Invariant measures and limiting shapes in sandpile models

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Summary

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This thesis is concerned with the study of three sandpile models: the CBTW model, the multiple addition sandpile model, and Zhang's sandpile model. As we discussed in Chapter 1, the initial motivation of the sandpile models is to study self-organized criticality. The simple rules of sandpile models make it possible to give rigorous treatment, which is the main work contained in the thesis. The members of the family of sandpile models have both similarity and difference with each other. In the following, we will give the comparison among the finite volume models related to this thesis.

First, each of the sandpile models consists of several basic elements: *configuration space*, *threshold value*, and *toppling rule*. In the BTW model, the height of a site can only take non-negative integer values, while in both the CBTW model and Zhang's model, it can take any non-negative real value. The threshold value is $2d$ in the BTW model in dimension d, it is 1 in the CBTW model and Zhang's model in every dimension. Both in the BTW model and the CBTW model, when a site topples, the toppled site loses 'threshold' amount of mass and each of its nearest neighbors receives $1/2d$ proportion of that amount; while in Zhang's model, the toppled site loses *all* its mass, each of its nearest neighbors receives $1/2d$ proportion of that amount. That amount depends on the height of the toppled site. Hence, both the BTW and the CBTW topplings are abelian but Zhang's topplings are not.

Second, the evolutions of the models are characterized by: *graph* Λ, which is a finite subset of lattice $\mathbb{Z}^d$ in all these three models; *the way of choosing the addition sites* and *the addition amounts*. In all these three sandpile models, at each time step, the addition site is chosen from Λ with uniform probability. In the BTW model, the addition amount is always 1. In the BTW- k model, the addition amount is always the fixed non-negative integer k and in the BTW-K model, it is a random number

Table 5.1. Table of sandpile models

	BTW	BTW-k	CBTW	Zhang
Graph	A finite subset $\Lambda \subset \mathbb{Z}^d$			
Configuration space	$\mathbb{N}_0^{\|\Lambda\|}$	$\mathbb{N}_0^{\|\Lambda\|}$	$[0,\infty)^{\|\Lambda\|}$	$[0,\infty)^{\|\Lambda\|}$
Threshold value	$2d$	$2d$	1	1
Toppling at x	$\eta(x) \to \eta(x) - 2d$;	$\eta(x) \to \eta(x) - 2d$;	$\eta(x) \to \eta(x) - 1$;	$\eta(x) \to 0$;
	$\eta(y) \to \eta(y) + 1$, if $\|y - x\| = 1$;	$\eta(y) \to \eta(y) + 1$, with $\|y - x\| = 1$;	$\eta(y) \to \eta(y) + 1/2d$, if $\|y - x\| = 1$;	$\eta(y) \to \eta(y) + \frac{1}{2d}\eta(x)$, if $\|y - x\| = 1$;
	$\eta(y) \to \eta(y)$, otherwise.			
Addition amount	1	k	R.v. uniformly on $[a,b]$ with $a,b \subset [0,1)$	
Addition site	R.v uniformly on Λ			

Remark: $\mathbb{N}_o$ − −−the set of natural numbers starting from 0; $|y - x| = 1$ − − − x, y are neighbors.

distributed on the set K of non-negative integers. In both the CBTW model and Zhang's model, the addition amount is a random variable uniformly distributed on an interval $[a, b]$, where a, b are any pair of real numbers satisfying $0 \leq a \leq b < 1$. During the evolution, all the addition sites and addition amounts are independent of each other in all these three models. Table 5.1 is an overview of the sandpile models studied in this thesis.

We now give a short summary of the results in this thesis. The mathematical treatment of the CBTW model is performed in Chapter 2. We first establish that the uniform measure μ on the so-called 'allowed configurations' is invariant under the dynamics. When $a < b$, we show with coupling ideas that starting from any initial configuration, the process converges in distribution to μ, which therefore is the unique invariant measure for the process. When $a = b$, that is, when the addition amount is non-random, and $a \notin \mathbb{Q}$, it is still the case that μ is the unique invariant probability measure, but in this case we use random ergodic theory to prove this; this proof proceeds in a very different way. Indeed, the coupling approach cannot work in this case since we also show the somewhat surprising fact that when $a = b \notin \mathbb{Q}$, the process does not converge in distribution at all starting from any initial configuration.

In Chapter 3, we give the formal definition of the multiple addition sandpile model. Our interests are again the convergence of the process and the uniqueness of the invariant measures. For a general graph $\Lambda \subset \mathbb{Z}^d$ and every non-negative integer k, the BTW-k process converges in distribution. For every graph $\Lambda \subset \mathbb{Z}^d$, we can find both infinitely many k and k' such that the BTW-k has a unique invariant measure, while the BTW-k' has many. In dimension 1, we get further results. Take $\Lambda = \{1, 2, \ldots, N\} \subset \mathbb{Z}$ and $k \in \mathbb{N}$ with $q = k \bmod (N + 1)$, in the BTW-k model the set of recurrent configurations can be divided into $\gcd(q, N + 1)$ different closed (under the process) subsets and the uniform measure on each of these subsets is invariant under the process. If K is a subset of $\mathbb{N}$ and $q_k = k \bmod (N + 1)$ for all $k \in K$, then the BTW-K model has $\gcd(N + 1, q_k, k \in K)$ different recurrent classes and the uniform measure on each of these recurrent classes is invariant under the process.

The results related to Zhang' model are presented in Chapter 4 and Chapter 5. In Chapter 4, we show that when $\Lambda \subset \mathbb{Z}$ with $|\Lambda| = N$, Zhang's model has a unique invariant measure for all $0 \leq a < b \leq 1$. Additionally, we also investigate the infinite volume Zhang's sandpile model in dimension $d \geq 1$. We study the stabilizability of initial configurations chosen according to some measure μ. We show that for a stationary ergodic measure μ with density ρ, for all $\rho < \frac{1}{2}$, μ is stabilizable; for all $\rho \geq 1$, μ is not stabilizable; for $\frac{1}{2} \leq \rho < 1$, when ρ is near to $\frac{1}{2}$ or 1, both possibilities can occur.

In Chapter 5, we turn to a rather different subject related to Zhang's model. We define a growth model in which the mass can split with the same rule as Zhang's topplings. The initial configuration contains a large mass $n > 1$ in the center and $h < 1$ at every other sites of $\mathbb{Z}^d$. When a site has mass at least 1, it is unstable and it can split. The mass can spread only by splittings. We specify the order of splittings. We point out that when $h < \frac{1}{2}$, it is robust and when $h > 1 - \frac{1}{2d}$, it is explosive. For $d \geq 2$, when we take the parallel toppling order, there exists constants $C_d < 1 - \frac{3}{4d+2}$ such that when $h > C_d$ and for large n, the splitting process cannot stop. we have that $1 - \frac{3}{4d+2} < 1 - \frac{1}{2d}$, then with the parallel splitting order, the interval of h that makes the splitting process does not stop is a bit larger than that with the general splitting order. With the parallel splitting order and for h that the splitting process does not stop, there are various limiting shapes including a diamond, a square and an octagon. For $h < 0$, we can find both outer and inner bounds for the set of toppled sites.