Invariant measures and limiting shapes in sandpile models

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Invariant measures and limiting shapes in sandpile models

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Contents

1 Introduction

Summary

This chapter starts with the explanation of the expression 'self-organized criticality', which is one of the motivations to study sandpile models. In the rest of this chapter, I will give a short introduction to the BTW sandpile model, the CBTW sandpile model and Zhang's sandpile model. The results on the CBTW sandpile model, Zhang's sandpile model and their associated questions are presented in Chapters 2, 3, 4 and 5.

1.1 Introduction

The sandpile models we are concerned with in this thesis are not real sandpiles with physical sand, but stochastic models based on the movement of sandpiles. The sandpile models are a very general family of models. In this thesis, we will discuss three of them: the BTW sandpile model, the continuous height abelian sandpile model, and Zhang's sandpile model.

The purpose of this chapter is to give an overview of the sandpile models. It intents to be readable for all the people. Therefore, the language in it is informal. In the following part of this chapter, we first talk about the motivation for sandpile models. The expression of 'self-organized criticality' will be explained. After that, we shall give short introductions to each model related to the thesis and discuss our main interests on them.

1.2 Criticality and self-organized criticality

In conventional equilibrium statistical mechanics, there are several physical quantities which influence the models, such as temperature, density, volume, etc. Some of these quantities can act as the parameters of the models. When we tune a parameter, at some special value a small change of the parameter may cause dramatic changes in the system from one phase to another, which is called a *phase transition*. In case there are phase transitions, the parameter space can be divided into several disjoint sets. As the parameter varies within one set, the system keeps looking similar but when the parameter moves from one set to another, the system suddenly looks dramatically different, i.e., a phase transition occurs. The simplest phase transition is *ice* to *water*, which occurs when the temperature is increased above 0°C. One may notice that the transition from ice to water is not spontaneous but lasts for a period of time, during which we can see the existence of both ice and water. In the literature of physics, this transition is called a *first-order* phase transition.

Both physicists and mathematicians are also interested in another kind of phase transitions, which happen instantaneously. When the parameter of the model is fixed on the boundary of two sets that divide the parameter space into separate phases, the model shows special scale-invariant phenomena. One observes *self-similar fractal* and *power-law behavior*, which are almost taken as the signature of criticality. When we talk about *criticality*, we always mean Self-similar fractal behavior means that no matter to which level we scale the lattice, the pictures we see are similar to each other. Power law behavior tells us that the probability distributions of certain objects obey a power law with some exponent that is of the form $p(s) \propto s^{-\alpha}$. For instance, in the 2-dimensional Bernoulli bound percolation model, the critical value is 1/2 (see [32]). At the critical point, it is believed that the probability that the origin is connected by open bonds to a site with distance at most s is proportional to $s^{-5/48}$, see [49].

In fact, many natural phenomena exhibit critical behavior, such as earthquakes, landscape formation, etc. Nature cannot tune the parameters for all these phenomena, hence the system must organize itself to criticality. A *self-organized critical* model is a model that can evolve to the critical state without tuning any parameter. The evolution of the model generates the external driving force which will eventually lead the system to reach a critical state. Since the main work in this thesis is about mathematical treatment about sandpile models, I would not talk too much about self-organized criticality. People's who are interested in this topic can refer to [29]. For mathematicians, the BTW sandpile model [5], the forest fire model [3] and the Bak-Sneppen model [4] are all interesting models of self-organized criticality.

1.3 The BTW sandpile model

The BTW sandpile model was named after Bak, Tang, and Wiesenfeld (see [5, 6]). Since its introduction, it has attracted attention both from physicists and mathematicians.

The BTW sandpile model is defined on a graph Λ, where Λ is a finite connected subset of the square lattice $\mathbb{Z}^d$ with open boundaries. For every site $x \in \Lambda$, the non-negative integer $\eta(x)$ denotes the number of grains on site x. The vector $(\eta(x))_{x\in\Lambda}$ is called a *configuration* on Λ. To each site $x \in \Lambda$, we associate an integer $h_c > 0$, which is called the *threshold* or *capacity* of the sites in Λ. When $\eta(x)$ is at least h_c, x is *unstable*; otherwise when $\eta(x) < h_c$, site x is *stable*. A configuration η is stable if and only if *all* sites x in Λ are stable. For convenience, in the BTW sandpile model we set h_c to be $2d$ for all x. Figure 1.1 are examples of a stable configuration and an unstable configuration in dimension 2. We denote by $\mathcal{X}^o_\Lambda$ and Ω^o_Λ the set of *all*

1	2	3
2	0	1
3	2	0

1	2	**6**
2	**4**	1
3	2	0

Figure 1.1. A 'stable configuration' (left), and an 'unstable configuration' (right).

configurations and the set of stable configurations in the BTW sandpile model, respectively.

When a configuration is unstable, grains are transported by *topplings*. During a toppling, the toppled site loses $2d$ grains and each of its nearest neighbors receives one. A toppling may make some of the neighboring sites become unstable. A toppling is *legal* if it operates on an unstable site, otherwise it is *illegal*. A boundary site has less than $2d$ neighbors, so when it topples, some of the grains escape from the system. Hence the total number of grains in the system decreases. This fact guarantees that *any* configuration in $\mathcal{X}^o_\Lambda$ can reach a stable one after a *finite* number of legal topplings. By t_x we mean the toppling operator at site x. Figure 1.2 is an example of a toppling. A crucial aspect of the BTW sandpile model is that the

1	2	6
2	**4**	1
3	2	0

$\rightarrow$

1	3	6
3	0	2
3	3	0

Figure 1.2. The BTW toppling of the unstable site in the center of a 3×3 square.

topplings are abelian, i.e., *commutative*. Hence if there are two unstable sites in a configuration, then no matter which one we topple first, we will get the same result. Let us look at an example in dimension 1. Let $\Lambda = \{1, 2, 3, 4\}$ and $(1, 1, 2, 3)$ be the configuration with site 3 and site 4 being unstable. One can check that

$$t_4 t_3(1, 1, 2, 3) = t_4(1, 2, 0, 4) = (1, 2, 1, 2)$$

and

$$t_3 t_4(1, 1, 2, 3) = t_3(1, 1, 3, 1) = (1, 2, 1, 2).$$

With the help of the abelian property, it follows that starting from any configuration $\eta \in \mathcal{X}^o_\Lambda$, no matter in which order we topple the unstable sites, we will get the *same* stable one eventually. If we denote the final stable configuration by ζ, the transition from η to ζ is called *stabilization* of η. By $\mathcal{S}^o$ we mean the stabilization operator such that $\mathcal{S}^o \eta = \zeta$ in the BTW sandpile model, which is well defined, see Theorem 2.1 of [41].

Another kind of operators used in the BTW sandpile model are the *addition* operators. For every $x \in \Lambda$, the addition operator $a_x : \Omega^o_\Lambda \to \Omega_\Lambda$ represents the addition of a grain at position x and stabilizing the obtained configuration From the abelian property of the toppling operators, it follows that the addition operators are also abelian. This is why the BTW sandpile model is also called the *abelian sandpile model* in many contexts. The abelian property is a standard tool to analyze the abelian sandpile model and related questions.

The dynamics of the BTW sandpile model is defined as a discrete time Markov chain with finite state space Ω^o_Λ. Starting from a stable configuration $\eta \in \Omega^o_\Lambda$, at every discrete time t, we first randomly choose a site from Λ with uniform probability, then add a grain to the chosen site and stabilize the obtained configuration. That is, with $\eta_0 = \eta$,

$$\eta_t = a_{X_t} \eta_{t-1}, \quad t = 1, 2, 3, \cdots, \tag{1.1}$$

with $X_1, X_2, \ldots$ a sequence of i.i.d. uniformly distributed random variables on Λ. A configuration $\eta \in \Omega^o_\Lambda$ is *recurrent* if starting from it, it will be visited by the process (1.1) infinitely often a.s. A distribution μ on Ω^o_Λ is stationary for the model, if for every measurable set $B \subset \Omega^o_\Lambda$,

$$\mu(B) = \sum_{x \in \Lambda} \mathbb{P}(X_1 = x) \mu\{\eta \in \Omega^o_\Lambda : a_x \eta \in B\}.$$

From a mathematical point of view, this is a very interesting model. In spite of the long-term interactions among grains of different sites, the rules of the model are simple enough to allow rigorous analysis. Dhar in his article [9] discusses many results about the BTW sandpile model, such as the number of recurrent configurations, the group structure of the addition operators, and the relation between the recurrent configurations and the *allowed* configurations. Mathematical treatments of these kind of results can be found in [41].

As we mentioned in Section 1.2, the motivation for the BTW sandpile model is to study the concept of self-organized criticality (SOC). In the article [5], Bak and his co-authors simulated the behavior of the BTW sandpile model. The simulations show that starting from any stable configuration, the model adjusts itself to a critical stationary state eventually.

In a series of papers [12, 13, 46, 47], a connection between SOC and conventional criticality was set up. From their arguments, the authors hinted that self-organized critical systems could also be considered as conventional critical systems with an internal driving force. In [41, 21], it was shown that in dimension 1 the infinite volume limit of the stationary density (average height under the stationary distribution) of the BTW sandpile model coincides with the critical density of the fixed-energy sandpile model (infinite volume sandpile, see Section 1.5.2). In the very recent papers [16, 17], it is shown that for a variety of graphs, the infinite volume limit of the stationary density is not the same as the critical density of the the fixed-energy system.

After the introduction of the BTW sandpile model, many natural modifications of the BTW sandpile model were introduced. One can define a new model by changing the grid, toppling rules or addition rules. There are a number of sandpile models, such as the sandpile model on a tree [38], the Manna sandpile model with stochastic toppling rules [37], Zhang's sandpile model with random addition amounts [50] and continuous height space, etc. For sandpile models, many researchers have focused on the following questions: uniqueness of the invariant measure, convergence of the distribution of the process, infinite volume version of the model (stabilizability of a configuration), see Section 1.5.2, and sandpile growth models (shapes of the set of toppled sites), see [23] or Chapter 5. In the following two sections, I will give a short introduction to the Continuous height abelian sandpile model [1, 31], Zhang's sandpile model [50, 20] and some associated questions.

1.4 The Continuous height abelian sandpile model

The first model considered in this thesis is the *Continuous height abelian sandpile model*, which is also called the *CBTW* sandpile model. It was first introduced in [1]. In the paper [27], the authors investigated another continuous height sandpile model but with dissipation on every site.

The CBTW sandpile model is different from the BTW sandpile model in four aspects. In the CBTW model: (1) the state space is continuous, $\mathcal{X}_\Lambda = [0,\infty)^{|\Lambda|}$ is the set of all configurations; (2) the threshold value h_c is 1 and $\Omega_\Lambda = [0,1)^{|\Lambda|}$ is the set of all stable configurations; (3) when an unstable site topples, it loses mass 1 and each of its nearest neighbors receives $\frac{1}{2d}$; (4) the addition amount is not 1 but could be *any* value in the interval $[0,1)$.

0.1	0	0.7
0.2	0.5	1
0.8	**1.3**	0

$\rightarrow$

0.1	0	0.7
0.2	0.75	1
1.05	0.3	0.25

Figure 1.3. The CBTW toppling of the unstable site in a 3×3 square.

Similar as the BTW topplings, the CBTW topplings are also abelian. Hence to every configuration $\eta \in \mathcal{X}_\Lambda$, there corresponds a *unique* $\zeta \in \Omega_\Lambda$ which is obtained from η by legal topplings. The transition from η to ζ is again called *stabilization*, denoted by $\mathcal{S}$. For a fixed $u \in [0,1)$ and $x \in \Lambda$, the addition operator A_x^u is defined as the joint effect of adding u amount of mass to a configuration at x and stabilizing the obtained configuration.

The dynamics of the model is characterized by the graph $\Lambda \subset \mathbb{Z}^d$ and an interval $[a,b] \subset [0,1)$, from which at every time step, the addition amount is uniformly chosen. Therefore sometimes we speak of the CBTW$(\Lambda,[a,b])$ model instead of the BTW model to emphasis the graph and addition interval. The evolution of the CBTW$(\Lambda,[a,b])$ is similar to that of the BTW model, and can be concisely described by the following Markov chain:

$$\eta_t = A_{X_t}^{U_t}\eta_{t-1}, \quad t = 1,2,3,\cdots,$$

where $\eta_0 \in \Omega_\Lambda$ is the initial configuration, $X_1, X_2, \ldots$ is a sequence of i.i.d. uniformly distributed random variables on Λ and $U_1, U_2, \ldots$ are i.i.d. and uniformly distributed on $[a,b]$, where $[a,b] \subset [0,1)$. The two sequences are also independent of each other.

For every CBTW configuration $\eta \in [0,\infty)^{|\Lambda|}$, we can write:

$$\eta = \frac{1}{2d}\overline{\eta} + \tilde{\eta},$$

where $\overline{\eta} = \sum_{x\in\Lambda}\lfloor 2d\eta(x)\rfloor\delta_x$ is the *integer part* and $\tilde{\eta} = \sum_{x\in\Lambda}[\eta(x) \bmod \frac{1}{2d}]\delta_x$ is the *fractional part*. The integer part itself is a BTW configuration. In Chapter 2, in order to study the effect of the CBTW additions, very often we study their effect on the integer part and fractional part individually. Since during a CBTW topplings, only multiples of $\frac{1}{2d}$ amount of mass are transported, topplings do not influence the fractional part of a configuration. Therefore, no matter whether we take the 'integer part' first and then perform a BTW toppling or we perform a CBTW toppling first and then take the 'integer part', we will get the same configuration. This fact is used to set up the relation between the BTW model and the CBTW model. When the addition A_x^u operates on η, sometimes the addition amount on the integer part is $\lfloor 2du \rfloor$ and sometimes is $\lfloor 2du \rfloor + 1$, depending on the value $\tilde{\eta}(x)$. This observation motivates the study of the models in Chapter 3.

For the CBTW model, we are interested in whether the process converges and whether there is a unique invariant measure. The answers to these two questions depend heavily on the choice of the interval $[a, b]$. The main tools used in the proof are Fourier transforms and ergodicity. More information about this model can be found in Chapter 2.

1.5 Zhang's sandpile model

Zhang's sandpile model was proposed by Zhang in 1989 [50]. As in the CBTW model, in Zhang's model $\mathcal{X}_\Lambda = [0, \infty)^{|\Lambda|}$ and $\Omega_\Lambda = [0, 1)^{|\Lambda|}$ are the set of all configurations and the set of stable configurations respectively. Zhang toppling rules are quite different from those of the BTW model. In Zhang's model, when a site topples, the toppled site loses *all* its mass and each of its nearest neighbors receives a $\frac{1}{2d}$ *proportion* of that amount. We denote by T_x the Zhang toppling at site x. One

0.1	0	0.7		0.1	0	0.7
0.2	0.5	1	→	0.2	0.825	1
0.8	**1.3**	0		1.125	0	0.325

Figure 1.4. Zhang's toppling of the unstable site in a 3×3 square.

may notice that in Zhang's model, the mass transported by a toppling depends on the current configuration. Therefore, the Zhang's topplings are *not* abelian. Hence different orders of toppling may result in different configurations. Let us consider an example in dimension 1. Take $\eta = (0, 5, 1, 0.2, 1.3, 5)$, then

$$T_4 T_5 \eta = T_4(0.5, 1, 0.2, 3.8, 0) = (0.5, 1, 2.1, 0, 1.9)$$

and

$$T_5T_4\eta = T_5(0.5, 1, 0.85, 0, 5.65) = (0.5, 1, 0.85, 2.825, 0).$$

The loss of the abelian property makes Zhang's model difficult to analyze. Whenever there are topplings, we need to point out the specific order used. There are two natural toppling orders:

1. Parallel: at every discrete time $t = 0, 1, 2, \ldots$, all the unstable sites topple simultaneously;
2. Markov: Associate to each site a rate 1 Poisson clock, when the clock rings at site x, we check that site and if it is unstable, we topple that site; otherwise, we do nothing.

1.5.1 The evolution of the model

In Zhang's model, for each $u \in [0, 1)$ and $x \in \Lambda$, $\mathcal{A}_x^u$ is the addition operator defined by adding an amount u to site x and then toppling all the unstable sites in parallel until a new stable configuration is reached (*stabilization*). Like the CBTW model, Zhang's model is also characterized by the graph $\Lambda \subset \mathbb{Z}^d$ and the addition interval $[a, b] \subset [0, 1)$. The evolution of Zhang's model is quite similar to that of the CBTW model. We use the same strategy to decide the addition sites and addition amounts. The only difference is that in Zhang's model, the unstable sites topple in Zhang's way.

In [50], Zhang showed by simulation that for a large lattice, as time goes by, the height variables in the configuration tend to concentrate around $2d$ discrete values, which he called 'quasi-units'. Based on the simulation results, it was conjectured that despite the different addition and topplings rules, for large volume, the stationary distribution of Zhang's model should have the same probability structure as that in the BTW model. This is quite interesting, since in Zhang's model the addition amount takes any real value in $[0, 1)$.

There was no mathematical rigorous treatments of this model until the article [20]. Most of the results in [20] are in dimension 1. Uniqueness of the stationary distribution was proved for a number of special cases. In Chapter 4, we generalize this result. We set up a coupling to show the uniqueness of the stationary distribution, which works for intervals $[a, b]$ with $0 \leq a < b \leq 1$ and all intervals $\Lambda \subset \mathbb{Z}$.

1.5.2 The infinite volume version of Zhang's sandpile model

The infinite volume version of a sandpile model is a fixed energy sandpile model. It was proposed to study the connection between 'organized criticality' and 'self-organized criticality (SOC)'.

The infinite volume versions of sandpile models are conventional parameterized equilibrium systems. We are interested in two phases: *stabilizable* and *unstabilizable*. What do 'stabilizable' and 'unstabilizable' mean in our model? Consider the following process. Fix an order of topplings and starting from an unstable configuration, topple the unstable sites according to the given order. There are two possibilities. One is that after an a.s. finite random time τ, the origin becomes stable and does not change any more, which we call *local fixation*. The other is that after any toppling at a site, that site will become unstable and topple again eventually. If the first possibility happens, we say that the initial configuration is *stabilizable* and if the second possibility happens, we say that the initial configuration is *unstabilizable*. If the initial configuration is distributed according to some translation invariant measure μ, μ is *stabilizable* if μ-almost every configuration is stabilizable; otherwise μ is *unstabilizable*. For a translation invariant measure, all the sites have the same density (expectation of height), which is a possible parameter of the system.

The infinite volume version of the BTW sandpile model was well studied for instance in [40, 21, 22]. In Section 4.4 of the thesis, we will study the infinite volume version of Zhang's model, where we choose the Markov order of toppling.

In Section 4.4, we observe first that if an invariant measure is stabilizable, then the density is conserved. We then conclude that a translation invariant measure with density larger than 1 is unstabilizable. Based on the fact that mass passing a bond in the last time stays in the system and that amount is at least $\frac{1}{2d}$, we can conclude that if an ergodic invariant measure is unstabilizable, the density is at least $\frac{1}{2}$. When the density is in $[\frac{1}{2}, 1)$ and near to $\frac{1}{2}$ or 1, examples show that both possibilities can happen, see Section 4.4.

1.5.3 A growth model with Zhang's toppling rules

In Chapter 5, a splitting game with Zhang's toppling rules is studied. The initial state is a configuration with mass $n \geq 1$ at the origin of $\mathbb{Z}^d$, and mass $h < 1$ at every other site, where n and h are real numbers. At every discrete time t, if there are unstable sites, we choose 'some' (one or many) among them and topple all of them in parallel. To make sure that the process evolves efficiently, we suppose that every unstable sites would be chosen in finite time; and every time, we choose at least one unstable site. One may not stop before all the piles have mass less than 1.

Initially, only the origin is unstable. At every time step, some mass is transported by topplings and the neighbors of the toppled sites may become unstable. We are interested in whether the mass will spread all over $\mathbb{Z}^d$ or only over a finite part of $\mathbb{Z}^d$. In the first case, how does it spread? And in the second case, what are the size and shape of the sets of toppled sites, depending on n and h?

A background h is *explosive* if for large n, the splitting process cannot stop a.s. and is *robust* if for every finite n, it stops at some finite time a.s. Under the parallel toppling order, when h is in the explosive region, we find that our model has various limiting shapes, including a diamond, a square and an octagon. But an octagon cannot occur in the BTW sandpile growth model [23]. In the proofs, we set up several cellular automata which generate certain limiting shapes. Then we define some maps that associate the height intervals in Zhang's growth model to certain states of the cellular automata. The correspondence is conserved after each time step. Therefore from the limiting shapes of the cellular automata, we can get the limiting shapes of Zhang's growth model. Furthermore in Chapter 5, we formulate some open questions.

1.6 List of publications

The left part of this thesis is based on the following articles:

- Kager, W., Liu, H. & Meester, R. (2010). Existence and Uniqueness of the Stationary Measure in the Continuous Abelian Sandpile. *Markov Process and Related Fields* **16**(1), pp. 185–204.
- Fey, A., Liu, H. & Meester. R (2009). Uniqueness of the stationary distribution and stabilizability in Zhang's sandpile model. *Electronic journal of Probability* **14**, pp. 895–911.
- Fey, A. & Liu, H. (2010). Limiting shapes for a nonabelian sandpile growth model and related cellular automata. Preprint available at `http://arxiv.org/abs/1006.4928`, to appear in *Journal of Cellular Automata*.
- Liu, H. (2010) The multiple addition sandpile model. *In preparation*

2 Existence and uniqueness of the stationary measure in the continuous abelian sandpile

Summary

This chapter defines the continuous height abelian sandpile model formally, which is called CBTW sandpile model in short in the whole chapter. Results about the existence and uniqueness of the invariant measure are included. The main techniques used in the main proof a re the couplings approach and random ergodic theory.

The content in this chapter is a reproduction of the article by W. Kager, H.Liu and R. Meester [31].

2.1 Introduction

The classical *BTW sandpile model* was introduced by Bak, Tang and Wiesenfeld in 1988 as a prototype model for what they called 'self-organized criticality' [6]. It has attracted attention from both physicists and mathematicians. In Dhar's paper [9], the model was coined the *abelian sandpile model* (ASM) because of the Abelian structure of the toppling operators. The BTW sandpile model is defined on a finite subset Λ of the d-dimensional integer lattice. At each site of Λ, we have a non-negative integer value that denotes the *height* or the *number of 'grains'* at that site. At each discrete time t, one grain is added to a random site in Λ; the positions of the additions are independent of each other. When a site has at least $2d$ grains, it is defined *unstable* and *topples*, that is, it loses $2d$ grains, and each of its nearest neighbours in Λ receives one grain. Neighboring sites that have received a grain, can now become unstable themselves, and also topple. This is continued until all heights are at most $2d-1$ again. This will certainly happen, since grains at the boundary of the system are lost. See [9, 41, 43] for many detailed results about this model.

As a variant of the BTW model, the *Zhang sandpile model* was introduced by Zhang [50] in 1989. The Zhang sandpile model has some of the flavor of the BTW sandpile, albeit with some important and crucial differences: (i) in Zhang's model, the height variables are continuous with values in $[0,\infty)$ and the threshold is (somewhat arbitrarily) set to 1 (we also speak of *mass* rather than of height sometimes); (ii) at each discrete time, a random amount of mass, which is uniformly distributed on an interval $[a,b] \subset [0,1)$, is added to a randomly chosen site; (iii) when a site has height larger than 1, it is defined unstable and topples. In this model this means that it loses *all* its mass and each of its nearest neighbors in Λ receives a $1/2d$ proportion of this mass.

Since the mass redistributed during a toppling depends on the current configuration, the topplings in Zhang's sandpile are not abelian. From simulations in [50], it is shown that in large volume under stationary measure, the height variables of this model concentrate on several discrete quasi-values', and in this sense, this model behaves similarly to the BTW sandpile model. Recently, this model as well as the infinite volume version of this model have been rigorously studied in [20, 19].

Here we discuss a model introduced in [1], which is continuous like Zhang's sandpile, but abelian like the BTW sandpile. The model is related to the deterministic model introduced in [24]. In our model, the threshold value of all sites is again set to 1. The only difference with Zhang's model, is that when a site topples, it does not lose all its mass, but rather only a total of mass 1 instead. Each neighbor in Λ then receives mass $1/2d$. Note that this is more similar to the BTW sandpile than Zhang's sandpile: in the BTW model the number of grains involved in a toppling does not depend on the actual height of the vertex itself either. In this sense, the newly defined model is perhaps a more natural analogue of the BTW model than Zhang's model. In this chapter we therefore call the new model the CBTW sandpile, where the 'C' stands for 'continuous'.

In this chapter, we study existence and uniqueness of the invariant probability measure for the CBTW sandpile. In the next section we formally define the model, set our notation and state our results. After that we study the relation between the CBTW and the BTW sandpile model, used mainly as a tool in the subsequent proofs.

2.2 Definitions, notation and main results

Let Λ be a finite subset of $\mathbb{Z}^d$ and let $\eta = (\eta(x), x \in \Lambda)$ be a configuration of heights, taking values in $\mathcal{X}_\Lambda = [0, \infty)^{|\Lambda|}$. Site x is called *stable* (in η) if $0 \le \eta(x) < 1$; if $\eta(x) \ge 1$, site x is called *unstable*. A configuration η is called stable if every site in Λ is stable. We define $\Omega_\Lambda = [0, 1)^{|\Lambda|}$ to be the collection of stable configurations.

By T_x, we denote the *toppling operator* associated to site x: if Δ is the $|\Lambda| \times |\Lambda|$ matrix

$$\Delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ -\frac{1}{2d} & \text{if } |y - x| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$T_x\eta := \eta - \Delta(x, \cdot).$$

Therefore, $T_x\eta$ is the configuration obtained from η by performing one toppling at site x. The operation of T_x on η is said to be *legal* if $\eta(x) \geq 1$, otherwise the operation is said to be *illegal*. It is easy to see that toppling operators commute, that is, we have

$$T_xT_y\eta = T_yT_x\eta. \tag{2.1}$$

This abelian property of the topplings implies a number of useful properties, the proofs of which are similar to the proof of Theorem 2.1 in [41] and are therefore not repeated here.

Proposition 2.1. *Let $\eta \in \mathcal{X}_\Lambda$ and suppose we start the sandpile dynamics with initial configuration η. Then we have that*

1. *the system evolves to a stable configuration after finitely many legal topplings;*
2. *the final configuration is the same for all orders of legal topplings;*
3. *for each $x \in \Lambda$, the number of legal topplings at site x is the same for all sequences of legal topplings that result in a stable configuration.*

It follows that for $\eta \in \mathcal{X}_\Lambda$, there is a unique stable configuration $\eta' \in \Omega_\Lambda$ reachable from η by a series of legal topplings. We define the *stabilization operator* $\mathcal{S}$ as a map from $\mathcal{X}_\Lambda$ to Ω_Λ such that

$$\mathcal{S}\eta := \eta'. \tag{2.2}$$

For $x \in \Lambda$, $u \in [0, 1)$, A_x^u denotes the *addition operator* defined by

$$A_x^u\eta = \mathcal{S}(\eta + u\delta_x), \tag{2.3}$$

where

$$\delta_x(y) = \begin{cases} 0 & \text{if } y \neq x, \\ 1 & \text{if } y = x, \end{cases}$$

that is, we add mass u at site x and subsequentially stabilize. According to Proposition 2.1, A_x^u is well-defined, and satisfies

$$A_x^uA_y^v\eta = A_y^vA_x^u\eta.$$

The evolution of the CBTW model can now concisely be described via

$$\eta_t = A_{X_t}^{U_t}\eta_{t-1}, \quad t = 1, 2, 3, \cdots, \tag{2.4}$$

where η_0 is the initial configuration, $X_1, X_2, \ldots$ is a sequence of i.i.d. uniformly distributed random variables on Λ and $U_1, U_2, \ldots$ are i.i.d. and uniformly distributed on $[a, b]$, where $[a, b] \subset [0, 1]$. The two sequences are also independent of each other.

For $\eta \in \Omega_\Lambda$ and $W \subseteq \Lambda$, $\eta|_W$ denotes the restriction of η to W. For every finite subset W of Λ, $\eta|_W$ is called a *forbidden sub-configuration* (FSC) if for all $x \in W$

$$\eta(x) < \frac{1}{2d}\#(\text{nearest neighbours of } x \text{ in } W),$$

where # denotes cardinality. A configuration $\eta \in \Omega_\Lambda$ is said to be *allowed* if it has no FSC's. This notion is parallel to the corresponding notion in the BTW sandpile model—see also below. It is well known that in the BTW model, the uniform measure on all allowed configurations is invariant under the dynamics (and is the only such probability measure). Let us therefore denote by $\mathcal{R}_\Lambda$ the set of all allowed configurations and by μ normalized Lebesgue measure on $\mathcal{R}_\Lambda$, that is, the measure which assigns mass such that for every measurable $B \subset \mathcal{R}_\Lambda$ we have

$$\mu(B) = \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(\mathcal{R}_\Lambda)}, \tag{2.5}$$

where $\mathrm{Vol}(\cdot)$ denotes Lebesgue measure.

For every initial probability measure ν, $\nu_t^{a,b}$ denotes the distribution of the process at time t. The first result (similar to the corresponding result in the deterministic model in [24]) states not only that μ is invariant for the process, but moreover that μ is in fact invariant for each individual transformation A_x^u:

Theorem 2.2. *For every $x \in \Lambda, u \in [0, 1)$, A_x^u is a bijection on $\mathcal{R}_\Lambda$ and μ is A_x^u-invariant. Hence, μ is invariant for the CBTW-sandpile model.*

When $a < b$, the situation is as in the traditional BTW model:

Theorem 2.3. *For every $0 \le a < b < 1$, μ is the unique invariant probability measure of the CBTW model and starting from every measure ν on Ω_Λ, $\nu_t^{a,b}$ converges exponentially fast in total variation to μ as t tends to infinity.*

In the case $a = b$ however, things are more interesting:

Theorem 2.4. *When $a = b \in [0, 1)$ but $a \notin \{\frac{l}{2d} : l = 0, 1, \ldots, 2d - 1\}$, for every initial configuration $\eta \in \Omega_\Lambda$, the distribution of the process at time t does not converge weakly at all as $t \to \infty$.*

Theorem 2.5. *When $a = b$ and $a \in \{\frac{l}{2d} : l = 0, 1, \ldots, 2d-1\}$, for every initial configuration $\eta \in \Omega_\Lambda$, the distribution of the process at time t converges exponentially fast in total variation to a measure μ_a^η as $t \to \infty$.*

In the proof of Theorem 2.5 below, we give an explicit description of the limiting distribution μ_a^η in terms of the uniform measure on the allowed configurations of the BTW model. In general, the limiting distribution depends both on the initial configuration η and the value of a. Finally we have

Theorem 2.6. *When $a \in [0, 1)$ is irrational and $a = b$, μ is the unique invariant (and ergodic) probability measure for the CBTW model.*

Hence, it is always the case that μ is the unique invariant probability measure for the CBTW model, except possibly if $a = b \in \mathbb{Q}$. However, convergence to this unique stationary measure only takes place when $a < b$, and not (apart from the obvious exceptional case when we start with μ) in the case $a = b$. In the proof of Theorem 2.3, we will use the fact that a is strictly smaller than b in order to construct a coupling. The proof of Theorem 2.6 is very different from the proof of Theorem 2.3 (although both results are about the uniqueness of the invariant measure). In the case $a = b$ we use ideas from random ergodic theory rather than a coupling. We could have done this also in the proof of Theorem 2.3 but then we would not have obtained the exponential convergence corollary. Theorem 2.6 is by far the most difficult to prove here, having no coupling approach at our disposal. In the next section, we discuss the relation between the CBTW and the BTW models. This relation will be used in the subsequent proofs of our results.

2.3 Relation between the CBTW model and the BTW model

We denote by $\mathcal{X}_\Lambda^o$ and Ω_Λ^o the collection of all height configurations and stable configurations in the classical BTW model, respectively. Furthermore, we use the notation t_x, a_x and $\mathcal{S}_\Lambda^o$ to denote the toppling operator at x, the addition operator at x and the stabilization operator in the BTW model, respectively.

In order to compare the CBTW and BTW models, it turns out to be very useful to define the *integer* and the *fractional* part of a configuration $\eta \in \mathcal{X}_\Lambda$.

Definition 2.7. *For a configuration $\eta \in \mathcal{X}_\Lambda$, $\overline{\eta}$ is defined by*

$$\overline{\eta}(x) = \lfloor 2d\eta(x) \rfloor, \tag{2.6}$$

and we call $\overline{\eta}$ the integer part *of η. We define $\tilde{\eta}$ by*

$$\tilde{\eta}(x) = \eta(x) \bmod \frac{1}{2d},$$

and we call $\tilde{\eta}$ the fractional part *of η.*

The identity

$$\eta = \frac{1}{2d}\overline{\eta} + \tilde{\eta} \tag{2.7}$$

clearly holds. The evolution of the integer part of η is closely related to a BTW sandpile, while the fractional part is invariant under topplings. These two features will make these definitions very useful in the sequel.

Clearly, $\eta(x) \geq 1$ if and only if $\overline{\eta}(x) \geq 2d$. The operation of taking integer parts commutes with legal topplings:

Lemma 2.8. *For $\eta \in \mathcal{X}_\Lambda$ such that $\eta(x) \geq 1$, we have*

$$\overline{T_x\eta} = t_x\overline{\eta}$$

and hence

$$\overline{\mathcal{S}\eta} = \mathcal{S}^o\overline{\eta}.$$

Proof. For $\eta \in \mathcal{X}_\Lambda$, $\eta(x) \geq 1$ implies $\overline{\eta}(x) \geq 2d$. From the toppling rule, we have

$$T_x\eta = \eta - \delta_x + \sum_{y\in\Lambda:|y-x|=1} \frac{1}{2d}\delta_y,$$

which implies that for $z \in \Lambda$,

$$\overline{T_x\eta}(z) = \lfloor 2d\eta(z) - 2d\delta_x(z) + \sum_{y\in\Lambda:|y-x|=1} \delta_y(z) \rfloor.$$

Since both $2d\delta_x(z)$ and $\sum_{y\in\Lambda:|y-x|=1} \delta_y(z)$ are integers, we get

$$\overline{T_x\eta}(z) = \lfloor 2d\eta(z) \rfloor - 2d\delta_x(z) + \sum_{y\in\Lambda:|y-x|=1} \delta_y(z)$$

which is equal to $(t_x\overline{\eta})(z)$. Therefore we obtain $\overline{T_x\eta} = t_x\overline{\eta}$. It now also follows that $\overline{\mathcal{S}\eta} = \mathcal{S}^o\overline{\eta}$. □

For a configuration $\xi \in \mathcal{X}^o_\Lambda$, let $C(\xi)$ be the set

$$C(\xi) := \{\eta \in \mathcal{X}_\Lambda : \overline{\eta} = \xi\},$$

that is, $C(\xi)$ is the set of all configurations which have ξ as their integer part. The sets $C(\xi), \xi \in \Omega^o_\Lambda$, partition Ω_Λ. Indeed it is easy to see that for $\xi, \xi' \in \Omega^o_\Lambda$ with $\xi \neq \xi'$, we have

$$C(\xi) \cap C(\xi') = \emptyset \tag{2.8}$$

and

$$\Omega_\Lambda = \bigcup_{\xi \in \Omega^o_\Lambda} C(\xi). \tag{2.9}$$

Before continuing to the next observation, we recall the definition of a BTW-forbidden sub-configuration. For $\xi \in \Omega^o_\Lambda$, $\xi|_W$ is a FSC if for all $x \in W$

$$\xi(x) < \#(\text{ nearest neighbours of } x \text{ in } W).$$

By $\mathcal{R}^o_\Lambda$, we denote the set of allowed configurations (that is, configurations without FSC's) in the BTW model, which is also the set of all recurrent configurations, see [41]. For η and its corresponding integer part $\overline{\eta}$, we have the following simple observation, the proof of which we leave to the reader.

Lemma 2.9. *For $\eta \in \Omega_\Lambda$, η is allowed if and only if $\overline{\eta}$ is allowed in BTW.*

Definition 2.10. *A configuration $\eta \in \Omega_\Lambda$ is called* reachable *if there exists a configuration $\eta' \in \mathcal{X}_\Lambda$ with $\eta'(x) \geq 1$ for all $x \in \Lambda$, such that*

$$\eta = \mathcal{S}\eta'.$$

We denote the set of all reachable configurations by $\mathcal{R}'_\Lambda$.

Theorem 2.11. $\mathcal{R}_\Lambda = \mathcal{R}'_\Lambda$.

Proof. Let $\eta \in \mathcal{R}_\Lambda$. By Lemma 2.9, $\overline{\eta}$ is BTW-allowed and therefore also recurrent (Theorem 5.4 in [41]). From [41, Theorem 4.1], we have that for every $x \in \Lambda$, there exists $n_x \geq 1$ such that

$$a_x^{n_x}\overline{\eta} = \overline{\eta}.$$

Therefore

$$\prod_{x\in\Lambda} a_x^{2dn_x}\overline{\eta} = \mathcal{S}^o(\overline{\eta} + \sum_{x\in\Lambda} 2dn_x\delta_x) = \overline{\eta}. \tag{2.10}$$

Now let

$$\xi = \overline{\eta} + \sum_{x\in\Lambda} 2dn_x\delta_x$$

be a configuration in $\mathcal{X}^o_\Lambda$; note that $\xi(x) \geq 2d$ for all $x \in \Lambda$. Let

$$\eta' = \frac{1}{2d}\xi + \tilde{\eta}. \tag{2.11}$$

Then $\eta' \in \mathcal{X}_\Lambda$ and

1. $\eta'(x) \geq 1$ for all $x \in \Lambda$;
2. $\overline{\eta'} = \xi$;
3. $\widetilde{\eta'} = \tilde{\eta}$.

We will now argue that $\mathcal{S}\eta' = \eta$. From Lemma 2.8 and (2.10) we have

$$\overline{\mathcal{S}\eta'} = \mathcal{S}^o\overline{\eta'} = \mathcal{S}^o\xi = \overline{\eta}.$$

Furthermore, since fractional parts are invariant under $\mathcal{S}$, we have

$$\widetilde{\mathcal{S}\eta'} = \widetilde{\eta'} = \tilde{\eta},$$

and hence $\mathcal{S}\eta' = \eta$.

For the other direction, if $\eta \in \mathcal{R}'_\Lambda$, there exists an $\eta' \in \mathcal{X}_\Lambda$ with $\eta'(x) \geq 1$ for all $x \in \Lambda$, and such that

$$\eta = \mathcal{S}\eta'. \tag{2.12}$$

Clearly, $\overline{\eta'}(x) \geq 2d$. By Lemma 2.8, we can now write

$$\overline{\mathcal{S}\eta'} = \mathcal{S}^o\overline{\eta'} = \prod_{x\in\Lambda} a_x^{\overline{\eta'}(x)-(2d-1)}\xi^{max}, \tag{2.13}$$

where $\xi^{max}(x) = 2d-1$, for all $x \in \Lambda$. This means that $\overline{\mathcal{S}\eta'}$ is a configuration obtained by additions to ξ^{max}. Since ξ^{max} is allowed for the BTW model, by Theorem 5.4 of [41], ξ^{max} is also recurrent. But then $\overline{\mathcal{S}\eta'}$ is also recurrent and therefore allowed. It then follows that $\mathcal{S}\eta'$ is allowed for the CBTW model. □

Corollary 2.12. *For $x \in \Lambda, u \in [0,1)$, $\mathcal{R}_\Lambda$ is closed under the operation of A_x^u.*

Proof. For $\eta \in \mathcal{R}_\Lambda$, by Theorem 2.11, there is a $\eta' \in \mathcal{X}_\Lambda$ with $\eta'(x) \geq 1$ for all $x \in \Lambda$, such that

$$\eta = \mathcal{S}\eta'.$$

By the abelian property of toppling operators,

$$A_x^u \eta = \mathcal{S}(\mathcal{S}\eta' + u\delta_x) = \mathcal{S}(\eta' + u\delta_x)$$

and $\eta' + u\delta_x$ is a configuration with only unstable sites. It follows that $A_x^u \eta \in \mathcal{R}'_\Lambda$, and hence $A_x^u \eta \in \mathcal{R}_\Lambda$. □

Lemma 2.13. *(1)* $\mathrm{Vol}(\mathcal{R}_\Lambda) = \det(\Delta)$*;*
(2) For every $\xi \in \mathcal{R}^o_\Lambda$*,* $\mu(C(\xi)) = \frac{1}{|\mathcal{R}^o_\Lambda|}$.

Proof. (1) We have

$$\mathrm{Vol}(\mathcal{R}_\Lambda) = \sum_{\xi \in \mathcal{R}^o_\Lambda} \mathrm{Vol}(C(\xi)).$$

For each $\xi \in \mathcal{R}^o_\Lambda$, $\mathrm{Vol}(C(\xi)) = (2d)^{-|\Lambda|}$, hence $\mathrm{Vol}(\mathcal{R}_\Lambda) = |\mathcal{R}^o_\Lambda|(2d)^{-|\Lambda|}$. In the BTW-sandpile model, it is well known that $|\mathcal{R}^o_\Lambda| = \det(\Delta^o)$, where Δ^o is the toppling matrix, see [41, Theorem 4.3]. Since $\Delta^o = 2d\Delta$, we have

$$\mathrm{Vol}(\mathcal{R}_\Lambda) = (2d)^{|\Lambda|} \det(\Delta)(2d)^{-|\Lambda|} = \det(\Delta).$$

(2) is immediate from the definitions. □

2.4 Proof of Theorem 2.2

We denote the sites in Λ by $x_1, x_2, \ldots, x_{|\Lambda|}$ and define the collection $\mathcal{L}$ as

$$\mathcal{L} = \Big\{[c,d) = \prod_{1 \leq k \leq |\Lambda|} [c_k, d_k) : 0 \leq c_k \leq d_k \leq \frac{1}{2d}\Big\},$$

where $c = (c_1, \ldots, c_{|\Lambda|})$ and $d = (d_1, \ldots, d_{|\Lambda|})$. Note that $\mathcal{L}$ is a π-system. For $\xi \in \mathcal{R}^o_\Lambda$, let

$$C(\xi, [c,d)) = \{\eta \in C(\xi) : \tilde{\eta} \in [c,d)\},$$

which is the set of configurations whose integer part is ξ and whose fractional part is in the interval $[c,d)$. Let

$$\mathcal{I} = \{C(\xi,[c,d)) : [c,d) \in \mathcal{L}, \xi \in \mathcal{R}^o_\Lambda\}. \tag{2.14}$$

$\mathcal{I}$ is also a π-system. In order to show that μ is A^u_x-invariant, it suffices to show that

$$\mu\{\eta : A^u_x\eta \in B\} = \mu(B) \tag{2.15}$$

for all $B \in \mathcal{I}$, and we will do that by direct calculation.

Let $B = C(\xi,[c,d))$ and $\zeta \in B$, and define the configuration η by

$$\tilde{\eta}(z) = \left(\zeta(z) - u\delta_x(z)\right) \bmod \frac{1}{2d}, \qquad z \in \Lambda, \tag{2.16}$$

and

$$\overline{\eta} = \begin{cases} (a_x^{-1})^{\lfloor 2du\rfloor}\xi & \text{if } \tilde{\zeta}(x) \geq u \bmod \frac{1}{2d}; \\ (a_x^{-1})^{\lfloor 2du\rfloor+1}\xi & \text{if } \tilde{\zeta}(x) < u \bmod \frac{1}{2d}. \end{cases} \tag{2.17}$$

We claim that this η is the unique $\eta \in \mathcal{R}_\Lambda$ such that $A^u_x\eta = \zeta$. To see this, first note that for any $\eta \in \mathcal{R}_\Lambda$, $\tilde{\eta}$ differs from $\widetilde{A^u_x\eta}$ only at the site x, and that

$$\widetilde{A^u_x\eta}(x) = (\eta(x) + u) \bmod \frac{1}{2d}.$$

This shows that for any $\eta \in \mathcal{R}_\Lambda$ such that $A^u_x\eta = \zeta$, its fractional part is given by (2.16). Now there are two possibilities:

1. $(\widetilde{A^u_x\eta})(x) = \tilde{\zeta}(x) \geq u \bmod \frac{1}{2d}$;
2. $(\widetilde{A^u_x\eta})(x) = \tilde{\zeta}(x) < u \bmod \frac{1}{2d}$.

A little algebra reveals that in the first case,

$$\overline{A^u_x\eta} = \mathcal{S}^o(\overline{\eta} + \lfloor 2du\rfloor\delta_x) = a_x^{\lfloor 2du\rfloor}\overline{\eta},$$

and in the second case,

$$\overline{A^u_x\eta} = \mathcal{S}^o(\overline{\eta} + (\lfloor 2du\rfloor + 1)\delta_x) = a_x^{\lfloor 2du\rfloor+1}\overline{\eta}.$$

In words, depending on the fractional part of ζ at site x, the addition of u to site x corresponds to either adding $\lfloor 2du\rfloor$ or $\lfloor 2du\rfloor + 1$ particles at x in the BTW model. It follows that any $\eta \in \mathcal{R}_\Lambda$ such that $A^u_x\eta = \zeta$ must have integer part given by (2.17). We conclude that for every $\zeta \in B$, the η defined by (2.16) and (2.17) is the unique $\eta \in \mathcal{R}_\Lambda$ such that $A^u_x\eta = \zeta$. It follows that A^u_x is a bijection.

Next we show that A_x^u preserves the measure μ. For this, note that the inverse image of B naturally partitions into two sets: if $x = x_i$, define $[c^1, d^1)$, $[c^2, d^2)$ to be intervals that differ from $[c, d)$ in the i^{th} coordinate only, to the effect that $[c_i^1, d_i^1) = [c_i, d_i) \cap [u \bmod \frac{1}{2d}, \frac{1}{2d})$ and $[c_i^2, d_i^2) = [c_i, d_i) \cap [0, u \bmod \frac{1}{2d})$. Note that $[c, d) = [c^1, d^1) \cup [c^2, d^2)$. From the above we have

$$\{\eta : A_x^u \eta \in C(\xi, [c^1, d^1))\} = C\Big((a_x^{-1})^{\lfloor 2du \rfloor} \xi, [c', d')\Big)$$

and

$$\{\eta : A_x^u \eta \in C(\xi, [c^2, d^2))\} = C\Big((a_x^{-1})^{\lfloor 2du \rfloor + 1} \xi, [c'', d'')\Big),$$

where $[c', d')$ and $[c'', d'')$ are intervals that differ from $[c, d)$ only in the i^{th} coordinate to the effect that $[c_i', d_i') = [c_i^1 - u \bmod \frac{1}{2d}, d_i^1 - u \bmod \frac{1}{2d})$ and $[c_i'', d_i'') = [c_i^2 + \frac{1}{2d} - u \bmod \frac{1}{2d}, d_i^2 + \frac{1}{2d} - u \bmod \frac{1}{2d})$. Finally,

$$\mathrm{Vol}\Big\{C\Big((a_x^{-1})^{\lfloor 2du \rfloor} \xi, [c', d')\Big)\Big\} = \mathrm{Vol}\Big\{C\Big(\xi, [c^1, d^1)\Big)\Big\}$$

and

$$\mathrm{Vol}\Big\{C\Big((a_x^{-1})^{\lfloor 2du \rfloor + 1} \xi, [c'', d'')\Big)\Big\} = \mathrm{Vol}\Big\{C\Big(\xi, [c^2, d^2)\Big)\Big\},$$

which implies that μ is A_x^u-invariant. □

2.5 Proof of Theorem 2.3

We prove Theorem 2.3 with a coupling. Let $\eta, \zeta \in \Omega_\Lambda$ be two initial configurations and let η_t, ζ_t be two copies of the CBTW sandpile starting from η, ζ respectively (hence $\eta = \eta_0$ and $\zeta = \zeta_0$). Addition amounts at time t are U_t^η, U_t^ζ respectively , the addition sites are X_t^η, X_t^ζ respectively. All these random quantities are independent of each other. In the proof, we will couple the two processes, and in the coupling, random variables will be written in 'hat'-notation; see below.

For every $x \in \Lambda$ and $t = 0, 1, \ldots$, let $D_t(x)$ be defined as

$$D_t(x) = \frac{1}{\lceil \frac{4}{b-a} \rceil}\Big(\eta_t(x) - \zeta_t(x)\Big). \tag{2.18}$$

We now define a coupling of the CBTW realizations starting from η and ζ respectively. The sites to which we add are copied from the η-process, that is, we define

$$\hat{X}_t^\eta = \hat{X}_t^\zeta = X_t^\eta \tag{2.19}$$

and write X_t (without any superscript) for the common value. The addition amount in the η-process is unchanged, that is,

$$\hat{U}_t^\eta = U_t^\eta,$$

but in the ζ-process we define

$$\hat{U}_t^\zeta = [U_t^\eta + D_0(X_t) - a] \bmod (b-a) + a.$$

Hence, in the coupling the η-process evolves as in the original version, but the ζ-process does not. It is not hard to see that this definition gives the correct marginals and that if $U_t^\eta \in [\frac{3a+b}{4}, \frac{a+3b}{4}]$, then

$$\hat{U}_t^\zeta = U_t^\eta + \frac{1}{\lceil \frac{4}{b-a} \rceil}(\eta_0(X_t) - \zeta_0(X_t)). \tag{2.20}$$

Let us now say that event O occurs if

1. between times 1 and $1 + |\Lambda| \lceil \frac{4}{b-a} \rceil$ (inclusive), all sites are chosen as addition sites exactly $\lceil \frac{4}{b-a} \rceil$ times;
2. between times 1 and $1 + |\Lambda| \lceil \frac{4}{b-a} \rceil$ (inclusive), the addition amounts in the η-process are all contained in the interval $[\frac{3a+b}{4}, \frac{a+3b}{4}]$.

If O occurs, then we claim that at time $1 + |\Lambda| \lceil \frac{4}{b-a} \rceil$, the two processes are in the same state. Indeed, by the abelian property we can obtain the configuration at time t by first adding all additions up to time t, and after that topple all unstable sites in any order. If O occurs, and we defer toppling to the very end, all heights will be the same in the two processes by construction, and hence after toppling they remain the same.

The probability that O occurs is uniformly bounded below, that is, uniformly in the initial configurations η and ζ. Indeed, all that is necessary is that all sites are equally often addition sites, and that the addition amounts are in the correct subinterval of $[a, b]$. These two requirements are independent of the starting configurations. Of course, the evolution of the coupling does depend on the initial configurations via the relations between the added amounts. That is where the subtlety of the present coupling lies.

Hence, to finish our construction, we first see whether or not O occurs. If it does we are done. If it does not, we start all over again, with the current configurations at time t as our new initial configurations, and $D_t(x)$ instead of $D_0(x)$. Continuing this way, we have a fixed positive probability for success at each trial and therefore the two processes will almost surely be equal eventually. Due to the fact that the success probability is uniformly bounded below, convergence will be exponentially fast. The result now follows by choosing η and ζ according to the distributions ν and μ, respectively. □

2.6 Proof of Theorems 2.4 and 2.5

We start with the proof of Theorem 2.4.

Proof of Theorem 2.4. Define the function g by

$$g(\eta) = \exp\Big(4d\pi i \sum_{x\in\Lambda} \eta(x)\Big).$$

It is easy to see that g is continuous and, of course, bounded. Denoting the distribution of the process at time t by ν_t, for ν_t to converge weakly to ν, say, it must be the case that

$$\int g d\nu_t \to \int g d\nu. \tag{2.21}$$

However, since at each iteration of the process, exactly one of the fractional parts is increased by a and subsequently taken modulo $1/2d$, we have that $\sum_{x\in\Lambda} \eta_t(x)$ mod $1/2d$ is ν_t-surely equal to $\left(\sum_{x\in\Lambda} \eta_0(x) + ta\right)$ mod $1/2d$, and hence the sequence of integrals on the left of (2.21) does not converge at all, unless a is a multiple of $1/2d$. □

We now move on to the proof of Theorem 2.5. In order to describe the limiting measure μ_a^η appearing in the statement of the theorem, we first introduce the following notation. For a measure ν on the space of height configurations $\mathcal{X}_\Lambda$, $\mathcal{S}\nu$ denotes the measure on the stable configurations Ω_Λ defined by

$$\mathcal{S}\nu(B) = \nu\{\eta \in \mathcal{X}_\Lambda : \mathcal{S}\eta \in B\},$$

for every measurable set $B \subset \Omega_\Lambda$.

Proof of Theorem 2.5. When $a = 0$, $\eta_t = \eta$ for all $t = 1, 2, \ldots$, and the limiting distribution μ_0^η is point mass at η.

So assume $l > 0$ and take $a = l/2d$. We first consider the case in which $\eta(x) = 0$ for all sites x, and proceed by a coupling between the BTW and the CBTW model. To introduce this coupling, let (X_t) denote a common sequence of (random) addition sites, and define

$$\theta_t = \sum_{s=1}^{t} a\delta_{X_s}, \qquad \theta_t^o = \sum_{s=1}^{t} \delta_{X_s}.$$

This couples the vector θ_t of all additions until time t in the CBTW model, with the vector θ_t^o of all additions until time t in the BTW model.

In order to arrive at the BTW configuration at time t, now we can first apply all additions, and then topple unstable sites as long as there are any. In doing so, we couple the topplings in the BTW model with those in the CBTW model: if we topple a site x in the BTW model, we topple the corresponding site in the CBTW model $l = 2da$ times. If we topple at x, then in the BTW model, x loses $2d$ particles, while all neighbors receive 1. In the CBTW model, x loses total mass $2da$ while all its neighbors receive a. Hence, the dynamics in the CBTW model is exactly the same as in the BTW model, but multiplied by a factor of a. In particular, since all the topplings in the BTW model were legal, all the topplings in the CBTW model must have been legal as well. Furthermore, when the BTW model has reached the stable configuration $\kappa_t = \mathcal{S}^o\theta^o_t$, then the corresponding configuration in the CBTW model is simply $a\kappa_t$. However, $a\kappa_t$ need not be stable in the CBTW model. Hence, in order to reach the CBTW configuration at time t, we have to stabilize $a\kappa_t$, leading to the stable configuration $\rho_t = \mathcal{S}\theta_t$.

From the results in [41], we know that the distribution of κ_t converges exponentially fast (as $t \to \infty$) in total variation to the uniform distribution on $\mathcal{R}^o_\Lambda$. Hence, ρ_t is exponentially close to $\mathcal{S}\nu_a$, where ν_a is the uniform measure on the set $\{a\xi : \xi \in \mathcal{R}^o_\Lambda\}$.

This settles the limiting measure in the case where we start with the empty configuration. If we start with configuration η in the CBTW model, we may (by abeliannes) first start with the empty configuration as above—coupled to the BTW model in the same way—and then add the 'extra' η to ρ_t at the end, and stabilize the obtained configuration. It follows that the height distribution in the CBTW model converges to $\mathcal{S}\nu^\eta_a$, where ν^η_a is the uniform measure on the set $\{\eta + a\xi : \xi \in \mathcal{R}^o_\Lambda\}$. □

2.7 Proof of Theorem 2.6

The proof of Theorem 2.6 is the most involved. It turns out that the viewpoint of random ergodic theory is very useful here, and we start by reformulating the sandpile in this framework; see [33] for a review of this subject.

We consider the CBTW with $a = b$, that is, with non-random additions at a randomly chosen site. As before, we denote by μ the uniform measure on $\mathcal{R}_\Lambda$. For every $x \in \Lambda$, we have a transformation $A^a_x : \mathcal{R}_\Lambda \to \mathcal{R}_\Lambda$ which we denote in this section by A_x (since a is fixed). Recall that each A_x is a bijection by Theorem 2.2. The system evolves by each time picking one of the A_x uniformly at random (among all A_x, $x \in \Lambda$) independently of each other. The product measure governing the choice of the subsequent transformations is denoted by $\mathbf{p}$, that is, $\mathbf{p}$ assigns probability $1/|\Lambda|$ to each transformation, and makes sure that transformations are chosen independently. The system has randomness in two ways: an initial distribution ν on $\mathcal{R}_\Lambda$ and the choice of the transformations. To account for this we sometimes work with the product measure $\nu \times \mathbf{p}$.

A probability measure ν on $\mathcal{R}_\Lambda$ is called *invariant* if

$$\nu(B) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \nu(A_x^{-1} B),$$

that is, ν preserves measure on average, not necessarily for each transformation individually. We call a bounded function g on $\mathcal{R}_\Lambda$ ν-*invariant* if

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} g \circ A_x(\eta) = g(\eta),$$

for ν-almost all $\eta \in \mathcal{R}_\Lambda$. We call a (measurable) subset B of $\mathcal{R}_\Lambda$ invariant if its indicator function is a ν-almost everywhere invariant function; this boils down to the requirement that up to sets of ν-measure 0, B is invariant under each of the transformations A_x individually. Finally, we call an invariant probability measure ν on $\mathcal{R}_\Lambda$ *ergodic* if any ν-invariant function is a ν-a.s. constant. These definitions extend the usual definitions in ordinary (non-random) ergodic theory. It is well known and not hard to show (see [33, Lemma 2.4]) that ν is ergodic if and only if every invariant set has ν-probability zero or one. As before, we denote by $\bar{\eta}$ and $\tilde{\eta}$ the integer and fractional parts of the configuration η.

Lemma 2.14. *Let λ be an invariant probability measure, B a λ-invariant set and C a set such that $\lambda(C \,\Delta\, B) = 0$. Then $\lambda(A_x^{-1} C \,\Delta\, B) = 0$ for all $x \in \Lambda$.*

Proof. From the assumptions and invariance of λ, it follows that the sets $A_x^{-1} B \,\Delta\, B$, $A_x^{-1}(B \setminus C)$ and $A_x^{-1}(C \setminus B)$ all have λ-measure 0. Now suppose $\omega \in A_x^{-1} C \setminus B$. Then either $\omega \in A_x^{-1} B \setminus B$ or else $\omega \in A_x^{-1}(C \setminus B)$. Since both these sets have λ-measure 0, $\lambda(A_x^{-1} C \setminus B) = 0$. Next suppose that $\omega \in B \setminus A_x^{-1} C$. Then either $\omega \in B \setminus A_x^{-1} B$ or else $\omega \in A_x^{-1}(B \setminus C)$. Again, since both these sets have λ-measure 0, $\lambda(B \setminus A_x^{-1} C) = 0$. □

A version of the following lemma is well known in ordinary ergodic theory, see e.g. [8, Proposition 5.4].

Lemma 2.15. *Let ν be an ergodic probability measure (in our sense) and let λ be an invariant probability measure which is absolutely continuous with respect to ν. Then λ is ergodic, and therefore $\lambda = \nu$.*

Proof. For $u \in \mathbb{Z}_{\geq 0}^{\Lambda}$, write

$$T_u := \prod_{x \in \Lambda} A_x^{u(x)}.$$

By commutativity, the order of the composition is irrelevant. Now suppose that B is a λ-invariant set. Define

$$C := \bigcap_{n=0}^{\infty} \bigcup_{u \in \mathbb{Z}_{\geq n}^{\Lambda}} T_u^{-1} B.$$

In words, C is the set of configurations η such that for all n, there is a u with $u(x) \geq n$ at every $x \in \Lambda$, for which $T_u \eta \in B$. Since $T_u(A_x \eta) = T_{u+\delta_x} \eta$, it is not difficult to see that $A_x \eta \in C$ if and only if $\eta \in C$. Hence $A_x^{-1} C = C$ for all $x \in \Lambda$. In particular, C is a ν-invariant set, so by ergodicity of ν, either $\nu(C) = 0$ or $\nu(C^c) = 0$. But since λ is absolutely continuous with respect to ν, this implies that either $\lambda(C) = 0$ or $\lambda(C^c) = 0$.

To prove that λ is ergodic, it therefore suffices to show that $\lambda(B) = \lambda(C)$, or equivalently, $\lambda(C \mathbin{\Delta} B) = 0$. To this end, we first claim that

$$(C \mathbin{\Delta} B) \subset \bigcup_{u \in \mathbb{Z}_{\geq 0}^{\Lambda}} (T_u^{-1} B \mathbin{\Delta} B). \tag{2.22}$$

Indeed, if $\eta \in C \setminus B$, there must be a $u \in \mathbb{Z}_{\geq 0}^{\Lambda}$ such that $T_u \eta \in B$, hence $\eta \in T_u^{-1} B \setminus B$. And if $\eta \in B \setminus C$, then there are only finitely many n such that $T_u \eta \in B$ with $u(x) = n$ for all $x \in \Lambda$. Hence $\eta \in B \setminus C$ implies that for some $n > 0$ we have $T_u \eta \notin B$ with $u(x) = n$ for all $x \in \Lambda$, and therefore $\eta \in B \setminus T_u^{-1} B$ for this particular u. This establishes (2.22).

To complete the proof, note that by repeated application of Lemma 2.14, it follows that for every $u \in \mathbb{Z}_{\geq 0}^{\Lambda}$, $\lambda(T_u^{-1} B \mathbin{\Delta} B) = 0$. Hence $\lambda(C \mathbin{\Delta} B) = 0$ by (2.22), and we conclude that λ is ergodic. Now it follows from Theorem 2.1 in [33] that $\nu \times \mathbf{p}$ and $\lambda \times \mathbf{p}$ are also ergodic (in the deterministic sense w.r.t. the skew product transformation). But since λ is absolutely continuous with respect to ν, $\lambda \times \mathbf{p}$ is absolutely continuous with respect to $\nu \times \mathbf{p}$, and hence must be equal to $\nu \times \mathbf{p}$ by ordinary (non-random) ergodic theory. It follows that $\nu = \lambda$. □

We will apply Lemma 2.15 with the uniform measure μ in the role of ν. Before we can do so, we must first show that μ is ergodic, which is an interesting result in its own right.

Theorem 2.16. *When $a = b \notin \mathbb{Q}$, the uniform measure μ on $\mathcal{R}_\Lambda$ is ergodic.*

Proof. We denote the sites in Λ by $x_1, x_2, \ldots, x_m$, and identify a configuration $\eta \in [0,\infty)^\Lambda$ with the point $(\eta(x_1), \ldots, \eta(x_m))$ in $[0,\infty)^m$. By Proposition 3.1 in [41], there exists a $n = (n_1, \ldots, n_m) \in \mathbb{Z}^m_{\geq 1}$ such that for all $i = 1, 2, \ldots, m$,

$$a_{x_i}^{n_i} \xi = \xi, \qquad \forall \xi \in \mathcal{R}^o_\Lambda.$$

Now let $\mathcal{A}$ be the rectangle

$$\Big[0, \frac{1}{2d} n\Big) = \Big[0, \frac{n_1}{2d}\Big) \times \Big[0, \frac{n_2}{2d}\Big) \times \cdots \times \Big[0, \frac{n_m}{2d}\Big),$$

and denote Lebesgue measure on $\mathcal{A}$ by λ. Write τ_i for the translation by a modulo $n_i/2d$ in the i^{th} coordinate direction on $\mathcal{A}$.

A subset D of $\mathcal{A}$ is called λ-*invariant* if $\lambda(\tau_i^{-1} D) = \lambda(D)$ for all τ_i. We claim that for any λ-invariant set D, either $\lambda(D) = 0$ or $\lambda(D^c) = 0$. Although this fact is probably well known, we give a proof for completeness. Write I_D for the indicator function of D, and for $k \in \mathbb{Z}^m$, denote by c_k the Fourier coefficients of I_D. Then

$$c_k = \int_{\mathcal{A}} I_D(\omega) f_k(\omega)\, d\lambda(\omega) = \int_D f_k(\omega)\, d\lambda(\omega),$$

where

$$f_k(\omega) = \prod_{j=1}^m \tfrac{2d}{n_j} e^{2\pi i \frac{2d}{n_j} k_j \omega_j}, \qquad \omega \in \mathcal{A}.$$

Now suppose that $k_j \neq 0$ for some $1 \leq j \leq m$. Then, changing coordinates by applying τ_j, we have that

$$c_k = \int_{\tau_j^{-1} D} f_k(\tau_j \omega)\, d\lambda(\tau_j \omega) = \int_D f_k(\omega) e^{2\pi i \frac{2d}{n_j} k_j a}\, d\lambda(\omega) = c_k\, e^{4d\pi i \frac{k_j}{n_j} a},$$

because Lebesgue measure is invariant under τ_j and D is λ-invariant. But since a is irrational, this implies that $c_k = 0$. It follows that $I_D = c_o$ λ-almost everywhere. Hence c_o is either 0 or 1, so that either $\lambda(D) = 0$ or $\lambda(D^c) = 0$.

Next note that $\mathcal{A}$ is composed of the cubes

$$C_k = \Big[\frac{k_1}{2d}, \frac{k_1+1}{2d}\Big) \times \Big[\frac{k_2}{2d}, \frac{k_2+1}{2d}\Big) \times \cdots \times \Big[\frac{k_m}{2d}, \frac{k_m+1}{2d}\Big),$$

where $k \in \mathbb{Z}^m$ with $0 \leq k_i < n_i$ for $i = 1, 2, \ldots, m$. Using this fact, we define a map $\psi : \mathcal{A} \to \mathcal{R}_\Lambda$ as follows. If ω is in the cube C_k, let

$$\xi_k = \prod\nolimits_{i=1}^m a_{x_i}^{k_i} \xi^{max},$$

where $\xi^{max}(x) = 2d - 1$ for all $x \in \Lambda$, and set

$$\psi(\omega) := \omega - \frac{1}{2d}k + \frac{1}{2d}\xi_k.$$

Thus, ψ simply translates the cube C_k onto the cube of configurations in $\mathcal{R}_\Lambda$ whose integer part is ξ_k. Notice that multiple cubes in $\mathcal{A}$ may be mapped by ψ onto the same cube in $\mathcal{R}_\Lambda$, but that $\psi : \mathcal{A} \to \mathcal{R}_\Lambda$ is surjective, because every allowed configuration of the BTW model can be reached from ξ^{max} after a finite number of additions (and subsequent topplings). Furthermore, it is easy to see that a translation τ_i on $\mathcal{A}$ corresponds to an addition A_{x_i} on $\mathcal{R}_\Lambda$, in the sense that

$$\psi(\tau_i\omega) = A_{x_i}\psi(\omega). \tag{2.23}$$

Now suppose that $B \subset \mathcal{R}_\Lambda$ is μ-invariant. Since μ is normalized Lebesgue measure on $\mathcal{R}_\Lambda$ and λ is Lebesgue measure on $\mathcal{A}$, we have that

$$\lambda(\psi^{-1}(B)) = \sum_k \lambda(\psi^{-1}(B) \cap C_k) = \mathrm{Vol}(\mathcal{R}_\Lambda) \sum_k \mu(B \cap \psi(C_k)),$$

where the sum is over all cubes in $\mathcal{A}$. Because $\mu(A_{x_i}^{-1}B \,\Delta\, B) = 0$, this gives

$$\lambda(\psi^{-1}(B)) = \mathrm{Vol}(\mathcal{R}_\Lambda) \sum_k \mu(A_{x_i}^{-1}B \cap \psi(C_k)) = \sum_k \lambda(\psi^{-1}(A_{x_i}^{-1}B) \cap C_k).$$

By (2.23), we have that

$$\psi^{-1}(A_{x_i}^{-1}B) = \tau_i^{-1}\psi^{-1}(B)$$

and it follows that $\lambda(\psi^{-1}(B)) = \lambda(\tau_i^{-1}\psi^{-1}(B))$, hence $\psi^{-1}(B)$ is a λ-invariant set. Therefore, either $\lambda(\psi^{-1}(B)) = 0$ or $\lambda(\psi^{-1}(B^c)) = 0$. From the construction it then follows that either $\mu(B) = 0$ or $\mu(B^c) = 0$. Therefore, μ is ergodic. □

The next lemma will be used to deal with the evolution of the joint distribution of the fractional parts. In order to state it we need a few definitions. Consider the unit cube $I_m = [0,1]^m$. Let a be an irrational number and let θ be the measure which assigns mass $1/m$ to each of the points $(a, 0, \ldots, 0), (0, a, 0, \ldots, 0), \ldots, (0, \ldots, 0, a)$. Denote by θ^{*n} the n-fold convolution of θ, where additions are modulo 1. Translation over the vector x (modulo 1 also) is denoted by τ_x. We define the measure μ_N^x by

$$\mu_N^x = \frac{1}{N}\sum_{n=0}^{N-1} \theta^{*n}\tau_x^{-1}.$$

In words, this measure corresponds to choosing $n \in \{0, 1, \ldots, N-1\}$ uniformly, and then applying the convolution of n independently chosen transformations with starting point x.

Lemma 2.17. *For all x, μ_N^x converges weakly to Lebesgue measure on I_m as $N \to \infty$.*

Proof. The setting is ideal for Fourier analysis. It suffices to prove (using Stone-Weierstrass or otherwise) that

$$\int f_k \, d\mu_N^x \to \int f_k \, d\lambda,$$

where λ denotes Lebesgue measure and

$$f_k(y) = e^{2\pi i k\cdot y} = e^{2\pi i \sum_{j=1}^m k_j y_j},$$

for $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ and $y \in I_m$.

When $k = (0, \ldots, 0)$, all integrals are equal to 1. For $k \neq (0, \ldots, 0)$, $\int f_k \, d\lambda = 0$, and hence it suffices to prove that for these k,

$$\int f_k \, d\mu_N^x \to 0.$$

For $n \geq 0$ and $l = (l_1, \ldots, l_m) \in \mathbb{Z}_{\geq 0}^m$ such that $l_1 + \cdots + l_m = n$,

$$\theta^{*n} \tau_x^{-1}\{x + la\} = \frac{1}{m^n}\binom{n}{l_1, l_2, \ldots, l_m}.$$

Hence, by the multinomial theorem,

$$\begin{aligned}\int f_k \, d\theta^{*n}\tau_x^{-1} &= \sum_{l_1+\cdots+l_m=n} \frac{1}{m^n}\binom{n}{l_1, l_2, \ldots, l_m} e^{2\pi i k\cdot x} \prod_{j=1}^m e^{2\pi i a k_j l_j} \\ &= f_k(x)\left(\sum_{j=1}^m \frac{1}{m} e^{2\pi i a k_j}\right)^n =: f_k(x)(\alpha_k)^n.\end{aligned}$$

Note that $|\alpha_k| \leq 1$ and $\alpha_k \neq 1$ because a is irrational. Therefore,

$$\int f_k \, d\mu_N^x = f_k(x)\frac{1}{N}\sum_{n=0}^{N-1}(\alpha_k)^n = f_k(x)\frac{1}{N}\frac{1-(\alpha_k)^N}{1-\alpha_k} \to 0,$$

as required. □

We will use Lemma 2.17 to understand the fractional parts in the sandpile, working on the cube $[0, 1/2d]^m$ for $m = |\Lambda|$ rather than on $[0, 1]^m$. Since topplings have no effect on the fractional parts of the heights, for the fractional parts it suffices to study the additions, and a point $x \in [0, 1/2d]^m$ corresponds to all fractional parts of the m sites in the system.

Proof of Theorem 2.6.. We have from Theorem 2.16 that the uniform measure μ on $\mathcal{R}_\Lambda$ is ergodic. Suppose now that ν is another invariant measure. We will show that $\mu(B) = 0$ implies $\nu(B) = 0$, and according to Lemma 2.15 it then follows that $\nu = \mu$.

Consider the "factor map" f defined via $f(\eta) = \tilde{\eta}$, that is, f produces the fractional part when applied to a configuration. The map f commutes with the random transformations we apply to η, in the sense that

$$f(A_x(\eta)) = \tilde{A}_x(f(\eta)),$$

where $\tilde{A}_x(\tilde{\eta})$ is the configuration of fractional parts that results upon adding a to the height at site x. Write $\tilde{\nu} = \nu f^{-1}$ and $\tilde{\mu} = \mu f^{-1}$. Because f commutes with the random transformations and ν is invariant, for any measurable subset $\tilde{B}$ of $[0, 1/2d]^{|\Lambda|}$ we have that

$$\nu(f^{-1}(\tilde{B})) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \nu(A_x^{-1} f^{-1}(\tilde{B})) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \nu(f^{-1}(\tilde{A}_x^{-1} \tilde{B})).$$

Therefore, we have

$$\tilde{\nu}(\tilde{B}) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \tilde{\nu}(\tilde{A}_x^{-1} \tilde{B}),$$

and hence $\tilde{\nu}$ is invariant. We claim that $\tilde{\mu} = \tilde{\nu}$. To prove this, it suffices to show that $\tilde{\mu}(R) = \tilde{\nu}(R)$ for any rectangle R. Since rectangles are $\tilde{\mu}$-continuity sets, by Lemma 2.17 and bounded convergence,

$$\tilde{\mu}(R) = \int \tilde{\mu}(R)\, d\tilde{\nu} = \int \lim_{N \to \infty} \int I_R \, d\mu_N^h \, d\tilde{\nu}(h) = \lim_{N \to \infty} \iint I_R \, d\mu_N^h \, d\tilde{\nu}(h).$$

Here μ_N^h is the analogue of the measure defined in Lemma 2.17 on the space $[0, 1/2d]^{|\Lambda|}$ rather than $[0,1]^m$; it corresponds to choosing $n \in \{0, 1, \ldots, N-1\}$ uniformly, and then applying n uniformly and independently chosen transformations to the fractional height configuration h. Thus we also have that

$$\begin{aligned} \iint I_R \, d\mu_N^h \, d\tilde{\nu}(h) &= \int \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{|\Lambda|^n} \sum_{y_1, \ldots, y_n \in \Lambda} I_R(\tilde{A}_{y_1} \cdots \tilde{A}_{y_n} h)\, d\tilde{\nu}(h) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{|\Lambda|^n} \sum_{y_1, \ldots, y_n \in \Lambda} \tilde{\nu}(\tilde{A}_{y_n}^{-1} \cdots \tilde{A}_{y_1}^{-1} R). \end{aligned}$$

By invariance of $\tilde{\nu}$, this last expression is equal to

$$\frac{1}{N} \sum_{n=0}^{N-1} \tilde{\nu}(R) = \tilde{\nu}(R),$$

and we conclude that $\tilde{\mu}(R) = \tilde{\nu}(R)$ for every rectangle R, hence $\tilde{\mu} = \tilde{\nu}$.

For the final step, let B be such that $\mu(B) = 0$ and write $\tilde{B} = f(B)$. We claim that then $\tilde{\mu}(\tilde{B}) = 0$. Indeed, the inverse image of $\tilde{B}$ is a collection of points of the form $x + (2d)^{-1}\eta$, where $x \in B$ and $\eta \in \mathbb{Z}^{\Lambda}$. If $\mu(B) = 0$, then also this collection of points has μ-measure 0. Since $\tilde{\mu} = \tilde{\nu}$, it follows that $\tilde{\nu}(\tilde{B}) = 0$, hence also $\nu(B) = 0$. This concludes the proof. □

3 The multiple addition sandpile model

Summary

In this chapter, we study a multiple addition sandpile model, which shows some similarity to the BTW sandpile model but also has some differences. At each time, the addition contains multiple grains instead of one grain. This modification leads to the new phenomenon that this model might have several different recurrent classes, depending on the addition amounts at each time step.

This chapter contains the results in [36] by H. Liu.

3.1 Definition and main results

The *BTW sandpile model* is a sandpile model named after Bak, Tang and Wiesenfeld and was first introduced in 1987 to study 'self-organized criticality' [5]. In Dhar's paper [9], this model is also called teh *abelian sandpile model*(ASM) because of the abelian property of the addition operators. The abelian property has become a basic tool to study the BTW model and related questions. An introduction to and the definition of the BTW model can be found in Section 1.3 of Chapter 1. Mathematical studies of this model and related questions are performed in many papers such as [21, 41, 27, 44, 22]. In this chapter, we denote by $\mathcal{X}^o_\Lambda$ the set of *all* configurations, and by Ω^o_Λ the set of *stable* configurations in the BTW model. We use $\mathcal{S}^o$ for the *stabilization* operator, a_x for the addition operator at site x, and t_x for the toppling operator at site x.

A configuration $\eta \in \Omega^o_\Lambda$ is *recurrent* if starting from η, η will be visited by the process infinitely often almost surely. By $\mathcal{R}^o_\Lambda$ we denote the set of all recurrent configurations in the BTW sandpile model. As we know from [9, 41, 43], $\mathcal{R}^o_\Lambda$ is the unique recurrent class, and no matter from which configuration in Ω^o_Λ we start, the process will converge in distribution to the uniform measure on $\mathcal{R}^o_\Lambda$.

In [31] and Chapter 2, we have studied a continuous height abelian sandpile model, which for abbreviation is also called the *CBTW model*. As was discussed in Section 2.3 of Chapter 2, for every continuous height configuration ξ, we can define $\overline{\xi}$ and $\tilde{\xi}$, which are called the *integer part* and the *fractional part* of ξ respectively (see Definition 2.7). And ξ has the identity form:

$$\xi = \frac{1}{2d}\overline{\xi} + \tilde{\xi}$$

When we study the result of an addition, very often we study the integer part and fractional part separately. For example, if we add mass $u \in [0,1)$ to ξ at position x, this will change both the integer and the fractional part. The integer part itself is a kind of sandpile model with integer values. But the addition amount is not fixed, sometimes it is $\lfloor 2du \rfloor$ and sometimes it is $\lfloor 2du \rfloor + 1$, depending on the value of $\tilde{\xi}(x)$. Both $\lfloor 2du \rfloor$ and $\lfloor 2du \rfloor + 1$ could be an integer larger than 1. This interesting observation motivates the model we will study in this chapter.

Now we start to give the formal definition of the new model. By $\mathbb{N}$, we mean the set of natural numbers. The addition amount in this model is always the same amount $k \in \mathbb{N}$. Starting from a configuration $\eta \in \Omega^o_\Lambda$, at every time step, we first randomly choose a site from Λ with uniform probability. Then we add $k \in \mathbb{N}$ grains simultaneously to the chosen site and stabilize the obtained configuration. The dynamics can be simply described by the following Markov chain:

$$\eta_t = a^k_{X_t}\eta_{t-1}, \quad t = 1, 2, \dots \tag{3.1}$$

where X_t is chosen from Λ with uniform probability. We call this model the BTW-k model. Under this definition, the BTW sandpile model is also named the BTW-1 model. We are interested in the number of recurrent classes of the BTW-k model.

In this chapter, we are going to study the convergence and the recurrent classes of the BTW-k model. The first result has a corresponding version in the BTW model.

Theorem 3.1. *For every $k \in \mathbb{N}$, for every initial configuration $\eta \in \Omega^o_\Lambda$, the distribution of the process at time t converges exponentially fast in total variation to a measure μ^η_k as $t \to \infty$.*

Before moving on to the next theorem, first let us look at an example. Let Λ be a 2×2 square and $k = 8$. Then we can see that both the set with

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and the set with

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

are closed and communicating classes for the BTW-8 model. For the 2×2 square in dimension 2, from [9, 43], there are in total 192 recurrent configurations which belong to the same recurrent class in the BTW model. Here in the BTW-8 model, there are 64 different recurrent classes (each containing 3 configurations).

For a general dimension d, we get:

Theorem 3.2. *For finite $\Lambda \subset \mathbb{Z}^d$, there exist infinitely many k and infinitely many k' such that BTW-k has a unique recurrent class, while BTW-k' has multiple recurrent classes.*

Since in dimension 1 the results of additions are better known, we can get more results. The following two theorems are in dimension 1.

For integers a, b, $\gcd(a, b)$ is the *greatest common divisor* of a and b, which is the largest positive integer that divides the numbers a and b without a remainder.

Theorem 3.3. *Let Λ be an interval of $\mathbb{Z}$ with $|\Lambda| = N$, $k \in \mathbb{N}$ and let $q = k \bmod (N+1)$. Then the set $\mathcal{R}^o_\Lambda$ can be divided into $\gcd(q, N+1)$ different closed (under the process) subsets and each of these subsets contains exactly $(N+1)/gcd(q, N+1)$ configurations; the uniform measure on each of these subsets is invariant under the process.*

Now, we define a new model. At each discrete time, instead of a fixed $k \in \mathbb{N}$, the addition amount is a random number chosen from a given subset $K \subset \mathbb{N}$ (K might contain infinitely many elements), where every number $k \in K$ can be chosen with positive probability. This new model is called BTW-K model. For every $k \in K$, let $q_k = k \bmod (N+1)$ and let $\gcd(N+1, q_k, k \in K)$ be the greatest common divisor of $(N+1)$ and all numbers $(q_k, k \in K)$. We then get:

Theorem 3.4. *Let Λ be an interval of $\mathbb{Z}$ with $|\Lambda| = N$ and K a subset of $\mathbb{N}$. Then the BTW-K model has $\gcd(N+1, q_k, k \in K)$ different recurrent classes and the uniform measure on each of these recurrent classes is invariant under the process.*

For this theorem, we give an example. Let $\Lambda = \{1, 2, 3, 4, 5\} \subset \mathbb{Z}$ and $K = \{2, 4\}$. Once can check that the set $\mathcal{R}^o_\Lambda$ is divided into 'two' (which equals $\gcd(6, 2, 4)$) subsets

$$\{(1,1,1,1,1), (1,0,1,1,1), (1,1,1,0,1)\}$$

and

$$\{(0,1,1,1,1), (1,1,0,1,1), (1,1,1,1,0)\}$$

both of which are closed and communicating classes for BTW-K.

A special case of Theorems 3.3 and 3.4 is the case when $gcd(N+1, q_k, k \in K) = N+1$, which means that $q_k = 0$ for all $k \in K$. In this specific case, every configuration $\eta \in \mathcal{R}^o_\Lambda$ is absorbing. Therefore, point mass on each single configuration in $\mathcal{R}^o_\Lambda$ is invariant under the process.

3.2 Preliminary results

The first result is about the recurrent configurations in the BTW-k model.

Lemma 3.5. *For every $\Lambda \in \mathbb{Z}^d$ and every $k \in \mathbb{N}$, the BTW-k model has the same set of recurrent configurations as the BTW model.*

Proof. From the proof of Lemma 3.23 in [43], we know that in the BTW model, if a configuration η is obtained from ζ by toppling every site $x \in \Lambda$ at least once, then η is allowed. From Theorem 3.27 in [43], an allowed configuration is also recurrent in the BTW model. Hence η is a recurrent configuration in the BTW model. This tells us that if we add enough grains to any configuration in Ω^o_Λ and make sure that every site in Λ topples at least once, then the new configuration is in $\mathcal{R}^o_\Lambda$. Starting from a configuration in $\mathcal{R}^o_\Lambda$, the process will stay in $\mathcal{R}^o_\Lambda$. Therefore in the BTW-k model, the process will also eventually enter and stay inside of $\mathcal{R}^o_\Lambda$. Hence all the recurrent configurations of the BTW-k model are contained in $\mathcal{R}^o_\Lambda$.

From Item 2 of Proposition 3.1 in [41], we know that for each $x \in \Lambda$, there is an integer $n_x \geq 1$ such that

$$a_x^{n_x}\eta = \eta, \quad \text{for all } \eta \in \mathcal{R}^o_\Lambda. \tag{3.2}$$

Therefore, $a_x^{kn_x}\eta = \eta$, so every configuration in $\mathcal{R}^o_\Lambda$ is also a recurrent configuration of the BTW-k model. □

In the following part of this section, some results in dimension 1 are presented. From many contexts such as [9, 41, 43], we know that, for an interval $\Lambda = \{1, 2, ..., N\}$, $\mathcal{R}^o_\Lambda$ has exactly $N+1$ elements, which are the configurations with at most one empty site. For convenience, we will label all the $N+1$ recurrent configurations according to the position of the empty sites. $\eta^1, \dots, \eta^N$ are configurations with site $1, \dots, N$ empty, respectively, and η^0 is the full configuration, which is the configuration in which every site has height 1.

For $\eta \in \{0,1\}^{|\Lambda|}$ and $x \in \Lambda$, define:

$$e^-(x,\eta) = \begin{cases} 0 & \text{if } \eta(y) = 1, \text{for all } y < x, \\ \max\{y < x : \eta(y) = 0\} & \text{otherwise,} \end{cases}$$

and

$$e^+(x,\eta) = \begin{cases} N+1 & \text{if } \eta(y) = 1, \text{for all } y > x, \\ \min\{y > x : \eta(y) = 0\} & \text{otherwise,} \end{cases}$$

which are the empty sites nearest to x to the left and right respectively. The following proposition tells us the results of one addition. A similar statement can be found on Page 5 of [43].

Proposition 3.6. *If $\eta(x) = 0$, then $a_x\eta = \eta + \delta_x$. Otherwise, when $\eta(x) = 1$, then for $z \in \Lambda$*

$$(a_x\eta)(z) = \begin{cases} \eta(z) & \text{if } z \notin [e^-(x,\eta), e^+(x,\eta)] \\ 0 & \text{if } z = e^-(x,\eta) + e^+(x,\eta) - x \\ 1 & \text{otherwise} \end{cases}$$

One can check this is true by yourself.

Lemma 3.7. *Restricted to $\mathcal{R}^o_\Lambda$, we have $a_x = a_1^x$, for all $x \in \Lambda$.*

Proof. We will show this by induction. First, when $x = 1$, it is obvious.

Now suppose that for $x \leq m$, we have $a_x = a_1^x$. If we add 2 grains to site m, site m becomes unstable and topples, and sites $m-1$ and $m+1$ receive one grain. Hence for every $\eta \in \mathcal{R}^o_\Lambda$, $a_m^2\eta = a_{m-1}a_{m+1}\eta$. Using the fact that for each $x = 1, \ldots, N$, a_x is a bijection from $\mathcal{R}^o_\Lambda$ to $\mathcal{R}^o_\Lambda$ and a_x^{-1} is well defined (see Proposition 3.1 of [41]), we get that for all $\eta \in \mathcal{R}^o_\Lambda$, $a_{m+1}\eta = a_{m-1}^{-1}a_m^2\eta = a_1^{-m+1+2m}\eta = a_1^{m+1}\eta$. Therefore, $a_{m+1} = a_1^{m+1}$. □

Lemma 3.8. $a_1\eta^l = \eta^{(l-1) \bmod (N+1)}$ *for $l = 0, \ldots, N$.*

Proof. If $l = 1$, $\eta^1(1) = 0$, hence $a_1\eta^1 = \eta^1 + \delta_1 = \eta^0$.

If $2 \leq l \leq N$, $e^-(1,\eta^l) = 0$ and $e^+(1,\eta^l) = l$, Proposition 3.6 tells us that $(a_1\eta^l)(z) = 0$, when $z = l-1$ and $(a_1\eta^l)(z) = 1$; otherwise, which shows that $a_1\eta^l = \eta^{l-1}$.

If $l = 0$, $e^-(1,\eta^0) = 0$ and $e^+(1,\eta^0) = N+1$. Again from Proposition 3.6, $a_1\eta^o = \eta^N$.

Combining these three items, we can conclude that $a_1\eta^l = \eta^{(l-1) \bmod (N+1)}$. □

By equation (3.2), there exists a neutral element e such that for every $\eta \in \mathcal{R}^o_\Lambda$, $e\eta = \eta$. From the above lemma, we get

Corollary 3.9. $a_1^{N+1} = e$.

3.3 Proofs of Theorems 3.1 and 3.2

In order to describe the limiting measure μ_k^η appearing in the statement of Theorem 3.1, we first introduce the following notation. For a measure ν on the space of height configurations $\mathcal{X}^o_\Lambda$, $\mathcal{S}^o\nu$ denotes the measure on the stable configurations Ω^o_Λ defined by

$$\mathcal{S}^o\nu(B) = \nu\{\eta \in \mathcal{X}^o_\Lambda : \mathcal{S}^o\eta \in B\},$$

for every measurable set $B \subset \Omega^o_\Lambda$.

The following proof is very similar to the proof of Theorem 2.5 in Chapter 2.

Proof of Theorem 3.1. For $k = 1$, it is the same as the BTW sandpile model.

So assume $k > 1$. We first consider the case in which $\eta(x) = 0$ for all sites x, and proceed by a coupling between the BTW and the BTW-k models. To introduce this coupling, let (X_t) denote a common sequence of (random) addition sites, and define

$$\theta^k_t = \sum_{s=1}^{t} k\delta_{X_s}, \qquad \theta^1_t = \sum_{s=1}^{t} \delta_{X_s}.$$

This couples the vector θ^k_t of all additions until time t in the BTW-k model, with the vector θ^1_t of all additions until time t in the BTW model.

In order to arrive at the BTW configuration at time t, now we can first apply all additions, and then topple unstable sites as long as there are any. In doing so, we couple the topplings in the BTW model with those in the BTW-k model: if we topple a site x in the BTW model, we topple the corresponding site in the BTW-k model k times.

In the BTW model, we can find a finite sequence of legal topplings $t_{x_1}, ..., t_{x_T}$ such that

$$t_{x_T} \cdots t_{x_1} \theta^1_t$$

is a stable configuration. Before toppling, we have $\theta^k_t = k\theta^1_t$. If we topple at x, then in the BTW model, x loses $2d$ particles, while all neighbors receive 1. In the BTW-k model, x loses $2dk$ grains while all its neighbors receive k grains. Therefore after each toppling, the configuration of the BTW-k model is still k times that that of the BTW model. Therefore $t^k_{x_T} \cdots t^k_{x_1}$ must be a sequence of legal topplings in the BTW-k model as well.

Hence the dynamics in the BTW-k model is exactly the same as in the BTW model, but multiplied by a factor of k. Furthermore, when the BTW model has reached the stable configuration $\kappa_t = \mathcal{S}^o\theta^1_t$, then the corresponding configuration in the BTW-k model is simply $k\kappa_t$. However, $k\kappa_t$ need not be stable in the BTW-k model. Hence, in order to reach the BTW-k configuration at time t, we have to stabilize $k\kappa_t$, leading to the stable configuration $\rho_t = \mathcal{S}^o\theta^k_t$.

From the results in [41], we know that the distribution of κ_t converges exponentially fast (as $t \to \infty$) in total variation to the uniform distribution on $\mathcal{R}^o_\Lambda$. Hence, ρ_t is exponentially close to $\mathcal{S}^o \nu_k$, where ν_k is the uniform measure on the set $\{k\xi : \xi \in \mathcal{R}^o_\Lambda\}$.

This settles the limiting measure in the case where we start with the empty configuration. If we start with configuration η in the BTW-k model, we may (by abelianess) first start with the empty configuration as above—coupled to the BTW model in the same way—and then add the 'extra' η to ρ_t at the end, and stabilize the obtained configuration. It follows that the height distribution in the BTW-k model converges to $\mathcal{S}^o \nu^\eta_k$, where ν^η_k is the uniform measure on the set $\{\eta + k\xi : \xi \in \mathcal{R}^o_\Lambda\}$. □

Now we turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. From Item 2 of Proposition 3.1 in [41], we know that for each $x \in \Lambda$, there exists a $n_x > 0$ such that for each $\eta \in \mathcal{R}^o_\Lambda$, $a_x^{n_x}\eta = \eta$. Take $C = \prod_{x\in\Lambda} n_x$.

We have for every $x \in \Lambda$ and for every $\eta \in \mathcal{R}^o_\Lambda$, $a_x^C \eta = \eta$. Therefore, when k' is a multiple of C, each $\eta \in \mathcal{R}^o_\Lambda$ is an absorbing state of the process. Now, we can conclude that point mass on each configuration in $\mathcal{R}^o_\Lambda$ is invariant under the process.

Now take k to be a multiple of C plus 1. Then for every $\eta \in \mathcal{R}^o_\Lambda$ and every $x \in \Lambda$,

$$a_x^k \eta = a_x^{mC+1}\eta = a_x \eta, \text{ for some } m \in \mathbb{N}$$

which means that each time, the new configuration is the same as adding only one grain. Therefore, by setting up a coupling such that at every time step we choose the same addition site in the BTW model and the BTW-k model, we can conclude that $\mathcal{R}^o_\Lambda$ is also the unique recurrent class for the BTW-k model, and the uniform measure on $\mathcal{R}^o_\Lambda$ is the unique invariant measure for the BTW-k model with $k = mC + 1$, for all non-negative integers m. □

3.4 Proofs of Theorems 3.3 and 3.4

First we will include a statement −− Bézout's Identity −− which can be found in many articles such as [30]. We will use it as a tool in the main proof.

Lemma 3.10. *If $r_1, \ldots, r_M$ are nonzero integers with greatest common divisor z, then there exist integers $b_1, \ldots, b_M$ (called Bézout numbers or Bézout coefficients) such that*

$$b_1 r_1 + \cdots + b_M r_M = z$$

Additionally, z is the smallest positive integer for which there are integer solutions $b_1, \ldots, b_M$ for the preceding equation.

One thing to remark is that some of the Bézouts coefficients might be negative.

The proofs of Theorem 3.3 and 3.4 are similar to each other. We will only give the proof of Theorem 3.4.

Proof of Theorem 3.4 Assume $\Lambda = \{1, \ldots, N\}$ without loss of generality.

Case 1: Every $k \in K$ is a multiple of $N+1$, so $q_k = 0$. In this case, from Corollary 3.9, we get for every $x = 1, \ldots, N$, for every $\eta \in \mathcal{R}^o_\Lambda$ and every $k \in K$,

$$a_x^k \eta = a_1^{kx} \eta = \eta.$$

Therefore every $\eta \in \mathcal{R}^o_\Lambda$ is an absorbing state of the model. We hence conclude that the point mass on each single configuration in $\mathcal{R}^o_\Lambda$ is an invariant measure for the model.

Case 2: For some $k \in K$, $q_k > 0$. We define $K' = \{k \in K : q_k > 0\}$.

Suppose $\gcd(N+1, q_k, k \in K') = L > 0$ and define $\frac{N+1}{L} = P$. For $m = 0, 1, \ldots, L-1$, define:

$$C_m = \{m, m+L, \ldots, m+(P-1)L\}$$

and

$$\mathcal{R}^{o,m}_\Lambda = \{\eta^l : l \in C_m\}.$$

We have for $m \neq m'$

$$\mathcal{R}^{o,m}_\Lambda \cap \mathcal{R}^{o,m'}_\Lambda = \emptyset \text{ and } \Cup_{0 \leq m \leq L-1} \mathcal{R}^{o,m}_\Lambda = \mathcal{R}^o_\Lambda.$$

The proof consists of three steps which are stated in the form of three lemmas.

Lemma 3.11. *For each $m = 0, 1, \ldots, L-1$, $\mathcal{R}^{o,m}_\Lambda$ is closed under all a_x^k with $x \in \Lambda$ and $k \in K$.*

Proof. We must show that for $\eta \in \mathcal{R}^{o,m}_\Lambda$,

$$a_x^k \eta \in \mathcal{R}^{o,m}_\Lambda \text{ for all } x = 1, \ldots, N \text{ and for all } k \in K.$$

Without loss of generality, we can restrict to the case $m = 0$. For all other m, we can use a similar argument.

We know that $\mathcal{R}_\Lambda^{o,0}$ is the set that contains all the configurations and only the configurations of the form η^l with $l \bmod L = 0$. Therefore for $\eta \in \mathcal{R}_\Lambda^{o,0}$, there is an $l \in \{0, 1, \dots, N\}$ such that $\eta = \eta^l$ with $l \bmod L = 0$. For $x \in \{1, \dots, N\}$, define l' by $a_x^k \eta =: \eta^{l'}$. If we can show that $l' \bmod L = 0$, then we can conclude that $\eta^{l'} \in \mathcal{R}_\Lambda^{o,0}$.

From Lemma 3.7, we have that $\eta^{l'} = a_1^{xk}\eta$. Lemma 3.8 tells us that $l' = (l - xk) \bmod (N+1)$. By computation,

$$\begin{aligned} l' &= (l - xk) \bmod (N+1) \\ &= \big(l - x \cdot k \bmod (N+1)\big) \bmod (N+1) \\ &= (l - xq) \bmod (N+1). \end{aligned} \tag{3.3}$$

Since L is a common divisor of $N+1$ and q,

$$l' \bmod L = \big((l - xq) \bmod (N+1)\big) \bmod L = (l - xq) \bmod L = l \bmod L = 0.$$

Therefore $\eta^{l'} \in \mathcal{R}_\Lambda^{o,0}$. □

The following lemma is the most involved part of this chapter. In the proof, we set up a special way of performing the additions. We make all the additions only to site 1 and we will use Bézout's Identity to find the number of times we need to add an amount k, for all $k \in K'$.

Lemma 3.12. *For each $m = 0, 1, \dots, L-1$, $\mathcal{R}_\Lambda^{o,m}$ is a communicating class.*

Proof. Without loss of generality, we still only discuss the case $m = 0$. We must show that for any $\eta, \eta' \in \mathcal{R}_\Lambda^{o,0}$, there is a t such that $\mathbb{P}_\eta(\eta_t = \eta') > 0$.

First from the assumption, we know $\gcd(N+1, k, k \in K') = L$ and K' is acountable. Then there exists a non-decreasing sequence of finite subsets $B_n \subset K'$ such that:

$$\lim_{w \to \infty} \gcd(N+1, k, k \in B_w) \downarrow L$$

and for every $w \in \mathbb{N}$, $\gcd(N+1, k, k \in B_w)$ is an integer of value at least L. Hence there must exist a $W < \infty$ such that $\gcd(N+1, k, k \in B_W) = L$.

Now we choose a special way of making additions. At each time step we only add to site 1 and the addition amounts are only chosen from B_W. We are going to find some non-negative integers $(b_k^0, k \in B_W)$ which are the numbers of times that the amount k is chosen, such that

$$a_1^{\sum_{k \in B_W} k b_k^0} \eta = \eta'.$$

From the definition of the BTW-k model, we know that at every time step, site 1 is chosen with positive probability and each amount $k \in B_W$ is also chosen with positive probability. Therefore the probability to realize η' by time $t = \sum_{k \in B_W} b_k^o$ is positive.

For $\eta, \eta' \in \mathcal{R}_\Lambda^{o,0}$, there exist integers i, j with $0 \leq i, j \leq P-1$ such that

$$\eta = \eta^{iL}, \eta' = \eta^{jL}.$$

From Lemma 3.8, we know that in case $i > j$, $a_1^{(i-j)L}\eta = \eta'$.

From Corollary 3.9, we have $e = a_1^{N+1}$. In order to show that there exist non-negative integers $(b_k^0, k \in B_W)$ such that $a_1^{\sum_{k \in B_W} k b_k^0}\eta = \eta'$, it is sufficient to show that there exist non-negative integers $(b_k^0, k \in B_W)$ and a non-negative integer f_0 satisfying the following equation:

$$(i-j)L + f_0(N+1) = \sum_{k \in B_W} b_k^0 k. \tag{3.4}$$

Now we divide both sides of equation (3.4) by L, it follows that equation (3.4) is equivalent to

$$(i-j) = \sum_{k \in B_W} \frac{q_k}{L} b_k^0 - \frac{N+1}{L} f_0.$$

From the assumption, we have $\gcd(\frac{N+1}{L}, \frac{q_k}{L}, k \in B_W) = 1$. From Bézout's Identity, we get that there exist integers $(b_k, k \in B_W)$ and an integer f such that

$$1 = \sum_{k \in B_W} \frac{q_k}{L} b_k + \frac{N+1}{L} f. \tag{3.5}$$

However, we are not sure yet that all the numbers we found are non-negative. We will now show that we do have non-negative integers in Equation (3.4).

For the obtained $(b_k, k \in B_W)$, there must exist integers $(c_k > 0, k \in B_W)$ such that for every $k \in B_W$, $b_k + \frac{N+1}{L} c_k > 0$ and $f - \sum_{k \in K'} \frac{q_k}{L} c_k < 0$. Equation (3.5) is the same as

$$1 = \sum_{k \in B_W} \frac{q_k}{L}\Big(b_k + \frac{N+1}{L} c_k\Big) + \frac{N+1}{L}\Big(f - \sum_{k \in B_W} \frac{q_k}{L} c_k\Big). \tag{3.6}$$

Take $b_k^0 = (i-j)\big(b_k + \frac{N+1}{L} c_k\big)$, which is positive, and take $f_0 = -(i-j)\big(f - \sum_{k \in B_W} \frac{q_k}{L} c_k\big)$, which is also positive.

A similar argument can be given for $i < j$. $\square$

Lemma 3.13. *The uniform measure $\mu_\Lambda^{o,m}$ on $\mathcal{R}_\Lambda^{o,m}$ is invariant.*

Proof. Since there are only P configurations in $\mathcal{R}_\Lambda^{o,m}$, if we can show that for every $x \in \{1, \dots, N\}$ and every $k \in \mathbb{N}$, a_x^k is a bijection from $\mathcal{R}_\Lambda^{o,m}$ to $\mathcal{R}_\Lambda^{o,m}$, then we can conclude the result directly. First, from Lemma 3.11, we know that $\mathcal{R}_\Lambda^{o,m}$ is closed under the operation a_x^k.

From Item 1 of Proposition 3.1 in [41], for every $x \in \Lambda$ and every $k \in \mathbb{N}$, a_x^k is a bijection from $\mathcal{R}_\Lambda^o$ to $\mathcal{R}_\Lambda^o$. Therefore there are the same number of configurations in the sets $a_x^k(\mathcal{R}_\Lambda^{o,m})$ and $\mathcal{R}_\Lambda^{o,m}$. Combing with the fact that $a_x^k(\mathcal{R}_\Lambda^{o,m}) \subset \mathcal{R}_\Lambda^{o,m}$, we get $a_x^k(\mathcal{R}_\Lambda^{o,m}) = \mathcal{R}_\Lambda^{o,m}$. Hence we can conclude that a_x^k is a bijection from $\mathcal{R}_\Lambda^{o,m}$ to $\mathcal{R}_\Lambda^{o,m}$. □

Lemma 3.11 and Lemma 3.12 together imply that there is a unique invariant measure on $\mathcal{R}_\Lambda^{o,1}$. Combining with Lemma 3.13, we know the uniform measure $\mathcal{R}_\Lambda^{o,m}$ is the unique invariant measure on $\mathcal{R}_\Lambda^{o,m}$. This completes the proof of Theorem 3.4.

4 Uniqueness of the stationary distribution and stabilizability in Zhang's sandpile model

Summary

This chapter studies Zhang's sandpile model. We show that Zhang's sandpile model $(N, [a, b])$ has a unique stationary distribution for all $0 \leq a < b \leq 1$. We define the infinite volume Zhang's sandpile model in dimension $d \geq 1$ and study the stabilizability of initial configurations chosen according to some measure μ.

The content of this chapter is based on the article by A. Fey, H. Liu and R. Meester [19] with some corrections.

4.1 Introduction

Zhang's sandpile model [50] is a variant of the more common abelian sandpile model [10], which was introduced in [6] as a toy model to study self-organized criticality. We define the model more precisely in the next section, but we start here with an informal discussion.

Zhang's model differs from the abelian sandpile model on a finite grid Λ in the following respects: The configuration space is $[0, 1)^\Lambda$, rather than $\{0, 1, \ldots, 2d-1\}^\Lambda$. The model evolves, like the abelian sandpile model, in discrete time through additions and subsequent stabilization through topplings of unstable sites. However, in Zhang's model, an addition consists of a continuous amount, uniformly distributed on $[a, b] \subseteq [0, 1]$, rather than one 'grain'. Furthermore, in a Zhang toppling of an unstable site, the entire height of this site is distributed equally among the neighbors, whereas in the abelian sandpile model, one grain moves to each neighbor irrespective of the height of the toppling site.

Since the result of a toppling depends on the height of the toppling site, Zhang's model is not abelian. This means that 'stabilization through topplings' is not immediately well-defined. However, as pointed out in [20], when we work on the line, and as long as there are no two neighboring unstable sites, topplings are abelian. When the initial configuration is stable, we will only encounter realizations with no two neighboring unstable sites, and we have a fortiori that the model is abelian.

In [20], when $\Lambda = \{1, 2, \ldots, N-1N\} \subset \mathbb{Z}$, the following main results were obtained. Uniqueness of the stationary measure was proved for a number of special cases: (1) $a \geq \frac{1}{2}$; (2) $N = 1$, and (3) $[a, b] = [0, 1]$. For the model on one site with $a = 0$, an explicit expression for the stationary height distribution was obtained. Furthermore, the existence of so called 'quasi-units' was proved for $a \geq 1/2$, that is, in the limit of the number of sites to infinity, the one-dimensional marginal of the stationary distribution concentrates on a single value $\frac{a+b}{2}$.

In the first part of this chapter, we prove, in dimension 1, uniqueness of the stationary measure for the general model, via a coupling which is much more complicated than the one used in [20] for the special case $a \geq 1/2$.

In Section 4.4, we study an infinite-volume version of Zhang's model in *any* dimension. A similar infinite-volume version of the abelian sandpile model has been studied in [21, 22, 40] and we will in fact also use some of the ideas from these papers.

For the infinite-volume Zhang model in dimension d, we start from a random initial configuration in $[0,\infty)^{\mathbb{Z}^d}$, and evolve it in time by Zhang topplings of unstable sites. We are interested in whether or not there exists a limiting stable configuration. Since Zhang topplings are not abelian, for a given configuration $\eta \in [0,\infty)^{\mathbb{Z}^d}$, for some sequence of topplings it may converge to a stable configuration but for others, it may not. Moreover we do not expect the final configuration – if there is one – to be unique. Therefore, we choose a random order of topplings, as follows. To every site we attach an independent rate 1 Poisson clock, and when the clock rings, we topple this site if it is unstable; if it is stable we do nothing. For obvious reasons we call this the *Markov toppling process*.

We show that if we choose the initial configuration according to a stationary ergodic measure μ with density ρ, then for all $\rho < \frac{1}{2}$, μ is stabilizable, that is, the configuration converges to a final stable configuration. For all $\rho \geq 1$, μ is not stabilizable. For $\frac{1}{2} \leq \rho < 1$, when ρ is near to $\frac{1}{2}$ or 1, both possibilities can occur.

4.2 Model definition and notation

In this section, we discuss Zhang's sandpile on $\Lambda = \{1, 2, \ldots, N\} \subset \mathbb{Z}$. We denote by $\mathcal{X}_N = [0,\infty)^N$ the set of all possible configurations in Zhang's sandpile model. We will use the symbols η and ξ to denote a configuration. We denote the value of a configuration η at site x by η_x, and refer to this value as the *height*, *mass* or *energy* at site x. We introduce a labeling of sites according to their height, as follows.

Definition 4.1. *For $\eta \in \mathcal{X}_N$, we call site x*

$$\begin{array}{lll} \textit{empty} & \textit{if} & \eta_x = 0, \\ \textit{anomalous} & \textit{if} & \eta_x \in (0, \frac{1}{2}), \\ \textit{full} & \textit{if} & \eta_x \in [\frac{1}{2}, 1), \\ \textit{unstable} & \textit{if} & \eta_x \geq 1. \end{array}$$

A site x is stable for η if $0 \leq \eta_x < 1$, and hence all the empty, anomalous and full sites are stable. A configuration η is called stable if all sites are stable, otherwise η is unstable. $\Omega_N = [0,1)^N$ denotes the set of all stable configurations.

By $T_x(\eta)$ we denote the (Zhang) toppling operator for site x, acting on η, which is defined as follows.

Definition 4.2. *For all $\eta \in \mathcal{X}_N$ such that $\eta_x \geq 1$, we define*

$$T_x(\eta)_y = \begin{cases} 0 & \textit{if } x = y, \\ \eta_y + \frac{1}{2}\eta_x & \textit{if } |y - x| = 1, \\ \eta_y & \textit{otherwise.} \end{cases}$$

For all η such that $\eta_x < 1, T_x(\eta) = \eta$, for all x.

In other words, the toppling operator only changes η if site x is unstable; in that case, it divides its energy in equal portions among its neighbors. We say in that case that site x *topples*. If a boundary site topples, then half of its energy disappears from the system. Every configuration in $\mathcal{X}_N$ can stabilize, that is, reach a final configuration in Ω_N, through finitely many topplings of unstable sites, since energy is dissipated at the boundary.

We define the $(N, [a,b])$ model as a discrete time Markov process with state space Ω_N, as follows. The process starts at time 0 from the configuration $\eta(0) = \eta$ with $\eta \in \Omega_N$. For every $t = 1, 2, \ldots$, the configuration $\eta(t)$ is obtained from $\eta(t-1)$ as follows: a random amount of energy $U(t)$, uniformly distributed on $[a,b]$, is added to a uniformly chosen random site $X(t) \in \{1, \ldots, N\}$, that is $\mathbb{P}(X(t) = j) = 1/N$ for all $j = 1, \ldots, N$. The random variables $U(t)$ and $X(t)$ are independent of each other and of the past of the process. We stabilize the resulting configuration through topplings (if it is already stable, then we do not change it), to obtain the new configuration $\eta(t)$. By $\mathbb{E}^\eta$ and $\mathbb{P}^\eta$, we denote expectation, probability with respect to this process, respectively.

4.3 Uniqueness of the stationary distribution

In Zhang's sandpile model, it is not obvious that the stationary distribution is unique, since the state space is uncountable. For the three cases: (1) $N = 1$, (2) $a \geq \frac{1}{2}$ and $N \geq 2$, (3) $a = 0, b = 1$ and $N \geq 2$, it is shown in [20] that the model has a unique stationary distribution, and in addition, in cases (2) and (3), for every initial distribution ν, the measure at time t, denoted by $\nu_t^{a,b,N}$, converges in total variation to the stationary distribution. In the case $N = 1$, there are values of a and b where we only have time-average total variation convergence, see Theorem 4.1 of [20].

In all these cases (except when $N = 1$) the proof consisted of constructing a coupling of two copies of Zhang's model with arbitrary initial configurations, in such a way that after some (random) time, the two coupled processes are identical. Each coupling was very specific for the case considered. In the proof for the case $a \geq \frac{1}{2}$ and $N \geq 2$, explicit use was made of the fact that an addition to a full site always causes a toppling. The proof given for the case $a = 0$ and $b = 1$ and $N > 1$ can be generalized to other values of b, but $a = 0$ is necessary, since in the coupling we used that additions can be arbitrarily small. In the special case $N = 1$, the model is a renewal process, and the proof relied on that.

To prove the following result, we will again construct a coupling of two copies of Zhang's model with arbitrary initial configurations, in such a way that after some (random) time, the two coupled processes are identical. Such a coupling will be called 'successful', as in [45]. Here is the main result of this section; note that only the case $a = b$ is not included.

Theorem 4.3. *For every $0 \leq a < b \leq 1$, and $N \geq 2$, Zhang's sandpile model $(N, [a, b])$ has a unique stationary distribution which we denote by $\mu^{a,b,N}$. Moreover, for every initial distribution on $[0, 1)^N$, the distribution of the process at time t converges exponentially fast in total variation to $\mu^{a,b,N}$, as $t \to \infty$.*

We introduce some notation. Denote by $\eta, \xi \in \Omega_N$ the initial configurations, and by $\eta(t), \xi(t)$ two independent copies of the processes, starting from η and ξ respectively. The independent additions at time t for the two processes starting from η, ξ are $U^\eta(t)$ and $U^\xi(t)$, addition sites are $X^\eta(t), X^\xi(t)$ respectively. Often, we will use 'hat'-versions of the various quantities to denote a coupling between the two processes. So, for instance, $\hat{\eta}(t), \hat{\xi}(t)$ denote coupled processes (to be made precise below) with initial configurations η and ξ respectively. By $\hat{X}^\eta(t)$ and $\hat{X}^\xi(t)$ we denote the addition sites at time t in the coupling, and by $\hat{U}^\eta(t)$ and $\hat{U}^\xi(t)$ the addition amounts at time t.

In this section, we will encounter configurations that are empty at some site x, $1 \leq x \leq N$, and full at all the other sites. We denote the set of these configurations by $\mathcal{E}_x$. By $\mathcal{E}_b$, we denote the set of configurations that have only one empty boundary site, and are full at all other sites, that is, $\mathcal{E}_b = \mathcal{E}_1 \cup \mathcal{E}_N$.

The coupling that we will construct is rather technical, but the ideas behind the main steps are not so difficult. In the first step, we make sure that the two copies of the process simultaneously reach a situation in which the N-th site is empty, and all other sites are full. In step 2, we make sure that the heights of the two copies at each vertex are within some small ϵ of each other. This can be achieved by carefully selecting the additions. Finally, in step 3, we show that once the heights at all sites are close to each other, then we can make the two copies of the process equal to each other by very carefully coupling the amounts of mass that we add each time.

In order to give the proof of Theorem 4.3, we need the following three preliminary results, the proofs of which will be given in Sections 4.3.1, 4.3.2 and 4.3.3, respectively.

In the following part of this section, for all $\epsilon > 0$, we take

$$t_\epsilon = 2\left\lceil \frac{2}{a+b} \right\rceil \cdot \left\lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}\left(\frac{2\epsilon}{N}\right) \right\rceil.$$

Lemma 4.4. *Let η and ξ be two configurations in $\mathcal{E}_N$. Consider couplings $(\hat{\eta}(t), \hat{\xi}(t))$ of the process starting at η and ξ respectively. Let, in such a coupling, T be the first time t with the property that*

$$\max_{1 \leq x \leq N} \mid \hat{\eta}_x(t) - \hat{\xi}_x(t) \mid < \epsilon \tag{4.1}$$

and

$$\hat{\eta}(t) \in \mathcal{E}_N, \hat{\xi}(t) \in \mathcal{E}_N. \tag{4.2}$$

There exists a coupling such that the event $\{T \leq t_\epsilon\}$ has probability at least $(2N)^{-t_\epsilon}$, uniformly in η and ξ.

In the following, take $\epsilon_{a,b} = \frac{b-a}{8}$

Lemma 4.5. *Starting from η, ξ in Ω_N, within $2(N+1)\left\lceil \frac{1}{a+b} \right\rceil + t_{\epsilon_{a,b}} + 2\left\lceil \frac{2}{a+b} \right\rceil + 1$ time steps, the two processes $\eta(t)$ and $\xi(t)$ are simultaneously in $\mathcal{E}_N$ with a probability bounded below by a positive constant that only depends on a, b, N.*

Lemma 4.6. *Let*

$$\epsilon_{a,b,N} = \frac{b-a}{6 + 16\prod_{l=1}^{N-1}(1 + 2^{N-2-l})}.$$

Consider couplings $(\hat{\eta}(t), \hat{\xi}(t))$ of the process starting at η and ξ respectively, with the property that

$$\max_{1 \le x \le N} \mid \eta_x - \xi_x \mid < \epsilon_{a,b,N}. \tag{4.3}$$

Let T' be the first time t with the property that $\hat{\eta}(t) = \hat{\xi}(t)$. There exists a coupling such that the event $\{T' < (N-1)\lceil \frac{1}{a+b} \rceil\}$ has probability bounded below by a positive constant that depends only on a, b and N.

We now present the coupling that constitutes the proof of Theorem 4.3, making use of the results stated in Lemma 4.4, Lemma 4.5 and Lemma 4.6.

Proof of Theorem 4.3. Take two probability distributions ν_1, ν_2 on Ω_N, and choose η and ξ according to ν_1, ν_2 respectively. We present a successful coupling $\{\hat{\eta}(t), \hat{\xi}(t)\}$, with $\hat{\eta}(0) = \eta$ and $\hat{\xi}(0) = \xi$. If we assume that both ν_1 and ν_2 are stationary, then the existence of the coupling shows that $\nu_1 = \nu_2 = \nu$. If we take $\nu_1 = \nu$ and ν_2 arbitrary, then the existence of the coupling shows that any initial distribution ν_2 converges in total variation to ν.

The coupling consists of three steps, and is described as follows.

- *step 1.* We evolve the two processes independently until they encounter a configuration in $\mathcal{E}_N$ simultaneously. From that moment on, we proceed to
- *step 2.* We use the coupling as described in the proof of Lemma 4.4. That amounts to choosing $\hat{X}^{\xi}(t) = \hat{X}^{\eta}(t) = X^{\eta}(t)$, and $\hat{U}^{\xi}(t) = \hat{U}^{\eta}(t) = U^{\eta}(t)$. As the proof of Lemma 4.4 shows, if $U^{\eta}(t)$ and $X^{\eta}(t)$ satisfy certain requirements for at most t_{ϵ} time steps, then we have that (4.1) and (4.2) occur, with $\epsilon = \epsilon_{a,b,N}$. If during this step, at any time step either $U^{\eta}(t)$ or $X^{\eta}(t)$ does not satisfy the requirements, then we return to step 1. But once we have (4.1) and (4.2) (which, by Lemma 4.4, has positive probability), then we proceed to
- *step 3.* Here, we use the coupling as described in the proof of Lemma 4.6. Again, we choose $\hat{X}^{\xi}(t) = \hat{X}^{\eta}(t) = X^{\eta}(t)$ and $\hat{U}^{\eta}(t) = U^{\eta}(t)$, but the dependence of $\hat{U}^{\xi}(t)$ on $U^{\eta}(t)$ is more complicated; the details can be found in the proof of Lemma 4.6. As the proof of Lemma 4.6 shows, if $U^{\eta}(t)$ and $X^{\eta}(t)$ satisfy certain requirements for at most $(N-1)\lceil \frac{1}{a+b} \rceil$ time steps, then we have that $\hat{\eta}(t) = \hat{\xi}(t)$ occurs, and from that moment on the two processes evolve identically. By Lemma 4.6, this event has positive probability. If during this step, at any time step, either $U^{\eta}(t)$ or $X^{\eta}(t)$ does not satisfy the requirements, we return to step 1.

In the coupling, we keep returning to step 1 until step 2 and subsequently step 3 are successfully completed, after which we have that $\hat{\eta}(t) = \hat{\xi}(t)$. Since each step is successfully completed with uniform positive probability, we a.s. need only finitely many attempts. Therefore we achieve $\hat{\eta}(t) = \hat{\xi}(t)$ in finite time, so that the coupling is successful. □

Now, we will proceed to give the proof of Lemma 4.4, Lemma 4.5 and Lemma 4.6.

4.3.1 Proof of Lemma 4.4

In this part, we couple two processes starting from $\eta, \xi \in \mathcal{E}_N$. The coupling consists of choosing the addition amounts and sites equal at each time step. For this coupling, we present an event that has probability $(2N)^{-t_\epsilon}$, with $t_\epsilon = 2\left\lceil \frac{2}{a+b} \right\rceil \cdot \left\lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}(\frac{2\epsilon}{N}) \right\rceil$, and which is such that if it occurs, then (4.1) and (4.2) are satisfied.

The event we need is that for t_ϵ time steps,

1. all additions are heavy;
2. the additions occur to site N until site N becomes unstable, then to site 1 until site 1 becomes unstable, then to site N again, etcetera.

The probability for an addition to be heavy is $\frac{1}{2}$ and the probability for the addition to occur to a fixed site is $\frac{1}{N}$. Therefore, the probability of this event is $(2N)^{-t_\epsilon}$.

Now we show that if this event occurs, then (4.1) and (4.2) are satisfied. Let $\hat{U}(t)$ be the addition amount at time t. Define a series of stopping times $\{\tau_k\}_{k\geq 0}$ by

$$\tau_0 = 0, \tau_k := \min\left\{t > \tau_{k-1} : \sum_{t=\tau_{k-1}+1}^{\tau_k} \hat{U}(t) \geq 1\right\} \text{ for } k \geq 1, \tag{4.4}$$

and write

$$S_k = \sum_{t=\tau_{k-1}+1}^{\tau_k} \hat{U}(t). \tag{4.5}$$

The times τ_k ($k > 0$) are such that in both configurations, *only* at these times an avalanche occurs. Indeed, for the first avalanche this is clear because we only added to site N, which was empty before we started adding. But whenever an avalanche starts at a boundary site, and all other sites are full, then every site topples exactly once and after the avalanche, the opposite boundary site is empty. Thus the argument applies to all avalanches.

Since we make only heavy additions,

$$\tau_k - \tau_{k-1} \leq \left\lceil \frac{2}{a+b} \right\rceil, \text{ for all } k. \tag{4.6}$$

After the k-th avalanche, the height $\hat{\eta}_y(\tau_k)$ is a linear combination of $S_1, \ldots, S_k$ and $\eta_1, \ldots, \eta_{N-1}$, which we write as

$$\hat{\eta}_y(\tau_k) = \sum_{l=1}^{k} A_{ly}(k) S_l + \sum_{m=1}^{N-1} B_{my}(k) \eta_m, \text{ for } 1 \leq y \leq N, \tag{4.7}$$

and a similar expression for $\hat{\xi}_y(\tau_k)$. From Proposition 3.7 of [20], we have that

$$B_{my}(k) \leq (1 - 2^{-\lceil \frac{3N}{2} \rceil}) \max_x B_{mx}(k-1).$$

By induction, we find

$$B_{my}(k) \leq (1 - 2^{-\lceil \frac{3N}{2} \rceil})^k$$

and hence

$$\begin{aligned} \max_{1 \leq y \leq N} \mid \hat{\eta}_y(\tau_k) - \hat{\xi}_y(\tau_k) \mid &\leq N(1 - 2^{-\lceil \frac{3N}{2} \rceil})^k \max_{1 \leq x \leq N} \mid \eta_x - \xi_x \mid \\ &\leq \frac{N}{2}(1 - 2^{-\lceil \frac{3N}{2} \rceil})^k, \end{aligned}$$

where we use the fact that $\eta, \xi \in \mathcal{E}_N$ implies $\max_{1 \leq x \leq N} \mid \eta_x - \xi_x \mid \leq \frac{1}{2}$.

For each $\epsilon > 0$, choose $k_\epsilon = \left\lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})} \left(\frac{2\epsilon}{N} \right) \right\rceil$. Then $\frac{N}{2}(1-2^{-\lceil \frac{3N}{2} \rceil})^{k_\epsilon} \leq \epsilon$, so that

$$\max_{1 \leq x \leq N} \mid \hat{\eta}_x(\tau_{k_\epsilon}) - \hat{\xi}_x(\tau_{k_\epsilon}) \mid < \epsilon,$$

and moreover, an even number of avalanches occurred, which means that at time τ_{k_ϵ}, both processes are in $\mathcal{E}_N$. By (4.6), $\tau_{k_\epsilon} \leq t_\epsilon = k_\epsilon \left\lceil \frac{2}{a+b} \right\rceil$. Thus, τ_{k_ϵ} is a random time T as in the statement of Lemma 4.4. □

4.3.2 Proof of Lemma 4.5

The proof of Lemma 4.5 consists of two stages, the results gained in which are stated as Lemma 4.7 and Lemma 4.8, respectively.

Lemma 4.7. *For all $\eta \in \Omega_N, \xi \in \Omega_N$, $\eta(t)$ and $\xi(t)$ are simultaneously in $\mathcal{E}_b$ within $2(N+1)\left\lceil \frac{1}{a+b} \right\rceil$ time steps with probability at least $\left(4N^3\right)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$.*

Review that $\epsilon_{a,b} = \frac{b-a}{8}$ and for all $\epsilon > 0$,

$$t_\epsilon = 2\left\lceil \frac{2}{a+b} \right\rceil \cdot \left\lceil \log_{(1-2^{-\lceil \frac{3N}{2} \rceil})}\left(\frac{2\epsilon}{N}\right) \right\rceil$$

Lemma 4.8. *For $\eta \in \mathcal{E}_b$, $\xi \in \mathcal{E}_b$. The event that within $t_{\epsilon_{a,b}} + 2\left\lceil \frac{2}{a+b} \right\rceil + 1$ steps, two processes are both in $\mathcal{E}_N$ has probability at least $\left(\frac{1}{8N}\right)\left(4N\right)^{-2\lceil \frac{2}{a+b} \rceil}(2N)^{-t_{\epsilon_{a,b}}}$*

To give the proof of Lemma 4.7, we first need the following two lemmas.

Lemma 4.9. *Let η be a configuration in Ω_N. The process starting from η visits $\mathcal{E}_N$ within $(N+1)\left\lceil \frac{1}{a+b} \right\rceil$ time steps with probability at least $(\frac{1}{2N})^{(N+1)\lceil \frac{1}{a+b} \rceil}$.*

Proof. We prove this by giving an explicit event realizing this, which has the mentioned probability. In this step, we always make *heavy* additions, that is, additions with value at least $(a+b)/2$.

First, starting from configuration η, we make heavy additions to site 1 until site 1 becomes unstable. Then an avalanche occurs and a new configuration with at least one empty site is reached. Denote the leftmost empty site by r_1. If $r_1 = N$ we are done. If $r_1 \neq N$, then it is easy to check that site $r_1 + 1$ is full. The total number of additions needed for this step is at most $2\left\lceil \frac{1}{a+b} \right\rceil$.

Then, if $r_1 \neq N$, we continue by making heavy additions to site $r_1 + 1$ until site $r_1 + 1$ becomes unstable. Then an avalanche starts from site $r_1 + 1$. During this avalanche, sites 1 to $r_1 - 1$ are not affected, site r_1 becomes full and we again reach a new configuration with at least one empty site, the leftmost of which is denoted by r_2. If $r_2 = N$ we are done. If not, note that $r_2 \geq r_1 + 1$ and that all sites $1, \ldots, r_2 - 1$ and $r_2 + 1$ are full. At most $\left\lceil \frac{1}{a+b} \right\rceil$ heavy additions are needed for this step.

If $r_2 \neq N$, we repeat this last procedure. After each avalanche, the leftmost empty site moves at least one site to the right, and hence, after the first step we need at most $N - 1$ further steps.

Hence, the total number of heavy additions needed for the above steps is bounded above by $(N+1)\left\lceil \frac{1}{a+b} \right\rceil$. Every time step, with probability $\frac{1}{N}$, a fixed site is chosen and with probability $\frac{1}{2}$, an addition is a heavy addition. Therefore, the probability of this event is at least $(2N)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$. □

Lemma 4.10. *Let $\xi(0) \in \mathcal{E}_b$, then $\xi(1) \in \mathcal{E}_b$ with probability at least $\frac{1}{N}$.*

Proof. Again, we give an explicit possibility with probability $\frac{1}{N}$. Starting in $\xi \in \mathcal{E}_b$, we make one addition to the site next to the empty boundary site. If this site does not topple, then of course $\xi(1)$ is still in $\mathcal{E}_b$. But if it does topple, then every full site will topple once, after which all sites will be full except for the opposite (previously full) boundary site. In other words, then $\xi(1)$ is also in $\mathcal{E}_b$. The probability that the addition site is the site next to the empty boundary site, is $\frac{1}{N}$. Thus, $\xi(1) \in \mathcal{E}_b$ with probability at least $\frac{1}{N}$. □

We can now give the proof of Lemma 4.7.

Proof of Lemma 4.7. Let t_b^ξ be the first time that the process is in $\mathcal{E}_b$. Take

$$T_b^\eta = \min\{t : t \geq t_b^\xi, \eta(t) \in \mathcal{E}_b\}.$$

By the same lemma, the probability that $0 \leq T_b^\eta - t_b^\xi \leq (N+1)\left\lceil \frac{1}{a+b} \right\rceil$ is at least $(2N)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$. Repeatedly applying Lemma 4.10 gives that the event that $\xi(t_b^\xi + 1) \in \mathcal{E}_b, \xi(t_b^\xi + 2) \in \mathcal{E}_b, \dots, \xi\left(t_b^\xi + (N+1)\left\lceil \frac{1}{a+b} \right\rceil\right) \in \mathcal{E}_b$, occurs with probability bounded below by $(\frac{1}{N})^{(N+1)\lceil \frac{1}{a+b} \rceil}$.

It follows that when $\xi(t) \in \mathcal{E}_b$, within at most $(N+1)\lceil \frac{1}{a+b} \rceil$ time steps, the two processes are in $\mathcal{E}_b$ simultaneously with probability at least $(\frac{1}{2N^2})^{(N+1)\lceil \frac{1}{a+b} \rceil}$. From Lemma 4.9, we know the event $\{t_b^\xi < (N+1)\left\lceil \frac{1}{a+b} \right\rceil\}$ has probability at least $\left(2N\right)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$. We can now conclude that starting from all η, ξ in Ω_N, the two processes are simultaneously in $\mathcal{E}_b$ within $2(N+1)\left\lceil \frac{1}{a+b} \right\rceil$ time steps with probability at least $(4N^3)^{-(N+1)\lceil \frac{1}{a+b} \rceil}$. □

Now we turn to the proof of Lemma 4.8. First, we get

Lemma 4.11. *Let η and ξ be two stable configurations with the properties that*

$$\max_{1 \leq x \leq N} \mid \eta_x - \xi_{(1+N-x)} \mid < \epsilon_{a,b} \tag{4.8}$$

and

$$\eta \in \mathcal{E}_N, \xi \in \mathcal{E}_1. \tag{4.9}$$

Within $2\left\lceil \frac{2}{a+b} \right\rceil + 1$ steps, the two processes are simultaneously in $\mathcal{E}_N$ with probability at least $\left(\frac{1}{8N}\right) \cdot (4N)^{-2\lceil \frac{2}{a+b} \rceil}$ uniformly in all stable configurations η, ζ with properties (4.8) and (4.9).

Proof. We will show this by setting up a coupling. At the beginning, we choose $\hat{X}^\eta(t) = X^\eta(t)$ and $\hat{U}^\eta(t) = U^\eta(t)$. We choose $\hat{X}^\xi(t) = 1 + N - X^\eta(t)$ and $\hat{U}^\xi(t) = \hat{U}^\eta(t)$.

In the coupling, we first need the event that in the process $\hat{\eta}(t)$:

1. all addition amounts are chosen from $[\frac{a+b}{2}, \frac{a+b}{2} + 2\epsilon_{a,b}]$;
2. additions are made to site 1 firstly till the moment that one more addition would make site 1 topple, then are made to site N till the moment one more addition would make site N unstable.

By simple calculation, to make this event occur, we need at most $2\left\lceil \frac{2}{a+b} \right\rceil$ times of such additions and the resulting configuration has the following properties.

$$\begin{aligned} \hat{\eta}_1(t) &\geq 1 - \frac{a+b}{2} - 2\epsilon_{a,b} \\ \hat{\eta}_N(t) &\geq 1 - \frac{a+b}{2} - 2\epsilon_{a,b} \\ \hat{\eta}_x(t) &= \eta_x, \text{ otherwise.} \end{aligned} \tag{4.10}$$

Then if we take T_1 be the first time with the properties (4.10), then the probability of the event $\{T_1 < 2\left\lceil \frac{2}{a+b} \right\rceil\}$ is at least $(4N)^{-2\lceil \frac{2}{a+b} \rceil}$. Meanwhile we get that in the coupling, the process $\hat{\xi}(t)$ has the property that

$$\begin{aligned} \hat{\xi}_1(T_1) &\geq 1 - \frac{a+b}{2} - 3\epsilon_{a,b} \\ \hat{\xi}_N(T_1) &\geq 1 - \frac{a+b}{2} - 3\epsilon_{a,b} \\ \hat{\xi}_x(T_1) &= \xi_x, \text{ otherwise.} \end{aligned} \tag{4.11}$$

Once the coupled processes reach the states with properties (4.10) and (4.11) simultaneously, we start to chose $\hat{X}^\eta(t) = \hat{X}^\xi(t) = X^\eta(t)$ and $\hat{U}^\eta(t) = \hat{U}^\xi(t) = U^\eta(t)$. When an amount from interval $[\frac{a+b}{2} + 3\epsilon_{a,b}, b]$ is added to site 1, site 1 will become unstable and an avalanche starts from site 1 in both processes. When this avalanche stops, both of the processes reach states in $\mathcal{E}_N$. The probability of this step is $\frac{1}{8N}$.

We can now conclude that starting from configurations η and ζ with properties (4.8) (4.9), the event that within $2\left\lceil \frac{2}{a+b} \right\rceil + 1$ steps, both processes are simultaneously in $\mathcal{E}_N$ happens with probability at least $\left(\frac{1}{8N}\right) \cdot (4N)^{-2\lceil \frac{2}{a+b} \rceil}$. □

Proof of Lemma 4.8.

If both η and ξ are in $\mathcal{E}_N$, the result is obvious.

If η and ξ are both in $\mathcal{E}_1$, we chose $\hat{X}^\eta(t) = \hat{X}^\xi(t) = X^\eta(t)$ and $\hat{U}^\eta(t) = \hat{U}^\xi(t) = U^\eta(t)$. We need the event that every time we make *heavy* additions to site 1. Then within at most $\left\lceil \frac{2}{a+b} \right\rceil$ time steps, an avalanche occurs in both process simultaneously. The processes will transfer from states in $\mathcal{E}_1$ to states in $\mathcal{E}_N$. Then the probability that the two coupled processes are simultaneously in $\mathcal{E}_N$ within $\left\lceil \frac{2}{a+b} \right\rceil$ time steps is at least $(2N)^{-\lceil \frac{2}{a+b} \rceil}$ uniformly for all $\eta \in \mathcal{E}_1$ and $\xi \in \mathcal{E}_1$.

Otherwise, without loss of generality, suppose $\eta \in \mathcal{E}_N, \xi \in \mathcal{E}_1$. Let $T^{\epsilon_{a,b}}$ be the first time t that the two processes reach the states with the properties (4.8) and (4.9). Using the similar argument as in the proof of Lemma 4.4, we can get the event $\{T^{\epsilon_{a,b}} < t_{\epsilon_{a,b}}\}$ happens with probability at least $(2N)^{-t_{\epsilon_{a,b}}}$ uniformly for all $\eta \in \mathcal{E}_N$ and $\xi \in \mathcal{E}_1$.

When the two processes reach the states with properties (4.8) and (4.9), we start to use the couplings in the proof of Lemma 4.11, then we reach the conclusion. □

Combing the results in Lemma 4.7 and Lemma 4.8, we can conclude the results in Lemma 4.5.

4.3.3 Proof of Lemma 4.6

As in the proof of Lemma 4.4, we will describe the coupling, along with an event that has probability bounded below by a constant that only depends on a, b and N, and is such that if it occurs, then within $(N-1)\left\lceil \frac{1}{a+b} \right\rceil$ time steps, $\hat{\eta}(t) = \hat{\xi}(t)$. First we explain the idea behind the coupling and this event, after that we will work out the mathematical details.

The idea is that in both processes we add the same amount, only to site 1, until an avalanche is about to occur. We then add slightly different amounts while still ensuring that an avalanche occurs in both processes. After the first avalanche, all sites contain a linear combination of the energy before the last addition, plus a nonzero amount of the last addition. We choose the difference D_1 between the additions that cause the first avalanche such that site N will have the same energy in both processes after the first avalanche, where $|D_1|$ is bounded above by a value that only depends on a, b and N. Site $N-1$ will become empty in both configurations, and the differences between the two new configurations on all other sites will be larger than those before this avalanche, but can be controlled.

When we keep adding to site 1, in the next avalanche only the sites $1, \ldots, N-2$ will topple. We choose the addition amounts such that after the second avalanche, site $N-1$ will have the same energy in both configurations. Since site N did not change in this avalanche, we now have equality on two sites. After the second avalanche, site $N-2$ is empty, and the configurations are still more different on all other sites, but the difference can again be controlled.

We keep adding to site 1 until, after a total of $N-1$ avalanches, the configurations are equal on all sites. After each avalanche, we have equality on one more site, and the difference increases on the non-equal sites. We deal with this increasing difference by controlling the maximal difference between the corresponding sites of the two starting configurations by the constant $\epsilon_{a,b,N}$, so that we can choose each addition of both sequences from a nonempty interval in $[a, b]$. The whole event takes place in finite time, and will therefore have positive probability.

Proof of Lemma 4.6. The coupling is as follows. We choose $\hat{X}^\eta(t) = X^\eta(t)$, and $\hat{U}^\eta(t) = U^\eta(t)$. We choose the addition sites $\hat{X}^\eta(t)$ and $\hat{X}^\xi(t)$ equal, and the addition amounts $\hat{U}^\eta(t)$ and $\hat{U}^\xi(t)$ either equal, or not equal but dependent. In the last case, $\hat{U}^\xi(t)$ is always of the form $a + (\hat{U}^\eta(t) + D - a) \bmod (b-a)$, where D does not depend on $\hat{U}^\eta(t)$. The reader can check that, for any D, $\hat{U}^\xi(t)$ is then uniformly distributed on $[a, b]$.

The event we need is as follows. First, all additions are heavy. Second, for the duration of $N-1$ avalanches, which is at most $(N-1)\lceil \frac{2}{a+b} \rceil$ time steps, the additions to η occur to site 1, and the amount is for every time step in a certain subinterval of $[a, b]$, to be specified next.

We denote $\frac{a+b}{2} = a''$ and recursively define

$$\epsilon_{k+1} = (1 + 2^{N-k-2})\epsilon_k,$$

with $\epsilon_1 = \epsilon_{a,b,N}$ with $\epsilon_{a,b,N} = \frac{b-a}{6+16\prod_{l=1}^{N-1}(1+2^{N-2-l})}$. Between the $(k-1)$-st and k-th avalanche, the interval is $[a'', a'' + 2\epsilon_k]$, and at the time where the k-th avalanche occurs, the interval is $[a'_k, b] = [a'' + 3\epsilon_k, b]$ of length $\frac{b-a'_k}{2} = \frac{b-a}{4} - \frac{3}{2}\epsilon_k$; see below.

The probability of at most $(N-1)\lceil \frac{2}{a+b} \rceil$ additions occurring to site 1, is bounded from below by $N^{-\lceil \frac{2}{a+b} \rceil (N-1)}$. Since ϵ_k is increasing in k, the probability of all addition amounts occurring in the appropriate intervals is bounded below by $(2\epsilon_1)^{(N-1)(\lceil \frac{2}{a+b} \rceil - 1)} \cdot (\frac{b-a}{4} - \frac{3\epsilon_{(N-1)}}{2})^{N-1}$. The probability of the event is bounded below by the product of these two bounds.

Now we define the coupling such that if this event occurs, then after $N-1$ avalanches, we have that $\hat{\eta}(t) = \hat{\xi}(t)$. In the remainder, we suppose that the above event occurs.

We start with discussing the time steps until the first avalanche. Suppose, without loss of generality, that $\eta_1 \geq \xi_1$. We make equal additions in $[a'', a'' + 2\epsilon_1]$, until the first moment t such that $\eta_1(t) > 1 - a'' - 2\epsilon_1$. We then know that $\xi_1(t) > 1 - a'' - 3\epsilon_1$. If we now choose the last addition for both configurations in $[a'_1, b] = [a'' + 3\epsilon_1, b]$, then both will topple.

Define

$$D_1 := \sum_{y=1}^{N-1} 2^{y-1}(\eta_y - \xi_y). \tag{4.12}$$

Let τ_1 be the time at which the first avalanche occurs. Then we choose for all $t < \tau_1$, $\hat{U}^\eta(t) = \hat{U}^\xi(t)$, and $\hat{U}^\xi(\tau_1) = a + \left(\hat{U}^\eta(\tau_1) + D_1 - a\right) \bmod (b-a)$. Since $D_1 < 2^{N-1}\epsilon_1 \leq \frac{b-a_1'}{4}$, when $\hat{U}^\eta(\tau_1) \in [\frac{3a_1'+b}{4}, \frac{a_1'+3b}{4}]$ (the middle half of $[a_1', b]$)

$$a_1' \leq \hat{U}^\eta(\tau_1) + D_1 < b. \tag{4.13}$$

So, $\hat{U}^\xi(\tau_1) = \hat{U}^\eta(\tau_1) + D_1$. Hence if $\hat{U}^\eta(\tau_1)$ is uniformly distributed on $[\frac{3a'+b}{4}, \frac{a'+3b}{4}]$, then $\hat{U}^\xi(\tau_1)$ is uniformly distributed on $[\frac{3a_1'+b}{4} + D_1, \frac{a_1'+3b}{4} + D_1] \subset [a_1', b]$.

Let $R_1 = \sum_{t=1}^{\tau_1 - 1} \hat{U}^\eta(t)$. Then at time τ_1, for $1 \leq x \leq N-2$ we have

$$\hat{\eta}_x(\tau_1) = \frac{1}{2^{x+1}}(\eta_1 + R_1 + \hat{U}^\eta(\tau_1)) + \frac{1}{2^x}\eta_2 + \cdots + \frac{1}{2}\eta_{x+1},$$

and

$$\hat{\eta}_{N-1}(\tau_1) = 0; \hat{\eta}_N(\tau_1) = \hat{\eta}_{N-2}(\tau_1),$$

and a similar expression for $\hat{\xi}_x(\tau_1)$. It follows that

$$\hat{\eta}_N(\tau_1) - \hat{\xi}_N(\tau_1) = -2^{(1-N)}D_1 + \sum_{y=1}^{N-1} 2^{y-N}(\eta_y - \xi_y) = 0$$

which means that the two coupled processes at time τ_1 are equal at site N.

After this first avalanche, the differences on sites $1, \ldots, N-3$ have been increased. Ignoring the fact that sites $N-2$ happen to be equal (to simplify the discussion), we calculate

$$\begin{aligned}
\max_{1\leq x\leq N} \mid \hat{\eta}_x(\tau_1) - \hat{\xi}_x(\tau_1) \mid &\leq \max_{1\leq x\leq N} \left\{ \frac{1}{2^{x+1}} \mid D_1 \mid + \max_{1\leq y\leq N} \mid \eta_y - \xi_y \mid \sum_{l=1}^{x+1} \frac{1}{2^l} \right\} \\
&\leq \max_{1\leq x\leq N} \left(\frac{2^{N-1}}{2^{x+1}}\right) \max_{1\leq y\leq N} \mid \eta_y - \xi_y \mid \\
&\leq (1 + 2^{N-3}) \max_{1\leq y\leq N} \mid \eta_y - \xi_y \mid \\
&\leq (1 + 2^{N-3})\epsilon_1 = \epsilon_2.
\end{aligned} \tag{4.14}$$

We wish to iterate the above procedure for the next $N-2$ avalanches. We number the avalanches $1, \ldots, N-1$, and define τ_k as the time at which the k-th avalanche occurs. As explained for the case $k = 1$, we choose all additions equal, except at times τ_k, where we choose $\hat{U}^{\xi}(\tau_k) - \hat{U}^{\eta}(\tau_k) = D_k$, with

$$D_k = \sum_{y=1}^{N-k} 2^{y-1}[\hat{\eta}_y(\tau_{k-1}) - \hat{\xi}_y(\tau_{k-1})]$$

and

$$|D_k| \leq 2^{N-k} \max_{1\leq x\leq N} |\hat{\eta}_x(\tau_{k-1}) - \hat{\xi}_x(\tau_{k-1})|.$$

The maximal difference between corresponding sites in the two resulting configurations has the following bound:

$$\begin{aligned}\max_{1\leq x\leq N} |\hat{\eta}_x(\tau_k) - \hat{\xi}_x(\tau_k)| &\leq \max_{1\leq x\leq N} \left\{ \frac{1}{2^{x+1}}|D_k| + \sum_{l=1}^{x+1} \frac{1}{2^l} \max_{1\leq y\leq N} |\hat{\eta}_y(\tau_{k-1}) - \hat{\xi}_y(\tau_{k-1})| \right\} \\ &\leq (1 + 2^{N-k-2}) \max_{1\leq y\leq N} |\hat{\eta}_y(\tau_{k-1}) - \hat{\xi}_y(\tau_{k-1})| \\ &\leq (1 + 2^{N-k-2})\epsilon_k = \epsilon_{k+1}. \end{aligned} \tag{4.15}$$

Hence for all k, $|D_k|$ is bounded from above by $\epsilon_{k+1}2^{N-k}$, where ϵ_{k+1} only depends on k, ϵ_k and N. With induction, we find,

$$|D_k| \leq 2^{N-k} \prod_{l=1}^{k-1}(1 + 2^{N-l-2})\epsilon_1 =: d_k.$$

We chose $\epsilon_1 = \epsilon_{a,b,N}$ such that $| D_{N-1} | \leq \frac{b-a'_{N-1}}{4}$. As the upper bound d_k is increasing in k, we get $| D_k | \leq \frac{b-a'_{N-1}}{4}$, for all $k = 1, ..., N - 1$. □

It follows from our proof that the convergence to the stationary distribution goes in fact exponentially fast. Indeed, every step of the coupling is such that a certain good event occurs with a certain minimal probability within a bounded number of steps. Hence, there exists a probability $q > 0$ and a number $M > 0$ such that with probability q, the coupling is successful within M steps, uniformly in the initial configuration. This implies exponential convergence.

4.4 Zhang's sandpile in infinite volume

4.4.1 Definitions and main results

In this section we work in general dimension d. We let $\mathcal{X} = [0, \infty)^{\mathbb{Z}^d}$ denote the set of infinite-volume height configurations in dimension d and $\Omega = [0, 1)^{\mathbb{Z}^d}$ the set of all stable configurations. For $x \in \mathbb{Z}^d$, the (Zhang) toppling operator T_x is defined as

$$T_x(\eta)_y = \begin{cases} 0 & \text{if } x = y, \\ \eta_y + \frac{1}{2d}\eta_x & \text{if } |y - x| = 1, \\ \eta_y & \text{otherwise.} \end{cases}$$

The infinite-volume version of Zhang's sandpile model is quite different from its abelian sandpile counterpart. Indeed, in the infinite-volume abelian sandpile model, it is shown that if a configuration can reach a stable one via some order of topplings, it will reach a stable one by every order of topplings and the final configuration as well as the number of topplings per site are always the same, see [22, 40, 21].

In Zhang's sandpile model in infinite volume, the situation is not nearly as nice. Not only does the final stable realization depend on the order of topplings, the very stabilizability itself also does. We illustrate this with some examples.

Consider the initial configuration (in $d = 1$)

$$\eta = (\ldots, 0, 0, 1.4, 1.2, 0, 0, \ldots).$$

We can reach a stable configuration in any order of the topplings, but the final configuration as well as the number of topplings per site depend on which unstable site we topple first. We can choose to start toppling at the left or right unstable site, or to topple the two sites in parallel (that is, at the same time); the different results are presented in Table 4.1.

start at site	toppling numbers	final configuration
left	$(\ldots, 0, 0, 1, 1, 0, 0, \ldots)$	$(\ldots, 0, 0.7, 0.95, 0, 0.95, 0, \ldots)$
right	$(\ldots, 0, 1, 2, 3, 1, 0, \ldots)$	$(\ldots, 0.5, 0.5, 0.525, 0, 0.525, 0.55 \ldots)$
parallel	$(\ldots, 0, 0, 1, 1, 0, 0, \ldots)$	$(\ldots, 0, 0.7, 0.6, 0.7, 0.6, 0, \ldots)$

Table 4.1. The three possible stabilizations of $(\ldots, 0, 0, 1.4, 1.2, 0, 0, \ldots)$

For a second example, let

$$\xi = (\ldots, 0.9, 0.9, 0, 1.4, 1.2, 0, 0.9, 0.9, \ldots).$$

This is a configuration that evolves to a stable configuration by some orders of topplings, but not by others. Indeed, if we start to topple the left unstable site first, we obtain the stable configuration

$$(\dots, 0.9, 0.9, 0.7, 0.95, 0, 0.95, 0.9, 0.9, \dots),$$

but if we topple the right unstable site first, after two topplings we reach

$$\xi' = (\dots, 0.9, 0.9, 1, 0, 1, 0.6, 0.9, 0.9, \dots).$$

By arguing as in the proof of the forthcoming Theorem 4.14, one can see that this configuration cannot evolve to a stable configuration.

In view of these examples, we have to be more precise about the way we perform topplings. In the present paper, we will use the *Markov toppling process*: to each site we associate an independent rate 1 Poisson process. When the Poisson clock 'rings' at site x and x is unstable at that moment, we perform a Zhang-toppling at that site. If x is stable, we do nothing. By $\eta(t)$, we denote the random configuration at time t. We denote by $M(x, t, \eta)$ the (random) number of topplings at site x up to and including time t.

Definition 4.12. *A configuration $\eta \in \mathcal{X}$ is said to be* stabilizable *if for every $x \in \mathbb{Z}^d$,*

$$\lim_{t\to\infty} M(x, t, \eta) < \infty$$

a.s. In that case we denote the limit random variable by $M(x, \infty, \eta)$.

Denote the collection of all stabilizable configurations by $\mathcal{S}$. It is not hard to see that $\mathcal{S}$ is shift-invariant and measurable with respect to the usual Borel sigma field. Hence, if μ is an ergodic stationary probability measure on $\mathcal{X}$, $\mu(\mathcal{S})$ is either 0 or 1.

Definition 4.13. *A probability measure μ on $\mathcal{X}$ is called* stabilizable *if $\mu(\mathcal{S}) = 1$.*

The next theorem is the main result in this section. When the density ρ of an ergodic translation-invariant measure μ is at least 1, μ is not stabilizable, and when it is smaller than $\frac{1}{2}$, μ is stabilizable. The situation when $\frac{1}{2} \le \rho < 1$ is not nearly as elegant. Clearly, when we take μ_ρ to be the point mass at the configuration with constant height ρ, then μ_ρ is stabilizable for all $\frac{1}{2} \le \rho < 1$. On the other hand, the following theorem shows that there are measures μ with density close to $\frac{1}{2}$ and close to 1 which are not stabilizable.

Theorem 4.14. *Let μ be an ergodic translation-invariant probability measure on $\mathcal{X}$ with $\mathbb{E}_\mu(\eta_0) = \rho < \infty$. Then*

1. *If $\rho \geq 1$, then μ is not stabilizable, that is, $\mu(\mathcal{S}) = 0$.*
2. *If $0 \leq \rho < \frac{1}{2}$, then μ is stabilizable, that is, $\mu(\mathcal{S}) = 1$.*
3. *For all $1/2 \leq \rho < d/(2d-1)$ and $(2d-1)/(2d) < \rho < 1$, there exists an ergodic measure μ_ρ with density ρ which is not stabilizable.*

Remark There is no obvious monotonicity in the density as far as stabilizability is concerned. Hence we cannot conclude from the previous theorem that for *all* $1/2 \leq \rho < 1$ there exists an ergodic measure μ_ρ which is not stabilizable.

4.5 Proofs for the infinite-volume sandpile

For an initial measure μ, $\mathbb{E}_\mu$ and $\mathbb{P}_\mu$ denote the corresponding expectation and probability measure in the stabilization process. We first show that no mass is lost in the toppling process.

Proposition 4.15. *Let μ be an ergodic shift-invariant probability measure on $\mathcal{X}$ with*

$$\mathbb{E}_\mu(\eta_0) = \rho < \infty$$

which evolves according to the Markov toppling process. Then we have

1. *for $0 \leq t < \infty$, $\mathbb{E}_\mu \eta_0(t) = \rho$,*
2. *if μ is stabilizable, then $\mathbb{E}_\mu \eta_0(\infty) = \rho$.*

Proof. We prove the item 1 via the well known mass transport principle. Let the initial configuration be denoted by η. Imagine that at time $t = 0$ we have a certain amount of mass at each site, and we color all mass white, except the mass at a special site x which we color black. Whenever a site topples, we further imagine that the black and white mass present at that site, are *both* equally distributed among the neighbors. So, for instance, when x topples for the first time, all its neighbors receive a fraction $1/(2d)$ of the original black mass at x, plus possibly some white mass. We denote by $B(y,t)$ the total amount of black mass at site y at time t. First, we argue that at any finite time t,

$$\sum_{y \in \mathbb{Z}^d} B(y,t) = \eta_x, \tag{4.16}$$

that is, no mass is lost at any finite time t. Indeed, had this not been the case, then we define t^* to be the infimum over those times t for which (4.16) is not true. Since mass must be subsequently passed from one site to the next, this implies that there is a path $(x = x_0, x_1, \ldots)$ of neighboring sites to infinity, starting at x such that the sites x_i all topple before time t^*, in the order given. Moreover, since t^* is the infimum, the toppling times t_i of x_i satisfies $\lim_{i\to\infty} t_i = t^*$. Hence, for any $\epsilon > 0$, we can find i_0 so large that for all $i > i_0$, $t_i > t^* - \epsilon$. Call a site open of its Poisson clock 'rings' in the time interval $(t^* - \epsilon, t^*)$, and closed otherwise. This constitutes an independent percolation process, and if ϵ is sufficiently small, the open sites do not percolate. Hence a path as above cannot exist, and we have reached a contradiction. It follows that no mass is lost at any finite time t, and we can now proceed to the routine proof of item 1 via mass-transport.

We denote by $X(x, y, t, \eta)$ the amount of mass at y at time t which started at x. From mass preservation, we have

$$\eta_x = \sum_{y\in\mathbb{Z}^d} X(x, y, t, \eta) \tag{4.17}$$

and

$$\eta_y(t) = \sum_{x\in\mathbb{Z}^d} X(x, y, t, \eta). \tag{4.18}$$

Since all terms are non-negative and by symmetry, this gives

$$\begin{aligned}\mathbb{E}_\mu \eta_0(0) &= \sum_{y\in\mathbb{Z}^d} \mathbb{E}_\mu X(0, y, t, \eta) \\ &= \sum_{y\in\mathbb{Z}^d} \mathbb{E}_\mu X(y, 0, t, \eta) = \mathbb{E}_\mu \eta_0(t).\end{aligned}$$

To prove item 2, we argue as follows. From item 1 we have that for every $t < \infty$, $\mathbb{E}_\mu \eta_0(t) = \rho$. Using Fatou's lemma we obtain

$$\mathbb{E}_\mu(\eta_0(\infty)) = \mathbb{E}_\mu(\lim_{t\to\infty} \eta_0(t)) \le \liminf_{t\to\infty} \mathbb{E}_\mu(\eta_0(t)) = \rho, \tag{4.19}$$

and therefore it remains to show that $\mathbb{E}_\mu(\eta_0(\infty)) \ge \rho$. This can be shown in the same way as Lemma 2.10 in [21], using the obvious identity

$$\eta_x(t) = \eta_x - L(x, t, \eta) + \frac{1}{2d} \sum_{|y-x|=1} L(y, t, \eta) \tag{4.20}$$

instead of (3) in [21], where $L(x, t, \eta)$ (for $0 \le t \le \infty$) denotes the total amount of mass that is lost from site x via topplings, until and including time t. □

Proposition 4.16. *Let $\eta(t)$ be obtained by the Markov toppling process starting from $\eta \in \mathcal{X}$. Let Λ be a finite subset of $\mathbb{Z}^d$, such that all sites in Λ toppled at least once before time t. Let β_Λ be the number of internal bonds of Λ. Then*

$$\sum_{x \in \Lambda} \eta_x(t) \geq \frac{1}{2d}\beta_\Lambda. \tag{4.21}$$

Proof. Let (x, y) be an internal bond of Λ. By assumption, both x and y topple before time t. Suppose that x is the last to topple among x and y. As a result of this toppling, at least mass $1/(2d)$ is transferred from x to y and this mass will stay at y until time t since y does not topple anymore before time t. In this way, we associate with each internal bond an amount of mass of at least $1/(2d)$, which is present in Λ at time t. Hence the total amount of mass in Λ at time t is at least $1/(2d)$ times the number of internal bonds. □

We can now prove Theorem 4.14.

Proof of Theorem 4.14. We prove item 1 first. Let μ be any ergodic shift-invariant measure with $\mathbb{E}_\mu(\eta_0) = \rho \geq 1$ and suppose μ is stabilizable. According to Proposition 4.15, we have

$$\mathbb{E}_\mu(\eta_0(\infty)) = \mathbb{E}_\mu(\eta_0) = \rho \geq 1, \tag{4.22}$$

which contradicts the assumption that $\eta(\infty)$ is stable.

For item 2, let μ be any ergodic shift-invariant probability measure on $\mathcal{X}$ with $\mathbb{E}_\mu(\eta_0) = \rho < \frac{1}{2}$, and suppose that μ is not stabilizable. We will now show that this leads to a contradiction.

Define $C_n(t)$ to be the event that before time t, every site in the box $[-n, n]^d$ topples at least once. Since μ is not stabilizable we have that $\mathbb{P}_\mu(C_n(t)) \to 1$ as $t \to \infty$. Indeed, if a configuration is not stabilizable, all sites will topple infinitely many times as can be easily seen.

Choose $\epsilon > 0$ such that $1 - \epsilon > 2\rho$. For this ϵ and every n, there exists a non-random time $T_n^\epsilon > 0$ such that for all $t > T_n^\epsilon$,

$$\mathbb{P}_\mu(C_n(t)) > 1 - \epsilon. \tag{4.23}$$

From Proposition 4.16 we have that at time $t \geq T_n^\epsilon > 0$, with probability at least $1 - \epsilon$, the following inequality holds:

$$\frac{\sum_{x \in [-n,n]^d} \eta_x(t)}{(2n+1)^d} \geq \frac{1}{2}\frac{(2n)^d}{(2n+1)^d}. \tag{4.24}$$

Therefore, we also have

$$\mathbb{E}_\mu\left(\frac{\sum_{x\in[-n,n]^d}\eta_x(t)}{(2n+1)^d}\right) \geq \frac{1}{2}(1-\epsilon)\frac{(2n)^d}{(2n+1)^d}.$$

Since $2\rho < 1$, we can choose n so large that

$$(1-\epsilon)\frac{(2n)^d}{(2n+1)^d} > 2\rho.$$

Using the shift-invariance of μ and the toppling process, for $t \geq T_n^\epsilon$, we find

$$\mathbb{E}_\mu\eta_0(t) = \mathbb{E}_\mu\left(\frac{\sum_{x\in[-n,n]^d}\eta_x(t)}{(2n+1)^d}\right) \geq \frac{1}{2}(1-\epsilon)\frac{(2n)^d}{(2n+1)^d} > \rho. \tag{4.25}$$

However, from Proposition 4.15, we have for any finite t that $\mathbb{E}_\mu\eta_0(t) = \mathbb{E}_\mu\eta_0(0) = \rho$.

Next we prove item 3, We start with $\rho > (2d-1)/(2d)$. To understand the idea of the argument, it is useful to first assume that we have an unstable configuration η on a *bounded* domain Λ (with periodic boundary conditions) with the property that $\eta_x \geq 1 - 1/(2d)$, for all $x \in \Lambda$. On such a bounded domain, we can order the topplings according to the time at which they occur. Hence we can find a sequence of sites $x_1, x_2, \ldots$ (not necessarily all distinct) and a sequence of times $t_1 < t_2 < \cdots$ such that the i-th toppling takes place at site $x_i \in \Lambda$ at time t_i. At time t_1, x_1 topples, so all neighbors of x_1 receive at least $1/(2d)$ from x_1. This means that all neighbors of x_1 become unstable at time t_1, and therefore they will all topple at some moment in the future. As a result, x_1 itself will also again be unstable after all its neighbors have toppled, and hence x_1 will topple again in the future.

In an inductive fashion, assume that after the k-th toppling (at site x_k at time t_k), we have that it is certain that all sites that have toppled so far, will topple again in the future, that is, after time t_k. Now consider the next toppling, at site x_{k+1} at time t_{k+1}. If none of the neighbors of x_{k+1} have toppled before, then a similar argument as for x_1 tells us that x_{k+1} will topple again in the future. If one or more neighbors of x_{k+1} have toppled before, then the inductive hypothesis implies that they will topple again after time t_{k+1}. Hence, we conclude that *all* neighbors of x_{k+1} will topple again which implies, just as before, that x_{k+1} itself will topple again. We conclude that each sites which topples, will topple again in the future, and therefore the configuration cannot be stabilized.

This argument used the fact that we work on a bounded domain, since only then is there a well-defined sequence of consecutive topplings. But with some extra work, we can make a similar argument work for the infinite-volume model as well, as follows.

Let $s_0 > 0$ be so small that the probability that the Poisson clock at the origin 'rings' before time s_0 is smaller than the critical probability for independent site percolation on the d-dimensional integer lattice. Call a site *open* if its Poisson clock rings before time s_0. By the choice of s_0, all components of connected open sites are finite. For each such component of open sites, we now order the topplings that took place between time 0 and time s_0. For each of these components, we can argue as in the first paragraphs of this proof, and we conclude that all sites that toppled before time s_0, will topple again at some time larger than s_0. We then repeat this procedure for the time interval $[s_0, 2s_0]$, $[2s_0, 3s_0], \ldots$, and conclude that at any time, a site that topples, will topple again in the future. This means that the configuration is not stabilizable. Hence, if we take a measure μ_ρ such that with μ_ρ-probability 1, all configurations have the properties we started out with, then μ_ρ is not stabilizable.

Next, we consider the case where $1/2 \leq \rho < d/(2d-1)$. Consider a measure μ_ρ whose realizations are a.s. 'checkerboard' patterns in the following way: any realization is a translation of the configuration in which all sites whose sum of the coordinates is even obtain mass 2ρ, and all sites whose sum of coordinates is odd obtain zero mass. Consider a site x with zero mass. Since all neighbors of x are unstable, these neighbors will all topple at some point. By our choice of ρ, x will become unstable precisely at the moment that the *last* neighbor topples. This follows from a simple computation. By an argument pretty much the same as in the first case, we now see that all sites that originally obtained mass 2ρ, have the property that after they have toppled, *all* their neighbors will topple again in the future, making the site unstable again. This will go on forever, and we conclude that the configuration is not stabilizable. Hence, μ_ρ is not stabilizable. □

Remark The arguments in case of parallel topplings are simpler: the case $\rho < 1/2$ can be done as above, while for *all* $\rho \geq 1/2$, the checkerboard pattern is preserved at all times, preventing stabilization.

5 Limiting shapes for a non-abelian sandpile growth model and related cellular automata

Summary

In this chapter, we present limiting shape results for a non-abelian variant of the abelian sandpile growth model (ASGM), some of which have no analog in the ASGM. One of our limiting shapes is an octagon. In our model, mass spreads from the origin by the toppling rule in Zhang's sandpile model. In our main proof, we introduce several cellular automata to mimic our growth model.

The content in this chapter is a reproduction of the article by A. Fey and H.Liu [18].

5.1 Introduction

We consider the following setup. Start with a pile of mass $n \geq 1$ at the origin of the rectangular grid $\mathbb{Z}^d$, and mass $h < 1$ at every other site, where n and h are real numbers. Mass may be moved by 'splitting piles', that is, one may take all the mass from one site, and divide it evenly among its neighbors. One can only split 'unstable' piles of mass at least 1, and one can not stop before all the piles have mass less than 1. We call $\mathbb{T}$ the - possibly random - set of sites where at least one split was performed.

Will the mass spread over all of $\mathbb{Z}^d$, or will $\mathbb{T}$ be a finite subset of $\mathbb{Z}^d$? In the first case, how does it spread? In the last case, what is the size and shape of this set, depending on h and n? How do these answers depend on the order of splitting?

This splitting game is related to abelian sandpile growth models. In fact, the 'splitting pile' rule is the same as the toppling rule in Zhang's sandpile model [50], a sandpile model that has not been considered as a growth model before. A notorious difficulty of moving mass by the Zhang toppling rule is the non-abelianness, that is, the end result depends on the order of topplings [20, 19].

In other studies of sandpile growth models [15, 23, 35], ample use was made of the freedom, by abelianness, to choose some convenient toppling order. In this way, information could be derived about limiting shapes, growth rates and about whether an explosion occurs or not. The term 'explosion' is introduced in [15]: if an explosion occurs, then the mass from the origin spreads over all of $\mathbb{Z}^d$, and every site topples infinitely often. For 'robust backgrounds', that is, values of h such that an explosion never occurs, one can examine the growth rate and existence of a limiting shape as $n \to \infty$. Roughly speaking, the growth rate is the radius of $\mathbb{T}$ as a function of n, and if the set $\mathbb{T}$, properly scaled, converges to a deterministic shape as $n \to \infty$, then that is the limiting shape.

In [35], the main topic is the rotor router model. In the rotor router model, the mass consists of discrete grains, so that only discrete values of h are possible. All sites except the origin start empty, and at all sites there is a router which points to a neighbor. Instead of splitting, a pile that has at least one grain may give a grain to the neighbor indicated by the router. The router then rotates to point to the next neighbor, for a cyclic ordering of all the neighbors. It was proved that the limiting shape is a sphere, and the growth rate is proportional to $n^{1/d}$. The proof makes use of properties of Green's function for simple random walk. In this paper, we will demonstrate that the method of [35] can be adapted for our model, resulting in a growth rate proportional to $n^{1/d}$ for all $h < 0$; see Theorem 5.6.

In the abelian sandpile model (for background, see [11, 41]), the mass also consists of discrete sand grains. Instead of splitting, a pile that consists of at least $2d$ grains may *topple*, that is, give one grain to each of its neighbors. Thus, $2d$ consecutive rotor router moves of one site equal one abelian sandpile toppling of that site. In [23], making use of this equivalence, it was shown that for the abelian sandpile growth model with $h \to -\infty$, the limiting shape is a sphere. Moreover, it was proved that for $h = 2d - 2$, the limiting shape is a cube. In [15], it was proved that for all $h \leq 2d - 2$, the growth rate is proportional to $n^{1/d}$.

All these proofs heavily rely on the abelianness, which is an almost routine technique for the abelian sandpile model, but fails to hold in our case. For instance, consider the case $d = 1$, $n = 4$, and $h = 0$. If we choose the parallel updating rule that is common in cellular automata, in each time step splitting every unstable site, then we end up with

$$\ldots, 0, 1/2, 3/4, 3/4, 0, 3/4, 3/4, 1/2, 0, \ldots$$

However, if we for example choose to split in each time step only the leftmost unstable site, then we end up with

$$\ldots, 0, 1/2, 1/2, 7/8, 3/4, 0, 3/4, 5/8, 0, \ldots$$

For arbitrary splitting order, it may even depend solely on the order if there occurs an explosion or not; see the examples in [19], Section 4.1. In this paper, we focus primarily on the parallel splitting order, but several of our results are valid for arbitrary splitting order.

Another complicating property of our model is that unlike the abelian sandpile model, we have no monotonicity in h nor in n. In the abelian sandpile growth model, it is almost trivially true that for fixed h, $\mathbb{T}$ is nondecreasing in n, and for fixed n, $\mathbb{T}$ is nondecreasing in h. For our growth model however, this is false, even if we fix the parallel splitting order. Consider the following examples for $d = 1$: For the first example, fix $h = 23/64$. Then if $n = 165/32 \approx 5.16\ldots$, we find that $\mathbb{T}$ is the interval $[-5,-4,\ldots,4,5]$. However, if $n = 167/32 \approx 5.22\ldots$, then $\mathbb{T} = [-4,-3,\ldots,3,4]$. For the second example, fix $n = 343/64$. Then if $h = 21/64 \approx 0.33\ldots$, we find that $\mathbb{T} = [-5,-4,\ldots,4,5]$, but if $h = 23/64 \approx 0.36\ldots$, then $\mathbb{T} = [-4,-3,\ldots,3,4]$.

One should take care when performing numerical simulations for this model. Since in each splitting, a real number is divided by $2d$, one quickly encounters rounding errors due to limited machine accuracy. All simulations presented in this paper were done by performing exact calculations in binary format.

This article is organized as follows: after giving definitions in Section 5.2, we present our results and some short proofs in Section 5.3. Our main result is that for h explosive and with the parallel splitting order, the splitting model exhibits several different limiting shapes as $t \to \infty$. We find a square, a diamond and an octagon. These results are stated in our main Theorem 5.3, which is proved in Section 5.4. Sections 5.5 and 5.6 contain the remaining proofs of our other results. Finally, in Section 5.7 we comment on some open problems for this model.

5.2 Definitions

In this section, we formally define the splitting model. While we focus primarily on the parallel order of splitting, some of our results are also valid for more general splitting order. Therefore, we give a general definition of the splitting model, with the splitting automaton (parallel splitting order) as a special case.

For $n \in [0,\infty)$ and $h \in (-\infty,1)$, η_n^h is the configuration given by

$$\eta_n^h(\mathbf{x}) = \begin{cases} n & \text{if } \mathbf{x} = \mathbf{0}, \\ h & \text{if } \mathbf{x} \in \mathbb{Z}^d \setminus \mathbf{0}. \end{cases}$$

For every $t = 0,1,2,\ldots$, and for fixed n and h, η_t is the configuration at time t, and the initial configuration is $\eta_0 = \eta_n^h$. We interpret $\eta_t(\mathbf{x})$ as the mass at site $\mathbf{x}$ at time t.

We now describe how η_{t+1} is obtained from η_t, for every t. Denote by $\mathcal{U}_t = \{\mathbf{x} : \eta_t(\mathbf{x}) \geq 1\}$ the set of all *unstable* sites at time t. $\mathcal{S}_{t+1}$ is a (possibly random) subset of $\mathcal{U}_t$. We say that $\mathcal{S}_{t+1}$ is the set of sites that *split* at time $t+1$. Then the configuration at time $t+1$ is for all $\mathbf{x}$ defined by

$$\eta_{t+1}(\mathbf{x}) = \eta_t(\mathbf{x})\,(1 - \mathbf{1}_{\mathbf{x}\in\mathcal{S}_{t+1}}) + \frac{1}{2d} \sum_{\mathbf{y}\in\mathcal{S}_{t+1}} \eta_t(\mathbf{y})\mathbf{1}_{\mathbf{y}\sim\mathbf{x}}.$$

The *splitting order* of the model determines how we choose $\mathcal{S}_{t+1}$, given $\mathcal{U}_t$, for every t. For example, if we have the parallel splitting order then we choose $\mathcal{S}_{t+1} = \mathcal{U}_t$ for every t. In this case, we call our model the *splitting automaton*. Some of our results are also valid for other splitting orders. In this paper we only consider splitting orders with the following properties: At every time t, $\mathcal{S}_{t+1}$ is non-empty unless $\mathcal{U}_t$ is empty, and for every $\mathbf{x}$ that is unstable at time t, there exists a finite time t_0 such that $x \in \mathcal{S}_{t+t_0}$. For example, we allow the random splitting order where $\mathcal{S}_{t+1}$ contains a single element of $\mathcal{U}_t$, chosen uniformly at random from all elements of $\mathcal{U}_t$. With this splitting order, at each time step a single site splits, randomly chosen from all unstable sites. This splitting order is valid because $\mathcal{U}_t$ increases slowly enough. Since only the neighbors of sites that split can become unstable, we have for any splitting order that $\mathcal{U}_{t+1} \subseteq \mathcal{U}_t + \partial\mathcal{U}_t$, where with $\partial\mathbb{X}$ for a set $\mathcal{X} \subset \mathbb{Z}^d$, we denote the set of sites that are not in $\mathbb{X}$, but have at least one neighbor in $\mathbb{X}$. But when every time step only a single site splits, at most $2d$ sites can become unstable, so that $|\mathcal{U}_{t+1}| \leq |\mathcal{U}_t| + 2d$.

We are interested in the properties of

$$\mathbb{T}_t = \bigcup_{0<t'\leq t} \mathcal{S}_{t'},$$

all the sites that split at least once until time t, as well as

$$\mathbb{T} = \lim_{t\to\infty} \mathbb{T}_t.$$

For a fixed splitting order, we say that η_n^h *stabilizes* if in the limit $t \to \infty$, for every $\mathbf{x}$, the total number of times that site $\mathbf{x}$ splits is finite. Note that if η_n^h does not stabilize, then every site splits infinitely often. We also remark that in order to show that η_n^h does not stabilize, it suffices to show that $\mathbb{T} = \mathbb{Z}^d$. Namely, if $\mathbb{T} = \mathbb{Z}^d$, then every site splits infinitely often. Otherwise, there is a site $\mathbf{x}$ and a time t such that $\mathbf{x}$ does not split at any time $t' > t$, but each of its neighbors $\mathbf{y}_i$, $i = 1, 2, \ldots, 2d$, splits at some time $t_i \geq t$. But then at time $\max_i t_i$, $\mathbf{x}$ is not stable, because it received at least mass $\frac{1}{2d}$ from each of its neighbors. Therefore, $\mathbf{x}$ must split again.

We call η_n^h *stabilizable* if η_n^h stabilizes almost surely (The "almost surely" refers to randomness in the splitting order). As defined in [15],

Definition 5.1. *The background h is said to be* robust *if η_n^h is stabilizable for all finite n; it is said to be* explosive *if there is a $N^h < \infty$ such that for all $n \geq N^h$, η_n^h is not stabilizable.*

Remark We expect the splitting model for every h to be either robust or explosive, independent of the splitting order (see Section 5.7.1). However, even for a fixed splitting order we cannot a priori exclude intermediate cases where the background is neither robust nor explosive. For example, since the splitting model is not monotone in n, it might occur for some h that for every n, there exist $n_1 > n_2 > n$ such that $\eta^h_{n_1}$ is stabilizable, but $\eta^h_{n_2}$ is not.

Finally, we give our definition of a limiting shape. In [15, 23, 35], limiting shape results were obtained for $\mathbb{T}$ in the limit $n \to \infty$, for robust background. In this paper however, we present limiting shape results for $\mathbb{T}_t$ with n fixed, parallel splitting order and explosive background. We study the limiting behavior in t rather than in n. Accordingly, we have a different definition for the limiting shape.

Let $\mathbf{C}$ denote the cube of radius 1/2 centered at the origin $\{\mathbf{x} \in \mathbb{R}^d : \max_i x_i \leq 1/2\}$. Then $\mathbf{x}+\mathbf{C}$ is the same cube centered at $\mathbf{x}$, and by $\mathcal{V}+\mathbf{C}$ we denote the volume $\bigcup_{\mathbf{x}\in\mathcal{V}}(\mathbf{x}+\mathbf{C})$.

Definition 5.2. *Let $\mathcal{V}_t$, with $t = 0, 1, \ldots$ be a sequence of sets in $\mathbb{Z}^d$. Let $\mathbf{S}$ be a deterministic shape in $\mathbb{R}^d$, scaled such that $\max_{x_i}\{\mathbf{x} \in \mathbf{S}\} = 1$. Let $\mathbf{S}_\epsilon$ and $\mathbf{S}^\epsilon$ denote respectively the inner and outer ϵ-neighborhoods of $\mathbf{S}$. We say that $\mathbf{S}$ is the* limiting shape *of $\mathcal{V}_t$, if there is a scaling function $f(t)$, and for all $\epsilon > 0$ there is a t^ϵ such that for all $t > t^\epsilon$,*

$$\mathbf{S}_\epsilon \subseteq f(t)\,(\mathcal{V}_t + \mathbf{C}) \subseteq \mathbf{S}^\epsilon.$$

If $\mathbf{S}$ is the limiting shape of $\mathbb{T}_t$, then we say that $\mathbf{S}$ is the limiting shape of the splitting automaton.

5.3 Main results

We have observed - see Figure 5.1 - that varying h has a striking effect on the dynamics of the splitting automaton. For large values of h, $\mathbb{T}_t$ appears to grow in time with linear speed, resembling a polygon, but which polygon depends on the value of h. Figures 5.1 and 5.3 support the conjecture that for the parallel splitting order, as $t \to \infty$, there are many possible different limiting shapes depending on h. Our main result, Theorem 5.3, is that there are at least three different polygonal limiting shapes, for three different intervals of h. For small values of h, the splitting model stabilizes; see Theorem 5.4. In between, there is a third regime that we were not able to characterize. It appears that for h in this regime, $\mathbb{T}_t$ keeps increasing in time, but does not have a polygonal limiting shape. We comment on this in Section 5.7.

Let $\mathbf{D}$ be the diamond in $\mathbb{R}^d$ with radius 1 centered at the origin; let $\mathbf{Q}$ be the square with radius 1 in $\mathbb{R}^2$ centered at the origin, and let $\mathbf{O}$ be the octagon in $\mathbb{R}^2$ with vertices (0,1) and $(\frac{5}{6}, \frac{5}{6})$, and the other six vertices follow from symmetry. See Figure 5.2 for these shapes.

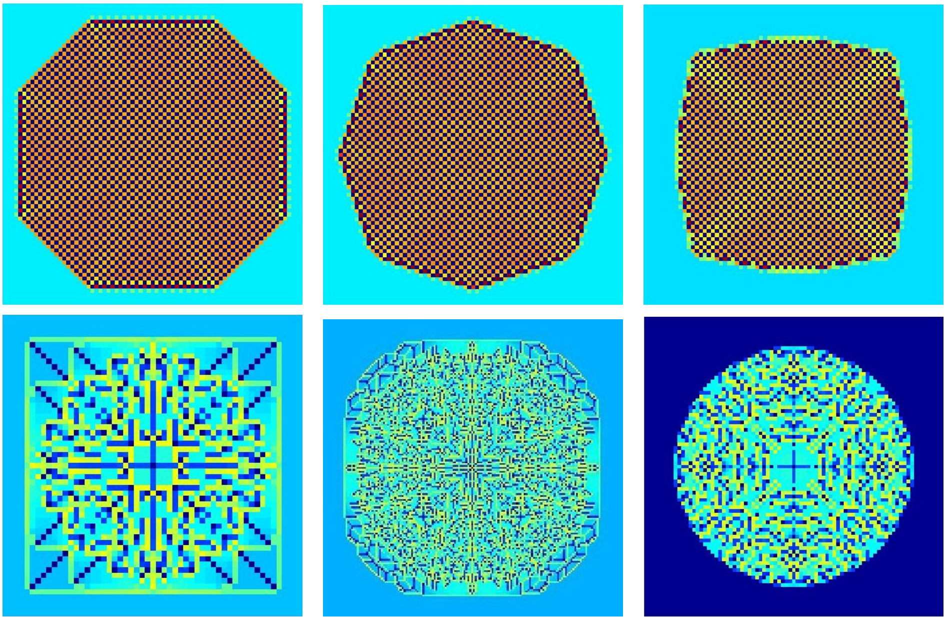

Figure 5.1. The splitting automaton for different values of h. "Warmer" color is larger mass; dark yellow, orange or red cells have mass ≥ 1. Top row: $h = 47/64 \approx 0.734$; $h = 1495/2048 \approx 0.73$ and $h = 727/1024 \approx 0.71$, each with $n = 8$ and after 50 time steps. Bottom row: $h = 1/2$ and $n = 256$; $h = 511/1024 \approx 0.499$ and $n = 2048$; $h = 0$ and $n = 2048$, each after the model stabilized.

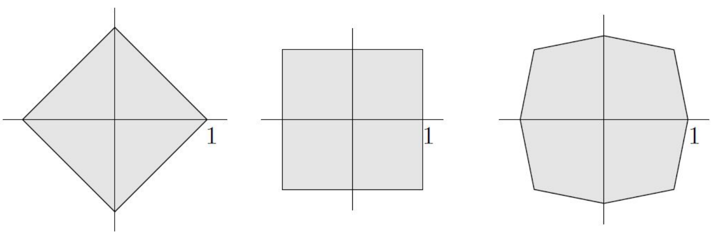

Figure 5.2. The diamond **D**, the square **Q**, and the octagon **O**.

Theorem 5.3.

1. *The limiting shape of the splitting automaton on $\mathbb{Z}^d$, for $1 - \frac{1}{2d} \le h < 1$ and $n \ge 1$, is the diamond* $\mathbf{D}$. *The scaling function is $f(t) = \frac{1}{t}$.*
2. *The limiting shape of splitting automaton on $\mathbb{Z}^2$, for $7/10 \le h < 40/57$ and $4 - 4h \le n \le 16 - 20h$, is the square* $\mathbf{Q}$. *The scaling function is $f(t) = \frac{2}{t}$.*
3. *The limiting shape of the splitting automaton on $\mathbb{Z}^2$, for $5/7 \le h < 13/18$ and $n = 3$, is the octagon* $\mathbf{O}$. *The scaling function is $f(t) = \frac{5}{3t}$.*

In the abelian sandpile growth model, $h = 2d-1$ is the only possible explosive background value, and our proof of part 1 also works for that situation. However, the second two parts have no analog in the abelian sandpile growth model.

Our proof uses the method of mimicking the splitting automaton with a finite state space cellular automaton; we consider our three explicit results as introductory examples for this method. We expect that with this method, many more limiting shape results can be obtained.

Next, we characterize several regimes of h for the splitting model on $\mathbb{Z}^d$.

Theorem 5.4. *In the splitting model on $\mathbb{Z}^d$,*

1. *The background is explosive if $h \ge 1 - \frac{1}{2d}$,*
2. *The background is robust if $h < \frac{1}{2}$,*
3. *In the splitting automaton, for $d \ge 2$, there exist constants $C_d < 1 - \frac{3}{4d+2}$ such that the background is explosive if $h \ge C_d$.*

We give the proof of the first part here, because it is a very short argument. The proof of parts 2 and 3 will be given in Section 5.5, where we give the precise form of C_d. We do not believe that this bound is sharp. From the simulations for $d = 2$ for example, a transition between an explosive and robust regime appears to take place at $h = 2/3$, while in our proof of part 3, $C_2 = 13/19 \approx 0.684$.

Proof of Theorem 5.4, part 1

If $n \ge 1$, then the origin splits, so then $\mathbb{T}$ is not empty. Now suppose that $\mathbb{T}$ is a finite set. Then there exist sites outside $\mathbb{T}$ that have a neighbor in $\mathbb{T}$. Such a site received at least $\frac{1}{2d}$, but did not split. For $h \ge 1 - \frac{1}{2d}$, this is a contradiction. □

Note that Theorem 5.4 does not exclude the possibility that there exists, for d fixed, a single critical value of h that separates explosive and robust backgrounds, independent of the splitting order. We only know this in the case $d = 1$, for which the first two bounds are equal.

We give another result that can be proved by a short argument:

Theorem 5.5. *In every splitting model on $\mathbb{Z}^d$, for every $n \geq 1$ and $h \geq 1 - \frac{1}{d}$, if the model stabilizes then $\mathbb{T}$ is a d-dimensional rectangle.*

Proof. Because $n \geq 1$, $\mathbb{T}$ is not empty. Suppose that $\mathbb{T}$ is not a rectangle. Then, as is not hard to see, there must exist a site that did not split, but has at least two neighbors that split. Therefore, its final mass is at least $h + \frac{1}{d}$, but strictly less than 1. This can only be true for $h < 1 - \frac{1}{d}$. □

In the case of the parallel splitting order, we additionally have symmetry. Thus, for $h \geq 1 - \frac{1}{d}$, $\mathbb{T}$ is a cube. We remark that the above proof also works for the abelian sandpile model, thus considerably simplifying the proof of Theorem 4.1 (first 2 parts) in [23].

Our final result gives information on the size and shape of $\mathbb{T}$ when $h < 0$. This theorem is similar to Theorem 4.1 of [35]. Let $\mathbf{B}_r$ denote the Euclidean ball in $\mathbb{R}^d$ with radius r, and let ω_d be the volume of $\mathbf{B}_1$.

Theorem 5.6.

1. *(Inner bound) For all $h < 1$,*

$$\mathbf{B}_{c_1 r - c_2} \subset \mathbb{T},$$

with $c_1 = (1-h)^{-1/d}$, $r = (\frac{n}{\omega_d})^{1/d}$ and c_2 a constant which depends only on d;

2. *(Outer bound) When $h < 0$, for every $\epsilon > 0$,*

$$\mathbb{T} \subset \mathbf{B}_{c_1' r + c_2'},$$

with $c_1' = (\frac{1}{2} - \epsilon - h)^{-1/d}$, $r = (\frac{n}{\omega_d})^{1/d}$ and c_2' a constant which depends only on ϵ, h and d.

5.4 Limiting shapes in the explosive regime

In this section, we will prove Theorem 5.3. Each part of this theorem is stated for h in a certain interval, and the first two parts for n in a certain interval. That means the theorem is stated for uncountably many possible initial configurations. However, we will show that we do not need to know all the exact masses to determine $\mathbb{T}_t$ for a certain t. For each part of the theorem, we will introduce a labeling of sites,

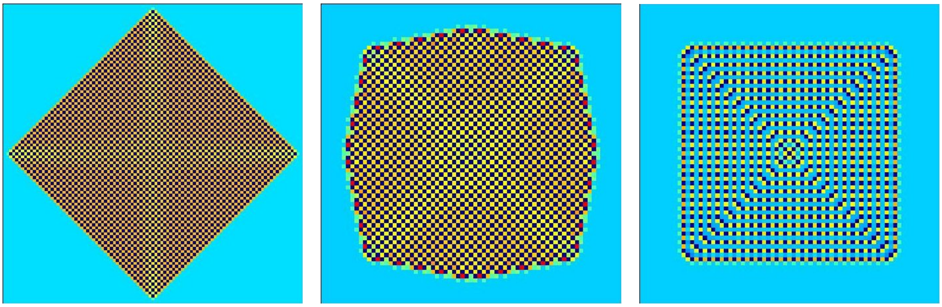

Figure 5.3. The splitting automaton after 50 time steps, with $n = 3$ and $h = \frac{3}{4} = 0.75$ (diamond), $h = \frac{23}{32} \approx 0.72$ (octagon), and $h = \frac{359}{512} \approx 0.701$ (square).

using only a finite number of labels. It will suffice to know the labels of all sites at time t, to determine $\mathbb{T}_{t'}$ for all $t' \geq t$.

We will see in each case that the time evolution in terms of the labels is a lot more enlightening than in terms of the full information contained in η_t. In each case, we can identify a certain recurrent pattern of the labels. Our limiting shape proofs will be by induction in t, making use of these recurrent patterns.

The label of a site at time t will depend on its own label at time $t-1$ plus those of its neighbors at time $t-1$. We will specify the labels at $t = 0$, and the transition rules for the labels. In other words, for each part of the theorem we will introduce a finite state space cellular automaton, that describes the splitting automaton for certain intervals of h and n in terms of the labels.

A cellular automaton is defined by giving its state space $\mathcal{S}$, its initial configuration ξ_0, and its transition rules. By $\xi_t(\mathbf{x})$, we denote the label of site $\mathbf{x}$ at time t. The state space will consist of a finite number of labels. The transition rules specify how the label of each cell changes as a function of its own current label and those of its neighbors. A cellular automaton evolves from the initial configuration in discrete time; each time step, all cells are updated in parallel according to the transition rules.

We use the following notation for the transition rules. Let s' and $s, s_1, s_2, \ldots$ denote labels in $\mathcal{S}$ (not necessarily all different). By

$$s \oplus s_1, s_2, \ldots, s_{2d} \longrightarrow s',$$

we mean that if a cell has label s, and there is a permutation of the labels of its $2d$ neighbors equal to $\{s_1, s_2, \ldots, s_{2d}\}$, then the label of this cell changes to s'. By $*$, we will denote an arbitrary label. For example, if we have a transition rule

$$s \oplus s_1, s_1, *, \dots, * \longrightarrow s',$$

then the label of a cell with label s will change into s' if at least two of its neighbors have label s_1, irrespective of the labels of the other neighbors.

We first give the proof for part 1, which is the simplest case. In fact, in this case the splitting model is equivalent to (1,d) bootstrap percolation: $\mathbb{T}_{t+1}$ is the union of $\mathbb{T}_t$ with all sites that have at least one neighbor in $\mathbb{T}_t$. The proof we give below will seem somewhat elaborate for such a simple case. That is because we use this case to illustrate our method of labels and cellular automata.

We will need the following observation, which can be proved by induction in t:

Lemma 5.7. *We call* $\mathbf{x}$ *an odd site if* $\sum_i x_i$ *is odd, otherwise we call* $\mathbf{x}$ *an even site. Then in the splitting automaton, even sites only split at even times, and odd sites only at odd times.*

Proof of Theorem 5.3, part 1

We begin by defining the diamond cellular automaton.

Definition 5.8.

The diamond cellular automaton *has state space* $\{e, \hbar, u\}$. *We additionally use the symbol* s *to denote* e *or* $\hbar$. *In the initial configuration, every cell has label* $\hbar$, *only the origin has label* u. *The transition rules are:*

1. $\hbar \oplus s, \dots, s \longrightarrow \hbar$
2. $s \oplus u, \dots, u \longrightarrow u$
3. $u \oplus s, \dots, s \longrightarrow e$
4. $\hbar \oplus u, *, \dots, * \longrightarrow u$

The above set of transition rules is sufficient to define the diamond cellular automaton, because, as we will demonstrate below, other combinations of cell and neighborhood labels do not occur.

Let $\mathbb{G}_t$, the growth cluster of the cellular automaton, be the set of all cells that do not have label $\hbar$ at time t. We will first prove the limiting shape result for $\mathbb{G}_t$, and then demonstrate that if $1 - \frac{1}{2d} \leq h < 1$, then $\mathbb{G}_t$ is the same set as $\mathbb{T}_t \cup \partial\mathbb{T}_t$ for every t.

Let $\mathcal{D}_r$ be the diamond $\{\mathbf{x} \in \mathbb{Z}^d : \sum_i |x_i| \leq r\}$. To prove the limiting shape result, we will show by induction in t that $\mathbb{G}_t = \mathcal{D}_t$, so that the limiting shape of $\mathbb{G}_t$ is $\mathbf{D}$, with scaling function $f(t) = \frac{1}{t}$.

Our induction hypothesis is that at time t, ξ_t is as follows (see figure 5.4): all sites $x \in \mathcal{D}_t$ have label u if $(\sum_i x_i - t) \mod 2 = 0$, and label e otherwise. All other sites have label $\hbar$. If this claim is true for all t, then we have $\mathbb{G}_t = \mathcal{D}_t$.

		u		
	u	e	u	
u	e	u	e	u
	u	e	u	
		u		

Figure 5.4. The induction hypothesis for the diamond cellular automaton at $t = 2$. Labels not shown are $\hbar$.

As a starting point, we take $t = 0$. At that time, the origin has label u, and all other cells have label $\hbar$. Therefore, the hypothesis is true at $t = 0$.

Now suppose the hypothesis is true at time t. Then all sites with label e have $2d$ neighbors with label u, therefore by the second transition rule they will have label u at time $t + 1$. All sites with label u have $2d$ neighbors with label e or $\hbar$, therefore by the third rule they will have label e at time $t + 1$. All sites in $\mathcal{D}_{t+1} \setminus \mathcal{D}_t$ have label $\hbar$ and at least one neighbor with label u. Therefore, by the fourth rule they will have label u at $t + 1$. Other labels do not change, by the first rule. This concludes the induction, and moreover shows that our set of transition rules is sufficient to define the diamond cellular automaton.

Finally, we show that if $1 - \frac{1}{2d} \leq h < 1$, then $\mathbb{G}_t$ is the same set as $\mathbb{T}_t \cup \partial\mathbb{T}_t$. To compare the configurations η_t and ξ_t, we give a mapping

$$\mathcal{M}_d : \{e, \hbar, u, s\} \to \mathbb{I},$$

where $\mathbb{I}$ is the set of intervals $\{[a, b) : a \leq b,\ a, b \in [0, \infty]\}$, that maps the state space of the diamond cellular automaton to the mass values of the splitting automaton:

$$\begin{aligned}
\mathcal{M}_d(e) &= 0 \\
\mathcal{M}_d(\hbar) &= h \\
\mathcal{M}_d(s) &= [0, 1) \\
\mathcal{M}_d(u) &= [1, \infty)
\end{aligned}$$

For a fixed diamond cellular automaton configuration ξ, define

$$\mathcal{M}_d^{\xi} = \{\eta : \eta(\mathbf{x}) \in \mathcal{M}_d(\xi(\mathbf{x})),\ \text{for all } \mathbf{x} \in \mathbb{Z}^d\}.$$

With this mapping, the initial configuration of the splitting automaton η_n^h is in $\mathcal{M}_d^{\xi_0}$, for all $n \geq 1$ and $1 - \frac{1}{2d} \leq h < 1$. We will show by induction in t that η_t is in $\mathcal{M}_d^{\xi_t}$ for all t. Suppose that at time t, η_t is in $\mathcal{M}_d^{\xi_t}$.

We check one by one the transition rules:

- (rule 1) If in the splitting automaton a site has mass h, and none of its neighbors split, then its mass does not change. This is true for all $h < 1$.
- (rule 2) If in the splitting automaton a stable site has $2d$ unstable neighbors, then its receives at least $2d$ times $\frac{1}{2d}$, therefore its mass will become at least 1. This is true for all h.
- (rule 3) If in the splitting automaton a site has mass at least 1, then it splits. If none of its neighbors splits, then its mass will become 0. This is true for all h.
- (rule 4) If in the splitting automaton a site with mass h has at least one neighbor that splits, it receives at least $\frac{1}{2d}$. Therefore, it will become unstable only if $1 - \frac{1}{2d} \leq h < 1$.

Therefore, if η_t is in $\mathbb{M}_d^{\xi_t}$, $1 - \frac{1}{2d} \leq h < 1$ and $n \geq 1$, then η_{t+1} is in $\mathbb{M}_d^{\xi_{t+1}}$. This completes the induction.

Finally, by the following observations:

- only the label u maps to mass 1 or larger, so a site is in $\mathbb{T}_t$ if and only if it has had label u at least once before t,
- the label of a site changes into another label if and only if at least one neighbor has label u,
- if a site does not have label $\hbar$ at time t, then it cannot get label $\hbar$ at any time $t' \geq t$,

we can conclude that $\mathbb{G}_t$ of the diamond cellular automaton is the same set as $\mathbb{T}_t \cup \partial\mathbb{T}_t$ for the splitting automaton with $1 - \frac{1}{2d} \leq h < 1$ and $n \geq 1$. □

We now give the proofs of the remaining two parts; note that in these next two proofs, we are in dimension 2. We will need more elaborate cellular automata, in which there are several labels for unstable sites. For example, it is important to know whether the mass of a splitting site is below or above $4(1-h)$: if its neighbor has mass h then in the first case it might not become unstable, but in the second case, it will.

Proof of Theorem 5.3, part 2

We begin by defining the square cellular automata.

Definition 5.9.

The square cellular automaton *has state space* $\{e, \hbar, p, m, m', c, d\}$. *We additionally use the symbol* s *to denote a label that is* e, $\hbar$ *or* p. *In the initial configuration, every cell has label* $\hbar$, *only the origin has label* d. *The transition rules are:*

1. $\hbar \oplus s, s, s, s \longrightarrow \hbar$
2. $p \oplus s, s, s, s \longrightarrow p$
3. $c \oplus *, *, *, * \longrightarrow c$
4. $m \oplus *, *, *, * \longrightarrow e$
5. $d \oplus *, *, *, * \longrightarrow c$
6. $m' \oplus *, *, *, * \longrightarrow c$
7. $\hbar \oplus d, s, s, s \longrightarrow m$
8. $\hbar \oplus m, s, s, s \longrightarrow p$
9. $p \oplus m, m, m', s \longrightarrow d$
10. $\hbar \oplus d, m, s, s \longrightarrow d$
11. $\hbar \oplus m, m, s, s \longrightarrow d$
12. $e \oplus d, d, c, p \longrightarrow m'$

The above set of transition rules is sufficient to define the cellular automaton, because, as we will demonstrate below, other combinations of cell and neighborhood labels do not occur.

Recall that the growth cluster $\mathbb{G}_t$ is the set of all cells that do not have label $\hbar$ at time t. To prove the limiting shape result for the growth cluster of the square cellular automaton, we use induction. Let $C_r \in \mathbb{Z}^2$ be the square $\{(i, j) : |i| \leq r, |j| \leq r\}$. Let ζ_r be the following configuration (see Figure 5.5):

- All sites in C_{r-1} have label c.
- The labels in $C_r \setminus C_{r-1}$ are d, if $(i - j) \mod 2 = 0$, and e otherwise.
- The labels outside C_r are p if they have a neighbor with label e, and $\hbar$ otherwise.

		p		p		
	d	e	d	e	d	
p	e	c	c	c	e	p
	d	c	c	c	d	
p	e	c	c	c	e	p
	d	e	d	e	d	
		p		p		

Figure 5.5. The configuration ζ_r, used in the induction hypothesis for the square cellular automaton, for $r = 2$. Labels not shown are $\hbar$.

Our induction hypothesis is that for every even t, $\xi_t = \zeta_{t/2}$. The initial configuration ξ_0 of the square cellular automaton is ζ_0. Now suppose that at some even time t, $\xi_t = \zeta_{t/2}$. Then by using the transition rules, one can check that at time ξ_{t+2} will be $\zeta_{t/2+1} = \zeta_{(t+2)/2}$. This completes the induction.

The shape result now follows: for every t, $C_{t/2} \subseteq \mathbb{G}_t \subseteq C_{t/2+1}$, so that the limiting shape of $\mathbb{G}_t$ is the square $\mathbf{Q}$, with scaling function $f(t) = \frac{2}{t}$.

Finally, to show that $\mathbb{G}_t$ for the square cellular automaton is equal to $\mathbb{T}_t \cup \partial\mathbb{T}_t$ for the splitting automaton with $4 - 4h \leq n \leq 16 - 20h$ and $7/10 \leq h < 40/57$, we give a mapping

$$\mathcal{M}_s : \{e, \hbar, p, m, m', c, d, s\} \rightarrow \mathbb{I},$$

that maps the state space of the square cellular automaton to the mass values of the splitting automaton:

$$\begin{aligned}
\mathcal{M}_s(e) &= 0\\
\mathcal{M}_s(\hbar) &= h\\
\mathcal{M}_s(s) &= [0,1)\\
\mathcal{M}_s(p) &= [h+\frac{1}{4},1)\\
\mathcal{M}_s(m) &= [1,4-4h)\\
\mathcal{M}_s(m') &= [0,12-15h)\\
\mathcal{M}_s(c) &= [0,16-20h)\\
\mathcal{M}_s(d) &= [4-4h,16-20h)
\end{aligned}$$

For all $h < 3/4$, these intervals are nonempty, moreover, $4-4h > 1$.

With this mapping, one may check that η_n^h is in $\mathcal{M}_s^{\xi_0}$. By induction in t, we will show that η_t is in $\mathcal{M}_s^{\xi_t}$ for all t. Suppose at time t, η_t is in $\mathbb{M}_s^{\xi_t}$.

We check one by one the transition rules:

- (rule 1) If in the splitting automaton a site has mass h, and none of its neighbors split, then its mass does not change. This is true for all $h < 1$.
- (rules 2-5) If an unstable site splits, then by Lemma 5.7, its neighbors do not split. Therefore, it will become empty. If a cell has split at least once, then from that time on it cannot receive sand from more than 4 neighbors before splitting itself. Therefore, no cell that split at least once can gain mass greater than $16-20h$. This is true for all $h < 1$.
- (rule 6) $h+\frac{1}{4}[4-4h,16-20h) \to [1,4-4h)$. This is true for all $h < 1$.
- (rule 7) $h+\frac{1}{4}[1,4-4h) \to [h+1/4,1)$. This is true for all $h < 1$.
- (rule 8) $[h+\frac{1}{4},1)+\frac{1}{2}[1,4-4h)+\frac{1}{4}[0,12-15h) \to [h+\frac{3}{4},6-\frac{23h}{4})$. We have that $[h+\frac{3}{4},6-\frac{23h}{4}) \subseteq [4-4h,16-20h)$ if $13/20 \le h \le 40/57$.
- (rule 9) $h+\frac{1}{2}[1,4-4h) \to [h+\frac{1}{2},2-h)$. We have that $[h+\frac{1}{2},2-h) \subseteq [4-4h,16-20h)$ if $7/10 \le h \le 14/19$.
- (rule 10) $h+\frac{1}{4}[4-4h,16-20h)+\frac{1}{4}[1,4-4h) \to [\frac{5}{4},5-5h)$. We have that $[\frac{5}{4},5-5h) \subseteq [4-4h,16-20h)$ if $11/16 \le h \le 11/15$.
- (rule 11) $\frac{1}{2}[4-4h,16-20h)+\frac{1}{4}[0,16-20h) \to [2-2h,12-15h)$. We have that $[2-2h,12-15h) \subseteq [0,12-15h)$ if $h < 1$.

Therefore, if η_t is in $\mathbb{M}_s^{\xi_t}$, $4-4h \le n \le 16-20h$ and $7/10 \le h \le 40/57$, then η_{t+1} is in $\mathbb{M}_s^{\xi_{t+1}}$. This completes the induction.

Finally, by the following observations:

- the labels m and d map to an interval in $[1, \infty)$, so a site is in $\mathbb{T}_t$ if it has had label m or d at least once before t,
- the label of a site with label $\hbar$ changes into another label if and only if at least one neighbor has label m or d,
- if a site does not have label $\hbar$ at time t, then it cannot get label $\hbar$ at any time $t' \geq t$,

we can conclude that if $4 - 4h \leq n \leq 16 - 20h$ and $7/10 \leq h \leq 40/57$, then $\mathbb{G}_t$ for the square cellular automaton is the same set as $\mathbb{T}_t \cup \partial\mathbb{T}_t$. □

For the final part, we first perform 8 time steps in the splitting automaton before we describe its further evolution as a finite state space cellular automaton. Otherwise, we would need many more labels and transition rules.

Proof of Theorem 5.3, part 3

We begin by defining the octagon cellular automaton.

Definition 5.10.

The octagon cellular automaton *has state space* $\{e, \hbar, p, m, d, d', d!, c, c', q, q'\}$. *We additionally use the symbol s to denote a label that is e, $\hbar$ or p, and the symbol u to denote a label that is any of the other.*

The initial configuration is given in the table below. We show only the first quadrant (left bottom cell is the origin). The rest follows by symmetry. All labels not shown are $\hbar$.

p							
e	$d!$	p	m				
c'	e	c	e	d'			
e	c	e	c	e	m		
c	e	c	e	c	p		
e	c	e	c	e	$d!$		
c	e	c	e	c'	e	p	

The transition rules are:

1. $\hbar \oplus s, s, s, s \longrightarrow \hbar$
2. $u \oplus *, *, *, * \longrightarrow e$
3. $\hbar \oplus m, s, s, s \longrightarrow p$
4. $\hbar \oplus d, s, s, s \longrightarrow m$
5. $\hbar \oplus d', s, s, s \longrightarrow m$
6. $\hbar \oplus d!, s, s, s \longrightarrow m$
7. $\hbar \oplus q, s, s, s \longrightarrow d$
8. $\hbar \oplus q, m, s, s \longrightarrow d!$
9. $\hbar \oplus q', m, s, s \longrightarrow d!$
10. $\hbar \oplus m, d', s, s \longrightarrow d'$
11. $\hbar \oplus d', d', s, s \longrightarrow d'$
12. $p \oplus d!, m, c, s \longrightarrow q'$
13. $p \oplus m, m, c, s \longrightarrow q$
14. $p \oplus d, m, c, s \longrightarrow q$
15. $e \oplus q', c, d', s \longrightarrow c$
16. $e \oplus d!, d!, c', s \longrightarrow c$

17. $e \oplus q, c, d!, s \longrightarrow c$
18. $e \oplus q', c, d!, s \longrightarrow c$
19. $e \oplus u, u, u, u \longrightarrow c$

except in the following case:

20. $e \oplus m, q, q, c \longrightarrow c'$

Our proof that the growth cluster of the octagon cellular automaton has **O** as limiting shape, is by induction. We will show that there is a pattern that repeats every 10 time steps. To describe this pattern, we introduce two sub-configurations that we call 'tile' and 'cornerstone'. They are given in Figure 5.6. We say the sub-configuration is at position (x, y), if the left bottom cell has coordinates (x, y).

p	$\hbar$	$\hbar$	$\hbar$	$\hbar$
e	$d!$	p	m	$\hbar$
c	e	c	e	q'

p	$\hbar$	$\hbar$	$\hbar$
e	d'	$\hbar$	$\hbar$
c	e	d'	$\hbar$
e	c	e	p

Figure 5.6. A 'tile' (left), and a 'cornerstone' (right).

To specify a configuration, we will only specify the cells $\mathbf{x} = (x, y)$ with $y \geq x \geq 0$; the rest follows by symmetry.

Definition 5.11. *We define the configuration* χ^i, *with* $i \in \{0, 1, \ldots\}$, *as follows:*

- *there is a cornerstone at position* $(5(i+1), 5(i+1))$,
- *for every* $j = 0, \ldots, i$, *there is a tile at position* $(5(i-j), 7+5i+j)$,
- *the leftmost tile is different, namely,* $\chi^i(0, 7+6i) = c'$,
- *for every cell* (x, y) *such that it is not in a tile or cornerstone, but every directed path from* (x, y) *to* $(0, 0)$ *intersects a tile or cornerstone,* $\chi^i(x, y) = \hbar$,
- *for every other cell* (x, y), $\chi^i(x, y) = e$ *if* $(x+y) \mod 2 = 0$, *and* c *otherwise.*

Our induction hypothesis now is: In the octagon cellular automaton, at time $5+10i$, $\xi_{5+10i} = \chi^i$. In words this says that every 10 time steps, an extra tile is formed.

The hypothesis can be verified for $i \leq 6$, by performing 65 time steps starting from the initial configuration ξ_0. We show the results of this computation in Figure 5.7, generated by a computer program of the octagon cellular automaton.

Suppose now that the hypothesis is true at time $5+10i$, with $i > 6$. We will construct $\xi_{5+10(i+1)}$ by performing 10 time steps starting from χ^i. Observe that the label of a cell at time $t+10$ depends only on its own label and that of all cells in $\mathbf{x}+\mathcal{D}_{10}$; we call this set of cells the '10-neighborhood' of $\mathbf{x}$. By the definition of χ^i, we have that for every $i > 5$ and for every $\mathbf{x}$, there exists a cell $\mathbf{y}$ such that the labeling

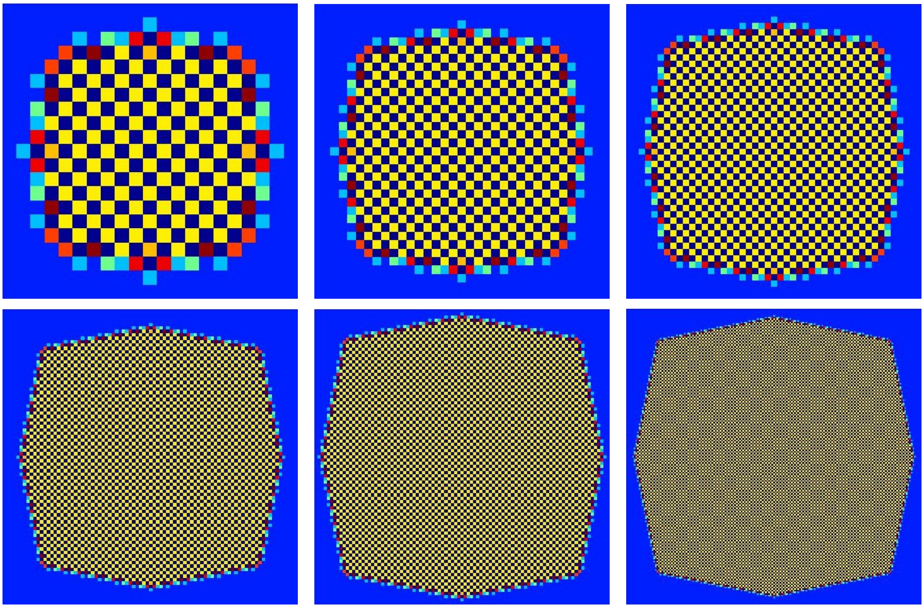

Figure 5.7. The octagon cellular automaton after 5, 15, 25, 55, 65 and 135 time steps, or equivalently, the configurations χ^0, χ^1, χ^2, χ^5, χ^6 and χ^{13}. Black = e, yellow = c, dark yellow = c', light blue = p, dark blue = $\hbar$, light green = m, orange = d', red = $d!$, dark red = q'.

of the 10-neighborhood of $\mathbf{x}$ in χ^i is identical to that of the 10-neighborhood of $\mathbf{y}$ in χ^5. Therefore, the label of $\mathbf{x}$ in χ^{i+1} will be identical to that of $\mathbf{y}$ in χ^6. Thus, we can construct $\xi_{5+10(i+1)}$ from χ^i, and we find that indeed, if $\xi_{5+10i} = \chi^i$ then $\xi_{5+10(i+1)} = \chi^{i+1}$.

Since $\mathbb{G}_t$ is nondecreasing in t, we have that for every t there is an i such that $\mathbb{G}_{5+10i} \subseteq \mathbb{G}_t \subseteq \mathbb{G}_{5+10(i+1)}$. The radius of $\mathbb{G}_{5+10i}$ is $9+6i$. This means that every 10 time steps, the radius increases by 6. We conclude that the limiting shape of $\mathbb{G}_t$ is the octagon $\mathbf{O}$, with scaling function $f(t) = \frac{5}{3t}$.

Finally, we prove that $\mathbb{G}_t$ of the octagon cellular automaton is equal to $\mathbb{T}_{t+8} \cup \partial\mathbb{T}_{t+8}$ if $n = 3$ and $h \in [5/7, 13/18)$.

We give a mapping

$$\mathcal{M}_o : \{e, \hbar, p, m, d, d', d!, c, c', q, q', s, u\} \to \mathbb{I},$$

that maps the state space of the octagon cellular automaton to the mass values of the splitting automaton: (if $1/2 < h < 946/1301 \approx 0.727$ then all intervals are nonempty, and moreover, the labels e, $\hbar$ and p map to an interval in $[0,1)$, while all other labels map to an interval in $[1,\infty)$):

$$\begin{aligned}
\mathcal{M}_o(e) &= 0\\
\mathcal{M}_o(\hbar) &= h\\
\mathcal{M}_o(p) &= [h+\frac{1}{4},1)\\
\mathcal{M}_o(s) &= [0,1)\\
\mathcal{M}_o(m) &= [1,4-4h)\\
\mathcal{M}_o(d) &= [4-4h,16-20h)\\
\mathcal{M}_o(d') &= [\frac{3}{8}+\frac{5h}{4},16-20h)\\
\mathcal{M}_o(d!) &= [\frac{1}{2}+\frac{5h}{4},16-20h)\\
\mathcal{M}_o(q) &= [1+h,60-80h)\\
\mathcal{M}_o(q') &= [\frac{21h}{16}+\frac{7}{8},60-80h)\\
\mathcal{M}_o(c) &= [1,60-80h)\\
\mathcal{M}_o(c') &= [1+\frac{h}{2},60-80h)\\
\mathcal{M}_o(u) &= [1,60-80h)
\end{aligned}$$

For every x, if $n=3$ and $h\in[5/7,13/18)$ then $\eta_8\in\mathcal{M}_o^{\xi_0}$. This can be verified by tedious, but straightforward inspection: In Figure 5.8 we give the configuration at $t=8$ for the splitting automaton with $n=3$ and $h\in[5/7,13/18)$.

$111+88388h$						
0	$675+128772h$	$108+89360h$	$81+95692h$			
$2610+128408h$	0	$1350+96824h$	0	$162+125848h$		
0	$4842+116632h$	0	$1572+112880h$	0	$81+95692h$	
$9423+99268h$	0	$5814+102920h$	0	$1350+96824h$	$108+89360h$	
0	$11700+98608h$	0	$4842+116632h$	0	$675+128772h$	
$14592+96512h$	0	$9423+99268h$	0	$2610+128408h$	0	$111+88388h$

Figure 5.8. η_8 multiplied by $4^8=65536$, for the splitting automaton with $n=3$ and $h\in(5/7,13/18)$. Masses not shown are $65536h$.

Next, we will prove by induction in t that η_{t+8} is in $\mathbb{M}_o^{\xi_t}$ for all t. We assume that for some t, η_{t+8} is in $\mathbb{M}_o^{\xi_t}$. By examining every transition rule, we can then show that as a consequence, η_{t+9} is in $\mathbb{M}_o^{\xi_{t+1}}$.

We check one by one the transition rules:

- (rule 1) If in the splitting automaton a site has mass h, and none of its neighbors split, then its mass does not change. This is true for all $h < 1$.
- (rule 2) If an unstable site splits, then by Lemma 5.7, its neighbors do not split. Therefore, it will become empty.
- (rule 3) $h + \frac{1}{4}[1, 4 - 4h) \to [h + 1/4, 1)$. This is true for all $h < 1$.
- (rule 4) $h + \frac{1}{4}[4 - 4h, 16 - 20h) \to [1, 4 - 4h)$. This is true for all $h < 1$.
- (rule 5) $h+\frac{1}{4}[\frac{3}{8}+\frac{5h}{4}, 16-20h) \to [\frac{3}{32}+\frac{21h}{16}, 4-4h)$. We have that $[\frac{3}{32}+\frac{21h}{16}, 4-4h) \subseteq [1, 4 - 4h)$ if $1 > h \geq 29/42 \approx 06905$.
- (rule 6) $h+\frac{1}{4}[\frac{1}{2}+\frac{5h}{4}, 16-20h) \to [\frac{1}{8}+\frac{21h}{16}, 4-4h)$. We have that $[\frac{1}{8}+\frac{21h}{16}, 4-4h) \subseteq [1, 4 - 4h)$ if $1 > h \geq 2/3$.
- (rule 7) $h+\frac{1}{4}[1+h, 60-80h) \to [\frac{5h}{4}+\frac{1}{4}, 15-19h)$. We have that $[\frac{5h}{4}+\frac{1}{4}, 15-19h) \subseteq [4 - 4h, 16 - 20h)$ if $0.7143 \approx 5/7 \leq h < 1$.
- (rule 8) $h + \frac{1}{4}[1 + h, 60 - 80h) + \frac{1}{4}[1, 4 - 4h) \to [\frac{5h}{4} + \frac{1}{2}, 16 - 20h)$. This is true for all $h < 1$.
- (rule 9) $h + \frac{1}{4}[\frac{21h}{16} + \frac{7}{8}, 60 - 80h) + \frac{1}{4}[1, 4 - 4h) \to [\frac{85h}{64} + \frac{7}{32}, 16 - 20h)$. We have that $[\frac{85h}{64} + \frac{7}{32}, 16 - 20h) \subseteq [\frac{5h}{4} + \frac{1}{2}, 16 - 20h)$ if $1 > h \geq 5/2$.
- (rule 10) $h + \frac{1}{4}[\frac{5h}{4} + \frac{3}{8}, 16 - 20h) + \frac{1}{4}[1, 4 - 4h) \to [\frac{21h}{16} + \frac{11}{32}, 5 - 5h)$. We have that $[\frac{21h}{16} + \frac{11}{32}, 5 - 5h) \subseteq [\frac{5h}{4} + \frac{3}{8}, 16 - 20h)$ if $1/2 \leq h \leq 11/15 \approx 0.7333$.
- (rule 11) $h+\frac{1}{2}[\frac{5h}{4}+\frac{3}{8}, 16-20h) \to [\frac{13h}{8}+\frac{3}{16}, 8-9h)$. We have that $[\frac{13h}{8}+\frac{3}{16}, 8-9h) \subseteq [\frac{5h}{4} + \frac{3}{8}, 16 - 20h)$ if $1/2 \leq h \leq 8/11 \approx 0.7273$.
- (rule 12) $[h+\frac{1}{4}, 1)+\frac{1}{4}[\frac{5h}{4}+\frac{1}{2}, 16-20h)+\frac{1}{4}[1, 4-4h)+\frac{1}{4}[1, 60-80h) \to [\frac{21h}{16}+\frac{7}{8}, 21-26h)$. We have that $[\frac{21h}{16} + \frac{7}{8}, 21 - 26h) \subseteq [\frac{21h}{16} + \frac{7}{8}, 60 - 80h)$ if $h \leq 13/18 \approx 0.7222$.
- (rule 13) $[h + \frac{1}{4}, 1) + \frac{1}{2}[1, 4 - 4h) + \frac{1}{4}[1, 60 - 80h) \to [1 + h, 18 - 22h)$. We have that $[1 + h, 18 - 22h) \subseteq [1 + h, 60 - 80h)$ if $h \leq 21/29 \approx 0.7241$.
- (rule 14) $[h+\frac{1}{4}, 1)+\frac{1}{4}[1, 4-4h)\frac{1}{4}[4-4h, 16-20h)+\frac{1}{4}[1, 60-80h) \to [\frac{7}{4}, 21-26h)$. We have that $[\frac{7}{4}, 21 - 26h) \subseteq [1 + h, 60 - 80h)$ if $h \leq 13/18 \approx 0.7222$.
- (rule 15) $\frac{1}{4}[\frac{21h}{16}+\frac{7}{8}, 60-80h)+\frac{1}{4}[1, 60-80h)+\frac{1}{4}[\frac{5h}{4}+\frac{3}{8}, 16-20h) \to [\frac{41h}{64}+\frac{9}{16}, 34-45h)$. We have that $[\frac{41h}{64} + \frac{9}{16}, 34 - 45h) \subseteq [1, 60 - 80h)$ if $0.6829 \approx 28/41 \leq h \leq 26/35 \approx 0.7429$.
- (rule 16) $\frac{1}{2}[\frac{5h}{4} + \frac{1}{2}, 16 - 20h) + \frac{1}{4}[1 + \frac{h}{2}, 60 - 80h) \to [\frac{3h}{4} + \frac{1}{2}, 23 - 30h)$. We have that $[\frac{3h}{4} + \frac{1}{2}, 23 - 30h) \subseteq [1, 60 - 80h)$ if $2/3 \leq h \leq 37/50 \approx 0.7400$.
- (rule 17) $\frac{1}{4}[1+h, 60-80h)+\frac{1}{4}[1, 60-80h)+\frac{1}{4}[\frac{5h}{4}+\frac{1}{2}, 16-20h) \to [\frac{9h}{16}+\frac{5}{8}, 34-45h)$. We have that $[\frac{9h}{16} + \frac{5}{8}, 34 - 45h) \subseteq [1, 60 - 80h)$ if $2/3 \leq h \leq 26/35 \approx 0.7429$.
- (rule 18) $\frac{1}{4}[\frac{21h}{16}+\frac{7}{8}, 60-80h)+\frac{1}{4}[1, 60-80h)+\frac{1}{4}[\frac{5h}{4}+\frac{1}{2}, 16-20h) \to [\frac{41h}{64}+\frac{19}{32}, 34-45h)$. We have that $[\frac{41h}{64} + \frac{19}{32}, 34 - 45h) \subseteq [1, 60 - 80h)$ if $0.6341 \approx 26/41 \leq h \leq 26/35 \approx 0.7429$.

- (rule 19) If in the splitting automaton a cell is empty, and all its neighbors split, then it gets mass at least 1. Since no cell has mass exceeding $60-80h$, that is the maximum mass that an empty cell can get.
- (rule 20) $\frac{1}{4}[1,4-4h)+\frac{1}{2}[1+h,60-80h)+\frac{1}{4}[1,60-80h) \to [1+\frac{h}{2},46-61h)$. We have that $46-61h \le 60-80h$ if $h \le 14/19 \approx 0.7368$.

In summary, all the rules are valid if $h \in [5/7,13/18]$. Therefore, if η_{t+8} is in $\mathbb{M}_o^{\xi_t}$ and $5/7 \le h \le 13/18$, then η_{t+9} is in $\mathbb{M}_o^{\xi_{t+1}}$. This completes the induction.

Finally, by the following observations:

- all sites in $\mathbb{G}_0$ are in $\mathbb{T}_8 \cup \partial\mathbb{T}_8$,
- only labels denoted as u map to values in $[1,\infty)$, so for all $t \ge 8$, a site is in $\mathbb{T}_t$ if and only if it is in $\mathbb{T}_8$ or it has had a label denoted as u at least once before t,
- the label of a site with label $\hbar$ changes into another label if and only if at least one neighbor has a label denoted as u,
- if a site does not have label $\hbar$ at time t, then it cannot get label $\hbar$ at any time $t' \ge t$,

We can conclude that if $n=3$ and $h \in [5/7,13/18]$, then $\mathbb{G}_t$ for the octagon cellular automaton is the same set as $\mathbb{T}_{t+8} \cup \partial\mathbb{T}_{t+8}$.

□

5.5 Explosive and robust regimes

In this section, we prove parts 2 and 3 of Theorem 5.4.

Proof of Theorem 5.4, part 2

We will prove that for all n, all $h < 1/2$ and all t, $|\mathbb{T}_t| \le \frac{n}{1/2-h}$, where by $|\mathbb{T}|$ we denote the cardinality of a set $\mathbb{T} \subset \mathbb{Z}^d$. It follows that

$$|\mathbb{T}| \le \frac{n}{1/2-h}, \tag{5.1}$$

so that for all $h < 1/2$, the background is robust.

Let m_0 be the total mass in $\mathbb{T}_t \cup \partial\mathbb{T}_t$ at time 0, and let m_t the total mass in $\mathbb{T}_t \cup \partial\mathbb{T}_t$ at time t. We have

$$m_0 = n + h|\mathbb{T}_t| + h|\partial\mathbb{T}_t|.$$

At time t, $\mathbb{T}_t \cup \partial\mathbb{T}_t$ contains a total mass of at least $\frac{1}{2d}$ times the number of internal edges in $\mathbb{T}_t \cup \partial\mathbb{T}_t$. Namely, consider a pair of sites $\mathbf{x}$ and $\mathbf{y}$ connected by an internal edge. Each time that one of them splits, a mass of at least $\frac{1}{2d}$ travels to the other one.

The number of internal edges in $\mathbb{T}_t \cup \partial\mathbb{T}_t$ is at least $d|\mathbb{T}_t|$. We demonstrate this by the following argument: Fix an ordering for the $2d$ edges connecting a site to its $2d$ neighbors, such that the first d edges of the origin are in the same closed orthant. For every site $\mathbf{x}$ in $\mathbb{T}_t$, all its edges are in $\mathbb{T}_t \cup \partial\mathbb{T}_t$. If for every $\mathbf{x} \in \mathbb{T}_t$ we count only the first d edges, then we count each edge in $\mathbb{T}_t \cup \partial\mathbb{T}_t$ at most once, and we arrive at a total of $d|\mathbb{T}_t|$.

Therefore, at least a mass of $d\frac{1}{2d}|\mathbb{T}_t| = \frac{1}{2}|\mathbb{T}_t|$ remains in $\mathbb{T}_t \cup \partial\mathbb{T}_t$. Moreover, since the sites in $\partial\mathbb{T}_t$ did not split, the mass h at every site in $\partial\mathbb{T}_t$ also remains in $\mathbb{T}_t \cup \partial\mathbb{T}_t$. Therefore, we have

$$m_t \geq \frac{1}{2}|\mathbb{T}_t| + h|\partial\mathbb{T}_t|.$$

Finally, we note that since up to time t no mass can have entered or left $\mathbb{T}_t \cup \partial\mathbb{T}_t$, we have $m_0 = m_t$. Putting everything together, we find

$$n + h|\mathbb{T}_t| + h|\partial\mathbb{T}_t| \geq \frac{1}{2}|\mathbb{T}_t| + h|\partial\mathbb{T}_t|,$$

from which the result follows. □

Proof of Theorem 5.4, part 3

We first give the proof for $d \geq 3$.

First, we need some notation. Denote by $\mathcal{D}_r \subset \mathbb{Z}^d$ the diamond $\mathcal{D}_r = \{x : \sum_i x_i \leq r\}$, and by $\mathcal{L}_r \subset \mathbb{Z}^d$ the layer $\{x : \sum_i x_i = r\}$. Denote $d_k = (k, k, \ldots k) \in \mathbb{Z}^d$. Let $\Gamma_{k,0} = d_k$, and for $i = 1 \ldots d$, let $\Gamma_{k,i}$ be the set of sites in $\mathcal{L}_{dk+i}$ that have i nearest neighbors in $\Gamma_{k,i-1}$. Observe that for every i, $\Gamma_{k,i}$ is not empty, and that $\Gamma_{k,d} = d_{k+1}$. For example, in dimension 3: $d_1 = (1,1,1)$, $\Gamma_{1,1} = \{(1,1,2),(1,2,1),(2,1,1)\}$, $\Gamma_{1,2} = \{(1,2,2),(2,1,2),(2,2,1)\}$, and $\Gamma_{1,3} = d_2 = (2,2,2)$.

Let $p_d = \frac{d!}{(2d)^d}\sum_{l=2}^{d}\frac{(2d)^l)}{l!}$, $q_d = \frac{d!}{(2d)^{d-1}}$, and h^* be defined by $q_d + p_d h^* = 2d(1-h^*)$, so that

$$h^* = \frac{q_d - 2d}{p_d + 2d}.$$

Finally, we define

$$C'_d = \max\{1 - \frac{1}{d}, h^*\},$$

and remark that $C'_d \leq 1 - \frac{3}{4d+2}$, with equality only in the case $d = 2$.

We will prove the following statement:

Lemma 5.12. *In the splitting automaton on $\mathbb{Z}^d$, if $h \geq C'_d$, and $n \geq 2d(1-h)$, then for every $k \geq 1$, at time $dk+2$, the sites in $\Gamma_{k,1}$ split.*

Theorem 5.4, part 3 follows from Lemma 5.12 combined with Theorem 5.5. Lemma 5.12 tells us that for every r, there is a site on the boundary of the cube $\{x : \max_i x_i \leq r\}$ that splits at least once. By Theorem 5.5 and by symmetry, all sites in this cube split at least once. Therefore, $\lim_{t\to\infty} \mathbb{T}_t = \mathbb{Z}^d$.

Proof. Note that at time t, no site outside $\mathcal{D}_t$ can have split yet, so if a site in $\Gamma_{k,i}$ splits at time $dk+i+1$, it does so for the first time. We will show that in fact the sites in $\Gamma_{k,i}$ do split at time $dk+i+1$. Since we take the parallel splitting order, by symmetry, all sites in $\Gamma_{k,i}$ distribute the same mass when they split; denote by $m(k,i)$ the mass distributed in the first split of a site in $\Gamma_{k,i}$.

We will prove the lemma by induction. For $k = 0$, the lemma is true, because we chose n large enough. Now suppose it is true for some value k. Then at time $dk+2$, the sites in $\Gamma_{k,2}$ receive $\frac{2}{2d}m(k,1) \geq \frac{1}{d}$, because they each have two neighbors in $\Gamma_{k,1}$. Since they did not split before, their mass is now at least $h + \frac{1}{d}$. Therefore, if $h \geq 1 - \frac{1}{d}$, they split at time $dk+3$. This condition is fulfilled because $C'_d \geq 1 - \frac{1}{d}$.

Continuing this reasoning, we find for all $i = 2, \ldots, d-1$ that at time $dk+i+1$, the sites in $\Gamma_{k,i}$ split, because each site in $\Gamma_{k,i}$ has i neighbors in $\Gamma_{k,i-1}$, so it receives mass $\frac{i}{2d}m(k,i-1) \geq \frac{i}{2d} \geq \frac{1}{d}$. We calculate, using that for $i = 2, \ldots, d$, we have $m(k,i) = h + \frac{i}{2d}m(k,i-1)$,

$$m(0,d) = q_d m(0,1) + p_d h \geq q_d + p_d h.$$

Recall that $\Gamma_{k,d} = \Gamma_{k+1,0}$. Therefore, at time $d(k+1)+1$ the sites in $\Gamma_{k+1,1}$ receive mass $\frac{m(0,d)}{2d}$. If $h + \frac{m(0,d)}{2d} \geq 1$, then the sites in $\Gamma_{k+1,1}$ split at time $d(k+1)+2$. This condition is fulfilled if $h \geq C'_d$. This completes the induction. Therefore, in Theorem 5.4, part 3, for $d \geq 3$ we can take $C_d = C'_d$. □

For $d = 2$, C'_d is equal to $1 - \frac{3}{4d+2} = 0.7$. In this case, we can take $C_2 = 13/19 = 0.684\ldots$:

Proposition 5.13. *In the splitting automaton with $d = 2$, the background is explosive if $h \geq 13/19$.*

Proof. In the proof of Theorem 5.4, part 3, we have proved that at time $2k+2$, sites $(k+1,k)$ and $(k,k+1)$ split, making use of the fact that at time $2k+1$, site (k,k) splits. We did not take into account that more sites in $\mathcal{L}_k$ might split at time $2k+1$.

We now choose $n \geq 64 - 84h$, so that at $t = 3$, sites $(0,2)$, $(1,1)$ and $(2,0)$ split for the first time. We prove by induction that if $h \geq 13/19$, then at time $2k$, the sites

$(k-1, k+1)$ and $(k+1, k-1)$ have mass at least 1, and site (k, k) has mass at least $2-h > 1$. With our choice for n, this is true for $k = 1$. Now suppose the hypothesis is true for k. This implies that at time $2k+1$, the sites sites $(k, k+1)$ and $(k+1, k)$ have mass at least $\frac{3h}{4} + \frac{3}{4}$. If this is at least equal to $4-4h$, then we obtain the induction hypothesis for $k+1$. Solving $\frac{3h}{4} + \frac{3}{4} \geq 4 - 4h$ gives $h \geq 13/19$. □

Remark We have extended this method further and obtained even smaller bounds for h. But as we increase the number of sites we consider, the calculations quickly become very elaborate, and the bound we obtain decreases very slowly. The smallest bound we recorded was 0.683.

5.6 The growth rate for $h < 0$.

In this section, we prove Theorem 5.6. The proof will follow closely the method used for the proof of Theorem 4.1 in [35], based on the estimates presented in [35], Section 2. An important ingredient to obtain the bounds is that after stabilization, every site has mass at most 1, so that the mass n starting from the origin, must have spread over a minimum number $\lfloor \frac{n}{1-h} \rfloor$ of sites. On the other hand, as shown in the proof of Theorem 3.1, part 2, we know: $|\mathbb{T}| \leq \frac{n}{\frac{1}{2}-h}$.

But crucial is the use of Green's functions and their asymptotic spherical symmetry, allowing to conclude more about the shape of the set of sites that split. Thus we can derive that $\mathbb{T}$ contains a ball of cardinality comparable to the coarse estimate $\lfloor \frac{n}{1-h} \rfloor$. Moreover, $\mathbb{T}$ is contained in a ball of cardinality close to $\lfloor \frac{n}{1/2-h} \rfloor$.

The method for $h < 0$ does not depend on abelianness or monotonicity, therefore it can be adapted to the splitting model with arbitrary splitting order.

We start with introducing some notation.

Denote by $\mathbb{P}_0, \mathbb{E}_0$ as the probability and expectation operator corresponding to the Simple Random Walk $\langle X(t) \rangle$ starting from the origin. For $d \geq 3$, define

$$g(z) = \mathbb{E}_0 \sum_{t=1}^{\infty} I_{X(t)=z}.$$

For $d = 2$, define

$$g_n(z) = \mathbb{E}_0 \sum_{t=1}^{n} I_{X(t)=z},$$

and

$$g(z) = \lim_{n\to\infty}[g_n(z) - g_n(0)].$$

Defining the operator Δ as

$$\Delta f(x) = \frac{1}{2d}\sum_{y\sim x} f(y) - f(x),$$

From [34], we have $\Delta g(z) = -1$ when x is the origin and $\Delta g(z) = 0$ for all other $x \in \mathbb{Z}^d$.

By $u(x)$, we denote the total mass emitted from x during stabilization. Then $\Delta u(x)$ is the net increase of mass at site x during stabilization. For all x, let $\eta_\infty(x)$ be the final mass at site x after stabilization. Since the final mass at each site is strictly less than 1, we have for all $x \in \mathbb{Z}^d$

$$\Delta u(x) + (n-h)\delta_{0,x} = \eta_\infty(x) - h < 1 - h, \tag{5.2}$$

with $\delta_{0,x} = 1$ if x is the origin and 0 for all other x.

Moreover, since the final mass at each site $x \in \mathbb{T}$ is in $[0,1)$, we have for all $x \in \mathbb{T}$

$$-h \le \Delta u(x) + (n-h)\delta_{0,x} = \eta_\infty(x) - h < 1 - h, \tag{5.3}$$

Proof for the inner bound:

For $x \in \mathbb{Z}^d$, $|x|$ is the Euclidean distance from x to the origin. Let

$$\tilde{\xi}_d(x) = (1-h)|x|^2 + (n-h)g(x) \text{ if } d \ge 2,$$

and let

$$\xi_d(x) = \tilde{\xi}_d(x) - \tilde{\xi}_d(\lfloor c_1 r \rfloor e_1),$$

with $e_1 = (1,0,0,\dots,0)$.

From Lemma 2.2 of [35], we have:

$$\xi_d(x) = O(1), \ x \in \partial\mathbf{B}_{c_1 r},$$

therefore there is a constant $C > 0$ such that $|\xi_d(x)| < C,\ x \in \partial\mathbf{B}_{c_1 r}$. Then

$$u(x) - \xi_d(x) \ge -\xi_d(x) > -C,\ x \in \partial\mathbf{B}_{c_1 r}.$$

Furthermore, using (5.2), we have for all $x \in \mathbb{Z}^d$,

$$\Delta(u - \xi_d) = \Delta u - \Delta \xi_d < 1 - h - (n-h)\delta_{0,x} - (1-h) - (n-h)\delta_{0,x} = 0.$$

Therefore, $u - \xi_d$ is superharmonic, which means that it reaches its minimum value on the boundary. Now, as in the proof of Theorem 4.1 of [35], the estimates in Lemmas 2.1 and 2.3 of [35] can be applied to conclude that there is a suitable constant c_2 such that $u(x)$ is positive for all $x \in \mathbf{B}_{c_1 r - c_2}$. □

The proof for the outer bound is more involved that that in [35], because Lemma 4.2 from [35], which is valid for the abelian sandpile growth model with $h \leq 0$, is not applicable for the splitting model. In essence, this lemma uses that if $u(x) > u(y)$ for some x and y, then the difference must be at least 1, because mass travels in the form of integer grains. Clearly, we have no such lower bound in the splitting model.

We note that in [15], a different proof for the outer bound appeared which is valid for the abelian sandpile growth model with $h < d$. Unfortunately, we cannot adapt this proof for the splitting model either. We will comment on this in Section 5.7.

We therefore first present some lemma's which we need to prove the outer bound.

Lemma 5.14. *Let $h < 0$, and take $x_0 \in \mathbb{T}$ adjacent to $\partial\mathbb{T}$. There is a path $x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_m = 0$ in $\mathbb{T}$ with*

$$u(x_{k+1}) > u(x_k) - \frac{2d}{2d-1}h, \qquad k = 0, \ldots, m-1.$$

Proof. We will first show that we can find a nearest neighbor path such that:

$$\frac{2d-1}{2d}u(x_{k+1}) - u(x_k) + \frac{1}{2d}u(x_{k-1}) \geq -h.$$

Let x_1 be the nearest neighbor of x_0 that loses the maximal amount of mass among all the nearest neighbors of x_0. If there is a tie, then we make an arbitrary choice. Because x_0 has at least one neighbor that does not split, $\Delta u(x_0) \leq \frac{2d-1}{2d}u(x_1) - u(x_0)$. Therefore:

$$\frac{2d-1}{2d}u(x_1) - u(x_0) \geq \Delta u(x_0) = \eta_\infty(x_0) - h \geq -h. \tag{5.4}$$

For $k \geq 1$ take $x_{k+1} \neq x_{k-1}$ to be the site that looses the maximal amount of mass among all the nearest neighbors of x_k except x_{k-1}. If there is a tie, then we make an arbitrary choice. It is always possible to choose x_{k+1}. As long as x_k is not the origin, then we get from (5.3) that:

$$\frac{2d-1}{2d}u(x_{k+1}) + \frac{1}{2d}u(x_{k-1}) - u(x_k) \geq \eta_\infty(x_k) - h \geq -h.$$

Thus we get a chain $\{x_k, k = 0, 1, \ldots\}$ of nearest neighbors, that possibly ends at the origin.

We rewrite

$$u(x_{k+1}) \geq \frac{2d}{2d-1}u(x_k) - \frac{1}{2d-1}u(x_{k-1}) - \frac{2d}{2d-1}h.$$

With this expression, and the fact that $u_0 < u_1$ by (5.4), it is readily derived by induction that $u(x_{k-1}) < u(x_k)$. Inserting this, we obtain $u(x_{k+1}) > u(x_k) - \frac{2d}{2d-1}h$, so that $u(x_k)$ is strictly increasing in k (recall that $h < 0$).

Now it is left to show that the chain does end at the origin. We derive this by contradiction: suppose the chain does not visit the origin. Then the chain cannot end, because there is always a new nearest neighbor that loses the maximal amount of mass among all the new nearest neighbors. But the chain cannot revisit a site that is already in the chain, because $u(x_k)$ is strictly increasing in k. But by (5.1), the chain cannot visit more than $\frac{n}{1/2-h}$ sites. Therefore, the chain must visit the origin. □

Define $Q_k(x) = \{y \in \mathbb{Z}^d : \max_i |x_i - y_i| \leq k\}$ as the cube centered at x with radius k. Let

$$u^{(k)}(x) = (2k+1)^{-d} \sum_{y \in Q_k(x)} u(y)$$

be the average loss of mass of the sites in cube $Q_k(x)$, and

$$\mathbb{T}^{(k)} = \{x : Q_k(x) \subset \mathbb{T}\}.$$

Lemma 5.15. $\Delta u^{(k)}(x) \geq \frac{k}{2k+1} - h - \frac{(n-h)}{(2k+1)^d}\mathbf{1}_{0 \in Q_k(x)}$, *for all* $x \in \mathbb{T}^{(k)}$.

Proof. From Proposition 5.3 of [19], we know for every x:

$$\sum_{y \in Q_k(x)} \eta_\infty(y) \geq \frac{1}{2d}(\text{number of internal bounds in } Q_k(x)). \tag{5.5}$$

Equation (5.2) tells that $\Delta u(y) = \eta_\infty(y) - h - (n-h)\delta_{0,y}$. Therefore

$$\Delta u^{(k)}(x) = \frac{1}{(2k+1)^d} \sum_{y \in Q_k(x)} [\eta_\infty(y) - h - (n-h)\delta_{0,y}].$$

Since $Q_k(x)$ has $2dk(2k+1)^{d-1}$ internal bounds, we get:

$$\Delta u^{(k)}(x) \geq \frac{k}{2k+1} - h - \frac{(n-h)}{(2k+1)^d}\mathbf{1}_{0\in Q_k(x)}.$$

□

Lemma 5.16. *For every $x \notin \mathbb{T}^{(k)}$,*

$$u(x) < a',$$

where a' depends only on k, d and h.

Proof. For $x \notin \mathbb{T}^{(k)}$, there is at least one site $y_0 \in Q_k(x)$ that does not split. For $l \geq 1$, take y_l as the nearest neighbor of y_{l-1} that loses the maximal amount of mass among all the neighbors of y_{l-1}. Since y_0 does not split, we have

$$\frac{1}{2d}\sum_{y\sim y_0} u(y) < 1-h.$$

Therefore $u(y_1) < 2d(1-h)$. For every $l > 1$, we have from (5.2) that $\frac{1}{2d}\sum_{y\sim y_l} u(y) < 1-h+u(y_l)$, therefore $u(y_{l+1}) < 2d(1-h) + 2du(y_l)$. We know there are at most $(2k+1)^d$ sites in $\{y_l\}_{l=0}$. Then:

$$\max_{x\in Q_k(x)} u(x) < (1-h)\left[(2d) + (2d)^2 + \cdots + (2d)^{(2k+1)^d}\right] < 2(1-h)(2d)^{(2k+1)^d},$$

so we can choose $a' = 2(1-h)(2d)^{(2k+1)^d}$. □

Proof of the outer bound:

First, we wish to find an upper bound for $u(x)$ for all x with $c_1'r - 1 < |x| \leq c_1'r$, that does not depend on n. If x is not in $\mathbb{T}^{(k)}$, then we use Lemma 5.16.

For $x \in \mathbb{T}^{(k)}$, take

$$\hat{\psi}_d(x) = (\frac{1}{2} - \epsilon - h)|x|^2 + (n-h)g(x) \text{ if } d \geq 2.$$

For a fixed small ϵ, we choose k such that

$$\frac{k}{2k+1} \geq \frac{1}{2} - \epsilon.$$

For the fixed chosen k, define

$$\tilde{\phi}_d(x) = \frac{1}{(2k+1)^d}\sum_{y\in Q_k(x)} \hat{\psi}_d(y).$$

Take

$$\phi_d(x) = \tilde{\phi}_d(x) - \tilde{\phi}_d(\lfloor c_1' r \rfloor e_1).$$

By calculation, we obtain $\Delta\phi_d(x) = \Delta\tilde{\phi}_d(x) = 1/2 - \epsilon - h - (n-h)\mathbf{1}_{0\in Q_k(x)}$. Then from Lemma 5.15, we know

$$\Delta(u^{(k)} - \phi_d) = \Delta u^{(k)} - \Delta\phi_d \geq 0, \forall x \in \mathbb{T}^{(k)}. \tag{5.6}$$

This shows that $u^{(k)} - \phi_d$ is subharmonic on $\mathbb{T}^{(k)}$. So, it takes its maximal value on the boundary. We combine this information with some lemma's:

- Lemma 2.4 of [35] gives that for all x, $\phi_d(x) \geq -a$ for some constant a depending only on d.
- Lemma 5.16 gives that for every $x \in \partial\mathbb{T}^{(k)}$, $u(x) < a'$.
- Finally, from Lemma 2.2 of [35], there is a $\tilde{c}_2$ which only depends on ϵ, d and h, such that for x with $c_1'r - 1 < |x| \leq c_1'r$, $\phi_d(x) \leq \tilde{c}_2$.

The first two lemma's imply that for $x \in \partial\mathbb{T}^{(k)}$, $u^{(k)}(x) - \phi_d(x) \leq a' + a$, an upper bound that does not depend on n. Therefore, since $u^{(k)} - \phi_d$ is subharmonic on $\mathbb{T}^{(k)}$,

$$u(x) - \phi_d(x) \leq a' + a, \forall x \in \mathbb{T}^k.$$

Combining this with the third lemma, we get:

$$u(x) \leq \tilde{c}_2 + a' + a, \forall x \in \mathbf{B}_{c_1'r} \cap \mathbb{T}^k.$$

Therefore, there is an upper bound for $u(x)$ that does not depend on n, for all $x \in \mathbf{B}_{c_1'r} \cap \mathbb{T}^k$. From Lemma 6.3, we know also for $x \notin \mathbb{T}^k$, $u(x) < a'$. Summarizing all, we obtain that for all x with $c_1'r - 1 < |x| \leq c_1'r$, $u(x) \leq \tilde{C}$, with $\tilde{C}$ a constant that does not depend on n.

To summarize, for all x with $c_1'r - 1 < |x| \leq c_1'r$, $u(x) \leq \tilde{C}$, with $\tilde{C}$ a constant that does not depend on n.

Now it remains to show that a site that splits, must lie at a bounded distance c_2' from $\mathbf{B}_{c_1'r}$. This follows from Lemma 5.14: From every site x_0 that splits, there is a path along which $u(x)$ increases by an amount of at least $-\frac{2d}{2d-1}h$ every step, and this path continues until the origin. Then along the way, this path must cross the boundary of $\mathbf{B}_{c_1'r}$, and there $u(x) \leq \tilde{C}$. Therefore, we can choose $c_2' = -\frac{(2d-1)\tilde{C}}{2dh}$. □

5.7 Open problems

Based on numerical simulations, we present some tantalizing open problems.

5.7.1 A critical h?

In Theorem 5.4, we give two regimes for h for which we know that the splitting model is explosive resp. robust. In between, there is a large interval for h where we can prove neither. We conjecture however that the t wo behaviors are separated by a single critical value of h, and that this value does not even depend on the splitting order. In dimension 2, our simulations indicate that this critical h is 2/3.

Conjecture 5.17.

1. *For the splitting model on $\mathbb{Z}^d$, there exists a $h_c = h_c(d)$ such that for all $h < h_c$, the model is robust, and for all $h \geq h_c$, the model is explosive.*
2. $h_c(2) = 2/3$.

5.7.2 The robust regime

We have proved Theorem 5.6 for all $h < 0$. We hoped to extend this result to all $h < 1/2$, by adapting the proof used in Section 3.1 of [15] for the abelian sandpile growth model (ASGM). However, the first step of this proof uses the fact that u_n is nondecreasing in n, where u_n is the total number of topplings that each site performs in stabilizing η_n^h. This follows from abelianness of the topplings. Since the splitting model is not abelian, we were not able to adapt this proof to work for our model. Nevertheless, we conjecture

Conjecture 5.18. *Theorem 5.6 holds for all $h < 1/2$.*

5.7.3 The explosive regime

We have only just started classifying the multitude of shapes of the splitting automaton, that one can observe by varying h. We are confident that our method is capable of generating many more limiting shape results. In some cases, we observe that varying n can make a difference, however, we expect the following to be true:

Conjecture 5.19. *For the splitting automaton on $\mathbb{Z}^d$, for every $h \in [1 - \frac{3}{4d+2}, 1)$ there exists a n_0 such that for every $n > n_0$, the limiting shape is a polygon, and depends only on h and d.*

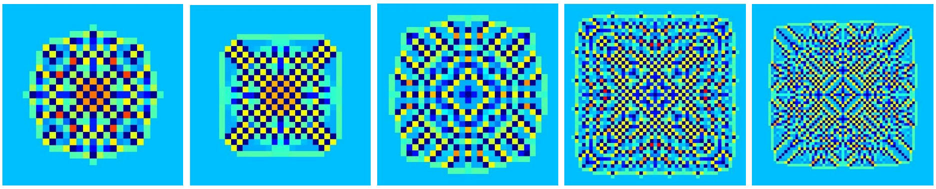

Figure 5.9. The splitting automaton with $h = 0.667$ and $n = 16$: From left to right, $t = 17$, $t = 24$, $t = 39$, $t = 76$, $t = 103$. In this case, a limiting shape may not exist.

This conjecture is reminiscent of Theorem 1 on threshold growth in [25], but the splitting automaton is not e quivalent to a two-state cellular automaton.

For smaller values of h, the behavior of the splitting automaton seems to be not nearly as orderly. In Figure 5.9, we show the behavior at $h = 0.667$, where we conjecture the model to be explosive. The shape of $\mathbb{T}$ seems to alternate between square and rounded. We are not sure whether a limiting shape exists for this value of h.

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Summary

Invariant measures and limiting shapes in sandpile models

This thesis is concerned with the study of three sandpile models: the CBTW model, the multiple addition sandpile model, and Zhang's sandpile model. As we discussed in Chapter 1, the initial motivation of the sandpile models is to study self-organized criticality. The simple rules of sandpile models make it possible to give rigorous treatment, which is the main work contained in the thesis. The members of the family of sandpile models have both similarity and difference with each other. In the following, we will give the comparison among the finite volume models related to this thesis.

First, each of the sandpile models consists of several basic elements: *configuration space*, *threshold value*, and *toppling rule*. In the BTW model, the height of a site can only take non-negative integer values, while in both the CBTW model and Zhang's model, it can take any non-negative real value. The threshold value is $2d$ in the BTW model in dimension d, it is 1 in the CBTW model and Zhang's model in every dimension. Both in the BTW model and the CBTW model, when a site topples, the toppled site loses 'threshold' amount of mass and each of its nearest neighbors receives $1/2d$ proportion of that amount; while in Zhang's model, the toppled site loses *all* its mass, each of its nearest neighbors receives $1/2d$ proportion of that amount. That amount depends on the height of the toppled site. Hence, both the BTW and the CBTW topplings are abelian but Zhang's topplings are not.

Second, the evolutions of the models are characterized by: *graph* Λ, which is a finite subset of lattice $\mathbb{Z}^d$ in all these three models; *the way of choosing the addition sites* and *the addition amounts*. In all these three sandpile models, at each time step, the addition site is chosen from Λ with uniform probability. In the BTW model, the addition amount is always 1. In the BTW- k model, the addition amount is always the fixed non-negative integer k and in the BTW-K model, it is a random number

Table 5.1. Table of sandpile models

	BTW	BTW-k	CBTW	Zhang
Graph	A finite subset $\Lambda \subset \mathbb{Z}^d$			
Configuration space	$\mathbb{N}_0^{\lvert\Lambda\rvert}$	$\mathbb{N}_0^{\lvert\Lambda\rvert}$	$[0,\infty)^{\lvert\Lambda\rvert}$	$[0,\infty)^{\lvert\Lambda\rvert}$
Threshold value	$2d$	$2d$	1	1
Toppling at x	$\eta(x) \to \eta(x) - 2d$;	$\eta(x) \to \eta(x) - 2d$;	$\eta(x) \to \eta(x) - 1$;	$\eta(x) \to 0$;
	$\eta(y) \to \eta(y) + 1$, if $\lvert y - x\rvert = 1$;	$\eta(y) \to \eta(y) + 1$, with $\lvert y - x\rvert = 1$;	$\eta(y) \to \eta(y) + 1/2d$, if $\lvert y - x\rvert = 1$;	$\eta(y) \to \eta(y) + \frac{1}{2d}\eta(x)$, if $\lvert y - x\rvert = 1$;
	$\eta(y) \to \eta(y)$, otherwise.			
Addition amount	1	k	R.v. uniformly on $[a,b]$ with $a, b \subset [0,1)$	
Addition site	R.v uniformly on Λ			

Remark: $\mathbb{N}_o$ − −−the set of natural numbers starting from 0; $\lvert y - x\rvert = 1$ − − − x, y are neighbors.

distributed on the set K of non-negative integers. In both the CBTW model and Zhang's model, the addition amount is a random variable uniformly distributed on an interval $[a, b]$, where a, b are any pair of real numbers satisfying $0 \leq a \leq b < 1$. During the evolution, all the addition sites and addition amounts are independent of each other in all these three models. Table 5.1 is an overview of the sandpile models studied in this thesis.

We now give a short summary of the results in this thesis. The mathematical treatment of the CBTW model is performed in Chapter 2. We first establish that the uniform measure μ on the so-called 'allowed configurations' is invariant under the dynamics. When $a < b$, we show with coupling ideas that starting from any initial configuration, the process converges in distribution to μ, which therefore is the unique invariant measure for the process. When $a = b$, that is, when the addition amount is non-random, and $a \notin \mathbb{Q}$, it is still the case that μ is the unique invariant probability measure, but in this case we use random ergodic theory to prove this; this proof proceeds in a very different way. Indeed, the coupling approach cannot work in this case since we also show the somewhat surprising fact that when $a = b \notin \mathbb{Q}$, the process does not converge in distribution at all starting from any initial configuration.

In Chapter 3, we give the formal definition of the multiple addition sandpile model. Our interests are again the convergence of the process and the uniqueness of the invariant measures. For a general graph $\Lambda \subset \mathbb{Z}^d$ and every non-negative integer k, the BTW-k process converges in distribution. For every graph $\Lambda \subset \mathbb{Z}^d$, we can find both infinitely many k and k' such that the BTW-k has a unique invariant measure, while the BTW-k' has many. In dimension 1, we get further results. Take $\Lambda = \{1, 2, \ldots, N\} \subset \mathbb{Z}$ and $k \in \mathbb{N}$ with $q = k \bmod (N+1)$, in the BTW-k model the set of recurrent configurations can be divided into $\gcd(q, N+1)$ different closed (under the process) subsets and the uniform measure on each of these subsets is invariant under the process. If K is a subset of $\mathbb{N}$ and $q_k = k \bmod (N+1)$ for all $k \in K$, then the BTW-K model has $\gcd(N+1, q_k, k \in K)$ different recurrent classes and the uniform measure on each of these recurrent classes is invariant under the process.

The results related to Zhang' model are presented in Chapter 4 and Chapter 5. In Chapter 4, we show that when $\Lambda \subset \mathbb{Z}$ with $|\Lambda| = N$, Zhang's model has a unique invariant measure for all $0 \leq a < b \leq 1$. Additionally, we also investigate the infinite volume Zhang's sandpile model in dimension $d \geq 1$. We study the stabilizability of initial configurations chosen according to some measure μ. We show that for a stationary ergodic measure μ with density ρ, for all $\rho < \frac{1}{2}$, μ is stabilizable; for all $\rho \geq 1$, μ is not stabilizable; for $\frac{1}{2} \leq \rho < 1$, when ρ is near to $\frac{1}{2}$ or 1, both possibilities can occur.

In Chapter 5, we turn to a rather different subject related to Zhang's model. We

define a growth model in which the mass can split with the same rule as Zhang's topplings. The initial configuration contains a large mass $n > 1$ in the center and $h < 1$ at every other sites of $\mathbb{Z}^d$. When a site has mass at least 1, it is unstable and it can split. The mass can spread only by splittings. We specify the order of splittings. We point out that when $h < \frac{1}{2}$, it is robust and when $h > 1 - \frac{1}{2d}$, it is explosive. For $d \geq 2$, when we take the parallel toppling order, there exists constants $C_d < 1 - \frac{3}{4d+2}$ such that when $h > C_d$ and for large n, the splitting process cannot stop. we have that $1 - \frac{3}{4d+2} < 1 - \frac{1}{2d}$, then with the parallel splitting order, the interval of h that makes the splitting process does not stop is a bit larger than that with the general splitting order. With the parallel splitting order and for h that the splitting process does not stop, there are various limiting shapes including a diamond, a square and an octagon. For $h < 0$, we can find both outer and inner bounds for the set of toppled sites.

Samenvatting (Dutch Summary)

Invariante maten en limietvormen voor zandhapen

Dit proefschrift bestudeert drie zandhoopmodellen: het CBTW model, het meervoudige toevoeging-zandhoopmodel, en Zhang's zandhoopmodel. Zoals vermeld in Hoofdstuk 1 is de oorspronkelijke motivatie van zandhoopmodellen het bestuderen van 'self-organized criticality'. De simpele regels van zandhoopmodellen maken het mogelijk om ze rigoreus te behandelen. De diverse zandhoopmodellen hebben overeenkomsten en verschillen. Hieronder geven we een vergelijking van de eindig volume-modellen uit dit proefschrift.

Ten eerste, ieder van de zandhoopmodellen bestaat uit een aantal basiselementen: de *configuratieruimte*, de *grenswaarde*, en de *topplingregel*. In het klassieke BTW model kan de hoogte van een punt alleen niet-negatieve integer waardes aannemen, terwijl in zowel het CBTW als Zhang's model de hoogte iedere niet-negatieve reële waarde kan aannemen. De grenswaarde is $2d$ in het BTW model in dimension d, 1 in het CBTW model en in Zhang's model in iedere dimensie. Zowel in het BTW model als in het CBTW model is het zo dat als een punt toppelt, dit punt de grenswaarde aan massa verliest en ieder van zijn buren een fractie $1/2d$ daarvan krijgt; maar in Zhang's model verliest het toppelende punt alle massa, en ieder van zijn naaste buren krijgt een fractie $1/2d$ daarvan. Die hoeveelheid hangt af van de hoogte van het topplende punt. Daarom zijn zowel BTW als CBTW topplings abels, maar Zhang's topplings zijn dat niet.

Ten tweede, de evolutie van de modellen is gekarakteriseerd door: de *graaf* Λ, een eindig deelverzameling van het rooster $\mathbb{Z}^d$ in alle drie deze modellen; *de manier waarop het toevoegpunt gekozen wordt* en *de toegevoegde hoeveelheid*. In alledrie deze zandhoopmodellen wordt iedere tijdstap het toevoegpunt gekozen uit Λ met uniforme kans. In het BTW model is de toegevoegde hoeveelheid altijd 1. In het BTW-k model is de toegevoegde hoeveelheid altijd een vast niet-negatief geheel getal k, en in het BTW-K model is het een toevallig getal verdeeld over een verzameling K van niet-negatieve gehele getallen. Zowel in het CBTW model en in Zhang's model is de toegevoegde hoeveelheid stochastisch, uniform verdeeld op een interval $[a, b]$, waar a, b twee reële getallen zijn met $0 \leq a \leq b < 1$. Tijdens de evolutie zijn alle toevoegpunten en toegevoegde hoeveelheden onafhankelijk van elkaar in alledrie deze modellen. Tabel 5.1 is een overzicht van de zandhoopmodellen die bestudeerd zijn in dit proefschrift.

We geven nu een korte samenvatting van de resultaten in dit proefschrift. De wiskundige behandeling van het CBTW model is in hoofdstuk 2. We leiden eerst af dat de uniforme maat μ op de zogenaamde 'toegestane configuraties' invariant is onder de dynamica. We laten met koppeling-ideeën zien dat als $a < b$, dan zal, startend van een willekeurige beginconfiguratie, het proces convergeren in verdeling naar μ. Daarom is dat de unieke invariante maat voor het proces. Als $a = b$, dat wil zeggen, als de toevoeging niet random is, en $a \notin \mathbb{Q}$, dan is het nog steeds zo dat μ de unieke invariante maat is, maar in dit geval gebruiken we random ergodentheorie om dit te bewijzen; dit bewijs gaat op een heel andere manier. Sterker nog, de koppeling-aanpak kan niet werken in dit geval omdat we ook het ietwat verrassende feit bewijzen dat als $a = b \notin \mathbb{Q}$, het proces helemaal niet in verdeling convergeert, startend van een willekeurige beginconfiguratie.

In Hoofdstuk 3 geven we formele definitie van het meervoudige toevoeging-zandhoopmodel. We zijn weer geinteresseerd in de convergentie van het proces en de uniciteit van de invariante maat. Voor een algemene graaf $\Lambda \subset \mathbb{Z}^d$ en elk niet-negatief geheel getal k, convergeert het BTW-k proces in verdeling. Voor elke graaf $\Lambda \subset \mathbb{Z}^d$ kunnen we oneindig veel k en k' vinden, zodanig dat het BTW-k model een unieke invariante maat heeft, terwijl het BTW-k' model er vele heeft. In dimensie 1 hebben we meer resultaten. Neem $\Lambda = \{1, 2, \ldots, N\} \subset \mathbb{Z}$ en $k \in \mathbb{N}$ met $q = k \bmod (N + 1)$. In het BTW-k model kan de set van recurrente configuraties verdeeld worden in $\gcd(q, N+1)$ verschillende gesloten (onder het proces) subsets, en de uniforme maat op deze subsets is invariant onder het proces. Als K een deelverzameling is van $\mathbb{N}$ en $q_k = k \bmod (N + 1)$ voor alle $k \in K$, dan heeft het BTW-K model $\gcd(N+1, q_k, k \in K)$ verschillende recurrente klasses, en de uniforme maat op elk van deze recurrente klasses is invariant onder het proces.

De resultaten gerelateerd aan Zhang' model staan in Hoofdstuk 4 en Hoofdstuk 5. In Hoofdstuk 4 laten we zien dat als $\Lambda \subset \mathbb{Z}$ met $|\Lambda| = N$, dan heeft Zhang's model een unieke invariant maat voor alle $0 \leq a < b \leq 1$. Verder onderzoeken we ook het oneindig volume Zhang's zandhoopmodel in dimensie $d \geq 1$. We bestuderen stabiliseerbaarheid van beginconfiguraties gekozen volgens een bepaalde maat μ. We laten zien dat voor een stationaire ergodische maat μ met dichtheid ρ, μ stabiliseerbaar is voor alle $\rho < \frac{1}{2}$; μ is niet stabiliseerbaar is voor alle $\rho \geq 1$; voor $\frac{1}{2} \leq \rho < 1$, als ρ dichtbij $\frac{1}{2}$ of 1 is, dan kunnen beide mogelijkheden voorkomen.

In Hoofdstuk 5 komen we bij een nogal verschillend onderwerp gerelateerd aan Zhang's zandhoopmodel. We definiëren een groeimodel waarin de massa kan splitsen met Zhang's toppleregel. De beginconfiguratie bevat een grote massa $n > 1$ in de oorsprong en $h < 1$ op ieder ander punt van $\mathbb{Z}^d$. Als een punt tenminste massa 1 heeft, dan is het instabiel en kan het splitsen met Zhang's toppelregel. De massa kan zich alleen verspreiden met splitsingen. Als $h < \frac{1}{2}$, dan is het model robuust en als $h > 1 - \frac{1}{2d}$, dan is het model explosief.

Als $d \geq 2$, en als we de parallelle toppelvolgorde kiezen, dan bestaan er constanten $C_d < 1 - \frac{3}{4d+2}$, zodanig dat als $h > C_d$ en n groot, dan stopt het splitsproces niet. Aangezien $1 - \frac{3}{4d+2} < 1 - \frac{1}{2d}$, is met de parallelle toppelvolgorde het interval van h waarvoor het splitsproces niet stopt een beetje groter dan met willekeurige splitsvolgorde. Met de parallelle topplevolgorde en voor h zodanig dat het splitsproces niet stopt, zijn er verscheidene limietvormen waaronder een ruit, een vierkant en een octagon. Voor $h < 0$ kunnen we binnen- en buitengrenzen vinden voor de set van sites die getoppeld hebben.

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