Chapter 6

Conclusion

After almost fifty years of research on deviation matrices and their applications to which this thesis adds another four years of intensive study, the main properties of the deviation matrix, various closed-form representations as well as several applications are known and have become common knowledge. However, as already indicated within the conclusions of the previous chapters there is still a wide range of possible extensions to the current state of the art of research. Whereas there is obviously an unlimited number of Markov processes for which the deviation matrix is not yet known in closed-form representation, every new closed-form representation allows for the computation of the value function and thus facilitates the application of policy improvement measures to the respective process. Furthermore, any new representation will lead to a broadening of the applicability of our series expansions presented in Chapter 3 and 4.

So far, a sufficient condition ensuring the convergence of the series expansion - regardless of whether we wish to approximate $\Pi^*$ or $D^*$ - is that there exists a finite number $N$ such that we can find $\delta \in (0, 1)$, which satisfies:

$$\|((Q^* - Q)D)^N\|_v < \delta.$$  

However, there exist various examples which highlight the fact that this condition is not a necessary one at all. Due to D. van der Laan is the following one. Suppose that

$$Q = \begin{pmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{pmatrix}, \quad Q^* = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$$
with the limiting matrices and the deviation matrix given by

\[
\Pi = \Pi^* = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \quad D = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]

then it holds

\[
\|((Q^* - Q)D)^n\|_v = 1, \quad \forall n \in \mathbb{N}_0,
\]

so that our previous condition is not satisfied. However, it holds for the remainder term

\[
\|\Pi^*((Q^* - Q)D)^n\|_v = 0, \quad \forall n \in \mathbb{N},
\]

and thus the series expansion converges and it holds \(H(1) = \Pi^*\). This simple example suggests to look for alternative conditions which are less strict than the one presented in Chapter 3 and 4 and prevent us from excluding processes to which the series expansion in fact does apply.

While the literature was so far restricted to the investigation of deviation matrices of homogeneous Markov processes it might be of interest to extend the findings provided in this thesis and the references mentioned herein to the more general case of time-nonhomogeneous Markov processes for which under some conditions the transition probability matrix as well as the limiting distribution are given (see, e.g., [1]).

A relatively unexplored area are the deviation matrices of Markov processes on general state space. It was shown in Section 1.6 of this thesis that an extension to continuous state space processes is possible. Moreover we presented the detailed derivation of the closed-form representation of the deviation matrix of a Brownian motion. Taking this derivation as a basis, the computation of deviation matrices for more general processes will soon be possible.

Based on this list of topics for further research there will surely be another fifty years in research related to the deviation matrix.