are open sets E_w and G_w such that

$$(x, x, w) \in E_w \times E_w \times G_w \subseteq \mu^{-1}[U_W].$$

Since $K \cap \overline{W}$ is compact, there is a finite set $F \subseteq K \cap \overline{W}$ such that the collection $\{G_w : w \in F\}$ covers $K \cap \overline{W}$. Let V_x be given by $\bigcap\{E_w : w \in F\}$. Then V_x is an open neighbourhood of x and clearly

$$V_x \times V_x \times \big(K \cap \overline{W}\big) \subseteq \mu^{-1}[U_W].$$

To see this, suppose $(y, z, u) \in V_x \times V_x \times (K \cap \overline{W})$. Then $u \in G_w$ for some $w \in F$. It follows that $y, z \in E_w$ and since $E_w \times E_w \times G_w \subseteq \mu^{-1}[U_W]$, it follows that $(y, z, u) \in \mu^{-1}[U_W]$. So the open cover $\mathcal{V}_W$ may be given by $\{V_x : x \in X\}$. ◀

Since K is compact, we may fix a finite subcollection $\mathcal{W}'$ of $\mathcal{W}$, such that $\mathcal{W}'$ is an open cover of K. Since $\mathcal{W}'$ is finite we may find an open cover $\mathcal{V}$ of X, such that $\mathcal{V}$ is a refinement of $\mathcal{V}_W$ for every $W \in \mathcal{W}'$.

We claim that the cover $\mathcal{V}$ is as required. To show this, suppose that $x, y \in V$ for some $V \in \mathcal{V}$. We define $f : X \to X$ by the formula $f(z) = \mu(y, x, z)$. Then f is continuous and $f(x) = \mu(y, x, x) = y$. To show that $f \restriction K$ is limited by $\mathcal{U}$, suppose $z \in K$. Then $z \in W$ for some $W \in \mathcal{W}'$. Since $x, y \in V \in \mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{V}_W$, we have

$$(y, x, z) \in V \times V \times \big(K \cap \overline{W}\big) \subseteq \mu^{-1}[U_W].$$

It follows that $f(z) = \mu(y, x, z) \in U_W$. Recall that $z \in W \subseteq U_W$ and therefore it follows that $\{z, f(z)\} \subseteq U_W$. Since z was an arbitrary element of K, this shows that $f \restriction K$ is limited by $\mathcal{U}$. □

The following provides a useful characterization of the forms of Ungar's Theorem for compact metric spaces. Using this reformulation, Corollary 4.2.2 is usually referred to as Ungar's Theorem, which explains our terminology.

4.2.5. PROPOSITION. *Let (X, ϱ) be a compact metric space. Then X satisfies (the weak form of) Ungar's Theorem if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that whenever $\varrho(x, y) < \delta$, there is a homeomorphism (continuous map) $f : X \to X$ mapping x onto y such that for all $z \in X$, $\varrho(z, f(z)) < \varepsilon$.*

PROOF. Suppose X satisfies (the weak form of) Ungar's Theorem and let $\varepsilon > 0$. Apply the fact that X satisfies (the weak form of) Ungar's Theorem to the cover $\{B_\varrho(x, \varepsilon) : x \in X\}$ and the compact set X to obtain a cover $\mathcal{V}$ with the desired properties. Since X is compact, there is a Lebesgue number δ for $\mathcal{V}$. If $\varrho(x, y) < \delta$, then $x, y \in V$ for some $V \in \mathcal{V}$ and this shows that δ is as required.

Conversely, suppose that X satisfies the $\varepsilon\delta$-condition and let $\mathcal{U}$ be an open cover of X and $K \subseteq X$ be a compact subset. We assume without loss of generality that $K = X$. Now we let ε be a Lebesgue number for the cover $\mathcal{U}$ and we find a suitable δ for this $\varepsilon > 0$. Let $\mathcal{V} = \{B_\varrho(x, \delta/2) : x \in X\}$. Now if $x, y \in V$ for some $V \in \mathcal{V}$, then $\varrho(x, y) < \delta$, so there is a homeomorphism (continuous map) $f : X \to X$ mapping x onto y such that $\varrho(z, f(z)) < \varepsilon$ for all $z \in X$. Since ε is a Lebesgue number for the cover $\mathcal{U}$, this means that f is limited by $\mathcal{U}$. □

4.3 Consequences of the weak form of Ungar's Theorem

In the previous section we have shown that there are natural spaces which satisfy the weak form of Ungar's Theorem. In this section we will prove some properties of spaces that satisfy the weak form of Ungar's Theorem. This provides us with useful techniques to prove that certain spaces are not retracts of coset spaces or Mal'tsev spaces.

Whenever X is a topological space, and $\mathcal{R}$ is a partition of X, then by $X/\mathcal{R}$ we denote the quotient space associated to $\mathcal{R}$ and $\pi : X \to X/\mathcal{R}$ is the corresponding quotient map. Whenever $x \in X$, by R_x we denote the unique element of $\mathcal{R}$ that contains x. Note that $R_x = \pi^{-1}\big[\pi(x)\big]$. We say that the partition $\mathcal{R}$ is an invariant partition if the following holds: for every continuous function $f : X \to X$ and for all $R, Q \in \mathcal{R}$, if $f[R] \cap Q \neq \emptyset$ then $f[R] \subseteq Q$. Examples of invariant partitions are $\mathcal{C}(X)$ and $\mathcal{PC}(X)$ where $\mathcal{C}(X)$ is the family of all components in X and $\mathcal{PC}(X)$ is the family of all path-components in X. We will write $\mathcal{C}$ and $\mathcal{PC}$ if X is clear from the context. In particular we use C_x (PC_x) for the (path)-component containing x, where $x \in X$. We will state many of our results for an arbitrary invariant partition $\mathcal{R}$ of some space X. For applications, we often use either $\mathcal{C}(X)$ or $\mathcal{PC}(X)$.

It should be pointed out that the quotient space $X/\mathcal{R}$ might fail to be Hausdorff, contrary to our convention that we only deal with Hausdorff spaces. If X is compact, then $X/\mathcal{C}$ is a Hausdorff space, since in compact spaces components and quasi-components coincide.

We have already mentioned the following result of AARTS and OVERSTEEGEN [1]; if a homogeneous space X is locally compact, separable and metrizable, then it is homeomorphic to the product $Z \times Y$, where Z is zero-dimensional and Y is connected. The proof in [1] can be divided into two steps; first it is proved that the quotient map $\pi : X \to X/\mathcal{Q}$ is open. Here $\mathcal{Q}$ is the invariant partition of X consisting of all quasi-components in X. Next it is noted that X is a coset space of the completely metrizable group $\mathcal{H}(X)$ which acts transitively on X. Applying a theorem of EFFROS [21] and a selection theorem of MICHAEL [44], it is proved that there is a homeomorphism of $Z \times Y$ onto X.

We have shown in §4.2 that if X is a locally compact and homogeneous metrizable space then it satisfies Ungar's Theorem. It follows from our next result that in this case the quotient mapping π of X onto $X/\mathcal{Q}$ is open. In fact, this is also true for spaces satisfying the *weak* form of Ungar's Theorem. Simple examples show that such spaces might fail to have a product structure as in the result of Aarts and Oversteegen; see Example 5.3.2.

4.3.1. THEOREM. *Suppose X satisfies the weak form of Ungar's Theorem and $\mathcal{R}$ is an invariant partition in X. Then $\pi : X \to X/\mathcal{R}$ is an open map.*

PROOF. Let $U \subseteq X$ be open. We will show that $\pi^{-1}\big[\pi[U]\big]$ is open. Assume to the contrary that this set is not open. Then there is an $x \in \pi^{-1}\big[\pi[U]\big]$ such that $V \not\subseteq \pi^{-1}\big[\pi[U]\big]$ for every neighbourhood V of x. The set A is given by $X \setminus \pi^{-1}\big[\pi[U]\big]$. By assumption we have $x \in \overline{A}$.

Since $\pi(x) \in \pi[U]$ we have that $U \cap R_x \neq \emptyset$. So we may choose $y \in U \cap R_x$. Let K be the compact set $\{y\}$. Let $\mathcal{U} = \{U, W\}$ where $W = X \setminus \{y\}$.

Since X satisfies the weak form of Ungar's Theorem and $x \in \overline{A}$, we may find a continuous function $f : X \to X$ with the property that $f(x) \in A$ and $\{f(y), y\} \subseteq U$. Let $f(x) = a$. By invariance of $\mathcal{R}$ it follows that $f(y) \in R_a$. However since $a \in A$ we have $a \notin \pi^{-1}\big[\pi[U]\big]$ and therefore

$R_a \cap \pi^{-1}\big[\pi[U]\big] = \emptyset$. In particular it follows that $R_a \cap U = \emptyset$. But we have just shown that $f(y) \in R_a \cap U$, which is a contradiction. □

Before proving the next result, we recall some dimension theory from ENGELKING [23]. If A, B is a pair of disjoint closed subsets of a space X, then a subset L of X is called a partition between A and B if there are disjoint open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $X \setminus L = U \cup V$. If X is a normal space, then by $\dim X$ we denote its covering dimension. We recall the following important facts;

4.3.2. THE COUNTABLE SUM THEOREM. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that $\dim F_i \leq n$ for $i = 1, 2, \ldots$, then $\dim X \leq n$.*

4.3.3. THEOREM ON PARTITIONS. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every sequence*

$$(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$$

of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

A sequence $\{(A_i, B_i) : i \in I\}$ of pairs of disjoint closed subsets of X is called essential if whenever $\{L_i : i \in I\}$ is a sequence of closed subsets of X such that for all i, L_i is a partition between A_i and B_i, the intersection $\bigcap_{i\in I} L_i$ is non-empty. A sequence of pairs of disjoint closed subsets of X is called inessential if it is not essential. So the theorem on partitions states that for $n \geq 0$, the covering dimension of X is less than or equal to n if and only if every sequence of length $n + 1$ of pairs of disjoint closed subsets of X is inessential.

If $\mathcal{A} = \{A_i : i \in I\}$ is a collection of subsets of X, then a collection $\mathcal{B} = \{B_i : i \in I\}$ of subsets of X is called a swelling of $\mathcal{A}$ if for all $i \in I$, $A_i \subseteq B_i$ and for every $n < \omega$ and $s \in I^n$ we have

$$A_{s(0)} \cap \ldots \cap A_{s(n-1)} \neq \emptyset \;\Leftrightarrow\; B_{s(0)} \cap \ldots \cap B_{s(n-1)} \neq \emptyset.$$

A swelling is called an open swelling if it consists of open subsets. It is a well-known fact that every finite collection of closed subsets of a normal space admits an open swelling, see for example ENGELKING [23, Theorem 3.1.1]. The following lemma is a corollary to this result; we give a sketch of the simple proof.

4.3.4. LEMMA. *Let $\mathcal{F}$ be a finite collection of closed subsets of a normal space X. Then there is an open cover $\mathcal{V}$ of X such that for all $F, G \in \mathcal{F}$ and $U, V \in \mathcal{V}$ the following holds*

$(*)$ *If $F \cap G = \emptyset$, $F \cap U \neq \emptyset$ and $G \cap V \neq \emptyset$ then $U \cap V = \emptyset$.*

PROOF. Let $\mathcal{U} = \{U_F : F \in \mathcal{F}\}$ be an open swelling of the family $\mathcal{F}$. For our purposes it suffices to know that whenever $F, G \in \mathcal{F}$ are disjoint then so are U_F and U_G. For every $x \in \bigcup \mathcal{F}$ we set $W_x = \bigcap\{U_F : x \in F \in \mathcal{F}\}$ and $G_x = \bigcup\{F \in \mathcal{F} : x \notin F\}$. One easily verifies that the cover $\mathcal{V}$ given by $\{W_x \setminus G_x : x \in \bigcup \mathcal{F}\} \cup \{X \setminus \bigcup \mathcal{F}\}$ satisfies property $(*)$. □

4.3.5. THEOREM. *Let X be a regular space which satisfies the weak form of Ungar's Theorem. Suppose that $\mathcal{R}$ is an invariant partition in X such that all elements of $\mathcal{R}$ are σ-compact. Let $n < \omega$ and consider the set A consisting of all points a in X such that* $\dim R_a \leq n$. *Then A is a closed subset of X.*

PROOF. Assume that $p \in \overline{A}$. We will show that $p \in A$. Since R_p is σ-compact and regular it is Lindelöf and therefore normal. It follows from the countable sum theorem (Theorem 4.3.2) that it suffices to show that any compact subset K of R_p satisfies $\dim K \leq n$.

Fix a compact set K in R_p. We will prove that every family of $n+1$ pairs of disjoint closed subsets of K is inessential (see Theorem 4.3.3). So let $\{(A_i, B_i) : 1 \leq i \leq n+1\}$ be such a family. Let $\mathcal{F}$ be the family consisting of all compact sets A_i and B_i for $1 \leq i \leq n+1$. The collection $\mathcal{F}$ is a family of closed subsets of the normal space βX, so we may apply the previous lemma to obtain an open cover $\mathcal{V}$ of βX with property $(*)$. Restricting the cover $\mathcal{V}$ to X, we obtain an open cover $\mathcal{U}$ of X with property $(*)$. In particular it follows that whenever $A_i \cap U \neq \emptyset$ and $B_i \cap V \neq \emptyset$ for some $U, V \in \mathcal{U}$ then $U \cap V = \emptyset$. Since X satisfies the weak form of Ungar's Theorem and the fact that $p \in \overline{A}$, there is a continuous map f of X which maps p onto a for some $a \in A$ and $f \restriction K$ is limited by $\mathcal{U}$. By invariance of $\mathcal{R}$, we have $f[R_p] \subseteq R_a$.

By compactness, the collection $\Gamma = \{(f[A_i], f[B_i]) : 1 \leq i \leq n+1\}$ is a family of $n+1$ pairs of closed subsets of R_a. We will show that it is also a collection of pairs of *disjoint* subsets of R_a. So let $f(z) \in f[A_i]$ and $f(w) \in f[B_i]$, where $z \in A_i$ and $w \in B_i$. Then there are $U, V \in \mathcal{U}$ such that $\{z, f(z)\} \subseteq U$ and $\{w, f(w)\} \subseteq V$. By $(*)$ it follows that $U \cap V = \emptyset$ so $f(z) \neq f(w)$. It follows that $f[A_i] \cap f[B_i] = \emptyset$.

Since Γ is a family of $n+1$ pairs of disjoint closed subsets of R_a and $a \in A$, it follows that it is an inessential family in R_a. By continuity of f, we conclude that the original family $\{(A_i, B_i) : 1 \le i \le n+1\}$ is inessential in K. Thus we have shown that $\dim K \le n$. This completes the proof. □

For applications of the previous theorem, note that components are always closed. So if X is σ-compact, then every element of $\mathcal{C}$ is σ-compact as well.

4.3.6. THEOREM. *Suppose X satisfies the weak form of Ungar's Theorem and $\mathcal{R}$ is an invariant partition in X. Let $R, Q \in \mathcal{R}$ such that $\overline{R} \cap Q \neq \emptyset$. Then $R \subseteq \overline{Q}$.*

PROOF. Let $R, Q \in \mathcal{R}$ with $\overline{R} \cap Q \neq \emptyset$. Fix $z \in R$ and let U be an arbitrary neighbourhood of z in X. Apply the fact that X satisfies the weak form of Ungar's Theorem to the compact set $K = \{z\}$ and the cover $\mathcal{U} = \{U, W\}$ of X where $W = X \setminus \{z\}$, to obtain a cover $\mathcal{V}$ with the desired properties. Pick $y \in \overline{R} \cap Q$. Since $\mathcal{V}$ covers X there is a set $V \in \mathcal{V}$ with $y \in V$. Since $y \in \overline{R}$, we have $R \cap V \neq \emptyset$, so let $x \in R \cap V$. By the properties of $\mathcal{V}$ we may find a continuous function $f : X \to X$ such that $f(x) = y$ and $\{z, f(z)\} \subseteq U$. By invariance of $\mathcal{R}$ it follows that $f[R] \subseteq Q$ and therefore it follows that $f(z) \in Q$. We have shown that $f(z) \in U \cap Q$. Since U was an arbitrary neighbourhood of z, we have shown that $z \in \overline{Q}$. Since z was arbitrary we have shown that $R \subseteq \overline{Q}$. □

4.3.7. COROLLARY. *Suppose X satisfies the weak form of Ungar's Theorem and $\mathcal{R}$ is an invariant partition in X. Let $R, Q \in \mathcal{R}$. The following are equivalent:*

(1) $\overline{R} \cap Q \neq \emptyset$,

(2) $\overline{Q} \cap R \neq \emptyset$,

(3) $\overline{R} = \overline{Q}$.

PROOF. It suffices to show equivalence of (1) and (3). It is clearly the case that (3) $\Rightarrow$ (1), so assume (1), i.e., $\overline{R} \cap Q \neq \emptyset$. By the previous theorem it follows that $R \subseteq \overline{Q}$ and thus $\overline{R} \subseteq \overline{Q}$. In particular $\overline{Q} \cap R \neq \emptyset$ and again it follows that $\overline{Q} \subseteq \overline{R}$. Thus $\overline{R} = \overline{Q}$. □

For compact metric spaces Theorem 4.3.5 also follows from a more general result on the existence of inverse limits in such spaces. We prove this result below. First we introduce inverse sequences. Suppose $\{X_n : n \in \omega\}$ is a collection of spaces and for every $n < m < \omega$, let $h_n^m : X_m \to X_n$ be a continuous map. If for every $n < k < m < \omega$, we have

$$h_n^m = h_k^m \circ h_n^k,$$

then we call $\{X_n, h_n^m\}_{n<m<\omega}$ an inverse sequence, in this case we also write $\{X_n, h_n^m\}$. Associated with an inverse sequence is a topological space X_ω which is defined as the following subset of the product space $Y = \prod\{X_n : n \in \omega\}$;

$$X_\omega = \big\{x \in Y : \text{ for all } n < m < \omega \ \big(h_n^m(x_m) = x_n\big)\big\}.$$

The topology of X_ω is the subspace topology that it inherits from Y; note that X_ω is a closed subset of Y. The space X_ω is called the inverse limit of the inverse sequence $\{X_n, h_n^m\}$. The following theorem is well-known;

4.3.8. THEOREM. *If X_ω is an inverse limit of an inverse sequence $\{X_n, h_n^m\}$ where every X_n is a separable metrizable space and $\dim X_n \leq k$ for all $n \in \omega$, then also $\dim X_\omega \leq k$.*

The previous theorem implies that our next result is more general than Theorem 4.3.5 for compact metric spaces and $\mathcal{R} = \mathcal{C}$. We will give the proof in several steps.

4.3.9. THEOREM. *Suppose (X, ϱ) is a compact metric space which satisfies the weak form of Ungar's Theorem. Let $A \subseteq X$ be a subset of X with $p \in \overline{A}$. Then there is a sequence $(C_n)_n$ of components of elements of A such that C_p is homeomorphic to the inverse limit of some inverse sequence $\{C_n, h_m^n\}_{m<n<\infty}$. Furthermore, there is a homeomorphism $z : C_p \to C_\infty$ such that for every $x \in C_p$ we have*

$$x = \lim_{n\to\infty} z(x)_n.$$

We fix a compact metric space (X, ϱ). Furthermore, we assume we have a collection $\{C_n : n \in \mathbb{N}\}$ of components in X and a component C in X. We will derive sufficient conditions for C to be an inverse limit of the C_n's. To define an inverse limit we will need maps h_m^n from C_n into C_m. These are defined using two collections of mappings $\{f_n : n \in \mathbb{N}\}$ and $\{g_n : n \in \mathbb{N}\}$.

We assume that for every $n \in \mathbb{N}$ we have that $f_n : C \to C_n$ and $g_n : C_n \to C$ are continuous maps. Let $m < n$ be natural numbers. We define $h_m^n : C_n \to C_m$ as follows (see Figure 4.1),

$$h_m^n = f_m \circ g_{m+1} \circ f_{m+1} \circ \cdots \circ f_{n-1} \circ g_n.$$

Note that for all $k < m < n$ we have $h_k^m \circ h_m^n = h_k^m$. So it follows that $\{C_n, h_m^n\}$ forms an inverse sequence. By C_∞ we denote the inverse limit of this sequence.

The following result on ε-commuting diagrams is due to MIODUSZEWSKI [52, Theorem 4].

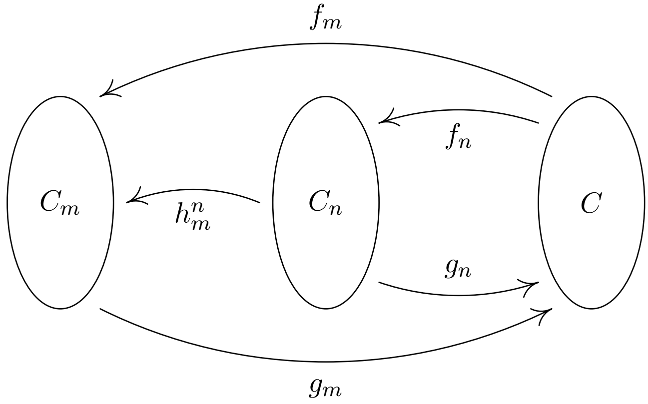

Figure 4.1: The mappings f, g and h.

4.3.10. THEOREM. *Let $(\varepsilon_n)_n$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Suppose further that for every $k < m < n$ we have*

(1) $\varrho(h^m_k f_m, h^n_k f_n) \leq \varepsilon_m$

(2) $\varrho(f_m, h^n_m f_n) \leq \varepsilon_m$

(3) $\varrho(g_n, g_m h^n_m) \leq \varepsilon_m$

(4) $\varrho(f_m, \mathrm{id}_C) \leq \varepsilon_m$ *and* $\varrho(g_n, \mathrm{id}_{C_n}) \leq \varepsilon_n$.

Then the component C in X and the inverse limit C_∞ are homeomorphic. Furthermore, there is a homeomorphism $z : C \to C_\infty$ such that for all $x \in C$, we have

$$x = \lim_{n\to\infty} z(x)_n.$$

PROOF. We will construct a homeomorphism $z : C \to C_\infty$. For $x \in C$ and $k < n$ we define

$$z^k_n(x) = h^n_k f_n(x).$$

By the first condition in the theorem it follows that the sequence $(z^k_n(x))^\infty_{n=k+1}$ forms a Cauchy-sequence in the component C_k. By compactness it follows that its limit exists, so we may define

$$z^k(x) = \lim_{n\to\infty} z^k_n(x).$$

Now note that for $k < m$ we have

$$\begin{aligned} h_k^m(z^m(x)) &= \lim_{n\to\infty} h_k^m h_m^n f_n(x) \\ &= \lim_{n\to\infty} h_k^n f_n(x) \\ &= \lim_{n\to\infty} z_n^k(x) = z^k(x). \end{aligned}$$

Thus it follows that if we define $z(x) = (z^k(x))_{k\in\mathbb{N}}$ then $z(x) \in C_\infty$. Conversely suppose that $x = (x_n)_{n\in N}$ is an element of C_∞. Then for every $m < n$ we have $x_m = h_m^n(x_n)$. By the third condition in the theorem it follows that the sequence $(g_n(x_n))_n$ is a Cauchy sequence in C. So by compactness we may define $w(x) = \lim_{n\to\infty} g_n(x_n)$. Note that by the fourth condition in the theorem, $\varrho(x_n, g_n(x_n)) \to 0$. Thus it follows that $w(x) = \lim_{n\to\infty} x_n$.

CLAIM 1. For every $x \in C$, $w(z(x)) = x$ and for $x \in C_\infty$ we have $z(w(x)) = x$.

PROOF OF CLAIM. We only prove that $w(z(x)) = x$. We note the following for $m \in \mathbb{N}$,

$$\varrho(z(x)_m, x) = \lim_{n\to\infty} \varrho(z_n^m(x), x) = \lim_{n\to\infty} \varrho(h_m^n f_n(x), x).$$

However by the second and the fourth condition in the theorem it follows that for every $n > m$,

$$\varrho(h_m^n f_n, \mathrm{id}_C) \leq 2\varepsilon_m.$$

So it follows that $\varrho(z(x)_m, x) \leq 2\varepsilon_m$. We have shown that the sequence $(z(x)_m)_m$ converges to x. Since this sequence also converges to $w(z(x))$, it follows that $w(z(x)) = x$. ◀

CLAIM 2. The functions $z : C \to C_\infty$ and $w : C_\infty \to C$ are continuous.

PROOF OF CLAIM. We will show that z is continuous. Since w is the inverse mapping of z, it follows from compactness that w is also continuous. To show that z is continuous, it suffices to show that z is coordinatewise continuous. So we fix $m \in \mathbb{N}$, $x \in X$ and $\varepsilon > 0$. Since $\varepsilon_n \to 0$, we may fix $k > m$ such that $\varepsilon_k < \varepsilon/3$. Since $h_m^k f_k$ is continuous, we further fix $\delta > 0$ such that whenever $\varrho(x, y) < \delta$, then $\varrho(h_m^k f_k(x), h_m^k f_k(y)) < \varepsilon/3$.

We claim that this δ is a required. So assume that $\varrho(x,y) < \delta$. By the first condition, we have for $n > k$;

$$\begin{aligned}\varrho(h_m^n f_n(x), h_m^n f_n(y)) &\leq \varrho(h_m^n f_n(x), h_m^k f_k(x)) + \varrho(h_m^k f_k(x), h_m^k f_k(y)) \\ &\quad + \varrho(h_m^k f_k(y), h_m^n f_n(y)) \\ &\leq \varepsilon_k + \varepsilon/3 + \varepsilon_k < \varepsilon.\end{aligned}$$

So it follows that $\varrho(z(x)_m, z(y)_m) = \lim_{n\to\infty} \varrho(h_m^n f_n(x), h_m^n f_n(y)) \leq \varepsilon$. ◄

From Claims 1 and 2 it follows that $z : C \to C_\infty$ is a homeomorphism. Furthermore, in the proof of Claim 1 we have shown that for all $x \in C$,

$$x = \lim_{n\to\infty} z(x)_n. \qquad \square$$

The following trivial lemma is crucial in the following results,

4.3.11. LEMMA. *Let $\xi_i : X \to X$ be continuous functions such that for every $i \leq N$, $\varrho(\xi_i, \mathrm{id}_X) \leq \varepsilon_i$. If $\xi = \xi_1 \circ \xi_2 \circ \cdots \circ \xi_N$ and $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_N$ then $\varrho(\xi, \mathrm{id}_X) \leq \varepsilon$.* □

4.3.12. LEMMA. *Suppose that (ζ_n) and (ε_n) are decreasing sequences of positive real numbers such that the following properties are satisfied for all $n \in \mathbb{N}$,*

(1) $\sum_{i=n}^{\infty} 3\zeta_i \leq 4\zeta_n \leq \varepsilon_n$

(2) *For all $x, y \in C$, if $\varrho(x,y) \leq 4\zeta_{m+1}$ then for every $k < m$*

$$\varrho(h_k^m f_m(x), h_k^m f_m(y)) \leq \varepsilon_m.$$

(3) $\varrho(f_n, \mathrm{id}_C) \leq \zeta_n$ *and* $\varrho(g_n, \mathrm{id}_{C_n}) \leq \zeta_n$.

Then the closeness conditions of Theorem 4.3.10 are satisfied.

PROOF. First we note that $\zeta_n \leq \varepsilon_n$ so by the third requirement it follows that condition (4) is satisfied. Now recall that h_m^n is defined by

$$h_m^n = f_m \circ g_{m+1} \circ f_{m+1} \circ \cdots \circ f_{n-1} \circ g_n.$$

It follows from the previous lemma that for every $m < n$,

$$\varrho(h_m^n, \mathrm{id}_{C_n}) \leq \sum_{i=m}^{n} 2\zeta_i \leq \sum_{i=m}^{\infty} 2\zeta_i.$$

Thus applying the lemma again, it follows that

$$\begin{aligned}\varrho(f_m, h^n_m f_n) &\leq \varrho(f_m, \mathrm{id}_C) + \varrho(\mathrm{id}_C, h^n_m f_n)\\ &\leq \zeta_m + \zeta_n + \sum_{i=m}^{\infty} 2\zeta_i\\ &\leq \sum_{i=m}^{\infty} 3\zeta_i \leq \varepsilon_m.\end{aligned}$$

This shows that condition (2) of theorem Theorem 4.3.10 is satisfied. Condition (3) follows similarly. We will go on to show that for every $k < m < n$ we have condition (1), i.e.,

$$\varrho(h^m_k f_m, h^n_k f_n) \leq \varepsilon_m.$$

We have just shown that

$$\varrho(\mathrm{id}_{C_n}, h^n_{m+1}) \leq \sum_{i=m+1}^{\infty} 2\zeta_i.$$

Thus it follows that

$$\varrho(\mathrm{id}_C, g_{m+1} h^n_{m+1} f_n) \leq \sum_{i=m+1}^{\infty} 3\zeta_i \leq 4\zeta_{m+1}.$$

It follows from the second requirement that

$$\begin{aligned}\varrho(h^m_k f_m, h^n_k f_n) &= \varrho(h^m_k f_m, h^m_k h^n_m f_n)\\ &= \varrho(h^m_k f_m, h^m_k f_m g_{m+1} h^n_{m+1} f_n)\\ &\leq \varepsilon_m.\end{aligned}$$

This completes the proof. □

The following proposition provides the necessary input. Whenever $A \subseteq X$, by C_A we denote the union of all components of points in A, thus $C_A = \bigcup\{C_a : a \in A\}$.

4.3.13. PROPOSITION. *Suppose (X, ϱ) is a compact metric space which satisfies the weak form of Ungar's Theorem. Suppose further that $A \subseteq X$. If $p \in \overline{C}_A \setminus C_A$ and (ε_n) is a decreasing sequence of positive real numbers. Then there is a sequence $\{C_n : n \in \mathbb{N}\}$ of components in C_A and collections $\{f_n : n \in \mathbb{N}\}$ and $\{g_n : n \in \mathbb{N}\}$ and a sequence (ζ_n) such that the conditions of lemma 4.3.12 are satisfied, where $C = C_p$.*

PROOF. The proof is by induction on n. Suppose that for every $j \le n$ the functions $f_j : C \to C_j$ and $g_j : C_j \to C$ and a number $\zeta_j < \varepsilon_j$ have been defined such that

(1) For all $x, y \in C$, if $\varrho(x,y) \le 4\zeta_j$ then for every $k < j-1$

$$\varrho(h_k^{j-1} f_{j-1}(x), h_k^{j-1} f_{j-1}(y)) \le \varepsilon_{j-1}.$$

(2) $\varrho(f_j, \mathrm{id}_C) \le \zeta_j$ and $\varrho(g_j, \mathrm{id}_{C_j}) \le \zeta_j$.

We will show how to find a component C_{n+1}, a real number ζ_{n+1} and functions f_{n+1} and g_{n+1} such that these conditions are still satisfied for $j = n+1$. We will always choose $\zeta_{n+1} = (1/4)^N$ for some $N \in \mathbb{N}$ and $\zeta_{n+1} < \zeta_n$.

The collection $\{h_k^n \circ f_n : k < n\}$ consists of uniformly continuous functions each with domain C. Therefore there is some $\delta > 0$ such that if $\varrho(x,y) < \delta$ then for every $k < n$,

$$\varrho(h_k^n f_n(x), h_k^n f_n(y)) \le \varepsilon_n.$$

We choose $\zeta_{n+1} < \zeta_n$ such that $\zeta_{n+1} = (1/4)^N$ for some $N \in \mathbb{N}$ and

$$4\zeta_{n+1} \le \min\{\varepsilon_{n+1}, \delta\}.$$

Since X satisfies the weak form of Ungar's Theorem and $p \in \overline{C}_A \setminus A$ we may find a component $C_{n+1} \subseteq C_A$ and continuous functions $f_{n+1} : C \to C_{n+1}$ and $g_{n+1} : C_{n+1} \to C$ such that

$$\varrho(f_{n+1}, \mathrm{id}_C) \le \zeta_{n+1} \qquad \text{and} \qquad \varrho(g_{n+1}, \mathrm{id}_{C_{n+1}}) \le \zeta_{n+1}.$$

This completes the induction step. It remains only to verify that for every $n \in \mathbb{N}$ we have

$$\sum_{i=n}^{\infty} 3\zeta_i \le 4\zeta_n.$$

Note that $\zeta_n = (1/9)^N$ for some $N \in \mathbb{N}$ and $\{\zeta_i : i > n\} \subseteq \{(1/9)^M : M > N\}$. So it follows that

$$\begin{aligned} \sum_{i=n+1}^{\infty} 3\zeta_i &\le 3 \cdot \sum_{i=N+1}^{\infty} \frac{1}{9^i} \\ &= \frac{3}{8} \cdot \frac{1}{9^N} \le \zeta_n. \end{aligned}$$

The desired inequality follows and this completes the proof. □

PROOF OF THEOREM 4.3.9. The proof follows from Theorem 4.3.10, Lemma 4.3.12 and the previous proposition. □

If we take $\mathcal{R} = \mathcal{C}$, then for compact spaces we can deduce Theorem 4.3.5 from Theorem 4.3.8 and Theorem 4.3.9. So Theorem 4.3.9 is more general than Theorem 4.3.5, and we can even improve this result. Whenever X and Y are spaces, then we write $X\tau Y$ if for every continuous function $f : A \to Y$, where A is a closed subset of X, there is an extension $\bar{f} : X \to Y$ of f. It is well-known that if X is a separable metric space, then $\dim X \leq n$ if and only if $X\tau\mathbb{S}^n$. So for compact spaces, we may restate Theorem 4.3.5 as follows; if X is a compact metrizable space which satisfies the weak form of Ungar's Theorem, then the set

$$\{x \in X : C_x\tau\mathbb{S}^n\}$$

is a closed subset of X. The following generalization of Theorem 4.3.8 was proved by RUBIN and SCHAPIRO [61]; if K is a simplicial complex and X is an inverse limit of spaces X_n such that $X_n\tau|K|$ for every n, then also $X\tau|K|$. So applying Theorem 4.3.9 we obtain the following more general result; if X is a compact metric space which satisfies the weak form of Ungar's Theorem and K is a simplicial complex, then the set

$$\{x \in X : C_x\tau|K|\},$$

is a closed subset of X. We now provide a more direct proof of this result which remains valid if we replace the simplicial complex K with an ANR. Recall that K is an ANR if for every separable metrizable space X with closed subset A, for every continuous function $f : A \to X$ there is a neighbourhood U of A such that f is continuously extendable over U. For more information on ANR's, see VAN MILL [46].

4.3.14. THEOREM. *Suppose (X, ϱ) is a compact metric space and K an* ANR. *If X satisfies the weak form of Ungar's Theorem and $\mathcal{R}$ is an invariant partition of X, then the subset E of X is closed where*

$$E = \{x \in X : R_x\tau K\}.$$

PROOF. To show that E is closed we fix $x \in \overline{E}$ and we will prove that $x \in E$. So let A be a closed subset of R_x and suppose that $f : A \to K$ is a continuous function. To show that f is continuously extendable over R_x, we will prove that it is homotopic to an extendable function $g : A \to K$. Note that $f[A]$ is compact and therefore there is some $\varepsilon > 0$ such that whenever $g : A \to K$ is such that $\varrho(f, g) \leq \varepsilon$, then f and g are homotopic. This is well-known, see for example VAN MILL [46, Theorem 4.1.1].

Since K is an ANR, there is a neighbourhood U of A in X and a continuous extension $F : U \to K$ of f. By compactness of A, we may find $\delta > 0$ such that whenever $a \in A$ and $y \in X$ with $\varrho(a, y) < \delta$, then $y \in U$ and

$$\varrho(f(a), F(y)) = \varrho(F(a), F(y)) < \varepsilon.$$

Now using the fact that X satisfies the weak form of Ungar's Theorem and $x \in \overline{E}$, we find a continuous function $h : X \to X$ such that $h(x) \in E$ and for all $a \in A$, $\varrho(a, h(a)) < \delta$. Let g be the restriction of $F \circ h$ to A. If $a \in A$, then $\varrho(a, h(a)) < \delta$ and therefore

$$\varrho(f(a), g(a)) = \varrho(f(a), F(h(a))) < \varepsilon.$$

So it follows that $\varrho(f, g) \leq \varepsilon$, and by construction f and g are homotopic. It remains to verify that g is continuously extendable over C_x. To see this, note that $h[A]$ is a closed subset of $R_{h(x)}$ and since $h(x) \in E$, the function $F \restriction h[A]$ is continuously extendable over $R_{h(x)}$. Since $h[R_x] \subseteq R_{h(x)}$, it follows that $g = F \circ h \restriction A$ is continuously extendable over R_x.

We have shown that the original function $f : A \to K$ is homotopic to a function $g : A \to K$, where g is continuously extendable over R_x. It now follows from the Borsuk Homotopy Extension Theorem (VAN MILL [46, §1.4]) that f is also continuously extendable over R_x. So $x \in E$ and this completes the proof. □

4.4 Mal'tsev continua with the fixed point property

A space X is said to have the fixed point property provided that for every continuous function $f : X \to X$, there is some $x \in X$ such that $f(x) = x$. In this section we will show that a Mal'tsev continuum with the fixed point property is locally connected. This will provide us with an example of a coset space which is not Mal'tsev.

Before proving the main result of this section, we introduce some connectivity properties. We shall use these properties to show that Mal'tsev continua with the fixed point property are locally connected. Furthermore, we shall prove in §4.5 that power homogeneous spaces are homogeneous with respect to these properties. This leads to several examples of separable metrizable spaces that are not power homogeneous in Chapter 5.

Recall that a space X is locally connected if for all $x \in X$ and every open neighbourhood U of x, there is a connected set $C \subseteq U$ such that $x \in \operatorname{Int} C$. If $x, y \in X$ and $A \subseteq X$, then we say that x and y are connected in A provided there is a connected set C containing x and y such that $C \subseteq A$. In particular, x and y are connected in X if and only if they belong to the same component of X. By convention, the empty set is a connected set.

4.4.1. DEFINITION. Let X be a topological space. If $x \in X$, then we say that the components of X are regularly locally connected at x if for every neighbourhood U of x there is a neighbourhood V of x such that for every component C of X, there is a connected set C' in X such that

$$V \cap C \subseteq C' \subseteq U.$$

We say that the components of X are regularly locally connected if for every $x \in X$, the components of X are regularly locally connected at x.

4.4.2. LEMMA. *Consider the following statements regarding a space X:*

(1) *X is locally connected.*

(2) *The components of X are regularly locally connected.*

(3) *The components of X are locally connected.*

For every space X we have $(1) \to (2)$ and $(2) \to (3)$. If X is connected then $(2) \to (1)$.

PROOF. $(1) \to (2)$. Suppose X is locally connected. Fix $x \in X$ and a neighbourhood U of x. By (1) there is a connected set A in X such that $A \subseteq U$ and $x \in \operatorname{Int} A$. Let $V = \operatorname{Int} A$. Suppose C is some component of X. If $C \cap V = \emptyset$ then there is nothing to prove. So assume that $C \cap V \neq \emptyset$. Since V is contained in the connected set A and C is a component, it follows that $A \subseteq C$. In particular we have $C \cap V = V \subseteq A \subseteq U$.

$(2) \to (3)$. Suppose that the components of X are regularly locally connected. To show that the components of X are locally connected, fix some component C of X. Let $x \in C$ and suppose $U \cap C$ is an arbitrary neighbourhood of x in C, where U is some neighbourhood of x in X. By (2) there is a neighbourhood V of x and a connected set C' such that $x \in V \cap C \subseteq C' \subseteq U$. Since C is a component and $x \in C \cap C'$, it follows that $C' \subseteq C$. So we have

$$x \in V \cap C \subseteq C' \subseteq U \cap C.$$

We have $x \in \mathrm{Int}_C(C')$ and $C' \subseteq U \cap C$ and this shows that C is locally connected.

If X is connected then $(2) \to (1)$. Let $x \in X$ and U be some neighbourhood of x in X. By (2) and the fact that X is a component, there is a neighbourhood V of x and a connected set C' such that $x \in V = X \cap V \subseteq C' \subseteq U$. So we have $x \in \mathrm{Int}\, C'$ and since x was arbitrary, this shows that X is locally connected. □

4.4.3. REMARK. Simple examples show that the conditions in the previous lemma are not equivalent. The space given in Example 5.4.2 satisfies (3) but not (2). Furthermore, if Z is a convergent sequence, then $Z \times \mathbb{I}$ satisfies (2) but not (1).

4.4.4. PROPOSITION. *Suppose X is a space and $x \in X$. The following are equivalent:*

(1) *The components of X are regularly locally connected at x.*

(2) *For every neighbourhood U of x there is a neighbourhood V of x such that for all $y, z \in V$ the following holds: If y and z are connected in X, then y and z are connected in U.*

PROOF. $(1) \to (2)$. The neighbourhood V is provided by (1). If $y, z \in V$ and y and z are connected in X, then there is a component C of X such that $y, z \in C$. By (1), there is a connected set C' such that $y, z \in V \cap C \subseteq C' \subseteq U$. It follows that y and z are connected in U.

$(2) \to (1)$. Suppose $x \in X$ and U is a neighbourhood of x in X. The neighbourhood V is provided by (2). Suppose C is a component of X. If $V \cap C = \emptyset$ then there is nothing to prove. So suppose $y \in V \cap C$. Let C' be given by

$$\bigcup \{K : K \text{ is connected } \& \ y \in K \ \& \ K \subseteq U\}.$$

Then C' is connected and $C' \subseteq U$. Furthermore, by (2) it follows that if $z \in V \cap C$, then $z \in C'$ and therefore $V \cap C \subseteq C'$. This completes the proof. □

A topological space X is connected in dimension n, abbreviated C^n, if for every $0 \leq m \leq n$, every continuous function $f : \mathbb{S}^m \to X$ can be extended to a continuous function $\bar{f} : B^{m+1} \to X$. A space X is called locally connected in dimension n, abbreviated LC^n, if for every $x \in X$ and for every neighbourhood U of x and for every $0 \leq m \leq n$, there exists a neighbourhood V of x such that every continuous function $f : \mathbb{S}^m \to V$ can be extended to a continuous function $\bar{f} : B^{m+1} \to U$. Note that a space X is path-connected if and only if it is C^0 and locally path-connected if and only if it is LC^0.

If X is a space, $n \in \omega$ and $x \in X$, then we say that X satisfies the implication $C^n \to LC^n$ locally at x if for every $0 \leq m \leq n$ and every neighbourhood U of x there is a neighbourhood V of x such that for every continuous function $f : \mathbb{S}^m \to V$ the following holds; if f can be extended to a continuous function from B^{m+1} into X, then f can be extended to a continuous function $\bar{f} : B^{m+1} \to U$. We say that a space satisfies the implication $C^n \to LC^n$ if it satisfies this implication locally at x for all $x \in X$. In particular every LC^n space satisfies the implication $C^n \to LC^n$. The converse of this statement is not true.

Suppose X satisfies the implication $C^0 \to LC^0$. If U is some open subset of X, then there is an open subset V of X such that whenever $x, y \in V$ are path-connected, then x and y are connected by a path which is contained in U. So intuitively we may think of the implication $C^0 \to LC^0$ as follows; if x and y are path-connected in X, then they are connected by arbitrarily 'small' paths in X. The well-known Mazurkiewicz Theorem (cf. VAN MILL [46, Theorem 1.5.22]) provides a link between local-connectedness and local-path connectedness in Polish spaces, this yields the following observation;

4.4.5. THEOREM. *Let X be a Polish space and assume that the components of X are regularly locally connected. Then X satisfies the implication $C^0 \to LC^0$.*

PROOF. Let $x \in X$ and fix an arbitrary neighbourhood U of x. Using Proposition 4.4.4, we obtain a neighbourhood V of x such that whenever $y, z \in V$ are connected in X, then y and z are connected in U. We claim that V is also a witness to the implication $C^0 \to LC^0$ at x for the neighbourhood U. So assume that $y, z \in V$ are path-connected (in X). Let $C \subseteq U$ be the component of y in U. Since y and z are path-connected in X, they are connected in X. By construction it follows that y and z are connected in U and therefore $z \in C$. Since U is an open subset of X, it follows easily that the components of U are regularly locally connected. It follows from Lemma 4.4.2 that C is locally connected. Since components are closed subsets, it follows that C is a closed subset of U and therefore it is a G_δ-subset of U. Since U is an open subset of X, it follows that C is a G_δ-subset of X and therefore C is Polish. So C is Polish, connected and locally connected. It follows from the Mazurkiewicz Theorem that there is a path in C connecting y and z and this is what we wanted to show. □

Suppose X is a space in which components and path-components coincide. If X satisfies the implication $C^0 \to LC^0$, then one verifies easily that the components of X are regularly locally connected. To see this, suppose we are given a neighbourhood U of some point x in X. Using the implication $C^0 \to LC^0$, we obtain a neighbourhood V of x. Now if $y, z \in V$ are connected in X, then by assumption they are path-connected in X. So by construction there is a path connecting y and z which is contained in U. In particular, this means that y and z are connected in U, which is what we wanted to show. So with the previous theorem, we obtain the following equivalence;

4.4.6. COROLLARY. *Suppose X is a Polish space in which components and path-components coincide. The following statements are equivalent;*

(1) *The components of X are regularly locally connected,*

(2) *X satisfies the implication $C^0 \to LC^0$.*

4.4.7. REMARK. Recall that a continuum X is called decomposable if it is the union of two proper subcontinua and indecomposable if it is not decomposable. A continuous image of the unit interval $\mathbb{I}$ is locally connected (see for example ENGELKING [22, Exercise 6.3.3(d)]). Furthermore, a non-degenerate indecomposable continuum is not locally connected by VAN MILL [46, Corollary 1.10.14]. So if X is a hereditarily indecomposable continuum, then the path-components in X consist of single points. Observe that such spaces satisfy the implication $C^0 \to LC^0$ trivially. Furthermore, since X is connected but not locally connected, it follows from Lemma 4.4.2 that the components of X are not regularly locally connected. For example, since the well-known pseudo-arc is a hereditarily indecomposable continuum, it satisfies (2) but not (1) of the previous corollary. So the requirement that components and path-components coincide is essential in this result.

4.4.8. THEOREM. *Let X be a compact Mal'tsev space in which all components have the fixed point property. Then the components of X are regularly locally connected.*

PROOF. We use the formulation provided by Proposition 4.4.4. So fix $x \in X$ and let U be some arbitrary neighbourhood of x. Fix a Mal'tsev function μ on X. Let G be a neighbourhood of x such that $\overline{G} \subseteq U$. We have

$$\overline{G} \times \Delta(X) \subseteq \mu^{-1}[U].$$

So we may find an open cover $\mathcal{W}$ of X such that

$$\bigcup\{\overline{G} \times W \times W : W \in \mathcal{W}\} \subseteq \mu^{-1}[U].$$

Since X is a Mal'tsev space it satisfies the weak form of Ungar's Theorem by Theorem 4.2.4. We apply this observation to the open cover $\mathcal{W}$ to obtain an open cover $\mathcal{V}$ of X such that whenever $y, z \in V \in \mathcal{V}$, there is a continuous function $f : X \to X$ such that $f(y) = z$ and f is limited by $\mathcal{W}$.

Now choose $V \in \mathcal{V}$ such that $x \in V$. We claim that $V \cap G$ is as required. To see this, let $y, z \in V \cap G$ and suppose that y and z are connected in X. Let C be the component of X containing y and z. By construction there is a continuous function $f : X \to X$ such that $f(y) = z$ and f is limited by $\mathcal{W}$. We define a function $g : X \to X$ by $g(w) = \mu(y, w, f(w))$. Then g is continuous. Since $f(y) = z$, we have $f[C] \subseteq C$ and therefore there is some $v \in C$ such that $f(v) = v$. It follows that $g(v) = \mu(y, v, f(v)) = \mu(y, v, v) = y$. Since we also have that $g(y) = \mu(y, y, f(y)) = f(y) = z$, the set $C' = g[C]$ is a connected set containing both y and z. We will show that $C' \subseteq U$.

So suppose that $w \in C$. Then $\{w, f(w)\} \subseteq W$ for some $W \in \mathcal{W}$. Since $y \in G$, it follows that $(y, w, f(w)) \in \mu^{-1}[U]$ and therefore

$$g(w) = \mu(y, w, f(w)) \in U.$$

Since w was an arbitrary element of C, we have shown that $C' = g[C] \subseteq U$ and this completes the proof. □

4.4.9. COROLLARY. *Every Mal'tsev continuum with the fixed point property is locally connected.*

PROOF. This follows from Theorem 4.4.8 and Lemma 4.4.2. □

Note that if C is a component of a compact Mal'tsev space X, then C is a Mal'tsev continuum. To see this, let μ be a Mal'tsev function on X. Since C^3 is a connected subset of X^3, it follows that $\mu[C^3]$ is also connected. If $x \in X$, then $\mu(x, x, x) = x$ and therefore it follows that $\mu[C^3] \subseteq C$, i.e., the restriction of μ to C^3 is a Mal'tsev function on C. It follows from the previous result that if C is a component in a compact Mal'tsev space and C has the fixed point property, then C is locally connected.

We have already mentioned the fact that non-degenerate indecomposable continua are not locally connected (see Remark 4.4.7) so we obtain the following corollary;

4.4.10. COROLLARY. *Suppose X is a non-degenerate indecomposable continuum with the fixed point property. Then X is not a Mal'tsev space and hence not a retract of a topological group.*

4.4.11. REMARK. The dyadic solenoid is an example of a topological group which is also an indecomposable continuum. This example shows that the fixed point property is essential in the previous result. The pseudo-arc is homogeneous and it is the unique non-degenerate metric continuum which is chainable and hereditarily indecomposable, see BING [14, 15]. Furthermore, the pseudo-arc has the fixed point property, and therefore it is not a Mal'tsev space by the previous corollary. So this is an example of a compact coset space which is not a Mal'tsev space.

4.5 Power homogeneous spaces

In this section we study some structural properties of power homogeneous spaces. The arguments are all of a similar type; we start with some local topological property. Next we show that it is preserved by taking products and projections. It follows that if X is a power homogeneous space, then either all points possess this property or no points at all. This type of argument has been carried out before, for example by VAN DOUWEN [19, §6] and ARHANGEL'SKIĬ [11].

4.5.1. LEMMA. *Let $X = \prod\{X_i : i \in I\}$ and $x = (x_i) \in X$. Suppose X is compact and I is finite. If for all $i \in I$, X_i satisfies the weak form of Ungar's Theorem locally at x_i then X satisfies the weak form of Ungar's Theorem locally at x.*

PROOF. Let $\mathcal{U}$ be an open cover of X. We may assume that $\mathcal{U}$ consists of basic open subsets of X and by compactness, we may assume that $\mathcal{U}$ is finite. We further assume that $I = \{1, \dots, n\}$ for some $n < \omega$. For $U \in \mathcal{U}$, we have $U = U_1 \times \dots \times U_n$. For now we fix $i \in I$. For every $y \in X_i$, we define

$$W_y = \bigcap\{U_i : y \in U_i \text{ and } U \in \mathcal{U}\}.$$

Since the set $\{U_i : U \in \mathcal{U}\}$ is a finite open cover of X_i, it follows that the collection $\mathcal{W}_i$ given by $\{W_y : y \in X_i\}$ is an open cover of X_i. Applying the fact that X_i satisfies the weak form of Ungar's Theorem locally at x_i, we find an open neighbourhood V_i of x_i which satisfies the desired properties with respect to the cover $\mathcal{W}_i$.

We let $V = V_1 \times \ldots \times V_n$ and $\mathcal{W}$ is defined as follows;

$$\mathcal{W} = \{W_1 \times \ldots \times W_n : \text{ for all } i \in I,\ W_i \in \mathcal{W}_i\}.$$

We will show that the open neighbourhood V of x in X satisfies the desired properties with respect to the cover $\mathcal{U}$, but first we show that $\mathcal{W}$ is a refinement of $\mathcal{U}$. So suppose $W \in \mathcal{W}$. Then W is of the form $W_{y_1} \times \ldots \times W_{y_n}$ where $y_i \in X_i$ for $i \in I$. Let $y = (y_1, \ldots, y_n)$. Since $\mathcal{U}$ is a cover of X, there is some $U \in \mathcal{U}$ such that $y \in U$. But then $y_i \in U_i$ for $i \in I$ and by definition of W_{y_i}, it follows that $W_{y_i} \subseteq U_i$ for all $i \in I$. It follows that $W \subseteq U$. Since W was arbitrary, this shows that $\mathcal{W}$ is a refinement of $\mathcal{U}$.

Now we go on to show that for every $y, z \in V$, there is a map $f : X \to X$ such that $f(y) = z$ and f is limited by $\mathcal{U}$. So let $y, z \in V$ be arbitrary. Then $y_i, z_i \in V_i$ for all $i \in I$, so by construction there are maps $f_i : X_i \to X_i$ such that $f_i(y_i) = z_i$ and f_i is limited by $\mathcal{W}_i$. We let $f : X \to X$ be the product of the f_i's. Then clearly f maps y onto z and f is limited by $\mathcal{W}$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$, it follows that f is also limited by $\mathcal{U}$ and this completes the proof. □

4.5.2. THEOREM. *Let $X = \prod\{X_i : i \in I\}$ and $x = (x_i) \in X$. If X is compact, then the following statements are equivalent;*

(1) *X satisfies the weak form of Ungar's Theorem locally at x,*

(2) *For all $i \in I$, X_i satisfies the weak form of Ungar's Theorem locally at x_i.*

PROOF. (1) → (2): If $i \in I$, then we may view X_i is a retract of X where x is mapped onto x_i. So if X satisfies the weak form of Ungar's Theorem locally at x, then it follows from Proposition 4.2.3 that for all $i \in I$, X_i satisfies the weak form of Ungar's Theorem locally at x_i.

(2) → (1): Suppose that for all $i \in I$, X_i satisfies the weak form of Ungar's Theorem locally at x_i. Let $\mathcal{U}$ be an open cover of X. We may assume that $\mathcal{U}$ consists of basic open subsets of X and by compactness we may assume that $\mathcal{U}$ is finite. Then there is a finite set of coordinates $J \subseteq I$ such that the members of $\mathcal{U}$ depend only on the coordinates in J. Let $Y = \prod\{X_i : i \in J\}$ and $Z = \prod\{X_i : i \in I \setminus J\}$. The restriction of $\mathcal{U}$ to Y is an open cover of Y and by Lemma 4.5.1, Y satisfies the weak form of Ungar's Theorem locally at the point $(x_i)_{i \in J}$. We may therefore find an open neighbourhood V of $(x_i)_{i \in J}$ in Y with the desired properties.

We will show that the neighbourhood $V \times Z$ of x in X satisfies the required properties. So let $p, q \in V \times Z$. By construction there is a map

$f : Y \to Y$ which maps p_J onto q_J such that f is limited by the restriction of $\mathcal{U}$ to Y. We let $g : Z \to Z$ be the constant map taking the value $q_{I \setminus J}$. The product $f \times g$ of the mappings f and g is a continuous map from X to X and it maps p onto q. Since f is limited by the restriction of $\mathcal{U}$ to Y and the members of $\mathcal{U}$ depend only on the coordinates in J, it follows that $f \times g$ is limited by $\mathcal{U}$ and this completes the proof. □

4.5.3. COROLLARY. *Suppose X is a power homogeneous and compact space. If X satisfies the weak form of Ungar's Theorem locally at some point, then it satisfies the weak form of Ungar's Theorem (at all points).*

4.5.4. LEMMA. *Suppose $r : X \to Y$ is a retraction and let $y \in Y$. If the components of X are regularly locally connected at y, then the components of Y are also regularly locally connected at y.*

PROOF. We use the characterization from Proposition 4.4.4; so let U be an arbitrary open neighbourhood of y in Y. Then $r^{-1}[U]$ is an open neighbourhood of y in X, so we may find an open neighbourhood V of y in X such that whenever $x, z \in V$ are connected in X then they are connected in $r^{-1}[U]$. We claim that $W = Y \cap V$ is as required. To see this, let $x, z \in W$ be arbitrary and suppose x, z are connected in Y. Then these points are also connected in X. Since $W \subseteq V$ it follows that x and z are connected in $r^{-1}[U]$. But then there is a connected set $C \subseteq r^{-1}[U]$ with $x, z \in C$. Then $r[C]$ is a connected set which contains x and z and $r[C] \subseteq U$. This completes the proof. □

4.5.5. LEMMA. *Let $X = \prod\{X_i : i \in I\}$ and suppose $x = (x_i) \in X$. The following are equivalent:*

(1) *The components of X are regularly locally connected at x.*

(2) *For every $i \in I$, the components of X_i are regularly locally connected at x_i.*

PROOF. (1) $\to$ (2): If $i \in I$, then we may view X_i is a retract of X where x is mapped onto x_i. So (1) $\to$ (2) follows from Lemma 4.5.4.

(2) $\to$ (1): Suppose U is an arbitrary neighbourhood of x in X. We may assume that U is a basic open subset of X and therefore there is a finite set $J \subseteq I$ such that $U = \prod\{U_i : i \in I\}$, where for every $i \in I$, U_i is an open neighbourhood of x_i and if $i \notin J$, then $U_i = X_i$. For every $j \in J$, we may find a neighbourhood V_j of x_j such that whenever C_j is a component of X_j, there is a connected set C'_j in X_j such that

$$C_j \cap V_j \subseteq C'_j \subseteq U_j.$$

We let $V = \prod\{V_i : i \in I\}$ where $V_i = X_i$ if $i \notin J$. We claim that V is as required. To see this, suppose C is a component of X. Then C is the product of components in the X_i's (see [22, Theorem 6.1.21]), so we may write $C = \prod\{C_i : i \in I\}$, where for every $i \in I$, C_i is a component of X_i. By construction, for every $j \in J$, there are connected sets C'_j in X_j such that $C_j \cap V_j \subseteq C'_j \subseteq U_j$. For $i \notin J$, we let $C'_i = C_i$ and we set $C' = \prod\{C'_i : i \in I\}$. Then C' is a connected subset of X (see [22, Theorem 6.1.15]). Furthermore, it is clear that $C \cap V \subseteq C' \subseteq U$. This completes the proof. □

4.5.6. COROLLARY. *Suppose X is power homogeneous and for some $x \in X$, the components of X are regularly locally connected at x. Then the components of X are regularly locally connected (everywhere).*

4.5.7. COROLLARY. *Suppose X is power homogeneous and connected. If X is locally connected at x for some $x \in X$, then X is locally connected (everywhere).*

PROOF. This follows from Lemma 4.4.2 and Corollary 4.5.6. □

The assumption that X is connected cannot be omitted from the previous corollary; see the remarks after Corollary 4.5.10.

4.5.8. PROPOSITION. *Suppose $r : X \to Y$ is a retraction, $y \in Y$ and $n \in \omega$. If X satisfies the implication $C^n \to LC^n$ at y, then Y also satisfies the implication $C^n \to LC^n$ at y.*

PROOF. Let U be an open neighbourhood of y and $0 \leq m \leq n$. We may find a neighbourhood V of y in X which satisfies the extension property with respect to the open subset $r^{-1}[U]$ of X. Then $W = Y \cap V$ is an open neighbourhood of y in Y. We will show that this W satisfies the required extension property. So assume that $f : \mathbb{S}^m \to W \subseteq V$ is a continuous function which can be extended to a function from B^{m+1} into Y. By construction, there is a continuous extension $\bar{f} : B^{m+1} \to r^{-1}[U]$. We let $g = r \circ \bar{f}$. Since $r(z) = z$ for all $z \in Y$, it follows that g is an extension of f. Since the range of f is contained in the set $r^{-1}[U]$, the range of g is contained in U and this completes the proof. □

4.5.9. PROPOSITION. *Suppose $X = \prod\{X_i : i \in I\}$ and $x = (x_i) \in X$. For $n \in \omega$, the following statements are equivalent;*

(1) *X satisfies the implication $C^n \to LC^n$ locally at x,*

(2) *For all $i \in I$, X_i satisfies the implication $C^n \to LC^n$ locally at x_i.*

PROOF. (1) $\rightarrow$ (2): This follows from the previous proposition and the fact that we may view X_i as a retract of X under some retraction which maps x onto x_i.

(2) $\rightarrow$ (1): Let U be an arbitrary open neighbourhood of x in X and $0 \leq m \leq n$. We may assume that U is a basic open subset of X. Then U depends on a finite set of coordinates $J \subseteq I$. For every $i \in J$, we let V_i be an open neighbourhood of x_i which satisfies the extension property with respect to U_i. For $i \in I \setminus J$, we let $V_i = X_i$ and we set $V = \prod\{V_i : i \in I\}$. Then V is an open neighbourhood of x in X and we claim that this neighbourhood satisfies the desired extension property with respect to U. To this end, suppose that $f : \mathbb{S}^m \to V$ is a continuous function which can be extended to a continuous function $g : B^{m+1} \to X$. For $i \in I$, we let $f_i = \pi_i \circ f$ and $g_i = \pi_i \circ g$. By construction, for every $i \in J$ we may find a continuous extension $\bar{f}_i$ of f_i from B^{m+1} into U_i. We let $\bar{f}$ be defined a follows for $t \in B^{m+1}$ and $i \in I$;

$$\bar{f}(t)_i = \begin{cases} \bar{f}_i(t) & \text{if } i \in J, \\ g_i(t) & \text{if } i \notin J. \end{cases}$$

Then $\bar{f}$ is a continuous extension of f and by construction the range of $\bar{f}$ is contained in the open set U. This completes the proof. □

4.5.10. COROLLARY. *Suppose X is power homogeneous and let $n \in \omega$. If X satisfies the implication $C^n \rightarrow LC^n$ locally at some point of X then X satisfies the implication $C^n \rightarrow LC^n$ (everywhere).*

It follows from the previous corollary that if a power homogeneous space is LC^n at some point, then it satisfies the implication $C^n \rightarrow LC^n$. Simple examples show that the implication $C^n \rightarrow LC^n$ in the conclusion cannot be replaced by just LC^n. For example, if Z is a convergent sequence then $Z \times \mathbb{I}$ is power homogeneous and LC^0 at some but not at all points. Also note that $Z \times I$ is locally connected at some but not all points, which shows that in Corollary 4.5.7, the assumption that X is connected cannot be omitted.

Chapter 5

Examples and counterexamples

5.1 Introduction

In this chapter we present some examples and counterexamples related to the topics discussed in this thesis. As an introduction we give a description of the well-known $\sin(1/x)$-continuum. This space is not a retract of a homogeneous compact space and in particular it is not power homogeneous. Furthermore, it is neither a retract of a coset space nor is it a Mal'tsev space.

In the first section of this chapter we give some examples of compact homogeneous spaces. The first example is due to VAN MILL [47]. It is an example of a compact space which is homogeneous under $\mathsf{MA} + \neg\mathsf{CH}$ but not homogeneous under CH. The original construction of Van Mill produced such a space which is infinite dimensional. It was shown by HART and RIDDERBOS [31] that the construction of Van Mill may be modified to obtain a zero-dimensional compact space with similar properties. We present the modification from [31].

Our next example is related to coset spaces. Recall that coset spaces are homogeneous and that if a space is homogeneous, separable, metrizable and locally compact then it is a coset space (see Corollary 4.2.2). FORD [25] gave an example of a homogeneous space which is not a coset space. Ford's example is neither metrizable nor locally compact. VAN MILL [48] gave an example of a metrizable homogeneous space which is not a coset space. Of course, this example cannot be locally compact, it is however σ-compact. We present an improvement of Ford's example in the other direction; we give an example of a compact homogeneous space which is not (a retract of) a coset space. The construction is similar to the construction of the first example, except that no additional set-theory is required.

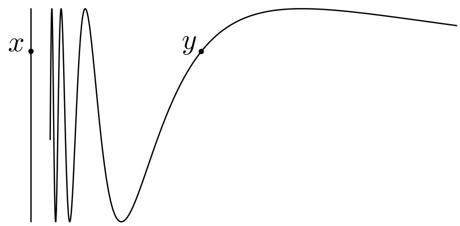

Figure 5.1: The $\sin(1/x)$-continuum

Recall that the $\sin(1/x)$-continuum is an example of a compact space which is not a retract of a compact homogeneous space. This is of interest because every space is a retract of a homogeneous space by USPENSKIĬ [69]. We give an example which is similar in spirit; the example is a metrizable space which is a retract of a compact homogeneous space but not a retract of a compact *metrizable* homogeneous space.

Next we present some applications of the results in §4.5. We first present an example of a compact space which is the quotient of some retract of a coset space. Since the example does not satisfy the weak form of Ungar's Theorem, it follows that this property is not preserved by quotient mappings. We shall also show that the example is not power homogeneous. The next examples are also not power homogeneous but they do satisfy the weak form of Ungar's Theorem. To prove non-power homogeneity, we examine some connectivity properties of the spaces under consideration.

In the final section of this chapter we study the difference between Δ-homogeneity and homogeneity in powers of spaces. We have shown that a space is power homogeneous if and only if it is Δ-power homogeneous. Furthermore if X is power homogeneous then $X^{\pi w(X)}$ is Δ-homogeneous. So the question arises whether in this case $X^{\pi w(X)}$ is also homogeneous. We do not know the answer to this question, but we will present an example of a power homogeneous space X such that X^2 is Δ-homogeneous but not homogeneous. Also, for every infinite cardinal κ, we give an example of a power homogeneous space X for which $\mathrm{hdeg}(X) = \kappa$; i.e., X^κ is homogeneous, but X^λ is not homogeneous for every $\lambda < \kappa$.

5.1.1. EXAMPLE. In this example we give a description of the well-known $\sin(1/x)$-continuum. It is the following subset of the plane, see also Figure 5.1;

$$\big\{(0, x) : x \in [-1, 1]\big\} \cup \big\{\big(x, \sin(1/x)\big) : x \in (0, 1]\big\}.$$

The $\sin(1/x)$-continuum is connected but not path-connected. Furthermore, it has the fixed point property. We use X to denote the $\sin(1/x)$-continuum, A is the closed subset $\{0\} \times [-1,1]$ and B is the complement of A in X. The points $x \in A$ and $y \in B$ are marked in Figure 5.1. If $f : X \to X$ is a continuous function such that $f(y) \in A$, then also $f[B] \subseteq A$ and since B is dense in X it follows that $f[X] \subseteq A$. This shows that X is not weakly $1\frac{1}{2}$-homogeneous, so by Corollary 4.1.4 the $\sin(1/x)$-continuum is not a retract of a compact homogeneous space.

Since X is not weakly $1\frac{1}{2}$-homogeneous, it is not a Mal′tsev space by Proposition 4.1.5. In particular X is not retral. To see that X is not a retract of a coset space we use Proposition 4.2.5. Note that any continuous function f which maps some point of B onto x moves every point of B into A. Using Proposition 4.2.5 it is clear that X is not a retract of a coset space.

Recall that by USPENSKIĬ [69], there is some homogeneous space Y such that $X \times Y$ is homeomorphic to Y. In particular, X is a retract of Y. So this space Y is an example of a homogeneous space which is not a coset space. Below we shall construct a compact homogeneous space which is not a coset space. ♢

5.2 Homogeneous compacta

We start this section by describing resolutions. The method of generating topologies by way of resolutions is due to FEDORČUK [24]. More information can be found in WATSON [75]. Suppose X is a space, $\{Y_x : x \in X\}$ is a collection of spaces and for each $x \in X$, $f_x : X \setminus \{x\} \to Y_x$ is a continuous function. Let Z be the space given by

$$\bigcup \big\{\{x\} \times Y_x : x \in X\big\}.$$

If $x \in X$, U_x is an open neighbourhood of x in X and W is an open subset of Y_x, then the subset $U_x \otimes W$ of Z is given by

$$\big(\{x\} \times W\big) \cup \bigcup \big\{\{x'\} \times Y_{x'} : x' \in U_x \cap f_x^{-1}[W]\big\}.$$

Using these sets, the following collection may serve as a basis for a topology on Z;

$$\big\{U_x \otimes W : \big(U_x \in \tau(X), x \in U_x\big) \text{ and } W \in \tau(Y_x)\big\}$$

Topologized in this way, the space Z is called the resolution of X at each point $x \in X$ into Y_x by the mapping f_x. In this context, we denote by π_0 the natural projection from Z onto X. It is easily verified that π_0 is continuous. The following results are well-known, see FEDORČUK [24] and WATSON [75] for details.

5.2.1. LEMMA. *Let Z be the resolution of X at each point $x \in X$ into Y_x be the mapping f_x. If X and each Y_x is compact, then so is Z.*

5.2.2. LEMMA. *Let Z be the resolution of X at each point $x \in X$ into Y_x be the mapping f_x. If X and each Y_x is zero-dimensional, then so is Z.*

We now come to the main theorem of this section. This theorem will provide us with some curious examples of homogeneous compact spaces.

5.2.3. THEOREM. *Suppose Y is a homogeneous compactum which satisfies the following conditions;*

(1) $w(Y) < \mathfrak{p}$,

(2) *There is a homeomorphism $\eta : Y \to Y$ such that the set $\{\eta^n(d) : n \in \omega\}$ is dense in Y for some $d \in Y$.*

(3) *Y is a product of second countable compacta.*

Then there are maps $f_x : \mathsf{C} \setminus \{x\} \to Y$ for every $x \in \mathsf{C}$, such that the resolution X of C at each point $x \in \mathsf{C}$ into Y by the mapping f_x is homogeneous. Furthermore, the π-weight of X is countable.

We shall prove this theorem in several steps. We fix a compact space Y as in the statement of the theorem, along with the map η and the point d; for each $n \in \omega$ we write $d_n = \eta^n(d)$. It is convenient to identify X with the product $\mathsf{C} \times Y$; we keep in mind that the topology on X is the resolution topology induced by the mappings f_x which will be defined below, it is not the product topology. We fix some notation.

We will denote elements of X by $\langle x, y\rangle$ where $x \in \mathsf{C}$ and $y \in Y$. Given $s \in 2^{<\omega}$, so s is a finite sequence of zeroes and ones, we put

$$[s] = \{x \in \mathsf{C} : s \subseteq x\}.$$

The family $\{[s] : s \in 2^{<\omega}\}$ is the canonical base for the topology of C. If $s \in 2^{<\omega}$ and $x \in \mathsf{C}$ then $s * x$ denotes the concatenation of s and x.

Given $x \in \mathsf{C}$ and $n \in \omega$ we put $U_{x,n} = [x \restriction n]$, the n^{th} basic neighbourhood of x, and $C_{x,n} = U_{x,n} \setminus U_{x,n+1}$. Note that $C_{x,n}$ is of the form $U_{y,n+1}$ for some suitably chosen $y \in \mathsf{C}$. For $x \in \mathsf{C}$, we define the map $f_x : \mathsf{C} \setminus \{x\} \to Y$ as follows, for $y \in \mathsf{C} \setminus \{x\}$;

$$f_x(y) = d_n \overset{\text{def}}{\iff} y \in C_{x,n}.$$

The sets of the form $U_{x,n} \otimes W$, where $x \in X$, $n \in \omega$ and W an open subset of Y, form a basis for the resolution topology on X. Note that x' is an element of the set $U_{x,n} \cap f_x^{-1}[W]$ if and only if $x' \in C_{x,m}$ for some $m \geq n$ with $d_m \in W$. So we have;

$$(\dagger) \qquad U_{x,n} \otimes W = \big(\{x\} \times W\big) \cup \bigcup \big\{C_{x,m} \times Y : m \geq n, d_m \in W\big\}.$$

The following lemma shows that the π-weight of X is countable;

5.2.4. LEMMA. *The family $\{[s] \times Y : s \in 2^{<\omega}\}$ is a π-base for X, hence $\pi w(X) = \aleph_0$.*

PROOF. If U is an non-empty open subset of X, then $U_{x,n} \otimes W \subseteq U$ for some suitably chosen x, n and W. Since the collection $\{d_m : m \geq n\}$ is dense in Y, it follows from (†) that $C_{x,m} \times Y \subseteq U$ for some $m \in \omega$. Since $C_{x,m} = [s]$ for some $s \in 2^{<\omega}$, this completes the proof. □

To prove homogeneity of X we will first show that points with the same second coordinate can be mapped onto each other by a homeomorphism. Recall that C is a topological group, by $+$ we denote the Boolean group operation. For $a \in \mathsf{C}$, we define a map $T_a : X \to X$ as follows, for $\langle x, y \rangle \in X$;

$$T_a(x, y) = \langle x + a, y \rangle.$$

We show that for all $a \in \mathsf{C}$, the map T_a is a homeomorphism. Note that if $x, x' \in \mathsf{C}$ and $a = x + x'$, then for $y \in Y$, $T_a(x, y) = \langle x', y \rangle$. So the mappings T_a show that points of X with the same second coordinate are equivalent.

5.2.5. LEMMA. *For each $a \in \mathsf{C}$, the map $T_a : X \to X$ is a homeomorphism.*

PROOF. Since addition is a group operation on the Cantor set C, it follows that T_a is a bijection. To prove that T_a and its inverse are continuous, we will show that for $x \in \mathsf{C}$, $n < \omega$ and $W \in \tau(Y)$,

$$T_a[U_{x,n} \otimes W] = U_{x+a,n} \otimes W.$$

We use (†); First note that $T_a[\{x\} \times W] = \{x + a\} \times W$. Next it is not hard to realize that $T_a[C_{x,m} \times Y] = C_{x+a,m} \times Y$. Using (†), these equalities clearly imply that the image of $U_{x,n} \otimes W$ under T_a is the set $U_{x+a,n} \otimes W$.

It now follows that both T_a and its inverse are continuous and this completes the proof. $\square$

The hard work will be in establishing that points with the same first coordinate are similar. We begin by showing that the special clopen sets $[s] \times Y$ are all homeomorphic and we give canonical homeomorphisms between them.

5.2.6. LEMMA. *Let $s, t \in 2^{<\omega}$, put $k = |t| - |s|$ and define $\xi_{s,t} : [s] \times Y \to [t] \times Y$ by $\xi_{s,t}(s * x, y) = \langle t * x, \eta^k(y) \rangle$. Then $\xi_{s,t}$ is a homeomorphism.*

PROOF. Note that $\xi_{s,t}$ is a bijection. Furthermore, note that $\xi_{s,t}^{-1} = \xi_{t,s}$ and that $\xi_{s,t} = \xi_{\emptyset,t} \circ \xi_{s,\emptyset}$. Therefore, it suffices to prove that for $t \in 2^{<\omega}$, the map $\xi_{\emptyset,t}$ is a homeomorphism. For ease of notation, we denote $\xi_{\emptyset,t}$ by ξ_t. We will prove that for $x \in \mathsf{C}$, $n < \omega$, $W \in \tau(Y)$ and $k = |t|$,

$$\xi_t[U_{x,n} \otimes W] = U_{t*x,n+k} \otimes \eta^k[W].$$

We make the following three observations;

(1) $\xi_t[\{x\} \times W] = \{t * x\} \times \eta^k[W]$,

(2) $\xi_t[C_{x,m} \times Y] = C_{t*x,m+k} \times Y$,

(3) $\big(m \geq n \text{ and } d_m \in W\big) \leftrightarrow \big(m + k \geq n + k \text{ and } d_{m+k} \in \eta^k[W]\big)$.

The proofs are straightforward and left to the reader, recall for (3) that $d_{m+k} = \eta^{m+k}(d) = \eta^k(d_m)$. Using (†), it follows that the image of $U_{x,n} \otimes W$ under ξ_t is the set $U_{t*x,n+k} \otimes \eta^k[W]$. Since the sets of the form $U_{x,n} \otimes W$ form a basis for X and sets of the form $U_{t*x,n+k} \otimes \eta^k[W]$ form a basis for $[t] \times Y$, it follows that both ξ_t and its inverse are continuous and this completes the proof. $\square$

For ease of notation we let e be the point of C with all coordinates zero and we abbreviate $U_{e,n}$ and $C_{e,n}$ by U_n and C_n respectively. We shall prove that for every homeomorphism $f : Y \to Y$ there is a homeomorphism $F : X \to X$ such that $F(e, y) = \langle e, f(y) \rangle$ for all $y \in Y$. This will complete the proof that X is homogeneous because it shows that all points of the form $\langle e, y \rangle$ are similar.

For $n \in \omega$, let $x_n \in \mathsf{C}$ be the point in C_n with all coordinates zero except for the n^{th}. Let $E = \{\langle x_n, d_n \rangle : n \in \omega\}$. Note that the collection $\{C_n \times Y : n \in \omega\}$ forms a family of pairwise disjoint open subsets of X

and $\langle x_n, d_n\rangle \in C_n \times Y$. It follows that E is a discrete subset of X. So the closure of E in X is homeomorphic to a compactification of ω. We aim at applying Theorem 2.5.5 and for this we first prove that the remainder of this compactification is homeomorphic to Y; this is the following lemma.

5.2.7. LEMMA. *$\overline{E} = (\{e\} \times Y) \cup E$.*

PROOF. First let $y \in Y$ be arbitrary and let $U_n \otimes W$ be a basic open neighbourhood of $\langle e, y\rangle$ in X. Since $\{d_m : m \geq n\}$ is dense in Y, it follows from (†) that $\langle x_m, d_m\rangle \in C_m \times Y \subseteq U_n \otimes W$ for some $m \geq n$. It follows that $\langle e, y\rangle \in \overline{E}$ and since y was arbitrary, this proves that $\{e\} \times Y \subseteq \overline{E}$.

Next consider an element of the form $\langle x, y\rangle$ where $x \neq e$ and $y \notin \{d_n : n \in \omega\}$. Since $x \neq e$, there is some $n < \omega$ such that $x \in C_n$ and therefore $C_n \times Y$ is an open neighbourhood of $\langle x, y\rangle$ in X. The set $(C_n \times Y)\setminus\{\langle x_n, d_n\rangle\}$ is an open neighbourhood of $\langle x, y\rangle$ which misses E and therefore $\langle x, y\rangle \notin \overline{E}$. This completes the proof. $\square$

Now fix an arbitrary homeomorphism f of Y. Since $\{e\} \times Y$ is homeomorphic to Y, we may think of f as a homeomorphism of $\{e\} \times Y$. We have just shown that $\{e\} \times Y$ is the remainder of the compactification $\overline{E}$ of E and E is homeomorphic to ω. Since $\{e\} \times Y$ is homeomorphic to Y, it is a product of second countable compacta and it follows from Proposition 2.5.6 that $\{e\} \times Y$ is a retract of E. Using the fact that $w(Y) < \mathfrak{p}$, we now apply Theorem 2.5.5 to obtain a permutation σ of ω such that $f \cup \sigma$ is a homeomorphism of $\overline{E}$, where we let σ act on E in the obvious way.

By Lemma 5.2.6, for every $n, n' \in \omega$, we may fix a homeomorphism $\zeta_{n,n'} : U_n \times Y \to U_{n'} \times Y$; we let s be the sequence of length n consisting of only 0's and t is such a sequence of length n'. We let $\zeta_{n,n'} = \xi_{s,t}$. Note that $\zeta_{n,n'}(x_n, d_n) = \langle x_{n'}, d_{n'}\rangle$. We now define $F : X \to X$ as follows for $\langle x, y\rangle \in X$,

$$F(x,y) = \begin{cases} \langle e, f(y)\rangle & \text{if } x = e, \\ \zeta_{n,n'}(x,y) & \text{if } x \in C_n \text{ and } \sigma(n) = n'. \end{cases}$$

The map F is clearly a bijection of X. To prove continuity of F and its inverse, it suffices to prove that F is continuous and open in all points of the set $\{e\} \times Y$. By compactness of X, it suffices to prove that F is an open mapping in all points of the form $\langle e, y\rangle$.

Note that since $\zeta_{n,n'}(x_n, d_n) = \langle x_{n'}, d_{n'}\rangle$, the restriction of F to $\overline{E}$ is in fact a bijection of $\overline{E}$. So by the choice of σ, the restriction of F to $\overline{E}$ is a homeomorphism of $\overline{E}$.

To show that F is open in points of the form $\langle e, y\rangle$, let $y \in Y$ be arbitrary and fix a basic open neighbourhood $U_n \otimes W$ of $\langle e, y\rangle$ in X. Note that $F(e, y) \in \{e\} \times Y$, so let $F(e, y) = \langle e, y'\rangle$. Since $F \upharpoonright \overline{E}$ is a homeomorphism of $\overline{E}$, we may find a basic open neighbourhood $U_{n'} \otimes W'$ of $\langle e, y'\rangle$ such that

$$\left(U_{n'} \otimes W'\right) \cap \overline{E} \subseteq F\left[U_n \otimes W\right].$$

We will prove that in fact $U_{n'} \otimes W'$ is contained in $F[U_n \otimes W]$. So let $\langle x, z\rangle \in U_{n'} \otimes W'$ be arbitrary. There are two cases two consider;

Case 1: $x = e$. In this case $\langle x, z\rangle \in \{e\} \times W'$ and since $\{e\} \times W' \subseteq \overline{E}$ it follows that $\langle x, z\rangle \in F[U_n \otimes W]$.

Case 2: $x \neq e$. In this case $x \in C_{m'}$ for some $m' < \omega$. It follows from (†) that $C_{m'} \times Y \subseteq U_{n'} \otimes W'$ and therefore $\langle x_{m'}, d_{m'}\rangle \in F[U_n \otimes W]$. By definition of F, it follows that $\langle x_m, d_m\rangle \in U \otimes W$, where $\sigma(m) = m'$. Again using (†), it follows that $C_m \times Y \subseteq U_n \otimes W$ and therefore $C_{m'} \times Y \subseteq F[U_n \otimes W]$. So in particular we have $\langle x, z\rangle \in F[U_n \otimes W]$.

We have shown that

$$U_{n'} \otimes W' \subseteq F\left[U_n \otimes W\right].$$

and this proves that the mapping F is a homeomorphism of X. We summarize the homogeneity of X in the following result;

5.2.8. COROLLARY. *The space X is homogeneous.*

PROOF. Let $\langle x, y\rangle$ and $\langle x', y'\rangle$ be arbitrary elements of X. Since Y is homogeneous, there is a homeomorphism f of Y such that $f(y) = y'$. We have just proved that there is a homeomorphism F of X such that $F(e, y) = \langle e, f(y)\rangle = \langle e, y'\rangle$. Furthermore, the map T_x maps $\langle x, y\rangle$ onto $\langle e, y\rangle$ and the map $T_{x'}$ maps the point $\langle e, y'\rangle$ onto $\langle x', y'\rangle$. So if we let $G = T_{x'} \circ F \circ T_x$, then G is a homeomorphism of X which maps $\langle x, y\rangle$ onto $\langle x', y'\rangle$. □

This proves Theorem 5.2.3. We now apply this theorem to provide an example of a compact space X which is homogeneous under $\mathsf{MA} + \neg\mathsf{CH}$ but inhomogeneous under CH. The next lemma provides the necessary input space Y;

5.2.9. LEMMA. *Let $Y = 2^{\omega_1 \times \mathbb{Z}}$. Then Y is a homogeneous compactum which satisfies* (2) *and* (3) *of Theorem 5.2.3.*

PROOF. It is clear that the space Y is a product of second countable compacta and that it is homogeneous. To complete the proof, we need to find an autohomeomorphism η and a point d such that $\{\eta^n(d) : n \in \omega\}$ is dense in Y. The map $\eta : Y \to Y$ is defined coordinatewise as follows (with $\alpha \in \omega_1$ and $i \in \mathbb{Z}$),

$$\eta(y)(\alpha, i) = y(\alpha, i+1).$$

We may think of points of Y as ω_1 by $\mathbb{Z}$ matrices. The action of η on such a matrix consists of shifting every row one step downwards. For $n < \omega$, $[-n, n]$ is the set $\{-n, -n+1, \ldots, n-1, n\}$.

To make the point d we take a countable dense subset Q of $2^{\omega_1 \times \mathbb{Z}}$ and we enumerate $Q \times \omega$ as $\{\langle q_k, n_k\rangle : k < \omega\}$. We define the point d by concatenating the restrictions $q_k \restriction \omega_1 \times [-n_k, n_k]$. First write $N_k = \sum_{j<k}\bigl(2 \cdot n_k + 1\bigr)$ for all k and then define, for each α and each n,

$$d(\alpha, n) = \begin{cases} 0 & n < 0, \\ q_k(\alpha, n - N_k - n_k) & N_k \leq n < N_{k+1}. \end{cases}$$

Next we verify that the point d has a dense positive orbit under η. Observe that it follows by construction that for all $k < \omega$ we have

$$\eta^{N_k + n_k}(d) \restriction (\omega_1 \times [-n_k, n_k]) = q_k \restriction (\omega_1 \times [-n_k, n_k]).$$

Now let an arbitrary basic open subset U of $2^{\omega_1 \times \mathbb{Z}}$ be given by a function $s : F \to 2$, where $F \subseteq \omega_1 \times \mathbb{Z}$ is finite. Thus U is given by

$$U = \{y \in 2^{\omega_1 \times \mathbb{Z}} : s \subseteq y\}.$$

Since F is finite, we may find n such that $F \subseteq \omega_1 \times [-n, n]$. The set Q was chosen dense in $2^{\omega_1 \times \mathbb{Z}}$, so there is a $k < \omega$ with $q_k \in U$ and $n_k = n$. It follows that

$$q_k \restriction (\omega_1 \times [-n_k, n_k]) \supseteq s$$

from which it follows that

$$\eta^{N_k + n_k}(d) \restriction (\omega_1 \times [-n_k, n_k]) \supseteq s.$$

This implies that $\eta^{N_k + n_k}(d) \in U$.

We find that the set $\{\eta^n(d) : n < \omega\}$ is dense in $2^{\omega_1 \times \mathbb{Z}}$, which means that we are done. □

5.2.10. Example. Now let X be the resolution of the Cantor set into $Y = 2^{\omega_1 \times \mathbb{Z}}$ at each point using the resolution mappings defined after the statement of Theorem 5.2.3. Then X is a compact space of countable π-weight.

Furthermore, it is zero-dimensional by Lemma 5.2.2. Since Y embeds as a closed subspace into X, we have $\aleph_1 = \chi(Y) \leq \chi(X)$. It is not hard to see that in fact the character of X is equal to $\aleph_1$.

Now assume $\mathsf{MA} + \neg\mathsf{CH}$. Then $\aleph_1 < \mathfrak{p}$, and since $w(Y) = \aleph_1$, it follows that X is homogeneous from Theorem 5.2.3. If we assume CH, then we have $\chi(X) = \aleph_1 > \aleph_0 = \pi w(X)$. It follows from Corollary 3.4.20 that in this case X is not homogeneous.

So X is an example of a compact and zero-dimensional space which is homogeneous under $\mathsf{MA} + \neg\mathsf{CH}$ but not homogeneous under CH. $\Diamond$

The next application of Theorem 5.2.3 provides an example of a compact homogeneous space which is not a coset space;

5.2.11. EXAMPLE. It is well known that the unit circle $\mathbb{S}^1$ contains a point d and a homeomorphism η such that $\{\eta^n(d) : n \in \omega\}$ is dense in $\mathbb{S}^1$. For example, if we view $\mathbb{S}^1$ as the set $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$, then for d we may take 1, i.e., e^0, and η may be given by $\eta(e^{i\theta}) = e^{i(\theta+1)}$. Since $w(\mathbb{S}^1) = \aleph_0 < \mathfrak{p}$, we may apply Theorem 5.2.3 to obtain a homogeneous compact space X which is the resolution of the Cantor set into $\mathbb{S}^1$ at each point.

We will use Theorem 4.3.1 to show that X does not satisfy the weak form of Ungar's Theorem; so in particular X is not a retract of a coset space and X is also not a Mal'tsev space. We will show that the projection $\pi : X \to X/\mathcal{C}$ is not an open mapping. The components of X are precisely the sets $\{x\} \times \mathbb{S}^1$, thus we may identify $X/\mathcal{C}$ with the set C. Consider a basic open set of the form $U_x \otimes W$ in X. Then

$$\pi[U_x \otimes W] = \{x\} \cup \{x' \in \mathsf{C} : x' \in U_x \cap f_x^{-1}[W]\}.$$

Then $\pi^{-1}\big[\pi[U_x \otimes W]\big]$ is given by the set

$$\big(\{x\} \times \mathbb{S}^1\big) \cup \bigcup \big\{\{x'\} \times \mathbb{S}^1 : x' \in U_x \cap f_x^{-1}[W]\big\}.$$

Whenever W is not dense in $\mathbb{S}^1$, this set is not open in the resolution topology. This follows from the observation that if $V \cap W = \emptyset$ and V is open in $\mathbb{S}^1$, then $U_x \otimes V$ is an open neighbourhood of some point of $\{x\} \times \mathbb{S}^1$, but

$$U_x \cap f_x^{-1}[V] \not\subseteq U_x \cap f_x^{-1}[W].$$

This shows that the projection map $\pi : X \to X/\mathcal{C}$ is not an open mapping. Since $\mathcal{C}$ is an invariant partition of X, it follows from Theorem 4.3.1 that X does not satisfy the weak form of Ungar's Theorem. In particular, X is a compact homogeneous space which is not a coset space. $\Diamond$

5.3 Compact metric spaces

Testing for the weak form of Ungar's Theorem provides a tool for showing that certain spaces are not retracts of coset spaces. We have already seen an example of this; the $\sin(1/x)$-continuum is not a retract of a coset spaces because it does not satisfy the weak form of Ungar's Theorem. This space is not a retract of any compact homogeneous space. The next example is better in this respect; it is a retract of some homogeneous compact space, but it is not a retract of a coset space.

5.3.1. EXAMPLE. We use the space X from Example 5.2.11; it is a homogeneous compactum which does not satisfy the weak form of Ungar's Theorem. We use the notation from §5.2; so e is the point of C with all coordinates zero. The sets U_n and C_n abbreviate $U_{e,n}$ and $C_{e,n}$ and for every $n < \omega$ we pick $x_n \in C_n$ and $d_n = \eta^n(d)$. The example is the space Y which is just the set $\overline{E}$ from §5.2. It is convenient to write Y as the union of A and B where

$$\begin{aligned} A &= \{e\} \times \mathbb{S}^1, \\ B &= \big\{(x_n, d_n) : n < \omega\big\}. \end{aligned}$$

The space Y inherits the topology of X, but this coincides with the topology that Y inherits from the usual Cartesian product of the Cantor set and the circle in the plane. One can easily verify this. It suffices to note that B is a discrete subspace of X, and $(U_e \otimes W) \cap Y = (U_e \times W) \cap Y$ whenever U_e is an open neighbourhood of e in C and W is an open subset of $\mathbb{S}^1$. This equality follows from (†) in §5.2.

The components of Y are A, which is a circle, and the points of B, so all components of Y are compact. Furthermore, the components in B are all of dimension 0 and A is a component of dimension 1. Since B is dense in Y, it follows from Theorem 4.3.5 that Y does not satisfy the weak form of Ungar's Theorem.

We will show that Y is a retract of X. We define the function $r : X \to Y$ as follows,

$$r(w, z) = \begin{cases} (w, z) & \text{if } w = e, \\ (x_n, d_n) & \text{if } w \in C_n. \end{cases}$$

We show that the function r is continuous. First note that $r^{-1}[(x_n, d_n)]$ is open in X since this set is given by $C_n \times \mathbb{S}^1$ and this is just the basic open subset $C_n \otimes \mathbb{S}^1$ of X.

Next we consider basic open subsets V of Y that intersect the set A. Suppose V is given by

$$\big(\{e\} \times W\big) \cup \big\{(x_m, d_m) : m \geq n \text{ and } d_m \in W\big\},$$

where $W \subseteq \mathbb{S}^1$ is open and $n < \omega$. Then $r^{-1}[V]$ is open in X; simply observe that

$$r^{-1}[V] = \big(\{e\} \times W\big) \cup \bigcup \big\{C_m \times \mathbb{S}^1 : m \geq n \text{ and } d_m \in W\big\},$$

and this is a basic open subset of X by (†) in §5.2.

So we have shown that $r^{-1}[B]$ is open for every $B \in \mathcal{B}$ for some basis $\mathcal{B}$ of Y. It follows that r is a retraction. Note that since Y is a retract of X, it follows once again that X does not satisfy the weak form of Ungar's Theorem. Since every compact homogeneous metric space is a coset space, the space Y is an example of a compact metric space which is a retract of a compact homogeneous space, but it is not a retract of a compact metrizable homogeneous space. ♢

We continue with two examples related to dimension theory. We proved in Theorem 4.3.5 that if a compact space X satisfies the weak form of Ungar's Theorem and the set A of points $a \in X$ for which $\dim C_a \leq n$ is dense in X, then $\dim C_x \leq n$ for every $x \in X$. We first present an example to show that we cannot replace the inequality in this result with equality.

5.3.2. EXAMPLE. The example is a subspace of the plane $\mathbb{R}^2$. The space X is given by

$$X = \{0\} \cup \bigcup_{n \in \mathbb{N}} [1/(2n+1), 1/2n].$$

Using Proposition 4.2.5 it is geometrically obvious that X satisfies the weak form of Ungar's Theorem. Furthermore X is homeomorphic to a retract of the space $Z \times \mathbb{I}$ where Z is the convergent sequence given by $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ and $\mathbb{I}$ is the usual unit interval. Since $(Z \times \mathbb{I})^\omega$ is homogeneous, it follows that X is a retract of a compact homogeneous metric space and so X is a retract of a coset space.

The set $\bigcup_{n \in \mathbb{N}} [1/(2n+1), 1/2n]$ is dense in X and this set consists of components all of dimension 1. The set $\{(0,0)\}$ is a component of dimension 0, showing that the inequalities in Theorem 4.3.5 cannot be replaced by equality. ♢

Recall from ARHANGEL′SKIĬ [8] that if some point in a power homogeneous space X has a clopen basis, then X is zero-dimensional. This statement cannot be generalized to higher dimensions. For example, the disjoint sum of $\mathbb{I}$ and $\mathbb{I}^2$ is power homogeneous, but it contains components of different dimensions. The next example is similar to the previous example. However, the previous example is not power homogeneous (by the result just mentioned) whereas we will show that the next example is power homogeneous. The next example shows again that the observation from [8] cannot be generalized to higher dimensions.

5.3.3. EXAMPLE. The example is $X \times \mathbb{I}$, where X is the space form the previous example. As before, Z is the convergent sequence. Below will prove that the space $X \times \mathbb{I}$ is power homogeneous. Note that the set A consisting of all points $a \in X \times \mathbb{I}$ such that $\dim C_a = 2$ is dense but $X \times \mathbb{I}$ also contains points which are contained in a component of dimension 1. $\diamondsuit$

By Q we denote the Hilbert cube $\mathbb{I}^\omega$. The following is our main observation,

5.3.4. PROPOSITION. *The spaces $Z \times Q$ and $X \times Q$ are homeomorphic.*

PROOF. We write $X = \{0\} \cup \bigcup_{n\in\mathbb{N}} I_n$ where $I_n = [1/(2n+1), 1/2n]$. For every $n \in \mathbb{N}$ we fix a homeomorphism $h_n : I_n \to \mathbb{I}$. We define a map $h : X \times Q \to Z \times Q$ as follows. For $(x, y) \in X \times Q$, $h(x, y) = (x, y)$ if $x = 0$ and $h(x, y) = (1/n, w)$ if $x \in I_n$ and w is given by

$$w_m = \begin{cases} y_m & \text{if } m < n, \\ h_n(x) & \text{if } m = n, \\ y_{m-1} & \text{if } m > n. \end{cases}$$

Thus the set $I_n \times Q$ is mapped onto $\{1/n\} \times Q$ and the interval I_n is mapped onto the n^{th} interval in Q. It is not hard to verify that h is a homeomorphism, and this completes the proof. $\square$

5.3.5. COROLLARY. *The space $(X \times \mathbb{I})^\omega$ is homogeneous.*

PROOF. By the previous proposition it follows that

$$(X \times \mathbb{I})^\omega \approx (X \times Q)^\omega \approx (Z \times Q)^\omega \approx Z^\omega \times Q.$$

This last space is the product of the Cantor set and the Hilbert cube and is therefore homogeneous. $\square$

5.4 Connectivity properties and power homogeneity

We now present an example of a compact metric space which is not power homogeneous. We will show that it satisfies the weak form of Ungar's Theorem in some but not in all points. In particular the example is not a retract of a coset space. We will construct the example however as a quotient of a space which is a retract of a coset space. So in particular, the weak form of Ungar's Theorem is not preserved by quotient mappings.

5.4.1. EXAMPLE. Let Z be the space $(\omega + 1) \times \mathbb{S}^1$. Note that $\omega + 1$ is homeomorphic to a converging sequence. In particular, since $(\omega + 1)^\omega$ is homeomorphic to the Cantor set, Z is a retract of the compact topological group $\mathsf{C} \times \mathbb{S}^1$. We define an equivalence relation $\sim$ on Z as follows;

$$(\alpha, e^{i\phi}) \sim (\beta, e^{i\psi}) \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \phi = \psi \bmod(2\pi) & \& \quad \alpha = \beta < \omega, \\ \phi = \psi \bmod(\pi) & \& \quad \alpha = \beta = \omega. \end{cases}$$

The condition $\phi = \psi \bmod(2\pi)$ in the first line of this definition does not really change anything to the circle, but we emphasize that we let ϕ and ψ range over all real numbers when writing down elements of $\mathbb{S}^1$. The example is $Z/\sim$; we will show that it satisfies the weak form of Ungar's Theorem is some but not in all points.

We start by fixing some notation. For $(\alpha, z) \in Z$, we will use $[(\alpha, z)]$ to denote its equivalence class under $\sim$. If $z = e^{i\phi} \in \mathbb{S}^1$, then its antipodal point z^- is given by $z^- = e^{i(\phi+\pi)}$. Note that $(z^-)^- = z$.

If $\alpha \in \omega + 1$, the we let $z_\alpha = [(\alpha, z)]$. Note that $z_\omega = \{(\alpha, z), (\alpha, z^-)\} = z^-_\omega$, but for $n \in \omega$, $z_n = \{(n, z)\}$. So in Z, the equivalence classes under $\sim$ either contain one point or two points.

The mapping $q : Z \to Z/\sim$ is the quotient mapping. For all $\alpha \in \omega + 1$, $q[\{\alpha\} \times \mathbb{S}^1]$ is homeomorphic to $\mathbb{S}^1$ and for $n < \omega$, the mapping $q \restriction \{n\} \times \mathbb{S}^1$ is even a homeomorphism. A picture of the quotient space $Z/\sim$ is given in Figure 5.2. Note in particular that for $z \in \mathbb{S}^1$, the sequences $(z_n)_{n\in\omega}$ and $(z^-_n)_{n\in\omega}$ both converge to the point z_ω. We will show that the quotient space $Z/\sim$ satisfies the weak form of Ungar's Theorem in some but not in all points.

There is a natural retraction r of $Z/\sim$ onto $q[\{\omega\} \times \mathbb{S}^1]$, which maps the point z_n onto z_ω.

It is geometrically obvious that $Z/\sim$ satisfies the weak form of Ungar's Theorem in some points. For example, note that $Z/\sim$ is homeomorphic to the disjoint sum of itself with a circle. We will show that for all points

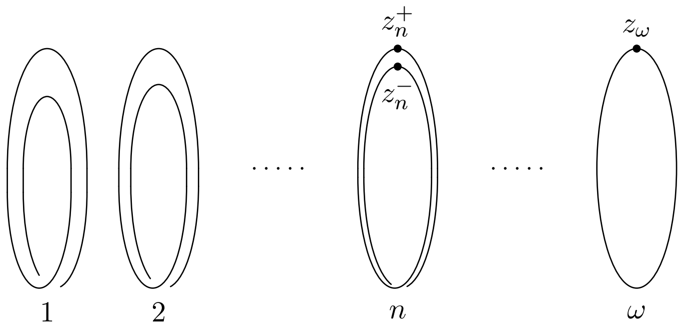

Figure 5.2: The space $Z/\!\sim$ from Example 5.4.1

in $Z/\sim$ of the form p_ω, the space $Z/\sim$ does not satisfy the weak form of Ungar's Theorem in p_ω.

To derive a contradiction we assume that Z satisfies the weak form of Ungar's Theorem in some point p_ω. We fix a real number ζ such that $0 < 4\zeta < \pi$. For $z = e^{i\phi} \in \mathbb{S}^1$, we define the set U_z as follows;

$$U_z = \{(e^{i\psi})_\omega : |\phi - \psi| < \zeta\}.$$

The set $r^{-1}[U_z]$ is an open subset of $Z/\!\sim$. Note that for $\alpha \in \omega + 1$, the set

$$\{z \in \mathbb{S}^1 : z_\alpha \in q^{-1}r^{-1}[U_z]\},$$

consists of two arcs in $\mathbb{S}^1$; an arc around z and an arc around z^-, both of diameter 2ζ. Let $\mathcal{U}$ be the open cover of $Z/\!\sim$ consisting of the sets $r^{-1}[U_z]$ for all $z \in \mathbb{S}^1$. Applying the assumption that $Z/\!\sim$ satisfies the weak form of Ungar's Theorem in p_ω, we obtain a continuous function $f : Z/\!\sim \to Z/\!\sim$ which is limited by $\mathcal{U}$ and which maps the point p_ω onto p_N for some $N < \omega$. This follows from the fact that the sequence $(p_n)_{n\in\omega}$ converges to p_ω.

Since $f(p_\omega) = p_N$, the component of p_ω, which is $q[\{\omega\} \times \mathbb{S}^1]$ is mapped into the component of p_N which is $q[\{N\} \times \mathbb{S}^1]$. Since these sets are both homeomorphic to the circle $\mathbb{S}^1$, we obtain a continuous mapping from the circle into the circle with some additional properties. We will deduce a contradiction from these properties. Let S denote the set $\{e^{i\phi} : \phi \in [0, \pi)\}$. The set S is obtained from $\mathbb{S}^1$ by identifying antipodal points. We give S the quotient topology and this makes S homeomorphic to $\mathbb{S}^1$. The set S is exactly a set of representatives for the equivalence classes of $q[\{\omega\} \times \mathbb{S}^1]$ and so we may view f as a continuous function from S into $\mathbb{S}^1$.

We fix the metric ϱ on $\mathbb{S}^1$ which measures the arc-distance between points of $\mathbb{S}^1$, so in particular $\varrho(z, z^-) = \pi$. We will define a new function

$g : S \to \mathbb{S}^1$ such that for every $z \in S$, $g(z) \in \{z^+, z^-\}$. Before we define g we prove the following claims,

CLAIM 1. For every $z \in S$, $\varrho(f(z), \{z^+, z^-\}) < 2\zeta$.

PROOF OF CLAIM. Let $z \in S$ be given. Since f is limited by $\mathcal{U}$, there is some $w \in \mathbb{S}^1$ such that $\{z_\omega, f(z_\omega)\} \subseteq r^{-1}[U_w]$. Since $z_\omega \in r^{-1}[U_w]$ we also have that $\{z_N, z_N^-\} \subseteq r^{-1}[U_w]$. Since we identify $\{N\} \times \mathbb{S}^1$ with $\mathbb{S}^1$, this means that $\{f(z), z, z^-\} \subseteq r^{-1}[U_w]$.

The metric ϱ measure the arc-distance between points in $\mathbb{S}^1$. Restricted to $\{N\} \times \mathbb{S}^1$, the set $r^{-1}[U_w]$ consists of two arcs; each of ϱ-diameter 2ζ. Since $2\zeta < \pi = \varrho(z, z^-)$ it is clear that one of these arcs contains z and one of them contains z^-. It follows that either $\varrho(z, f(z)) < 2\zeta$ or $\varrho(z^-, f(z)) < 2\zeta$. ◀

CLAIM 2. For every $z \in \mathbb{S}^1$, either $\varrho(f(z), z) \geq 2\zeta$ or $\varrho(f(z), z^-) \geq 2\zeta$.

PROOF OF CLAIM. Suppose not. Then by the triangular inequality it follows that

$$\pi = \varrho(z, z^-) \leq \varrho(f(z), z) + \varrho(f(z), z^-) < 4\zeta < \pi.$$

This is a contradiction. ◀

We now define $g : S \to \mathbb{S}^1$ as follows,

$$g(z) = \begin{cases} z & \text{if } \varrho(f(z), z) < 2\zeta, \\ z^- & \text{if } \varrho(f(z), z^-) < 2\zeta. \end{cases}$$

The claims establish that g is well-defined. Since f is continuous, it follows that g is continuous. But now let $\sigma : \mathbb{S}^1 \to \mathbb{S}^1$ be the antipodal mapping and $D = g[S]$. By compactness D is closed in $\mathbb{S}^1$. However, $D \cap \sigma[D] = \emptyset$ and $\mathbb{S}^1 = D \cup \sigma[D]$. By connectedness of $\mathbb{S}^1$ this is a contradiction. So it follows that $Z/\sim$ does not satisfy the weak form of Ungar's Theorem in the point p_ω.

It follows from Theorem 4.5.2 that $Z/\sim$ is not power homogeneous. Since $Z/\sim$ does not satisfy the weak form of Ungar's Theorem, it is not a retract of a coset space and in particular, it is not retral. It follows that this space is also not a Mal'tsev space, since every compact Mal'tsev space is retral by SIPACHEVA [66]. ◊

We now provide applications of Corollary 4.5.6 and Corollary 4.5.7.

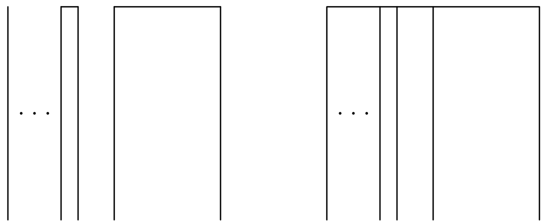

Figure 5.3: The spaces X and Y from Example 5.4.2

5.4.2. EXAMPLE. We let Z be the convergent sequence in the plane given by $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. Let X be the union of $Z \times \mathbb{I}$ and the set $\{[1/2n, 1/2n-1] \times \{1\} : n \in \mathbb{N}\}$, see Figure 5.3. Note that X is a compact metric space. It is not hard to see that the components of X are not regularly locally connected at the point $(0,0)$. Since the components of X are regularly locally connected at the point $(1,0)$, it follows from Corollary 4.5.6 that X is not power homogeneous.

We slightly modify the space X to give an example of a connected space which is not power homogeneous; let Y be the subset of $\mathbb{R}^2$ given by $(Z \times \mathbb{I}) \cup (\mathbb{I} \times \{1\})$, see Figure 5.3. The space Y is locally connected at some but not all points, so it follows from Corollary 4.5.7 that Y is not power homogeneous. ◊

It is not hard to realize that non-power homogeneity of the space X in the previous example also follows from Corollary 4.5.10 and the fact that it is not homogeneous with respect to the implication $C^0 \to LC^0$. This is not surprising since components and path-components in this space coincide. So by Corollary 4.4.6 the fact that the components of X are not regularly locally connected is equivalent to the statement that X does not satisfy the implication $C^0 \to LC^0$. In the next example we provide generalizations of the space X from the previous example. We will use Corollary 4.5.10 to show that the spaces in the next example are not power homogeneous.

5.4.3. EXAMPLE. For a locally compact space X, we shall use αX to denote its one-point compactification. For example, $\alpha\mathbb{N}$ is homeomorphic to a convergent sequence. For $n \in \omega$, we let $Y_n = \mathbb{N} \times \mathbb{S}^n$ and $Z_n = \mathbb{N} \times B^{n+1}$. So Y_n is a subspace of Z_n. Now we let X_n be the space given by

$$\big((\alpha Y_n) \times \mathbb{I}\big) \cup \big((\alpha Z_n) \times \{0\}\big).$$

Note that X_0 is homeomorphic to the space X from Example 5.4.2. See Figure 5.4 for X_1. The verification that X_n satisfies the implication

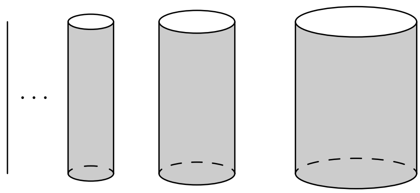

Figure 5.4: The space X_1 from Example 5.4.3

$C^n \to LC^n$ in some but not in all points is left to the reader. It follows from Corollary 4.5.10 that for $n \in \omega$, the space X_n is not power homogeneous. $\Diamond$

5.5 Power homogeneous compacta

We have shown in Theorem 3.2.9 that a space is Δ-power homogeneous if and only if it is power homogeneous. Furthermore, if X is power homogeneous then $X^{\pi w(X)}$ is Δ-homogeneous by Theorem 3.2.5. This raises the natural question whether in this case $X^{\pi w(X)}$ is also homogeneous, i.e., is $\mathrm{hdeg}(X) \leq \pi w(X)$ for power homogeneous spaces X? It follows from Theorem 3.2.9 that if X is power homogeneous and $\mathrm{hind}(X) < \mathrm{cf}(\pi w(X))$, then $\mathrm{hdeg}(X) \leq \pi w(X)$. So we may ask ourselves whether we can provide some upper bound for the homogeneity index of power homogeneous spaces. Of course, the homogeneity index, $\mathrm{hind}(X)$, is always bounded by the cardinality of X. So it follows from Van Douwen's theorem that for power homogeneous spaces X, we have

$$(*) \qquad \mathrm{hind}(X) \leq 2^{\pi w(X)}.$$

Although it seems that this bound is not very precise, it is sharp. This follows from an example provided by VAN MILL [45]. It is shown in [45] that there is a rigid compact metric space X for which $X \times X$ is homogeneous. So the π-weight of X is countable and since X is rigid, we have $\mathrm{hind}(X) = |X| = \mathfrak{c}$. Since $X \times X$ is homogeneous, we also have that $\mathrm{hdeg}(X) = 2$, so the 'gap' between the homogeneity degree and the homogeneity index is very large. This example also seems to suggest that if we are looking for a 'good' upper bound on the homogeneity degree of X, then it is not a good idea to include the homogeneity index.

The following example demonstrates that although Δ-power homogeneity is equivalent to power homogeneity, the first power in which Δ-

homogeneity occurs might be strictly smaller than the first power in which homogeneity occurs.

5.5.1. EXAMPLE. For every positive integer r, ORSATTI and RODINÒ [53] have provided an example of a compact and connected topological group Y such that for all $n, m \in \mathbb{N}$:

$$Y^n \approx Y^m \text{ iff } n \equiv m \operatorname{mod}(r).$$

Furthermore, if λ is an infinite cardinal number, then Y may be chosen to be of weight λ. Let $r = 2$ and let Y be the corresponding group of countable weight. The example X is the space $Y \oplus Y^2$. Note that X is a compact and metrizable space.

Let n be a natural number. Since Y is a group, Y^n is homogeneous. Furthermore, since Y is connected, the space X^n consists of 2^n clopen components and every component of X^n is homeomorphic to Y^m for some natural number m. It follows from the properties of Y that every component of X^n is either homeomorphic to Y or it is homeomorphic to Y^2.

The space X^2 consists of four components. Two of these components are homeomorphic to Y and the other two are homeomorphic to Y^2. Since $Y \not\approx Y^2$, the space X^2 is not homogeneous. However, the diagonal in X^2 is contained in the components of X^2 which are homeomorphic to Y^2. Since Y^2 is homogeneous, it follows that X^2 is Δ-homogeneous. So X is an example of a compact space for which X^2 is Δ-homogeneous but not homogeneous.

In the same way one verifies easily that X^n is Δ-homogeneous if and only if n is an even natural number and for every natural number n, the space X^n is not homogeneous. Since $\operatorname{hind}(X) = 2$, it follows from Theorem 3.2.9 that X^ω is homogeneous. In fact, X^ω is homeomorphic to the product of the Cantor set 2^ω and Y^ω, so it is even a topological group. $\Diamond$

The following example demonstrates that for any infinite cardinal κ, there is a power homogeneous space X such that the homogeneity degree of X is κ. So there are no set-theoretic restrictions on the homogeneity degree of power homogeneous spaces. We first prove the following simple lemma;

5.5.2. LEMMA. *If κ is an infinite cardinal number and Y is the disjoint sum of X and X^κ, then Y^κ is homeomorphic to the product of 2^κ and X^κ.*

PROOF. Let $Y = X \oplus X^\kappa$. Then $Y^\kappa = (X \oplus X^\kappa)^\kappa \approx X^\kappa \times (\{pt\} \oplus X^\kappa)^\kappa \approx (X^\kappa \oplus X^\kappa)^\kappa \approx 2^\kappa \times X^\kappa$. $\square$

5.5.3. EXAMPLE. Let κ be any infinite cardinal number and let X be the disjoint sum of the spaces $\{0,1\}^\kappa$ and $\{0,1\}$. Then X^κ is homeomorphic to $\{0,1\}^\kappa$ by the previous lemma and therefore $\mathrm{hdeg}(X) \leq \kappa$. If $\mu < \kappa$, then X^μ contains points of character κ but also points of character μ, and this means that X^μ is not homogeneous. It follows that $\mathrm{hdeg}(X) = \kappa$. $\Diamond$

Open Problems

We have seen in Chapter 3 that in the presence of either homogeneity or power homogeneity, one can prove cardinal restrictions that are not true for arbitrary spaces. There are still many unsolved problems in this field and these can be found in Arhangel′skiĭ [9]. In this section we list some open problems related to the topics discussed in this thesis.

Let X be a compact and homogeneous space and consider the following inequalities, where ϕ is some cardinal function;

$$(*) \qquad \chi(X) \leq \phi(X),$$

$$(**) \qquad |X| \leq 2^{\phi(X)}.$$

By Arhangel′skiĭ's Theorem we always have that $(*) \to (**)$ and it follows from the Čech-Pospišil Theorem that $(**) + \mathsf{GCH} \to (*)$. If $\phi = t$ then $(**)$ is De la Vega's Theorem from [74] and if $\phi = \pi\chi$, then Van Mill's example from [49] shows that $(*)$ might fail if GCH fails. For compact spaces, the π-character is always less than or equal to the tightness and this raises the question whether $(*)$ can fail if $\phi = t$ and whether $(**)$ is true for $\phi = \pi\chi$;

1. QUESTION. Is $\chi(X) \leq t(X)$ for compact homogeneous spaces X?

2. QUESTION. Suppose X is a homogeneous and compact space. Is it the case that

$$|X| \leq 2^{\pi\chi(X)}\ ?$$

These questions may also be asked for power homogeneous spaces. Since $|X| \leq 2^{t(X)}$ for compact power homogeneous spaces X, the second question is only interesting if $\pi\chi(X) < t(X)$. We have seen in §3.4 that in this case, such spaces are homogeneous with respect to tightness and character.

We have seen that if X is power homogeneous then $X^{\pi w(X)}$ is Δ-homogeneous. This fact yields a quick proof of Van Douwen's Theorem. We have strengthened this cardinal inequality in Theorem 3.3.8 and proved that the π-weight in Van Douwen's Theorem may be replaced by $c(X)\pi\chi(X)$. We also observed after Proposition 3.1.2, that if X^κ is Δ-homogeneous where $\kappa = c(X)\pi\chi(X)$, then the fact that $|X| \leq 2^\kappa$ has a more direct proof. This raises the following question;

3. QUESTION. Suppose X is power homogeneous and let $\kappa = c(X)\pi\chi(X)$. Is X^κ Δ-homogeneous?

A positive answer to this question, would improve Theorem 3.2.5. The following question asks for an improvement of this result in a different direction. Since power homogeneity of a space is equivalent to Δ-power homogeneity, the following question seems natural;

4. QUESTION. Suppose X is power homogeneous. Is $X^{\pi w(X)}$ homogeneous?

Let X be a compact and power homogeneous metrizable space. If the previous question has a positive answer, then X^ω is homogeneous. If this is the case, then X^ω is a coset space and satisfies Ungar's Theorem by Corollary 4.5.3. So in this case, the space X satisfies the weak form of Ungar's Theorem. We have also proved that if X satisfies the weak form of Ungar's Theorem locally at some point, then X satisfies the weak form of Ungar's Theorem (everywhere). We do not know whether we can drop the assumption that X satisfies the weak form of Ungar's Theorem at some point;

5. QUESTION. Let X be a compact and power homogeneous metrizable space. Does X satisfy the weak form of Ungar's Theorem?

To answer this question, it is natural to consider the Polish group $\mathcal{H}(X^\omega)$ which acts transitively on the set D where D is the type of the diagonal in X^ω. The space X is a retract of D, so if D is a coset space, then X satisfies the weak form of Ungar's Theorem. It follows from the Effros Theorem that D is a coset space if and only if it is Polish and this is the case if and only if it is a G_δ in X^ω. It was proved by Ryll-Nardzewski [63] that the set D, being a type in X^ω, is a Borel subset of X^ω. Of course, being G_δ is much stronger than just Borel.

Finally we come to the logical relation between retracts of coset spaces and Mal'tsev spaces. In §4.4 we have encountered an example of a coset space which is not a Mal'tsev space. Of course, Mal'tsev spaces may fail to be coset spaces since Mal'tsev spaces need not be homogeneous. However, compact Mal'tsev spaces are retral and therefore they are at least retracts of coset spaces. But what about arbitrary Mal'tsev spaces;

6. QUESTION. Is every Mal'tsev space a retract of a coset space?

Notes

This section contains notes to the text.

► §**2.2:**
The terms Δ-homogeneity and Δ-power homogeneity were introduced by Ridderbos [59]. The property $wD(\kappa)$ was introduced in Juhász, Nyikos and Szentmiklóssy [36].

► §**2.3:**
Proposition 2.3.8 was proved by Šapirovskiĭ [64, Theorem 3]. Theorem 2.3.10 is due to Ross and Stone [60].

► §**2.4:**
Proposition 2.4.1 is due to Arhangel′skiĭ [5, 2.2.2]. The proof of Theorem 2.4.2 is taken from Arhangel′skiĭ [5, 2.2.20], the result is due to Šapirovskiĭ [65, Theorem 1]. Lemma 2.4.3 is referred to as 'transitivity of the character' by Arhangel′skiĭ [7, Proposition 1.1]. For the origin of this inequality, see Engelking [22, 3.1.E]. Corollary 2.4.4 is due to Šapirovskiĭ [64, Theorem 4]. Proposition 2.4.10 was proved by Arhangel′skiĭ [5, Theorem 2.2.4]. Corollary 2.4.13 is essentially Tall [67, Lemma 2.21]. Corollary 2.4.14 and Proposition 2.4.15 were noted by Juhász, Nyikos and Szentmiklóssy [36].

► §**2.5:**
The term *pseudo P-point* is mentioned in Juhász, Nyikos and Szentmiklóssy [36, Problem 3.12].

► §**3.1:**
Proposition 3.1.1 was noted independently by Ridderbos [56, Proposition

2.2.7] and De la Vega [73, Theorem 1.14]. Proposition 3.1.2 was obtained by Carlson and Ridderbos [18]. Corollary 3.1.4 is due to Arhangel′skiĭ, Van Mill and Ridderbos [12].

▶ §**3.2:**
The results of this section can be found in Ridderbos [58, 59]. Theorem 3.2.1 is similar to [60, Theorem 4], see also [22, Exercise 2.7.12]. This result can also be obtained as an application of Corollary 3.1.4. Theorem 3.2.6 was proved by Van Douwen [19].

▶ §**3.3:**
The results of this section can be found in Arhangel′skiĭ, Van Mill and Ridderbos [12], Carlson and Ridderbos [18] and Ridderbos [59, 57]. Theorem 3.3.8 answers Van Mill [49, Remark 2.7] and also Carlson [17, Question 4.8]. Van Mill proved this result for compact spaces in [49, Corollary 2.6]. For regular spaces, this inequality was observed by Ridderbos [59] and Carlson [17] proved that regularity may be replaced with weaker separation axioms like 'quasi-regular' or 'Urysohn'. Finally, the case for Hausdorff spaces was proved by Carlson and Ridderbos [18]. Proposition 3.3.11 is due to Arhangel′skiĭ [10, Theorem 2.3]. The proof presented here is taken from Arhangel′skiĭ, Van Mill and Ridderbos [12, Corollary 2.4].

▶ §**3.4:**
Arhangel′skiĭ's question about the size of homogeneous countably tight compact spaces can be found in Arhangel′skiĭ [4] and a positive answer was conjectured by him in [7, Conjecture 1.18]. The proof of Theorem 3.4.12 presented here can be found in Arhangel′skiĭ, Van Mill and Ridderbos [12]. Corollary 3.4.14 is due to Ridderbos [59], this results answers Question 4.10 from [12]. For homogeneous regular spaces, this was proved by De la Vega [73]. Corollary 3.4.15 answers Arhangel′skiĭ [10, Problem 3.9]. Corollary 3.4.20 for homogeneous compacta is due to Van Mill [47].

▶ §**3.5:**
The results of this section can be found in Ridderbos [57]. This section generalizes results from Juhász, Nyikos and Szentmiklóssy [36], Van Mill [49] and Bella [13]. Theorem 3.5.3 generalizes Van Mill [49, Theorem 3.2]. Using the results of this section, it may also be proved that if X is a power homogeneous compact space which does not contain a copy of $\beta\kappa$ where $\kappa = c(X)$, then $\chi(X) \leq c(X)$. This is a generalization of [13, Corollary 4];

for details see Ridderbos [57]. Corollary 3.5.6 gives a consistent answer to Arhangel′skiĭ [10, Problem 3.17].

▶ §**4.1:**
Mal′tsev functions were introduced by Mal′tsev [41] and Uspenskiĭ has shown that much of the behaviour of topological groups generalizes to Mal′tsev spaces, see for example Uspenskiĭ [70, 71], Reznichenko and Uspenskiĭ [55]. The notion of $1^1\!/_2$-homogeneity is due to Motorov, we refer to Arhangel′skiĭ [6] for more information.

▶ §**4.2:**
Theorem 4.2.1 is due to Van Mill [48] and Theorem 4.2.4 is due to Van Mill and Ridderbos [51].

▶ §**4.3:**
The results of this section can be found in Van Mill and Ridderbos [50]. The countable sum theorem (Theorem 4.3.2) and the theorem on partitions (Theorem 4.3.3) can be found in Engelking [23, 3.18 & 3.2.6]. A proof of Theorem 4.3.8 can be found in [23, 1.13.4]. A proof of the fact that $\dim X \leq n$ if and only if $X \tau \mathbb{S}^n$ for separable metrizable spaces X can be found in [23, 1.9.3]

▶ §**4.4:**
The term 'regularly locally connected' was introduced by Van Mill and Ridderbos [51]. The results of this section can be found in [51].

▶ §**5.2:**
Theorem 5.2.3 is essentially due to Van Mill [47] who used the space $\mathbb{S}^{\omega_1}$ for Y to give an example under $\mathsf{MA} + \neg\mathsf{CH}$ of a homogeneous compactum X with $\pi w(X) < \chi(X)$. Hart and Ridderbos [31] used 2^{ω_1} for Y to give an example with similar properties which is in addition zero-dimensional. Example 5.2.11 can be found in Van Mill and Ridderbos [50, Example 4.1].

▶ §**5.3:**
The examples of this section can be found in Van Mill and Ridderbos [50, §4 & §5].

▶ **Open Problems:**
Question 1 is due to Arhangel′skiĭ. It can be found in [4] and a positive

answer was conjectured by him in [7, Conjecture 1.17]. Question 2 can be found in De la Vega [73, Question 5.5]. This question also appears in Van Mill [49] in a slightly different way. Question 3 was asked by Carlson and Ridderbos [18]. Question 4 was asked by Ridderbos in [58]. Question 6 can be found in Van Mill and Ridderbos [51, Question 4.1].

Bibliography

[1] J. M. Aarts and L. G. Oversteegen, *The product structure of homogeneous spaces*, Indag. Math. (N.S.) **1** (1990), 1–5.

[2] A. V. Arhangel′skiĭ, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl. **10** (1969), 951–955.

[3] A. V. Arhangel′skiĭ, *Bicompacta that satisfy the Suslin condition hereditarily. Tightness and free sequences*, Dokl. Akad. Nauk SSSR **199** (1971), 1227–1230.

[4] A. V. Arhangel′skiĭ, *Martin's axiom and the structure of homogeneous bicompacta of countable tightness*, Dokl. Akad. Nauk SSSR **226** (1976), 1249–1252.

[5] A. V. Arhangel′skiĭ, *The structure and classification of topological spaces and cardinal invariants*, Uspekhi Mat. Nauk **33** (1978), 29–84, 272.

[6] A. V. Arhangel′skiĭ, *Cell structures and homogeneity*, Mat. Zametki **37** (1985), 580–586, 602.

[7] A. V. Arhangel′skiĭ, *Topological homogeneity. Topological groups and their continuous images*, Uspekhi Mat. Nauk **42** (1987), 69–105, 287.

[8] A. V. Arhangel′skiĭ, *On power homogeneous spaces*, Topology Appl. **122** (2002), 15–33, Proceedings of the International Conference on Topology and its Applications.

[9] A. V. Arhangel′skiĭ, *Some open problems on homogeneous compacta*, in *Paul Erdős and his mathematics, II (Budapest, 1999)*, *Bolyai Soc. Math. Stud.*, vol. 11, 15–31, János Bolyai Math. Soc., Budapest (2002).

[10] A. V. Arhangel′skiĭ, *A weak algebraic structure on topological spaces and cardinal invariants*, Topology Proc. **28** (2004), 1–18, Spring Topology and Dynamical Systems Conference.

[11] A. V. Arhangel′skiĭ, *Homogeneity of powers of spaces and the character*, Proc. Amer. Math. Soc. **133** (2005), 2165–2172 (electronic).

[12] A. V. Arhangel′skiĭ, J. van Mill and G. J. Ridderbos, *A new bound on the cardinality of power homogeneous compacta* (2005), to appear in the Houston Journal of Mathematics.

[13] A. Bella, *Remarks on the cardinality of a power homogeneous space*, Comment. Math. Univ. Carolin. **46** (2005), 463–468.

[14] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. **15** (1948), 729–742.

[15] R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. **1** (1951), 43–51.

[16] L. E. J. Brouwer, *On the structure of perfect sets of points*, Proc. Akad. Amsterdam **12** (1910), 785–794.

[17] N. A. Carlson, *Non-regular power homogeneous spaces*, Topology Appl. **154** (2007), 302–308.

[18] N. A. Carlson and G. J. Ridderbos, *Partition relations and power homogeneity* (2007), preprint.

[19] E. K. van Douwen, *Nonhomogeneity of products of preimages and π-weight*, Proc. Amer. Math. Soc. **69** (1978), 183–192.

[20] E. K. van Douwen, *The integers and topology*, in *Handbook of set-theoretic topology*, 111–167, North-Holland, Amsterdam (1984).

[21] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55.

[22] R. Engelking, *General topology*, *Sigma Series in Pure Mathematics*, vol. 6, Heldermann Verlag, Berlin, second ed. (1989), translated from the Polish by the author.

[23] R. Engelking, *Theory of dimensions finite and infinite*, *Sigma Series in Pure Mathematics*, vol. 10, Heldermann Verlag, Lemgo (1995).

[24] V. Fedorčuk, *Bicompacta with noncoinciding dimensionalities*, Dokl. Akad. Nauk SSSR **182** (1968), 275–277.

[25] L. R. Ford, Jr., *Homeomorphism groups and coset spaces*, Trans. Amer. Math. Soc. **77** (1954), 490–497.

[26] Z. Frolík, *Homogeneity problems for extremally disconnected spaces*, Comment. Math. Univ. Carolinae **8** (1967), 757–763.

[27] P. M. Gartside, E. A. Reznichenko and O. V. Sipacheva, *Mal′tsev and retral spaces*, Topology Appl. **80** (1997), 115–129.

[28] J. Gerlits, *On subspaces of dyadic compacta*, Studia Sci. Math. Hungar. **11** (1976), 115–120.

[29] A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. **29** (1967), 343–356.

[30] Hart, J. E. and K. Kunen, *Bohr compactifications of discrete structures*, Fund. Math. **160** (1999), 101–151.

[31] Hart, K. P. and G. J. Ridderbos, *A note on an example by van Mill*, Topology Appl. **150** (2005), 207–211.

[32] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*, Die Grundlehren der mathematischen Wissenschaften, Bd. 115, Academic Press Inc., Publishers, New York (1963).

[33] M. Ismail, *Cardinal functions of homogeneous spaces and topological groups*, Math. Japon. **26** (1981), 635–646.

[34] I. Juhász, *Cardinal functions in topology—ten years later*, *Mathematical Centre Tracts*, vol. 123, Mathematisch Centrum, Amsterdam, second ed. (1980).

[35] I. Juhász, *Cardinal functions*, in *Recent progress in general topology*, 417–441, North-Holland, Amsterdam (1992).

[36] I. Juhász, P. Nyikos and Z. Szentmiklóssy, *Cardinal restrictions on some homogeneous compacta*, Proc. Amer. Math. Soc. **133** (2005), 2741–2750 (electronic).

[37] I. Juhász and Z. Szentmiklóssy, *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. **116** (1992), 1153–1160.

[38] O.-H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

[39] K. Kunen, *Set theory*, *Studies in Logic and the Foundations of Mathematics*, vol. 102, North-Holland Publishing Co., Amsterdam (1980), An introduction to independence proofs.

[40] K. Kunen, *Weak P-points in* $\mathbf{N}^*$, in *Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978)*, *Colloq. Math. Soc. János Bolyai*, vol. 23, 741–749, North-Holland, Amsterdam (1980).

[41] A. I. Mal′tsev, *On the general theory of algebraic systems*, Mat. Sb. N.S. **35(77)** (1954), 3–20.

[42] D. Marker, *Model theory*, *Graduate Texts in Mathematics*, vol. 217, Springer-Verlag, New York (2002), An introduction.

[43] M. V. Matveev, *Cardinal* $\mathfrak{p}$ *and a theorem of Pelczynski* (2000), online version on: `www.arXiv.org`, `arXiv:math.GN/0006197`.

[44] E. Michael, *Selected selection theorems*, The American Mathematical Monthly **63** (1956), 233–238.

[45] J. van Mill, *A rigid space* X *for which* $X \times X$ *is homogeneous; an application of infinite-dimensional topology*, Proc. Amer. Math. Soc. **83** (1981), 597–600.

[46] J. van Mill, *The infinite-dimensional topology of function spaces*, *North-Holland Mathematical Library*, vol. 64, North-Holland Publishing Co., Amsterdam (2001).

[47] J. van Mill, *On the character and* π*-weight of homogeneous compacta*, Israel J. Math. **133** (2003), 321–338.

[48] J. van Mill, *A note on Ford's example*, Topology Proc. **28** (2004), 689–694.

[49] J. van Mill, *On the cardinality of power homogeneous compacta*, Topology Appl. **146/147** (2005), 421–428.

[50] J. van Mill and G. J. Ridderbos, *Notes on retracts of coset spaces*, Bull. Polish Acad. Sci. Math. **53** (2005), 169–179.

[51] J. van Mill and G. J. Ridderbos, *Retral spaces and continua with the fixed point property*, Comm. Math. Univ. Car. **47** (2006), 661–668.

[52] J. Mioduszewski, *Mappings of inverse limits*, Colloq. Math. **10** (1963), 39–44.

[53] A. Orsatti and N. Rodinò, *Homeomorphisms between finite powers of topological spaces*, Topology Appl. **23** (1986), 271–277.

[54] R. Pol, *Short proofs of two theorems on cardinality of topological spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **22** (1974), 1245–1249.

[55] E. A. Reznichenko and V. V. Uspenskiĭ, *Pseudocompact Mal′tsev spaces*, Topology Appl. **86** (1998), 83–104, special issue on topological groups.

[56] G. J. Ridderbos, *Notes on cardinal functions and homogeneity* (2004), master Thesis, Vrije Universiteit, Amsterdam (unpublished).

[57] G. J. Ridderbos, *Cardinality restrictions on power homogeneous T_5 compacta* (2006), to appear in Studia Sci. Math. Hungar.

[58] G. J. Ridderbos, *A characterization of power homogeneity* (2006), to appear in Topology Appl.

[59] G. J. Ridderbos, *On the cardinality of power homogeneous Hausdorff spaces*, Fund. Math. **192** (2006), 255–266.

[60] K. A. Ross and A. H. Stone, *Products of separable spaces*, Amer. Math. Monthly **71** (1964), 398–403.

[61] L. R. Rubin and P. J. Schapiro, *Limit theorem for inverse sequences of metric spaces in extension theory*, Pacific J. Math. **187** (1999), 177–186.

[62] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.

[63] C. Ryll-Nardzewski, *On Borel measurability of orbits*, Fund. Math. **56** (1964), 129–130.

[64] B. Šapirovskiĭ, *Canonical sets and character. Density and weight in bicompacta*, Dokl. Akad. Nauk SSSR **218** (1974), 58–61.

[65] B. Šapirovskiĭ, *π-character and π-weight in bicompacta*, Dokl. Akad. Nauk SSSR **223** (1975), 799–802.

[66] O. V. Sipacheva, *Compacta with a continuous Mal′tsev operation and retracts of topological groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1991), 33–36, 104.

[67] F. D. Tall, *Normality versus collectionwise normality*, in *Handbook of set-theoretic topology*, 685–732, North-Holland, Amsterdam (1984).

[68] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. **212** (1975), 393–400.

[69] V. V. Uspenskiĭ, *For any X, the product $X \times Y$ is homogeneous for some Y*, Proc. Amer. Math. Soc. **87** (1983), 187–188.

[70] V. V. Uspenskiĭ, *Continuous images of Lindelöf topological groups*, Dokl. Akad. Nauk SSSR **285** (1985), 824–827.

[71] V. V. Uspenskiĭ, *The Mal′tsev operation on countably compact spaces*, Comment. Math. Univ. Carolin. **30** (1989), 395–402.

[72] V. V. Uspenskiĭ, *Topological groups and Dugundji compact spaces*, Mat. Sb. **180** (1989), 1092–1118, 1151.

[73] R. de la Vega, *Homogeneity properties on compact spaces* (2005), Doctoral Thesis, University of Wisconsin-Madison.

[74] R. de la Vega, *A new bound on the cardinality of homogeneous compacta*, Topology Appl. **153** (2006), 2118–2123.

[75] S. Watson, *The construction of topological spaces: planks and resolutions*, in *Recent progress in general topology*, 673–757, North-Holland, Amsterdam (1992).

[76] S. W. Williams and H. Zhou, *Order-like structure of monotonically normal spaces*, Comment. Math. Univ. Carolin. **39** (1998), 207–217.

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Samenvatting (in Dutch)

Machtshomogeniteit in de Topologie

In paragraaf 1 van Hoofdstuk 1 wordt uitgelegd wat een topologische ruimte is en wanneer een ruimte machtshomogeen genoemd wordt. We zullen hieronder kort de hoofdpunten uit deze paragraaf herhalen en vervolgens zal er worden uitgelegd waarom deze machtshomogene ruimten zo'n bijzondere rol spelen in dit proefschrift.

Met het woord *ruimte* bedoelen we eigenlijk *topologische ruimte*. Een topologische ruimte bestaat uit twee dingen: aan de ene kant hebben we een wiskundig object en aan de andere kant leggen we een bepaalde structuur vast op dit object. Het wiskundig object heet dan een ruimte en de structuur heet een topologie. De leden van de ruimte heten *punten* en de topologie geeft aan hoe de punten uit onze ruimte onderling aan elkaar gerelateerd zijn. Het bestuderen van objecten met een bepaalde structuur komt veelvuldig in de wiskunde voor. Als we onze structuur een *meetkunde* noemen dan verkrijgen we een meetkundige ruimte. Topologische en meetkundige ruimten lijken veel op elkaar: men kan grofweg stellen dat een topologie verkregen wordt uit een meetkunde door een bepaald deel van de structuur weg te gooien.

Zoals gezegd, worden punten aan elkaar gerelateerd door een topologie. Op deze manier legt een topologie bepaalde eigenschappen van punten vast. Om aan te geven dat deze eigenschappen voortkomen uit een topologische structuur, gebruiken we hiervoor de term *topologische eigenschappen*. Het kan voorkomen dat bepaalde punten dezelfde topologische eigenschappen hebben, maar dit hoeft niet altijd het geval te zijn. In het bijzondere geval dat alle punten in een ruimte exact dezelfde topologische eigenschappen bezitten, noemen we de ruimte *homogeen*. We kunnen dit ook anders zeggen: een ruimte heet homogeen als deze er overal hetzelfde uitziet.

We maken nu de stap van homogene ruimten naar machtshomogene ruimten. Zoals te lezen valt in Hoofdstuk 1, kunnen we uit een gegeven ruimte nieuwe ruimten maken. Zo kunnen we het vierkant en de kubus verkrijgen uit een lijnstukje. We zeggen dat het vierkant het *product* is van twee lijnstukjes en we schrijven (lijnstuk $\times$ lijnstuk) = vierkant. Op dezelfde manier is een kubus het product van drie lijnstukjes en we schrijven dan ook: (lijnstuk $\times$ lijnstuk $\times$ lijnstuk) = kubus. Gegeven een ruimte X kunnen we dus een heel stel nieuwe ruimten maken, namelijk $X \times X$, $X \times X \times X$, etc. In plaats van dit telkens uit te schrijven, gebruiken we hiervoor de notatie X^2 en X^3. Bij elk gegeven getal κ hebben we nu een nieuwe ruimte X^κ gemaakt. We merken op dat κ ook oneindig kan zijn wat correspondeert met het nemen van een oneindig product. Een ruimte van de vorm X^κ heet een machtsruimte van X. Een ruimte X heet *machtshomogeen* als er een machtsruimte van X te vinden is zodanig dat deze machtsruimte homogeen is.

In dit proefschrift houden we ons bezig met de vraag hoe groot een ruimte kan zijn. Met de grootte van een ruimte bedoelen we hier het *aantal* punten waaruit een ruimte bestaat. De tweepuntsruimte bestaat bijvoorbeeld uit precies twee punten en het lijnstukje bestaat uit oneindig veel punten, zie Figuur 1.1 op pagina 2. In het algemeen tellen we het aantal elementen van een ruimte met behulp van *kardinaalfuncties*. Een kardinaalfunctie kent een getal toe aan een ruimte aan de hand van enkele topologische eigenschappen. Er zijn verschillende kardinaalfuncties te bedenken die allemaal verschillende eigenschappen uitdrukken. Op deze manier krijgen we dus een aantal getallen. Het blijkt dat we met dit soort getallen kunnen bepalen hoe groot een ruimte is. Meestal is het moeilijk om de exacte grootte te bepalen van een ruimte maar in plaats daarvan kunnen we wel vaak achterhalen hoe groot een ruimte *maximaal* kan zijn. Uiteraard zijn we er altijd naar op zoek om deze afschattingen zo optimaal mogelijk te maken.

Het is over het algemeen makkelijker om de grootte van homogene ruimten te bepalen dan om de grootte van inhomogene ruimten vast te stellen. We illustreren dit verschijnsel aan de hand van het volgende voorbeeld. Stelt u zich een schaapherder in de lente voor, bijvoorbeeld in Dwingeloo. Zoals u weet is het voor deze man uiterst belangrijk om te weten hoeveel schapen er rondlopen in zijn kudde. Omdat er in de lente veel jonge schapen worden geboren, zal hij niet altijd weten uit hoeveel schapen zijn kudde bestaat zonder dat hij ze daadwerkelijk allemaal gaat tellen. Dit zal veel tijd vergen en deze tijd kan de herder uitsparen als de kudde gedurende een jaar homogeen is. Immers, als elk schaap dezelfde eigenschappen heeft, dan volstaat het om het aantal lammetjes van één schaap te tellen. Vanwege homogeniteit van de kudde weet de herder dan ook meteen het aantal lammetjes van alle andere schapen. Verder weet de herder hoeveel schapen er in de winter waren en met deze twee getallen berekent hij snel het aantal schapen in zijn kudde. Bijvoorbeeld, als er in de winter 100 schapen waren en elk schaap heeft 3 lammetjes, dan bestaat de kudde op dat moment uit 400 schapen, want $3 \times 100 + 100 = 400$.

Het verschijnsel uit het voorgaande voorbeeld doet zich ook in de topologie voor; wanneer men weet dat een bepaalde ruimte homogeen is, dan wordt het een stuk gemakkelijker om het aantal punten in een ruimte te tellen. Het feit dat het tellen van het aantal punten in een homogene ruimte ons zo gemakkelijk afgaat, komt doordat een homogene ruimte zich vrij netjes gedraagt. Qua netheid liggen de machtshomogene ruimten ergens tussen homogene ruimten en algemene ruimten in. We kunnen ons dan ook afvragen of bepaalde resultaten die gelden voor homogene ruimten ook gelden voor machtshomogene ruimten. In 1978 was Eric van Douwen de eerste persoon die zichzelf deze vraag stelde en hij heeft aangetoond dat dit inderdaad het geval is. Het tellen van machtshomogene ruimten is niet direct net zo gemakkelijk als het tellen van homogene ruimten. Het probleem zit 'm erin dat machtsruimten altijd groter worden naarmate men kijkt naar hogere machten. Verder geldt er dat als een ruimte machtshomogeen is, dat het dan nog best heel lang kan duren voordat een machtsruimte homogeen wordt. Als dit het geval is, dan is de machtsruimte wellicht veel groter dan de ruimte waarmee we begonnen waren en dan is het dus niet zo handig om het aantal elementen van deze zeer grote ruimte te gaan tellen. In plaats daarvan bedacht Eric van Douwen een manier om de homogeniteit van een machtsruimte te gebruiken, om op een efficiënte manier het aantal elementen van de originele ruimte te tellen.

In dit proefschrift worden de resultaten van E. van Douwen aangescherpt en verder worden recente resultaten voor homogene ruimten uitgebreid naar machtshomogene ruimten. Hiertoe introduceren we het begrip Δ-homogeniteit. Een ruimte van de vorm X^κ heet Δ-homogeen als alle punten uit de diagonaal van X^κ op elkaar kunnen worden afgebeeld door een homeomorfisme van X^κ. Het blijkt dat een ruimte X machtshomogeen is dan en slechts dan als X^κ Δ-homogeen is voor zekere κ. Verder tonen we aan dat er in dit geval zelfs geldt dat $X^{\pi w(X)}$ Δ-homogeen is. Dit is een soort reflectiestelling voor machtshomogeniteit: gegeven dat een zekere (mogelijk zeer hoge) macht van X homogeen is, volgt er dat een lagere macht van X Δ-homogeen is. Een belangrijk gevolg van dit resultaat is de Stelling van Van Douwen.

De la Vega heeft in [74] bewezen dat de kardinaliteit van homogene compacte ruimten X begrensd wordt door $2^{t(X)}$. In dit proefschrift laten we zien dat dit resultaat ook geldt voor machtshomogene compacte ruimten. Ter afsluiting van Hoofdstuk 3 bestuderen we erfelijk normale ruimten. We tonen aan dat machtshomogene erfelijk normale compacta voldoen aan het eerste aftelbaarheidsaxioma, gegeven dat $\mathfrak{c} < 2^{\aleph_1}$. Dit generaliseert dergelijke resultaten voor homogene ruimten.

In Hoofdstuk 4 worden retracten van coset ruimten en gerelateerde ruimten bestudeerd. We introduceren een topologische eigenschap genaamd 'de zwakke vorm van de Stelling van Ungar' en we bewijzen dat retracten van coset ruimten en Mal′tsev ruimten aan deze eigenschap voldoen. De zwakke vorm van de Stelling van Ungar is een eigenschap die bepaalde gevolgen heeft voor de topologische structuur van een ruimte. Zo geldt er bijvoorbeeld het volgende: als een compacte ruimte X voldoet aan de zwakke vorm van de Stelling van Ungar, dan is de verzameling die bestaat uit alle punten x zodanig dat x bevat is in een component van dimensie kleiner of gelijk aan n, een gesloten verzameling. Verder bewijzen we dat elke compacte Mal′tsev ruimte met de dekpuntseigenschap lokaal samenhangend is.

In de laatste sectie van Hoofdstuk 4 keren we terug naar machtshomogene ruimten. In tegenstelling tot de resultaten uit Hoofdstuk 3, zijn de resultaten uit deze sectie bijzonder geschikt om te bewijzen dat bepaalde metrische ruimten niet machtshomogeen zijn. We bewijzen onder andere het volgende resultaat voor samenhangende machtshomogene ruimten X: als X érgens lokaal samenhangend is, dan is X overal lokaal samenhangend. In Hoofdstuk 5 presenteren we enkele voorbeelden en tegenvoorbeelden van homogene en machtshomogene ruimten waarbij de resultaten uit eerdere hoofdstukken worden toegepast.

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