Solutions for Games with Restricted Cooperation

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SOLUTIONS FOR GAMES WITH RESTRICTED COOPERATION

VRIJE UNIVERSITEIT

Solutions for Games with Restricted Cooperation

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Chapter 1

Introduction

1.1 Cooperative games and restricted cooperation

Starting in 1944 with the publication of 'Theory of Games and Economic Behavior' by John von Neumann and Oscar Morgenstern, cooperative games have been studied now for 65 years. As counterpart of the subject of conflicting interests studied by non-cooperative games, von Neumann and Morgenstern introduced the notion of cooperative game as a particular type of non-cooperative games in which commitments are fully binding and enforceable (Harsanyi, 1966). Although cooperative games can be considered as a special case of non-cooperative games, historically cooperative games are formulated in a form that abstract away from describing the negotiation process and enforcement procedures explicitly. Instead it concentrates on the possibilities for agreement and studies questions like 'what coalitions will form?' and 'how will the payoff to a coalition that forms be divided between its members?'.

In this thesis we follow this approach and consider the standard situation that each coalition of players can reach a certain payoff by cooperating together. It is profitable for a group of players to join together if the payoff that can be obtained to the coalition of all players of the group is at least as high as when they all stay single or form several subgroups. After a coalition is formed, the next question is to agree on a distribution of the total payoff of the coalition amongst its members. Within cooperative game theory it is widely accepted that the payoffs that can be obtained by every subcoalition when operating on its own, are taken into account in obtaining a distribution of the total payoff of the coalition. A procedure that gives a distribution of the total payoff of a coalition taking into account the payoffs of all subcoalitions is called a solution. One of the main goals of cooperative game theory is the construction of a 'fair' solution. What does the word fair mean here? There are various definitions of this notion and the number of different approaches to this notion grows in the number of various solutions.

A possible notion of the 'fairness' of a distribution is to evaluate fairness by some distance measure between the payoffs to any subcoalition of players assigned by the solution and the payoffs that these subcoalitions can attain on their own. The traditional assumption in studying this type of problems is that every coalition is feasible and can form to attain their payoff. When defining fairness by some distance measure, this means that information about the payoffs of every subcoalition can be used. However, in many real life situations not every group of players has the opportunity to cooperate and to collect their own payoff. We say that we deal with cooperative games with restricted cooperation when not all coalitions can form. The reason of restrictions on the collection of feasible coalitions can be various, for instance restrictions induced by law, restrictions on the maximum number of players that are allowed to cooperate, restrictions because their is no full communication between players or restrictions because players need consent of their superiors to form coalitions with others. These type of situations can easily occur, but such situations are ignored by traditional cooperative game theory in which it is assumed that every group of players can form. Modelling situations of real life, a typical question is which properties have to be taken into account and which properties of the particular situation can be ignored. The question whether it is sufficient to model only the most important properties or to complicate the model including more specific properties, is not easy to answer. Within the framework of restricted cooperation the question is whether the standard model will be used, assuming that all coalitions may form, or a more specific model taking into account that not all coalitions are feasible. To answer this question we have to consider whether or not a more extended model provides us with a better prediction of the outcome. When a more complicated model leads to new and important information about the outcome, then it might be useful to use this model.

Myerson (1977) is one of the first studies in which restricted cooperation is taken into account. He considered a situation in which communication between agents is modelled by a non-directed graph on the set of agents. A group of players is able to communicate and thus feasible if and only if the corresponding subgraph is connected. The paper constructs and axiomatizes a solution for this situation of restricted cooperation. Even in case the graph on the group of all players is connected and thus the coalition of all players, the grand coalition, is feasible, Myerson's result shows that it is important to take into account the restricted cooperation. In fact, the model shows that the distribution of the total payoff of the grand coalition not only depends on how much payoff the subsets of players can collect, but also on the structure of the collection of feasible coalitions. His contribution has inspired many authors to study models with restricted cooperation and to develop solutions for these models.

A collection of coalitions is union stable when for every two coalitions in the collection with non-empty intersection, also the union of the two coalitions is in the collection. The collection of coalitions that are connected within a graph is union stable. Another interesting and simple property of a collection of coalitions is union closedness. We say that a collection of coalitions is union closed if the union of every pair of coalitions from this collection is also member of this collection. An example of a situation that results in a union closed collection of feasible coalitions is when agents are organized is some hierarchical structure represented by a directed graph, referred to as games with a permission structure. In such a situation, a player is a predecessor (or direct superior) of another player, when there is a (directed) edge from the former to the latter player. A coalition of players is feasible, when every player in the coalition has (some of) its predecessors (if any) in the coalition. Every player with at least one predecessor needs permission of some of (a subset of) its predecessors to cooperate with other players.

Notice that the class of games with union closed collection of feasible coalitions is a subclass of games with union stable collection of feasible coalitions. Of course, the latter class is a subclass of the class of games without restrictions on the collection of feasible coalitions. Properties of solutions that are satisfied for every game in a class of games, remain true for every game in a subset of this class. On the other hand, every uniqueness theorem that characterizes a solution on a class of games as the unique solution satisfying some properties, is weaker than the equivalent theorem that characterizes a solution by the same properties on a subset of this class. So, uniqueness theorems provide more information about the nature of a solution when the class of games becomes smaller.

Most research in the field of games with restricted cooperation are either concerned with the class of union stable collections of feasible coalitions or they consider just a particular case within the smaller class of games with a union closed collection of feasible coalitions, for instance games with a union closed system induced by a permission structure, or games with an antimatroid as the collection of feasible coalitions. Uniqueness theorems are either stated on the class of union stable collections of feasible coalitions or on a subclass of games with a union closed collection of feasible coalitions. This thesis is mostly concerned with games on union closed systems. On the one hand, solutions are considered and characterized on the general class of games with union closed systems, on the other hand specific properties of solutions and algorithms to compute solutions, in particular the nucleolus, are considered on specific subclasses of the class of games on union closed systems.

1.2 Outline of the thesis

Chapter 2 contains known facts about cooperative games. In this chapter properties of solutions and the solution concepts core, Shapley value, (pre)nucleolus and (pre)kernel are given. The notion of Davis-Maschler consistency, which plays an important role within this thesis, is given. Also basic notions and definitions of graph theory are introduced. Finally, the concept of restricted game is discussed. In most papers about games with restricted cooperation a solution is obtained from a solution concept for a standard cooperative game without restrictions. The standard approach is to define a modified game without restrictions, called the restricted game, in which in some way payoffs for the non-feasible coalitions are induced by the payoffs of their feasible subsets. Then the solution for the game without restrictions is used to obtain a solution for the original game with restrictions.

The other seven chapters of the thesis are divided into two parts. Part I consists of the chapters 3-6 and deals mainly with properties and the characterizations of solutions for games with restricted cooperation. Part II contains the chapters 7-9 and is mostly devoted to algorithms for the computation of the nucleolus for games with a permission structure (chapters 7, 8) or some other particular feature (chapter 9).

In chapter 3 games on union closed systems are described. The class of such games is a subclass of the class of games with restricted cooperation, namely that the collection of feasible sets is closed under union, but generalizes the classes of games with a permission structure and games on antimatroids. In this chapter an important notion for games on union closed systems is introduced: the superior graph. This graph can be constructed for every set system which is closed under union. Then, for every game with union closed system, a modified game with a permission structure can be defined by using the superior graph of the union closed system. Properties of the core, the Shapley value, the prenucleolus and the prekernel are considered on the class of games with union closed systems.

Also chapter 4 deals with games on union closed systems. In this chapter two new solutions are defined and axiomatized. The first of them (the superior rule) is based on the superior graph and is defined as the conjunctive permission value on this graph. The second (the union rule) is based on the approach which is similar to the approach described by Myerson (1977) applying the Shapley value directly to a modified game.

Chapter 5 differs from the other chapters in the sense that this chapter does not deal with restricted cooperation. In this chapter a class of new solutions for standard cooperative games is introduced. This class is based on properties which hold for both the prekernel and the prenucleolus. The prekernel is the maximal (with respect to inclusion) solution on this class and the prenucleolus the minimal. All other solutions in the class are intermediate between the prenucleolus and the prekernel. Each of these solutions can be described by a positive integer k. To characterize these solutions the known properties of reconfirmation and converse consistency are generalized to k-reconfirmation and k-converse consistency. Also a generalization of Kohlberg's theorem is formulated and proved.

Chapter 6 deals with a special subclass of games with a permission structure, the class of peer-group games. A peer-group game is a game with a permission structure induced by a tree as the underlying graph, and with an additive game as the underlying game. Without restricted cooperation, the class of additive games is trivial and the only efficient and individually rational solution is to give every player its own contribution. Therefore the interesting question is to analyse the impact of the permission structure on the distribution of the total payoff among the players. Moreover, the class of peer-group games is worthwile to consider because it has various interesting economic applications. In this chapter a characterization of the Shapley value on the class of peer-group games is given and also some interesting properties of the nucleolus are discussed.

The second part of the thesis is devoted to computational aspects in cooperative game theory, in particular algorithms to compute the nucleolus are given. The nucleolus is a very specific solution because its definition is not constructive. It is possible to prove that for every cooperative game the nucleolus exists and consists of only one point, but for the general class of games, algorithms to find the nucleolus do not exist. There is no simple way to construct the nucleolus for an arbitrary cooperative game, but it is possible to design algorithms for particular classes of games.

Chapter 7 presents a polynomial time algorithm for finding the nucleolus of games with a disjunctive permission structure, under some conditions on the game and the directed graph. In particular the graph has only one top-player (that is a node without superior) and is acyclic.

In chapter 8 this algorithm is modified to a polynomial time algorithm for situations that allow for the more general case that the directed graph has more top-players, but the underlying cooperative game is restricted to be additive. In fact it is shown that under these conditions the game decomposes into several subgames satisfying the conditions of chapter 7.

In chapter 9 a special class of games is considered: the class of co-insurance games. In principle every non-negative monotone game can be considered as a co-insurance game and this consideration gives new tools to describe arbitrary non-negative monotone cooperative games. Also a new class of games is considered in this chapter: the class of veto-removed games. Each game in this class can be presented as the Davis-Maschler reduced game of a veto-rich game. By investigation of this class it is possible to find new properties of some monotone games. The chapter concludes with a simple algorithm to compute the prenucleolus of a veto-removed game.

Chapter 2

Preliminaries

2.1 TU-games and solutions

2.1.1 Basic definitions

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathcal{N}$ is a finite set of $n = |N|$ players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. Here $\mathcal{N}$ is some universal set of players. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. For simplicity, for a single player i we often denote its worth $v(\{i\})$ by $v(i)$. We denote the collection of all characteristic functions on N by $\mathcal{G}^N$. The collection of all TU-games (for universal player set $\mathcal{N}$) is denoted by $\mathcal{G}(= \mathcal{G}(\mathcal{N}))$. For a general introduction to TU-games, see for instance Peleg and Sudhölter (2003) or Pechersky and Yanovskaya (2004).

A TU-game (N, v) is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. It is straightforward that every monotone game (N, v) is non-negative ($v(S) \geq 0$ for all $S \subseteq N$). It is 0-*monotone* if $v(T) \geq v(S) + \sum\limits_{i \in T \setminus S} v(\{i\})$ for all $S \subseteq T \subseteq N$. A TU-game (N, v) is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$. It is *convex (concave)* if $v(S) + v(T) \leq (\geq) v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

For a finite set A, $|A|$ denotes the cardinality of A. A payoff vector for (N, v) with $n = |N|$ is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to every $i \in N$. For $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$. Further x_S denotes the $|S|$-dimensional vector with components x_i, $i \in S$. A payoff vector is efficient if $x(N) = v(N)$ and it is individually rational if $x_i \geq v(i)$ for every $i \in N$. The set of efficient payoff vectors $X(N, v)$ of game (N, v) is given by

$$X(N, v) = \{x \in \mathbb{R}^n | x(N) = v(N)\}$$

and the imputation set $I(N, v)$ of game (N, v) is given by

$$I(N, v) = \{x \in \mathbb{R}^n | x(N) = v(N) \text{ and } x_i \geq v(i) \text{ for every } i \in N\},$$

i.e., $I(N,v)$ is the set of all efficient and individually rational payoff vectors.

Let $\mathcal{C} \subseteq \mathcal{G}$ be a class of games. A (set-valued) solution F on $\mathcal{C}$ assigns a set $F(N,v) \subset \mathbb{R}^n$ of payoff vectors to every characteristic function $v \in \mathcal{C}$. A solution F on $\mathcal{G}$ is said to be single-valued if it assigns to every $(N,v) \in \mathcal{G}$ a unique payoff vector. Although a solution assigns to every game a set of payoff vectors, for a single-valued solution f we often consider $f(N,v) \in \mathbb{R}^n$ as the element in the singleton instead of the singleton itself.

In some chapters the set of players N will be fixed throughout the chapter. In that case we often denote a TU-game (N,v) just by its characteristic function v and will refer to v as a game. Also in those chapters we often write a solution as $F(v)$ or $f(v)$.

2.1.2 Basic properties of solutions

In this subsection we recall some well-known properties of solutions on the class $\mathcal{G}$ of all TU games. We first recall some definitions. For $i \in N$ and permutation π of N, the game $(N, \pi v)$ is defined by $(\pi v)(\pi S) = v(S)$ for all $S \subseteq N$. Two players $i, j \in N$ are *symmetric* players in game (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i,j\}$. A player $i \in N$ is a *null* player in game (N,v) if $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N \setminus \{i\}$. For two games (N,v) and (N,w) the game $(N, v+w)$ is the sum of the two games if $(v+w)(S) = v(S) + w(S)$ for every $S \subseteq N$. For $\alpha > 0$ and $\beta \in \mathbb{R}^N$, the strategic transformation $(N, \alpha v + \beta)$ of game (N,v) is given by $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for every $S \subseteq N$.

Let $\mathcal{C} \subseteq \mathcal{G}$ be a class of games. Then a solution F is said to have some property from the list below on a subclass $\mathcal{C} \subseteq \mathcal{G}$ when this property holds for every game $(N,v) \in \mathcal{C}$. Solution F on a class $\mathcal{C} \subseteq \mathcal{G}$ is

- *non-empty* if $F(N,v) \neq \emptyset$;
- *efficient* if $\sum_{i \in N} x_i(N,v) = v(N)$ for any $x \in F(N,v)$;
- *anonymous* if $F_{\pi(i)}(\pi N, \pi v) = F_i(N,v)$ for every $i \in N$ and every injection $\pi : N \to \mathcal{N}$.
- *symmetric* or satisfies the *equal treatment property*, if $x_i(N,v) = x_j(N,v)$ for each $x \in F(N,v)$ when i and j are symmetric in (N,v);
- *covariant* if it is covariant under strategic transformation:

  $$F(N, \alpha v + \beta) = \alpha F(N,v) + \beta$$

  for every $\alpha > 0$ and $\beta \in \mathbb{R}^N$;
- *continuous* if $x_n \in F(N, v_n)$ and $x_n \to x$ when $n \to \infty$ implies that $x \in F(N,v)$, where $(N, v_n)_{n=1}^{\infty}$ is a sequence of games in $\mathcal{C}$, $v_n \to v$ and $(N,v) \in \mathcal{C}$;
- has the *null player property* if $x_i = 0$ for every $x \in F(N,v)$ when i is a null player in (N,v).

A single-valued solution F on $\mathcal{C}$ is

- *additive* if $F_i(N, v+w) = F_i(N, v) + F_i(N, w)$ for every two games $(N, v),\ (N, w) \in \mathcal{C}$ such that also $(N, v + w) \in \mathcal{C}$.

For more properties we refer to Peleg and Sudhölter (2003) or Pechersky and Yanovskaya (2004).

2.1.3 The core

The most well-known set-valued solution on $\mathcal{G}$ is the core assigning to every $(N, v) \in \mathcal{G}$ the set

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subset N\},$$

i.e., it is the set of all imputations that are stable in the sense that no coalition can do better by separating from the grand coalition.

A collection $\mathcal{B} = \{S_1, \ldots, S_m\}$ of subsets of N is said to be a *balanced collection* when the system of equations

$$\sum_{j=1}^{m} \lambda_j e^{S_j} = e^N \tag{2.1.1}$$

has a positive solution denoted by $\lambda_j^{\mathcal{B}}$, $j = 1, \ldots, m$, where, for $S \subseteq N$, $e^S \in \mathbb{R}^n$ is given by $e_i^S = 1$ when $i \in S$ and $e_i^S = 0$ otherwise. A collection is said to be a *minimal balanced collection* if it is balanced and does not contain any balanced subcollection. A game (N, v) is *balanced* if

$$\sum_{j=1}^{m} \lambda_j^{\mathcal{B}}\ v(S_j) \leq v(N)$$

for every minimal balanced collection $\mathcal{B} = \{S_1, \ldots, S_m\}$. The core of (N, v) is non-empty if and only if the game is balanced, see e.g. Bondareva (1962) or Shapley (1967).

It is well-known that every convex game is balanced and thus has a non-empty core, see Shapley (1971) and Ichiishi (1981).

2.1.4 The Shapley value

The two most well-known single-valued solutions are the *Shapley value* and the *nucleolus*. The nucleolus has been introduced by Schmeidler (1969) and will be given in the next section. Here we describe the Shapley value, defined and characterized in Shapley (1953).

Definition 2.1.1 *The Shapley value Sh assigns to every game $(N,v) \in \mathcal{G}$ the payoff vector $Sh(N,v)$ given by*

$$Sh_i(N,v) = \sum_{\{S \subseteq N, i \in S\}} \frac{(|N|-|S|)!(|S|-1)!}{|N|!} \left(v(S) - v(S \setminus \{i\})\right), \ i \in N.$$

Theorem 2.1.2 (Shapley, 1953) *On the class $\mathcal{G}$ of all TU games there is a unique single-valued solution with the properties efficiency, additivity, symmetry and the null player property. This solution is the Shapley value.*

There are various other characterizations of the Shapley value, see Young (1985), Hart and Mas-Colell (1988), Feltkamp (1995), van den Brink (2001), Hamiache (2001).

To give an alternative definition of the Shapley value we first define the notion of *marginal vector.* Let $\pi = (i_1, i_2, \ldots, i_{|N|})$ be some permutation of the player set N. Then *marginal vector* of a game (N,v) for the permutation π is the payoff vector m^π given by

$$m^\pi_{i_k}(N,v) = v(i_1, \ldots, i_k) - v(i_1, \ldots, i_{k-1}), \ k \in N.$$

The Shapley value of a game (N,v) is the average of all its marginal vectors $m^\pi(N,v)$.

The Shapley value is equal to average of all marginal vectors: for every TU-game $(N,v) \in \mathcal{G}$,

$$Sh_i(N,v) = \frac{1}{|N|!} \sum_\pi m^\pi_i(N,v), \ i \in N.$$

The Shapley value of (N,v) can also be obtained by solving an optimization problem.

Theorem 2.1.3 (Keane, 1969) *For every TU-game $(N,v) \in \mathcal{G}$,*

$$Sh(N,v) = \arg \min_{x \in X(N,v)} \sum_{S \subset N} (|S|-1)!(|N|-|S|-1)! \left(v(S) - x(S)\right)^2.$$

So the Shapley value $Sh(N,v)$ gives us the linear characteristic function $x(S)$, $S \subseteq N$, with $x(N) = v(N)$, which is most close to the characteristic function $v(S)$, $S \subseteq N$ with respect to the distance measure used in Theorem 2.1.3.

2.2 The nucleolus and the kernel

In this section we consider several solution concepts that play a central role in this thesis. In the first subsection we consider the set-valued solutions prekernel and kernel. These solutions have been introduced in Davis and Maschler (1965) and Maschler, Peleg and Shapley (1972), and select for every game (N,v) a subset of the set of efficient payoff vectors and the imputation set respectively. In the second subsection we consider the single-valued solutions prenucleolus and nucleolus, introduced in Schmeidler (1969). These solutions select for every game (N,v) a single element from the prekernel, respectively the kernel, of the game.

2.2.1 The prekernel and the kernel

To define the prekernel and the kernel of a game (N, v), we first introduce the notion of *complaint*. For a payoff vector $x \in \mathbb{R}^n$, the complaint of player $i \in N$ against another player $j \in N$ is given by

$$s_{ij}(x) = \max_{\{S \subseteq N | i \in S,\ j \notin S\}} (v(S) - x(S)).$$

So, the complaint of i against j at payoff vector x is the highest increase of his payoff that player i can possibly realize by cooperating with a set of players not containing j.

The *prekernel* assigns to every $(N, v) \in \mathcal{G}$ the set of payoff vectors

$$PK(N, v) = \{x \in X(N, v) | s_{ij}(x) = s_{ji}(x) \text{ for all } i, j \in N\},$$

i.e., the prekernel of (N, v) is the set of all efficient payoff vectors such that for each pair of players i and j the complaint of i against j is equal to the complaint of j against i. The prekernel is non-empty for every game (N, v). In particular we show in the next subsection that the prenucleolus is a single-valued solution and that for each (N, v) the prenucleolus vector belongs to the prekernel.

The *kernel* assigns to every $(N, v) \in \mathcal{G}$ the set of payoff vectors

$$K(N, v) = \{x \in I(N, v)|\ [s_{ij}(x) = s_{ji}(x)] \text{ or }$$

$$[s_{ij}(x) > s_{ji}(x) \text{ and } x_j = v(j)] \text{ for all } i, j \in N\},$$

i.e., the kernel of (N, v) is the set of all imputations such that for each pair of players i and j the complaint of i against j is at least equal to the complaint of j against i, with equality whenever j gets more than its individual worth $v(j)$. The subsets of the prekernel and the kernel that belong to the core coincide.

Theorem 2.2.1 (Maschler, Peleg and Shapley, 1979) *For every game $(N, v) \in \mathcal{G}$ it holds*

$$PK(N, v) \cap C(N, v) = K(N, v) \cap C(N, v).$$

Also we have the following result about the prekernel (and kernel) for convex games.

Theorem 2.2.2 (Maschler, Peleg and Shapley, 1972) *When $(N, v) \in \mathcal{G}$ is convex, then the prekernel $PK(N, v)$ coincides with the kernel $K(N, v)$ and consists of only one point.*

Finally we mention that both the prekernel and the kernel satisfy the symmetry and covariance properties.

2.2.2 The prenucleolus and the nucleolus

For a game $(N, v) \in \mathcal{G}$ and payoff vector $x \in \mathbb{R}^n$, the *excess* $e(S, x)$ of coalition $S \subseteq N$ is defined by

$$e(S, x) = v(S) - x(S).$$

Further, let $E(x)$ be the $(2^n - 2)$-component vector that is composed of the excesses of all coalitions $S \subset N,\ S \neq \emptyset$, in a non-increasing order, so $E_1(x) \geq E_2(x) \geq \ldots \geq E_{2^n-2}(x)$. Both the prenucleolus and the nucleolus are single-valued solutions. The prenucleolus $PN(N, v)$ of game (N, v) is the unique efficient payoff vector which lexicographically minimizes the vector-valued function $E(\cdot)$ over the set of efficient payoff vectors. Formally,

$$PN(N, v) = x \text{ such that } x \in X(N, v)$$
$$\text{and } E(x) \preceq_L E(y) \text{ for all } y \in X(N, v),$$

where $\preceq_L$ denotes the lexicographic order of vectors. So the prenucleolus $PN(N, v)$ gives us the linear characteristic function $x(S)$, $S \subseteq N$, with $x(N) = v(N)$, which is most close to the characteristic function $v(S)$, $S \subseteq N$ with respect to the distance measured by the lexicographic ordering of the differences between x and v. This shows that the Shapley value and the prenucleolus can be defined in a similar way and that the difference is only in the measure of distance between x and v.

The nucleolus $Nuc(N, v)$ of a game (N, v) is the unique imputation which lexicographically minimizes the vector-valued function $E(\cdot)$ over the imputation set, so

$$Nuc(N, v) = x \text{ such that } x \in I(N, v)$$
$$\text{and } E(x) \preceq_L E(y) \text{ for all } y \in I(N, v).$$

There are several known properties of the prenucleolus and the nucleolus. The first two properties show that the prekernel and the kernel of a game (N, v) are non-empty sets.

Properties 2.2.3 *Let* $(N, v) \in \mathcal{G}$. *Then*
1. $PN(N, v) \in PK(N, v)$,
2. $Nuc(N, v) \in K(N, v)$,
3. If $C(N, v) \neq \emptyset$ *then* $PN(N, v) = Nuc(N, v)$ *and* $Nuc(N, v) \in C(N, v)$.

Also the prenucleolus and nucleolus are symmetric and covariant solutions, moreover both satisfy the null player property.

The next theorem gives an elegant characterization of the prenucleolus in terms of balanced collections of coalitions. For game (N, v), we define for every real value α and payoff vector x the collection $\mathcal{B}(\alpha, x)$ as the collection of nonempty subsets of N with excess at least equal to α,

$$\mathcal{B}(\alpha, x) = \{\emptyset \neq S \subset N \mid e(S, x) \geq \alpha\}.$$

Theorem 2.2.4 (Kohlberg, 1971) *For game $(N, v) \in \mathcal{G}$, let $x \in X(N, v)$ be efficient. Then $x = PN(N, v)$ if and only if for every value α the collection $\mathcal{B}(x, \alpha)$ is either balanced or empty.*

One distinctive feature of the prenucleolus and the nucleolus is their computational complexity. It is hard to compute the prenucleolus for arbitrary cooperative game but it can be easier in some special case. There are various papers about computational aspects of nucleolus: Solymosi and Raghavan (1994), Potters, Reijnierse and Ansing (1996), Kuipers (1996), Kuipers, Solymosi and Aarts (2000) and others.

2.2.3 The Davis-Maschler consistency

For a given game $(N, v) \in \mathcal{G}$, and a payoff vector x, Davis and Maschler (1965) defined for every $S \subset N$ a reduced game on the player set S as the characteristic function that assigns to every coalition $T \subseteq S$ the worth that these players can obtain after the players that are not in S have left the game with payoffs $x(i), i \notin S$, and leave cooperation possibilities for the players in S.

Formally, for a TU game $(N, v) \in \mathcal{G}$ and a payoff vector x the Davis-Maschler *reduced game* (S, v_S^x) on the player set $S \subset N$ with respect to x is defined as

$$v_S^x(T) = \begin{cases} v(N) - x(N \setminus S), & \text{if } T = S, \\ \max_{Q \subseteq N \setminus S} \; (v(T \cup Q) - x(Q)) & \text{for } T \subset S. \end{cases}$$

The consistency of set-valued solution was defined by Peleg (1986).

For a solution F on a subclass $\mathcal{C} \subseteq \mathcal{G}$, we now define the following consistency properties with respect to the Davis-Maschler reduced games. A (set-valued) solution F on a subclass $\mathcal{C} \subseteq \mathcal{G}$

– is *consistent* if for any game $(N, v) \in \mathcal{C}$, for every $S \subset N$ and for every $x = (x_S, x_{N \setminus S}) \in F(N, v)$, it holds that the Davis-Maschler reduced game $(S,\ v_S^x)$ on player set S belongs to $\mathcal{C}$ and

$$x_S \in F(S, v_S^x). \tag{2.2.2}$$

– is *converse consistent* when for every $x \in X(N, v)$, $x_{\{i,j\}} \in F(\{i, j\}, v^x_{\{i,j\}})$ for all $i, j \in N$ implies that $x \in F(N, v)$.

– satisfies the *reconfirmation property*[1] if $x \in F(N, v)$, $S \subset N$, $y_S \in F(S, v_S^x)$ implies that $(y_S, x_{N \setminus S}) \in F(N, v)$.

[1]This concept appears in Balinsky and Young (1982) in their study of appointment problems as a component of the property thay call "uniformity"

In the literature there are also other definitions of the notion of reduced game, for instance the Hart-Mas-Colell reduced game, see Hart and Mas-Colell (1989). For these types of reduced games notions similar properties can be defined, but in this thesis we only consider the Davis-Maschler reduced game and its consistency properties.

For some subclass $\mathcal{C} \subseteq \mathcal{G}$, let $\mathcal{F}$ be a collection of solutions satisfying a set of properties including consistency. When $\mathcal{F}$ contains a single-valued solution F (thus the set $F(N,v)$ is a singleton set for every $(N,v) \in \mathcal{C}$), then F is *minimal* with respect to inclusion, i.e., there is no other solution $G \in \mathcal{F}$ such that $G(N,v) \subseteq F(N,v)$ for every $(N,v) \in \mathcal{C}$. Reversely, if the collection of solutions $\mathcal{F}$ satisfying a set of properties, including consistency, contains a solution that is also converse consistent, then this solution is maximal with respect to inclusion within $\mathcal{F}$, i.e., when $F, G \in \mathcal{F}$ and G is converse consistent, then $F(N,v) \subseteq G(N,v)$ for every $(N,v) \in \mathcal{C}$. These observations have proved to be useful to construct axiomatic characterizations of the prenucleolus and the prekernel. An axiomatization of the prenucleolus was constructed in 1975 by Sobolev, who showed that the prenucleolus is the unique single-valued, anonymous and covariant solution on $\mathcal{G}$ with the consistency property.

Theorem 2.2.5 (Sobolev, 1975) *For an infinite universal set of players $\mathcal{N}$ there is a unique single-valued solution on $\mathcal{G}$ satisfying covariance, anonymity and consistency. This is the prenucleolus.*

This theorem was improved by Orshan (1994) by using a slightly weaker axiom.

Theorem 2.2.6 (Orshan, 1994) *For an infinite universal set of players $\mathcal{N}$ there is a unique single-valued solution on $\mathcal{G}$ satisfying covariance, symmetry, and consistency. This is the prenucleolus.*

A characterization of the prekernel as the unique non-empty, convariant, consistent and converse consistent solution was constructed by Peleg (1986).

Theorem 2.2.7 (Peleg, 1986) *For an arbitrary universal set of players N there is a unique solution on $\mathcal{G}$ satisfying nonemptiness, covariance, symmetry, consistency, and converse consistency. This is the prekernel.*

Notice that in Theorem 2.2.6 single-valuedness implies nonemptiness. So, comparing Theorem 2.2.6 with Theorem 2.2.7 shows that the charactizations of the prenucleolus and the prekernel differ one from another only by a single axiom: single-valuedness in the characterization of the prenucleolus is replaced by converse consistency in the characerization of the prekernel. Also notice that it is known from Sobolev (1975) that consistency implies efficiency, so both the prenucleolus and the prekernel are efficient.

Also there is an axiomatization of the prenucleolus of just another type:

Theorem 2.2.8 (Orshan and Sudhölter, 2003) *There is a unique solution on $\mathcal{G}$ satisfying non-emptiness, covariance, symmetry, and reconfirmation. This is the prenucleolus.*

2.3 Graphs and set systems

2.3.1 Directed graphs

A directed graph or *digraph* is a pair (N, D) where $N \subset \mathbb{N}$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. Given (N, D) and $S \subseteq N$, the digraph $(S, D(S))$ is the subgraph on S given by $D(S) = \{(i, j) \in D | i, j \in S\}$. In the thesis we often simply refer to D for a digraph (N, D) and to $D(S)$ for the subgraph $(S, D(S))$. For $i \in N$ the nodes in $S_D(i) := \{j \in N \mid (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) := \{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i.

For given D on N, a *path* between i and j in N is a sequence of distinct nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$, and $\{(i_k, i_{k+1}), (i_{k+1}, i_k)\} \cap D \neq \emptyset$ for $k = 1, \ldots, m-1$. For each player $i \in N$, define the sets $\widehat{S}(i)$ and $\widehat{P}(i)$ as follows: $j \in \widehat{S}(i)$ if and only if there is a path from i to j and $j \in \widehat{P}(i)$ if and only if there is a path from j to i. Also, for every set $U \subset N$, define

$$\widehat{S}(U) = \bigcup_{i \in U} \widehat{S}(i),$$

$$\widehat{P}(U) = \bigcup_{i \in U} \widehat{P}(i).$$

A set of nodes $T \subseteq N$ is *connected* in digraph D if there is a path between any two nodes in T that only uses arcs between nodes in T, i.e., if for every $i, j \in T$ there is a path $(i_1, \ldots, i_m)$ between i and j such that $\{i_1, \ldots, i_m\} \subseteq T$. A *component* in D is a maximally connected set T of nodes, i.e. T is connected and $T \cup \{i\}$ is not connected for every $i \in N \setminus T$. A path $(i_1, \ldots, i_m)$ between i and j in D is a *directed path* if $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, m-1$. A directed path $(i_1, \ldots, i_m)$, $m \geq 1$, in D is a *cycle* if $(i_m, i_1) \in D$. We call digraph D *acyclic* if it does not contain any cycle. Note that acyclicity of a digraph D implies that D is irrefexive, i.e., $(i, i) \notin D$ for all $i \in N$.

A digraph is called *quasi-strongly connected* if there exists a node $i_0 \in N$, such that for every $j \neq i_0$ there is a directed path from i_0 to j. Note that this implies that N is connected. When D is acyclic then i_0 is the unique node in N having no predecessors and i_0 is called the *top-node* of the digraph. The collection of all acyclic, quasi-strongly connected digraphs on N is denoted by $\mathcal{D}^N$. A digraph $D \in \mathcal{D}^N$ is a rooted directed tree with root i_0 if there is precisely one path from the top-node i_0 to every other node. Node $j \in N$ is a *complete subordinate* of node $i \in N$ in $D \in \mathcal{D}^N$ if every directed path from the top-node i_0 to node j contains node i. We denote the set of complete subordinates of node i by $\overline{S}_D(i)$, i.e.,

$$\overline{S}_D(i) = \left\{ j \in N \,\middle|\, \begin{array}{c} i \in \{h_1, \ldots, h_{t-1}\} \text{ for every sequence of nodes} \\ (h_1, \ldots, h_t) \text{ such that } h_1 = i_0, h_{k+1} \in S_D(h_k), \\ k \in \{1, \ldots, t-1\}, \text{ and } h_t = j \end{array} \right\}.$$

Player $i \in N$ is *complete superior* of player $j \in N$ if j is a complete subordinate of i. We denote the set of complete superiors of node i by $\overline{P}_D(i)$.

2.3.2 Set systems

For some finite set N, a *set system* $\Omega \subseteq 2^N$ is an arbitrary collection of subsets of N. Unless mentioned otherwise, in this thesis it is assumed that $\emptyset,\ N \in \Omega$. Deleting this assumption gives more general systems. A set system Ω is *union closed* if for every pair $S, T \in \Omega$ it is true that $S \cup T \in \Omega$. A set system Ω is *union stable* if for every pair $S, T \in \Omega$ with $S \cap T \neq \emptyset$ it is true that $S \cup T \in \Omega$ (see Algaba et al. (2000, 2001)). Clearly, every union closed system is also union stable. A famous example of a union stable system that is not union closed is the collection of all connected subsets in a (directed) graph. Examples of union closed systems are *antimatroids* and *conjunctive or disjunctive feasible sets* for games with a permission structure.

A set system Ω is an *antimatroid* if, besides containing the empty set, it is union closed and for every $S \in \Omega$ there exists $i \in S$ such that $S \setminus \{i\} \in \Omega$. An antimatroid is normal if every player belongs to at least one feasible coalition, which by union closedness implies that N is feasible. Properties of antimatroids have been discussed in Dilworth (1940) and in Edelman and Jamison (1985).

Conjunctive and disjunctive feasible sets are defined within the framework of a directed graph. For a directed graph (N, D), a set $S \subseteq N$ is a *conjunctive feasible* set if for every $i \in S$ the set of all predessessors of i is contained in S. A set $S \subseteq N$ is a *disjunctive feasible* set if for every $i \in S$ at least one of the predecessors of i is also in S. The next theorem gives some properties of these sets of feasible coalitions. Parts of this theorem can be found in the next papers: Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) (items 1 and 3), and Gilles, Owen and van den Brink (1992) (item 2).

Theorem 2.3.1 *Let (N, D) be a directed graph. Then the following holds*
1. The collection of disjunctive feasible sets and the collection of conjunctive feasible sets are both antimatroids.
2. The intersection of two conjunctive feasible sets is also a conjunctive feasible set.
3. If a set system $\Omega \subseteq 2^N$ is closed under union and closed under intersection, then there exists a directed graph (N, D) such that Ω coincides with the collection of all conjunctive feasible sets for (N, D).

2.4 Restricted games

In many practical situations not all coalitions of players are able to cooperate together and collect their total payoff. Such situations with restricted cooperation are modelled by a triple (N, v, Ω), with v a characteristic function on 2^N and Ω a set system of all feasible coalitions. A subset S of players can only cooperate together and realize $v(S)$ if $S \in \Omega$. Under the assumption made in subsection 2.3.2 that $\emptyset,\ N \in \Omega$, the grand coalition N can always cooperate and realise $v(N)$.

To analyse situations (N, v, Ω) with restricted cooperation it is natural to try to construct a new game (N, r) where $r(S)$ is the profit that coalition S can collect with restrictions on the cooperation. So, for every feasible coalition $S \in \Omega$ we have that $r(S) = v(S)$. Depending on the properties of Ω, there are various approaches to define $r(S)$ for non-feasible coalitions S.

In case that Ω is a union closed system it is possible to consider for any non-feasible set S the maximal feasible subset. So, let $\{T \in \Omega | T \subseteq S\}$ be the collection of all feasible subsets of S. Since Ω is union closed it follows that the union of all the sets in this collection

$$\sigma(S) = \bigcup_{\{T \in \Omega | T \subseteq S\}} T \in \Omega.$$

Clearly, $\sigma(S)$ is the maximal feasible subset of S in the sense that $T' \subseteq \sigma(S)$ for every $T' \in \{T \in \Omega | T \subseteq S\}$. Then it is natural to define

$$r(S) = v(\sigma(S)),$$

i.e., the payoff that the players in S can realise together is the value of its maximal feasible subset $\sigma(S)$. The new game (N, r) reduces the situation (N, v, Ω) of restricted cooperation to a standard TU game and we can consider the usual solutions for TU games to predict the outcome of (N, v, Ω).

There are many papers following this approach for particular types of union closed systems. Games on antimatroids have been considered in Algaba, Bilbao, Borm and López (2001) and in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004). Other particular situations of union closed systems are induced by games with a *permission structure*. In such a situation the restricted cooperation is induced by a digraph (N, D) in which the set Ω is the union closed system consisting of the collection of all conjunctive or disjunctive feasible sets. The situation in which Ω is the collection of all disjunctive feasible sets of a digraph (N, D) has been considered in Gilles and Owen (1994) and in van den Brink (1997). Gilles *et al.* (1992) and van den Brink and Gilles (1996) analyse situations in which Ω is the collection of all conjunctive feasible sets.

Typically a somewhat different approach is practised when Ω is a union stable set system. Since union stable systems are more general than union closed systems, in such situations a non-feasible set does not need to have a unique maximal feasible subset. When Ω is a union stable set system, the set

$$\sigma(S) = \bigcup_{\{T \in \Omega | T \subseteq S\}} T$$

is the union of a collection of sets $\{T_1, \ldots, T_m\}$ with the properties that (i) for every $T' \in \{T \in \Omega | T \subseteq S\}$ there exists a j such that $T' \subseteq T_j$ and (ii) $T_j \cap T_k = \emptyset$ for every $j \neq k$. Then we define

$$r(S) = \sum_{j=1}^{m} v(T_j)$$

as the sum of the worths of the sets in the collection $\{T_1, \ldots, T_m\}$. Situations with union stable systems have been considered in Algaba *et al.* (2000, 2001).

A particular situation of union stable systems are induced by games with a *communication structure*. In such a situation the restricted cooperation is induced by an undirected graph (N, L) with L a set of undirected edges on the set of players N. A coalition S can only cooperate together and realise its value $v(S)$ if S is connected in the graph (N, L). So, Ω is the collection of all connected coalitions. A set $T \subseteq S$ is called a component of S if T is a maximal connected subset of S in the subgraph $(S, L(S))$, where $L(S) \subseteq L$ is the collection of edges such that both nodes of the edge are in S. A subset T of S is a maximal connected subset of S if it is connected in $(S, L(S))$ and $T \cup \{i\}$ is not connected in $(S, L(S))$ for every $i \in S \setminus T$. Now, $\sigma(S)$ is the union of the components of S and $r(S)$ is the sum of the worths of the components of S. Games with communication structure have been introduced by Myerson (1977), therefore the game (N, r) is called the *Myerson restricted* game. After their introduction games with communication structures have been studied and various solutions for games with communication structure have been developed, see for instance Owen (1986), Borm, Owen and Tijs (1992), van den Nouweland (1993), and Hamiache (1999).

Properties of solutions

Chapter 3

Games on union closed systems

3.1 Introduction

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn its worth. In the literature various restrictions on coalition formation are developed. For a survey we refer to Bilbao (2000). In this chapter, which is based on van den Brink, Katsev and van der Laan (2009a), we assume that the set of feasible coalitions is closed under union, meaning that for any pair of feasible coalitions also their union is feasible. By convention we assume that every union closed set of feasible coalitions contains the empty coalition $\emptyset$. Some examples of cooperation structures that yield union closed systems are the following.

Example 3.1.1 Suppose that only coalitions of a minimal size k are feasible. Then the set of coalitions $\Omega = \{S \subseteq N \mid |S| \geq k\} \cup \{\emptyset\}$ for some $k \in \{1, \ldots |N|\}$ is closed under union.

Example 3.1.2 To give a more general example, consider the situation where the player set N is partitioned in a coalition structure $\mathcal{P} = \{P^1, \ldots, P^m\}$ of nonempty coalitions such that for every element $P^k,\ k \in \{1, \ldots, m\}$ there is a quota $q_k \in \{1, \ldots, |P^k|\}$ meaning that a coaliton $S \subseteq N$ can form if for every $k = 1, \ldots, m$, S contains at least q_k players from P^k. So, given such a *majority cooperation situation* $(N, v, \{P^1, \ldots, P^m\}, \{q_1, \ldots, q_m\}\}$ with $\{P^1, \ldots, P^m\}$ being a partition of N and $q_k \in \{1, \ldots, |P^k|\}$ for all $k \in \{1, \ldots, m\}$, the set of feasible coalitions is given by

$$\Omega = \{S \subseteq N \mid |S \cap P_k| \geq q_k \text{ for all } k \in \{1, \ldots, m\}\} \cup \{\emptyset\}.$$

Obviously, if $\min\{|S \cap P_k|, |T \cap P_k|\} \geq q_k$ for all $k \in \{1, \ldots, m\}$, then $|(S \cup T) \cap P_k| \geq q_k$ for all $k \in \{1, \ldots, m\}$, and thus Ω is closed under union.

By definition antimatroids are closed under union. Thus, the games considered in this chapter generalize the games on antimatroids as considered in Algaba, Bilbao, van den

Brink and Jiménez-Losada (2003, 2004), and therefore also the *games with a permission structure* of Gilles, Owen and van den Brink (1992), van den Brink and Gilles (1996), Gilles and Owen (1994) and van den Brink (1997). Note that the two examples given above yield union closed structures that are not antimatroids[1]. Therefore to deal with such type of situations we need a more general approach.

This chapter is organized as follows. Section 3.2 deals with a definition of a game on a union closed system, the superior graph and solutions for games on union closed systems. In section 3.3 some properties of the core, least core and Shapley value for monotone games on union closed systems is considered. Section 3.4 deals with the prekernel of the restricted game.

3.2 Games on union closed systems

3.2.1 Union closed systems

In this chapter we consider tuples (N, v, Ω), where (N, v) is a TU-game and $\Omega \subseteq 2^N$ is a collection of subsets of the player set N. We call such a tuple a *game with restricted cooperation.* In such a game the collection of subsets Ω restricts the cooperation possibilities of the players in N. A set $S \subseteq N$ of players can only attain its value $v(S)$ if $S \in \Omega$. When $S \notin \Omega$, then the players are not able to cooperate, so that $v(S)$ can not be realised. In this chapter we consider sets of feasible coalitions that are closed under union. By convention we assume that every union closed set of feasible coalitions contains the empty set $\emptyset$ and the grand coalition N. We say that a coalition $S \in 2^N$ is feasible when $S \in \Omega$.

Definition 3.2.1 *A collection $\Omega \subseteq 2^N$ is a union closed system of coalitions if*
1. $\emptyset, N \in \Omega$,
2. If $S,\ T \in \Omega$, then $S \cup T \in \Omega$.

Example 3.2.2
1. The collection $\Omega = \{\emptyset, N\}$ is union closed.

2. The collection $\Omega = 2^N$ is union closed.

3. Consider a game with permission structure (see chapter 2). For both approaches the collection of feasible coalitions is an antimatroid and so a union closed system (see Algaba, Bilbao, van den Brink and Jiménez-Losada 2004).

4. A graph game is given by a TU-game (N, v) and an undirected graph (N, L) with the player set N as its nodes and L a collection of edges. Following Myerson (1977), the

[1]These structures do not satisfy accessibility. In Example 3.1.1 this can be seen since for $S \in \Omega$ with $|S| = k$ there is no $i \in S$ such that $S \setminus \{i\} \in \Omega$. In Example 3.1.2 this can be seen since taking a coalition $S \in \Omega$ with $|S \cap P_k| = q_k$ for all $k \in \{1, \ldots, m\}$, there is no $i \in S$ such that $S \setminus \{i\} \in \Omega$.

players in a set $S \subseteq N$ can only cooperate when S is connected in the graph (see also chapter 2). So, S is feasible if and only if S is connected. Since the union of two connected coalitions S and T does not need to be connected, the collection of all feasible coalitions induced by a graph game might not be union closed. $\square$

Since in this chapter we take the player set N fixed, we denote a game with limited cooperation just by the pair (v, Ω) with v the characteristic function and Ω the collection of feasible coalitions. In the sequel of this chapter we make the following assumption.

Assumption 3.2.3 *For every game with limited cooperation* (v, Ω) *it holds that* v *is monotone and* $\Omega \subset 2^N$ *is a union closed system.*

We denote the collection of all union closed systems in 2^N by $\mathcal{C}^N$ and the collection of monotone games with player set N by $\mathcal{G}^N_m$. Recall that we denote $x(S) = \sum_{i \in S} x_i$, $S \subseteq N$.

3.2.2 The superior graph

The *superior graph* of a union closed system $\Omega \in \mathcal{C}^N$ assigns an arc from player $i \in N$ to player $j \in N$, $j \neq i$, if player i is in every feasible coalition that contains player j. So, the arc can be seen as some kind of dominance relation in the sense that i dominates player j if j always needs player i to be in a feasible coalition.

Definition 3.2.4 *For two players* $i, j \in N$, $i \neq j$, *player* i *is a superior of player* j *in* $\Omega \in \mathcal{C}^N$, *if* $i \in S$ *for every* $S \in \Omega$ *such that* $j \in S$. *In that case player* j *is a subordinate of* i.

The next corollary is straightforward for $\Omega \in \mathcal{C}^N$.

Corollary 3.2.5 *If* i *is a superior of* j *in* Ω *and* k *is a superior of* i *in* Ω *then* k *is a superior of* j *in* Ω.

Definition 3.2.6 *For* $\Omega \in \mathcal{C}^N$, *the superior graph of* Ω *is the directed graph* (N, D^Ω) *with set of arcs*

$$D^\Omega = \{(i,j) \in N \times N \mid i \text{ is a superior of } j\}.$$

Example 3.2.7
1. If $\Omega = \{\emptyset, N\}$, then for every $S \in \Omega$ and for every $i, j \in N$ it holds that $i \in S$ when $j \in S$. So every $i \in N$ is a superior of every $j \in N$ and thus (N, D^Ω) is complete, i.e., $D^\Omega = \{(i,j) \in N \times N \mid i,j \in N,\ i \neq j\}$.

2. If $\Omega = 2^N$, then $S = \{i\} \in \Omega$ for every $i \in N$, and thus $D^\Omega = \emptyset$.

3. Let (N, D) be a directed graph representing a permission structure and let Ω be the union closed system of feasible coalitions under the conjunctive approach. Then (N, D^Ω) is the transitive closure of (N, D).

4. Let (N, D) be an acyclic, quasi-strongly connected, directed graph, i.e., (N, D) has a unique top node, say i_0, and there exists a directed path from this top node i_0 to every other node. Let Ω be the union closed system of feasible coalitions under the disjunctive approach. Then

$$D^\Omega = \{(i, j) \in N \times N \mid i \text{ is on every path from } i_0 \text{ to } j\}.$$

□

Having constructed the superior graph (N, D^Ω) of a union closed system Ω, we consider now the set of feasible coalitions of the permission structure (N, D^Ω) according to the conjunctive approach and we denote this collection of coalitions by Σ^Ω. Notice that this set is again a union closed system. The next two propositions give results on the relation between Ω and Σ^Ω.

Proposition 3.2.8 *Let $\Omega \in \mathcal{C}^N$ be a union closed system. Then $\Omega \subseteq \Sigma^\Omega$.*

Proof. Let $S \in \Omega$. By definition of superior it holds that S includes all superiors of i for every $i \in S$. On the other hand it holds that $(j, i) \in D^\Omega$ if and only if j is a superior of i, $i \in S$. It follows that S is feasible for the permission structure D^Ω according to the conjunctive approach. Hence $\Omega \subseteq \Sigma^\Omega$. □

Proposition 3.2.9 *Let Ω_1 and Ω_2 be two union closed systems such that $\Omega_1 \subseteq \Omega_2$. Then $\Sigma^{\Omega_1} \subseteq \Sigma^{\Omega_2}$.*

Proof. Suppose j is a superior of i in Ω_2. Since $S \in \Omega_1$ implies that $S \in \Omega_2$, it follows that j is also a superior of i in Ω_1. So, $(j, i) \in D^{\Omega_1}$ if $(j, i) \in D^{\Omega_2}$. Further, for $S \in \Sigma^{\Omega_k}$, $k = 1, 2$, we have that $j \in S$ when $i \in S$ and $(j, i) \in D^{\Omega_k}$. Hence $S \in \Sigma^{\Omega_1}$ implies that $S \in \Sigma^{\Omega_2}$. □

For $i \in N$, $\Omega \in \mathcal{C}^N$ define

$$S_i^\Omega = \{j \in N \mid j = i \text{ or } i \text{ is a superior of } j\},$$

i.e., $S_i^\Omega \subseteq N$ denotes the set containing player i and all subordinates of i in Ω. Then the next proposition says that when Ω is a union closed system, for every $i \in N$ the complement of i and all it subordinates is in Ω.

Proposition 3.2.10 *When $\Omega \in \mathcal{C}^N$, then $N \setminus S_i^\Omega \in \Omega$ for every $i \in N$.*

Proof. Let U be the union of all feasible sets not containing i. Since $\Omega \in \mathcal{C}^N$, it follows that $U \in \Omega$. Further, by definition of U we have that $i \notin U$. Consider a player $j \notin U$ with $j \neq i$. It holds that any feasible set without i does not contain j. So i is a superior of j and thus $j \in S_i^\Omega$. Hence $N \setminus U \subseteq S_i^\Omega$. On the other hand, consider some player $j \in S_i^\Omega$. If $j = i$ then $j \notin U$ by definition of U. If $j \neq i$, then any feasible set containing j also contains i. Hence $j \notin U$, which shows that $S_i^\Omega \subseteq N \setminus U$. Hence $N \setminus S_i^\Omega = U \in \Omega$. $\square$

3.2.3 Solutions for games on a union closed system

For a system $\Omega \in \mathcal{C}^N$, we define the function $\sigma_\Omega \colon 2^N \to \Omega$ by

$$\sigma_\Omega(S) = \bigcup_{\{U \in \Omega \mid U \subseteq S\}} U,$$

so $\sigma_\Omega(S)$ is the maximal feasible subset of S. By union closedness this maximal feasible subset is unique. For the tuple (N, v, Ω), the *restricted* game $r_{v,\Omega} \in \mathcal{G}^N$ is defined by

$$r_{v,\Omega}(S) = v(\sigma_\Omega(S)).$$

The restricted game assigns to each coalition $S \subseteq N$ the worth of its maximal feasible subset. Notice that when v is monotone, it holds that for every $\Omega \in \mathcal{C}^N$ also the restricted game $r_{v,\Omega}$ is monotone since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$.

Example 3.2.11
1. If $\Omega = \{\emptyset, N\}$ then $\sigma_\Omega(N) = N$ and $\sigma_\Omega(S) = \emptyset$ for all $S \neq N$. So, $r_{v,\Omega}(N) = v(N)$ and $r_{v,\Omega}(S) = 0$ for every $S \neq N$. Thus the restricted game $r_{v,\Omega}$ is a multiple of the unanimity game of N, being a game in which every player is a veto-player.

2. If $\Omega = 2^N$ then $\sigma_\Omega(S) = S$ and $r_{v,\Omega}(S) = v(S)$ for every $S \subseteq N$. The restricted game $r_{v,\Omega}$ coincides with v.

3. If Ω is the set of conjunctive feasible coalitions of some permission structure then $\sigma_\Omega(E) = E \setminus \widehat{S}_D(N \setminus E)$, and $r_{v,\Omega}$ is the conjunctive restriction. Similar for the disjunctive case.

A solution for games on union closed systems is a correspondence G that assigns a set of payoff vectors $G(v, \Omega)$ to every $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. Let $F \colon \mathcal{G}^N \to \mathbb{R}^n$ be a solution for TU-games, assigning a set of payoff vectors $F(v)$ to every $v \in \mathcal{G}^N$. In this chapter we only consider solutions for games on union closed systems that for each $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$, assigns the set of payoff vectors $F(r_{v,\Omega})$ of a solution $F \colon \mathcal{G}^N \to \mathbb{R}^n$, i.e., a solution for games on a union closed system assigns the set of payoff vectors that is assigned by a solution on $\mathcal{G}^N$ to the restricted game $r_{v,\Omega}$. For ease of notation we denote $F(v,\Omega) = F(r_{v,\Omega})$.

When we take as solution F the core of a game, then we obtain

$$Core(v,\Omega) = Core(r_{v,\Omega}).$$

Recall that (from chapter 2) $X(N,v)$ is the set

$$X(N,v) = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = v(N)\},$$

i.e., $X(N,v)$ is the set of efficient payoff vectors of $v \in \mathcal{G}^N$.

Proposition 3.2.12 *For every $v \in \mathcal{G}^N_m$ and $\Omega \in \mathcal{C}^N$ we have*

$$Core(v,\Omega) = \{x \in X(N,v) \mid x(S) \geq v(S) \text{ for every } S \in \Omega \text{ and } x_j \geq 0 \text{ for every } j \in N\}.$$

Proof. First, since $r_{v,\Omega}(S) = v(\sigma_\Omega(S))$ it follows that

$$Core(v,\Omega) = Core(r_{v,\Omega}) = \{x \in \mathbb{R}^n \mid x(N) = v(\sigma_\Omega(N)), x(S) \geq v(\sigma_\Omega(S)),\ S \subset N\}.$$

Since $N \in \Omega$ we have that $\sigma_\Omega(N) = N$ and thus $v(\sigma_\Omega(N)) = v(N)$. Hence $x \in X(N,v)$ for every $x \in Core(v,\Omega)$. Further, when the single player set $\{j\} \in \Omega$, then $\sigma_\Omega(\{j\}) = \{j\}$ and thus $v(\sigma_\Omega(\{j\})) = v(\{j\}) \geq 0$, since $v \in \mathcal{G}^N_m$. Otherwise $\sigma_\Omega(\{j\}) = \emptyset$ and thus $v(\sigma_\Omega(\{j\})) = v(\emptyset) = 0$. Hence for every $x \in Core(v,\Omega)$ we have that $x_j \geq v(\sigma_\Omega(\{j\})) \geq 0$ for every $j \in N$. Finally, since $\sigma_\Omega(S) \subseteq S$, $v \in \mathcal{G}^N_m$ and $x_j \geq 0$ for every $j \in N$, it follows that $x(S) \geq v(S)$ for every $S \in \Omega$ implies that $x(S) \geq v(\sigma_\Omega(S))$ for every $S \subset N$. □

The proposition says that a payoff vector x is in the core of (v,Ω) if and only if x is nonnegative, efficient for v, and yields a total payoff of at least $v(S)$ to the members of any feasible coalition S.

For a monotone game v on two different union closed systems Ω_1 and Ω_2 we write

$$(v,\Omega_1) \preceq (v,\Omega_2) \text{ if and only if } \Omega_1 \subseteq \Omega_2.$$

Since $v \in \mathcal{G}^N_m$ we have that $r_{v,\Omega_1}(S) \leq r_{v,\Omega_2}(S)$ for every $S \in 2^N$ if $(v,\Omega_1) \preceq (v,\Omega_2)$. Therefore the next proposition is straightforward.

Proposition 3.2.13 *Let v be monotone and Ω_1, Ω_2 be two union closed systems such that $(v,\Omega_1) \preceq (v,\Omega_2)$. Then $Core(v,\Omega_2) \subseteq Core(v,\Omega_1)$.*

Since $r_{v,\Omega} = v$ and thus $Core(v, \Omega) = Core(v)$ when $\Omega = 2^N$, this yields the following corollary.

Corollary 3.2.14 *If $v \in \mathcal{G}_m^N$ has a non-empty core, then $Core(v, \Omega) \neq \emptyset$ for any $\Omega \in \mathcal{C}^N$.*

For $x \in X(N, v)$, the *excess* $e_v(x)$ of x is defined by

$$e_v(x) = \max_{\{S \in 2^N \mid S \neq \emptyset, N\}} v(S) - x(S),$$

i.e., for any coalition $S \neq \emptyset, N$, its payoff $x(S)$ is at least equal to its own worth $v(S)$ minus the excess $e_v(x)$ with equality for at least one of these coalitions. We further define the *gain* $e(v)$ of the game (N, v) as the largest negative excess, thus

$$e(v) = \max_{x \in X(N,v)} - e_v(x).$$

Notice that $e_v(x) \leq 0$ when $x \in Core(N, v)$ and $e(v) \geq 0$ if and only if $Core(N, v) \neq \emptyset$.

Also the next proposition is straightforward.

Proposition 3.2.15 *Let v be monotone and Ω_1, Ω_2 be two union closed systems such that $(v, \Omega_1) \preceq (v, \Omega_2)$. Then $e(r_{v,\Omega_1}) \geq e(r_{v,\Omega_2})$.*

Further, the next corollary follows immediately from Proposition 3.2.8 and Proposition 3.2.13 and from Proposition 3.2.15 respectively.

Corollary 3.2.16 *Let (v, Ω) be a monotone game on a union closed system. Then*
1. $Core(v, \Sigma^\Omega) \subseteq Core(v, \Omega)$.
2. $e(r_{v,\Omega}) \geq e(r_{v,\Sigma^\Omega})$.

The first result in the corollary implies that for finding an element in the core of (v, Ω) we can restrict ourselves to the problem to find an element of the core of (v, Σ^Ω). Although this core is smaller, it might be easier to find a core element because of the structure of Σ^Ω as the set of feasible coalitions in the superior graph according to the conjunctive approach.

We conclude this subsection by defining the *least core* $LC(v)$ of a game $v \in \mathcal{G}_m^N$. The least core was introduced by Maschler, Peleg and Shapley (1979), see e.g. also Einy, Holzman and Monderer (1999), as the set

$$LC(v) = \{x \in X(N, v) | x(S) \geq v(S) + e(v) \text{ for every } S \neq \emptyset, N\}.$$

Observe that $LC(v) \subseteq Core(v)$ if $Core(v) \neq \emptyset$, with $LC(v) = Core(v)$ when $e(v) = 0$. We also have that $Nuc(v) \in LC(v)$ and that $LC(v) \subseteq I(N, v)$ when $v \in \mathcal{G}_m^N$. Notice that when $(v, \Omega_1) \preceq (v, \Omega_2)$ we have that $r_{v,\Omega_1}(S) \leq r_{v,\Omega_2}(S)$ for every $S \in 2^N$, but also that $e(r_{v,\Omega_1}) \geq e(r_{v,\Omega_2})$. Hence Proposition 3.2.13 does not hold for the least core.

3.3 Some properties of solutions for monotone games on union closed systems

In this section we consider the relation between the payoffs of some player j and its superior i for several solutions for monotone games on union closed systems, in particular we consider the core, nucleolus and Shapley value. It should be noticed that $r_{v,\Omega}(\{j\}) = 0$ when j has a superior, because $\{j\}$ is not feasible when j has a superior and thus $r_{v,\Omega}(\{j\}) = v(\emptyset) = 0$. We first consider some properties of the core and least core of the restricted game. Recall that the least core of a monotone game is contained in the imputation set of the game, so for the (monotone) restricted game $r_{v,\Omega}$ it holds for every $x \in LC(r_{v,\Omega})$ that $x \in I(N, r_{v,\Omega})$ and thus that $x_j \geq 0$ for every j, because either j is feasible in Ω and thus $x_j \geq v(\{j\}) \geq 0$ or j is not feasible and $x_j \geq r_{v,\Omega}(\{j\}) = v(\emptyset) = 0$.

In this section the players i and j are two fixed players such that i is a superior of j in Ω. Given i and j, we define for a vector x with $x_j > 0$ and $0 \leq a \leq x_j$, the vector x^a by

$$\begin{cases} x_i^a = x_i + a, & \\ x_j^a = x_j - a, & \\ x_k^a = x_k & \text{when } k \neq i, j. \end{cases} \tag{3.3.1}$$

Clearly, since $x_j^a = x_j - a \geq 0 = r_{v,\Omega}(\{j\})$ we have that $x^a \in I(N, r_{v,\Omega})$ when $x \in I(N, r_{v,\Omega})$. Moreover, for $S \subset N$

$$\begin{cases} x^a(S) = x(S) + a > x(S) & i \in S,\ j \notin S, \\ x^a(S) = x(S) - a < x(S) & j \in S,\ i \notin S, \\ x^a(S) = x(S) & \text{otherwise.} \end{cases}$$

So, for every $S \in \Omega$ it is true that $x^a(S) \geq x(S)$ because i is a superior of j and thus there does not exist $S \in \Omega$ with $j \in S$ and $i \notin S$.

Proposition 3.3.1 *Let (v, Ω) be a monotone game on a union closed system and, for a vector x and two players i and j such that i is a superior of j, let x^a be as defined in equation (3.3.1). Then*
(i) if $x \in Core(v, \Omega)$, then $x^a \in Core(v, \Omega)$ for all $a \in (0, x_j]$.
(ii) if $x \in LC(v, \Omega)$ and $x_i < x_j$, then $x^a \in LC(v, \Omega)$ for all $a \in (0, \frac{1}{2}(x_j - x_i)]$.

Proof. To prove (i), recall from Proposition 3.2.12 that

$$Core(v, \Omega) = \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for every } S \in \Omega \text{ and } x_j \geq 0 \text{ for every } j \in N\}.$$

Clearly, for every $S \in \Omega$ we have that $x^a(S) \geq x(S) \geq v(S)$. Further, we have for every $k \neq j$ that $x_k^a \geq x_k \geq 0$ and that $x_j^a = x_j - a \geq 0$. Since i is a superior of j and thus $\{j\} \notin \Omega$, it follows that $x^a \in Core(v, \Omega)$.

To prove (ii), notice that $x^a_i \leq x^a_j$ for all $a \in (0, \frac{1}{2}(x_j - x_i)]$. Suppose that x^a is not in $LC(v, \Omega)$. Then there exists a coalition $S \subset N$ such that

$$x^a(S) - r_{v,\Omega}(S) < e(r_{v,\Omega}). \tag{3.3.2}$$

Since $x \in LC(v, \Omega)$ we have that

$$x(S) - r_{v,\Omega}(S) \geq e(r_{v,\Omega}).$$

Hence $x(S) > x^a(S)$, implying that S contains j but not i. Let $T = S \backslash \{j\}$ and $S' = T \cup \{i\}$. Then

$$x^a(S) - r_{v,\Omega}(S) = x^a(S) - v(\sigma_\Omega(T))$$

because $i \notin S$ and thus $j \notin \sigma_\Omega(S)$. Hence

$$x^a(S) - r_{v,\Omega}(S) = x^a(S) - v(\sigma_\Omega(T)) \geq x^a(T) + x^a_j - v(\sigma_\Omega(T \cup \{i\})) \geq$$
$$x^a(T) + x^a_i - v(\sigma_\Omega(T \cup \{i\})) = x^a(S') - r_{v,\Omega}(S'),$$

where the second inequality follows because $x^a_i \leq x^a_j$. So, with equation (3.3.2) it follows that

$$x^a(S') - r_{v,\Omega}(S') \leq x^a(S) - r_{v,\Omega}(S) < e(r_{v,\Omega})$$

which contradicts that $x(S') - r_{v,\Omega}(S') \geq e(r_{v,\Omega})$, which must hold when $x \in LC(v, \Omega)$, since $x^a(S') > x(S')$ because $i \in S'$ and $j \notin S'$. □

From Part (i) of Proposition 3.3.1 we obtain the following corollary.

Corollary 3.3.2 *If $Core(r_{v,\Omega}) \neq \emptyset$, then there exists $x \in Core(r_{v,\Omega})$ such that $x_j = 0$ for every j that has a superior.*

The next two propositions state that for a monotone game on a union closed system (v, Ω) both the nucleolus and the Shapley value, assigns a payoff to some player i that is at least as high as the payoff to player j when i is a superior of j. It should be noticed that when v is monotone, $Nuc_j(v, \Omega) \geq 0$ for all j because $Nuc(v, \Omega)$ is in the least core of $r_{v,\Omega}$ and thus also in $I(N, r_{v,\Omega})$.

Proposition 3.3.3 *Let (v, Ω) be a monotone game on a union closed system. Then for every two players i and j such that i is a superior of j it holds that $Nuc_i(v, \Omega) \geq Nuc_j(v, \Omega)$.*

Proof. Let $w \in \mathcal{G}^N$ be a game such that for every $S \subseteq N \setminus \{i, j\}$ it holds that $w(S \cup \{i\}) \geq w(S \cup \{j\})$. Then we know from Peleg and Südholter (2003, Theorem 5.3.5) that $x_i \geq x_j$ for every x in the prekernel of w. Since the nucleolus of a game is in the prekernel of a game, it is sufficient to show that for every $S \subseteq N \setminus \{i, j\}$ it holds that $r_{v,\Omega}(S \cup \{i\}) \geq r_{v,\Omega}(S \cup \{j\})$ when i a superior of j. Indeed, in that case we have that

$$r_{v,\Omega}(S \cup \{i\}) = v(\sigma_\Omega(S \cup \{i\})) \geq v(\sigma_\Omega(S)) = v(\sigma_\Omega(S \cup \{j\})) = r_{v,\Omega}(S \cup \{j\}),$$

where the second equality follows from the fact that $i \notin S$ and there does not exist a feasible set containing j but not i. □

Proposition 3.3.4 *Let (v, Ω) be a monotone game on a union closed system. Then for every two players i and j such that i is a superior of j it holds that $Sh_i(v, \Omega) \geq Sh_j(v, \Omega)$.*

Proof. Notice that the Shapley value is the average of all marginal vectors m^π. Let S, T, R be a partition of the set $N \setminus \{i, j\}$ and let π be a permutation in which first the players of S enter, then i, then the players in T, then j and then the players in R. Further π' is the permutation such that $\pi(k) = \pi'(k)$ for all $k \neq i, j$, $\pi(i) = \pi'(j)$ and $\pi(j) = \pi'(i)$, i.e., first the players in S enter in the same order as in π, then j, then the players in T in the same order, then i and then the players of R in the same order. We consider the marginal contributions of i and j in the two permutations. In permutation π the marginal contribution m_i^π of i is given by

$$m_i^\pi = r_{v,\Omega}(S \cup \{i\}) - r_{v,\Omega}(S) = v(\sigma_\Omega(S \cup \{i\})) - v(\sigma_\Omega(S)) \geq 0,$$

whereas the marginal contribution of j in π' is given by

$$\begin{aligned} m_j^{\pi'} &= r_{v,\Omega}(S \cup \{j\}) - r_{v,\Omega}(S) = \\ &= v(\sigma_\Omega(S \cup \{j\})) - v(\sigma_\Omega(S)) = v(\sigma_\Omega(S)) - v(\sigma_\Omega(S)) = 0, \end{aligned}$$

because $i \notin S$ and there does not exist a feasible contribution containing j but not i. The marginal contribution of j in π is given by

$$m_j^\pi = r_{v,\Omega}(N \setminus R) - r_{v,\Omega}((N \setminus R) \setminus \{j\}) = v(\sigma_\Omega(N \setminus R)) - v(\sigma_\Omega((N \setminus R) \setminus \{j\}))$$

and the marginal contribution of i in π' is

$$m_i^{\pi'} = r_{v,\Omega}(N \setminus R) - r_{v,\Omega}((N \setminus R) \setminus \{i\}) = v(\sigma_\Omega(N \setminus R)) - v(\sigma_\Omega((N \setminus R) \setminus \{i\})).$$

Since

$$\sigma_\Omega((N \setminus R) \setminus \{i\}) = \sigma_\Omega((N \setminus R) \setminus \{i, j\}) \subseteq \sigma_\Omega((N \setminus R) \setminus \{j\}),$$

where the first equality follows from the fact that there is no feasible set without i containing j, it follows that

$$v(\sigma_\Omega((N \setminus R) \setminus \{i\})) \leq v(\sigma_\Omega((N \setminus R) \setminus \{j\})).$$

Hence $m_i^\pi \geq m_j^{\pi'}$ and $m_i^{\pi'} \geq m_j^\pi$. Since this holds for every π and corresponding π', it follows that the Shapley payoff to i is at least as big as the Shapley payoff to j. □

3.4 The prekernel of the restricted game

In this section we give a sufficient condition to guarantee that the prekernel and the core of a monotone game on a union closed system (v, Ω) have at most one point in common. Of course, when such a point exists, then it is the nucleolus of (v, Ω).

Arin and Feltkamp (1997) proved that the kernel of a game (N, v) consists of only one point (and coincides with the nucleolus), when the game is veto-rich and $I(N, v)$ is non-empty. Clearly, when in game (v, Ω), there exists a player $i \in N$ such that $i \in S$ for every $S \in \Omega$, then i is a veto-player in the restricted game $r_{v,\Omega}$. For $v \in \mathcal{G}_m^N$ we then have that $I(N, r_{v,\Omega}) \neq \emptyset$ and thus the kernel of $r_{v,\Omega}$ has the nucleolus of $r_{v,\Omega}$ as its unique element. On the other hand it is well-known that for every game (N, v) with $|N| \leq 3$, the intersection of the prekernel and the core consists of at most one point. In this section we generalize these results. To do so, we first introduce some new notions.

Definition 3.4.1 *For two players $i, j \in N$, $i \neq j$, player i is a strong superior of player j in $\Omega \in \mathcal{C}^N$ if i is a superior of j and j is not a superior of i.*

Definition 3.4.2 *A player $i \in N$ is a free player in $\Omega \in \mathcal{C}^N$ if i has no superiors; player $i \in N$ is a weak free player in $\Omega \in \mathcal{C}^N$ if i has no strong superior.*

Notice that a free player is also a weak free player and that a weak free player i is a superior of j when j is a superior of i. For $\Omega \in \mathcal{C}^N$, we denote the set of weak free players by W_Ω. The next proposition gives three properties of the set W_Ω.

Proposition 3.4.3
1. For every player $j \notin W_\Omega$, there is a player $i \in W_\Omega$, such that i is a strong superior of j.
2. When j is a superior of a player $i \in W_\Omega$, then i is a superior of j.
3. When j is a superior of a player $i \in W_\Omega$, then $j \in W_\Omega$.

Proof.
1. Consider some player $i_0 \in N$. If i_0 is not in W_Ω, then i_0 has a strong superior, say i_1. Then, either $i_1 \in W_\Omega$ and thus i_0 has a strong superior in W_Ω, or not. In the latter case i_1 has a strong superior, say i_2. When i_2 is not in W_Ω, it also has a strong superior. Continuing this we get a sequence of players $i_0, i_1, i_2, \ldots, i_m$ such that for $h = 1, \ldots, m-1$, player i_{h+1} is a strong superior of i_h and thus $i_h \notin W_\Omega$ and either $i_m \in W_\Omega$ or $m \geq 2$ and $i_m = i_k$ for some $k = 0, \ldots, m-2$. In the latter case, by Corollary 3.2.5 every pair i_j, i_ℓ with $j, \ell \in \{k, k+1, \ldots, m-1\}$ are superiors of each other, contradicting that i_{h+1} is strong superior of i_h, $h = k, \ldots, m-1$. Hence every next player in the sequence is different from all preceding players. Since the number of players is finite, this case can not happen and thus within a finite number of steps some player $i_m \in W_\Omega$ is generated. By Corollary 3.2.5 i_m is a superior of i_0. When i_0 is a superior of i_m, then again by Corollary 3.2.5 we have that i_0 is a superior of i_1, contradicting that i_1 is a strong superior of i_0. Hence $i_m \in W_\Omega$ and is a strong superior of i_0.

2. By definition, i is a superior of j, since otherwise j is a strong superior of i, which contradicts that $i \in W_\Omega$.
3. Suppose $j \notin W_\Omega$. Then by the first property, j has a strong superior k in W_Ω. By Corollary 3.2.5 player k is also a superior of i, and thus by property 2 we have that player i is also a superior of k. However this implies that also j is a superior of k, contradicting that k is a strong superior of j. □

The first property yields the following corollary.

Corollary 3.4.4 *For every $\Omega \in \mathcal{C}^N$, $W_\Omega \neq \emptyset$.*

Next, for $i \in W_\Omega$, let the set $T_\Omega(i)$ be given by

$$T_\Omega(i) = \{j \in N \mid j = i \text{ or } j \text{ is a superior of } i\}.$$

Notice that also i is a superior of j for every $j \in T_\Omega(i) \setminus \{i\}$ because $i \in W_\Omega$, and thus $T_\Omega(i) \subseteq S_i^\Omega = \{j \in N | j = i \text{ or } i \text{ is a superior of } j\}$. Let $\mathcal{T}_\Omega$ be the collection of sets defined by

$$\mathcal{T}_\Omega = \{T_\Omega(i) \mid i \in W_\Omega\}.$$

The next proposition describes the set W_Ω.

Proposition 3.4.5 *The collection $\mathcal{T}_\Omega$ is a partition of the set W_Ω.*

Proof. First, by property 3 of Proposition 3.4.3 we have that $j \in W_\Omega$ when $j \in T_\Omega(i)$ for some $i \in W_\Omega$ and thus $T_\Omega(i) \subseteq W_\Omega$. Next, let $R \subseteq W_\Omega \times W_\Omega$ be the binary relation on W_Ω defined by $(j,i) \in R$ if and only if $j \in T_\Omega(i)$. It is sufficient to show that this relation is an equivalence relation on W_Ω, i.e., the relation is reflexive, symmetric and transitive. First, by definition $(i,i) \in R$ for all $i \in W_\Omega$, so R is reflexive. Second, for $j \neq i$, when $(j,i) \in R$, then j is a superior of i. By property 2 of Proposition 3.4.3 then also i is a superior of j and thus $(i,j) \in R$, showing that R is symmetric. Third, when $(k,j) \in R$ and $(j,i) \in R$, then k is a superior of j and j of i and thus, by Corollary 3.2.5, also k is a superior of i. Hence, $(k,i) \in R$ and thus R is transitive. Since R is an equivalence relation, it follows that the sets $T_\Omega(i)$, $i \in W_\Omega$, are equivalence classes of W_Ω and thus the collection $\mathcal{T}_\Omega$ partitions W_Ω. □

Proposition 3.4.5 implies that $j \in T_\Omega(i)$ if and only if $i \in T_\Omega(j)$. When, for two different agents $i, j \in W_\Omega$, i is not a superior of j, then $T_\Omega(i)$ and $T_\Omega(j)$ are two different equivalence classes.

Proposition 3.4.6 *The partition $\mathcal{T}_\Omega$ of the set W_Ω has the property that, for every $i \in N$, when $j \in W_\Omega$ is a superior of i, then every $k \in T_\Omega(j)$ is a superior of i.*

Proof. For $i \in W_\Omega$ the proposition follows from Proposition 3.4.5, because $T_\Omega(i) = T_\Omega(j)$ when j is a superior of i. Let $i \notin W_\Omega$ and $j \in W_\Omega$ be a superior of i. Then every $k \neq j$ in $T_\Omega(j)$ is a superior of j and thus a superior of i by Corollary 3.2.5. □

Proposition 3.4.7 *Let (v, Ω) be a game on a union closed system. When $\mathcal{T}_\Omega$ consists of only one set, then every player in W_Ω is a veto-player in the restricted game $r_{v,\Omega}$.*

Proof. First, when $\mathcal{T}_\Omega$ consists of only one set, say T, then, by Proposition 3.4.5, $T = W_\Omega$. So, $T_\Omega(i) = W_\Omega$ for every $i \in W_\Omega$ and thus by definition of $T_\Omega(i)$ and Proposition 3.4.6 every player $k \in W_\Omega$ is a superior to every other i in W_Ω. Moreover, by property 1 of Proposition 3.4.3 every player not in W_Ω has a player i in W_Ω as its superior, and thus, again by Proposition 3.4.6, every player in W_Ω is a superior of every player not in W_Ω. So, every player in W_Ω is a superior of every other player in N, so that every $S \in \Omega$ contains all players in W_Ω. □

Notice that $T_\Omega(i) = \{i\}$ when i is a free player. So, every free player i gives a single element equivalence class $T_\Omega(i) = \{i\}$ in the partition $\mathcal{T}_\Omega$ of W_Ω. When there is a free player i and $\mathcal{T}_\Omega$ consists of only one set, then $W_\Omega = \{i\}$. In the sequel we call the number of sets in $\mathcal{T}_\Omega$ the *weak free player cardinality* of Ω. Since by Corollary 3.4.4 the set of weak free players is not-empty, this cardinality is at least one. It follows from Proposition 3.4.7 that $r_{v,\Omega}$ is a veto-rich game when this cardinality is equal to one. Then the next corollary follows from Arin and Feltkamp (1997).

Corollary 3.4.8 *Let (v, Ω) be a game on a union closed system. Then the kernel $K(r_{v,\Omega})$ contains $Nuc(r_{v,\Omega})$ as its unique element when the weak free player cardinality is one.*

To generalize this, we recall Kohlberg's theorem from chapter 2.

For game $v \in \mathcal{G}^N$, a payoff vector $x \in \mathbb{R}^n$ and real number α, let $\mathcal{B}(\alpha, x)$ be the collection of coalitions given by

$$\mathcal{B}(\alpha, x) = \{S \in N \mid e(S, x) \geq \alpha\}.$$

Theorem 3.4.9 *(***Kohlberg, 1979***) For game $v \in \mathcal{G}^N$, a payoff vector x is in $PN(v)$ if and only if for any real number α the collection of coalitions $\mathcal{B}(\alpha, x)$ is either balanced or empty.*

In Katsev and Yanovskaya (2009, see also chapter 7 in this thesis) an analogue of this theorem for the prekernel is proved in terms of 2-balancedness. We first give the notion of k-balancedness for $2 \leq k \leq |N|$.

Definition 3.4.10 *A collection $\mathcal{S}$ of coalitions $S \in 2^N$ is k-balanced if for every coalition $K \subseteq N$ with $|K| = k$ the collection*

$$\mathcal{S}_K = \{S' \subset K \mid S' = S \cap K, S \in \mathcal{S}\}$$

is balanced on K.

Theorem 3.4.11 *(***Katsev and Yanovskaya, 2009***) For $v \in \mathcal{G}^N$, a payoff vector x is in $PK(v)$ if and only if for any real number α the collection of coalitions*

$$\mathcal{B}(\alpha, x) = \{S \in N \mid e(S, x) \geq \alpha\}$$

is either 2-balanced or empty.

Recall from the standard definition of balancedness that when a collection $\mathcal{S}_K$ is balanced on K, then there exist strictly positive weights $\lambda_T^{\mathcal{S}_K}$, $T \in \mathcal{S}_K$, such that for every $i \in K$ the total weight of the sets $T \in \mathcal{S}_K$ that contain i is equal to one. From this the following corollary follows immediately.

Corollary 3.4.12 *Let $K = \{i, j\} \subseteq N$ be a two-player coalition and $\mathcal{S}$ be a collection of coalitions $S \in 2^N$ such that $\mathcal{S}_K$ is balanced on K. When $\mathcal{S}$ contains a set T such that $i \in T$ and $j \notin T$, then $\mathcal{S}$ contains a coalition T' such that $j \in T'$ and $i \notin T'$.*

Also notice that a k-balanced collection $\mathcal{S}$ is balanced when $k = n$. Moreover it should be noticed that when $|N| = 3$, any 2-balanced collection is also balanced. The next lemma generalizes this fact and will be used to prove the main result of this section.

Lemma 3.4.13 *For a union closed system Ω with weak free player cardinality of at most three, let $\mathcal{B} \subset 2^N$ be a 2-balanced collection that only contains feasible sets in Ω and singletons. Then $\mathcal{B}$ is balanced.*

Proof. Let $c \in \{1, 2, 3\}$ be the weak free player cardinality of Ω. Without loss of generality, let the players be numbered in such way that $W_\Omega \supset \{1, \ldots, c\}$ and that $T_\Omega(k)$, $k = 1, \ldots, c$, are the equivalence classes of $\mathcal{T}_\Omega$. By property 2 of Proposition 3.4.3, every player $j \neq k$ in $T_\Omega(k)$ has player k as its superior. Also, by property 1 of Proposition 3.4.3 and by Proposition 3.4.6, every player $j \in N \setminus W_\Omega$ has at least one of the players k, $k \in W_\Omega$ as one of its superiors. For $k \in W_\Omega$, suppose that there exists j in the set

$$S_k^\Omega \setminus \{k\} = \{i \in N | k \text{ is a superior of } i\}$$

such that there is some T in $\mathcal{B}$ containing k, but not j. Take $K = \{k, j\}$. By the 2-balancedness of $\mathcal{B}$ the collection $\{S \cap K \mid S \in \mathcal{B}\}$ is balanced on K. So, by Corollary 3.4.12 there exists a set $T' \in \mathcal{B}$ such that $j \in T'$ and $k \notin T'$. Since $\mathcal{B}$ only contains feasible sets and singletons, and k is a superior of j, it follows that $T' = \{j\}$. Let

$$S_k = \bigcap_{\{S \in \mathcal{B} | k \in S\}} S, \quad k \in W_\Omega.$$

From above it follows that $\{j\} \in \mathcal{B}$ for every $j \notin \cup_{k \in W_\Omega} S_k$. Now, let

$$\mathcal{B}' = \{U \in \mathcal{B} \mid U \cap W_\Omega \neq \emptyset\}$$

and consider the collection of subsets of W_Ω given by

$$\mathcal{B}'' = \{W_\Omega \cap U \mid U \in \mathcal{B}'\}.$$

This is a balanced collection on W_Ω. This is trivial when $c = 1$ and follows by the 2-balancedness of $\mathcal{B}$ when $c = 2$. When $c = 3$ this follows from the fact that every 2-balanced collection on a three player set is balanced. So, for $U \in \mathcal{B}'$, there are weights $\lambda_U^{\mathcal{B}}$ such that

$$\sum_{\{U \in \mathcal{B}' | k \in U\}} \lambda_U^{\mathcal{B}} = 1, \quad k \in W_\Omega.$$

Since every feasible set has a nonempty intersection with W_Ω, this yields weight $\lambda_U^{\mathcal{B}} > 0$ for every feasible set $U \in \mathcal{B}$. Moreover,

$$\sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}} = 1, \quad \text{for every } j \in \cup_{k=1,\ldots,c} S_k,$$

since if $j \in S_k$ for some $k = 1, \ldots, c$, then the collection of sets from $\mathcal{B}'$ containing j coincides with the collection of sets from $\mathcal{B}$' containing k. Finally, consider some $j \in N \setminus (\cup_{k=1,\ldots,c} S_k)$. Recall that such a player j has at least one of the players from the set W_Ω as one of its superiors, say player k. So, when j is contained in some set $U \in \mathcal{B}'$, then also $k \in U$. Moreover, there exists at least one $U \in \mathcal{B}'$ containing k and not j, otherwise $j \in S_k$. Therefore,

$$\sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}} < 1 \text{ for every } j \in N \setminus (\cup_{k=1,\ldots,c} S_k),$$

i.e., the total weight of the feasible sets containing such a player j is less than one. However, for every such j we also have that the singleton $\{j\} \in \mathcal{B}$. This yields weight $\lambda_{\{j\}}^{\mathcal{B}} = 1 - \sum_{\{U \in \mathcal{B}' | j \in U\}} \lambda_U^{\mathcal{B}}$ for every singleton set $\{j\} \in \mathcal{B}$, $j \in N \setminus (\cup_{k=1,\ldots,c} S_k)$. Since for every $j \in \cup_{k=1,\ldots,c} S_k$, every set in $\mathcal{B}$ containing j also contains one of the players from $\{1, \ldots, c\}$, there are no other singletons in $\mathcal{B}$. So, we have determined weights for all sets in $\mathcal{B}$ satisfying that

$$\sum_{\{S \in \mathcal{B} | j \in S\}} \lambda_S^{\mathcal{B}} = 1, \text{ for every } j \in N,$$

and thus $\mathcal{B}$ is balanced. □

Before we give the main result, we formulate and prove the following lemma.

Lemma 3.4.14 *Let $\mathcal{B}$ be a balanced collection and x an efficient payoff vector. Then*

$$\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} x(S) = v(N),$$

where $\lambda_S^{\mathcal{B}}$ is the weight of S, $S \in \mathcal{B}$.

Proof. By balancedness we have that $\sum_{\{S\in\mathcal{B}|i\in S\}} \lambda_S^{\mathcal{B}} = 1$ for every $i \in N$. So,

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} x(S) = \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} \sum_{i\in S} x_i = \sum_{i\in N} x_i \sum_{\{S\in\mathcal{B}|i\in S\}} \lambda_S^{\mathcal{B}} = \sum_{i\in N} x_i = v(N),$$

where the last equation follows from the fact that x is efficient. □

Next we formulate the main result of this section.

Theorem 3.4.15 *Let (v,Ω) be a monotone game on a union closed system. Then the intersection of $PK(v,\Omega)$ and $Core(v,\Omega)$ consists of at most one point if the weak free player cardinality of Ω is at most equal to three.*

Proof. Clearly, the statement of the theorem is true when $Core(v,\Omega) = \emptyset$. So, we only consider the case that $Core(v,\Omega) \neq \emptyset$. Then $PN(v,\Omega) = Nuc(v,\Omega)$ and lies in the core. Suppose there is a payoff vector $y \in PK(v,\Omega) \cap Core(v,\Omega)$ with $y \neq x = Nuc(v,\Omega)$. Since $y \neq PN(v,\Omega)$, according to Kohlberg's theorem there is some α for which $\mathcal{B}(\alpha,y)$ is not balanced. Since $x = PN(v,\Omega)$, also according to Kohlberg's theorem we have that $\mathcal{B}(\alpha,x)$ is balanced and thus $\mathcal{B}(\alpha,x) \neq \mathcal{B}(\alpha,y)$. Since for α big enough we have that $\mathcal{B}(\alpha,x) = \mathcal{B}(\alpha,y) = \emptyset$, there exists some value α with the properties that
(i) $\mathcal{B}(\alpha,x) \neq \mathcal{B}(\alpha,y)$ and
(ii) for every $\beta > \alpha$ it is true that either $\mathcal{B}(\beta,x) = \mathcal{B}(\beta,y)$ or both $\mathcal{B}(\alpha,x) = \mathcal{B}(\beta,x)$ and $\mathcal{B}(\alpha,y) = \mathcal{B}(\beta,y)$.

For a coalition S and payoff vector x, let $e(S,x) = r_{v,\Omega}(S) - x(S) = v(\sigma_\Omega(S)) - x(S)$ be the excess of coalition S at x in the restricted game $r_{v,\Omega}$ and let α^* be a value satisfying the two properties (i) and (ii). Now, suppose that there exists $S \in \mathcal{B}(\alpha^*,x)$ such that $e(S,x) < e(S,y)$. Then, for $\beta = e(S,y) > e(S,x) \geq \alpha^*$, we have that $S \in \mathcal{B}(\beta,y)$ and $S \notin \mathcal{B}(\beta,x)$. So, $\mathcal{B}(\beta,x) \neq \mathcal{B}(\beta,y)$ and $\mathcal{B}(\alpha^*,x) \neq \mathcal{B}(\beta,x)$, which contradicts that property (ii) holds for α^*. Hence

$$e(S,x) \geq e(S,y) \text{ for every } S \in \mathcal{B}(\alpha^*,x). \tag{3.4.3}$$

Further, for $S \in \mathcal{B}(\alpha^*,x)$, let λ_S be the weight of S in the balanced system of collection $\mathcal{B}(\alpha^*,x)$. Since both x and y are efficient, it follows with Lemma 3.4.14 that

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,x) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S(r_{v,\Omega}(S) - x(S)) =$$

$$= \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S r_{v,\Omega}(S) - r_{v,\Omega}(N)$$

and analogously

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,y) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S r_{v,\Omega}(S) - r_{v,\Omega}(N).$$

So,

$$\sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,x) = \sum_{\{S|S\in\mathcal{B}(\alpha^*,x)\}} \lambda_S e(S,y).$$

With inequalities (3.4.3) this implies $e(S,y) = e(S,x)$ for every $S \in \mathcal{B}(\alpha^*,x)$ and thus $\mathcal{B}(\alpha^*,x) \subseteq \mathcal{B}(\alpha^*,y)$.

Now, suppose that also the collection $B(\alpha^*,y)$ is balanced. Then by the same reasoning as above we obtain that $e(S,x) = e(S,y)$ for every $S \in \mathcal{B}(\alpha^*,y)$ and thus also $\mathcal{B}(\alpha^*,y) \subseteq \mathcal{B}(\alpha^*,x)$, which contradicts that $\mathcal{B}(\alpha^*,x) \neq \mathcal{B}(\alpha^*,y)$. Hence $B(\alpha^*,y)$ is not balanced.

On the other hand, by Theorem 3.4.11 we have that $\mathcal{B}(\alpha^*,y)$ is 2-balanced, since $y \in PK(v,\Omega)$. So, $\mathcal{B}(\alpha^*,y)$ is 2-balanced, but not balanced. Then, according to Lemma 3.4.13, $\mathcal{B}(\alpha^*,y)$ contains a non-feasible coalition S with $|S| > 1$. By definition of $\sigma_\Omega(S)$ and Ω being union closed, we have that $r_{v,\Omega}(T) = 0$ for every $T \subseteq S \setminus \sigma_\Omega(S)$. Then for every $i \in S \setminus \sigma_\Omega(S)$ it follows that

$$e(S,y) = r_{v,\Omega}(S) - y(S) = r_{v,\Omega}(\sigma_\Omega(S)) - \sum_{j\in\sigma_\Omega(S)} y_j - \sum_{h\in S\setminus\sigma_\Omega(S)} y_h \leq$$

$$e(\sigma_\Omega(S),y) - y_i = e(\sigma_\Omega(S),y) + e(\{i\},y),$$

because $y \in Core(r_{v,\Omega})$ and thus $y_h \geq r_{v,\Omega}(\{h\}) = 0$ for all $h \in S \setminus \sigma_\Omega(S)$. Since both $e(\sigma_\Omega(S),y) \leq 0$ and $e(\{i\},y) \leq 0$ (again because $y \in Core(r_{v,\Omega})$), it follows that

$$e(S,y) \leq e(\sigma_\Omega(S),y) \text{ and } e(S,y) \leq e(\{i\}),y)$$

so that both $\sigma(S) \in \mathcal{B}(\alpha^*,y)$ and $\{i\} \in \mathcal{B}(\alpha^*,y)$ for every $i \in S \setminus \sigma_\Omega(S)$. However, then also the collection $\mathcal{B}(\alpha^*,y) \setminus \{S\}$ is 2-balanced and not balanced. Let $NF = \{T \in \mathcal{B}(\alpha^*,y) \mid T \text{ is non-feasible and } |T| > 1\}$. Repeating the reasoning above for every $T \in NF$ it follows that $\mathcal{B}' = \mathcal{B}(\alpha^*,y) \setminus NF$ is 2-balanced and not balanced. However, since $\mathcal{B}'$ only consists of feasible sets and singletons, this contradicts Lemma 3.4.13. So, there is no $y \in PK(v,\Omega) \cap Core(v,\Omega)$ with $y \neq x = Nuc(v,\Omega)$. □

Chapter 4

Axiomatizations of two types of Shapley values for games on union closed systems

4.1 Introduction

In this chapter, which is based on van den Brink, Katsev and van der Laan (2009b), we also as in the previous chapter deal with games on union closed systems.

We define and axiomatize two solutions for games on union closed systems, one is based on games with a permission structure, the other on the approach of Myerson (1977, 1980) for communication graph games and conference structures. Both solutions generalize the Shapley value in the sense that both are equal to the Shapley value when the union closed system is the power set of player set N. First, recall from chapter 3 that for every union closed system the corresponding *superior graph* is the directed graph that is obtained by putting an arc from player i to player j if every feasible coalition containing player j also contains player i. Then we consider the game with permission structure of the original game on this superior graph, and define the superior rule as the conjunctive permission value of the corresponding game with permission structure, see Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996). We also give an axiomatization of this superior rule.

Second, we apply the method of Myerson (1977, 1980) to define another solution for games on union closed systems which generalizes the Shapley value for games on antimatroids, see Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). First, we define a modified or restricted game (as in Chapter 2) in which any feasible coalition earns its own worth. By union closedness, every nonfeasible coalition has a unique largest feasible subset. The restricted game assigns to any nonfeasible coalition the worth of this largest feasible subset. Then the *union rule* for games on union closed systems is defined as the Shapley value of this restricted game. We provide an axiomatization for this solution.

This chapter is organized as follows. Section 4.2 is a preliminary section that describes games with a permission structure. In Section 4.3 the superior rule is defined and axiomatized. In Section 4.4 the union rule is defined and axiomatized. The axioms discussed in Sections 4.3 and 4.4 all concern a fixed union closed system. In Section 4.5 several issues concerning a variable union closed system are discussed, and a comparison with Myerson (1980)'s conference structures is given. Section 4.6 contains concluding remarks.

4.2 Cooperative games with a permission structure

A game with a permission structure on N describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate within a coalition.

A tuple (v, D) with $v \in \mathcal{G}^N$ a TU-game and $D \in \mathcal{D}^N$ a digraph on N is called a *game with a permission structure*. In this chapter we follow the *conjunctive approach* as introduced in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996) in which it is assumed that a player needs permission from all its predecessors in order to cooperate with other players. Therefore a coalition is feasible only if for each player in the coalition all its predecessors are also in the coalition. So, for permission structure D the set of *conjunctive feasible coalitions* is given by

$$\Phi_D^c = \{E \subseteq N \,|\, P_D(i) \subseteq E \text{ for all } i \in E\}\,.$$

For any $E \subseteq N$, let $\overline{\sigma}_D^c(E) = E \setminus \widehat{S}_D(N \setminus E)$ be the largest conjunctive feasible subset of E in D.

Given the tuple (v, D) with $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, under the conjunctive permission structure the induced *restricted* game $r \colon 2^N \to \mathbb{R}$ is given by

$$r(S) = v(\overline{\sigma}_D^c(S)) \text{ for all } S \subseteq N. \tag{4.2.1}$$

The *conjunctive permission value* φ^c is the (single-valued) solution that assigns to every game with a permission structure the Shapley value of the restricted game, i.e.

$$\varphi^c(v, D) = Sh(r).$$

These games with a permission structure and the conjunctive permission value are generalized to games on antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004). In this chapter we consider a further generalization to games on union closed systems.

4.3 The superior rule

A (single-valued) solution for games on union closed systems is a function f that assigns a payoff distribution $f(v, \Omega) \in \mathbb{R}^N$ to every $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. Recall that $\mathcal{C}^N$ denotes the collection of all union closed systems in 2^N. In this section we introduce and axiomatize a solution for games on union closed systems that is based on the conjunctive permission value of a digraph associated with the union closed system.

Recall the definition of the superior graph which has been given in the previous chapter.

Definition 4.3.1 *For two players $i, j \in N$, $i \neq j$, player i is a* superior *of player j in $\Omega \in \mathcal{C}^N$, if $i \in S$ for every $S \in \Omega$ with $j \in S$. In that case we call player j a* subordinate *of player i. For $\Omega \in \mathcal{C}^N$, the superior graph of Ω is the directed graph $D^\Omega \in \mathcal{D}^N$ with*

$$D^\Omega = \{(i, j) \in N \times N \mid i \text{ is a superior of } j \text{ in } \Omega\}.$$

Now we define the superior rule *SUP* as the solution for games on union closed systems which assigns to every (v, Ω) the conjunctive permission value of the game v with permission structure D^Ω, i.e.

$$SUP_i(v, \Omega) = \varphi_i^c(v, D^\Omega) \text{ for all } i \in N.$$

Next we give an axiomatization of the superior rule as a solution for games on union closed systems. The axioms are generalizations of axioms used to axiomatize the conjunctive permission value in van den Brink and Gilles (1996) and the Shapley value for games on poset antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). First, efficiency states that the total sum of payoffs equals the worth of the 'grand' coalition.

Axiom 4.3.2 (Efficiency) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, $\sum_{i \in N} f_i(v, \Omega) = v(N)$.*

Additivity is a straightforward generalization of the well-known additivity axiom for TU-games.

Axiom 4.3.3 (Additivity) *For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f(v + w, \Omega) = f(v, \Omega) + f(w, \Omega)$.*

Next we introduce a generalization of the inessential player property stating that a null player in v whose subordinates in Ω are all null players in v, earns a zero payoff. We say that player $i \in N$ is *inessential* in (v, Ω) if $v(E \cup \{j\}) = v(E)$ for all $j \in \{i\} \cup S_{D^\Omega}(i)$ and $E \subseteq N \setminus \{j\}$. For $v \in \mathcal{G}^N$, $\Omega \in \mathcal{C}^N$, we denote by $I(v, \Omega)$ the set of all inessential players in (v, Ω).

Axiom 4.3.4 (Inessential player property) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f_i(v, \Omega) = 0$ for all $i \in I(v, \Omega)$.*

A player $i \in N$ is *necessary* in game v if $v(E) = 0$ for all $E \subseteq N \setminus \{i\}$. The next axiom generalizes the necessary player property (which holds for monotone TU-games) in a straightforward way, stating that a necessary player in a monotone game earns at least as much as any other player, irrespective of the coalitions in the union closed system.

Axiom 4.3.5 (Necessary player property) *For every monotone game $v \in \mathcal{G}_m^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f_i(v,\Omega) \geq f_j(v,\Omega)$ for all $j \in N$, when $i \in N$ is a necessary player in v.*

Finally, structural monotonicity is generalized using the superior graph, stating that whenever player i is a superior of player j in the union closed system and the game is monotone, then player i earns at least as much as player j.

Axiom 4.3.6 (Structural monotonicity) *For every monotone game $v \in \mathcal{G}_M^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f_i(v,\Omega) \geq f_j(v,\Omega)$ if $i \in N$ and $j \in S_{D^\Omega}(i)$.*

The five axioms above characterize the superior rule for games on union closed systems.

Theorem 4.3.7 *A solution f for cooperative games on union closed systems is equal to the superior rule SUP if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.*

Proof

By efficiency of the conjunctive permission value (i.e. $\sum_{i\in N} \varphi_i^c(v,D) = v(N)$ for every $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$) we have that $\sum_{i\in N} SUP_i(v,\Omega) = \sum_{i\in N} \varphi_i^c(v,D^\Omega) = v(N)$, showing that the superior rule satisfies efficiency. Additivity, the inessential player property, the necessary player property and structural monotonicity follow from the corresponding axioms of the conjunctive permission value for games with a permission structure, see van den Brink and Gilles (1996).

To prove uniqueness, suppose that solution f satisfies the five axioms. Let v_0 be the *null game* given by $v_0(E) = 0$ for all $E \subseteq N$. The inessential player property then implies that $f_i(v_0,\Omega) = 0$ for all $i \in N$.

Next, consider a union closed system Ω and the game $w_T = c_T u_T$, $c_T > 0$, $T \subseteq N$, $T \neq \emptyset$. We distinguish the following three cases with respect to $i \in N$:

1. If $i \in T$, then the necessary player property implies that there exists a $c^* \in \mathbb{R}$ such that $f_i(w_T,\Omega) = c^*$ for all $i \in T$, and $f_i(w_T,\Omega) \leq c^*$ for all $i \in N \setminus T$.

2. If $i \in N \setminus T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) \neq \emptyset$, then structural monotonicity implies that $f_i(w_T,\Omega) \geq f_j(w_T,\Omega)$ for every $j \in T \cap (\{i\} \cup S_{D^\Omega}(i))$, and thus with case 1 that $f_i(w_T,\Omega) = c^*$.

3. If $i \in N \setminus T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) = \emptyset$, then the inessential player property implies that $f_i(w_T, \Omega) = 0$.

From 1 and 2 it follows that $f_i(w_T, \Omega) = c^*$ for $i \in T \cup P_{D^\Omega}(T)$. Efficiency and 3 then imply that $\sum_{i \in N} f_i(w_T, \Omega) = |T \cup P_{D^\Omega}(T)|c^* = c_T$, implying that c^*, and thus $f(w_T, \Omega)$, is uniquely determined.

Next, consider (w_T, Ω) with $w_T = c_T u_T$ for some $c_T < 0$ (and thus we cannot apply the necessary player property and structural monotonicity since w_T is not monotone). Since $-w_T = -c_T u_T$ with $-c_T > 0$, and $v_0 = w_T + (-w_T)$, it follows from additivity of f that $f(w_T, \Omega) = f(v_0, \Omega) - f(-w_T, \Omega) = -f(-w_T, \Omega)$ is uniquely determined because $-w_T$ is monotone.

Finally, since every characteristic function $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{T \subseteq N} \Delta_v(T) u_T$ (with $\Delta_v(T)$ the *Harsanyi dividend* of coalition T, see Harsanyi (1959)), additivity uniquely determines $f(v, \Omega) = \sum_{T \subseteq N} f(\Delta_v(T) u_T, \Omega)$ for any $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. □

We end this section by showing logical independence of the five axioms stated in Theorem 4.3.7.

1. The solution that assigns to every game on union closed system simply the Shapley value of game v, i.e. $f(v, \Omega) = Sh(v)$, satisfies efficiency, additivity, the inessential player property and the necessary player property. It does not satisfy structural monotonicity.

2. For $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$, let $\overline{v} \in \mathcal{G}^N$ be given by $\overline{v}(E) = v(\bigcup_{i \in E}(\{i\} \cup S_{D^\Omega}(i)))$ for all $S \subseteq N$. The solution $f(v, \Omega) = Sh(\overline{v})$ satisfies efficiency, additivity, the inessential player property and structural monotonicity. It does not satisfy the necessary player property.

3. The equal division solution given by $f_i(v, \Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and structural monotonicity. It does not satisfy the inessential player property.

4. The equal division over essential players, given by

$$f_i(v, \Omega) = \begin{cases} \frac{v(N)}{|N \setminus I(v,\Omega)|} & \text{if } i \in N \setminus I(v, \Omega) \\ 0 & \text{if } i \in I(v, \Omega), \end{cases}$$

satisfies efficiency, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy additivity.

5. The zero solution given by $f_i(v, \Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy efficiency.

4.4 The union rule

In this section we introduce and axiomatize the *union rule* as a solution for games on union closed systems. This rule is defined similar as the Myerson rule for conference structures in Myerson (1980) and the Shapley value for games on antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). The union rule U assigns to every (v,Ω) the Shapley value of the restriced game $r_{v,\Omega}$, i.e.

$$U_i(v,\Omega) = Sh_i(r_{v,\Omega}) \text{ for all } i \in N.$$

This solution is different from the superior rule as illustrated in the following example.

Example 4.4.1 Consider the unanimity game $v = u_{\{1\}}$ and union closed system $\Omega = \{\emptyset, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ on $N = \{1,2,3\}$. The superior graph of Ω is $D^\Omega = \{(1,2),(1,3)\}$. Therefore, the superior rule equals $SUP(v,\Omega) = (1,0,0)$.

On the other hand, the restricted game is given by $r_{v,\Omega}(\{1\}) = r_{v,\Omega}(\{2\}) = r_{v,\Omega}(\{3\}) = r_{v,\Omega}(\{2,3\}) = 0$, $r_{v,\Omega}(\{1,2\}) = r_{v,\Omega}(\{1,3\}) = r_{v,\Omega}(\{1,2,3\}) = 1$, and thus the union rule equals $U(v,\Omega) = Sh(r_{v,\Omega}) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

From the axioms that are used to characterize the superior rule in Theorem 4.3.7, the union rule satisfies all the axioms except the inessential player property. The union rule not satisfying the inessential player property is illustrated by the following example.

Example 4.4.2 Consider the union closed system $\Omega = \{\emptyset, \{1,2\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}$ and the game $v = u_{\{3\}}$ on $N = \{1,2,3,4\}$. The superior graph is given by $D^\Omega = \{(1,2),(3,4)\}$, and $I(v,\Omega) = \{1,2,4\}$. However, the restricted game is $r_{v,\Omega} = u_{\{1,3\}} + u_{\{3,4\}} - u_{\{1,3,4\}}$, and thus $U(v,\Omega) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$.

The union rule satisfies a weaker axiom requiring zero payoffs for inessential players only in games where the worth of any coalition equals the worth of its largest feasible subset[1].

Axiom 4.4.3 (Inessential player property for union closed games) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$ such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$, $f_i(v,\Omega) = 0$ for every $i \in I(v,\Omega)$.*

To characterize the union rule we add one more axiom which states that the payoffs only depend on the worths of feasible coalitions.

Axiom 4.4.4 (Independence of irrelevant coalitions) *For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f(v,\Omega) = f(w,\Omega)$ whenever $v(S) = w(S)$ for all $S \in \Omega$.*

[1] Note that the union rule satisfies the stronger property requiring zero payoffs for all null players in games v such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$.

For $\Omega \in \mathcal{C}^N$ and $T \subseteq N$, we define $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ as the set of feasible coalitions containing coalition T. In the proof of uniqueness in Theorem 4.4.6 we use the following lemma.

Lemma 4.4.5 *For every $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$, there exist numbers $\delta_H \in \mathbb{R}$, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H \in \Omega_T} \delta_H u_H$.*

Proof

Consider $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$. If $T \in \Omega$ then $T \in \Omega_T$ and we have $\delta_T = c$ and $\delta_H = 0$ for all $H \in \Omega_T \setminus \{T\}$. If $T \notin \Omega$, then define

$$\mathcal{T}^1 = \{H \in \Omega \mid T \subset H \text{ and there is no } Z \in \Omega \text{ such that } T \subset Z \subset H\}$$

and, recursively, for $k = 2, \ldots$

$$\mathcal{T}^k = \Big\{H \in \Omega \mid T \subset H \text{ and for every } Z \in \Omega \text{ such that } T \subset Z \subset H \text{ it holds that } Z \in \bigcup_{p=1}^{k-1} \mathcal{T}^p\Big\}.$$

Since N is finite there exists an $M < \infty$ such that $\mathcal{T}^k \neq \emptyset$ for all $k \in \{1, \ldots M\}$, $\mathcal{T}^{M+1} = \emptyset$ and $\bigcup_{k=1}^M \mathcal{T}^k = \Omega_T$. Since by definition $\mathcal{T}^k \cap \mathcal{T}^l = \emptyset$ for all $k, l \in \mathbb{N}$, we have that $\mathcal{T}^1, \ldots, \mathcal{T}^M$ is a partition of the set $\{H \in \Omega \mid T \subset H\}$ of feasible coalitions containing non-feasible coalition T. (Note that this set equals Ω_T since $T \notin \Omega$.) Then $\delta_H = c$ for all $H \in \mathcal{T}^1$ and, recursively for $k = 2, \ldots, M$, the numbers δ_H, $H \in \mathcal{T}^k$, are determined by

$$\delta_H + \sum_{\{Z \subset H \mid Z \in \bigcup_{l=1}^{k-1} \mathcal{T}^l\}} \delta_Z = c.$$

□

Weakening, in Theorem 4.3.7, the inessential player property by requiring only the inessential player property for union closed games, and adding independence of irrelevant coalitions, characterizes the union rule.

Theorem 4.4.6 *A solution f for cooperative games on union closed systems is equal to the union rule U if and only if it satisfies efficiency, additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions.*

Proof

We first prove that U satisfies the five axioms. Let $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$.

1. By efficiency of the Shapley value and since $\sigma_\Omega(N) = N$, we have that $\sum_{i\in N} U_i(v,\Omega) = \sum_{i\in N} Sh_i(r_{v,\Omega}) = v(N)$, showing that U satisfies efficiency.

2. Additivity of the Shapley value and the fact that $r_{v,\Omega}(S) + r_{w,\Omega}(S) = v(\sigma_\Omega(S)) + w(\sigma_\Omega(S)) = (v+w)(\sigma_\Omega(S)) = r_{v+w,\Omega}(S)$ for all $S \subseteq N$, imply for $i \in N$ that $U_i(v,\Omega) + U_i(w,\Omega) = Sh_i(r_{v,\Omega}) + Sh_i(r_{w,\Omega}) = Sh_i(r_{v+w,\Omega}) = U_i(v+w,\Omega)$, showing that U satisfies additivity.

3. U satisfying the inessential player property for union closed games follows directly from the null player property of the Shapley value.

4. Let v be a monotone game on N. Since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$, by monotonicity of v we have that $r_{v,\Omega}$ is a monotone game on N. The necessary player property then follows from the necessary player property of the Shapley value.

5. If $v(S) = w(S)$ for all $S \in \Omega$, then $r_{v,\Omega} = r_{w,\Omega}$, showing that the union rule U satisfies independence of irrelevant coalitions.

To prove uniqueness, let $\Omega \in \mathcal{C}^N$. We first consider $v = cu_T$ for some $c \in \mathbb{R}$ and $\emptyset \neq T \subseteq N$. We distinguish two cases.
1. Let $T \in \Omega$, i.e. T is feasible. Then $r_{cu_T,\Omega} = cu_T$. From the necessary player property it follows that there exists a $c^* \in \mathbb{R}$ such that $f_i(cu_T,\Omega) = c^*$ for all $i \in T$. Since $i \in N \setminus T$ is a null player in cu_T, and $cu_T(E) = cu_T(\sigma_\Omega(E))$ for all $E \subseteq N$ if $T \in \Omega$, the inessential player property for union closed games implies that $f_i(cu_T,\Omega) = 0$ for all $i \in N \setminus T$. Then efficiency implies that $c^* = f_i(cu_T,\Omega) = \frac{c}{|T|}$ for all $i \in T$, and thus $f(cu_T,\Omega)$ is determined.
2. Suppose that $T \notin \Omega$, i.e. T is not feasible. Let $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ be the collection of feasible subsets of N that contain T. Note that $T \notin \Omega_T$ since $T \notin \Omega$. By Lemma 4.4.5 there exist numbers δ_H, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H\in\Omega_T} \delta_H u_H$. Since $cu_T(E) = r_{cu_T,\Omega}(E)$ for all $E \in \Omega$, by independence of irrelevant coalitions it then follows that $f(cu_T,\Omega) = f(r_{cu_T,\Omega},\Omega) = f(\sum_{H\in\Omega_T} \delta_H u_H,\Omega)$. By additivity we then have that

$$f(cu_T,\Omega) = f\left(\sum_{H\in\Omega_T} \delta_H u_H, \Omega\right) = \sum_{H\in\Omega_T} f(\delta_H u_H,\Omega). \tag{4.4.2}$$

Since all $H \in \Omega_T$ are feasible in Ω, we know from case 1 that $f(\delta_H u_H,\Omega)$ is uniquely determined for every $H \in \Omega_T$. Thus, with (4.4.2) also $f(cu_T,\Omega)$ is uniquely determined. It further follows that additivity uniquely determines
$f(v,\Omega) = \sum_{T\subseteq N} f(\Delta_v(T)u_T,\Omega)$ for any $v \in \mathcal{G}^N$. □

The following example illustrates that the superior rule does not satisfy independence of irrelevant coalitions.

Example 4.4.7 Consider the tuple (v,Ω) of Example 4.4.2, and let game w be the restriction of v on Ω. Obviously, $r_{v,\Omega} = r_{w,\Omega}$. However, since the superior graph is given by $D^\Omega = \{(1,2),(3,4)\}$, we have that $\overline{r}^c_{v,D^\Omega} = u_{\{3\}} = v$ and $\overline{r}^c_{w,D^\Omega} = u_{\{1,3\}}+u_{\{3,4\}}-u_{\{1,3,4\}} = w$, and thus $SUP(v,\Omega) = (0,0,1,0)$ and $SUP(w,\Omega) = (\frac{1}{6},0,\frac{2}{3},\frac{1}{6})$.

We end this section by showing logical independence of the five axioms stated in Theorem 4.4.6.

1. The superior rule satisfies efficiency, additivity, the inessential player property for union closed games and the necessary player property. It does not satisfy independence of irrelevant coalitions.

2. The solution that assigns to every game on union closed system the weighted Shapley of the restricted game $r_{v,\Omega}$ for some exogenous weight system $\omega \in \mathbb{R}^N$ with $\omega_i \neq \omega_j$ for some $i,j \in N$, satisfies efficiency, additivity, the inessential player property for union closed games and independence of irrelevant coalitions. It does not satisfy the necessary player property.

3. The equal division solution given by $f_i(v,\Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and independence of irrelevant coalitions. It does not satisfy the inessential player property for union closed games.

4. The equal division over non-null players, given by

$$f_i(v,\Omega) = \begin{cases} \frac{v(N)}{|N \setminus Null(v,\Omega)|} & \text{if } i \in N \setminus \text{Null}(v,\Omega) \\ 0 & \text{if } i \in \text{Null}(v,\Omega), \end{cases}$$

   where $\text{Null}(v,\Omega)$ denotes the set of null players in the restricted game $r_{v,\Omega}$, satisfies efficiency, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy additivity.

5. The zero solution given by $f_i(v,\Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy efficiency.

4.5 Irrelevant players and fairness

As mentioned before in Section 4.3, we can define and axiomatize the superior rule and the union rule also if we do not assume that the 'grand coalition' N is feasible in Definition 3.2.1. By condition 2 in that definition, the players that do not belong to the largest feasible subset of N do not belong to any feasible coalition. We refer to these players as *irrelevant* players. For such a union closed system Ω we denote by $R(\Omega) = \{i \in N \mid \text{there is an } S \in \Omega \text{ with } i \in S\}$ the set of *relevant* players, i.e. the players that belong to at least one feasible coalition. Then we can define the superior rule and the union rule by applying these two rules to the game and union closed system restricted to $R(\Omega)$, and assign the payoff zero to all irrelevant players. The corresponding two rules can be axiomatized similar as done before in this chapter, by adapting the axioms in a similar way (i.e. distinguishing between relevant and irrelevant players), and adding the axiom which states that irrelevant players get zero payoff.

Axiom 4.5.1 (Irrelevant player property) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, $f_i(v, \Omega) = 0$ for all $i \in N \setminus R(\Omega)$.*

As mentioned in the preliminaries, Myerson (1980) characterized the Myerson rule for conference structures by component efficiency and fairness. Although a conference structure[2] is any set of feasible coalitions on N, i.e. any subset of 2^N, by Myerson (1980)'s definition of connectedness all singletons are connected and thus earn their own worth in the restricted game. So, even singletons that are not feasible, in the sense that they do not belong to the conference structure, earn their worth in the restricted game. Note that in our approach we took $r_{v,\Omega}(\{i\}) = v(\{i\})$ only if $\{i\}$ is feasible, and $r_{v,\Omega}(\{i\}) = 0$ otherwise. Alternatively, in line with Myerson (1980) we could always take $r_{v,\Omega}(\{i\}) = v(\{i\})$ irrespective of whether $\{i\}$ is feasible or not.

Because of the definition of connectedness, and thus the restricted game, in Myerson (1980), it does not matter whether a conference structure does or does not contain $\{i\}$ for any $i \in N$. Consequently, a conference structure $\mathcal{F}$ yields the same Myerson rule payoffs as conference structure $\mathcal{F} \cup \{\{i\} \mid i \in N\}$. Considering the subclass of conference structures where all singletons are feasible, i.e. $\{i\} \in \mathcal{F}$ for all $i \in N$, the proof that there is a unique solution satisfying component efficiency and fairness is similar to that in Myerson (1980)[3]. However, for union closed systems typically we do not have $\{i\} \in \Omega$ for all $i \in N$, since the unique union closed system satisfying this property is $\Omega = 2^N$. Therefore, we only require the conditions in Definition 3.2.1.

Continuing our comparison with conference structures, we now discuss a fairness axiom for union closed systems similar as the one for conference structures. However, while applying fairness to conference structures any coalition can be deleted from the set of feasible coalitions, for union closed systems we can only delete coalitions such that the remaining set of feasible coalitions is still union closed. (Similar restrictions on deleting feasible coalitions hold for other types of structures satisfying specific properties.) In other words, we can only delete coalitions that are not the union of other feasible coalitions.

[2]For any conference structure, two players are called connected if there is a feasible coalition that contains both players. Moreover, also all players are defined to be connected with themselves. A component in the conference structure then is a maximally connected set of players. Component efficiency states that the sum of payoffs over all players in one component equals the worth of that component in the game.

[3]Allowing $\{i\} \notin \mathcal{F}$ for some $i \in N$ the axiomatization can be stated adding the irrelevant player property.

Definition 4.5.2 *Let $\Omega \in \mathcal{C}^N$. A coalition $T \in \Omega$ is a* basis *coalition in Ω if there do not exist $U, V \in \Omega$ with $T = U \cup V$.*

Alternatively we can say that a coalition $T \in \Omega \in \mathcal{C}^N$ is a basis coalition in Ω if $\Omega \setminus \{T\} \in \mathcal{C}^N$.

Axiom 4.5.3 (Fairness) *For every game $v \in \mathcal{G}^N$, every union closed system $\Omega \in \mathcal{C}^N$ and every basis coalition $S \in \Omega$, $f_i(v,\Omega) - f_i(v, \Omega \setminus \{S\}) = f_j(v,\Omega) - f_j(v,\Omega \setminus \{S\})$ for all $i, j \in S$.*

The superior rule does not satisfy fairness, as can be seen by comparing the game $v = u_{\{2\}}$ on union closed systems $\Omega = \{\emptyset, \{1\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ and $\Omega \setminus \{\{2,3\}\}$ (on $N = \{1,2,3\}$), where $D^\Omega = \{(2,3)\}$ and $D^{\Omega\setminus\{\{2,3\}\}} = \{(1,2)\}$, and thus $SUP_2(v,\Omega) - SUP_2(v,\Omega \setminus \{\{2,3\}\}) = 1 - \frac{1}{2} = \frac{1}{2} \neq 0 = SUP_3(v,\Omega) - SUP_3(v,\Omega \setminus \{\{2,3\}\})$.

The union rule satisfies fairness. However, not allowing all coalitions to be deleted from the set of feasible coalitions restricts the applicability of the fairness axiom to characterize solutions.

Besides fairness being a weaker axiom on union closed systems than on arbitrary systems, also component efficiency is weak since, by union closedness, it boils down to just efficiency and the irrelevant player property. Efficiency, fairness and the irrelevant player property do not characterize the union rule for games on union closed systems. Another solution that satisfies these axioms on the class of games on union closed systems is the *modified union rule* where we take two disjoint coalitions of equal cardinality and in case both are feasible we subtract a fixed amount, say 1, from the payoff of all players in one coalition and give it to all players in the other coalition. Formally, take two disjoint coalitions $S, T \subseteq N$ with $|S| = |T|$. Then the (S,T)-union rule is the rule $\overline{U}^{(S,T)}$ given by

$$\overline{U}^{(S,T)}(v,\Omega) = \begin{cases} U(v,\Omega) & \text{if } \{S,T\} \not\subset \Omega \\ \tilde{U}^{(S,T)}(v,\Omega) & \text{otherwise,} \end{cases}$$

where

$$\tilde{U}_i^{(S,T)}(v,\Omega) = \begin{cases} U_i(v,\Omega) + 1 & \text{if } i \in S \\ U_i(v,\Omega) - 1 & \text{if } i \in T \\ U_i(v,\Omega) & \text{otherwise.} \end{cases}$$

Note that the axioms discussed in the previous sections (see Theorems 4.3.7 and 4.4.6) all are applied to a fixed union closed system Ω. Applying axioms like fairness requires that we allow to change the set of feasible coalitions. This type of axiomatizations will be studied in future research.

4.6 Concluding remarks

In this chapter we have introduced two generalizations of the Shapley value to games on union closed systems. The superior rule is based on the conjunctive permission value of an associated game with permission structure, while the union rule is based on the Shapley value of the restricted game. We axiomatized both rules such that they differ in only one axiom. Both rules satisfy efficiency, additivity, the inessential player property for union closed games and the necessary player property. We obtain an axiomatization of the superior rule by strengthening the inessential player property for union closed games to the stronger inessential player property. We obtain an axiomatization of the union rule by adding independence of irrelevant coalitions.

Chapter 5

Between the prekernel and the prenucleolus

5.1 Introduction

The prekernel and the prenucleolus are the most popular solutions for games with transferable utilities that are consistent with respect to the Davis–Maschler reduced games. In fact, both possess, besides the Davis–Maschler consistency, the very attractive properties of nonemptiness, covariance and symmetry (equal treatment property). The prenucleolus is a single-valued solution and so it is a minimal solution with respect to set inclusion within the class of solutions satisfying these properties. The prekernel also satisfies converse consistency and therefore it is the maximal solution with respect to set inclusion possessing the other mentioned properties. Thus, the prekernel and the prenucleolus can be considered as the two "extremal" non-empty consistent solutions satisfying covariance and symmetry. As noticed in Chapter 2, both solutions are also efficient.

Most of the other consistent solutions for TU games do not possess at least one of the properties above or are not defined on the class $\mathcal{G}$ of all TU games. For instance, the positive core does not satisfy symmetry and the Dutta-Ray egalitarian solution does not satisfy covariance and is also only defined on the class of convex TU games. Only in Orshan and Sudhölter (2003) a sophisticated solution is given that satisfies all the properties above and is not equal to the prenucleolus.

In this chapter, based on Katsev and Yanovskaya (2009), we describe a class of solutions that satisfy nonemptiness, covariance and symmetry and that have the prenucleolus and the prekernel as its extreme cases. The class contains a countable number of solutions, indexed by a natural number not less than 2. Denoting by PKN_k the solution indexed by k, $k \geq 2$, then PKN_k has the property that

$$PKN_k(N, v) = PN(N, v) \text{ if } k \geq n = |N|.$$

In case $k < n$ the solution has the property that $x \in PKN_k(N, v)$ if for each coalition $S \subset N$ with $|S| = k$, $x_S = PN(S, v_S^x)$, where (S, v_S^x) is the Davis-Maschler reduced game on player set S with respect to x.

The class of solutions can be described by using lexicographical minimization. Recall that the prenucleolus of each TU game (N, v) is determined as the payoff vector which minimizes lexicographically over the set of all efficient payoff vectors the vectors of the excesses disposed in a weakly decreasing order. The solution PKN_k can be determined analogously by the lexicographical minimization of the vectors of excesses by comparing any pair of efficient payoff vectors that differ from each other in at most k coordinates. In case $k = 2$ we obtain $PKN_2(N, v) = PK(N, v)$, and for $k \geq n$ it follows that $PKN_k(N, v) = PN(N, v)$. The class of solutions can be characterized by the concept of k-balancedness.

This chapter is organized as follows. Section 5.2 is devoted to the optimization approach to define the prekernel, the prenucleolus and the new collection of solutions for TU games. Section 5.3 deals with the characterization of the class of solutions by using the notion of k-balancedness. Section 5.4 describes the consistency properties of the solutions. Finally, Section 5.5 contains an example of a non-trivial TU-game, for which $PKN_k(N, v)$, $2 < k < n$, does not coincide with either $PN(N, v)$ or $PK(N, v)$.

5.2 The between prekernel-prenucleolus solutions

This section is devoted to the construction of a collection of solutions on the class $\mathcal{G}$ of all TU games. For any TU game (N, v), a solution in this collection contains the prenucleolus in its solution set of payoff vectors. On the other hand, the solution set of payoff vectors is always contained in the prekernel.

In Chapter 2 the prenucleolus has been defined as the efficient payoff vector that solves the optimization problem of the lexicographical maximization of the vector of excesses. In contrast, the prekernel has been defined without optimization properties. Nevertheless, also the prekernel has an optimization property that can be used to define this solution in a similar way as the prenucleolus. For $S \subset N$, a vector $y_S \in \mathbb{R}^{|S|}$ with components y_j, $j \in S$, is called a *transfer* for S if

$$\sum_{j \in S} y_j = 0.$$

For a vector $x \in \mathbb{R}^n$ and a transfer y_S, we denote by $x||y_S$ the vector with components

$$(x||y_S)_j = \begin{cases} x_j + y_j & \text{if } j \in S, \\ x_j & \text{otherwise,} \end{cases}$$

i.e., the vector $x||y_S$ is the vector x with the components $j \in S$ replaced by $x_j + y_j$. In case $S = \{i, j\}$ is a two player set, for ease of notation we denote a transfer y_S be y_{ij} and $x||y_S$ by $x||y_{ij}$. The next proposition characterizes the prekernel as the set of solutions of a lexicographical optimization problem. Recall from Chapter 2 that for a payoff vector $x \in \mathbb{R}^n$ and a game (N, v), $E(x)$ denotes the $(2^n - 2)$-dimensional vector of the excesses of the coalitions S, $S \neq N, \emptyset$, in non-increasing order.

Proposition 5.2.1 *For a game $(N, v) \in \mathcal{G}$, a payoff vector $x \in X(N, v)$ belongs to the prekernel $PK(N, v)$ if and only if*

$$E(x) \preceq_L E(x||y_{ij}) \text{ for every pair } i, j \in N \text{ and every transfer } y_{ij}. \tag{5.2.1}$$

Proof. The "if" part. Suppose that a vector $x \in X(N, v)$ satisfies (5.2.1), but does not belong to the prekernel. Hence, there are players i, j such that $s_{ij}(x) \neq s_{ji}(x)$. Suppose $s_{ij}(x) > s_{ji}(x)$, thus

$$\max_{\{S|S \ni i, S \not\ni j\}} (v(S) - x(S)) > \max_{\{S|S \ni j, S \not\ni i\}} (v(S) - x(S)).$$

Then for any positive ε and transfer y_{ij} with $y_i = \varepsilon$, it follows that

$$e(x||y_{ij}, S) \begin{cases} < e(x, S), & \text{if } i \in S, j \notin S, \\ > e(x, S), & \text{if } j \in S, i \notin S, \\ = e(x, S), & \text{otherwise.} \end{cases} \tag{5.2.2}$$

Therefore, for sufficiently small ε,

$$E(x||y_{ij}) \preceq_L E(x),$$

which contradicts (5.2.1).

The "only if" part. Consider payoff vector $x \in PK(N, v)$. We show that for every two players $i, j \in N$ and every transfer y_{ij} it holds that

$$E(x) \preceq_L E(x||y_{ij}). \tag{5.2.3}$$

Without loss of generality we can assume that $y_i > 0$. Consider the first non-equal component of vectors $E(x)$ and $E(x||y_{ij})$. Notice that for $S \subset N$ such that either $i, j \in S$ or $i, j \notin S$ the excess does not change: $e(x, S) = e(x||y_{ij}, S)$. If $i \in S$ and $j \notin S$ then $e(x, S) > e(x||y_{ij}, S)$ and if $i \notin S$ and $j \in S$ then $e(x, S) < e(x||y_{ij}, S)$. Since $x \in PK(N, v)$, the maximal excess for coalitions with player i and without player j coincides with the maximal excess for coalitions with j and without i. So, the first non-equal component of the vectors $E(x)$ and $E(x||y_{ij})$ corresponds to a coalition with j and without i. It follows that $E(x) \preceq_L E(x||y_{ij})$. □

Proposition 5.2.1 shows that the prekernel, analogous to the prenucleolus, consists of vectors which lexicographically minimizes the vector of excesses. However, the domains of the two optimizations are different. For the prenucleolus the domain is the set of all efficient payoff vectors, whereas a vector $x \in X(N, v)$ belongs to the prekernel, if its excess vector is lexicographically minimal on the set of payoff vectors differing from x only in two components. So, to check whether or not an efficient payoff vector belongs to the perkernel, it is sufficient to check whether its vector of excesses is lexicographically minimal on the domain set

$$D(x) = \{z \in X(N, v) | z = x||y_{ij} \text{ for some } i, j \in N \text{ and transfer } y_{ij}\}.$$

Clearly, $D(x) \subset X(N, v)$.

The characterization of Proposition 5.2.1 can be generalized to obtain a set of payoff vectors such that every x in this set is lexicographically minimal on the domain of all payoff vectors that differ from x in at most a given number of components. This leads to a class of new solutions, namely a solution for each number of different components. For number of different components equal to k, $k \geq 2$, we denote the corresponding solution on the class $\mathcal{G}$ of all games by PKN_k.

Definition 5.2.2 For $k \geq 2$, the solution PKN_k on $\mathcal{G}$ is given by

$$PKN_k(N, v) = \begin{cases} PK(N, v) \text{ if } k = 2, \\ PN(N, v), \text{ if } k \geq n, \end{cases}$$

and for $2 < k < n$, $PKN_k(N, v)$ is defined by

$$x \in PKN_k(N, v) \iff E(x) \preceq_L E(x||y_S)$$

for any transfer $y_S, |S| = k$.

From Definition 5.2.2 it follows immediately that for each game $(N, v) \in \mathcal{G}$

$$PKN_{k_2}(N, v) \subseteq PKN_{k_1}(N, v) \text{ if } k_1 < k_2.$$

We denote the class of all solutions PKN_k, $k \geq 2$, by $\mathcal{PKN}$. Other properties and the characterizations of the solutions in this class are discussed in the following sections.

5.3 Characterization of the class $\mathcal{PKN}$ by balancedness

First, note that all solutions PKN_k, $k \geq 2$ satisfy covariance and symmetry. Covariance of the solutions follows from Definition 5.2.2, because the solutions are defined by lexicographical minimization between excess vectors. Since the excess vectors are homogeneously transformed under strategical transformations, the lexicographical relation is covariant with respect to strategic transformation. Also symmetry of follows from the definition. In fact Definition 5.2.2 shows that for every game $(N, v) \in \mathcal{G}$ and every $k \geq 2$ it is true that $PKN_k(N, v) \subset PK(N, v)$ and thus the symmetry of PKN_k follows from the symmetry of the prekernel.

Recall that $\mathcal{G}_N \subset \mathcal{G}$ is the class of games on player set N, and let

$$\mathcal{G}_n = \bigcup_{\{N||N| \leq n\}} \mathcal{G}^N$$

be the class of all games with at most n players. Then, for every $k \geq 2$, by Definition 5.2.2 the solution PKN_k is single-valued on the class $\mathcal{G}_k$ of games with at most k players. The solution PKN_k can be characterized by balancedness, similar to Kohlberg's characterization of the prenucleolus. To do so, first the notion of k-balancedness is defined.

Definition 5.3.1 For $k \geq 2$, a collection $\mathcal{S}$ of coalitions $S \subset N$ is *k-balanced* if for every coalition $K \subset N$ with $|K| = k$ the collection

$$\mathcal{S}_K = \{S' \subset K \mid S' \neq \emptyset, S' = S \cap K,\ S \in \mathcal{S}\}$$

is balanced on K.

It is easy to see that k-balancedness implies l-balancedness for $l < k$.

Theorem 5.3.2 *For $(N, v) \in \mathcal{G}$, a vector $x \in X(N, v)$ belongs to $PKN_k(N, v)$ if and only if for every real number α the collection of coalitions*

$$\mathcal{B}(\alpha, x) = \{\emptyset \neq S \subset N \mid e(S, x) \geq \alpha\}$$

is either k-balanced, or empty.

Proof. The "only if" part. Let $x \in PKN_k(N, v)$, $K \subset N$, be an arbitrary subset of N such that $|K| = k$. Then for any transfer y_K,

$$E(x) \preceq_L E(x||y_K). \tag{5.3.4}$$

Since $\sum_{k \in K} y_k = 0$, for every coalition $S \subset N$ it holds that

$$e(x, S) > e(x||y_K, S) \implies y(S \cap K) > x(S \cap K).$$

Therefore, relation (5.3.4) implies that for an arbitrary α the system of inequalities

$$x(S \cap K) \geq x||y_K(S \cap K), \text{ for all } S \in \mathcal{B}(\alpha, x), \tag{5.3.5}$$

does not have a solution if $x_K \neq x||y_K$. Since $x(K) = x||y_K(K)$ the unsolvedness of the system of inequalities (5.3.5) is equivalent to either k-balancedness of the collection of coalitions $\mathcal{B}(\alpha, x)$ or its emptiness.

The "if" part. For a vector $x \in X(N, v)$, let the collection $\mathcal{B}(\alpha, x)$ be either k-balanced, or empty for every real number α. Let α be a number such that $\mathcal{B}(\alpha, x) \neq \emptyset$. Then for every coalition K with $|K| = k$ and transfer y_K such that $x||y_K \neq x_K$, it is impossible that $x||y_K(S \cap K) \geq x(S \cap K)$ for all $S \in \mathcal{B}(\alpha, x)$. So, there exists a coalition $T_{K,\alpha} \in \mathcal{B}(\alpha, x)$ such that $x||y_K(T_{K,\alpha} \cap K) < x(T_{K,\alpha} \cap K)$.

Take $\alpha_1 = \max_{S \subset N} (v(S) - x(S))$. Then, either $x(S) = x||y_K(S)$ for all $S \in \mathcal{B}(\alpha_1, x)$, or there is a coalition $T_{K,\alpha_1} \in \mathcal{B}(\alpha_1, x)$ such that $x||y_K(T_{K,\alpha_1}) < x(T_{K,\alpha_1})$. In the latter case $E(x) \preceq_L E(x||y_K)$. In case $x(S) = x||y_K(S)$ for every $S \in \mathcal{B}(\alpha_1, x)$, then define α_2 as the second highest excess

$$\alpha_2 = \max_{\{S | S \notin \mathcal{B}(\alpha_1, x)\}} (v(S) - x(S)).$$

Again either $x(S) = x||y_K(S)$ for all $S \in \mathcal{B}(\alpha_2, x)$, or there is a coalition $T_{K,\alpha_2} \in \mathcal{B}(\alpha_2, x)$ such that $x||y_K(T_{K,\alpha_2}) < x(T_{K,\alpha_2})$, and thus $E(x) \preceq_L E(x||y_K)$. Continuing this procedure we obtain either $y_K \equiv x_K$, or $E(x) \preceq_L E(x||y_K)$. □

5.4 Consistency properties

In this section it is shown that PKN_k is consistent on the class $\mathcal{G}$ of all games for every $k \geq 2$. Whether or not PKN_k is also converse consistent or satisfies the reconfirmation property depends on the number of players in a game. In particular, by definition of PKN_k these properties are true on the class of all games with at most k players. In order to give analogous properties for these solutions, we define 'k-modifications' of the last two consistency axioms in a similar way as k-balancedness was defined as a modification of balancedness.

Definition 5.4.1 A solution F on the class $\mathcal{G}$ of all games satisfies *k-converse consistency*, if $x \in X(N,v)$ and $x_K \in F(K, v_K^x)$ for every $K \subset N$ with $|K| = k$, imply that $x \in F(N,v)$.

It is clear that converse consistency is equivalent to 2-converse consistency, and that k-converse consistency implies l-converse consistency for every $l > k$. On the other hand, this property has no implications on the subclass $\mathcal{G}_k$ of games with at most k players. So, trivially the k-converse consistency property is satisfied on $\mathcal{G}_k$.

Definition 5.4.2 A solution F on the class $\mathcal{G}$ of all players satisfies the *k-reconfirmation property*, if $x \in F(N,v)$ and $y_S \in F(S, v_S^x)$ for $S \subset N$ with $|S| \leq k$, imply that $(x_{N\setminus S}, y_S) \in F(N,v)$.

Evidently, the l-reconfirmation property implies the k-reconfirmation property for $k < l$. Consistency of the solution PKN_k follows easily from Theorem 5.3.2.

Proposition 5.4.3 *For every $k \geq 2$, the solution PKN_k is consistent on the class $\mathcal{G}$ of all games.*

Proof. Let $(N,v) \in \mathcal{G}$ and $x \in PKN_k(N,v)$. If $k \geq n = |N|$, then $x = PN(N,v)$, and consistency of PKN_k follows from that of the prenucleolus.

Next, consider the case $k < n$. By Theorem 5.3.2 the collection $\mathcal{B}(\alpha, x)$ is k-balanced or empty for every α. Therefore, the collection $\mathcal{B}^T(\alpha, x)$ defined by

$$\mathcal{B}^T(\alpha, x) = \{S \subset T \mid S \neq \emptyset, S = S' \cap T,\ S' \in \mathcal{B}(\alpha, x)\}$$

is also k-balanced for every $T \subset N$.

For the Davis-Maschler reduced game (T, v_T^x) on player set $T \subset N$, let $\mathcal{B}_{T,x}(\alpha, x)$ be the collection of proper subsets of T with excess at x in the reduced game v_T^x at least equal to α. Then

$$\mathcal{B}_{T,x}(\alpha, x) = \mathcal{B}^T(\alpha, x).$$

This equality between the two collections and the "if" part of the proof of Theorem 5.3.2 gives the relation $x_T \in PKN_k(T, v_T^x)$. □

For $k > 2$, the solutions PKN_k do not satisfy the converse consistency property. However, they satisfy the l-converse consistency for $l \geq k$.

Proposition 5.4.4 *For every $k \geq 2$, the solution PKN_k is l-converse consistent on the class $\mathcal{G}$ of all games for $l \geq k$.*

Proof. By Definition 5.4.1 it is sufficient to prove this property for games with number of players $n > k$. For $(N, v) \in \mathcal{G}$ with $n = |N| > k$, let $x \in X(N, v)$ be an efficient payoff vector such that $x_K = PKN_k(K, v_K^x) = PN(K, v_K^x)$ for every coalition $K \subset N$ with $|K| = k$. Then,

$$E^K(x_K) \preceq_L E^K(y_K) \tag{5.4.6}$$

for any payoff vector $y_K \in X(K, v_K^x)$, where $E^K(x)$ is the ordered vector of excesses at x of the nonempty, proper subsets of K with respect to the Davis-Maschler reduced game (K, v_K^x). From (5.4.6) it follows that for every K with $|K| = k$ it is true that

$$E(x) \preceq_L E(x||y_K),$$

where $E(x)$ is the vector of excesses at x with respect to (N, v). From this it follows that $x \in PKN_k(N, v)$. □

Proposition 5.4.5 *For every $k \geq 2$, the solution PKN_k satisfies the k-reconfirmation property on the class $\mathcal{G}$ of all games.*

Proof. For $(N, v) \in \mathcal{G}$ with $|N| \leq k$, PKN_k is single-valued by Definition 5.2.2, and by Proposition 5.2.1 it is consistent. Therefore the solution PKN_k satisfies the k-reconfirmation property for games with $|N| \leq k$. Consider now a game (N, v) with $|N| > k$ and let $x \in PKN_k(N, v)$. Consider the Davis-Maschler reduced game (S, v_S^x) of (N, v) on a player set S with $|S| \leq k$. Then by consistency of PKN_k it is true that $x_S \in PKN_k(S, v_S^x)$. However, the solution $PKN_k(S, v_S^x)$ is single-valued and thus $x_S = PKN_k(S, v_S^x)$. From this the k-reconfirmation property of PKN_k follows. □

Note that the properties in the Propositions 5.4.3-5.4.5 are indeed defined on the class $\mathcal{G}$ of all games and for any $k \geq 2$.

5.5 Example

For two- and three-person games the prekernel coincides with the prenucleolus. Therefore, the solutions PKN_k also coincide with this solution for all $k \geq 2$.

For four-person games the solution $PKN_4(N, v) = PN(N, v)$, and $PKN_3(N, v)$ is the set of all payoff vectors $x \in X(N, v)$ such that for every three person coalition $T \subset N$, $x_T \in PN(T, v_T^x)$ and thus $PKN_3(N, v) = PN(N, v)$. So, $PKN_k(N, v) = PK(N, v)$ for $k = 2$ and $PKN_k(N, v) = PN(N, v)$ for $k \geq 3$.

It is an open question to find the minimum number, say n^*, for which there exists a game (N, v) with n^* players and having the property that for some $k < n$ the solution

$PKN_k(N,v)$ contains at least two vectors (and thus is not equal to $PN(N,v)$) and is also a proper subset of $PK(N,v)$. In this section we construct an 11-person game (N,v) such that for some k the solution $PKN_k(N,v)$ differs from both the prekernel and the prenucleolus. This shows that the minimum number n^* is at most equal to 11.

We consider two games, $\Gamma_1 = (N_1, v_1)$ and $\Gamma_2 = (N_2, v_2)$, with $|N_1| = 5$ and $|N_2| = 6$. To define their characteristic functions v_1 and v_2, denote $N_1 = \{1,2,3,4,5\}$ and $N_2 = \{1',2',3',4',5',6'\}$. Further, denote $S_1 = \{1,2,3\} \subset N_1$, $S_2 = \{4,5\} \subset N_1$ and $S_3 = \{1',2',3',4'\} \subset N_2$, $S_4 = \{5',6'\} \subset N_2$. We now take the characteristic functions given by

$$
\begin{array}{ll}
v_1(N_1) = v_2(N_2) = 6, & \\
v_1(\{i,j,k\}) = 3, & \text{if } i,j \in S_1,\ k \in S_2,\\
v_2(\{i,j,k\}) = 3, & \text{if } i,j \in S_3,\ k \in S_4,\\
v_1(S) = v_2(T) = 0 & \text{for any other } S \subset N_1,\ T \subset N_2.
\end{array}
$$

Both games Γ_1 and Γ_2 are a modification of games known from Davis and Maschler (1965). In game Γ_1 the players 1, 2 and 3 are symmetric and the players 4 and 5 are symmetric, in game Γ_2 the players $1'$, $2'$, $3'$ and $4'$ are symmetric and the players $5'$ and $6'$ are symmetric. From this observation it easily follows that the prekernels of these games are given by

$$
PK(N_1,v_1) = \{x \in X(N_1,v_1) | x = (t,t,t,3-\frac{3t}{2},3-\frac{3t}{2}),\ 0 \le t \le \frac{3}{2}\} \tag{5.5.7}
$$

and

$$
PK(N_2,v_2) = \{y \in X(N_2,v_2) | y = (\tau,\tau,\tau,\tau,3-2\tau,3-2\tau),\ 0 \le \tau \le \frac{3}{2}\}. \tag{5.5.8}
$$

In the sequel, for $x \in X(N_j, v_j)$ and $\alpha \in \mathbb{R}$, let $\mathcal{B}^j(\alpha, x)$ denote the collection of nonempty, proper subsets S of N_j with excess $v_j(S) - x(S)$ at least equal to α, $j = 1,2$. Further, for $x \in X(N,v)$, let $\mathcal{S}_1(N,v,x)$ be the collection of nonempty, proper subsets S of N with the highest excess $v(S) - x(S)$ at x. For $x \in PK(N_1,v_1)$, it holds that

$$
\mathcal{S}_1(N_1,v_1,x) =
$$

$$
\begin{cases}
\{\{i,j,k\} \subset N_1 | i,j \in S_1,\ k \in S_2\} & \text{if } x_1 < \frac{3}{2},\\
\{\{i,j,k\} \subset N_1 | i,j \in S_1,\ k \in S_2\} \cup \{\{k\} \mid k \in S_2\} & \text{if } x_1 = \frac{3}{2}.
\end{cases} \tag{5.5.9}
$$

The payoff vector $x = (\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{4},\frac{3}{4})$ corresponding with $t = \frac{3}{2}$ is the unique payoff vector in the prenucleolus and thus the collection $\mathcal{S}_1(N_1,v_1,x)$ is balanced for $t = \frac{3}{2}$. For every value $t < \frac{3}{2}$ the collection of coalitions $\mathcal{S}_1(N_1,v_1,x)$ given in (5.5.9) is not balanced and also not 4-balanced. So, $x = (\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{4},\frac{3}{4})$ is also the unique element in $PKN_4(N_1,v_1)$. However, for every $x \in PK(N_1,v_1)$ the collection $\mathcal{S}_1(N_1,v_1,x)$ is 3-balanced and moreover, it is straightforward to verify that for every α and for every $x \in PK(N_1,v_1)$ the collection $\mathcal{B}^1(\alpha,x)$ is 3-balanced. Hence, by Theorem 5.3.2, every $x \in PK(N_1,v_1)$ also belongs to $PKN_3(N_1,v_1)$. So, $PKN_k(N_1,v_1) = PK(N_1,v_1)$ for $k \le 3$ and $PKN_k(N_1,v_1) = PN(N_1,v_1)$ for $k \ge 4$.

We now consider the second game. First, for every $y \in PK(N_2, v_2)$ the collection $\mathcal{S}_1(N_2, v_2, y)$ of the coalitions with the highest excess is given by

$$\mathcal{S}_1(N_2, v_2, y) = \{\{i, j, k\} \in N_2 | i, j \in S_3,\ k \in S_4\}$$

and this collection is balanced. Next, for $x \in X(N, v)$, let $\mathcal{S}_2(N, v, y)$ denote the collection of coalitions with the second highest excess at x. So, for (N_2, v_2) and $y \in X(N_2, v_2)$,

$$\mathcal{S}_2(N_2, v_2, y) = \arg \max_{S \notin \mathcal{S}_1(N_2, v_2, y)} v_2(S) - y(S).$$

For $y \in PK(N_2, v_2)$ we have that

$$\mathcal{S}_2(N_2, v_2, y) = \begin{cases} \{\{i\} \mid i \in S_3\} & \text{if } y_1 \in [0, 1), \\ \{\{k\} \mid k \in S_4\} & \text{if } y_1 \in (1, \frac{3}{2}), \\ \{\{i\} \mid i \in N_2\} & \text{if } y_1 = 1. \end{cases} \tag{5.5.10}$$

It follows that the collection $\mathcal{S}_1(N_2, v_2, y) \cup \mathcal{S}_2(N_2, v_2, y)$ of the coalitions with the highest and second highest excess is balanced if and only if $y_1 = 1$, so only for the unique prenucleolus payoff vector $y = (1, 1, 1, 1, 1, 1)$ corresponding to $\tau = 1$. For every other $\tau \neq 1$ the corresponding collection $\mathcal{S}_1(N_2, v_2, y) \cup \mathcal{S}_2(N_2, v_2, y)$ is k-balanced for $k \leq 5$ (but not balanced). Hence, by Theorem 5.3.2, every $y \in PK(N_2, v_2)$ also belongs to $PKN_5(N_2, v_2)$. So, $PKN_k(N_2, v_2) = PK(N_2, v_2)$ for $k \leq 5$ and $PKN_k(N_2, v_2) = PN(N_2, v_2)$ for $k \geq 6$.

We now construct an eleven person game as the composite game of the two games $\Gamma_1 = (N_1, v_1)$ and $\Gamma_2 = (N_2, v_2)$. In general, for two games (N_1, v_1) and (N_2, v_2) with $N_1 \cap N_2 = \emptyset$, we say that the game $\Gamma = (N, v)$ is the *composite* game of Γ_1 and Γ_2 if $N = N_1 \cup N_2$ and

$$v(Q) = v_1(Q \cap N_1) + v_2(Q \cap N_2), \quad Q \subseteq N.$$

Then we have the following lemma.

Lemma 5.5.1 *Let (N, v) be the composite game of the two games (N_1, v_1) and (N_2, v_2). Then for every $x \in PK(N, v)$ and every number α the collections of coalitions*

$$\{S \subset N_i \mid e(S, v) \geq \alpha\}, \quad i = 1, 2,$$

are either empty, or two-balanced.

Proof. Let $x \in PK(N, v)$ and let $S = S_1 \cup S_2$ be an arbitrary subset of N with $S_1 \subset N_1$ and $S_2 \subset N_2$. Then $e(S, x) = e(S_1, x) + e(S_2, x)$ and thus $e(S_1, x) \geq \alpha$ implies that $e(S, x) \geq \alpha + e(S_2, x)$. Let $i \in S_1$ and $j \in N_1 \setminus S_1$. Since

$$e(S, x) \leq s_{ij}(x) = \max_{\{Q | i \in Q,\ j \notin Q\}} e(Q, x)$$

and $x \in PK(N,v)$, there exists a coalition $T \subset N$ with $j \in T$ and $i \notin T$ such that

$$e(T,x) \geq e(S,x) \geq \alpha + e(S_2,x).$$

Let $T = T_1 \cup T_2,\ T_i \subseteq N_i,\ i = 1,2$. Then

$$e(T,x) = e(T_1,x) + e(T_2,x) \geq \alpha + e(S_2,x). \tag{5.5.11}$$

Note that inequality (5.5.11) holds for every coalition $S_2 \subset N_2$ and thus for $S_2 = T_2$ as well. So, inequality (5.5.11) implies $e(T_1,x) \geq \alpha$.

So we have proved that if there is a coalition $S_1 \subset N_1$ such that $i \in S_1, j \notin S_1$ with $e(S_1,x) \geq \alpha$ then there is another coalition $T_1 \subset N_1$ such that $i \notin T_1, j \in T_1$ with $e(T_1,x) \geq \alpha$. Because this holds for every pair of players $i,j \in N_1$, it follows that the collection $\{S \subset N_1 \,|\, e(S,x) \geq \alpha\}$ is either empty, or 2-balanced. Analogously we can show the same for the collection $\{S \subset N_2 \,|\, e(S,x) \geq \alpha\}$. □

We now consider the 11 player composite game (N,v) of the games (N_1,v_1) and (N_2,v_2) considered above and show that

$$x(N_1) = x(N_2) = 6$$

for every $x \in PK(N,v)$. Assume the contrary. Then for some payoff vector $x \in PK(N,v)$ it holds that $x(N_1) \neq 6$.

First, consider the case $x(N_1) > 6$ and thus $x(N_2) < 6$. Take coalitions $T \in \arg\max_{S \subseteq N_1} e(S,x)$ and $U \in \arg\max_{S \subseteq N_2} e(S,x)$. We show that $e(T,x) \geq 0$. Suppose the contrary that $e(T,x) < 0$. Since $v_2(N_2) = 6$, the inequality $x(N_1) > 6$ implies that $e(N_2,x) = v(N_2) - x(N_2) > 0$ and thus $e(U,x) > 0$. For arbitrary players $i \in T,\ j \in U$ consider a coalition R with $i \in R,\ j \notin R$ and $e(R,x) = s_{ij}(x)$. Let $R = R_1 \cup R_2$ with $R_i \subseteq N_i,\ i = 1,2$. Then

$$s_{ij}(x) = e(R,x) = e(R_1,x) + e(R_2,x) \leq e(T,x) + e(U,x) < e(U,x) \leq s_{ji}(x).$$

This contradiction proves the inequality $e(T,x) \geq 0$.

Next, since $PK(N,v)$ is symmetric and for every j, the players in $S_j,\ j = 1,2,3,4$, are symmetric players, we have that any vector $x \in PK(N,v)$ satisfies that $x_i = x_k$ when i and k both belonging to a same subset S_j. Let $x_i = t$ be the payoff to the three players i in S_1. Then the two players in S_2 both have payoff $\frac{1}{2}(x(N_1) - 3t)$. We now consider three possibilities.

a. When $t > 0$, then $e(T,x) \geq 0$ for $T \subset N_1$ is only possible when $T = \{4,5\}$ or $T = \{i,4,5\}$ with $i \in S_1$. So, the players $4,5$ belong to every coalition $T \subset N_1$ with the highest excess. However, by Lemma 5.5.1 there must exist a coalition $Q \subset N_1$ with $4 \notin Q$ or $5 \notin Q$, such that $e(Q,x) = e(T,x)$. This excludes $t > 0$.

b. If $t = 0$ then $\max_{S \subset N_1}\ e(S,x) = 0$ and every coalition $T \subset N_1$ with $e(T,x) = 0$ is a subset of S_1. Again by Lemma 5.5.1 this is impossible.

c. Finally, when $t < 0$, then similar to the arguments given above we obtain that the coalition $T \subset N_1$ with maximal excess can be only of the type $T = \{1, 2, 3\}$ or $T = \{1, 2, 3, k\}$ with $k \in S_2$. Again, since the players $1, 2, 3$ belong to all such coalitions, this case is also impossible. This contradicts that $x(N_1) > 6$.

Second we consider the case that $x(N_1) < 6$. Then $x(N_2) > 6$, and $e(N_2, x) < 0$. Similar to the case $x(N_1) > 6$ we now obtain that $e(U, x) \geq 0$ for $U \in \arg\max_{S \subset N_2} e(S, x)$. Now, let $x_i = \tau$ be the payoff to each of the four symmetric players $i \in S_3$. Then, similar as above we obtain that $U = S_4$ if $\tau \geq 0$ and $U = S_3$ if $\tau < 0$. By Lemma 5.5.1 both cases are impossible. This proves that $x(N_1) = x(N_2) = 6$ when $x \in PK(N, v)$.

Because $x(N_1) = 6 = v_1(N_1)$ and $x(N_2) = 6 = v_2(N_2)$, for every coalition $S = S_1 \cup S_2 \subset N$ with $S_i \subseteq N_i$, $i = 1, 2$, the following equality holds

$$e(S, x) = e_1(S_1, x) + e_2(S_2, x),$$

where $e_i(\cdot, \cdot)$, $i = 1, 2$, denotes the excess in (N_i, v_i). This implies that

$$PK(N, v) = PK(N_1, v_1) \times PK(N_2, v_2).$$

With the representations (5.5.7) and (5.5.8 it follows that every $x \in PK(N, v)$ is of the form

$$(t, t, t, 3 - \frac{3t}{2}, 3 - \frac{3t}{2}, \tau, \tau, \tau, \tau, 3 - 2\tau, 3 - 2\tau)$$

for some $t \in [0, \frac{3}{2}]$ and some $\tau \in [0, \frac{3}{2}]$. From this it is straightforward to verify that $PK(N, v) \subset C(N, v)$ and thus $e(S, x) \leq 0$ for every $S \subset N$ and $x \in PK(N, v)$. Therefore, when $e(S, x) \geq \alpha$ for some coalition $S \subset N$, then $\alpha \leq 0$ and also

$$e(S \cap N_1, x) \geq \alpha \text{ and } e(S \cap N_2, x) \geq \alpha.$$

From this it follows that for every $\alpha \leq 0$ the k-balancedness of $\mathcal{B}(\alpha, x)$ is equivalent to k-balancedness of the collections $\{S \cap N_i \mid S \in \mathcal{B}(\alpha, x)\}$, $i = 1, 2$. Now with Theorem 5.3.2 we obtain that

$$PKN_k(N, v) = PKN_k(N_1, v_1) \times PKN_k(N_2, v_2)$$

for all $k = 2, 3, \ldots$. Thus for the composite eleven person game (N, v) we obtain

$$PKN_2(N, v) = PKN_3(N, v) = PK(N, v),$$

$$PKN_4(N, v) = PKN_5(N, v) =$$
$$\{x \in X(N, v) | x_i = \frac{3}{2},\ i \in S_1;\ x_i = \frac{3}{4},\ i \in S_2;$$
$$\text{and for some } \tau \in [0, \frac{3}{2}],\ x_i = \tau,\ i \in S_3,\ x_i = 3 - 2\tau,\ i \in S_4\},$$

and finally

$$PKN_k(N,v) = PN(N,v), \text{ for all } k \geq 6,$$

with $x = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1, 1, 1, 1)$ the unique element in $PN(N,v)$. From this it follows that $PK(N,v) \subset PKN_4(N,v) = PKN_5(N,v) \subset PN(N,v)$, showing that the minimum number of players that is needed to construct a game (N,v) such that there is some k for which $PKN_k(N,v)$ differs from both $PK(N,v)$ and $PN(N,v)$ is at most eleven.

Chapter 6

Peer-group games

6.1 Introduction

In this chapter we consider the class of peer-group games as introduced in Brânzei, Fragnelli and Tijs (2002). Although the class of peer group games is rather specific, it contains many interesting applications (see Brânzei *et al.* (2002)). A peer group situation is a triple (N, a, D) with N a set of $n = |N|$ players, $a \in \mathbb{R}^n_+$ a nonnegative vector of real numbers and D a rooted directed tree on N. The directed graph represents a hierarchical structure, yielding a collection Ω of permission sets as the collection of feasible sets. Because D is a tree, the disjunctive and conjunctive feasible sets coincide, so that coalition $S \in \Omega$ if and only if for every $i \in S$ also the predecessor is contained in S. As before, for $S \subseteq N$, let $\sigma(S)$ be the maximal feasible subset. Notice that every non-empty feasible set S contains the top player (the root of D) and that every player from the top to a player $i \in S$ is also contained in S.

For peer group situation (N, a, D), the associated peer-group game is the game (N, r) with characteristic function r defined by

$$r(S) = \sum_{i \in \sigma(S)} a_i, \quad S \subseteq N, \tag{6.1.1}$$

i.e., the worth of coalition S is the sum of the values a_i of the players i in the maximal feasible subset of S. Notice that $r(S) = 0$ if S does not contain the top player and thus the top player is a veto player in (N, r). Because of this and monotonicity of game (N, r), this game has a non-empty core.

The class of peer-group games on player set N will be denoted by $\mathcal{G}^N_p$, i.e., $(N, r) \in \mathcal{G}^N_p$ if and only if there exists a peer group situation (N, a, D) such that $r(S) = \sum_{i \in \sigma(S)} a_i$ for every $S \subseteq N$. The class of peer-group games over all collections of players is denoted by $\mathcal{G}_p$. For a (cooperative game) solution F on $\mathcal{G}_p$ we denote (similar as before)

$$F(N, a, D) = F(N, r),$$

i.e., the set of payoff vectors assigned by solution F to the peer-group situation (N, a, D) is the set of payoff vectors assigned by F to the restricted game (N, r). In the sequel we assume without loss of generality that the root of D is numbered player 1.

Our standard interest is to find a fair distribution of the total worth $v(N) = \sum_{i \in N} a_i$ amongst the players in N. Because a player can not generate any worth without its superiors, it is reasonable that a part of the contribution a_i of player i will be assigned to each of its superiors. We say that any distribution with this property is a fair distribution. The Shapley value of (N, a, D) is fair and has the property that the contribution a_i of every player i is divided equally between player i and its superiors. This equal division looks reasonable because every superior of i has to agree in order to realize its player's ith contribution a_i. It makes therefore the Shapley value a very natural candidate solution for this class of games. In this chapter we give an axiomatization of the Shapley value on the class of peer-group games. We further consider the nucleolus. It is interesting to notice that on the class of peer-group games the Shapley value and the nucleolus have similar properties.

This chapter is organized as follows. In Section 6.2 it is shown that for certain sets of players the Davis-Maschler reduced game of a peer-group game is also a peer-group game itself. Further some weak type of Davis-Maschler consistency on the class of peer-group games is introduced. In Section 6.3 it is shown that the Shapley value satisfies this weak type of Davis-Maschler consistency. Using this consistency property an axiomatization of the Shapley value is constructed. Section 6.4 shows that the nucleolus satisfies a monotonicity property on the class of peer-group games.

6.2 PGG-Davis-Maschler consistency

A well-known difference between the Shapley value and prenucleolus is that the latter value satisfies the Davis-Maschler consistency property and that the Shapley value does not. Notice however that the Shapley value satisfies other consistency properties, such as the Hart-Mas-Colell consistency property (see Hart and Mas-Colell (1989)) and the Sobolev consistency (see Sobolev (1973)). In the next section we show that on the class of peer-group games the Shapley value also satisfies some weak type of Davis-Maschler consistency, called *PGG-Davis-Maschler consistency*, where PGG refers to the class of peer-group games. Before we introduce this consistency property, we first introduce some notation and state a proposition.

In the following, for tree (N, D), let L_D be the set of leaves of the tree, i.e., $L_D \subset N$ is the set of players without successors in (N, D). For $i \in N$, let N_{-i} denote the set $N \setminus \{i\}$ and let S_i denote the set containing i itself and all subordinates of i in D, i.e., $j \in S_i$ if $j = i$ or i is on the path from the top player 1 to j. Further, let l_i denote the length of the path from top player 1 to i, i.e., l_i is the number of superiors of i (number of players $k \neq i$ on the path from 1 to i). A set $S \subset N$ is a branch of (N, D) if $S = S_i$ for some $i \neq 1$. Finally, recall that for game (N, v) and $S \subset N$, the game $(N \setminus S, v^x_{N \setminus S})$ denotes the Davis-Maschler reduced game on the set $N \setminus S$ with respect to payoff vector $x \in \mathbb{R}^{|N|}$. Also recall that for graph (N, D) and $T \subset N$, $(T, D(T))$ denotes the subgraph of (N, D) on T.

Proposition 6.2.1 *Let (N, a, D) be a peer-group situation, (N, r) the corresponding peer group game and $x \in Core(N, r)$ a payoff vector. Let $S \subset N$ be a branch and k the unique predecessor of S in (N, D). Then the Davis-Maschler reduced game $(N \setminus S, r^x_{N\setminus S})$ is the peer-group game corresponding to $(N \setminus S, a', D(N \setminus S))$ with $a'_k = a_k + \sum_{i\in S} a_i - x(S)$ and $a'_j = a_j$ for every $j \in N \setminus S$, $j \neq k$.*

Proof. We will show that the weight of each player except k remains without changes. After it the new weight of player k can be found from the efficiency condition as $a'_k = a_k + r^x_{N\setminus S}(N \setminus S) - r(N \setminus S)$. Let us note that $x \in Core(N, r)$ and the set $N \setminus S$ is feasible. So if $a'_j = a_j$ for every $j \in N \setminus S$, $j \neq k$, then

$$x(N \setminus S) \geq r(N \setminus S) \Leftrightarrow r^x_{N\setminus S}(N \setminus S) \geq r(N \setminus S) \Rightarrow a'_k \geq a_k.$$

Let us show that $a'_j = a_j$ for every $j \in N \setminus S$, $j \neq k$. By definition of the Davis-Mashler reduced game it is sufficient to consider the case that $S = \{i\}$ for some $i \in L_D$. So, let i be a leave and k the unique predecessor of i in (N, D). By definition of the Davis-Maschler reduced game we have for $U \subset N_i$ that

$$v^x_{N\setminus S}(U) = \max\{v(U), v(U \cup \{i\}) - x_i\}.$$

If $k \in U$ and all superiors of k belong to U then $v(U \cup \{i\}) = v(U) + a_i$ and

$$v^x_{N\setminus S}(U) = \max\{v(U), v(U) + a_i - x_i\} = v(U) + \max\{0, a_i - x_i\}.$$

If (i) does not hold, then $v(U \cup \{i\}) = v(U)$ and $v^x_{N\setminus S}(U) = v(U)$. From this it follows that the reduced game $(N_{-i}, r^x_{N_{-i}})$ is the peer-group game associated to the peer-group situation $(N_{-i}, a', D(N_{-i}))$ with $a'_k = a_k + \max\{0, a_i - x_i\}$ and $a'_j = a_j$ for each $j \in N_{-i}$, $j \neq k$. □

The proposition states that the reduced game of a peer-group game is again a peer-group game when a branch in the graph (N, D) is removed from the game and the graph. Moreover, it shows that the peer-group situation inducing the reduced game is given by the subgraph $(N \setminus S, D(N \setminus S))$ of (N, D) and that for each player $i \in N \setminus S$ the contribution in the new peer-group situation is equal to its contribution in the original peer-group situation, except for the unique predecessor of the branch in the tree (N, D). We now give the PGG-Davis-Maschler property for a solution F defined on the class of peer group games $\mathcal{G}_p$.

Axiom 6.2.2 (PGG-Davis-Maschler consistency) *For every peer-group game $(N, r) \in \mathcal{G}_p$ and for every $i \in L_D$, it holds that*

1. $(N_{-i}, r^{F(N,r)}_{N_{-i}}) \in \mathcal{G}_p$
2. $F_j(N_{-i}, r^{F(N,r)}_{N_{-i}}) = F_j(N, r)$ *for all $j \in N_{-i}$.*

Notice that the PGG-Davis-Maschler consistency only requires consistency when a single player from the set L_D of players without successors is removed, so it is a weak type of Davis-Maschler consistency on the special class of peer-group games.

6.3 The Shapley value on the class of peer-group games

From Gilles, Owen and van den Brink (1992) it follows that the Shapley value of a peer-group situation (N, a, D) is given by

$$Sh_i(N, a, D) = \sum_{j \in S_i} \frac{a_j}{l_j + 1}, \quad i \in N \tag{6.3.2}$$

so the contribution a_j of player j is divided equally amongst player j itself and all its superiors. The next theorem shows that the Shapley value is PGG-Davis-Maschler consistent on the class of peer-group games.

Theorem 6.3.1 *The Shapley value is PGG-Davis-Maschler consistent on the class $\mathcal{G}_p$ of peer-group games on all player sets.*

Proof. Let (N, a, D) be a peer-group situation and $i \in L_D$. So, player i has no successors and $S_i = \{i\}$. Since D is a tree, player i has a unique predecessor, we denote this player by k. Further, for ease of notation we denote $r' = r_{N_{-i}}^{Sh(N,r)}$, i.e., r' is the characteristic function of the Davis-Maschler reduced game when removing player i with respect to the Shapley value $Sh(N, a, D) = Sh(N, r)$, with r the characteristic function associated to (N, a, D) as defined in equation (6.1.1).

It is evident that for an arbitrary peer-group game the Shapley value belongs to the core. So from Proposition 6.2.1 we have that (N_{-i}, r') is the peer-group game corresponding to a peer-group situation $(N_{-i}, a', D(N_{-i})$ with $a'_j = a_j$ for all $j \in N_{-i} \setminus \{k\}$. Moreover, from the proposition we know that

$$a'_k = a_k + a_i - x_i.$$

With $Sh_i(N, a, D) = \frac{a_i}{l_i + 1}$ it follows that

$$a'_k = a_k + \frac{a_i l_i}{l_i + 1}.$$

Now, to prove the consistency property that the payoffs of the players in N_{-i} do not change when $i \in L_D$ is removed, consider some player $t \in N_{-i}$. According to (6.3.2) we have

$$Sh_t(N, r) = \sum_{j \in S_t} \frac{a_j}{l_j + 1}.$$

If t is not a player on the path from top player 1 to i then the set S_t and the weights $a_j,\ j \in S_t$, are the same in the two peer-group situations (N, a, D) and (N_{-i}, a', D') and thus

$$Sh_t(N_{-i}, r') = Sh_t(N, r).$$

When t belongs to the path from top player 1 to i, then, by definition, $i \in S_t$ in the graph (N, D) and

$$Sh_t(N,r) = \sum_{j \in S_t} \frac{a_j}{l_j + 1} = \sum_{j \in S_t, j \neq i} \frac{a_j}{l_j + 1} + \frac{a_i}{l_i + 1}.$$

Let S'_t denote the set of subordinates of t (including t itself) in the peer-group situation (N_{-i}, a', D'). Then $S'_t = S_t \setminus \{i\}$, $k \in S'_t$ (because k is the predecessor of i) and in the peer group situation (N_{-i}, a', D'), all numbers a'_j of the players $j \in S'_t$, $j \neq k$, are the same as in (N, a, D), while the contribution of k changes from a_k to $a'_k = a_k + \frac{a_i l_i}{l_i + 1}$. Hence

$$Sh_t(N_{-i}, r') = \sum_{j \in S'_t} \frac{a'_j}{l'_j + 1} = \sum_{j \in S'_t, j \neq k} \frac{a_j}{l_j + 1} + \frac{a_k + \frac{a_i l_i}{l_i + 1}}{l_k + 1}.$$

Using the fact that $l_i = l_k + 1$ it follows that

$$Sh_t(N_{-i}, r') = \sum_{j \in S'_t} \frac{a_j}{l_j + 1} + \frac{\frac{a_i l_i}{l_i + 1}}{l_i} = \sum_{j \in S_t} \frac{a_j}{l_j + 1} = Sh_t(N, r).$$

Hence $Sh_t(N_{-i}, r') = Sh_t(N, r)$ for every $t \in N_{-i}$. □

Next we give an axiomatization of the Shapley value on the class $\mathcal{G}_p$ of peer-group games. Therefore we formulate several axioms of a solution F on the class of peer group games $\mathcal{G}_p$. Recall that $(N, r) \in \mathcal{G}_p$ if r is the characteristic function of a peer group situation (N, a, D). Therefore, with slight abuse of notation, the axioms are formulated in terms of the peer group situations.

Applying the necessary player property of Chapter 4 to peer group games states that (i) when there is only one player with a positive weight, then this player earns at least as much as any other player and, (ii) the top player always earns at least as much as any other player. Here we split this axiom in these two properties.

Axiom 6.3.2 (Weak Veto Property) *For every peer group situation (N, a, D) such that for some $i \in N$ the contributions $a_j = 0$ for all $j \neq i$, it holds that $F_i(N, a, D) \geq F_j(N, a, D)$ for all $j \neq i$.*

Axiom 6.3.3 (Top-monotonicity) *For every peer group situation (N, a, D), it holds that $F_1(N, a, D) \geq F_j(N, a, D)$.*

Note that top-monotonicity is also implied by structural monotonicity of Chapter 4 (which requires that any player with successors earns at least as much as its successors if the game is monotone.)

Axiom 6.3.4 (Independence of Non-Subordinates) *For every pair (N, a, D) and (N, a^*, D) such that for some player $i \in N$, $a_j = a_j^*$ for all $j \neq i$, it holds that $F_k(N, a, D) = F_k(N, a^*, D)$ for each player $k \neq i$ which is not a superior of i.*

This property means that the payoff of a player only depends on his own contribution and the contributions of all its subordinates. Similar as it is shown in van den Brink (2004) that independence on higher valuations for auction game solutions is satisfied by any game solution that satisfies strong monotonicity of Young (1985), it can be shown that independence of non-subordinates for peer group games is implied by strong monotonicity.

Together with efficiency and PGG-Davis-Maschler consistency the three axioms above characterize the Shapley value.

Theorem 6.3.5 *A solution F on $\mathcal{G}_p$ is equal to the Shapley value if and only if it satisfies efficiency, PGG-Davis-Maschler consistency, weak veto property, top-monotonicity and independence of non-subordinates.*

Proof. The Shapley value has all these five properties, it is efficient on the class of all games, by Theorem 6.3.1 it has PGG-Davis-Maschler consistency and the other three properties can easily be verified from (6.3.2). In particular notice that the weak veto property is true: in case there is only one positive weight a_i, player i and all his predecessors get an equal share in a_i and all the other players get nothing.

To prove uniqueness, consider some peer-group situation (N, a, D) and a player $i \in L_D$. By independence of non-subordinates it follows that this player gets the same payoff as in a peer group situation (N, a^*, D) with $a_i^* = a_i$ and $a_j^* = 0$ for all $j \neq i$. Let (N, r^*) be the game associated to (N, a^*, D). Then (N, r^*) is a veto-rich game with player i and all its superiors as the set of veto players. Consider a player $j \in L_D \setminus \{i\}$, i.e., $j \neq i$ is also a player without successors and $a_j^* = 0$. Clearly, j is not a superior of i and thus is a non-veto player in (N, r^*). Now, consider the peer-group situation (N, a', D) with $a'_k = 0$ for every $k \in N$. Again by independence of subordinates we obtain that $F_j(N, a^*, D) = F_j(N, a', D)$. Further, since every contribution is zero, by the weak veto property every player receives the same payoff in the peer-group situation (N, a', D), and by efficiency it follows that this payoff is 0. So $F_j(N, a^*, D) = F_j(N, a', D) = 0$. The PGG-Davis-Maschler consistency requires that the payoffs of the players in N_{-j} do not change when removing player j from (N, a^*, D). Then, repeating the reasoning above for a new player $k \in N_{-j}$, $j \neq i$, having no successor in $(N_{-j}, D \setminus \{(j', j)\}$, where j' is the predecessor of j in (N, D), we get $F_k(N, a^*, D) = 0$. Continuing in this way, we get that $F_h(N, a^*, D) = 0$ for every non-veto player h, i.e., for every h not equal to i or one of the superiors of i. Next, consider player i and let i_1 be its predecessor in (N, D). By the weak veto player property it follows that $F_i(N, a^*, D) \geq F_{i_1}(N, a^*, D)$. Next, consider the Davis-Maschler reduced game $(N \setminus \{i\}, (r^*)^{F(N,v)}_{N_{-i}})$ of the game (N, r^*) associated with (N, a^*, D). From the PGG-Davis-Maschler consistency it follows that every player $j \in N_{-i}$ has the same payoff in both games. By definition of the Davis-Maschler reduced game we have that

$$(r^*)^{F(N,v)}_{N_{-i}}(S) = \max\{v(S), v(S \cup \{i\}) - F_i(N, v)\},$$

so $(r^*)^{F(N,v)}_{N_{-i}}(S) = 0$ if $i_1 \notin S$ and $(r^*)^{F(N,v)}_{N_{-i}}(S) = a^*_i - F_i(N, r^*)$ if $i_1 \in S$. It follows that $(N \setminus \{i\}, (r^*)^{F(N,v)}_{N_{-i}})$ is the game associated with the peer group game $(N_{-i}, a^*_{-i}, D \setminus \{(i_1, i)\})$, with a^*_{-i} the $n-1$ vector given by $a^*_j = 0$ for every $j \in N_{-i} \setminus \{i_1\}$ and $a^*_{i_1} = a^*_i - F_i(N, r^*)$. So, we have a peer-group situation in which only player i_1 has a non-zero contribution and, by the same reasoning as above, we obtain that $F_{i_1}(N, r^*) \geq F_{i_2}(N, r^*)$, where i_2 is the predecessor of i_1. By repeating this procedure we conclude that

$$F_{i_0}(N, r^*) \geq F_{i_1}(N, r^*) \geq F_{i_2}(N, r^*) \geq \cdots \geq F_{i_{l_i}}(N, r^*),$$

where l_i is the number of superiors of i, i_k is the predecessor of i_{k-1}, $k = 1, \ldots, l_i$ with $i_0 = i$ and $i_{l_i} = 1$. On the other hand, top-monotonicity requires that $F_1(N, r^*) \geq F_i(N, r^*)$. So, in the peer-group situation (N, a^*, D), player i and all its superiors have the same payoff. Since all other players get zero payoff, it follows by efficiency that $F_i(N, a, D) = F_i(N, a^*, D) = F_i(N, r^*) = \frac{a_i}{l_i+1}$. This shows that for each value satisfying the five properties the payoff of a player i without successors is equal to $\frac{a_i}{l_i+1}$. Together with PGG-Davis-Maschler consistency this determines the payoff $F_j(N, a, D)$ for all $j \in N$. □

We now show the logical independence of the five properties of Theorem 6.3.5 by giving five alternative solutions on the class of peer-group games, each solution satisfying four of the five properties.

1. Let F on $\mathcal{G}_p$ be given by $F_i(N, a, D) = 0$ for every (N, a, D) and $i \in N$. Then F satisfies all properties except efficiency.

2. The solution F on $\mathcal{G}_p$ given by $F_i(N, v) = \frac{a_i}{2}$ for every $i \neq 1$ and $F_i(N, v) = \sum_{j \in N} \frac{1}{2} a_j$ for top player $i = 1$, satisfies all properties except PGG-Davis-Maschler consistency.

3. The solution F on $\mathcal{G}_p$ given by $F_i(N, a, D) = 0$ for every $i \neq 1$ and $F_1(N, a, D) = v(N)$, satisfies all properties except weak veto property.

4. Let F on $\mathcal{G}_p$ be given by $F_i(N, v) = a_i$ for every $i \in N$. This solution satisfies all properties except the weak top-player property.

5. The solution F on $\mathcal{G}_p$ given by $F(N, a, D) = Nuc(N, a, D)$ satisfies all properties except the independence of nonsubordinates property.

6.4 Monotonicity of solutions

In the previous section we have seen that the nucleolus differs from the Shapley value on the class of peer-group games in the sense that it satisfies all the axioms of the Shapley value except the independence of non-subordinates property. In this section we consider monotonicity of a solution F on the class of peer-group games.

Axiom 6.4.1 *(Strong Monotonicity) For every two peer-group situations (N, a, D) and (N, b, D) with $b_i \geq a_i$ for all $i \in N$ it holds that $F_i(N, b, D) \geq F_i(N, a, D)$ for every $i \in N$.*

From formula (6.3.2) it follows immediately that the Shapley value is strong monotone on the class of peer-group games. For the nucleolus no monotonicity properties are known for the class of all cooperative games (Young, 1985). In Hokari (2000) and Arin and Feltkamp (2005) it is proved that also on the subclasses of convex games and veto-rich games the nucleolus does not have monotonicity properties. In this section we show that, like the Shapley value, also the nucleolus is strong monotone on the class of peer-group games. To prove this result we use the next algorithm to compute the nucleolus of a peer-group game. The algorithm is just a modification of the algorithm given in Brânzei, Solymosi and Tijs (2005), the proof that the algorithm indeed finds the nucleolus is similar.

Let (K, b, E) be a peer-group situation with top player 1, and for $i \in K$, let $S_i(K, E)$ be the set of players containing i and all its subordinates in (K, E). For a player $i \in K \setminus \{1\}$, define

$$\tau_i(K, b, E) = \frac{\sum_{j \in S_i(K,E)} b_j}{|S_i(K, E)| + 1}.$$

Algorithm 6.4.2
Step 0 *Set $K = N$, $b = a$ and $E = D$.*
Step 1 *Find a player $i \in K \setminus \{1\}$, such that $\tau_i(K, b, E) = \min_{k \in K} \tau_k(K, b, E)$.*

Step 2 *Set $x_j = \tau_i(K, b, E)$ for every $j \in S_i(K, E)$. When $K \setminus S_i(K, E) = \{1\}$, set $x_1 = \sum_{j \in N} b_j - \sum_{j \neq 1} x_j$ and stop. Otherwise go to Step 3.*

Step 3 *Let h be the predecessor of i in (K, E). Define $N' = K \setminus S_i(K, E)$, $a'_h = b_h + \tau_i(K, E)$, $a'_i = b_i$ for every $i \in N' \setminus \{h\}$ and $D' = D(N')$, where $D(N')$ is the set of arcs of D on the set N'.*

Step 4 *Set $K = N'$, $b = a'$ and $E = D'$ and return to Step 1.*

In any Step 2 the payoff is determined for at least one player $i \in K$. So, the algorithm ends within at most $n - 1 = |N| - 1$ applications of Step 2. The last time that Step 2 is performed, also the payoff of top player 1 is determined. It follows from Brânzei, Solymosi and Tijs (2005) that the payoff vector x generated by the algorithm is the nucleolus of the peer-group situation (N, a, D).

The algorithm will be used below to compare the payoffs of two peer-group situations that differ in only one a_i. Notice that the algorithm subsequently chooses a player $i \neq 1$ in the set of remaining players and then determines the payoffs for the chosen player i and all its subordinates in the set of remaining players. In other words, the algorithm generates a sequence of non-overlapping subsets of N, determines in each iteration the payoffs of the players in the chosen subset at this iteration, until in Step 2 only player 1 is left. When in Step 1 there are several players in $K \setminus \{i\}$ with minimal value $\tau_i(K, b, E)$, then each of these players might be chosen. Hence the sequence of subsets generated by the algorithm does not need to be unique. We now have the following definition, which will be used to prove the subsequent theorem.

Definition 6.4.3 *Two peer-group situations (N, a, D) and (N, b, D) are algorithmic equivalent when at any iteration of the algorithm the same subset of players can be chosen in both situations.*

Theorem 6.4.4 *The nucleolus is strong monotone on the class of peer-group games.*

It is sufficient to prove that $F_k(N, b, D) \geq F_k(N, a, D)$ for all $k \in N$ for two peer-group situations (N, b, D) and (N, a, D) with $b_i > a_i$ for some $i \in N$ and $b_j = a_j$ for all $j \neq i$. Then, strong monotonicity follows for every (N, b, D) and (N, a, D) with $b \geq a$ by repeating this result for every $i = 1, \ldots, n$. To prove the theorem, we first give two lemmas. The first lemma shows that the theorem is true for two situations (N, b, D) and (N, a, D) that are algorithmic equivalent, so that the subsets can be chosen in such a way that in both situations the same sequence is generated. The second lemma considers the case when the two sitations are not algorithmic equivalent.

Lemma 6.4.5 *Let (N, a, D) and (N, b, D) be two algorithmic equivalent peer-group situations with $b_i > a_i$ for some $i \in N$ and $b_j = a_j$ for $j \neq i$. Then $Nuc_j(A, b, D) \geq Nuc_j(N, a, D)$ for every $j \in N$.*

Proof. For $n = |N| = 1$, the lemma is true by efficiency of the nucleolus. Next consider $n = 2$. So, take $N = \{1, 2\}$ and $D = \{(1, 2)\}$. It is well-known (and follows straightforward from the algorithm) that for such a peer-group situation (N, a, D) the nucleolus is given by $(a_1 + \frac{a_2}{2}, \frac{a_2}{2})$ and thus the nucleolus payoffs are increasing in the vector (a_1, a_2). We now proceed by induction and assume that the lemma is true for $k = 1, 2, \ldots, n-1$, i.e., for every pair (K, a, D) and (K, b, D) with $|K| \leq n - 1$. Suppose the lemma is not true for two situations with n players. Let i be the player with $b_i > a_i$.

Since the two situations are algorithmic equivalent, it is possible to choose in both situations the same set $S_k(N, D)$ for some $k \in N \setminus \{1\}$ in Step 1 of the first iteration.

Suppose that $i \notin S_k(N, D)$. Then $\tau_k(N, b, D) = \tau_k(N, a, D)$ and every player $h \in S_k(N, D)$ gets the same nucleolus payoff in Step 2 of the first iteration. Further, in Step 3, two new peer group situations (N', b', D') and (N', a', D') are generated with $N' = N \setminus S_k(N, D)$, $D' = D(N')$ and with $b'_i > a'_i$ and $b'_j = a'_j$ for every $j \in N \setminus S_k(N, D)$, $j \neq i$.

Since $|N \setminus S_k(N,D)| < |N| = n$, by the induction hypothesis we have that the strong monotonicity holds for these two situations and thus also for (N,b,D) and (N,a,D), which contradicts that $i \notin S_k(N,D)$.

Hence, player i is in the set $S_k(N,D)$ chosen at the first iteration. So, with $S = S_k(N,D)$,

$$\tau_k(N,b,D) = \frac{\sum_{j \in S} b_j}{|S|+1} > \frac{\sum_{j \in S} a_j}{|S|+1} = \tau_k(N,a,D),$$

and in Step 2 every player $j \in S_k(N,D)$ gets a higher payoff in (N,b,D) than in (N,a,D). Moreover, after deleting the set $S_k(N,D)$, in Step 3, two new peer-group situations (N',b',D') and (N',a',D') are generated with $N' = N \setminus S_k(N,D)$, $D' = D(N')$ and $b'_h = b_h + \tau_k(N,b,D) > a_h + \tau_k(N,a,D) = a'_h$ for the predecessor h of k, and $b'_j = b_j = a_j = a'_j$ for every $j \in N \setminus S_k(N,D)$, $j \neq h$. So we get two peer-group games which differ for only one player, but with a smaller number of players. So, again by the induction hypothesis, the strong monotonicity holds for these two situations and thus also for (N,b,D) and (N,a,D). □

Lemma 6.4.6 *For some $i \in N$ and $\alpha > 0$, let (N,a,D) and $(N,a+\alpha e^i,D)$ be a pair of peer group situations, where $e^i_i = 1$, $e^i_j = 0$ for $j \neq i$. Then, for some $t \geq 1$, there are numbers $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_t = \alpha$ such that the two peer-group situations $(N,a+\beta e^i,D)$ and $(N,a+\gamma e^i,D)$ are algorithmic equivalent when $\beta,\gamma \in [\alpha_l,\alpha_{l+1}]$ for some $l = 0,\ldots,t-1$.*

Proof. Let (N,r^β) denote the peer-group game associated to the peer-group situation $(N,a+\beta e^i,D)$, $\beta \in [0,\alpha]$, and, for ease of notation, denote $\tau_k(\beta) = \tau_k(N,a+\beta e^i,D)$, $k \in N \setminus \{1\}$. For $k \neq 1$, let $A_k \subseteq [0,\alpha]$ be the set of values of β for which $\tau_k(\beta)$ is minimal in the first step of the algorithm. So, $A_k \subseteq [0,\alpha]$ is the set of values such that the set $S_k(N,D)$ can be chosen in Step 1 of the algorithm applied to $(N,a+\beta e^i,D)$. Consider the set A_k for some $k \neq 1$. Since, for every $k \neq 1$, $\tau_k(\beta)$ is continuous in β, it follows that A_k is closed. Next, suppose that $\beta_1,\beta_2 \in A_k$ and for some $\lambda \in [0,1]$, consider the convex combination $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$. Since $\tau_h(\beta)$ is linear in β for every $h \neq 1$, it follows that $\beta \in A_k$. So, every non-empty set A_k is closed and convex, i.e., A_k is either empty, or it is some segment $[c,d] \subseteq [0,\alpha]$. Further, consider two non-empty sets A_k and $A_{k'}$ with non-empty intersection $A_k \cap A_{k'}$. Recall that β only affects the contribution of player i. So, if $i \notin S_k(N,D)$ and $i \notin S_{k'}(N,D)$, then neither $\tau_k(.)$ nor $\tau_{k'}(.)$ depends on β, implying that $A_k = A_{k'}$. When i belongs to one of the sets, say k, then $\tau_k(.)$ is increasing in β, while $\tau_{k'}$ is constant in β. Hence, the sets A_k and $A_{k'}$ can have only one point in common. Finally, when i is in both sets, then both $\tau_k(.)$ and $\tau_{k'}(.)$ are increasing in β, but with different speed because of the difference in the denominator. Again, the sets A_k and $A_{k'}$ can have only one point in common. So, the two sets A_k and $A_{k'}$ either have empty intersection, or are equal to each other, or have precisely one point in common. Moreover, such a common point is a boundary point of both sets. Hence, the collection of sets A_k,

$k \in N \setminus \{1\}$, divides the interval $[0, \alpha]$ into at most $n-1$ subsets, each subset an interval itself, and with the property that the intersection of the two subsets is either empty, or the subset itself, or consists of one point, being a boundary point (end point) of both subsets. When two or more subsets coincide, we just choose one of the players k , i.e., then for every value β in this interval, the same set $S_k(N, D)$ in chosen in the first iteration of the algorithm.

Now, consider some interval A_k. Then for each $\beta \in A_k$, it is possible to choose the set $S_k(N, D)$ in Step 1 of the algorithm (in fact, it is the only possible choice if β does not belong to some other $A_{k'}$). Then, in Step 3 of the algorithm the set $S_k(N, D)$ is removed from the set of players and a new peer group situation on the remaining set $K = N \setminus S_k(N, D)$ of players is obtained. Then, at the next iteration, the possible sets that can be chosen at Step 1 depends on the values of $\beta \in A_k$. Repeating the reasoning used above, the interval A_k is subdivided into a collection of sets A_h, $h \in K \setminus \{1\}$, again with the properties that each subset is an interval itself, and that the intersection of two subsets is either empty, or the subset itself, or consists of one point, being a boundary point (end point) of both subsets.

Repeating this, at each iteration each interval is subdivided in a finite collection of subsets. Since the algorithm terminates within at most $n-1$ iterations, it follows that the interval $[0, \alpha]$ is subdivided into a finite number of subsets, such that each subset is an interval itself, and that the intersection of two subsets is either empty, or the subset itself, or consists of one point, being a boundary point (end point) of both subsets. Moreover, by construction, for two values β and γ within the same interval, we can choose the same sequence of sets when applying the algorithm, i.e., the two peer-group situations $(N, a + \beta e^i, D)$ and $(N, a + \gamma e^i, D)$ are algorithmic equivalent. □

Proof of Theorem 6.4.4. It is sufficient to prove that $Nuc_j(N, b, D) \geq Nuc_j(N, a, D)$, when $b = a + \alpha e^i$ for some player $i \in N$ and $\alpha > 0$. By Lemma 6.4.6, for some $t \geq 1$, the interval $[0, \alpha]$ can be subdivided into subintervals $[\alpha_l, \alpha_{l+1}]$, $l = 0, \ldots, t-1$, such that two games $(N, a + \beta e^i, D)$ and $(N, a + \gamma e^i, D)$ are algorithmic equivalent for two values β, γ within a same subinterval. So, by Lemma 6.4.5, all payoffs are nondecreasing on every interval $[\alpha_l, \alpha_{l+1}]$ and thus also nondecreasing on $[0, \alpha]$. □

Notice that each end point of an interval (except 0 and α) is also the end point of an adjacent interval, so that at each end point the algorithm yields the same nucleolus payoffs for the two different sequences of subsets chosen in Step 1 of the algorithm. So, for each value β in the first interval $[\alpha_0, \alpha_1]$, the algorithm generates the same sequence of sets. Then, at the end point α_1, two different sequences are possible, of course both sequences generate the same nucleolus payoffs. So, at this end point it is possible to switch to the alternative sequence and then this sequence is applied in the next interval.

The next proposition says that when only the contribution a_i of player i increases, the nucleolus payoffs of player i and all its superiors stricly increase and the payoff of player i increases with at least the same amount as the increase of every other player.

Proposition 6.4.7 *Let (N, a, D) and (N, b, D) be two peer-group situations such that $b_i > a_i$ for some i and $b_j = a_j$ for every $j \neq i$. Then*
1. $Nuc_j(N, b, D) > Nuc_j(N, a, D)$ for $j = i$ and every superior j of i.
2. $Nuc_i(N, b, D) - Nuc_i(N, a, D) \geq Nuc_j(N, b, D) - Nuc_j(N, a, D)$ for every $j \neq i$.

Proof. By Lemma 6.4.6 it is sufficient to consider the case that (N, a, D) and (N, b, D) are algorithmic equivalent. Then, for both situations it happens that at some iteration of the algorithm, say with player set $K \subset N$, tree $E = D(K)$ and vectors $a', b' \in \mathbb{R}^{|K|}$ obtained through the updates in the Steps 3 at the preceding iterations, a player $k \in K \setminus \{i\}$ is chosen with $i \in S_k(K, E)$. Clearly,

$$\tau_k(K, b', E) = \tau_k(K, a', E) + \frac{b_i - a_i}{|S_k(K, E)| + 1}.$$

So,

$$Nuc_j(N, b, D) = Nuc_j(N, a, D) + \frac{b_i - a_i}{|S_k(K, E)| + 1} >$$
$$Nuc_j(N, a, D) \text{ for every } j \in S_k(K, E).$$

Moreover, in the next update in Step 3, the contributions of the predecessor of k are increased with $\tau_k(K, b', E)$ and $\tau_k(K, a', E)$, respectively, meaning that in (N, b, D) all the players in $K \setminus S_k(K, E)$ together receive $\frac{b_i - a_i}{|S_k(K,E)|+1}$ more than in (N, a, D). So, every player in $S_k(K, E)$, including player i itself, gets a higher payoff of $\frac{b_i - a_i}{|S_k(K,E)|+1}$, and all players in $K \setminus S_k(K, E)$ get together the same higher amount. Since the latter set contains at least player 1, this proves the second assertion of the theorem.

To prove the first assertion, notice that $S_k(K, E)$ contains all superiors from i on the path from k to i and that in Step 3 of the algorithm after k has been chosen, the contribution of the predecessor of k, say player h, gets in (N, b, D) an increase which is $\frac{b_i - a_i}{|S_k(K,E)|+1}$ higher than in (N, a, D). When, at one of the following iterations, a set S is chosen containing player h, then each player in this set gets a higher payoff equal to this amount divided by $|S| + 1$. Also, there is the same difference in the update of the contributions of the predecessor of S in Step 3 of the algorithm. Continuing in this way, it follows that all superiors of player i get a higher payoff in (N, b, D) than in (N, a, D). □

Notice that the set $S_k(K, E)$ including player i may also contain subordinates of i, so the fact that each superior gets a higher payoff, does not mean that also subordinates can get a higher payoff (and also other players, being neither superior nor subordinate).

Computation of the nucleolus

Chapter 7

Computation of the nucleolus for a class of disjunctive games with a permission structure

7.1 Introduction

In this chapter, which is based on van den Brink, Katsev and van der Laan (2008a), an algorithm is given to find in polynomial time the nucleolus of a game with permission structure in which the collection of feasible coalitions is given by the collection of disjunctive feasible sets.

A special subclass of games with a permission structure arises from peer group situations, as introduced in Brânzei *et al.* (2002). A *peer group situation* is a triple consisting of a set of players, a hierarchical structure represented by a rooted directed tree, and for each player a real number representing his potential individual (economic) contribution to the society of all players. This yields an associated TU-game being the additive game in which the worth of any coalition is equal to the sum of the individual potentials of its members. In a rooted directed tree there is one top node (not having a predecessor), while any other node has precisely one predecessor. So, in case the hierarchical structure on the player set is a rooted directed tree, the conjunctive approach and the disjunctive approach coincide and we deal with a peer-group game. These peer group games have many interesting applications, such as polluted river games (see Ni and Wang (2007)), sequencing games (see Curiel, Pederzoli, and Tijs (1989)), dual airport games (see Littlechild and Owen (1973)) and auction games (see Graham, Marshall and Richard (1990)), see also Brânzei *et al.* (2002). Clearly, in a peer group game the worth of a coalition is the sum of the individual potentials of the members of the largest feasible subset of the coalition. Since the top player is always in this set when he belongs to the coalition, and the largest feasible set is the empty set for any coalition not containing the unique top player, it follows that the top player is a veto player, i.e., any coalition not containing the (veto) top player has zero worth in the restricted game.

In Arin and Feltkamp (1997) an exponential time algorithm has been given to compute the nucleolus for veto-rich games. In Brânzei *et al.* (2005) a polynomial time algorithm is given to compute the nucleolus of a peer group game. In this chapter the Arin-Feltkamp algorithm is changed to compute the nucleolus of the restricted game induced by more general situations, including peer group situations and information market situations (see Muto *et al.* (1989)) as special cases. The generalization concerns both the hierarchical graph structure (by allowing for any digraph having one top node and no directed cycles) and the class of unrestricted TU-games (by allowing any game satisfying a so-called weak digraph monotonicity condition and a weak digraph concavity condition). The algorithm finds the nucleolus in polynomial time.

The chapter is organized as follows. Section 7.2 is a preliminary section in which the permission situation is described. Also some facts about the nucleolus are given. In Section 7.3 the properties of weak digraph monotonicity are introduced and weak digraph concavity and some examples satisfying these conditions are presented. In Section 7.4 some properties of essential and feasible coalitions are given. These properties are crucial for the algorithm given in Section 7.5. In Section 7.6 the complexity of the algorithm is discussed. Finally, Section 7.7 contains some concluding remarks.

7.2 Preliminaries

7.2.1 Games with permission structure

In this chapter it is assumed that the players in a TU-game are part of a hierarchical structure that is represented by a directed graph, refered to as a *permission structure*, such that some players need permission from other players before they are allowed to cooperate within a coalition. A triple (N, v, D) with (N, v) a TU-game and (N, D) a digraph with the player set N as the set of nodes is called a *game with permission structure*. In this chapter we assume that $D \in \mathcal{D}^N$ (recall from Chapter 2 that $\mathcal{D}^N$ denotes the collection of all acyclic, quasi-strongly connected digraphs on N) and (without loss of generality) that $i_0 = 1$ is its unique top-node[1].

Assumption 7.2.1 (N, D) *is acyclic and quasi-strongly connected with* $P_D(1) = \emptyset$ *(and thus* $P_D(i) \neq \emptyset$ *for every* $i \neq 1$*).*

As noticed in Chapter 2 two possible approaches can be distinguished: the conjunctive and disjunctive approach. In this chapter the *disjunctive* approach as developed in Gilles and Owen (1994) and van den Brink (1997) will be considered. In this approach a player $i \neq 1$ needs permission to cooperate of at least one of its predecessors. Therefore a coalition is feasible if and only if it contains the top-player 1 and for every other player in

[1]This implies that $1 \in N$. Later we consider reduced games on proper subsets of $N' \subset N$, but the top-player 1 always belongs to N'.

the coalition at least one of its predecessors is also in the coalition. So, for digraph (N, D), the set of disjunctive feasible coalitions is given by

$$\Phi_D^d = \{S \subseteq N \,|\, P_D(i) \cap S \neq \emptyset \text{ for all } i \in S \setminus \{1\}\}\,.$$

For any $S \subseteq N$, let $\sigma^d(S) = \bigcup_{\{T \in \Phi_D^d | T \subseteq S\}} T$ be the largest disjunctive feasible subset of S in D.[2] By Assumption 7.2.1 it follows that for every $i \neq 1$, there is at least one directed path from 1 to i. As a consequence it follows that for every $S \subseteq N$ with $\sigma^d(S) \neq \emptyset$, the subgraph $(\sigma^d(S), D(\sigma^d(S))$ is acyclic and quasi-strongly connected with node $1 \in \sigma^d(S)$ as its unique top-node.

Given the triple (N, v, D) with $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, under the disjunctive permission structure the induced *restricted* game $r^d\colon 2^N \to \mathbb{R}$ is given by

$$r^d(S) = v(\sigma^d(S)) \text{ for all } S \subseteq N. \tag{7.2.1}$$

Since player 1 is the top-node it holds that $r^d(S) = 0$ when $1 \notin S$, i.e., the restricted game is a *veto-rich* game with respect to the top-player 1. If D is a rooted directed tree (with node 1 as its root), then $|P_D(i)| = 1$ for all $i \neq 1$ and the conjunctive and disjunctive approach coincide. In this case the triple (N, v, D) is a peer group situation when the game (N, v) is a non-negative additive game (see the previous chapter).

7.2.2 Essential coalitions and nucleolus

In a game (N, v), a coalition S is called *inessential* if it has a partition $\{S_1, \ldots, S_r\}$ with $r \geq 2$, such that $v(S) \leq \sum_{j=1}^r v(S_j)$. Coalitions which are not inessential are called *essential*. Notice that single player coalitions are always essential. It is straightforward to observe that for an inessential coalition S it holds that

$$e(S, x) \leq \sum_{j=1}^{r} e(S_j, x), \text{ for all } x \in \mathbb{R}^n.$$

Therefore the core, and thus also the nucleolus, is independent of inessential coalitions, as was noticed by Huberman (1980). In fact, in any n player game there are at most $(2n-2)$ coalitions which actually determine the nucleolus, see Brune (1983) and Reijnierse and Potters (1998). Although, as noticed by Brânzei *et al.* (2005), identifying these coalitions is no less laborious as computing the nucleolus itself, in the following we state some facts for games with non-empty core which will appear to be useful later on. We denote

$$e^*(N, v) = \min_{\{S \subset N | S \neq \emptyset\}} -e(S, x) \text{ at } x = Nuc(N, v),$$

i.e., $e^*(N, v)$ the the minimal negative excess at the nucleolus of game (N, v). Clearly, $e^*(N, v) \geq 0$ if and only if $Core(N, v) \neq \emptyset$.

[2] Every coalition having a unique largest feasible subset follows from Φ_D^d being closed under union.

Lemma 7.2.2 *If $e^*(N,v) > 0$, then every coalition $S \subset N$ with $-e(S,x) = e^*(N,v)$ at $x = Nuc(N,v)$ is essential.*

Proof. Suppose $S \subset N$ with $-e(S,x) = e^*(N,v)$ is inessential. Then there is a partition $\{S_1,\ldots,S_m\}$ such that $e^*(N,v) = -e(S,x) \geq \sum_{j=1}^m -e(S_j,x)$. Since $e^*(N,v) > 0$ there must be at least one $j \in \{1,\ldots,m\}$ such that $-e(S_j,x) < -e(S,x)$, which contradicts that $e^*(N,v) = \min_{\{S\subset N|S\neq\emptyset\}} -e(S,x)$. □

For the next lemma, let $B = \{S_1,\ldots,S_m\}$ be a balanced collection of coalitions and let $\mathcal{B}$ denote the set of all balanced collections, excluding the balanced collection $\{N\}$ having the grand coalition N as its single element.

Lemma 7.2.3 *If $e^*(N,v) \geq 0$ then*

$$e^*(N,v) = \min_{B\in\mathcal{B}} \frac{v(N) - \sum_{j=1}^m \lambda_j^B v(S_j)}{\sum_{j=1}^m \lambda_j^B},$$

with λ_j^B, $j = 1,\ldots,m$, the solution of the system (2.1.1) for the balanced collection $B = \{S_1,\ldots,S_m\}$.

Proof. Let $B = \{S_1,\ldots,S_m\}$ be a balanced collection with $\lambda_1^B,..,\lambda_m^B$ as the corresponding solution of system (2.1.1). Observe that for every $i \in N$ it holds that $\sum_{\{j|i\in S_j\}} \lambda_j^B = 1$ and thus for every $x \in \mathbb{R}^n$ and $S \subset N$ we have that $x(S) = \sum_{i\in S} x_i = \sum_{i\in S}\sum_{\{j|i\in S_j\}} \lambda_j^B x_i$. Hence,

$$\sum_{j=1}^m \lambda_j^B x(S_j) = \sum_{j=1}^m \sum_{i\in S_j} \lambda_j^B x_i = \sum_{i\in N} \sum_{\{j|i\in S_j\}} \lambda_j^B x_i = x(N)$$

and thus at $x = Nuc(N,v)$ we have that the convex combination $\sum_{j=1}^m \frac{\lambda_j^B}{\sum_h \lambda_h^B} \cdot (-e(S_j,x))$ of all negative excesses $-e(S_j,x)$, $j = 1,\ldots,m$, is equal to

$$\sum_{j=1}^m \frac{\lambda_j^B}{\sum_h \lambda_h^B} \cdot (x(S_j) - v(S_j)) = \frac{v(N) - \sum_{j=1}^m \lambda_j^B\ v(S_j)}{\sum_h \lambda_h^B}.$$

Since every $-e(S_j,x) \geq e^*(N,v)$, $j = 1,\ldots,m$, also its convex combination is at least equal to $e^*(N,v)$, which shows that

$$e^*(N,v) \leq \frac{v(N) - \sum_{j=1}^m \lambda_j^B\ v(S_j)}{\sum_h \lambda_h^B}. \tag{7.2.2}$$

Finally, from Kohlberg's theorem (Theorem 7.2.2) we know that there exists a balanced collection $B = \{S_1,\ldots,S_m\}$ with $e^*(N,v) = -e(S_j,x)$ for all j. For such a balanced collection inequality (7.2.2) holds with equality, which proves the lemma. □

The next two corollaries follow immediately.

Corollary 7.2.4 *Let $B = \{S_1, \ldots, S_m\}$ be a balanced collection with weights λ_j^B, $j = 1, \ldots, m$, satisfying*

$$e^*(N,v) = \frac{v(N) - \sum_{j=1}^{m} \lambda_j^B v(S_j)}{\sum_{j=1}^{m} \lambda_j^B}. \tag{7.2.3}$$

Then at $x = Nuc(N,v)$ we have that $-e(S_j,x) = e^*(N,v)$, $j = 1, \ldots, m$.

Proof. As shown in the proof of Lemma 7.2.3, the right-hand side of equation (7.2.3) is a convex combination of the numbers $-e(S_j,x)$. Therefore, for each j, $e^*(N,v) \leq -e(S_j,x)$ must hold with equality. □

Corollary 7.2.5 *If $e^*(N,v) > 0$, then for any balanced collection $B = \{S_1, \ldots, S_m\}$ satisfying $e^*(N,v) = \frac{v(N)-\sum_{j=1}^{m} \lambda_j^B v(S_j)}{\sum_{j=1}^{m} \lambda_j^B}$, it holds that any set S_j is essential.*

Proof. This follows immediately from Lemma 7.2.2 and Corollary 7.2.4. □

Arin and Feltkamp (1997) propose an algorithm to find the nucleolus of a veto-rich game, i.e., a game (N,v) such that there exists (at least one) veto player being a player i such that $v(S) = 0$ when $i \notin S$. The algorithm makes use of the fact that for veto-rich games the kernel contains precisely one payoff vector, and thus the nucleolus is this unique element of the kernel. For an element x in the kernel they first show that for any player j it holds that $x_j = 0$ if there exists $S \subseteq N \setminus \{j\}$ such that $v(S) \geq v(N)$. The algorithm is initiated by setting $x_j = 0$ for all these players and setting A_0 as the set of these players. Observe that this set does not contain the set of veto players (unless it is the null-game and all players get zero payoff). It is also shown that the nucleolus payoff x_j is positive for all other players $j \in N \setminus A_0$ (thus including all veto players). After this initialisation the algorithm iteratively determines the payoffs of the other players as follows. Let i be an arbitrarily chosen veto player. Then, at each step t, let A_{t-1} be the set of players for which the payoffs have been determined already and let B_t be the collection of all sets S such that $i \in S$ and $(N \setminus A_{t-1}) \setminus S \neq \emptyset$. Then

$$q_t = \min_{S \in B_t} \; q_t(S), \tag{7.2.4}$$

where $q_t(S) = \frac{v(N)-v(S)-x(A_{t-1}\setminus S)}{|(N\setminus A_{t-1})\setminus S|+1}$ is determined, and for any player $j \in \cap\{S \in B_t | q_t(S) = q_t\}$ the nucleolus is set equal to $x_j = q_t$.

At any step t the payoff of at least one player is determined, so in at most $n-1$ steps all payoffs x_j, $j \neq i$ are determined. As soon as all these payoffs are determined, the payoff x_i of the chosen veto player i is set equal to $v(N) - x(N \setminus \{i\})$. In this chapter we modify this algorithm to find the nucleolus of restricted games arising from games with a permission structure in which players in a cooperative TU-game belong to a hierarchical structure that is represented by a directed graph in polynomial time.

7.3 Weak digraph monotonicity and concavity

The algorithm to be presented in Section 7.5 holds for games with a permission structure (N, v, D) with $D \in \mathcal{D}^N$ satisfying the next two conditions. First, we say that a game with permission structure (N, v, D) satisfies *weak digraph monotonicity* if

$$[S \subseteq N \text{ and } S \in \Phi_D^d] \Rightarrow v(S) \leq v(N). \tag{7.3.5}$$

Observe that weak digraph monotonicity weakens monotonicity in two respects, namely (i) the monotocity condition $v(S) \leq v(T)$ if $S \subseteq T$ only has to hold for $T = N$ and (ii) for sets S are feasible given the disjunctive permission structure on the digraph D. Second, we say that a game with permission structure (N, v, D) satisfies *weak digraph concavity* if

$$[S \cup T = N \text{ and } S, T \in \Phi_D^d] \Rightarrow v(S) + v(T) \geq v(S \cap T) + v(N). \tag{7.3.6}$$

Observe that also this property weakens the concavity of a game in two respects, namely that the concavity condition only has to hold for sets that S and T satisfying (i) $S \cup T = N$ and (ii) S and T are feasible given the disjunctive permission structure on D. So, for both properties the adjunctive *'weak'* means that the inequality conditions are only required for $T = N$, respectively $S \cup T = N$, and the adjunctive *'digraph'* means that the inequality conditions are only required for feasible sets with respect to the permission structure. Monotonicity is a condition satisfied by most of the games that arise from economic or social situations, so this is certainly the case for weak digraph monotonicity. Although concavity is a strong condition for profit games[3], weak digraph concavity is considerably weaker and is also satisfied by several interesting classes of profit games with permission structure. We give some examples.

Example 7.3.1 Generalised peer group situations
It is obvious that peer group situations (N, v, D) satisfies weak digraph monotonicity. Further, for any feasible S and T such that $S \cup T = N$ it follows that $S \cap T$ is feasible (since D is a rooted tree) and

$$v(S) + v(T) = \sum_{i \in S} a_i + \sum_{i \in T} a_i = \sum_{i \in S \cap T} a_i + \sum_{i \in N} a_i = v(S \cap T) + v(N).$$

So, (N, v, D) also satisfies weak digraph concavity.

(N, v, D) is a *generalised peer group situation* when $D \in \mathcal{D}^N$ is an acyclic and quasi-strongly connected digraph (and v is again a nonnegative additive game). Clearly, any generalised peer group situation satisfies weak digraph monotonicity and weak digraph concavity. It now might happen that $S \cap T$ is not feasible under the disjunctive approach. Then the weak digraph concavity condition might hold with strict inequality. □

[3] Given our nucleolus concept in which the maximum excess $v(S) - x(S)$ is minimized, in this chapter we deal with profit games.

Example 7.3.2 Generalised information market situation
Let $\mathcal{S} = \{S_1, \ldots, S_K\}$ be a collection of K (nonempty) subsets of N, and α_k, $k = 1, \ldots, K$, be positive numbers. Then the game (N, v) given by

$$v(S) = \sum_{\{k|S_k \cap S \neq \emptyset\}} \alpha_k, \quad S \subseteq N \tag{7.3.7}$$

will be considered.

Further, let $D \in \mathcal{D}^N$ be any digraph satisfying $(1, j) \in D$ for all $j \in \{2, \ldots, n\}$. So, $j = 1$ is the top-player and $S \subseteq N$ is feasible if and only if $1 \in S$. Now, the restricted game (N, r) is given by $r^d(S) = 0$ if $1 \notin S$ and

$$r^d(S) = \sum_{\{k|S_k \cap S \neq \emptyset\}} \alpha_k, \text{ if } 1 \in S.$$

The game (N, r) is an information game as introduced in Muto *et al.* (1989). Obviously, (N, v, D) satisfies weak digraph monotonicity. Further, for any feasible S and T such that $S \cup T = N$ we have that $S \cap T$ is feasible and

$$v(S) + v(T) = \sum_{\{k|S_k \cap S \neq \emptyset\}} \alpha_k + \sum_{\{k|S_k \cap T \neq \emptyset\}} \alpha_k =$$

$$\sum_{\{k|S_k \cap (S \cap T) \neq \emptyset\}} \alpha_k + \sum_{\{k|S_k \cap N \neq \emptyset\}} \alpha_k = v(S \cap T) + v(N),$$

where the last but one equality follows since $S \cup T = N$. Thus (N, v, D) also satisfies weak digraph concavity. In fact, this condition is satisfied for any $D \in \mathcal{D}^N$. In case $S \cap T$ is not feasible the condition might hold with strict inequality.

Observe that also any game with permission structure (N, v, D) where v is the sum of an additive game and a game as given above in equation (7.3.7), satisfies the conditions of weak digraph monotonicity and weak digraph concavity. □

Example 7.3.3 Market situation
Let the set N consist of one seller, say player 1, having one item for sale, and $n-1$ buyers, and let a_j be the nonnegative surplus of trade between the seller and buyer j, $j = 2, \ldots, n$. Then the market game is given by (N, v) with $v(S) = 0$ if $1 \notin S$ and

$$v(S) = \max_{j \in S \setminus \{1\}} a_j, \text{ if } 1 \in S.$$

Further, let $D \in \mathcal{D}^N$ be any digraph satisfying $(1, j) \in D$ for all $j \in \{2, \ldots, n\}$. So, $S \subseteq N$ is feasible if and only if $1 \in S$. Then for any feasible S and T such that $S \cup T = N$ it follows that

$$v(N) = \max[v(S),\ v(T)]$$

and

$$v(S \cap T) \leq \min[v(S),\ v(T)].$$

Hence

$$v(S) + v(T) = \max[v(S),\ v(T)] + \min[v(S),\ v(T)] \geq v(N) + v(S \cap T).$$

Therefore (N, v, D) satisfies weak digraph concavity. Clearly, it also satisfies weak digraph monotonicity. □

7.4 Essential and feasible coalitions

In this subsection several results of essential and feasible coalitions for games with permission structure (N, v, D) will be proved. It will be used later on to prove that the algorithm of Section 7.5 will indeed find the nucleolus of the restricted game. The first lemma does not yet require the two conditions (7.3.5) and (7.3.6) and says that any essential coalition with at least two elements is feasible.

Lemma 7.4.1 *If $S \subseteq N$ with $|S| \geq 2$ is essential in the restricted game (N, r), then S is feasible.*

Proof. Suppose that S is not feasible. Then $r^d(S) = r^d(\sigma^d(S))$ with $\sigma^d(S) \subset S$. Since $r^d(\{j\}) = 0$ for all $j \in S \setminus \sigma^d(S)$, it holds that $r^d(S) = r^d(\sigma^d(S)) + \sum_{j \in S \setminus \sigma^d(S)} r^d(\{j\})$, implying that S is not essential. □

Assume that (N, v, D) satisfies condition (7.3.5). Then it follows that the restricted game (N, r) is a weak monotone ($r^d(S) \leq r^d(N)$ for all $S \subseteq N$) veto-rich game (with veto player 1) and therefore the core contains the payoff vector $(r^d(N), 0, \ldots, 0)^\top \in \mathbb{R}^n$ and thus is not empty. (Observe that $r^d(N) = v(N)$.) Hence, $Nuc(N, r)$ is in the core of (N, r) and independent of inessential coalitions. From now on the following assumption will be made.

Assumption 7.4.2 *N is essential in the restricted game (N, r).*

In fact, when (N, v, D) is weak digraph monotone, this assumption is without loss of generality. If N is inessential then there exists a partition $\{S_1, \ldots, S_m\}$ such that (i) $r^d(N) \leq \sum_{j=1}^{m} r^d(S_j)$, (ii) S_1 is essential, and (iii) $1 \in S_1$. Because of (iii) we have that $S_2, \ldots, S_m$ are not feasible and thus $r^d(S_j) = 0$ for $j = 2, \ldots, m$. Together with weak digraph monotonicity this implies that $r^d(N) = r^d(S_1)$. So, according to Arin and Feltkamp (1997), the nucleolus assigns a zero payoff to every player not in S_1, and we can restrict ourselves to the subgame and subgraph on the essential coalition S_1 containing player 1. For N essential, also observe that according to Arin and Feltkamp (1997) the nucleolus assigns positive payoff to any player in N. Since the assumption that N is essential in the game (N, r) implies that $r^d(N) > r^d(S)$ for every $S \subset N$, we have the following lemma.

Lemma 7.4.3 *If a game with permission structure (N, v, D) with $D \in \mathcal{D}^N$ satisfies condition (7.3.5), then $e^*(N, r) > 0$.*

Proof. Since $C(N, r) \neq \emptyset$ we have that $e^*(N, r) \geq 0$. Hence, according to Lemma 7.2.3 it holds that

$$e^*(N, r) = \min_{B \in \mathcal{B}} \frac{r^d(N) - \sum_{j=1}^{m} \lambda_j^B r^d(S_j)}{\sum_{j=1}^{m} \lambda_j^B},$$

with λ_j^B, $j = 1, \dots, m$, the solution of the system (2.1.1) for the balanced collection B. Since $r^d(S_j) = 0$ when $1 \notin S_j$, we obtain that

$$e^*(N, r) = \min_{B \in \mathcal{B}} \frac{r^d(N) - \sum_{\{j | 1 \in S_j\}} \lambda_j^B r^d(S_j)}{\sum_{j=1}^{m} \lambda_j^B}.$$

Since the collection $\{N\}$ does not belong to $\mathcal{B}$, any S_j in a balanced collection B is a real subset of N and thus $r^d(S_j) < r^d(N)$ for any S_j, because N is essential. Since $\sum_{\{j|1 \in S_j\}} \lambda_j^B = 1$ by the definition of balancedness, it follows that $r^d(N) - \sum_{\{j|1 \in S_j\}} \lambda_j^B r^d(S_j) > 0$ for any $B \in \mathcal{B}$, which proves the lemma. □

Similar as in Arin and Feltkamp (1997), in the sequel we denote for $S \subset N$ and the restricted game (N, r),

$$\tau(S, r) = \frac{r^d(N) - r^d(S)}{|N \setminus S| + 1}.$$

In the following, $\Omega^D = \Phi_D^d \setminus \{N\}$ denotes the collection of all feasible coalitions not equal to N. We now have the following lemmas.

Lemma 7.4.4 *Let a game with permission structure (N, v, D) satisfy condition (7.3.5). Then*

$$e^*(N, r) = \min_{S \in \Omega^D} \tau(S, r).$$

Proof. According to Kohlberg's theorem there exists a balanced collection $\{S_1, \dots, S_m\}$ such that $-e(S_k, x) = e^*(N, r)$ for all $k = 1, \dots, m$. Since $e^*(N, r) > 0$ by Lemma 7.4.3, according to Corollary 7.2.5 we have that any S_j is essential. Without loss of generality, let $1 \in S_1$. Then we have that either $S_1 = \{1\}$ and thus feasible, or $|S_1| > 1$ and thus feasible according to Lemma 7.4.1. Denote $U = S_1$. Now, consider $j \notin U$. Since the collection is balanced, there must be a coalition $S_k \neq S_1 = U$ containing j, but not 1. Then S_k is essential, but not feasible. Hence it follows with Lemma 7.4.1 that $|S_k| = 1$ and thus $S_k = \{j\}$. Now, let λ_U^B and λ_j^B, $j \notin U$, be the corresponding weights. Then $\lambda_U^B = \lambda_j^B = 1$, $j \notin U$. Further $r^d(\{j\}) = 0$ for all $j \notin U$ since $\{j\}$ is not feasible. Substituting these values in (7.2.3) gives $e^*(N,r) = \frac{r^d(N)-\lambda_U^B r^d(U)}{|N\setminus U|+1} = \tau(U,r)$, showing that there exists a coalition $U \in \Omega^D$ satisfying $e^*(N,r) = \tau(U,r)$. Next, consider any $S \in \Omega^D$. Then $B = \{S\} \cup \{\{j\} \mid j \notin S\}$ is a balanced collection with corresponding weights $\lambda_S^B = \lambda_j^B = 1$, $j \notin S$. Since $1 \in S$ (because S is feasible), it follows that $r^d(\{j\}) = 0$ for all $j \notin S$. Hence with Lemma 7.2.3 we obtain that $e^*(N,r) \leq \frac{r^d(N)-\lambda_S^B r^d(S) - \sum_{j\notin S} \lambda_j^B r^d(j)}{|N\setminus S|+1} = \frac{r^d(N)-r^d(S)}{|N\setminus S|+1} = \tau(S,r)$. □

Lemma 7.4.5 *Let a game with permission structure (N, v, D) satisfy condition (7.3.5), let $U \in \Omega^D$ be such that $\tau(U,r) = e^*(N,r)$, and let $y \in \mathbb{R}^n$ be such that $y(U) = r^d(U) + \tau(U,r)$ and $y_j = \tau(U,r)$ for all $j \notin U$. Then $x = Nuc(N,r)$ satisfies $x(U) = y(U)$ and $x_j = y_j$ for all $j \notin U$.*

Proof. First, observe that

$$y(N) = y(U) + \sum_{j \notin U} y_j = r^d(U) + (|N \setminus U| + 1)\tau(U,r) = r^d(N),$$

so y is efficient. Next, observe that U is feasible and thus $1 \in U$. Hence for any $j \notin U$, the singleton coalition $\{j\}$ is not feasible and thus $r^d(\{j\}) = 0$. Therefore the excesses for the coalitions $U \in \Phi_D^d$ and the singletons $\{j\}$, $j \notin U$, at y are equal to $e(U,y) = -\tau(U,r) = e(\{j\},y)$, $j \notin U$. Now, suppose that $x = Nuc(N,r)$ does not satisfy $x(U) = y(U)$ and $x_j = y_j$. Then

$$\min[-e(U,x),\ \min_{j \notin U} -e(\{j\},x)] < \tau(U,r),$$

contradicting that $\tau(U,r) = e^*(N,r) = \min_{\{S \subset N, S \neq \emptyset\}} \ -e(S,x)$. □

The two lemmas above show that as soon as a coalition $U \in \Omega^D$ has been found with $\tau(U,r) = \min_{S\in\Omega^D} \ \tau(S,r)$, the nucleolus values of all players $j \notin U$ have been found and that these values are equal to $\tau(U,r)$. This gives us the basic idea for the algorithm in the next section. In the sequel, denote $\tau^*(r) = \min_{S\in\Omega^D} \ \tau(S,r)$. In the first step the algorithm searches for a coalition $U_1 \in \Omega^D$ satisfying

$$\tau(U_1,r) = \tau^*(r) \text{ and } |U_1| = \max_{\{U\in\Omega^D | \tau(U,r)=\tau^*(r)\}} |U|, \tag{7.4.8}$$

i.e., any other feasible set $U \neq N$ satisfying $\tau(U,r) = \tau^*(r)$ contains at most the same number of players as U_1. This gives nucleolus payoffs $\tau^*(r) = \tau(U_1,r)$ to any player $j \notin U_1$ and in the next step the algorithm continues with a search on a reduced set of players U_1. The details of the algorithm will be given in the next section. In the remaining of this section we give several results with respect to a set U_1 satisfying condition (7.4.8). These results will be used in Section 7.5 to prove that the algorithm indeed finds the nucleolus. Observe that the results above only require weak digraph monotonicity. The next results require both weak digraph monotonicity and weak digraph concavity.

Lemma 7.4.6 *Let a game with permission structure* (N, v, D) *satisfy conditions (7.3.5) and (7.3.6) and, for a coalition* U_1 *satisfying condition (7.4.8), let* $\{T_1,\ T_2\}$ *be a partition of* $N \setminus U_1$*. Then at least one of the two coalitions* $U_1 \cup T_1$*,* $U_1 \cup T_2$ *is not feasible.*

Proof. Suppose that both sets $U_1 \cup T_1$ and $U_1 \cup T_2$ are feasible. Then we have that

$$\frac{|T_2|+1}{|T_1|+|T_2|+2}\tau(U_1 \cup T_1, r) + \frac{|T_1|+1}{|T_1|+|T_2|+2}\tau(U_1 \cup T_2, r) =$$

$$= \frac{r^d(N) - r^d(N \setminus T_2)}{|T_1|+|T_2|+2} + \frac{r^d(N) - r^d(N \setminus T_1)}{|T_1|+|T_2|+2} =$$

$$= \frac{2r^d(N) - r^d(N \setminus T_1) - r^d(N \setminus T_2)}{|T_1|+|T_2|+2} \leq \frac{r^d(N) - r^d(U_1)}{|T_1|+|T_2|+2},$$

where the last inequality follows from condition (7.3.6) for the sets $N \setminus T_j$, $j = 1, 2$, since $r^d(N) = v(N)$, $r^d(U_1) = v(U_1)$ by the feasibility of U_1, and for $i \in \{1, 2\}$, $i \neq j$, it follows that $r^d(N \setminus T_j) = r^d(U_1 \cup T_i) = v(U_1 \cup T_i)$ because of the feasibility of $U_1 \cup T_i$. Further since $r^d(U_1) = v(U_1) \leq v(N) = r^d(N)$ because of condition (7.3.5), it follows that

$$\frac{r^d(N) - r^d(U_1)}{|T_1|+|T_2|+2} \leq \frac{r^d(N) - r^d(U_1)}{|T_1|+|T_2|+1} = \tau(U_1, r).$$

So, $\tau(U_1, r)$ is at least equal to the given convex combination of $\tau(U_1 \cup T_1, r)$ and $\tau(U_1 \cup T_2, r)$, implying that for at least one i, $i = 1, 2$, it holds that

$$\tau(U_1 \cup T_i, r) \leq \tau(U_1, r).$$

This contradicts condition (7.4.8). □

The next proposition says that for a set U_1 satisfying condition (7.4.8) the complement $N \setminus U_1$ is connected and that the collection of all successors of players in U_1 contains precisely one player not in U_1. For $T \subseteq N$, let $S_D(T) = \cup_{i \in T}\ S_D(i)$ denote the set of successors of all players of T in the digraph (N, D).

Proposition 7.4.7 *Let a game with permission structure* (N, v, D) *satisfy conditions (7.3.5) and (7.3.6) and let* U_1 *be a coalition satisfying condition (7.4.8). Then:*

1. *The set* $N \setminus U_1$ *is connected,*
2. $|S_D(U_1) \cap (N \setminus U_1)| = 1$*.*

Proof. 1. To prove 1, suppose $N \setminus U_1$ consists of at least two components. Let T_1 be one of the components and denote $T_2 = N \setminus (U_1 \cup T_1)$. The fact that both $U_1 \cup T_i$, $i = 1, 2$, are feasible will be shown.

To do so, let i be any player in T_1. By quasi-strongly connectedness of (N, D), there exists a directed path $(i_1, i_2, \ldots, i_m)$ from $i_1 = 1$ to $i_m = i$. Let i_k, $1 \leq k < m$, be the last player in the path not in T_1, thus $i_k \in U_1 \cup T_2$ and $i_{k+1}, \ldots, i_m \in T_1$. Since $(i_k, i_{k+1}) \in D$, $i_k \in T_2$ contradicts that T_1 is a component of $N \setminus U_1$. Hence $i_k \in U_1$. Since U_1 is feasible, $1 \in U_1$ and there is a path $(j_1, \ldots, j_\ell)$ from $j_1 = 1$ to $j_\ell = i_k$ with $j_r \in U_1$ for all $r = 1, \ldots, \ell$. Hence for any $i \in T_1$ there is a path $(j_1, \ldots, j_\ell, i_{k+1}, \ldots, i_m)$ from 1 to i only containing nodes in $U_1 \cup T_1$. This shows that $U_1 \cup T_1$ is feasible. Similarly it follows that $U_1 \cup T_2$ is feasible. This contradicts Lemma 7.4.6, which proves the first statement.

2. To prove 2, assume that there are two players $i_1, i_2 \in S_D(U_1) \cap (N \setminus U_1)$, $i_1 \neq i_2$. For any player $i \in N \setminus U_1$, let $\widetilde{S}_D(i)$ be defined as the subset of $N \setminus U_1$ such that node $j \in N \setminus U_1$ belongs to $\widetilde{S}_D(i)$ if and only if $j = i$ or there is a directed path from node i to node j that only consists of nodes in $N \setminus U_1$. Since (N, D) is acyclic by assumption, it follows that $i_1 \notin \widetilde{S}_D(i_2)$ or $i_2 \notin \widetilde{S}_D(i_1)$ (or both). Suppose $i_2 \notin \widetilde{S}_D(i_1)$. Now the partition of $N \setminus U_1$ into two non-empty sets $T_1 = \widetilde{S}_D(i_1)$ and $T_2 = (N \setminus U_1) \setminus T_1$ will be considered and a contradiction by using Lemma 7.4.6 will be obtained. Since there is a directed path from node 1 to $i_1 \in T_1$ consisting of nodes in $U_1 \cup \{i_1\}$, and from $i_1 \in T_1$ to any other node in T_1 consisting of nodes in T_1, for each $j \in U_1 \cup T_1$ there is a path in $U_1 \cup T_1$ from 1 to j, and thus $U_1 \cup T_1$ is feasible.

Next consider $U_1 \cup T_2$. For a node $j \in T_2$, let $(i_1, i_2, \ldots, i_m)$ be a path from $i_1 = 1$ to $i_m = j$ and let i_k, $1 \leq k < m$, be the last player in the path not in T_2, thus $i_k \in N \setminus T_2 = U_1 \cup T_1$. Then $i_k \in U_1$, because $i_k \in T_1 = \widetilde{S}_D(i_1)$ contradicts that $j \notin T_1$. Since U_1 is feasible, there is a path $(j_1, \ldots, j_\ell)$ from $j_1 = 1$ to $j_\ell = i_k$ with $j_r \in U_1$ for all $r = 1, \ldots, \ell$. Hence for any $j \in T_2$ there is a path $(j_1, \ldots, j_\ell, i_{k+1}, \ldots, i_m)$ from 1 to j only containing nodes in $U_1 \cup T_2$. This shows that $U_1 \cup T_2$ is feasible. Hence the existence of two players in $S_D(U_1) \cap (N \setminus U_1)$ contradicts Lemma 7.4.6, which proves the second statement. □

For U_1 satisfying condition (7.4.8), let i_1 be the unique node in $S_D(U_1) \cap (N \setminus U_1)$ i.e., i_1 is the unique successor of U_1 in $N \setminus U_1$. Since $1 \in U_1$, this implies that any path from node 1 to a player $j \in N \setminus U_1$ has node i_1 as the first player on the path not in U_1. Together with the connectedness of $N \setminus U_1$ (see Proposition 7.4.7) this gives the following corollary.

Corollary 7.4.8 *Let a game with permission structure (N, v, D) satisfy conditions (7.3.5) and (7.3.6) and let U_1 be a coalition satisfying condition (7.4.8). Then the subgraph $(N \setminus U_1, D(N \setminus U_1))$ of (N, D) on $N \setminus U_1$ is also a quasi-strongly connected, acyclic directed graph with one top-node (node i_1).*

7.5 An algorithm for computing the nucleolus

Since disjunctive restricted games are veto-rich games the exponential time algorithm of Arin and Feltkamp (1997) can be applied to find the nucleolus of the restricted game (N, r) of a game with permission structure (N, v, D) that satisfies conditions (7.3.5) and (7.3.6).

However, instead of directly applying the algorithm of Arin and Feltkamp to (N, r), in this section a modified version will be given of the algorithm which finds the nucleolus in polynomial time by making use of the hierarchical structure given by the digraph (N, D). In particular the hierarchical structure reduces at each step t the minimization problem to find q_t (see (7.2.4)), because it is sufficient to consider the feasible sets.

Let node 1 be the unique top node in (N, D), and thus 1 is a veto player in the restricted game (N, r). We again assume that (N, v, D) satisfies the conditions (7.3.5) and (7.3.6) and that N is essential in (N, r). Further for the reduced game with permission structure (U_k, v_k, D_k) defined in iteration $k-1$ at Step 3 of the algorithm given below, the set Ω^{D_k} denotes the set of all feasible coalitions not equal to U_k in the digraph (U_k, D_k). Also, for $i \in U_k$, we denote by $S_{D_k}(i)$ and $P_{D_k}(i)$ the set of successors and predecessors in (U_k, D_k) respectively. Then the algorithm proceeds as follows.

Algorithm

Step 1 Set $k = 0$, $U_0 = N$, $v_0 = v$, $D_0 = D$ and $r_0 = r$. Go to Step 2.

Step 2 Find $U_{k+1} \subset U_k$ satisfying condition (7.4.8) with respect to the game with permission structure (U_k, v_k, D_k), i.e.,

$$\tau(U_{k+1}, r_k) = \tau^*(r_k) \text{ and } |U_{k+1}| = \max_{\{U \in \Omega^{D_k} | \tau(U, r_k) = \tau^*(r_k)\}} |U|,$$

where $\tau^*(r_k) = \min_{U \in \Omega^{D_k}} \tau(U, r_k)$ with $\tau(U, r_k) = \frac{r_k(U_k) - r_k(U)}{|U_k \setminus U| + 1}$. Assign $y_j = \tau^*(r_k)$ to every player $j \in U_k \setminus U_{k+1}$. Go to Step 3.

Step 3 If $U_{k+1} = \{1\}$ then Go to Step 4. If $U_{k+1} \neq \{1\}$, let i_{k+1} be the unique top-player of the subgraph $(U_k \setminus U_{k+1}, D_k(U_k \setminus U_{k+1}))$ of the digraph (U_k, D_k) restricted to $U_k \setminus U_{k+1}$. Define the game (U_{k+1}, v_{k+1}) by setting for every $U \subseteq U_{k+1}$,

$$v_{k+1}(U) = \begin{cases} v_k(U) & \text{if } P_{D_k}(i_{k+1}) \cap U = \emptyset \\ v_k(U \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| & \text{else,} \end{cases} \tag{7.5.9}$$

digraph (U_{k+1}, D_{k+1}) given by

$$(i,j) \in D_{k+1} \text{ if } \begin{cases} (i,j) \in D_k \text{ or} \\ i \in P_{D_k}(i_{k+1}) \text{ and } j \in S_{D_k}(U_k \setminus U_{k+1}) \cap U_{k+1} \end{cases} \tag{7.5.10}$$

and let r_{k+1} be the restricted game of $(U_{k+1}, v_{k+1}, D_{k+1})$. Set $k = k + 1$. Goto Step 2.

Step 4 Assign $y_1 = v(N) - \sum_{j \in N \setminus \{1\}} x_j$. Stop.

In every step of the algorithm, for $U_{k+1} \subset U_k$ satisfying condition (7.4.8) with respect to (U_k, v_k, D_k), any player in $U_k \setminus U_{k+1}$ receives payoff $\tau(U_{k+1}, r_k)$. Observe that at any iteration the new found set U_{k+1} is essential in (U_{k+1}, r_{k+1}). If not, there exists an essential subset S of U_{k+1} with $r_{k+1}(S) = r_{k+1}(U_k)$, yielding payoff $y_j = 0$ for all $j \in U_{k+1} \setminus S$. This contradicts that all players get positive payoff (because it is assumed that N is essential). Since in any iteration the payoff of at least one player is determined, in at most $n-1$ iterations the algorithm stops with $U_{k+1} = \{1\}$ and player 1 getting what is left from $v(N)$ after all other players received their payoffs as determined by the algorithm. (Note that player 1 belongs to the player set of every game (U_k, D_k) that appears in the algorithm.) In the remaining of this section we show that the algorithm indeed yields the nucleolus.

Let K be such that $U_{K+1} = \{1\}$. To show that the algorithm is well-defined, it is needed that the results of Section 7.4 hold for every game (U_k, r_k), $k = 1, \ldots, K$. This is shown in the next two lemmas. The first lemma states that, for any $k = 0, 1, \ldots, K-1$, the digraph (U_{k+1}, D_{k+1}) is acyclic and quasi-strongly connected with $i = 1$ as its unique top-node.

Lemma 7.5.1 *The digraph (U_{k+1}, D_{k+1}) satisfies Assumption 7.2.1 for any $k = 0, 1, \ldots, K-1$.*

Proof. Since (N, D) satisfies Assumption 7.2.1, the statement is true for $k = 0$. We now proceed by induction and suppose that the statement is true for $j = 0, \ldots, k$, $k < K-1$. Then it remains to show that the statement is true for $j = k+1$. By the induction hypothesis it follows that (U_k, D_k) is acyclic and quasi-strongly connected and has $i = 1$ as its unique top node. So, for any $j \neq 1$ in U_{k+1} there is a directed path $i_1, \ldots, i_m$ in (U_k, D_k) with $i_1 = 1$ and $i_m = j$. If any node i_k, $k = 2, \ldots, m-1$, in this path is in U_{k+1}, then this path also exists in (U_{k+1}, D_{k+1}). Otherwise, for any node i_h on the path not in U_{k+1}, there exist two (not necessarily different) nodes i_r, i_s on the path with $r \leq h \leq s$ such that $i_{r-1}, i_{s+1} \in U_{k+1}$ and $i_r, i_s \notin U_{k+1}$. Then by (7.5.10) it follows that $(i_{r-1}, i_{s+1}) \in D_{k+1}$. Hence there is a directed path from $i = 1$ to $i = j$ in (U_{k+1}, D_{k+1}), showing (U_{k+1}, D_{k+1}) is quasi-strongly connected with node 1 as top node. Because in (U_{k+1}, D_{k+1}) there can only be a directed path from node i to node j if there is a directed path from i to j in (U_k, D_k), the acyclicity of (U_{k+1}, D_{k+1}) follows immediately from the fact that (U_k, D_k) is acyclic. □

The next lemma shows that any game (U_k, v_k, D_k), $k = 0, 1, \ldots, K$, satisfies the conditions of weak digraph monotonicity and weak digraph concavity. Again the proof is by induction, where Proposition 7.4.7 is used to show the weak digraph monotonicity.

Lemma 7.5.2 *Let a game with permission structure (N, v, D) satisfy conditions (7.3.5) and (7.3.6). Then the game with permission structure (U_k, v_k, D_k) satisfies these conditions on the player set U_k for every $k = 0, \ldots, K$.*

Proof. The proposition will be proved by induction on k. For $k = 0$ both conditions (7.3.5) and (7.3.6) are satisfied by assumption. Proceeding by induction, assume that these conditions are satisfied for $j = 0, \ldots, k$, $k < K - 1$. By Lemma 7.5.1 the digraph (U_k, D_k) satisfies Assumption 7.2.1. So, the game (U_k, v_k, D_k) satisfies all conditions of Proposition 7.4.7.

To show that condition (7.3.5) holds for $(U_{k+1}, v_{k+1}, D_{k+1})$, it is sufficient to show that $[U \subseteq U_{k+1}$ and U feasible in $(U_{k+1}, D_{k+1})] \Rightarrow v_{k+1}(U) \leq v_{k+1}(U_{k+1})$. Since $P_{D_k}(i_{k+1}) \cap U_{k+1} \neq \emptyset$, it follows that

$$v_{k+1}(U_{k+1}) = v_k(U_{k+1} \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}|.$$

Next, let $U \subseteq U_{k+1}$ be a feasible subset of U_{k+1} in (U_{k+1}, D_{k+1}). Two cases will be considered, either $P_{D_k}(i_{k+1}) \cap U \neq \emptyset$ or $P_{D_k}(i_{k+1}) \cap U = \emptyset$. In the latter case it follows that (i) $v_{k+1}(U) = v_k(U)$ and (ii) there is an arc between two nodes i and j of U in the digraph (U_{k+1}, D_{k+1}) if and only if there is also an arc between i and j in (U_k, D_k). Hence, U is also feasible in (U_k, D_k) and thus $v_{k+1}(U) = v_k(U) = r_k(U)$. Moreover, $\tau(U, r_k) = \frac{r_k(U_k) - r_k(U)}{|U_k \setminus U| + 1} \geq \tau(U_{k+1}, r_k)$ and thus $r_k(U_k) - r_k(U) \geq (|U_k \setminus U| + 1)\tau(U_{k+1}, r_k)$. Hence

$$\begin{aligned} v_{k+1}(U) &= r_k(U) \leq r_k(U_k) - (|U_k \setminus U| + 1)\tau(U_{k+1}, r_k) \\ &< v_k(U_k) - |U_k \setminus U_{k+1}|\tau(U_{k+1}, r_k) = v_{k+1}(U_{k+1}). \end{aligned}$$

In case $P_{D_k}(i_{k+1}) \cap U \neq \emptyset$, we obtain from applying Proposition 7.4.7 to (U_k, v_k, D_k), that $U \cup (U_k \setminus U_{k+1})$ is feasible in (U_k, D_k). From this it follows that

$$\begin{aligned} v_{k+1}(U) &= v_k(U \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| \\ &\leq v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = v_{k+1}(U_{k+1}) \end{aligned}$$

because condition (7.3.5) holds for (U_k, v_k, D_k).

Next condition (7.3.6) will be considered, i.e., we have to show that $[S \cup T = U_{k+1}$ and S, T feasible in $(U_{k+1}, D_{k+1})] \Rightarrow v_{k+1}(S) + v_{k+1}(T) \geq v_{k+1}(S \cap T) + v_{k+1}(U_{k+1})$. Since $S \cup T = U_{k+1}$ it follows that $P_{D_k}(i_{k+1}) \cap S \neq \emptyset$ or $P_{D_k}(i_{k+1}) \cap T \neq \emptyset$ (or both). First the case that both intersections are nonempty and thus also $P_{D_k}(i_{k+1}) \cap (S \cap T) \neq \emptyset$ will be considered. Then $S' = S \cup (U_k \setminus U_{k+1})$, $T' = T \cup (U_k \setminus U_{k+1})$ are feasible in (U_k, D_k) and $S' \cup T' = U_k$, and thus it follows from condition (7.3.6) for (U_k, v_k, D_k) that

$$\begin{aligned} v_{k+1}(S) + v_{k+1}(T) &= v_k(S') + v_k(T') - 2\tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| \geq \\ &\geq v_k(S' \cap T') + v_k(U_k) - 2\tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = \\ &= v_k((S \cap T) \cup (U_k \setminus U_{k+1})) + v_k(U_k) - \\ &\quad -2\tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = \\ &= v_{k+1}(S \cap T) + v_{k+1}(U_{k+1}), \end{aligned}$$

where the last equality follows from the fact that $v_{k+1}(S \cap T) = v_k((S \cap T) \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}|$ and $v_{k+1}(U_{k+1}) = v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}|$. In case one of the sets S and T has a nonempty intersection with $P_{D_k}(i_{k+1})$ and thus $P_{D_k}(i_{k+1}) \cap (S \cap T) = \emptyset$, suppose without loss of generality that $T \cap P_{D_k}(i_{k+1}) = \emptyset$. Then $S' = S \cup (U_k \setminus U_{k+1})$ and T are feasible in (U_k, D_k), $S' \cup T = U_k$ and thus it follows from condition (7.3.6) for (U_k, v_k, D_k) that

$$\begin{aligned} v_{k+1}(S) + v_{k+1}(T) &= v_k(S') + v_k(T) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| \geq \\ &\geq v_k((S' \cap T)) + v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = \\ &= v_{k+1}(S \cap T) + v_{k+1}(U_{k+1}), \end{aligned}$$

where the last equality follows from the fact that $v_k(S' \cap T) = v_k(S \cap T) = v_{k+1}(S \cap T)$ and $v_k(U_k) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| = v_{k+1}(U_{k+1})$. □

In the remaining of this section the fact that, for $k = 1, \ldots, K$, the game (U_{k+1}, r_{k+1}) is the Davis-Maschler reduced game of the game (U_k, r_k) with respect to the nucleolus will be shown. For a game (N, v), let $T \subset N$ be a nonempty coalition and $y \in \mathbb{R}^n$ a payoff vector. Then (see also the preliminary chapter) the Davis-Maschler reduced game on T at y is the game (T, v_T^y) given by $v_T^y(T) = v(N) - x(N \setminus T)$ and $v_T^y(S) = \max_{Q \subseteq N \setminus T} \ (v(S \cup Q) - y(Q))$, $S \subset T$, $S \neq T$. Observe that in the definition of the reduced game only the values y_j of the players $j \in N \setminus T$ appear. In the following, let (U_{k+1}, r'_k) denote the Davis-Maschler reduced game of the game (U_k, r_k) on the set U_{k+1} at y with $y_j = \tau^*(r_k) = \tau(U_{k+1}, r_k)$ for $j \in U_k \setminus U_{k+1}$.

We first show the following lemma on the largest disjunctive feasible subset of a coalition U in the digraph (U_k, D_k). In the sequel we denote this set by $\sigma_k^d(U)$. Observe that for $U \subseteq N$ it follows that $\sigma_0^d(U) = \sigma^d(U)$.

Lemma 7.5.3 *For the game with permission structure (U_k, v_k, D_k), let $U_{k+1} \subset U_k$ and $i_{k+1} \notin U_{k+1}$ be the set and node as obtained in iteration k of the algorithm for $k = 0, \ldots, K-1$. Then for each $U \subseteq U_{k+1}$ it follows that*

1. $\sigma_{k+1}^d(U) = \sigma_k^d(U)$ *if* $S_{D_k}(\sigma_k^d(U)) \subset U_{k+1}$*;*
2. $\sigma_{k+1}^d(U) = \sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1})$ *if* $i_{k+1} \in S_{D_k}(\sigma_k^d(U))$*.*

Proof. 1. Consider $U \subseteq U_{k+1}$ with $S_{D_k}(\sigma_k^d(U)) \subset U_{k+1}$. Clearly, then $\sigma_k^d(U)$ is feasible in (U_{k+1}, D_{k+1}) and thus $\sigma_k^d(U) \subseteq \sigma_{k+1}^d(U)$. Next, suppose that there exists some player $i \in \sigma_{k+1}^d(U) \setminus \sigma_k^d(U)$. Then there is a path $(a_0, a_1, \ldots, a_l)$ such that (i) $a_0 = 1$, (ii) $a_l = i$, (iii) $a_t \in U$ for all $t = 1, \ldots, l-1$, and (iv) $(a_t, a_{t+1}) \in D_{k+1}$ for all $t = 0, \ldots, l-1$. If $(a_t, a_{t+1}) \in D_k$ for all $t = 0, \ldots, l-1$, then $i \in \sigma_k^d(U)$ and there is a contradiction with our assumption that $i \in \sigma_{k+1}^d(U) \setminus \sigma_k^d(U)$. So, there must exist a $t \in \{0, \ldots, l-1\}$ such that $(a_t, a_{t+1}) \notin D_k$. By definition of digraph D_{k+1} it holds that $a_t \in P_{D_k}(i_{k+1})$, which contradicts $S_{D_k}(\sigma_k^d(U)) \subset U_{k+1}$. Hence $\sigma_{k+1}^d(U) = \sigma_k^d(U)$.

2. Consider $U \subseteq U_{k+1}$ with $i_{k+1} \in S_{D_k}(\sigma_k^d(U))$. If there is a player $i \in \sigma_{k+1}^d(U)$ then there is a path $(a_0, a_1, \ldots, a_l)$ such that (i) $a_0 = 1$, (ii) $a_l = i$, (iii) $a_t \in U$ for all $t = 1, \ldots, l-1$,

and (iv) $(a_t, a_{t+1}) \in D_{k+1}$ for all $t = 0, \ldots, l-1$. We show that these four conditions also describe all elements of

$$\sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1}).$$

If $(a_t, a_{t+1}) \in D_k$ for all $t = 0, \ldots, l-1$, then $i \in \sigma_k^d(U)$. Since $U \subseteq U_{k+1}$, it follows that $i \in \sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1})$. Otherwise, if $(a_t, a_{t+1}) \in D_{k+1} \setminus D_k$ for some t, then $a_t \in P_{D_k}(i_{k+1})$ and $a_{t+1} \in S_{D_k}(U_k \setminus U_{k+1})$. So there is a path from a_t to a_{t+1} which contains only elements from $U_k \setminus U_{k+1}$. In the path $(a_0, a_1, \ldots, a_l)$, replace the arc (a_t, a_{t+1}) by this path from a_t to a_{t+1}. Continuing in this way, it is possible to change each arc in the path $(a_0, a_1, \ldots, a_l)$ that belongs to $D_{k+1} \setminus D_k$ by a path which consists only of elements from $U_k \setminus U_{k+1}$. So, we have a path from 1 to i which consists only of elements from $U \cup (U_k \setminus U_{k+1})$, implying that $i \in \sigma_k^d(U \cup (U_k \setminus U_{k+1}))$. Since $i \notin U_k \setminus U_{k+1}$, we conclude that $i \in \sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1})$. So, in both cases it follows that $i \in \sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1})$ and therefore

$$\sigma_{k+1}^d(U) = \sigma_k^d(U \cup (U_k \setminus U_{k+1})) \setminus (U_k \setminus U_{k+1}).$$

□

The next lemma shows that the game (U_{k+1}, r_{k+1}) is the Davis-Maschler reduced game of the game (U_k, r_k) with respect to the nucleolus.

Lemma 7.5.4 *Let a game with permission structure (N, v, D) satisfy conditions (7.3.5) and (7.3.6). Then, for $k = 0, \ldots, K-1$, the game (U_{k+1}, r_{k+1}) is equal to the Davis-Maschler reduced game (U_{k+1}, r'_k) of the game (U_k, r_k) on U_{k+1} at y with $y_j = \tau^*(r_k)$ for $j \in U_k \setminus U_{k+1}$.*

Proof. For coalition $T \subseteq U_{k+1}$, two cases will be considered, namely whether or not $S_{D_k}(\sigma_k^d(T)) \subset U_{k+1}$. In case $S_{D_k}(\sigma_k^d(T)) \subset U_{k+1}$, Lemma 7.5.3 implies that $\sigma_{k+1}^d(T) = \sigma_k^d(T)$.
Further, since $P_{D_k}(i_{k+1}) \cap \sigma_k^d(T) = \emptyset$ it follows by equation (7.5.9) in Step 3 of the algorithm that $v_{k+1}(T) = v_k(T)$ and thus $r_{k+1}(T) = r_k(T)$ because $\sigma_{k+1}^d(T) = \sigma_k^d(T)$. On the other hand, for the Davis-Mashler reduced game (U_{k+1}, r'_k) it holds for any $T \subset U_{k+1}$ that

$$r'_k(T) = \max_{Q \subseteq U_k \setminus U_{k+1}} (r_k(T \cup Q) - y(Q)) = r_k(T),$$

because for any $Q \subseteq U_k \setminus U_{k+1}$ it follows that

$$r_k(T \cup Q) = v_k(\sigma_k^d(T \cup Q)) = v_k(\sigma_k^d(T)) = r_k(T),$$

where the second equality follows since for any pair $j \in (T \setminus \sigma_k^d(T) \cup Q)$ and $i \in \sigma_k^d(T)$, it holds that $(i, j) \notin D_k$ and thus $\sigma_k^d(T \cup Q) = \sigma_k^d(T)$. Hence $r'_k(T) = r_k(T) = r_{k+1}(T)$.

In case $S_{D_k}(\sigma_k^d(T))$ is not a subset of U_{k+1} it follows that $P_{D_k}(i_{k+1}) \cap \sigma_k^d(T) \neq \emptyset$, because i_{k+1} is the unique successor of U_{k+1} in $U_k \setminus U_{k+1}$. So, by equation (7.5.9) in Step 3 of the algorithm we have that

$$r_{k+1}(T) = v_{k+1}(\sigma_{k+1}^d(T)) = v_k(\sigma_{k+1}^d(T) \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}|.$$

From Lemma 7.5.3 it follows that $\sigma_{k+1}^d(T) \cup (U_k \setminus U_{k+1}) = \sigma_k^d(T \cup (U_k \setminus U_{k+1}))$ and so

$$r_{k+1}(T) = v_k(\sigma_k^d(T \cup (U_k \setminus U_{k+1}))) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}| =$$

$$r_k(T \cup (U_k \setminus U_{k+1})) - \tau(U_{k+1}, r_k)|U_k \setminus U_{k+1}|.$$

To show that $r_{k+1}(T) = r'_k(T)$ it remains to prove that the right-hand term in the equation

$$r'_k(T) = \max_{Q \subseteq U_k \setminus U_{k+1}} (r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|)$$

obtains its maximum for $U_k \setminus U_{k+1}$. To do so, denote $\overline{Q} = U_k \setminus U_{k+1}$, $V = T \cup \overline{Q}$ and, for $Q \subseteq \overline{Q}$, denote $W = U_{k+1} \cup Q$. Then (because of Lemma 7.5.3) the sets $\sigma_k^d(V) = \sigma_{k+1}^d(T) \cup \overline{Q}$ and $\sigma_k^d(W) = \sigma_k^d(U_{k+1} \cup Q) \supseteq U_{k+1}$ are feasible and satisfy $\sigma_k^d(V) \cup \sigma_k^d(W) = U_k$. By Lemma 7.5.2 the game with permission structure (U_k, v_k, D_k) satisfies weak digraph concavity and thus

$$r_k(V) + r_k(W) = v_k(\sigma_k^d(V)) + v_k(\sigma_k^d(W)) \geq v_k(U_k) + v_k(\sigma_k^d(V) \cap \sigma_k^d(W)) =$$

$$v_k(U_k) + v_k(\sigma_k^d(V \cap W)) = r_k(U_k) + r_k(V \cap W),$$

where the second equality follows from the fact that $\sigma_k^d(V \cap W) = \sigma_k^d(V) \cap \sigma_k^d(W)$ because of the graph structure. With $V \cap W = (T \cup \overline{Q}) \cap (U_{k+1} \cup Q) = T \cup Q$ this yields

$$r_k(T \cup \overline{Q}) - r_k(T \cup Q) \geq r_k(U_k) - r_k(U_{k+1} \cup Q) >$$

$$\frac{r_k(U_k) - r_k(U_{k+1} \cup Q)}{|\overline{Q}| - |Q| + 1}(|\overline{Q}| - |Q|) =$$

$$\tau(U_{k+1} \cup Q, r_k)(|\overline{Q}| - |Q|) \geq \tau(U_{k+1}, r_k)(|\overline{Q}| - |Q|)$$

by definition of U_{k+1}. Hence

$$r_k(T \cup \overline{Q}) - \tau(U_{k+1}, r_k)|\overline{Q}| \geq r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|,$$

for all $Q \subseteq \overline{Q}$, which shows that indeed

$$r_k(T \cup \overline{Q}) - \tau(U_{k+1}, r_k)|\overline{Q}| = \max_{Q \subseteq U_k \setminus U_{k+1}} (r_k(T \cup Q) - \tau(U_{k+1}, r_k)|Q|) .$$

□

We now have the following proposition.

Proposition 7.5.5 *Given a game with permission structure (N, v, D) satisfying the conditions (7.3.5) and (7.3.6), the algorithm described by the Steps 1-4 above yields the nucleolus of (N, r).*

Proof. In iteration $k = 0$ the algorithm assigns in Step 2 the value $\tau^*(r_0) = \tau(U_1, r_0) = \tau(U_1, r)$ to any player $j \in U_0 \setminus U_1 = N \setminus U_1$. According to Lemma 7.4.5, $\tau^*(r_0)$ is the nucleolus value of the players in $N \setminus U_1$. Applying Lemma 7.5.4 for $k = 0$, the game (U_1, r_1) is the Davis-Maschler reduced game of the game (N, r) with respect to the nucleolus values $y_j = \tau^*(r_0)$ of the players not in U_1. Since the nucleolus satisfies the Davis-Maschler reduced game consistency property, the nucleolus values of the reduced game (U_1, r_1) are equal to the nucleolus values of the players of U_1 in the game (N, r). In iteration $k = 1$ the algorithm assigns in Step 2 the value $\tau^*(r_1)$ to any player $j \in U_1 \setminus U_2$. According to Lemma 7.4.5, $\tau^*(r_1)$ is the nucleolus value of the players in $U_1 \setminus U_2$ in the game (U_1, r_1), and hence it is also the nucleolus value of these players in the game (N, r). Continuing this reasoning we have that in any iteration k, the algorithm assigns in Step 2 the value $\tau^*(r_k)$ to any player $j \in U_k \setminus U_{k+1}$, which is the nucleolus value of the players in $U_k \setminus U_{k+1}$ in the game (N, r). At the final iteration K we have that $U_{K+1} = \{1\}$ and player 1 gets its nucleolus value in Step 4 of the algorithm. □

7.6 Complexity of the algorithm

For arbitrary veto-rich games the algorithm of Arin and Feltkamp (1997) to compute the nucleolus is an exponential time algorithm of the order $\mathcal{O}(n.2^{n-1})$. Branzei *et al.* (2005) argue that applying the algorithm to the specific case of a peer group game the complexity reduces to a polynomial time algorithm of order $\mathcal{O}(n^3)$. They show that the algorithm given in their paper to find the nucleolus of a peer group game is a polynomial time algorithm of order $\mathcal{O}(n^2)$. In this section we show that the algorithm given in the previous section to find the nucleolus of the more general restricted game of a game with disjunctive permission structure is a polynomial time algorithm of order $\mathcal{O}(n^4)$. We first define the concept of a *good set* in a digraph.

Definition 7.6.1 *For a digraph (N, D) with $D \in \mathcal{D}^N$, a set $T \subset N$ is a good set, when*
(i) there is a unique top node in the subgraph $(T, D(T))$ of (N, D) and for any other node i in T there is a path from this unique top node to node i that only contains nodes in T,
(ii) the set $N \setminus T$ is connected, and
(iii) only the top node in $(T, D(T))$ has predecessors in $N \setminus T$.

We now have the following lemma.

Lemma 7.6.2 *In any iteration k of the algorithm, the set $U_k \setminus U_{k+1}$ is a good set.*

Proof. Applying Corollary 7.4.8 to (U_k, D_k) we have that the subgraph of (U_k, D_k) restricted to $U_k \setminus U_{k+1}$ is a connected, acyclic directed graph with one top node, so condition (i) holds. Next, denote $T_k = U_k \setminus U_{k+1}$. Then $U_k \setminus T_k = U_{k+1}$. Therefore condition (ii) holds, because U_{k+1} is feasible in (U_k, D_k) and thus connected in (U_k, D_k). Further, by applying the second statement of Proposition 7.4.7 to (U_k, D_k) we have that U_{k+1} has only one successor in $T_k = U_k \setminus U_{k+1}$. Let this only successor be node j in T_k. Since the digraph (U_k, D_k) is acyclic and quasi-strongly connected, there is a path from top node 1 in (U_k, D_k) to any other node in U_k, so also to any node in T_k. Since j is the only successor of U_{k+1} in T_k, any path from 1 to some node $h \in T_k$ must contain the node j. Moreover, the path from j to h can not contain nodes not in T_k, otherwise U_{k+1} has more than one successor in T_k. Hence, j is also a top node in T_k such that for any other node in T_k there is a path from j to this node that only contains nodes in T_k. □

Lemma 7.6.2 implies that in Step 2 of the algorithm the set U_{k+1} that we must find is such that its complement $U_k \setminus U_{k+1}$ is a good set. Conversely, when $\mathcal{T}_k$ is the collection of all good sets in (U_k, D_k), then the search for U_{k+1} can be restricted to sets in the collection $U_k \setminus T_k$, $T_k \in \mathcal{T}_k$. The next lemma says that in a game with permission structure (N, v, D) there is precisely one good set for any player $j \in N$. Applying this to (U_k, D_k) this means that at iteration k of the algorithm the number of good sets is equal to $|U_k|$. Observe that j itself is a singleton good set if j has no successors.

Lemma 7.6.3 *Let (N, D) be a digraph with $D \in \mathcal{D}^N$. Then for any node $j \in N$ there is exactly one good set T such that j is the unique top node in T.*

Proof. Recall from Section 2.3 that the set $\overline{S}_D(j)$ of all complete subordinates of j is the set of nodes i such that any path from top node 1 in (N, D) to node i contains node j. It is straightforward to verify that $\overline{S}_D(j)$ is a good set having node j as its unique top node. Next, suppose that there are two good sets with j as their unique top node, say T_1 and T_2 and, without loss of generality, suppose that $T_1 \setminus T_2$ is non-empty. Consider some node $h \in T_1 \setminus T_2$. By definition of a good set we know that any path from top player 1 to the player h contains the node j. However, $N \setminus T_2$ does not contain j and so there is no path from top node 1 to h in $N \setminus T_2$, contradicting condition (ii) of Definition 7.6.1. □

We are now ready to consider the complexity of the algorithm.

Proposition 7.6.4 *The complexity of the algorithm is of order $\mathcal{O}(n^4)$.*

Proof. First, in iteration k we have to find all good sets in U_k. To find the good set with some player j in U_k as its unique top node, delete player j from U_k. Then the good set consists of player j and all nodes in U_k that are no longer connected to player 1 when player j is deleted. Since U_k contains at most $n-1$ nodes not equal to 1, this requires at most $\mathcal{O}(n^2)$ actions to find the good set of node j. So, it requires at most $\mathcal{O}(n^3)$ actions to find all $n-1$ good sets of all players $j \neq 1$. Next, at each iteration k we need to calculate the number $\tau(U_k \setminus T, r_k)$ for any good set T. For this we need at most $O(n-1)m_k$ actions, where

m_k is the number of actions to find all values $v_k(U)$, $U \subseteq U_k$, in Step 3 of iteration $k-1$. Clearly $m_0 = 1$. Further, from equation (7.5.9) in Step 3 of the algorithm it follows that we need m_{k-1} actions to find $v_k(U)$ if $P_{D_k}(i_{k+1}) \cap U = \emptyset$. Otherwise m_{k-1} actions are needed to calculate $v_k(U) = v_{k-1}(U \cup (U_{k-1} \setminus U_k))$ and $\mathcal{O}(1)$ actions are needed for calculating $\tau(U_k, r_{k-1})|U_{k-1} \setminus U_k|$ and for substraction, because $\tau(U_k, r_{k-1})$ was already found before. Hence $m_k = m_{k-1} + \mathcal{O}(1)$. Together with $m_0 = 1$ this yields that $m_k \leq \mathcal{O}(n)$. Since the number of iterations is at most equal to n, it follows that the complexity of the algorithm is given by $n \cdot (\mathcal{O}(n^3) + \mathcal{O}((n-1)m_k)) = \mathcal{O}(n^4)$. □

Chapter 8

An algorithm for computing the nucleolus of disjunctive additive games with an acyclic permission structure

8.1 Introduction

This chapter, which is based on van den Brink, Katsev and van der Laan (2008b) also deals (like Chapter 7) with the nucleolus of the restricted game of a game with permission structure (N, v, D) under the disjunctive approach.
A polynomial time algorithm is provided to compute the nucleolus of the restricted game induced by situations in which the associated TU-game is additive, but in which we allow for any acyclic permission structure, so allowing for more than one top player. This considerably widens the applications, for example we can consider situations where sellers can sell objects to buyers through a directed network of intermediaries. Whereas quasi-strongly connected networks only can be used for situations with one seller, weakening this by only requiring that the network is acyclic, allows to study such situations with more than one seller. The algorithm presented in this chapter computes the nucleolus in polynomial time through a number of iterations. In each iteration a subgame with an acyclic, quasi-strongly connected permission structure is considered and the algorithm developed in the previous chapter is used to compute the nucleolus for this subgame.
This chapter is organized as follows. In Section 8.2 we provide the algorithm to find the nucleolus for disjunctive non-negative additive games with an acyclic permission structure. Section 8.3 shows that the algorithm indeed computes the nucleolus for this class of games. In Section 8.4 we discuss some properties of the algorithm, while Section 8.5 discusses its complexity, showing that it indeed finds the nucleolus in polynomial time. Section 8.6 illustrates the algorithm with an example of a market situation where sellers can sell objects to buyers through a directed network of intermediaries.

8.2 A polynomial time algorithm for the nucleolus based on quasi-strongly connectedness

In this part we consider a non-negative additive game with acyclic permission structure (N, v, D). Since $\sigma^d(S) = \cup\{T \in \Phi_D^d \mid T \subseteq S\} \subseteq S$ for any $S \subseteq N$, also in this case we have that $r^d(S) = v(\sigma^d(S)) \leq \sum_{i \in S} a_i$. Therefore the payoff vector $x \in \mathbb{R}_+^n$ given by $x_i = a_i$ for all $i \in N$ is in the core and thus the core is nonempty and contains the nucleolus. Since in this chapter we only consider the disjunctive approach we will denote the largest disjunctive feasible subset of coalition S simply by $\sigma(S)$, and the disjunctive restricted game by r.

We first show that for any non-negative additive game with acyclic permission structure (N, v, D), there exists a subset $K \subseteq N$ with the properties that (i) the subgraph $(K, D(K))$ is an acyclic, quasi-strongly connected permission structure and (ii) $\sum_{j \in K} x_j = \sum_{j \in K} a_j$ for any payoff vector $x \in \mathbb{R}^n$ in the core of the restricted game (N, r), so also for the nucleolus. These properties play an important role in the algorithm to compute the nucleolus of the restricted game obtained from a non-negative additive game with acyclic permission structure (N, v, D). In fact the nucleolus is obtained by computing the nucleolus of a sequence of smaller non-negative additive games with quasi-strongly connected permission structures. By definition the properties hold for $K = N$ when the graph has only one top player (see Chapter 7). So, in the sequel of this section we only consider the case that $|T_D| \geq 2$. The first lemma is obvious and stated without proof.

Lemma 8.2.1 *For every acyclic permission structure (N, D) it holds: j is a (complete) superior of i if and only if i is a complete subordinate of j.*

For a top player $t \in T_D$, let the set U^t be defined by $\overline{S}_D(t) \cup \{t\}$, i.e., U^t contains top player t together with all its complete subordinates. Observe that $T_D \setminus U^t \neq \emptyset$, because $|T_D| \geq 2$ and $(T_D \setminus \{t\}) \cap U^t = \emptyset$. Further, for $i \in \overline{S}_D(t)$, define

$$U_i = \cup \{H(p) \mid p \text{ is a directed path from } t \text{ to } i\},$$

i.e., the set $U_i \subset N$ is the union of all players on all directed paths from top player t to its complete subordinate i. Then we have the next lemma.

Lemma 8.2.2 *Let (N, D) be an acyclic permission structure, $t \in T_D$ a top player, and $i \in \overline{S}_D(t)$ a complete subordinate of t. Then $U_i \subseteq U^t$.*

Proof. By definition we have that $t \in U^t$. Since t is a complete superior of i, it follows that t is a complete superior of any $h \in U_i \setminus \{t\}$. Suppose not. Then for some $h \in U_i \setminus \{t\}$, there is a path from a top player $t' \neq t$ to h, and so also a path from t' to i, contradicting that t is a complete superior of i. By Lemma 8.2.1 we have that any h in U_i is a complete subordinate of t, and thus $h \in U^t$. □

Since $U_i \subseteq U^t$ for all $i \in U^t = \overline{S}_D(t) \cup \{t\}$ (see Lemma 8.2.2) it follows that

$$U^t = \begin{cases} \{t\} & \text{if } \overline{S}_D(t) = \emptyset, \\ \cup_{\{i \in \overline{S}_D(t)\}} U_i & \text{otherwise.} \end{cases}$$

So, U^t is the union of all sets U_i of the complete subordinates of top player t when t has at least one complete subordinate, and $U^t = \{t\}$ otherwise.
The next proposition shows the existence of a subset $K \subseteq N$ such that the subgraph $(K, D(K))$ is an acyclic, quasi-strongly connected permission structure.

Proposition 8.2.3 *Let (N, D) be an acyclic permission structure. Then for every $t \in T_D$ it holds that the subgraph $(U^t, D(U^t))$ is an acyclic, quasi-strongly connected permission structure with t its unique top node.*

Proof. First, when $\overline{S}_D(t) = \emptyset$, then $U^t = \{t\}$ and the statement is true. Otherwise, let i be a complete subordinate of t. Obviously t is the unique top-player in the subgraph $(U_i, D(U_i))$. Further, the subgraph $(U_i, D(U_i))$ is acyclic and quasi-strongly connected. Acyclicity follows from the acyclicity of D and the fact that $D(U_i) \subseteq D$. Quasi-strongly connectedness follows from acyclicity of D and the fact that U_i is the union of all directed paths from t to i. Then the result follows because $U^t = \cup_{\{i \in \overline{S}_D(t)\}} U_i$. □

The next lemma states that when N is partitioned in two disjunctive feasible sets, for both sets it holds that the total payoff of its players in every core payoff vector of the restricted game is equal to their own value.

Lemma 8.2.4 *Let $A, B \in \Phi_D^d$ be two disjunctive feasible coalitions in a non-negative additive game with acyclic permission structure (N, v, D) such that $A \cap B = \emptyset$ and $A \cup B = N$. Then $x(A) = \sum_{i \in A} a_i$ and $x(B) = \sum_{i \in B} a_i$ for every core element $x \in C(N, r)$.*

Proof. By definition of the restricted game (N, r) and feasibility of A we have that $r(A) = v(\sigma(A)) = v(A) = \sum_{i \in A} a_i$ and $r(B) = v(\sigma(B)) = v(B) = \sum_{i \in B} a_i$. Now, let $x \in C(N, r)$. Then $x(A) \geq r(A)$ and $x(B) \geq r(B)$. From the second inequality we obtain that $x(A) = r(N) - x(B) \leq \sum_{i \in N} a_i - \sum_{i \in B} a_i = \sum_{i \in A} a_i = r(A)$. Hence $x(A) = r(A) = \sum_{i \in A} a_i$. Analogous $x(B) = r(B) = \sum_{i \in B} a_i$. □

We now state the final result of this section.

Proposition 8.2.5 *Let (N, D) be an acyclic permission structure and $t \in T_D$. Then $x(U^t) = \sum_{i \in U^t} a_i$ and $x(N \setminus U^t) = \sum_{i \in N \setminus U^t} a_i$ for every core element $x \in C(N, r)$.*

Proof. By definition we have that U^t is disjunctive feasible. To show that also $N \setminus U^t$ is disjunctive feasible, consider a player $i \in N \setminus U^t$. Since $i \neq t$ and i is also not a complete subordinate of t, there is a path from $T_D \setminus \{t\}$ to i. Hence $N \setminus U^t$ is disjunctive feasible. So, both U^t and $N \setminus U^t$ are disjunctive feasible and thus the proposition follows by applying Lemma 8.2.4 to $A = U^t$ and $B = N \setminus U^t$. □

From Propositions 8.2.3 and 8.2.5 it follows that for any $t \in T_D$ it holds that $U^t \subset N$ satisfies both properties that $(U^t, D(U^t))$ is an acyclic, quasi-strongly connected permission structure and $\sum_{j \in U^t} x_j = r(U^t) = \sum_{j \in U^t} a_j$ for every $x \in C(N, r)$, so also for the nucleolus. This property is now used to compute the nucleolus of the restricted game (N, r) obtained from a non-negative additive game with acyclic permission structure (that is not necessarily quasi-strongly connected) in a finite number of steps. At each step we compute the nucleolus of a smaller additive game with an acyclic, quasi-strongly connected permission structure by applying the $\mathcal{O}(n^4)$ algorithm of Chapter 7 for such games. This algorithm is an adaptation of the algorithm of Arin and Feltkamp (1997), which computes the nucleolus of veto-rich games in exponential time. Note that permission games with an acyclic, quasi-strongly connected permission structure are indeed veto-rich games. We introduce one more notation. Let (N, D) be an acyclic permission structure, $t \in T_D$ be one of the top players and $K = N \setminus U^t$. Then the set $D^K \in \mathcal{D}^K$ on the set of players K is given by

$$(i, j) \in D^K \text{ if and only if } (i, j) \in D \text{ and } P_D(j) \cap U^t = \emptyset. \tag{8.2.1}$$

So, for two players $i, j \in K$, (i, j) is an arc in D^K if and only if (i, j) is an arc in D and j does not have a predecessor in U^t. Stated differently, D^K contains all arcs in $D(K)$, except the arcs (i, j) such that j is a successor of top player t or of one of its complete subordinates. Finally, we assume that the players are enumerated (labeled) by the numbers $1, 2, \ldots, n$ in such a way that for any $i, j \in \{1, \ldots, n\}$ it holds that $i < j$ if $(i, j) \in D$. From the theory on acyclic directed graphs it is well-known that such an enumeration exists. Observe that this assumption implies that node $1 \in T_D$.

Algorithm

Step 1 Set $k = 1$, $N_1 = N$, $D_1 = D$ and $t_1 = 1$. Go to Step 2.

Step 2 Take r_k the the restricted game of the non-negative additive game with acyclic, quasi-strongly connected permission structure $(U^{t_k}, v_k, D_k(U^{t_k}))$ with

$$v_k(U) = v(U) \text{ for all } U \subseteq U^{t_k}. \tag{8.2.2}$$

Go to Step 3.

Step 3 Apply the (polynomial time) algorithm of chapter 7 to find the nucleolus of the restricted game (U^{t_k}, r_k). Assign $y_i = Nuc_i(U^{t_k}, r_k)$ to every $i \in U^{t_k}$. Go to Step 4.

Step 4 If $U^{t_k} = N_k$ then Stop. Otherwise, go to Step 5.

Step 5 Define $N_{k+1} = N_k \setminus U^{t_k}$ and $D_{k+1} \in \mathcal{D}^{N_{k+1}}$ by $D_{k+1} = D_k^{N_{k+1}}$ (i.e., $D_{k+1} = D^K$ as defined in formula (8.2.1) with $D = D_k$ and $K = N_k \setminus U^{t_k}$). Define $t_{k+1} \in T_{D_{k+1}}$ as the top player in D_{k+1} with the lowest label ($t_{k+1} \leq h$ for every $h \in T_{D_{k+1}}$). Consider the set $U^{t_{k+1}}$ consisting of t_{k+1} and all its complete subordinates in the graph (N_{k+1}, D_{k+1}). Set $k = k + 1$ and return to Step 2.

8.3 The algorithm works

In this section we prove that the algorithm indeed finds the nucleolus of the original non-negative additive game with acyclic permission structure (N, v, D). As a first observation, according to Proposition 8.2.5 it must hold that $\sum_{i \in U^1} Nuc_i(N, r) = \sum_{i \in U^1} a_i$, so the total payoff $\sum_{i \in U^1} a_i$ assigned in the first iteration of the algorithm to the players in U^1 is indeed equal to the total payoff that the players in U^1 attain at the $Nuc(N, r)$. Of course, we still have to prove that the individual payoffs assigned at the first iteration are the individual payoffs for the players in U^1 in $Nuc(N, r)$, and subsequently for the payoffs assigned at any next iteration. This will be proved by using the Davis-Maschler reduced game property.

Generally, the Davis-Maschler reduced game property is not true for the nucleolus. But it is true for the prenucleolus and in the situation which is considered the prenucleolus coincides with the nucleolus.

In the sequel we will denote the characteristic function of the Davis-Maschler reduced game with respect to the nucleolus $x = Nuc(N, v)$ and coalition $T \subset N$ just by v', if there is no confusion[1]. Recall that for a game $v \in \mathcal{G}^N$ and a coalition $T \subset N$, the *subgame* $v_T \in \mathcal{G}^T$ is given by $v_T(S) = v(S)$ for all $S \subseteq T$. We now have the following proposition with respect to U^t for some $t \in T_D$. For notational simplicity, in the following we denote $U^t = K$.

Proposition 8.3.1 *For a non-negative additive game with acyclic permission structure (N, v, D) with $|T_D| \geq 2$, let $t \in T_D$ be a top player and $K = U^t$. Then the Davis-Maschler reduced game (K, r') of the restricted game (N, r) with respect to the nucleolus $x \in \mathbb{R}^n$ and the set K, coincides with the subgame (K, r_K) of (N, r).*

Proof. First, observe that K is feasible and thus $r_K(K) = \sum_{i \in K} a_i$. On the other hand $r'(K) = r(N) - x(N \setminus K) = \sum_{i \in N} a_i - \sum_{i \in N \setminus K} a_i = \sum_{i \in K} a_i$, where the first equality is by definition of the reduced game and the second equality by Proposition 8.2.5. So, $r_K(K) = r'(K) = \sum_{i \in K} a_i$.

Next, consider a set $U \subset K$. By definition of the Davis-Maschler reduced game we have

$$r'(U) = \max_{S \subseteq N \setminus K} (r(U \cup S) - x(S)). \tag{8.3.3}$$

[1] In general, the Davis-Maschler reduced game property is stated for an arbitrary solution. Since we apply it here to compute the nucleolus, we only state it in terms of this particular solution.

We first show that $r'(U) = r'(\sigma(U))$. Therefore it is sufficient to show that for any $S \subseteq N \setminus K$ it holds that $\sigma(U \cup S) = \sigma(\sigma(U) \cup S)$, because for any $S \subseteq N \setminus K$ the value of $r(U \cup S)$ in equation (8.3.3) is equal to the worth of $\sigma(U \cup S)$ in game v. Since $\sigma(U) \subseteq U$, it is evident that $\sigma(\sigma(U) \cup S) \subseteq \sigma(U \cup S)$. Suppose that the inclusion does not hold the other way around, i.e., there exists some player $i \in \sigma(U \cup S) \setminus \sigma(\sigma(U) \cup S)$. For this player there is a path p from T_D to i consisting of players in $\sigma(U \cup S)$ only, and there does not exist a path from T_D to i consisting of players in $\sigma(\sigma(U) \cup S)$ only. Consider a path $p = (p_1, \ldots, p_m)$ from T_D to i with $H(p) \subseteq \sigma(U \cup S)$. We distinguish the following two cases:
(i). Suppose that p is a directed path from t to i. Since $P_D(j) \subset K$ for all $j \in K$, there is a $k \in \{2, \ldots, m\}$ such that $\{p_1, \ldots, p_k\} \subseteq U$ and $\{p_{k+1}, \ldots, p_m\} \subseteq S$. Since $(p_1, \ldots, p_k)$ is a directed path in U, we have that $\{p_1, \ldots, p_k\} \subseteq \sigma(U)$. Thus, $H(p) \subseteq \sigma(U) \cup S$. But then $H(p) \subseteq \sigma(\sigma(U) \cup S)$ since p is a directed path with $p_1 = t$.
(ii). Suppose that p is a directed path from another top-player $t' \neq t$ to i. Since players in $K = U^t$ do not have predecessors in $N \setminus K$, we have that $U \cap H(p) = \emptyset$. Thus, $H(p) \subseteq \sigma(S)$, and so $H(p) \subseteq \sigma(\sigma(U) \cup S)$.
From (i) and (ii) we conclude that $\sigma(U \cup S) = \sigma(\sigma(U) \cup S)$. Hence we have that $r'(U) = r'(\sigma(U))$ and it remains to prove that $r_K(U) = r'(U)$ for any feasible $U \subset K$.

Observe that $r_K(U) = \sum_{i \in U} a_i$ when U is feasible. To find $r'(U)$, we first show that for finding the maximum in (8.3.3), it is sufficient to consider only sets S such that $U \cup S$ is feasible. If $U \cup S$ is not feasible, then

$$
\begin{aligned}
& r(U \cup S) - x(S) = r(\sigma(U \cup S)) - x(S) \leq \\
& r\left(U \cup \left(\sigma\left(U \cup S\right) \setminus U\right)\right) - x\left(\sigma\left(U \cup S\right) \setminus U\right) - x\left(S \setminus \left(\sigma\left(U \cup S\right) \setminus U\right)\right) \leq \\
& r\left(U \cup \left(\sigma\left(U \cup S\right) \setminus U\right)\right) - x\left(\sigma\left(U \cup S\right) \setminus U\right),
\end{aligned}
$$

where the first inequality follows from $\sigma(U \cup S) \setminus U \subseteq S$ and $\sigma(U \cup S) \setminus U \neq S$ if $U \cup S$ is not feasible. So, in case that $U \cup S$ is not feasible, replacing set S by set $\sigma\left(U \cup S\right) \setminus U$ does not decrease $r(U \cup S) - x(S)$ in formula (8.3.3), and thus this expresion is maximized by a coalition S such that $U \cup S$ is feasible.

By definition of r', it now follows that there is some $S \subseteq N \setminus K$ such that $U \cup S$ is feasible and

$$r'(U) = r(U \cup S) - x(S) = \sum_{i \in U \cup S} a_i - x(S). \tag{8.3.4}$$

Further, since $U \cup S$ is feasible and $U \subset K = U^t$, we have that also $K \cup S$ is feasible. So, by the fact that $x \in C(N, r)$, we have that $x(K \cup S) = x(K) + x(S) \geq \sum_{i \in K \cup S} a_i$. By Proposition 8.2.5 we have that $x(K) = a(K)$ and thus $x(S) \geq \sum_{i \in S} a_i$. It follows from equation (8.3.4) that

$$r'(U) = \sum_{i \in U} a_i + \sum_{i \in S} a_i - x(S) \leq \sum_{i \in U} a_i = r(U). \tag{8.3.5}$$

From (8.3.5) and the fact that by definition of the reduced game $r'(U) \geq r(U)$, we conclude that

$$r'(U) = r(U) = \sum_{i \in U} a_i.$$

This proves that $r'(U) = r_K(U)$ for all $U \subset K = U^t$. □

In the first iteration the algorithm finds the nucleolus of (U^1, r_1). Clearly, the restricted game (U^1, r_1) of (U^1, v_1, D_1) is equal to the subgame (U^1, r_{U^1}) of (N, r), which is equal to the Davis-Maschler reduced game according to the proposition above. So, with the Davis-Maschler reduced game Property 2.2.2, the proposition above shows that in the first iteration the algorithm indeed computes the nucleolus payoffs of the players in U^1 in game (N, r). For $t \in T_D$ and $K = N \setminus U^t$, the next proposition shows that the Davis-Maschler reduced game with respect to the nucleolus and the set K coincides with the game (K, r_2), where r_2 is the restricted game of the non-negative game with the reduced acyclic permission structure (K, v_K, D^K) on the set of players K, where (K, v_K) is the subgame of (N, v) on K and D^K as given in formula (8.2.1) for $t = 1$, i.e., r_2 is the restricted game used in the second iteration of the algorithm.

Proposition 8.3.2 *For a non-negative additive game with acyclic permission structure (N, v, D) with $|T_D| \geq 2$, let $t \in T_D$ and $K = N \setminus U^t$. Then the Davis-Maschler reduced game (K, r') of the restricted game (N, r) with respect to the nucleolus payoff vector $x \in \mathbb{R}^n$ and the set K, coincides with the restricted game (K, r_2) of the non-negative additive game with acyclic permission structure (K, v_K, D^K) with D^K as defined in formula (8.2.1).*

Proof. By definition, $P_{D^K}(j) \subseteq K$ for all $j \in K$, so K is feasible in the reduced graph (K, D^K). Hence $r_2(K) = \sum_{i \in K} a_i$. On the other hand, $r'(K) = r(N) - x(N \setminus K) = r(N) - x(U^t) = \sum_{i \in N} a_i - \sum_{i \in U^t} a_i = \sum_{i \in K} a_i$, where the first equality is by definition of the reduced game and the second equality by Proposition 8.2.5. So, $r_2(K) = r'(K)$.

Next, for a set $S \subset K$ we consider two cases.

(i). First, suppose that $P_D(S) \cap U^t = \emptyset$. For all $U \subseteq U^t$ it holds that $\sigma(U \cup S) = \sigma(U) \cup \sigma(S)$, since $P_D(U) \cap S = \emptyset$. So,

$$r(S \cup U) - x(U) = v(\sigma(U) \cup \sigma(S)) - x(U) =$$

$$v(\sigma(U)) + v(\sigma(S)) - x(U) = r(S) + r(U) - x(U) \leq r(S),$$

where the second equality follows from v being an additive game and the (last) inequality follows from the nucleolus being a core element. Hence

$$r'(S) = \max_{U \subseteq U^t} (r(S \cup U) - x(U)) = r(S).$$

On the other hand, for all $i, j \in S$ we have that $(i, j) \in D^K$ if $(i, j) \in D$, because no player $j \in S$ has a predecessor in U^t. Hence, $\sigma(S) = \sigma_K(S)$, where $\sigma_K(S)$ is the largest feasible subset of S in (K, D_K). So,

$$r'(S) = r(S) = v(\sigma(S)) = v(\sigma_K(S)) = r_2(S).$$

(ii). Secondly we consider the case that $P_D(S) \cap U^t \neq \emptyset$. Now, let $S' \subseteq S$ be given by $S' = \sigma(U^t \cup S) \setminus U^t$. By definition of σ, $U^t \cup S'$ is the maximal feasible subset of $U^t \cup S$ and thus

$$r(U^t \cup S) - x(U^t) = v(U^t \cup S') - x(U^t) = \sum_{i \in U^t \cup S'} a_i - \sum_{i \in U^t} a_i = \sum_{i \in S'} a_i,$$

where the second equality follows by Proposition 8.2.5. Hence we have that

$$r'(S) = \max_{U \subseteq U^t} \; (r(S \cup U) - x(U)) \geq r(U^t \cup S) - x(U^t) = \sum_{i \in S'} a_i. \tag{8.3.6}$$

We now show that this holds with equality. First, recall from the proof of Proposition 8.3.1 that $\sigma(U \cup S) = \sigma(\sigma(U) \cup S)$ for every $U \subset U^t$ and $S \subset K$. Hence

$$r(U \cup S) - x(U) = r(\sigma(U) \cup S) - x(U) \leq r(\sigma(U) \cup S) - x(\sigma(U)),$$

so that $\max_{U \subseteq U^t} \; (r(S \cup U) - x(U))$ will be attained by a feasible set U. Suppose there exists a feasible $U \subseteq U^t$ with $r(U \cup S) - x(U) > \sum_{i \in S'} \; a_i$. Then, with $S_U \subseteq S$ given by $S_U = \sigma(U \cup S) \setminus U$, it follows in an analogous way as for $U = U^t$ above, that $\sigma(U \cup S) = U \cup S_U$, and thus

$$r(U \cup S) - x(U) = v(U \cup S_U) - x(U) = \sum_{i \in U \cup S_U} a_i - x(U) =$$

$$\sum_{i \in S_U} a_i + \sum_{i \in U} a_i - x(U) \leq \sum_{i \in S_U} a_i \leq \sum_{i \in S'} a_i, \tag{8.3.7}$$

where the first inequality follows because U is feasible and the nucleolus lies in the core, and the second inequality because $U \subseteq U^t$ and thus $S_U = \sigma(U \cup S) \setminus U \subseteq \sigma(U^t \cup S) \setminus U^t$. From equations (8.3.6) and (8.3.7) it follows that

$$r'(S) = \sum_{i \in S'} \; a_i = v(S').$$

It remains to prove that also $r_2(S) = v(S')$. By definition we have that $r_2(S) = v_K(\sigma_K(S)) = v(\sigma_K(S))$, where $\sigma_K(S)$ is the maximal feasible subset of S in graph (K, D^K). So, it remains to show that $\sigma_K(S) = S'$.

We first show that $\sigma_K(S) \subseteq S' = \sigma(U^t \cup S) \setminus U^t$. Consider $i \in \sigma_K(S)$. For such a player i there is a path p from T_{D^K} to i which only contains elements of S. We consider two cases. If p is a path from a top $t' \in T_D$, then $i \in \sigma(S) \subseteq S'$. For the case that p is a path from a top t' in $T_{D^K} \setminus T_D$, then t' is a top in (K, D^K) but not in (N, D) and thus $P^D(t') \cap U^t \neq \emptyset$, implying that in (N, D) there is a path p' from t to t'. Hence, the path p'' consisting of the path p' from t to t' and the path p from t' to i is a path in (N, D) from T_D to i. Since $H(p') \setminus \{t'\} \subseteq U^t$ and $H(p) \subseteq S$, we have that the set of nodes

$H(p'') \subseteq U^t \cup S$ and thus $H(p'') \subseteq \sigma(U^t \cup S)$. So $i \in \sigma(U^t \cup S)$ and we can conclude that $i \in \sigma(U^t \cup S) \setminus U^t = S'$.

Next we show the reverse that $S' \subseteq \sigma_K(S)$. Let $i \in S'$. Then $i \in \sigma(U^t \cup S)$ and thus there is path p from T_D to i that consists of elements of $U^t \cup S$. Again there are two cases. If $H(p) \subseteq S$, then $i \in \sigma(S) \subseteq \sigma_K(S)$. Otherwise, p consists of two subpaths p' in U^t and p'' in S that are connected to each other by a link from the last node of p' to the first node of p''. Let j be that last node of path p'' that has a predecessor in U^t. Then, by construction, j is a top-node in game (K, D^K) and thus the part of p'' from j to i is a path in (K, D_K) from T_{D^K} to i. So $i \in \sigma_K(S)$. □

By repeated application of the propositions above it follows that the algorithm of Section 3 computes the nucleolus of (N, r).

Proposition 8.3.3 *For a non-negative additive game with acyclic permission structure (N, v, D), the algorithm described in Section 8.2 finds the nucleolus of (N, r) within a finite number of iterations.*

Proof. In the first iteration the algorithm finds the nucleolus of the restricted game (U^1, r_1) of (U^1, v_1, D_1), which is equal to the subgame (U^1, r_{U^1}) of (N, r). By Proposition 8.3.1 and the Davis-Maschler reduced game property it follows that $Nuc_i(U^1, r_1) = Nuc_i(N, r)$ for all $i \in U^1$. If $U^1 = N$, the algorithm ends with the nucleolus in one iteration. Otherwise, the algorithm continues in iteration 2 with the restricted game $(N \setminus U^1, r_2)$ of $(N \setminus U^1, v, D_2)$. By Proposition 8.3.2 this restricted game is equal to the DM reduced game on $N \setminus U^1$ with respect to the nucleolus and so the nucleolus payoffs of the game $(N \setminus U^1, r_2)$ are equal to the nucleolus payoffs of the players in $N \setminus U^1$ in the game (N, r). Repeating the arguments of the first iteration, in the second iteration the algorithm computes the nucleolus payoffs of the game $(N \setminus U^1, r_2)$ for the players $i \in U^{t_2}$, which thus is equal to their nucleolus payoffs in the game (N, r). Subsequentially in each iteration k the algorithm computes the nucleolus payoffs of the set of players in U^{t_k}. Since the number of players is reduced with at least one in each iteration of the algorithm, the algorithm ends within at most n iterations. □

8.4 Properties of the algorithm

In this section we first show several properties of the algorithm. From these properties we then obtain an interesting property of the nucleolus of a non-negative additive game with permission structure, namely that each coalition consisting of a *free player* and its complete subordinates distributes its own value among themselves. A *free player* is a player that does not have a complete superior. (Recall from Chapter 2 that by $\overline{P}_D(i)$ we denote the set of all complete superiors of node i in $D \in \mathcal{D}^N$.)

Definition 8.4.1 *A player $i \in N$ in an acyclic permission structure (N, D) is a* **free player** *if $\overline{P}_D(i) = \emptyset$.*

Example 8.4.2 Consider the permission structure (N, D) with $N = \{1, 2, 3, 4, 5, 6\}$ and $D = \{(1, 3), (1, 4), (2, 4), (3, 5), (3, 6), (4, 6)\}$. This permission structure has four free players. Two of them are the two top players 1 and 2. Besides the top players the two other free players are 4 and 6.

In the sequel we denote the set of free players in (N, D) by F_D. Notice that $T_D \subseteq F_D$. For every non-top player i, the set $\mathcal{P}_i$ denotes the collection of all paths from T_D to i. For a path $p = (i_1, \ldots, i_{m-1}, i_m) \in \mathcal{P}_i$ (so $i_1 \in T_D$ and $i_m = i$), in the sequel $H_i(p)$ denotes the set of players $H(p) \setminus \{i\}$, i.e., $H_i(p)$ is the set of all players on the path p except player i itself. We now give two lemmas. The first one is obvious and stated without proof.

Lemma 8.4.3 *For every acyclic permission structure (N, D), if j is a (complete) superior of i and k is a (complete) superior of j then k is a (complete) superior of i.*

The second lemma states that for any free player $i \in F_D \setminus T_D$ there exist (at least) two paths p and q in $\mathcal{P}_i$ such that $H_i(p) \cap H_i(q) = \emptyset$, i.e., for any non-top free player i there exist two disjunct paths (except for i itself) from T_D to i. Recall that it is assumed that the players are enumerated (labeled) by the numbers $1, 2, \ldots, n$ in such a way that for any $i, j \in \{1, \ldots, n\}$ it holds that $i < j$ if $(i, j) \in D$.

Lemma 8.4.4 *Let (N, D) be an acyclic permission structure and let i be a player in $F_D \setminus T_D$. Then there exist (at least) two paths p and q in $\mathcal{P}_i$, such that $H_i(p) \cap H_i(q) = \emptyset$.*

Proof. For any two different paths p^h and p^k in $\mathcal{P}_i$, define $m_{hk} = \max\{j | j \in H_i(p^k) \cap H_i(p^h)\}$ with the convention that $m_{hk} = 0$ if $H_i(p^k) \cap H_i(p^h) = \emptyset$, i.e., m_{hk} is the highest labeled player that is on both paths. Further, define $m = \min_{h,k}\ m_{hk}$. Suppose $m \geq 1$, i.e., there exist two paths, say p^1 and p^2 with $H_i(p^1) \cap H_i(p^2) \neq \emptyset$, such that (i) m is the highest labeled common node of p^1 and p^2 and (ii) $m_{hk} \geq m$ for any two paths p^h and p^k in $\mathcal{P}_i$. Since i is a free player, and thus $\overline{P}_D(i) = \emptyset$, there is no player $j \neq i$ that is on all paths in $\mathcal{P}_i$. Therefore we must have a third path, say $p^3 \in \mathcal{P}_i$, such that $m \notin H_i(p^3)$. Because of (ii) we have that $m_{13} > m$ and $m_{23} > m$, so for both paths p^1 and p^2 it holds that they have a node in common with p^3 with a higher label than m. For $j = 1, 2$, define $m_j = \min\{s > m | s \in H_i(p^j) \cap H_i(p^3)\}$, so m_j is the lowest labeled common node on the paths p^j and p^3 higher than m. Thus $m_1 > m$ and $m_2 > m$ and also $m_1 \neq m_2$, otherwise $m_1 = m_2 \in H_i(p^1) \cap H_i(p^2)$, contradicting (i). Without loss of generality, suppose that $m_1 > m_2$. Now, let p^4 be the path in $\mathcal{P}_i$ that is equal to p^3 from T_D to node m_2 and equal to p^2 from m_2 to i. Then $m_{14} < m$, because p^4 coincides with p^2 from node m_2 to i and the highest labeled common node of p^1 and p^2 is node $m < m_2$, and p^4 coincides with p^3 from T_D to m_2 and the smallest labeled common node of p^1 and p^3 higher or equal to m is node $m_1 > m_2$. However, $m_{14} < m$ contradicts (ii). So it follows that $m = 0$, which proves that there exist two paths in $\mathcal{P}_i$ that only have node i in common. $\square$

Recall that in the first iteration player 1 is chosen to be the top and that the nucleolus payoffs of the players in U^1 are computed. In the second iteration the algorithm continues with the non-negative additive game with permission structure (K, v_K, D^K), where $K = N \setminus U^1$, (K, v_K) the subgame of (N, v) on K, and D^K the permission structure as obtained in formula (8.2.1). The next lemma states that $j \neq 1$ is a free player in (K, D^K) if and only if it is a free player in (N, D).

Lemma 8.4.5 *Let (N, D) be an acyclic permission structure, $K = N \setminus U^1$ and D^K the permission structure as defined in formula (8.2.1). Then $F_{D^K} = F_D \setminus \{1\}$.*

Proof. First, observe that U^1 consists of player 1 and all its complete subordinates, so by Lemma 8.2.1, U^1 does not contain any of the players in $F_D \setminus \{1\}$, and thus $F_D \setminus \{1\} \subseteq K$. We first prove that $F_D \setminus \{1\} \subseteq F_{D_K}$. When $i \in T_D \setminus \{1\}$, then obviously $i \in T_{D^K}$. So, consider $i \in F_D \setminus T_D$. According to Lemma 8.4.4, there are two different paths in (N, D) from T_D to i. Let p be such a path. We consider two cases.

1. The path p is completely in K and $P_D(j) \cap U^1 = \emptyset$ for any $j \in H(p)$. Then any link $(h, k) \in D$ on the path is also a link in D_K and $p' = p$ is a path in (K, D_K) from $T_D \setminus \{1\} \subseteq T_{D^K}$ to i.

2. There is some $h \in H(p)$ such that $P_D(h) \cap U^1 \neq \emptyset$. With $A(p)$ being the set of all players in $H(p)$ having a predecessor in U^1, let k be the player in $A(p)$ such that $k \geq h$ for all $h \in A(p)$, i.e., k is the player in $A(p)$ with the highest label. Then, by formula (8.2.1), any link $(i, k) \in D(K)$ is deleted to obtain D^K, so $k \in T_{D^K}$. Let p' be the path consisting of the part of path p from k to i. Then p' is a path from T_{D^K} to i in $(K, D(K))$. Observe that $p' = (i)$ with $i \in T_{D^K}$ if $k = i$.

It follows that every path p from T_D to i in (N, D) gives a path p' from T_{D^K} to i in $(K, D(K))$. When for some p the path p' reduces to the single player path $p' = (i)$ (when $k = i$ in case 2), then i becomes a top player in $(K, D(K))$ and every path from T_D to i reduces to the single element i. Otherwise, i has two different paths in $(K, D_K))$ when i has two different paths in (N, D), because $H(p') \subseteq H(p)$ for every path p from T_D to i. So $i \in F_{D^K}$ when $i \in F_D \setminus \{1\}$.

Second, we prove the reverse inclusion that $F_{D^K} \subseteq F_D \setminus \{1\}$. To do so we show that a node $i \neq 1$ which is not free in (N, D) is also not free in (K, D_K). Let $i \notin F_D \setminus \{1\}$, so i has a complete superior in (N, D). When there is a complete superior in U^1 then, by Lemma 8.4.3, also 1 is a complete superior of i. Then $i \in U^1$ and thus i is not in K. It remains to consider the case that $\overline{P}_D(i) \subset K$. Let k be a player in $\overline{P}_D(i)$. For p a path from T_D to i, let p' be the part of the path from k to i. Then there is no player $h > k$ on the path p' that has a predecessor in U^1, otherwise there is in (N, D) a path from 1 to the predecessor of h in U^1, then to h and then to i, contradicting that k is a complete superior of i in the graph (N, D). So, when $(j, l) \in D$ is a link on p', then also $(j, l) \in D_K$ and thus p' is a path in (K, D_K). Since this holds for every path p from T_D to i in (N, D), it follows that k is also a complete superior of i in (K, D_K), and thus i is not free in (K, D_K). □

The next lemma states that a player is chosen as top in one of the iterations of the algorithm if and only if it is a free player in (N, D). Let A_D denote the set of players that is chosen as top in one of the iterations of the algorithm of Section 8.2.

Lemma 8.4.6 *Let (N, v, D) be a non-negative additive game with acyclic permission structure. Then $A_D = F_D$.*

Proof. The proof follows by repeated application of Lemma 8.4.5. In iteration 1, player 1 is chosen as top. So, player 1 belongs both to A_D and F_D. In iteration 2, player t_2 is chosen as top. This player is determined in Step 5 of the previous iteration as the top player with the lowest label in (N_2, D_2) with $N_2 = N_1 \setminus U^1 = N \setminus U^1$ and $D_2 = D(N_2)$. By Lemma 8.4.5 this top player belongs to $F_{D_2} = F_D \setminus \{1\}$. From repeated application of the lemma it follows that the top player t_k in iteration k is a top player of $F_{D_k} = F_{D_{k-1}} \setminus \{t_{k-1}\} = F_D \setminus \{t_1, \ldots, t_{k-1}\}$, where (N_k, D_k) is the graph at iteration k. It follows that succeedingly all players of F_D are chosen as top in increasing order of their label. □

Finally we show that the set of complete subordinates of the chosen top player t_k in the graph (N_k, D_k) in iteration k is equal to the set of complete subordinates of the free player t_k in (N, D). Moreover we have that $(U^{t_k}, D_k(U^{t_k})) = (U^{t_k}, D(U^{t_k}))$, i.e., the subgraph on U^{t_k} of (N_k, D_k) is equal to the subgraph of U^{t_k} of (N, D).

Lemma 8.4.7 *Let (N, v, D) be a non-negative additive game with acyclic permission structure and let t_k be the chosen top-player in iteration k of the algorithm. Then $\overline{S}_{D_k}(t_k) = \overline{S}_D(t_k)$. Moreover, $(U^{t_k}, D_k(U^{t_k})) = (U^{t_k}, D(U^{t_k}))$.*

Proof. From the proof of Lemma 8.4.5 it follows that when $k \in N_2 = N_1 \setminus U^1$ is a complete superior of i in $(N_1, D_1) = (N, D)$, then k is also a complete superior of i in (N_2, D_2). With Lemma 8.2.1 it follows reversely that $i \in \overline{S}_{D_2}(k)$ when $i \in \overline{S}_{D_1}(k)$. On the other hand, when i is not a complete subordinate of k in (N_1, D_1), then it is also not in (N_2, D_2) because $D_2 \subset D_1$ and either $P_{D_2}(i) = \emptyset$ or there is a path from T_{D_2} to i in (N_2, D_2) without k. So, for any $k \in N_2$ we have that $\overline{S}_{D_2}(k) = \overline{S}_{D_1}(k)$. The first statement of the lemma follows by repeating these arguments for all the remaining top players at any iteration of the algorithm. To show the second statement, let $k \in N_2$ be a complete superior of a player $i \in N_2$. From the last part of the proof of Lemma 8.4.5 it follows that any link on a path from k to i in (N_1, D_1) is also a link in (N_2, D_2). So, $(U^{t_2}, D_2(U^{t_2})) = (U^{t_2}, D_1(U^{t_2})) = (U^{t_2}, D(U^{t_2}))$. The result follows from repeating this at any next iteration of the algorithm. □

We now come to the main result of this section. In Section 8.3 we have seen that the algorithm of Section 8.2 at any iteration k computes the nucleolus payoffs in the restricted game (N, r) of the players in U^{t_k}, where U^{t_k} is the set of players consisting of the chosen top t_k in iteration k and all its complete subordinates in (N_k, D_k). Moreover, the total payoff of the players in U^{t_k} is equal to the sum of their values a_i, $i \in U^{t_k}$. Therefore the next proposition follows from the lemmas above without further proof. It states that for any free player the nucleolus distributes the total contributions of this player and its complete subordinates among themselves.

Proposition 8.4.8 *Let (N, v, D) be a non-negative additive game with acyclic permission structure. Then $\sum_{i \in \overline{S}_D(k)} Nuc_i(N, r) = \sum_{i \in \overline{S}_D(k)} a_i$ for every $k \in F_D$.*

Finally, when the structure of the graph (N, D) is known ex ante, in particular the set F_D is known ex ante and also for each $k \in F_D$ its set of complete subordinates, it follows without further proof from the lemmas above that the algorithm of Section 8.2 reduces to $|F_D|$ applications of the algorithm of Chapter 7. To state this result, for $k \in F_D$, let $\overline{r}_k$ be the restricted game of $(\overline{S}_D(k), v_k, D(\overline{S}_D(k)))$, where v_k is the non-negative additive game on $\overline{S}_D(k)$ given by $v_k(S) = \sum_{i \in S} a_i$, $S \subseteq \overline{S}_D(k)$. Then we have the following proposition.

Proposition 8.4.9 *Let (N, v, D) be a non-negative additive game with acyclic permission structure. Then $Nuc_i(N, r) = Nuc_i(\overline{S}_D(k), r_k)$, $i \in \overline{S}_D(k)$, $k \in F_D$.*

From this last proposition we can conclude that for any $k \in F_D$, the nucleolus values in the game (N, r) of the players in $\overline{S}_D(k)$ can be computed by applying the algorithm of Chapter 7 to $(\overline{S}_D(k), v_k, D(\overline{S}_D(k)))$.

8.5 Complexity of the algorithm

When the structure of the game is known ex ante we can apply Proposition 8.4.9 and apply the algorithm of Chapter 7 to any $(\overline{S}_D(k), v_k, D(\overline{S}_D(k)))$, $k \in F_D$. Since the complexity of this algorithm is of $\mathcal{O}(n^4)$, the complexity reduces to $\mathcal{O}(|F_D|a^4)$, where $a = max_{k \in F_D} |\overline{S}_D(k)|$, because we have $|F_D|$ problems and the kth problem has complexity $\mathcal{O}(|\overline{S}_D(k)|^4)$. In particular, we have that the complexity is of $\mathcal{O}(n^4)$ when $|F_D| = 1$ (and thus $|\overline{S}_D(1)| = n$) and of $\mathcal{O}(n)$ when $|F_D| = n$ (and thus $|\overline{S}_D(k)| = 1$ for all k). Clearly, in the latter (extreme) case we have that (N, D) is the empty graph and (N, v, D) reduces to the additive game, so that every player i gets its own value a_i.

Typically in practice the structure is not known in advance. Also, although we assumed in the previous sections that the players are enumerated by the numbers $1, 2, \ldots, n$ in such a way that for any $i, j \in \{1, \ldots, n\}$ it holds that $i < j$ if $(i, j) \in D$, in practice such an enumeration will not be known in advance. So, to perform the algorithm at each iteration k first a top node t_k in (N_k, D_k) has to be found and its corresponding set U^{t_k} of complete subordinates in (N_k, D_k). For the complexity of this search, let us consider the first iteration. It is evident that we can find the collection T_D of top nodes in at most n^2 actions, because we just have to consider each pair of players once. Then we enumerate the top nodes from 1 to k_1, where $k_1 = |T_D|$ is the number of top nodes in (N, D). To find the set U^1 of all complete subordinates of top node 1 we can proceed as follows. First, assign label 1 to every successor of node 1. Next, assign label 1 to every successor of every node with label 1 and continue to do this. So, every player in the set $\widehat{S}_D(1)$ of subordinates of top player 1 gets a label 1. Clearly, this requires at most $\mathcal{O}(n^2)$ actions. Next, repeat this procedure for every other top node, so for every player in the set $\widehat{S}_D(j)$ of top node j, $j = 1, \ldots, k_1$. So, a node can receive multiple labels. The set U^1 consists of all nodes that only receive label 1 and can be found in at most $k_1\mathcal{O}(n^2) < \mathcal{O}(n^3)$ actions. In fact this procedure gives in at most $\mathcal{O}(n^3)$ actions all top nodes $j \in T_D$ and their sets U^j of complete subordinates.

We can now use the result of Proposition 8.4.9 to modify the algorithm, namely instead of adapting the graph after the first iteration according to Step 5, we first apply k_1 times the algorithm of Chapter 7, namely to subgame with permission structure $(U^k, v_k, D(U^k))$ for every top player $k \in T_D$. The complexity of each application is given by $\mathcal{O}(|U^k|^4)$.

After having eliminated all top players and their complete subordinates we now adapt the graph on the remaining set of players $N \setminus (\cup_{k \in T_D} U^k)$ analogously as described in Step 5 of the algorithm. Let M be this set, then for each player $j \in M$ we consider whether or not j has a predecessor in (N, D) belonging to $\cup_{k \in T_D} U^k$. If so, all edges (h, j) in the subgraph $(M, D(M))$ are deleted and j becomes a top node in the remaining graph. Observe that such a top node in the new graph is a free player in (N, D). This requires at most $\mathcal{O}(n^2)$ actions and yields also the set of top nodes in the remaining graph. Next, repeating the procedure as described above for the set of new top nodes, in $\mathcal{O}(m^3)$ actions, where $m = |M|$, the sets U^j of complete subordinates of the new top nodes can be found. Then we apply again the algorithm of Chapter 7 to each new top node j and its set of complete subordinates U^j. After that we apply the procedure of finding the new graph and new set of top players for the remaining nodes and so on.

Summarizing, starting the algorithm the number of actions to find the set of top nodes and their sets of complete subordinates is of $\mathcal{O}(n^3)$. This has to be repeated at most $n - 1$ times to find the set of all free players (each free player is a top node at some stage) and their sets of complete subordinates. So, the number of actions to find all free players and their sets of complete subordinates is (at most) of $\mathcal{O}(n^4)$. For each free player $k \in F_D$ the complexity of the algorithm of Chapter 7 is of $\mathcal{O}(|U^k|^4)$. Since $\sum_{k=1}^{s} |U^k| = n$, where $s = |F_D|$, the total complexity of applying the algorithm s times, namely for each free player, is of $\mathcal{O}(n^4)$. So, both the complexity of finding all free players and their sets of complete subordinates and the total complexity to find the nucleolus payoffs for every free player with its set of complete subordinates, is given by $\mathcal{O}(n^4)$. Hence the total complexity of the algorithm of Section 8.2 to find all nucleolus payoffs is of $\mathcal{O}(n^4)$, showing that the algorithm finds the nucleolus in polynomial time.

8.6 An example

In this section we illustrate the computation of the nucleolus for non-negative additive disjunctive games with a permission structure by giving an example concerning a market situation where sellers can sell objects to buyers through a (directed) network of intermediaries. First, we give a simple example without intermediaries[2].

Example 8.6.1 Consider a situation where there is one seller (player 1) and one buyer (player 2) who can realize a non-negative surplus $a > 0$ from trade. The corresponding assignment game on $N = \{1, 2\}$ is given by $v(\{1\}) = v(\{2\}) = 0$ and $v(\{1, 2\}) = a$. Note

[2]This is a special case of the assignment game, introduced by Shapley and Shubik (1972).

that the restricted game (N, r) on the permission structure $D = \{(1, 2)\}$ (or $D = \{(2, 1)\}$) is the same as (N, v). Clearly, the nucleolus of this game yields an equal division of the surplus, i.e., $Nuc(N, v) = Nuc(N, r) = (\frac{a}{2}, \frac{a}{2})$. But as soon as there are two or more sellers (say players $1, \ldots, n-1\}$, such that the buyer (player n) can realize the surplus a with any one of the sellers and cannot generate more surplus by trading with more sellers (for example, the sellers all own one item of a good for which they have reservation value zero, and the buyer wants only one item of the good and is prepared to pay at most a for it) then the characteristic function of the assignment game on $N = \{1, \ldots n\}$ is $v(S) = a$ if $n \in S$ and $|S| \geq 2$, and $v(S) = 0$ otherwise. Again the restricted game on the permission structure $D = \{(i, n) \mid i \in \{1, \ldots n-1\}$ (i.e., the permission structure where all sellers are predecessor of the buyer) is the same as (N, v). Now, it is clear that $Nuc_n(N, v) = Nuc_n(N, r) = a$ and $Nuc_i(N, v) = Nuc_i(N, r) = 0$ for all $i \in \{1, \ldots n-1\}$ since this is the unique core payoff vector. This also follows immediately from Proposition 8.4.8 by observing that r is the restricted game of the non-negative additive game with $a_i = 0$ for $i = 1, \ldots, n_1$ and $a_n = a$ and that player n is a free player in (N, D) and thus receives its own value. So, similar as in a linear Bertrand price competition game, as soon as there is more than one seller, the surplus fully goes to the buyer. (A similar story holds if there is only one seller but more buyers.) □

Next, we consider an example of a market situation as described above, but buyers and sellers may not be able to trade directly with each other, but need intermediaries to connect them.

Example 8.6.2 Consider a market situation with two sellers (players 1 and 2) and four buyers (players 7, 8, 9 and 10) who cannot trade directly with each other but need intermediaries. Consider the permission structure D on $N = \{1, \ldots, 10\}$ given by $D = \{(1, 3), (1, 4), (2, 4), (2, 5), (2, 6), (3, 7), (4, 7), (4, 8), (5, 9), (6, 9), (6, 10)\}$, see Figure 1. For every buyer-seller pair that wants to make a deal, it is sufficient to use only one of the intermediaries they are both connected with. For example, for seller 1 and buyer 7 it is necessary and sufficient to use either intermediary 3 or intermediary 4, while seller 1 and buyer 8 need intermediary 4 to trade. Suppose that each seller owns at least four items of the good and each buyer wants one item. Buyer $i \in \{7, 8, 9, 10\}$ is prepared to pay $a_i > 0$ for the item. The reservation value of the sellers and all intermediaries is zero. This can be modelled as the game with permission structure (N, v, D) with $N = \{1, \ldots, 10\}$, D as given above, and v the non-negative additive game with $a_i = 0$ for $i = 1, \ldots, 6$ and a_i for $i = 7, \ldots, 10$, so that $v(S) = \sum_{i \in S} a_i = \sum_{i \in S \cap \{7,8,9,10\}} a_i$ for all $S \subseteq N$.

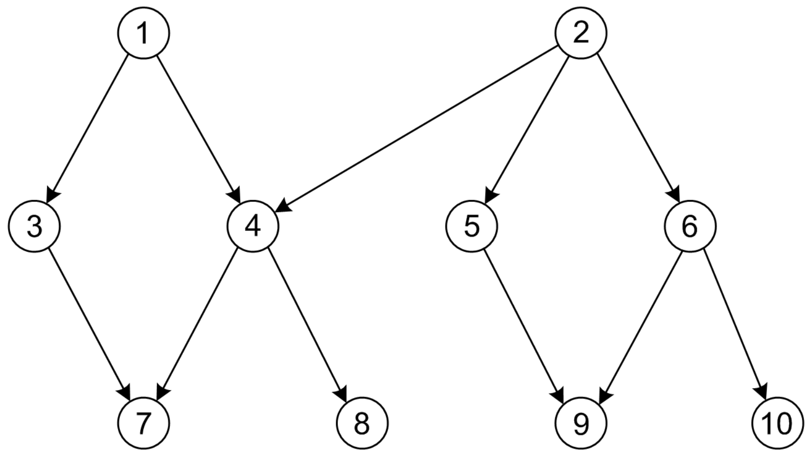

Figure 1.

The nucleolus of the restricted game (N, r) can be computed using the algorithm of Section 8.2 and the properties of Section 8.4. Notice that the players 1 and 2 are top players, and that the set of free players is given by $F_D = \{1, 2, 4, 7\}$. Using Proposition 8.4.9 it follows that we can find the nucleolus of (N, r) by considering the four subgames with permission structure given in Figure 2, namely one subgame for each of the four free players.

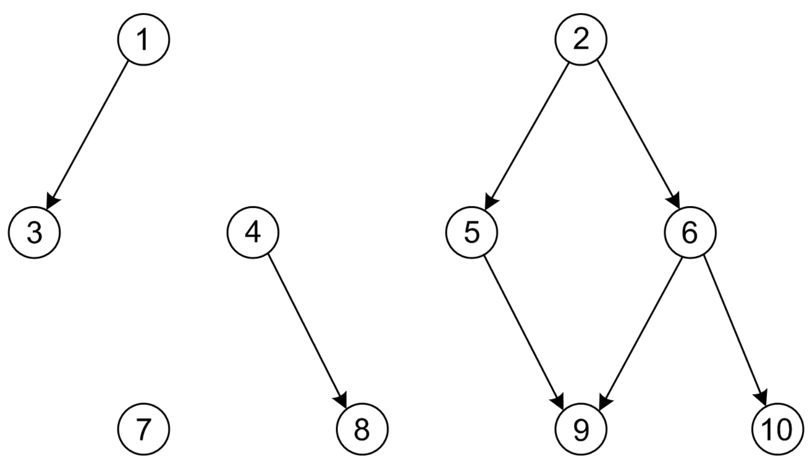

Figure 2

The nucleoli for the four subgames are as follows.

1. Consider the game with permission structure (N_1, v_1, D_1) given by $N_1 = \{1, 3\}$, $v_1 = v|_{N_1}$ and $D_1 = D|_{N_1} = \{(1, 3)\}$. Since v_1 is the null game assigning worth zero to all coalitions in N_1, it follows that $Nuc_1(N_1, r_1) = Nuc_3(N_1, r_1) = 0$.

2. For the game with permission structure (N_2, v_2, D_2) given by $N_2 = \{2,5,6,9,10\}$, $v_2 = v|_{N_2}$, and $D_2 = D|_{N_2} = \{(2,5),(2,6),(5,9),\ (6,9),(6,10)\}$, we have that $v_2(S) = \sum_{i \in S \cap \{9,10\}} a_i$ for all $S \subseteq N$. Then the restricted game is given by

$$r_2(S) = \begin{cases} a_9 & \text{if } S \in \{\{2,5,9\},\{2,6,9\},\{2,5,6,9\}\} \\ a_{10} & \text{if } S \in \{\{2,6,10\},\{2,5,6,10\}\} \\ a_9 + a_{10} & \text{if } S \in \{\{2,6,9,10\},\{2,5,6,9,10\}\} \\ 0 & \text{otherwise.} \end{cases}$$

Applying the algorithm of Chapter 7 we find that $Nuc_2(N_2, r_2) = \frac{a_9}{2} + \frac{a_{10}}{3}$, $Nuc_5(N_2, r_2) = 0$, $Nuc_6(N_2, r_2) = \frac{a_{10}}{3}$, $Nuc_9(N_2, r_2) = \frac{a_9}{2}$ and $Nuc_{10}(N_2, r_2) = \frac{a_{10}}{3}$.

3. For the game with permission structure (N_3, v_3, D_3) given by $N_3 = \{4,8\}$, $v_3 = v|_{N_3}$ and $D_3 = D|_{N_3} = \{(4,8)\}$, we have that $v_3(S) = \sum_{i \in S \cap \{8\}} a_i$ for all $S \subseteq N$. So, the restricted game is given by

$$r_3(S) = \begin{cases} a_8 & \text{if } S = \{4,8\} \\ 0 & \text{otherwise.} \end{cases}$$

We find that $Nuc_4(N_3, r_3) = Nuc_8(N_3, r_3) = \frac{a_8}{2}$.

4. The game with permission structure (N_4, v_4, D_4) is given by $N_4 = \{7\}$, $v_4 = v|_{N_4}$ and $D_4 = \emptyset$. Clearly $v_4(S) = r_4(S) = a_7$ for $S = \{7\}$, and thus $Nuc_7(N_4, r_4) = a_7$.

By Proposition 8.4.9 we then have
$Nuc(N, r) = (0, \frac{a_9}{2} + \frac{a_{10}}{3}, 0, \frac{a_8}{2}, 0, \frac{a_{10}}{3}, a_7, \frac{a_8}{2}, \frac{a_9}{2}, \frac{a_{10}}{3})$. □

In the example above, there was only one level of intermediaries, but the algorithm also works if there are more levels of intermediaries between buyers and sellers.

Chapter 9

On 1-convexity and nucleolus of co-insurance games

9.1 Introduction

In this chapter we consider the situation of an agent who wants to insure itself against certain risks. In many practical situations the risks are too large to be insured by only one insurance company, for example environmental pollution risk. As a result, several companies share the liability and premium. In such a risk sharing situation two important practical questions arise: which premium the insurance companies have to charge and how should the companies split the risk and the premium keeping themselves as much competitive as possible and at the same time obtaining a fair division? In Fragnelli and Marina (2004) the problem is approached from a game theoretical point of view through the construction of a cooperative game, the so-called co-insurance game. In this chapter we show that a co-insurance game possesses several interesting properties that allow to study the non-emptiness and the structure of the core, and to construct an efficient algorithm for computing the nucleolus. This chapter is based on Driessen, Fragnelli, Katsev and Khmelnitskaya (2009).

The interest in the class of co-insurance games is not only because they reflect well defined actual economic situations but also by the fact that any arbitrary nonnegative monotone cooperative game may be represented in the form of a co-insurance game. This allows to glance into the nature of nonnegative monotone games from another angle and by that to discover new properties and peculiarities.

This chapter is organized as follow: Section 9.2 is a preliminary section, in Section 9.3 the class of co-insurance games is described and in Section 9.4 the algorithm for computing the nucleolus of a co-insurance game is given.

9.2 Preliminaries

For a game $(N, v) \in \mathcal{G}^N$ we consider the vector $m^v \in \mathbb{R}^n$ of marginal contributions to the grand coalition, the so-called *marginal worth vector*, defined as

$$m_i^v = v(N) - v(N\backslash\{i\}), \text{ for all } i \in N,$$

and the *gap* vector $g^v \in \mathbb{R}^{2^N}$ defined as

$$g^v(S) = \begin{cases} \sum_{i\in S} m_i^v - v(S), & S \subseteq N,\ S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases}$$

i.e., the gap vector measures for every coalition $S \subseteq N$ the total coalitional surplus of the marginal contributions to the grand coalition over its worth. In fact, $g^v(S) = -e(S, m^v)$, with $e(S, m^v)$ the excess of S in game (N, v) at vector $x = m^v$.

It is easy to check that in any game $(N, v) \in \mathcal{G}^N$, the vector m^v yields an upper bound for payoff vectors in the core of (N, v), namely $x_i \leq m_i^v$, for any $x \in C(N, v)$ and all $i \in N$. In particular, the condition $v(N) \leq \sum_{i\in N} m_i^v$ is a necessary (but not sufficient) condition for non-emptiness of the core of an arbitrary game (N, v), i.e., a strictly negative gap of the grand coalition $g^v(N) < 0$ implies $C(N, v) = \emptyset$.

Proposition 9.2.1 *For every convex game $(N, v) \in \mathcal{G}_N$ it holds that*

$$g^v(N) \geq 0 \textit{ and } g^v(N) \geq g^v(S) \textit{ for all } S \subset N.$$

Proof. The inequality $g^v(N) \geq 0$ follows directly from the nonemptiness of the core of any convex game. Next notice that for any $S \subset N$,

$$g^v(N) - g^v(S) = \sum_{i\in N\backslash S} \big[v(N) - v(N\backslash\{i\}\big] - \big[v(N) - v(S)\big].$$

Denote elements of $N\backslash S$ by $i_1, i_2, \dots i_{n-s}$, i.e., $N\backslash S = \{i_1, i_2, \dots i_{n-s}\}$. Then,

$$v(N) - v(S) = \big[v(N) - v(N\backslash\{i_1\})\big] +$$
$$\big[v(N\backslash\{i_1\}) - v(N\backslash\{i_1, i_2\})\big] + \dots + \big[v(S \cup \{i_{n-s}\}) - v(S)\big].$$

Next, notice that the convexity of (N, v) yields that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for every $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. Applying successively $n - s$ times this inequality, we obtain that for all $S \subseteq N$, $g^v(N) - g^v(S) \geq 0$. □

Next we give the definition of a *1-convex* game, see for instance Driessen (1985). A game $v \in \mathcal{G}_N$ is 1-convex if

$$0 \leq g^v(N) \leq g^v(S), \text{ for all } S \subseteq N, \ S \neq \emptyset. \tag{9.2.1}$$

As it is shown in Driessen and Tijs [35] and Driessen [32], every 1-convex game has a nonempty core. In a 1-convex game (N, v), for every efficient vector $x \in \mathbb{R}^n$, the inequalities $x_i \leq m_i^v$, for all $i \in N$, guarantee that $x \in C(N, v)$. In particular, the characterizing property of a 1-convex game is that the replacement of any single coordinate m_i^v in the vector m^v by the amount of $v(N) - g^v(N \backslash \{i\})$ places the resultant vector $\bar{m}^v(i) = \{\bar{m}_j^v(i)\}_{j \in N}$, given by

$$\bar{m}_j^v(i) = \begin{cases} v(N) - g^v(N \backslash \{i\}) = m_i^v - g^v(N), & j = i, \\ m_j^v, & j \neq i, \end{cases}$$

into the core $C(N, v)$. Moreover, in a 1-convex game the set of vectors $\{\bar{m}^v(i)\}_{i \in N}$ creates a set of extreme points of the core, which in turn coincides with their convex hull, i.e., $C(N, v) = \text{Conv}(\{\bar{m}^v(i)\}_{i \in N})$. Besides, the nucleolus $Nuc(N, v)$ occupies the central position in the core coinciding with the barycenter of the core vertices, and is given by

$$Nuc_i(N, v) = \frac{1}{n} \sum_{i \in N} \bar{m}^v(i) = m_i^v - \frac{g^v(N)}{n}, \ i \in N. \tag{9.2.2}$$

So, the nucleolus coincides with the equal allocation of nonseparable contribution the amount of $g^v(N)$ over the players, i.e., every player gets its marginal contribution to the grand coalition minus an equal share in the gap $g^v(N)$ of the grand coalition. That presents a special property of the class of 1-convex games. Although the nucleolus is defined as a solution to a lexicographical optimization problem that, in general, is difficult to compute, for 1-convex games it appears to be linear and thus simple to determine.

By definition of 1-convexity (9.2.1) and from Proposition 9.2.1 we easily obtain the next proposition.

Proposition 9.2.2 *A convex game* $(N, v) \in \mathcal{G}^N$ *is 1-convex if and only if*

$$g^v(N) = g^v(S), \text{ for all } S \subseteq N, \ S \neq \emptyset.$$

In the next section we define the so-called co-insurance game. It appears that this game is closely related to the well-known bankruptcy game, see e.g. Aumann and Maschler (1985) and O'Neill (1982). A bankruptcy problem is a pair (E, d) where $E \in \mathbb{R}_+$ is the estate and $d \in \mathbb{R}_+^n$ a vector of (positive) claims with $d(N) = \sum_{i \in N} d_i > E$, i.e., the total claim of the creditors is bigger than the remaining estate. Aumann and Maschler (1985) defined the corresponding *bankruptcy game* as the game in $\mathcal{G}^N$ with characteristic function $v_{E,d}$ given by

$$v_{E,d}(S) = \begin{cases} \max\{E - d(N \backslash S), 0\}, & S \subseteq N, \ S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

To conclude this preliminary section, recall that a player i is a *veto-player* in the game $(N, v) \in \mathcal{G}^N$ if $v(S) = 0$, for every $S \subseteq N \setminus i$. (For notational convenience, in this chapter we often write $\{i\}$ just as i.) A game $(N, v) \in \mathcal{G}^N$ is a *veto-rich* game if it has at least one veto-player.

9.3 Co-insurance games and their core

Consider the problem in which a risk is evaluated to be too heavy for a single insurance company, but it can be insured by a finite set N of companies that share a given risk $\mathcal{R}$ and premium Π. First, it is assumed that every company $i \in N$ expresses the valuation of a random variable X (that is, the expected claim payment and security considerations with respect to insurable risks X) through a real-valued nonnegative functional $H_i(X)$ on the class of random variables. For any subset $S \subseteq N$ of companies, let $\mathcal{A}(S) = \{(X_i)_{i\in S} \mid \sum_{i\in S} X_i = \mathcal{R}\}$ represent the (non-empty) set of feasible decompositions of risk $\mathcal{R}$. Second, by hypothesis, it is supposed, for every $S \subseteq N$, that an optimal decomposition of the risk exists, so that $\min\{\sum_{i\in S} H_i(X_i) \mid (X_i)_{i\in S} \in \mathcal{A}(S)\} := \mathcal{P}(S)$ is well-defined. Here $\mathcal{P}(S)$ can be seen as the evaluation of the optimal decomposition of the risk $\mathcal{R}$ by the companies in coalition S as a whole. From this, together with $H_i(0) = 0$ for all $i \in N$, it follows that the real-valued set function $\mathcal{P}$ is nonnegative and non-increasing, i.e., for all $S \subseteq T \subseteq N$, $0 \leq \mathcal{P}(T) \leq \mathcal{P}(S)$.

For a given premium Π and an evaluation function $\mathcal{P}\colon 2^N \to \mathbb{R}$, Fragnelli and Marina (2004) define the associated *co-insurance game* $(N, v_{\Pi,\mathcal{P}}) \in \mathcal{G}_N$ by the characteristic function

$$v_{\Pi,\mathcal{P}}(S) = \begin{cases} \max\{0, \Pi - \mathcal{P}(S)\}, & S \subseteq N,\ S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

By definition, the co-insurance game $(N, v_{\Pi,\mathcal{P}})$ is nonnegative and from the fact that $\mathcal{P}$ is non-increasing, it easily follows that the game is monotone, i.e., for all $S \subseteq T \subseteq N$, $0 \leq v_{\Pi,\mathcal{P}}(S) \leq v_{\Pi,\mathcal{P}}(T)$.

In the framework of the co-insurance game, we consider the evaluation function $\mathcal{P}$ being fixed, while the premium Π is a variable quantity varying from small up to sufficiently large amounts. In order to avoid trivial situations, let the premium Π be large enough so that $\Pi > \mathcal{P}(N)$. The following facts are already shown in Fragnelli and Marina (2004).

1. If the premium Π is small enough in that $\Pi \leq \max_{i\in N} \mathcal{P}(N\backslash\{i\})$, then the co-insurance game is balanced. In particular it holds that $C(N, v_{\Pi,\mathcal{P}})$ contains the efficient allocation $\xi = X(N, v_{\Pi,\mathcal{P}})$, with $\xi_{i^*} = v_{\Pi,\mathcal{P}}(N)$ for some $i^* \in \arg\max_{i\in N} \mathcal{P}(N\backslash\{i\})$, and $\xi_i = 0$ for all $i \neq i^*$.

2. If $\Pi > \bar{\alpha}_{\mathcal{P}} = \sum_{i\in N} \big[\mathcal{P}(N\backslash\{i\}) - \mathcal{P}(N)\big] + \mathcal{P}(N)$, then $C(N, v_{\Pi,\mathcal{P}}) = \emptyset$.

3. For all $\Pi \leq \bar{\alpha}_{\mathcal{P}}$,

   $C(v_{\Pi,\mathcal{P}}) \neq \emptyset$

when $\mathcal{P}$ satisfies the reduced concavity condition

$$\mathcal{P}(S) - \mathcal{P}(S \cup \{i\}) \geq \mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N), \text{ for every } i \in N \backslash S \text{ and } S \subseteq N. \quad (9.3.3)$$

To ensure strictly positive worth $v_{\Pi,\mathcal{P}}(S) > 0$ for every coalition $S \subseteq N$, $S \neq \emptyset$, we assume that the premium Π is strictly bounded from below by the critical number $\underline{\alpha}_{\mathcal{P}} = \max_{i \in N} \mathcal{P}(\{i\})$. For all $\Pi \geq \underline{\alpha}_{\mathcal{P}}$, we have

$$m_i^{v_{\Pi,\mathcal{P}}} = v_{\Pi,\mathcal{P}}(N) - v_{\Pi,\mathcal{P}}(N \backslash \{i\}) = \mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N), \text{ for every } \ i \in N, \quad (9.3.4)$$

and for any $S \subseteq N$, $S \neq \emptyset$,

$$g^{v_{\Pi,\mathcal{P}}}(S) = \sum_{i \in S} m_i^{v_{\Pi,\mathcal{P}}} - v_{\Pi,\mathcal{P}}(S) = \sum_{i \in S} \big[\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)\big] + \mathcal{P}(S) - \Pi. \quad (9.3.5)$$

In what follows we distinguish the two cases $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$ and $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$.

We first consider the case $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$. Then, for $\Pi = \bar{\alpha}_{\mathcal{P}}$, the nonemptiness of the core $C(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is equivalent to 1-convexity of the co-insurance game $(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$.

Theorem 9.3.1 *Let $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$ and $\Pi = \bar{\alpha}_{\mathcal{P}}$. Then the following equivalences hold:*

(i) The co-insurance game $(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is balanced.

(ii) The core $C(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is a singleton and coincides with the marginal worth vector $m^{v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}}$.

(iii) The evaluation function $\mathcal{P}$ meets the so-called 1-concavity condition

$$\mathcal{P}(S) - \mathcal{P}(N) \geq \sum_{i \in N \backslash S} \big[\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)\big], \text{ for all } S \subseteq N, \ S \neq \emptyset. \quad (9.3.6)$$

(iv) The co-insurance game $(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is 1-convex.

Proof. From (9.3.5) it follows that for all $\Pi \geq \underline{\alpha}_{\mathcal{P}}$,

$$\bar{\alpha}_{\mathcal{P}} = \sum_{i \in N} \big[\mathcal{P}(N \backslash \{i\}) - \mathcal{P}(N)\big] + \mathcal{P}(N) = g^{v_{\Pi,\mathcal{P}}}(N) + \Pi.$$

By hypothesis $\bar{\alpha}_{\mathcal{P}} \geq \underline{\alpha}_{\mathcal{P}}$, therefore, applying the last equality to $\Pi = \bar{\alpha}_{\mathcal{P}}$, we obtain that

$$g^{v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}}(N) = 0. \quad (9.3.7)$$

Since for any game $(N, v) \in \mathcal{G}^N$, the marginal worth vector m^v provides an upper bound for the core, a game v with zero gap $g^v(N) = 0$ can possess at most one core allocation coinciding with m^v, which is $m^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}}$ in case of the co-insurance game $(N, v_{\bar{\alpha}_\mathcal{P},\mathcal{P}})$. Next notice that the 1-concavity condition (9.3.6) is equivalent to

$$\sum_{i\in S}\big[\mathcal{P}(N\backslash\{i\})-\mathcal{P}(N)\big] \geq \sum_{i\in N}\big[\mathcal{P}(N\backslash\{i\})-\mathcal{P}(N)\big]+\mathcal{P}(N)-\mathcal{P}(S), \tag{9.3.8}$$

for every $S \subseteq N$, $S \neq \emptyset$. This is the same as that the marginal worth vector $m^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}}$ satisfies the core constraints

$$\sum_{i\in S} m_i^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}} \geq \bar{\alpha}_\mathcal{P} - \mathcal{P}(S) = v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}(S), \text{ for all } S \subseteq N \ S \neq \emptyset.$$

It follows that the marginal worth vector $m^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}} \in C(N, v_{\bar{\alpha}_\mathcal{P},\mathcal{P}})$, if and only if the evaluation function $\mathcal{P}$ satisfies the 1-concavity condition (9.3.6). Moreover, because of (9.3.5), the inequality (9.3.8) is equivalent to

$$g^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}}(N) \leq g^{v_{\bar{\alpha}_\mathcal{P},\mathcal{P}}}(S), \text{ for all } S \subseteq N, \ S \neq \emptyset,$$

which is together with (9.3.7) equivalent to 1-convexity of the co-insurance game $(N, v_{\bar{\alpha}_\mathcal{P},\mathcal{P}})$.
□

Remark 9.3.2 Notice that our 1-concavity condition (9.3.6) is weaker than the condition of reduced concavity (9.3.3) used in Fragnelli and Marina (2004).

Theorem 9.3.3 *If for some fixed premium $\Pi^* \geq \underline{\alpha}_\mathcal{P}$, the co-insurance game $v_{\Pi^*,\mathcal{P}}$ is 1-convex, then for every premium Π, $\underline{\alpha}_\mathcal{P} \leq \Pi \leq \Pi^*$, the corresponding co-insurance game $(N, v_{\Pi,\mathcal{P}})$ is 1-convex.*

Proof. For all $\Pi \geq \underline{\alpha}_\mathcal{P}$, due to (9.3.5) it holds that for every $S \subseteq N$, $S \neq \emptyset$, the gap $g^{v_{\Pi,\mathcal{P}}}(S)$ is a decreasing linear function of the variable Π, while the difference $g^{v_{\Pi,\mathcal{P}}}(S) - g^{v_{\Pi,\mathcal{P}}}(N)$ is constant for all Π. Whence, it follows that if for some fixed premium $\Pi^* \geq \underline{\alpha}_\mathcal{P}$ the co-insurance game $v_{\Pi^*,\mathcal{P}}$ is 1-convex, i.e., for all $S \subseteq N$, $S \neq \emptyset$, the inequality (9.2.1) holds, then this inequality remains valid for every premium $\underline{\alpha}_\mathcal{P} \leq \Pi \leq \Pi^*$, i.e., all games $v_{\Pi,\mathcal{P}}$ appear to be 1-convex as well. □

The next theorem follows easily from Theorem 9.3.1 and Theorem 9.3.3.

Theorem 9.3.4 *Let $\bar{\alpha}_\mathcal{P} \geq \underline{\alpha}_\mathcal{P}$. If the evaluation function $\mathcal{P}$ satisfies the 1-concavity condition* (9.3.6)*, then for any premium $\underline{\alpha}_\mathcal{P} \leq \Pi \leq \bar{\alpha}_\mathcal{P}$,*

(i) the corresponding co-insurance game $(N, v_{\Pi,\mathcal{P}})$ is 1-convex.

(ii) the core $C(N, v_{\Pi,\mathcal{P}}) \neq \emptyset$.

(*iii*) *the nucleolus* $Nuc(N, v_{\Pi,\mathcal{P}})$ *is the barycenter of the core* $C(N, v_{\Pi,\mathcal{P}})$ *and is given by*

$$Nuc_i(N, v_{\Pi,\mathcal{P}}) = \mathcal{P}(N\backslash\{i\}) - \mathcal{P}(N) + \frac{\Pi - \bar{\alpha}_{\mathcal{P}}}{n}, \ i \in N.$$

Proof. The first statement follows directly from Theorem 9.3.1 and Theorem 9.3.3. Next recall the already mentioned above results obtained in Driessen and Tijs [35] and Driessen [32], stating that every 1-convex has a nonempty core and its nucleolus being the barycenter of the core is given by the formula (9.2.2). These facts, together with (9.3.4) and (9.3.5), complete the proof. □

In words, the third statement of Theorem 9.3.4 means that the nucleolus of these co-insurance games is a linear function of the variable premium such that each incremental premium is shared equally among the insurance companies. Geometrically, the nucleoli payoffs follow a straight line to end up at the marginal worth vector yielding payoff $\mathcal{P}(N\backslash\{i\}) - \mathcal{P}(N)$ to player $i \in N$.

Remark 9.3.5 The statement of Theorem 9.3.4 remains in force if the 1-concavity condition (9.3.6) for the evaluation function $\mathcal{P}$ is replaced by any one of the equivalent conditions given by Theorem 9.3.1, in particular if $C(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}) \neq \emptyset$ or if the co-insurance game $(N, v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}})$ is 1-convex.

Remark 9.3.6 There is an alternative way to get this expression for the nucleolus. It is possible to use the approach of Quant *et al.* (2005). In this paper the authors describe the class of compromise stable games and present a method to calculate the nucleolus for such kind of games. It is possible to show that in this situation a co-insurance game is compromise stable.

We now turn to the case that $\bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}}$. In this case it turns out that, even if the co-insurance game $v_{\bar{\alpha}_{\mathcal{P}},\mathcal{P}}$ is 1-convex, for the co-insurance game $v_{\Pi,\mathcal{P}}$ corresponding to premium $\Pi < \bar{\alpha}_{\mathcal{P}}$ the 1-convexity may be lost immediately. This happens when lowering the premium, the co-insurance worth of at least one coalition remains at zero level (see proof of Theorem 9.3.3). For instance, consider the following example.

Example 9.3.7 Take $N = \{1, 2, 3\}$ and let the evaluation function $\mathcal{P}$ be given by $\mathcal{P}(\{1\}) = 5$, $\mathcal{P}(\{2\}) = 4$, $\mathcal{P}(\{3\}) = 3$, $\mathcal{P}(\{1, 2\}) = \mathcal{P}(\{1, 3\}) = \mathcal{P}(\{2, 3\}) = 2$, and $\mathcal{P}(\{1, 2, 3\}) = 1$. Obviously, $4 = \bar{\alpha}_{\mathcal{P}} < \underline{\alpha}_{\mathcal{P}} = 5$.

If the premium $\Pi = 4$, then the corresponding 3-person co-insurance game $(N, v_{4,\mathcal{P}})$ is given by $v_{4,\mathcal{P}}(\{1\}) = v_{4,\mathcal{P}}(\{2\}) = 0$, $v_{4,\mathcal{P}}(\{3\}) = 1$, $v_{4,\mathcal{P}}(\{1, 2\}) = v_{4,\mathcal{P}}(\{1, 3\}) = v_{4,\mathcal{P}}(\{2, 3\}) = 2$ and $v_{4,\mathcal{P}}(\{1, 2, 3\}) = 3$. This is a 1-convex game with gap $g^{v_{4,\mathcal{P}}}(\{1, 2, 3\}) = 0$. Therefore the unique core allocation is the marginal worth vector $m^{v_{4,\mathcal{P}}} = (1, 1, 1)$.

If the premium $\Pi = 3$, then the corresponding 3-person co-insurance game is given by $v_{3,\mathcal{P}}(\{1\}) = v_{3,\mathcal{P}}(\{2\}) = v_{3,\mathcal{P}}(\{3\}) = 0$, $v_{3,\mathcal{P}}(\{1, 2\}) = v_{3,\mathcal{P}}(\{1, 3\}) = v_{3,\mathcal{P}}(\{2, 3\}) = 1$

and $v_{3,\mathcal{P}}(\{1,2,3\}) = 2$. This is a symmetric 1-convex game with constant gap $g^{v_{3,\mathcal{P}}}(S) = 1$ for all non-empty coalitions $S \subseteq N$. Its core is the triangle with the three extreme points $(1,1,0)$, $(1,0,1)$, $(0,1,1)$.

For any premium $2 \leq \Pi < 3$, the corresponding co-insurance game $(N, v_{\Pi,\mathcal{P}})$ is zero-normalized and symmetric with worths 0, $\Pi - 2$ and $\Pi - 1$ for the one-player, two-player and three-player coalitions, respectively. In this case the 1-convexity fails because the gap of singletons equal to 1 is strictly less than the gap of the full player set amounting to $4 - \Pi$.

Remark 9.3.8 Notice that the well-known bankruptcy game given in Aumann and Maschler (1985) is an example of the co-insurance game. Indeed, if for each insurance company $i \in N$ there exists a fixed "claim" $d_i \geq 0$ such that $\mathcal{P}(S) = \sum_{i \in N \setminus S} d_i$, for all $S \subseteq N$, $S \neq \emptyset$, then the co-insurance game reduces to a bankruptcy game. In the bankruptcy framework, the particular evaluation function $\mathcal{P}$ is nonnegative and non-increasing with $\mathcal{P}(N) = 0$. Therefore, $\bar{\alpha}_{\mathcal{P}} = \sum_{i \in N} d_i$, $\underline{\alpha}_{\mathcal{P}} = \sum_{i \in N} d_i - \min_{i \in N} d_i$, and thus $\underline{\alpha}_{\mathcal{P}} \leq \bar{\alpha}_{\mathcal{P}}$. In the bankruptcy setting Theorem 9.3.4 expresses the fact that the nucleolus provides equal losses to all creditors (insurance companies) with respect to their individual claims, if the estate (premium) varies between $\sum_{i \in N} d_i - \min_{i \in N} d_i$ and $\sum_{i \in N} d_i$, which well agrees with the Talmud rule for bankruptcy situations studied exhaustively in Aumann and Maschler (1985).

9.4 Algorithm for computing the nucleolus

The nucleolus of a co-insurance game is easy to compute in case it is a linear function of the premium as stated in Theorem 9.3.4. In this section we introduce an algorithm that quite easily computes the nucleolus of a co-insurance game when it is not linear in the premium. To do so, we first uncover the relation between a class of co-insurance games, in particular bankruptcy games, and the class of Davis-Maschler reduced games of monotone veto-rich games with respect to the nucleolus obtained by deleting a veto-player. Second, we provide an algorithm for computing the nucleolus for games which are such Davis-Maschler reduced games.

In the following we denote $\mathcal{G}_m^N$ as the class of monotone games in N and, for a finite subset $N \subset \mathbb{N} \setminus \{0\}$, we denote $N_0 = N \cup \{0\}$. Further, Let $\mathcal{G}_+^{N_0}$ be the class of nonnegative games with player set N_0 and characteristic function v_0 satisfying $v_0(N_0) \geq v_0(S)$, for all $S \subseteq N_0$. Notice that $\mathcal{G}_m^{N_0} \subset \mathcal{G}_+^{N_0}$. In the following we only consider games in $\mathcal{G}_+^{N_0}$ with player 0 being a veto-player. Further, the class $\mathcal{V}^N$ is the subclass of games $(N, v) \in \mathcal{G}^N$, such that (N, v) is the Davis-Maschler reduced game of a monotone, veto-rich game $(N_0, v_0) \in \mathcal{G}_m^{N_0}$ with respect to the nucleolus payoff, obtained by deleting the veto-player 0. Notice that every game $(N, v) \in \mathcal{V}^N$ is balanced, what follows from the balancedness of monotone veto-rich games (see Arin and Feltkamp (1997)) and the fact that the Davis-Maschler reduced game inherits the core property. Moreover, since the nucleolus gives a payoff to a veto-player that is at least the same as to any other player (see Arin and Feltkamp (1997)), and because the worth of the grand coalition in any nontrivial monotone game is strictly positive, the nucleolus payoff to a veto-player is strictly positive in every nontrivial monotone veto-rich game.

With every monotone game $(N, v) \in \mathcal{G}_m^N$ we associate the following monotone veto-rich game $(N_0, v_0) \in \mathcal{G}_m^{N_0}$ defined by

$$v_0(S) = \begin{cases} 0, & S \not\ni 0, \\ v(S\backslash\{0\}), & S \ni 0, \end{cases} \quad \text{for all } S \subseteq N_0. \tag{9.4.9}$$

For a game $(N, v) \in \mathcal{G}^N$ and $a \in \mathbb{R}_+$ we define the game $(N, v^{-a}) \in \mathcal{G}^N$ by the characteristic function

$$v^{-a}(S) = \max\{0, v(S) - a\}, \text{ for all } S \subseteq N. \tag{9.4.10}$$

Notice that $(N, v^{-a}) \in \mathcal{G}_m^N$ if $(N, v) \in \mathcal{G}_m^N$.

Theorem 9.4.1 *It holds that*

(i) every game $(N, v) \in \mathcal{V}^N$ can be presented in the form of a co-insurance game $(N, v_{\Pi,\mathcal{P}}) \in \mathcal{G}^N$.

(ii) if $(N, v_{\Pi^*,\mathcal{P}}) \in \mathcal{V}^N$, then also $(N, v_{\Pi,\mathcal{P}}) \in \mathcal{V}^N$ for every premium $\Pi \leq \Pi^*$.

Proof. (i). Consider $(N, v) \in \mathcal{V}^N$. By definition of $\mathcal{V}^N$ there exists an $(N_0, v_0) \in \mathcal{G}_m^{N_0}$ such that (N, v) is the Davis-Maschler reduced game with respect to the nucleolus derived from (N_0, v_0) by deleting veto-player 0. Let $x = Nuc(N_0, v_0)$ be the nucleolus payoff vector of the game (N_0, v_0). By definition of the Davis-Maschler reduced game it holds that

$$v(S) = \begin{cases} 0, & S = \emptyset, \\ v_0(N_0)) - x_0, & S = N, \\ \max\{v_0(S), v_0(S \cup \{0\}) - x_0\} = & \\ \qquad \max\{0, v_0(S \cup \{0\}) - x_0\}, & S \subset N, S \neq \emptyset. \end{cases}$$

Take some positive $k > \frac{v_0(N_0)}{x_0} - 1$. Set $\Pi = kx_0$ and

$$\mathcal{P}(S) = (k+1)x_0 - v_0(S \cup \{0\}), \text{ for all } S \subseteq N, \ S \neq \emptyset.$$

It follows that

$$v(S) = \max\{0, \Pi - \mathcal{P}(S)\}, \text{ for all } S \subseteq N, \ S \neq \emptyset.$$

(ii). First, recall that any co-insurance game $(N, v_{\Pi^*,\mathcal{P}})$ is monotone. Consider a monotone game $(N, v) \in \mathcal{G}_m^N$ together with its associated veto-rich game $(N_0, v_0) \in \mathcal{G}_m^{N_0}$. Set $A = x_0$, where $x = Nuc(N_0, v_0)$. It is shown in Arin and Feltkamp (1997) that $\frac{v(N)}{n+1} < A \leq v(N)$. Further it follows easily that $v^{-A} \in \mathcal{G}^N$ is the Davis-Maschler reduced game of the game (N_0, v_0) with respect to x obtained by deleting the veto-player 0. So, by definition of $\mathcal{V}^N$, $(N, v^{-A}) \in \mathcal{V}^N$. Notice that if $A = v(N)$, $v^{-A} \equiv 0$.

Next we show that if $A < v(N)$, then for every B, $A \le B \le v(N)$, also $(N, v^{-B}) \in \mathcal{V}^N$. Recall that also (N, v^{-A}) is monotone, so the above procedure can also be applied to (N, v^{-A}). Doing so, we get another monotone game, say $(N, v^{-A^1}) \in \mathcal{V}^N$, with $A^1 > A$ when $A < v(N)$. We show that $(N, v^{-B}) \in \mathcal{V}^N$, for all $A \le B \le A^1$. Consider $0 \le C \le A$ and apply the above procedure for every game $(N, v^{-C}) \in \mathcal{G}^N_m$. For $C = 0$ we start with (N, v) and obtain the monotone game $(N, v^{-A}) \in \mathcal{V}^N$. For $C = A$ we start with (N, v^{-A}) and obtain the monotone game $(N, v^{-A^1}) \in \mathcal{V}^N$. Due to the continuity of the nucleolus, we obtain (N, v^{-B}) while C varies between 0 and A. Hence $(N, v^{-B}) \in \mathcal{V}^N$.

When $A^1 < v(N)$ then applying the above procedure to the monotone game (N, v^{-A^1}) we obtain a game $(N, v^{-A^2}) \in \mathcal{V}^N$ with $A^2 > A^1$ and so on. Since on each step k, $A^k - A^{k-1} > \frac{v(N)-A^{k-1}}{n+1}$, any number $A \le B \le v(N)$ can be reached within at most $\frac{B-A}{v(N)-B}(n+1)$ steps. Therefore, for every B, $A \le B \le v(N)$, $(N, v^{-B}) \in \mathcal{V}^N$.

Now, the proof follows from item (i) of the theorem and the fact that for any co-insurance game $(N, v_{\Pi,\mathcal{P}})$ and for any $a \in \mathbb{R}_+$, the game $(N, v^{-a}_{\Pi,\mathcal{P}})$ is also a co-insurance game with premium equal to $\Pi - a$, i.e., $v^{-a}_{\Pi,\mathcal{P}} = v_{\Pi-a,\mathcal{P}}$. □

We now present an algorithm for constructing a payoff vector, say $x \in \mathbb{R}^N$, of a game $(N, v) \in \mathcal{V}_N$. For a game $(K, w) \in \mathcal{G}^K$, define for every $S \subset K$ the number $\tau(S, w)$ by

$$\tau(S,w) = \begin{cases} \frac{v(K)-v(S)}{|K|-s+1}, & S \neq \emptyset, \\ \\ \frac{v(K)}{|K|}, & S = \emptyset, \end{cases}$$

and define $\tau^*(w) = \min_{S \subset K} \; \tau(S, w)$.

Algorithm 9.4.2
Step 0 Set $(K, w) = (N, v)$.

Step 1 Find a coalition $S \subset K$ such that $\tau(S, w) = \tau^*(w)$ and $|S| \le |S'|$ for every other $S' \subset K$ with $\tau(S', w) = \tau^*(w)$.

Step 2 For $i \in N \backslash S$, set $x_i = \tau(S, w)$. When $S = \emptyset$, stop. Otherwise, go to Step 3.

Step 3 Construct the Davis-Maschler reduced game $(S, w^S_x) \in \mathcal{G}^S$. Set $(K, w) = (S, w^S_x)$ and return to Step 1.

Notice that in any Step 2 the payoff x_i is determined for at least one player i. So, the algorithm ends within at most $n = |N|$ applications of Step 2.

Theorem 9.4.3 *For any veto-removed game $(N, v) \in \mathcal{V}^N$, Algorithm 9.4.2 yields the nucleolus payoff vector: $x = Nuc(N, v)$.*

The proof of Theorem 9.4.3 is obtained by comparing the payoff vector $x \in \mathbb{R}^n$ generated by Algorithm 9.4.2 with the vector $y \in \mathbb{R}^{n+1}$ of the associated monotone veto-rich game $(N_0, v_0) \in \mathcal{G}_m^{N_0}$ generated by Algorithm 9.4.4 below and therefore it will be given later. Algorithm 9.4.4 is conceptually close to the algorithm for computing the nucleolus for veto-rich games presented in Arin and Feltkamp (1997).

Algorithm 9.4.4
Step 0 Set $(K, w) = (N_0, v_0)$.

Step 1 Find a coalition $S \subset K$, $S \ni 0$, such that $\tau(S, w) = \min_{\{T \subset K | T \ni 0\}} \tau(T, w)$.

Step 2 For $i \in K \backslash S$, set $y_i = \tau(S, w)$. When $S = \{0\}$, set $y_0 = v_0(N_0) - \sum_{i \in N} y_i$ and stop. Otherwise go to Step 3.

Step 3 Construct the Davis-Maschler reduced game $(S, w_y^S) \in \mathcal{G}^S$. Set $(K, w) = (S, w_y^S)$ and return to Step 1.

Again, notice that in any Step 2 the payoff y_i is determined for at least one player i and that y_0 is determined in Step 2 as soon as every y_i, $i \neq 0$, has been determined. So, the algorithm ends within at most $n = |N|$ applications of Step 2.

Theorem 9.4.5 *For any veto-rich game $(N_0, v_0) \in \mathcal{G}_+^{N_0}$, Algorithm 9.4.4 yields the nucleolus payoff vector: $y = Nuc(N_0, v_0)$.*

Proof. Let $(N_0, v_0) \in \mathcal{G}_+^{N_0}$ with veto-player 0 and denote $x = Nuc(N_0, v_0)$. We show that there is a partition $\mathcal{T}$ of N_0 with the following properties:

1. Every $S \in \mathcal{T}$ is a coalition with maximal excess $e(S, x) = v_0(S) - x(S)$ at $x = Nuc(N_0, v_0)$.
2. Every coalition $S \in \mathcal{T}$ not containing veto-player 0 is a singleton.

We show the existence of such a partition by using the corollary of Kolberg's theorem saying that the collection of coalitions with maximal excess values with respect to the nucleolus is balanced. Therefore there exists a set $S \subseteq N_0$ with maximal excess and containing the veto-player 0. We show that every singleton $\{i\}$, $i \notin S$, also has maximal excess value. Suppose $i \notin S$. Then, again from balancedness it follows that there exists a set $T \subseteq N_0$ with maximal excess value, such that $i \in T$ and $0 \notin T$. If $|T| > 1$ then it follows with $v_0(T) = 0$ when $0 \notin T$, that

$$e(T, x) = v_0(T) - x(T) = -x(T) = -x(\{i\}) - x(T \setminus \{i\}) =$$
$$= e(\{i\}, x) + e(T \setminus \{i\}, x).$$

Since the core of a monotone veto-rich game in $\mathcal{G}_+^{N_0}$ is non-empty, all excesses with respect to the nucleolus payoff vector x are non-positive. So $e(T \setminus \{i\}, x) \leq 0$ and thus $e(\{i\}, x) \geq e(T, x)$. Hence the singleton set $\{i\}$ has maximal excess value, which shows that

$$\mathcal{T} = \{S\} \cup \{\{i\},\ i \notin S\}$$

is a partition of N_0 satisfying the properties 1 and 2. Notice that this partition consists of a single set S containing the veto-player 0 and several singletons.

Denote $e^*(v_0) = \max_{T \subset N} \; e(T, x)$. Then it follows that

$$e^*(v_0) \cdot (|N \setminus S| + 1) = e(S, x) + \sum_{i \in N \setminus S} e(\{i\}, x) =$$

$$= v_0(S) - x(S) - \sum_{i \in N \setminus S} x(i) = v_0(S) - x(N) = v_0(S) - v_0(N).$$

So,

$$e^*(v_0) = \frac{v_0(S) - v_0(N)}{|N \setminus S| + 1}$$

and also for the set S containing the veto-player in the partition it holds that

$$e(S, x) = e^*(v_0) = \frac{v_0(S) - v_0(N)}{|N \setminus S| + 1}.$$

Now, consider a set $S' \ni 0$ such that $e(S', x)$ is not maximal. Then

$$v_0(S') - v_0(N) = e(S', x) + \sum_{i \in N \setminus S'} e(\{i\}, x) < e^*(v_0) + \sum_{i \in N \setminus S'} e(\{i\}, x)$$

$$\leq e^*(v_0) \cdot (|N \setminus S'| + 1).$$

Form this inequality it follows that

$$e^*(v_0) > \frac{v_0(S') - v_0(N)}{|N \setminus S'| + 1}.$$

This shows that the following two conditions are equivalent:
1. Coalition $S \ni 0$ has maximal excess with respect to the nucleolus payoff x.
2. $\frac{v_0(S) - v_0(N)}{|N \setminus S| + 1} = \max_{T \subset N} \frac{v_0(T) - v_0(N)}{|N \setminus T| + 1}$.

Since $\frac{v_0(S) - v_0(N)}{|N \setminus S| + 1} = -\tau(S, v_0)$, we conclude that coalition S with veto-player 0 has maximal excess with respect to the nucleolus payoff x if and only if $\tau(S, v_0) = \min_{T \subset N} \; \tau(T, v_0)$. So Step 1 of the Algorithm 9.4.4 gives us a set S containing veto-player 0, which has maximal excess value with respect to the nucleolus payoff x.

At Step 2 of Algorithm 9.4.4 player $i \notin S$ is assigned value $y_i = \tau(S, v_0)$. So

$$y_i = \tau(S, v_0) = -e^*(v_0) = -(v_0(\{i\}) - x_i) = x_i$$

and thus any player $i \notin S$ is assigned its nucleolus payoff.

Because of the Davis-Maschler consistency of the prenucleolus (and the fact that the prenucleolus of (N_0, v_0) coincides with the nucleolus), it follows that the nucleolus payoffs of the players in the Davis-Maschler reduced game (S, w_S^y) obtained in Step 3 is the same as the nucleolus payoffs of the players in S in the game (N_0, v_0). It only remains to show that the Davis-Maschler reduced game (S, w_S^y) of a game $(K, w) \in \mathcal{G}_+^K$ with K containing veto-player 0 is itself a veto-rich game and belongs to the class $\mathcal{G}_+^S$.

First, notice that $y_i = x_i$ for every $i \notin S$, so the Davis-Maschler reduced game (S, w_S^y) obtained in Step 3 is the Davis-Maschler reduced game (S, w_S^x), i.e., it is the Davis-Maschler reduced game with respect to the nucleolus payoffs. By definition of S we have that $0 \in S$. Take $T \subset S \setminus \{0\}$. Then

$$w_S^x(T) = \max_{U \subseteq K \setminus S} \{w(T \cup U) - x(U)\} = \max_{U \subseteq K \setminus S} \{0 - x(U)\} = 0,$$

because $T \cup U \subseteq K \setminus S$ for every $U \subseteq K \setminus S$ and 0 is veto-player in (K, w). So, player 0 is also veto-player in (S, w_S^x). Further it is evident that (S, w_S^x) is a non-negative game. So, to show that $(S, w_S^x) \in \mathcal{G}_+^S$, it remains to show that $w_S^x(S) \geq w_S^x(U)$ for each $U \subset S$. When $U \subseteq S \setminus \{0\}$ then $w_S^x(U) = 0 \leq w_S^x(S)$. So, it remains to consider the sets U with $0 \in U$. By Step 1 of Algorithm 9.4.4 we have that

$$\tau(S, w) \leq \tau(T', w)$$

for every $T' \subset K$ containing player 0. For $U \subset K$ with $0 \in U$ and $T \subseteq K \setminus S$, denote $T' = U \cup T$. Then $T' \subset K$ and contains 0 and thus

$$\tau(T', w) = \frac{w(K) - w(T')}{|K \setminus T'| + 1} = \frac{w(K) - w(U \cup T)}{|K \setminus (U \cup T)| + 1} \geq \tau(S, w).$$

Hence, also

$$\frac{w(K) - w(U \cup T)}{|K \setminus S| - |T|} \geq \tau(S, w),$$

because $|S| \geq |U| + 1$. From this it follows that

$$w(K) - \tau(S, w)|K \setminus S| \geq w(U \cup T) - \tau(S, w)|T|$$

and thus

$$w(K) - x(K \setminus S) \geq w(U \cup T) - x(T),$$

because by Step 2 of Algorithm 9.4.4, $x_i = y_i = \tau(S, w)$ for every $i \in K \setminus S$. Hence, by definition of the Davis-Maschler reduced game, $w_x^S(U) \leq w_x^S(S)$ for every $U \subset S$. $\square$

The next result follows immediately from Theorem 9.4.5.

Proposition 9.4.6 *For any veto-rich game $(N_0, v_0) \in \mathcal{G}_+^{N_0}$ it holds that*

$$\min_{S \subset N_0 | S \ni 0} \tau(S, v_0) \leq \frac{v_0(N_0) - x_0}{n}$$

with equality if and only if $x_i = x_j$ for all $i, j \neq 0$, where $x = Nuc(N_0, v_0)$ and $n = |N| = |N_0| - 1$.

Proof. Consider a set $U \subset N_0$ containing 0, such that

$$\tau(U, v_0) = \min_{\{S \subset N_0 | S \ni 0\}} \tau(S, v_0)$$

and let $i \in N_0 \setminus U$. By Algorithm 9.4.4, $x_i = \tau(U, v_0)$. Moreover, because the singleton $\{i\}$ has maximal excess $e(\{i\}, x)$, for every $j \in N_0$ it holds that $x_j \geq x_i = \tau(U, v_0)$. So

$$v_0(N_0) - x_0 = \sum_{j \in N_0 \setminus \{0\}} x_j \geq (|N_0| - 1)\tau(U, v_0),$$

which shows the inequality. In case the property holds with equality, then $e(\{i\}, x)$ is maximal for every singleton $\{j\}$, $j \in N_0 \setminus \{0\}$ and so $x_j = x_i$ for all $i, j \neq 0$. □

Next, consider a game $(N, v) \in \mathcal{V}^N$. Because of the definition of $\mathcal{V}^N$, there is a veto-rich game $(N_0, v_0) \in \mathcal{G}_+^{N_0}$ such that (N, v) is the Davis-Maschler reduced game of (N_0, v_0) for coalition N with respect to the nucleolus $Nuc(N_0, v_0)$. Since (N, v) is the Davis-Maschler reduced game of (N_0, v_0) and because the consistency property, it follows that $Nuc(N, v)$ coincides with $Nuc(N_0, v_0)$ on the set N. Since Algorithm 9.4.4 yields $Nuc(N_0, v_0)$, to prove Theorem 9.4.3 it is sufficient to show that the payoff vector x computed by Algorithm 9.4.2 for game (N, v) coincides on the set N with the payoff vector y computed by Algorithm 9.4.4 for game (N_0, v_0).

Proof of Theorem 9.4.3. For game (N_0, v_0), let T_0 denote a subset of N_0 containing 0. We first prove that either in Step 1 of the Algorithm the set $S = \emptyset$ is chosen and the algorithm yields the nucleolus, or that

$$\min_{T \subset N} \tau(T, v)) = \min_{T_0 \subset N_0} \tau(T_0, v_0).$$

Using that $y = Nuc(N_0, v_0)$ is the nucleolus payoff computed by Algorithm 9.4.4, by definition of the Davis-Maschler reduced game we have that $v(S) = \max\{v_0(S \cup \{0\}) - y_0,\ 0\}$, so $v(S) = v_0(S \cup \{0\}) - y_0$ or $v(S) = 0$. Let $S \subset N$ be such that $\tau(S, v) = \min_{T \subset N} \tau(T, v)$. Suppose S is not empty and $v(S) = 0$. Then

$$\tau(S, v)) = \frac{v(N) - v(S)}{|N \setminus S| + 1} = \frac{v(N)}{|N \setminus S| + 1} \geq \frac{v(N)}{|N|} = \tau(\emptyset, v).$$

So, such a set S cannot be chosen in Step 1 of Algorithm 9.4.2, because $\tau(\emptyset, v) \leq \tau(S, v)$ and $|\emptyset| < |S|$. It follows that in Step 1 either $S = \emptyset$ is chosen or a set S such that $v(S) = v_0(S \cup \{0\}) - y_0$.

Now, consider game (N_0, v_0) and let $S_0 \subset N_0$ (so $0 \in S_0$) such that $\tau(S_0, v_0) = \min_{T_0 \subset N_0} \tau(T_0, v_0)$. Similar as above, suppose $S = S_0 \setminus \{0\}$ is not empty and $v(S) = \max\{v_0(S \cup \{0\}) - y_0,\ 0\} = 0 > v_0(S_0) - y_0$. Then

$$\tau(S_0, v_0) = \frac{v_0(N_0) - v_0(S_0)}{|N_0 \setminus S_0| + 1} > \frac{v_0(N_0) - y_0}{|N \setminus S| + 1} \geq \frac{v_0(N_0) - y_0}{|N|},$$

which contradicts Proposition 9.4.6. So, also in Step 1 of Algorithm 9.4.4 a set S_0 is chosen such that either $S = S_0 \setminus \{0\} = \emptyset$ or S satisfies $v(S) = v_0(S_0) - y_0$. From this it follows that to find both $\min_{T \subset N} \ \tau(T, v)$ and $\min_{T_0 \subset N_0} \ \tau(T_0, v_0)$ it is sufficient to consider the same collection of subsets of N, namely the collection consisting of the empty set and the sets S satisfying $v(S) = v_0(S \cup \{0\}) - y_0$. We now consider both possiblities.

First, let S be such that $v(S) = v_0(S \cup \{0\}) - y_0$. Then it follows with $v_0(N_0) = v(N) + y_0$ and $v_0(S_0) = v(S) + y_0$ that

$$\tau(S_0, v_0) = \frac{v_0(N_0) - v_0(S_0)}{|N_0 \setminus S_0| + 1} = \frac{v(N) - v(S)}{|N \setminus S| + 1} = \tau(S, v). \tag{9.4.11}$$

Second, we consider the case that $S = \emptyset$. Then $S_0 = S \cup \{0\} = \{0\}$ and

$$\tau(S_0, v_0) = \tau(\{0\}, v_0) = \frac{v_0(N_0) - v_0(\{0\})}{|N_0|}.$$

Now, we have again two possibilities, namely
(i) $\tau(\{0\}, v_0) = \min_{T_0 \subset N_0} \ \tau(T_0, v_0)$ and
(ii) $\tau(\{0\}, v_0) > \min_{T_0 \subset N_0} \ \tau(T_0, v_0)$.

In case (i) Algorithm 9.4.4 terminates at the first iteration and in Step 2 all players $i \neq 0$ get the same value $y_i = \tau(\{0\}, v_0)$. Because of the coincidence of the nucleolus payoffs on N in the games (N_0, v_0) and (N, v), it follows that every player $i \in N$ has the same nucleolus payoff in game (N, v) and thus, by efficiency, every player $i \in N$ gets nucleolus payoff $\frac{v(N)}{|N|}$ in the game (N, v). Hence, with $i \in N$,

$$\tau(\emptyset, v) = \frac{v(N)}{|N|} = y_i = \frac{v_0(N_0) - v_0(\{0\})}{|N_0|} = \tau(\{0\}, v_0).$$

It follows that

$$\tau(\emptyset, v) = \tau(\{0\}, v_0) \leq \min_{T \subset N} \ \tau(T \cup \{0\}, v_0) = \min_{T \subset N} \ \tau(T, v),$$

where the last equality follows from equation (9.4.11). So, we conclude that in this case $\min_{T_0 \subset N_0} \ \tau(T_0, v_0) = \min_{T \subset N} \ \tau(T, v)$.

We now consider case (ii): $\tau(\{0\}, v_0) > \min_{T_0 \subset N_0} \ \tau(T_0, v_0)$. Then there is a non-empty set $S \subset N$ such that

$$\tau(S \cup \{0\}) = \min_{T_0 \subset N_0} \ \tau(T_0, v_0).$$

Then, according to equation (9.4.11) we have that $\tau(S_0, v_0) = \tau(S, v)$ and by Proposition 9.4.6

$$\tau(S, v) = \min_{T_0 \subset N_0} \ \tau(T_0, v_0) \leq \frac{v_0(N_0) - y_0}{|N_0| - 1} = \frac{v(N)}{|N|} = \tau(\emptyset, v).$$

where the second equality is because (N, v) is the reduced game of (N_0, v_0). So, either $\tau(\emptyset, v) = \tau(S, v)$ or $\tau(\emptyset, v) > \tau(S, v)$. If $\tau(\emptyset, v) = \tau(S, v)$ then

$$\min_{T_0 \subset N_0} \ \tau(T_0, v_0) = \frac{v_0(N_0) - y_0}{|N_0| - 1} = \frac{v_0(N_0) - y_0}{|N|},$$

and from Proposition 9.4.6 it follows that $Nuc_i(N, v) = \frac{v(N)}{|N|}$. On the other hand, $\tau(\emptyset, v) = \min_{T \subset N} \ \tau(T, v)$, so in Step 1 of Algorithm 9.4.2 the empty set is chosen and Algorithm 9.4.2 indeeds yields the nucleolus. Otherwise, when $\tau(\emptyset, v) > \tau(S, v)$, then there is a non-empty set S such that $\tau(S, v) = \min_{T \subset N} \ \tau(T, v)$ and it follows with equation (9.4.11) that $\tau(S, v) = \tau(S \cup \{0\}, v_0)$. So, we conclude that either in Step 1 of Algorithm the set $S = \emptyset$ is chosen, or that $\min_{T \subset N} \ \tau(T, v) = \min_{T_0 \subset N_0} \ \tau(T_0, v_0)$.

To conclude the proof, it remains to consider the situation that in Step 1 of Algorithm 9.4.2 a non-empty set $S \subset N$ is chosen. From the results above we obtain that we can choose coalition $S \cup \{0\}$ in Step 1 of Algorithm 9.4.4 and that $\tau(S, v) = \tau(S_0, v_0)$. So, at the first iteration both algorithms do the same in the Steps 2: set $x_i = y_i = \tau(S, v) = \tau(S_0, v_0)$ for every $i \in N \setminus S = N_0 \setminus (S \cup \{0\})$.

At Step 3 of Algorithm 9.4.2 the reduced game (S, v_x^S) is obtained and in Algorithm 9.4.4 the reduced game for $S \cup \{0\}$ of (N_0, v_0). It is evident that (S, v_x^S) is the Davis-Maschler reduced game of the reduced game obtained in Algorithm 9.4.4 for $S \cup \{0\}$, because both games are Davis-Maschler reduced games of (N_0, v_0) with respect to the nucleolus. So, when returning to Step 1, we get in the next iteration the same situation as in the previous iteration. Again there is a veto-rich game and a veto-removed game and the latter is the Davis-Maschler reduced game of the first with respect to the nucleolus. So, repeating the reasonings above, at any iteration the two algorithms assign the same payoff to a subset of players in N. $\square$

Notice that the proof above shows that Algorithm 9.4.2 indeed generates the nucleolus payoff of a game $(N, v) \in \mathcal{V}^N$. There is no need to use Algorithm 9.4.4 and neither it is needed to know the game (N_0, v_0). Since the second algorithm is only used to prove the working of Algorithm 9.4.2, it is only needed to know that (N, v) is the Davis-Maschler reduced game of a game (N_0, v_0).

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Samenvatting

Oplossingen voor Spelen onder Restricties voor Samenwerking

Speltheorie bestudeert situaties waarin meerdere agenten met mogelijk conflicterende doelen optreden. De niet-coöperatieve speltheorie houdt zich voornamelijk bezig met het modelleren van de strategische interactie tussen de agenten. In de coöperatieve speltheorie worden de onderliggende strategische procedures buiten beschouwing gelaten. In plaats daarvan wordt een coöperatief spel gevormd door een verzameling van spelers en een karakteristieke functie die voor elke coalitie van spelers een (maximaal te bereiken) waarde aangeeft. Gegeven een spel concentreert de coöperatieve speltheorie zich op vragen zoals 'welke coalitie wordt gevormd?' en met name de vraag 'hoe wordt de waarde van de gevormde coalitie verdeeld over de leden van de coalitie?'. Een oplossing voor coöperatieve spelen is een functie die voor elk spel een uitbetaling aan elk van de spelers toekent.

De karakterisering van oplossingen door middel van axioma's is een centraal thema in de coöperatieve speltheorie. Een oplossing wordt gekarakteriseerd door een stelsel van axioma's (op een deelklasse van spelen) als het de enige oplossing is die voldoet aan alle axioma's op de betreffende deelklasse. Verschillende stelsels van axioma's leiden in het algemeen tot verschillende oplossingen. Veel oplossingen voldoen aan het efficiëntie axioma, d.w.z. in elk spel wordt de waarde van de grote coalitie, dit is de coalitie die alle spelers bevat, in zijn geheel verdeeld over alle spelers. Twee bekende efficiënte oplossingen op de verzameling van alle spelen zijn de 'Shapley value' en de 'nucleolus'. Andere axioma's zorgen ervoor dat de uitbetaling aan de diverse spelers in het algemeen afhangt van de waarden van alle mogelijke coalities. De waarden van de deelcoalities zijn dus van invloed op de uiteindelijke verdeling van de waarde van de grote coalitie over de spelers. Als voor twee spelers alle coalities met de eerste speler en zonder de tweede speler, hogere waarden hebben dan de overeenkomstige coalities maar waarin de eerste speler is vervangen door de tweede, dan zal in vrijwel elke oplossing de eerste speler een uitbetaling krijgen die minstens zo hoog is als de uitbetaling van de tweede speler. De uitbetalingen volgens de 'Shapley value' worden in hoge mate bepaald door de marginale bijdragen van de spelers, waarbij de marginale bijdrage van een speler aan een coalitie het verschil is tussen de waarde van de coalitie minus de waarde van de coalitie zonder de speler. Een kenmerk van de nucleolus is dat daarbij het minimum van het kleinste surplus over alle niet-lege deelcoalities wordt gemaximaliseerd, waarbij het surplus van een coalitie is gedefiniëerd als de totale uitbetaling aan de spelers van de coalitie minus de waarde van de coalitie.

Behalve de oplossingen zoals hierboven beschreven, kijkt men in de coöperatieve speltheorie ook vaak naar verzamelingen van uitbetalingsvectoren voor een spel die aan bepaalde eigenschappen voldoen, zoals de core en de kernel van een spel. De core is de meest bekende verzameling van uitbetalingsvectoren en bestaat uit alle efficiënte uitbetalingsvectoren, zodanig dat voor iedere coalitie geldt dat de som van de uitbetalingen aan alle spelers in die coalitie minstens gelijk is aan de waarde van de coalitie. Met andere woorden, een uitbetalingsvector in de core van een spel is een efficiënte uitbetalingsvector die voldoet aan groepsstabiliteit, d.w.z. geen enkele coalitie kan een hogere uitbetaling krijgen door zich af te splitsen van de grote coalitie en de eigen waarde te genereren.

In de standaard coöperatieve speltheorie wordt aangenomen dat elke coalitie van spelers kan worden gevormd en de waarde van die coalitie kan realiseren. In veel (economische) situaties is dit echter niet het geval. Het is bijvoorbeeld in veel situaties niet mogelijk dat twee spelers die niet over de mogelijkheden beschikken om met elkaar te kunnen communiceren, wel samen in een coalitie zitten zonder andere spelers via welke zij indirect zouden kunnen communiceren. In zo'n situatie kunnen alleen die coalities worden gevormd waarin elk tweetal spelers in een coalitie met elkaar kunnen communiceren. In bijvoorbeeld hiërarchische situaties komt het voor dat een speler alleen met anderen kan samenwerken in een coalitie als deze coalitie ook minstens één van zijn superieuren (of wellicht alle superieuren) bevat. In 1977 was Nobelprijswinnaar Myerson één van de eersten die dergelijke situaties met beperkte mogelijkheden voor coalitievorming modelleerde en een axiomatisch onderbouwde oplossing introduceerde. Deze oplossing geeft een verdeling van de totale uitbetaling over de spelers, die niet alleen afhangt van de waarden van de coalities, maar ook van de collectie van mogelijke coalities die kunnen worden gevormd.

Myerson modelleerde beperkte communicatie tussen spelers door middel van een graaf op de verzameling van spelers. De mogelijke coalities zijn dan de coalities die verbonden zijn in de graaf. Deze collectie is stabiel onder vereniging, d.w.z. voor elk tweetal coalities in de collectie met een niet-lege doorsnijding geldt dat ook de vereniging in de collectie zit. Een andere interessante eigenschap is geslotenheid onder vereniging, d.w.z. voor elk tweetal coalities in de collectie (dus ook twee disjuncte coalities) geldt dat ook de vereniging in de collectie zit. Een voorbeeld van een collectie die gesloten is onder vereniging is de collectie die wordt verkregen in de eerder genoemde hiërarchische situatie waarin spelers toestemming van superieuren nodig hebben.

Omdat geslotenheid onder vereniging meer eisen oplegt aan de collectie dan stabiliteit onder vereniging, is elke collectie die gesloten is onder vereniging ook stabiel onder vereniging. Aan de ene kant betekent dit dat elke oplossing voor een spel met een collectie van mogelijke coalities die stabiel is onder vereniging, ook kan worden toegepast op een spel met een collectie die gesloten is onder vereniging. De andere kant van de medaille is echter dat een stelsel van axioma's dat een unieke oplossing karakteriseert op de verzameling van spelen met alle collecties die stabiel zijn onder vereniging, niet altijd een unieke oplossing karakteriseert op de kleinere klasse van spelen met alle collecties die gesloten zijn onder vereniging. De reden is dat de kleinere klasse minder restricties oplegt en dat er dus in het algemeen meer eisen moeten opgelegd om de oplossing uniek te bepalen. Uniciteit op een klasse van spelen geeft dus meer informatie over de eigenschappen van een oplossing naarmate de klasse kleiner is.

Axiomatiseringen van oplossingen zijn behulpzaam om een gemotiveerde keuze te kunnen maken welke oplossing in een bepaalde situatie het meest geschikt is. Behalve deze motivatie is het ook van belang om de oplossing voor bepaalde spelen te kunnen berekenen. Hiervoor is het van belang om algoritmen te ontwikkelen die een oplossing in polynomiale tijd berekent. Hoewel het voor de meeste oplossingen onmogelijk is om deze voor alle spelen in polynomiale tijd te kunnen berekenen, is het voor bepaalde deelklassen van spelen wel mogelijk om polynomiale algoritmen te ontwikkelen. Dit is essentieel indien men de oplossing wil toepassen in situaties met een 'groot' aantal spelers.

De belangrijkste thema's van dit proefschrift zijn de karakterisering van oplossingen en het introduceren van algoritmen voor het berekenen van oplossingen voor spelen met beperkingen op de mogelijk te vormen coalities, waarbij de collectie van mogelijke coalities gesloten onder vereniging is. We noemen dit spelen die gesloten zijn onder vereniging.

In hoofdstuk 2 worden de belangrijkste begrippen en oplossingsconcepten uit de coöperatieve speltheorie geïntroduceerd. Daarnaast wordt ook het concept van gerestricteerd spel besproken. In de meeste publicaties over spelen met beperkte coalitie mogelijkheden wordt eerst een gerestricteerd spel gedefiniëerd. Dit is een afgeleid spel waarin alle coalities kunnen worden gevormd, maar waarin de waarden van de coalities die in het oorspronkelijke spel niet kunnen worden gevormd, worden afgeleid van de waarden van de coalities die wel mogelijk zijn. Vervolgens wordt een bekend oplossingsconcept toegepast op het nieuwe afgeleide spel. De andere zeven hoofdstukken zijn verdeeld in twee delen. Deel I bevat de hoofdstukken 3-6 en gaat voornamelijk over oplossingen voor spelen die gesloten zijn onder vereniging. Deel II bevat de hoofdstukken 7-9 en gaat voornamelijk over algoritmen voor het berekenen van de nucleolus voor spelen met een hiërarchische structuur op de verzameling van spelers.

In hoofdstuk 3 wordt voor een spel dat gesloten is onder vereniging eerst de zogenaamde 'superior' graaf geïntroduceerd. De superior graaf associeert met elke collectie van coalities die gesloten is onder verening een hiërarchische structuur. Vervolgens wordt voor het spel met de geassocieerde hiërarchische structuur het bijbehorende gerestricteerde spel afgeleid. Bekende oplossingen voor het gerestricteerde spel geven dan een oplossing voor het oorspronkelijke spel dat gesloten is onder vereniging. In hoofdstuk 4 worden twee nieuwe oplossingen voor spelen die gesloten onder vereniging zijn geïntroduceerd en gekarakteriseerd. De eerste oplossing is gebaseerd op de in hoofdstuk 3 geïntroduceerde superior graaf, in de tweede oplossing wordt de Shapley value toegepast op een standaard spel dat is afgeleid van het spel dat gesloten is onder vereniging door aan elke coalitie de waarde van de (uniek bepaalde) maximaal mogelijke deelverzameling van de coalitie toe te wijzen.

Hoofdstuk 5 wijkt af van de andere hoofstukken in die zin dat in dit hoofdstuk standaard spelen worden beschouwd, waarbij alle coalities mogelijk zijn. Er wordt een nieuwe klasse van oplossingen geïntroduceerd en gekarakteriseerd. Deze klasse van oplossingen is gebaseerd op eigenschappen van de prekernel en de prenucleolus en heeft deze twee bekende oplossingen als extreme elementen. Elke oplossing geeft voor elk spel een verzameling van uitbetalingsvectoren, zodanig dat voor elk spel de prenucleolus een uitbetaling in deze verzameling is, en omgekeerd deze verzameling een deelverzameling van de prekernel is.

In hoofdstuk 6 worden eigenschappen van enkele bekende oplossingen gegeven voor de klasse van zogenaamde 'peer-group' spelen. Een peer-group spel is een additief spel (d.w.z. de waarde van een coalitie is de som van de waarden van de agenten in de coalitie) met een hiërarchische structuur weergegeven door een 'boom': een gerichte graaf met één speler als top en waarin elke andere speler precies één voorganger (superieur) heeft. Een coalitie behoort tot de verzameling van mogelijke coalities als voor elke speler, behalve de top speler, geldt dat ook zijn superieur in de coalitie zit. Het hoofdstuk geeft een karakterisering van de Shapley waarde op deze klasse van spelen, ook worden enkele interessante eigenschappen van de nucleolus gegeven.

In het tweede deel van dit proefschrift zijn voornamelijk algoritmen voor het berekenen van de nucleolus voor specifieke klassen van spelen met beperkte mogelijkheden voor samenwerking ontwikkeld. Er bestaat geen algoritme om de nucleolus te vinden voor een willekeurig spel, maar voor specifieke klassen kunnen wel algoritmen worden ontwikkeld. In hoofdstuk 7 wordt een polynomiaal algoritme geïntroduceerd voor het berekenen van de nucleolus in een spel met als hiërarchische structuur een acyclische gerichte graaf met één top, en waarbij een coalitie mogelijk is als voor elke speler (behalve de top) in de coalitie geldt dat ook minstens één van zijn superieuren in de coalitie aanwezig is. In hoofdstuk 8 wordt aangetoond dat dit algoritme ook kan worden toegepast in situaties dat er meerdere top spelers in de hiërarchische structuur aanwezig zijn, waarbij dan wel meer voorwaarden aan het spel zijn gesteld.

In hoofdstuk 9 tenslotte staat een speciale klasse van spelen centraal, de klasse van zogenaamde 'co-insurance' spelen. Een dergelijk spel geeft een situatie weer waarin een groot risico wordt verzekerd door meerdere verzekeringsmaatschappijen, die zowel het risico als de premie onderling verdelen. Het blijkt dat elk niet-negatief monotoon spel behoort tot de klasse van co-insurance spelen, hetgeen nieuwe inzichten geeft in de klasse van niet-negatieve monotone spelen. In dit hoofdstuk wordt ook het concept van een 'veto-removed' spel geïntroduceerd, een dergelijk spel ontstaat als uit een spel met veto spelers één van de veto spelers wordt verwijderd. Door de klasse van veto-removed spelen te onderzoeken worden nieuwe eigenschappen voor monotone spelen gevonden. In dit hoofdstuk wordt ook een algoritme gegeven om de prenucleolus van een veto-removed spel te vinden.