On Nonseparable Erdos Type Spaces

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Summary

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According to the Oxford English Dictionary, *Topology* is the branch of mathematics concerned with those properties of figures and surfaces which are independent of size and shape and are unchanged by any deformation that is continuous, neither creating new points nor fusing existing ones; hence, with those of abstract spaces that are invariant under homeomorphic transformations. A figure or a set with a topological structure is called a *(topological) space*.

The results collected in this dissertation deal with *Erdős type spaces*, that are in addition nonseparable. Such spaces are subspaces of a real Banach space ℓ^p_μ, where μ denotes an infinite cardinal number and $p \geq 1$. The space ℓ^p_μ is the natural generalization of the Banach space ℓ^p (or ℓ^p_ω with ω the set of natural numbers including zero) with vectors in $\mathbb{R}^\mu$, equipped with the topology generated by the p-norm. It is nonseparable in case μ is an uncountable cardinal number. To be precise, a space $\mathscr{E}_\mu$ is said to be an Erdős type space if it can be written as:

$$\mathscr{E}_\mu = \Big\{ x \in \ell^p_\mu : x_\alpha \in E_\alpha \text{ for each } \alpha \in \mu \Big\},$$

where $(E_\alpha)_{\alpha \in \mu}$ is a collection consisting of subsets of the real numbers, $\mathbb{R}$.

The space $\mathfrak{E}$ is the Erdős type space known as *Erdős space* with $p = 2$, $\mu = \omega$ and $E_\alpha = \mathbb{Q}$ for every $\alpha \in \omega$, hence the subspace of all vectors in Hilbert space ℓ^2 for which all co-ordinates are rational. With $p = 2$, $\mu = \omega$ and convergent sequences $E_\alpha = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ one obtains *complete Erdős space*, denoted by $\mathfrak{E}_\mathrm{c}$. Both spaces were introduced in 1940 by Paul Erdős, who showed that they have the peculiar properties that they are one-dimensional, homeomorphic to their own squares and totally disconnected. Hence the square of $\mathfrak{E}$ is one-dimensional, as opposed to the plane, the two-dimensional square of a (one-dimensional) line. Therefore, both $\mathfrak{E}$ and $\mathfrak{E}_\mathrm{c}$ serve as important examples in dimension theory and they can be found in every textbook on that topic.

Despite the peculiar properties of for instance complete Erdős space, this space appears in different branches of mathematics. It has surfaced in dynamics as the end-point set of Julia sets of certain exponential mappings (certain fractals) and in topology as the end-point set of the Lelek fan.

In geometry it appears as the end-point set of the universal separable $\mathbb{R}$-tree. These results are established by Kawamura, Oversteegen and Tymchatyn.

The space $\mathfrak{E}_c$ also surfaced in functional analysis as the line-free weakly closed subgroups of reflexive Banach spaces (Dobrowolski, Grabowski and Kawamura) and more recently in descriptive set theory: if $\mathscr{I}$ is a Polishable F_σ-ideal on ω, then $\mathscr{I}$ endowed with its (unique) Polish topology is homeomorphic to $\mathfrak{E}_c$ if and only if it is not zero-dimensional. All these results about $\mathfrak{E}_c$ are a consequence of the topological characterization of $\mathfrak{E}_c$ by Dijkstra and van Mill. In addition, the space $\mathfrak{E}_c^\omega$ is characterized topologically by Dijkstra. The unexpected result that $\mathfrak{E}_c$ and its infinite power $\mathfrak{E}_c^\omega$ are not homeomorphic is due to Dijkstra, van Mill and Steprāns.

Dijkstra and van Mill also characterized $\mathfrak{E}$ in topological terms. As a consequence of their characterizations, Erdős space is found to be homeomorphic to certain homeomorphism groups of topological manifolds.

So far, all results concern separable spaces. In particular, represented as an Erdős type space, we have looked primarily at subspaces of ℓ^p_ω. In this dissertation, Erdős type spaces in nonseparable ℓ^p-spaces are studied, that is, ℓ^p_μ with μ uncountable. Recall that the *weight* of a space is given by the smallest infinite cardinality of a basis for its topology. The weight of a space X is denoted by $w(X)$. The *local weight* of a space X is defined as

$$lw(X) = \min\{w(U) : U \text{ a non-empty open subset of } X\}.$$

The cardinal invariants weight and local weight, as well as whether or not an Erdős type space is zero-dimensional, can be derived from the sets E_α, as shown in Chapter 4. Note that $lw(\mathfrak{E}) = w(\mathfrak{E}) = lw(\mathfrak{E}_c) = w(\mathfrak{E}_c) = \omega$.

The core of this dissertation is formed by Chapters 5 and 6, where classification theorems are proved concerning certain Erdős type spaces. The main result in Chapter 5 asserts that an Erdős type space $\mathscr{E}_\mu$ with local weight λ and weight κ, is homeomorphic to $\mathfrak{E}_c \times (\lambda_D)^\omega \times \kappa_D$ if and only if every E_α is a zero-dimensional G_δ-subset of $\mathbb{R}$ and $\mathscr{E}_\mu$ has dimension greater than zero. In this classification for instance κ_D denotes the set of cardinality κ equipped with the discrete topology. Hence, one-dimensional topologically complete Erdős type spaces that are constructed with zero-dimensional sets E_α, are homeomorphic to the product of the separable complete Erdős space and a number of zero-dimensional spaces of cardinality depending on the cardinal invariants weight and local weight of the Erdős type space under consideration.

The countably infinite power of a one-dimensional complete Erdős type space is not homeomorphic to any complete Erdős type space, by which we extend the result mentioned above by Dijkstra, van Mill and Steprāns.

In Chapter 6 we prove a classification theorem for Erdős type spaces that bear an analogy with Erdős space. A consequence of the characterization of $\mathfrak{E}$ by Dijkstra and van Mill is that a one-dimensional Erdős type space $\mathscr{E}_\omega$ constructed with zero-dimensional $F_{\sigma\delta}$-subsets of $\mathbb{R}$, of which infinitely many are of the first category in themselves, is homeomorphic to $\mathfrak{E}$. The main result in Chapter 6 extends this and deals with Erdős type spaces $\mathscr{E}_\mu$ constructed with sets E_α of which infinitely many are of the first category in themselves. Such a space $\mathscr{E}_\mu$, with local weight λ and weight κ, is homeomorphic to $\mathfrak{E} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$ if and only if every E_α is a zero-dimensional $F_{\sigma\delta}$-subset of $\mathbb{R}$ and $\mathscr{E}_\mu$ has dimension greater than zero.

In Chapters 7 and 8 we consider two generalizations of the representations of complete Erdős space mentioned above. Chapter 7 deals with ideals generated by submeasures on uncountable cardinals, for which we investigate the link with nonseparable complete Erdős type spaces. Finally, Chapter 8 is about representations of Erdős type spaces as end-point sets of certain nonseparable $\mathbb{R}$-trees and universality properties.