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Blowup in two geometric flows

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Chapter 1

Introduction

In this thesis we investigate the blowup behavior of certain geometric flows. We do this using three different methods:

- formal matched asymptotics,
- moving mesh method,
- maximum principle for partial differential equations (PDEs).

In this Chapter we introduce the subjects of geometric flows and singularity formation (blowup) and discuss the methods used in this thesis. In doing so, we set some terminology and introduce some concepts we use. In the Sections 1.3 and 1.4 we discuss the results of this thesis and give final considerations on these results.

1.0.1 Geometric flows

The subject of geometric analysis has gained increasing attention in modern mathematics. This interplay between geometry and analysis has helped to solve some long-standing problems of which the Poincaré conjecture is probably the most famous. We refer to the overview article [llm] and references therein, for more on geometric analysis (and its successes) in the last couple of decades. For more on the geometrization and Poincaré conjectures we refer to the standard overviews [mt] [rq] [tm] and to the more PDE oriented [lko] and references therein.

The relation between geometry and analysis is straightforward through differential equations. In this field of mathematics, geometric structures can be expressed in terms of differential equations. As an example we mention the theorem of Bonnet which relates the immersion of a surface in Euclidean space with a PDE on the metric and the second fundamental form of this surface (see Subsection 2.2.1).

Evolution equations play an important role in geometric analysis. One of the first examples of an evolution equation in geometric analysis is the harmonic map heat flow, introduced by Eells and Sampson in [44]. Consider a map between two manifolds and a corresponding energy that measures a certain geometric difference between the two spaces. Finding a minimizer of this energy can help to compare geometric structures on these manifolds. In [44] the authors suggested to study the harmonic map heat flow to find such minimizers. This flow, which we give and introduce explicitly in Section 1.2, is the equation under consideration in Chapter 4.

Inspired by the harmonic map heat flow of [44], Hamilton suggested in [62] to study the so-called Ricci flow. Just as the harmonic map heat flow evolves maps between manifolds towards “more interesting maps” (minimizers), the Ricci flow takes a metric on an immersion and evolves it towards a “better metric”. The idea of Hamilton was to prove the Poincaré conjecture, which states that every simply-connected closed 3-manifold is homeomorphic to the 3-sphere. Hamilton introduced this equation as a variant of the gradient flow corresponding to the energy that measures the total scalar curvature.
over the manifold. Later Perelman showed that the Ricci flow is a gradient-like flow. Equations such as the harmonic map heat flow and the Ricci flow, which minimize a certain geometric structure or are driven otherwise by a geometric structure, are called geometric flows. While the harmonic map heat flow is used to compare the geometries of two manifolds, by flowing the maps between these manifolds, the Ricci flow is used to study the geometry of one manifold by flowing its metric. A geometric flow, such as the Ricci flow, that evolves the metric of a manifold is called an intrinsic geometric flow.

Other examples of geometric flows are the mean curvature flow, the inverse mean curvature flow and the Willmore flow. These equations evolve, driven by a curvature, immersions of a manifold into another manifold. This means that not only the intrinsic geometry of the immersion is changed but also its extrinsic geometry. These flows are called extrinsic flows.

All these geometric flows are nonlinear partial differential equations and can exhibit singularities in their evolution. Understanding the structure of these singularities can give crucial information on the topology or geometry of the problem at hand. The solution to the Poincaré conjecture is an example of this strategy.

For more on geometric analysis or the role of geometry in PDEs we refer to [lll] and [mn].

1.0.2 Singularities in nonlinear PDEs

It is not only in geometric analysis that singularities play an important role. In many natural systems some form of breakdown or blowup occurs, which can be described with singularity formation, see for instance [45] and references therein for the role of singularities in natural sciences and [75] for the role of singularities in nonlinear PDEs.

On the other hand, it is also important to know whether an equation can develop singularities, or not, in the case that one does not expect any blowup behavior. Typical for nonlinear differential equations is the possibility of blowup, even if one starts from smooth initial data. For singularities in PDEs we refer to the overview articles [52], [75] and [23]. Whereas [52] is an exposition of singularities and their relating problems in different nonlinear equations, [75] and [23] are more general overviews putting the role of singularities in nonlinear PDEs in their context.

A simple example of a nonlinear ODE exhibiting blowup behavior is

\[ u_t = u^2, \quad \text{with} \quad u(t = 0) > 0. \] (1.1)

Since the solution of (1.1) is given by

\[ u = \frac{1}{T - t}, \quad \text{with} \quad T := \frac{1}{u(t = 0)}, \] (1.2)

we see immediately that the function \( u \) is well-defined on the time interval \([0, T)\). At the finite time \( T \), however, the function has reached infinity and we say that the solution has blown up.

A typical example of a PDE that exhibits singularity formation is

\[ u_t = \Delta u + u^p, \] (1.3)

with \( u \) a function from \( \mathbb{R}^n \) to \( \mathbb{R} \) that decays at infinity, with non-negative initial data. It is known (see [49], [87] and references therein) that for exponent \( p \) greater than 1 there
are basically two scenarios. If $1 < p \leq 1 + \frac{2}{n}$ the solution blows up, unless $u \equiv 0$. If $p > 1 + \frac{2}{n}$ the solution may blow up, depending on the size of the initial condition.

Concerning blowup in differential equations there are several standard questions one could ask (see also [11] and [52]).

1. Does blowup occur?
The first step in answering this question is to agree on what to call blowup. The concept of blowup is directly related to the choice of function space for the solutions of the equation. We say that a solution blows up if we can not continue the solution in the chosen function space. In this thesis we only consider function spaces of smooth functions and classical solutions. A function going to infinity (at a particular point) is an example of blowup, but one could also think of a (higher order) derivative going to infinity.

If one has determined whether blowup can occur the follow-up question is if it has to occur. Consider equation (1.3) for example. It is known that for $1 < p \leq 1 + \frac{2}{n}$ a non-trivial solution always blows up. If $p > 1 + \frac{2}{n}$, the equation can certainly create a singularity, but this is dependent on the initial conditions. If the initial condition is small, blowup will not take place. For large enough initial conditions the solution has to blow up. The exact conditions for the solution to blow up can be of importance. See for instance [49], [87] and references therein.

2. When does the solution blow up?
If one has determined that the solution blows up, the next question is the time of blowup. The examples (1.1) and (1.3) are equations where the blowup occurs in finite time, but there exist also equations where this happens in infinite time. In Chapter 4 we investigate the blowup behavior of solutions that blow up in infinite time. In Chapters 2 and 3 we are concerned with the question whether a particular blowup scenario happens in finite or infinite time.

3. Where does blowup occur?
The regions in the domain where the solution blows up can be isolated points, intervals or the whole domain itself. In the case of $p > 1$ and positive radial solutions, equation (1.3) blows up in the origin, for any dimension (see [49]). See [52] and reference therein for more on the different blowup patterns of (1.3). In this thesis we are only concerned with blowup scenarios in single points.

4. How does the solution blow up?
This concerns the asymptotic behavior of the solution, such as the blowup rate and the blowup profile, as $t$ approaches the blowup time. The blowup rate is the speed at which the function (or any of its derivatives) diverges, as $t$ approaches the blowup time, and $x$ approaches the blowup point. The limit profile is the function at blowup time. Although the function at blowup time does not live in our chosen function space, we can look at the limit process of the solution towards the limit profile in the regular parts.

Consider equation (1.3). This equation can exhibit different kinds of blowup scenarios where the blowup rate is different. In [53] it is shown, using the scale invariance of the equation, that the blowup is self-similar whenever $1 < p < \frac{n+2}{n-2}$ or $n \leq 2$. 

This means that for $u$, the solution to (1.3), we have

$$u(a, t) \sim (T - t)^{-1 + p}, \quad \text{for } t \to T,$$

and $u$ a blowup point. See [53] and references therein for the precise asymptotics.

In [64] a scenario of quasi-stationary blowup is given for equation (1.5). That is, the function $u$ at the blowup point $a$ goes faster to infinity than the self-similar rate $(T - t)^{-1 + p}$. See also [88] and references therein.

In the next Subsection we discuss scale invariance in a PDE and the two different blowup behaviors (self-similar and quasi-stationary) using another example. In this thesis, both equations under consideration are scale invariant and have quasi-stationary singularities. In the case of finite time blowup in geometric flows the rate of blowup of the singularities is of importance if one wants to continue the flow after blowup. This leads to the following question.

How does one continue after blowup?

To gain geometric or topological information one often needs to know how to continue the flow after finite time blowup. This is one of the major issues in the proof of the Poincaré conjecture through the Ricci flow. See, for instance, the standard overviews [29], [76], [92] and references therein or the more PDE oriented overview [104].

Before one can ask this question we need to know if continuation is even possible. In the case of (1.3), for instance, we have that for certain $p > 1$, radial solutions and non-negative initial condition, the solution after blowup has to equal infinity.

In this case we say that the blowup is complete and there is no way to continue the flow after blowup time.

On the other hand there exist also cases where the blowup can be continued with classical solutions for $t > T$, with $T$ the blowup time. We refer to [52] and references therein. In this thesis we do not study this question for the flows under considerations.

In these investigations we are mainly concerned with questions 2 and 4. In Chapters 2 and 3 we investigate whether the Willmore flow can blow up in finite time and, if so, how fast. In Chapter 4 we prove the precise asymptotics of the infinite time blowup rate of the harmonic map heat flow in some particular scenarios.

**1.0.3 Example**

To introduce some terminology we briefly discuss an example of a geometric flow on a curve that exhibits different kinds of blowup behavior.

Consider a closed curve $\gamma$ immersed in $\mathbb{R}^2$. Hence, the curve can have self-intersections as long as its derivative is well-defined. The length $L$ of this curve is given by

$$L = \int_{\gamma} d\mu_\gamma,$$

with $d\mu_\gamma$ the volume form on $\gamma$. Let $\gamma_t$ be a family of curves parameterized by $t$ such that the displacement of the curves is given by $\partial_t \gamma_t = VN$, with $N$ the normal on the curve
and $V$ some unknown function on the curve. The first variation of the integral (1.5) is given by
\[ \frac{\partial}{\partial t} L = - \int_{\gamma_t} V k d\mu, \] (1.6)
with $k$ the curvature of the curve (see for instance [50] or Section 2.2). We see that the choice $V = k$ monotonically decreases the length. The $L^2$-gradient flow corresponding to the length integral (1.5) is given by
\[ \partial_t \gamma_t = k N. \] (1.7)
This evolution is the so-called curve shortening flow. It is the one-dimensional version of the mean curvature flow of an isometric immersion from one manifold to another. Let the immersed curve be (locally) convex in each point. Hence, the curvature on the curve is either non-positive or non-negative. If the curve is convex we can rewrite the equation to
\[ k_t = k^2 (k_{\text{app}} + k), \quad \text{for } \theta \in [0, 2\pi], \] (1.8)
with periodic boundary conditions on the curvature $k$ and with $m$ the rotation index of the curve (see for instance [50] or [5]). This equation is an example of an extrinsic geometric flow as its evolution is driven by its curvature and changes the immersion. In this Subsection we discuss some known results on this flow to explain the concepts of self-similarity and quasi-stationary solutions.

One particular solution of (1.7) and (1.8) is the circle, whose radius $R$ evolves as $R = \sqrt{2(T - t)}$. This solution is a shrinking circle that vanishes into a point. As the curvature of a circle is given by the inverse of its radius, we see that the curvature blows up.

Consider equation (1.7). Every closed embedded curve in a plane, that solves (1.7), eventually becomes convex and ultimately shrinks into a point ([50] and [54]). While the closed curve shrinks to a point it approaches a circle, meaning that the ratio of the minimal curvature on the curve with the maximal curvature approaches 1. As the curve shrinks to a point, the maximal curvature, which can be seen as the curvature of the curve in the limit, tends to infinity. The evolution of such a curve equals, asymptotically, the evolution of the shrinking circle. Hence, the blowup rate of the curvature is of order $(T - t)^{-1/2}$.

Consider a convex curve (not necessarily embedded) that creates a singularity in finite time $T$. Let this singularity be created by the curvature $k$ going to infinity somewhere on the curve. Since the curve is convex we can use equation (1.8) as the curve shortening flow. Consider the rescaling of the curvature $k(\theta, t) = \frac{1}{\mu} \kappa(\theta, t)$, such that $\kappa$ stays bounded as $t \to T$. Hence, $\mu$ is some function going to zero as $t \to T$. If we substitute the rescaled curvature into (1.8), we have
\[ \mu^2 \kappa_t - \mu \kappa_{\mu} = \kappa^2 (k_{\text{app}} + \kappa). \] (1.9)
Consider the case that $\kappa_t = 0$ and $\mu = (T - t)^{1/2}$. This scale for $\mu$ is the unique scale for which $\mu \kappa$ is constant. Then (1.9) simplifies to
\[ \kappa_{\text{app}} + \kappa - \frac{1}{2} \kappa^{-1} = 0. \] (1.10)
Every curve with separable curvature $k$ given by $k(\theta, t) = (T - t)^{-1/2} \kappa(\theta)$ and $\kappa$ obeying (1.10) is a curve that evolves according to the curve shortening flow and vanishes into a point without changing its shape. The circle, with $\kappa = 1/\sqrt{2}$, is an example of such a curve (see [1] for a classification of all solutions of (1.10)). Because the curve does not change its shape, the rate given by $\mu = \sqrt{T - t}$ is called the self-similar rate and a solution $k(\theta, t) = (T - t)^{-1/2} \kappa(\theta)$ with $\kappa$ obeying (1.10) is called a self-similar solution.

In the case that the solution is not separable and $\kappa \neq 0$, but stays bounded, we can approximate equation (1.9), in the limit $t \to T$, with

$$-\mu \kappa \kappa = \kappa^2 (\kappa \kappa + \kappa).$$

Whenever $\mu \sim \sqrt{T - t}$, we still say that the solution blows up with a self-similar rate. This is the case in the examples of closed embedded curves that, asymptotically, evolve as the circle into a point. Consider now the case that $\mu$ goes faster to zero than in the self-similar case, such that $\mu \mu \to 0$ in the limit $t \to T$. Equation (1.9) can then be approximated with

$$\kappa^2 (\kappa \kappa + \kappa) = 0,$$

which is the time-independent part of equation (1.9). Just as in the self-similar case, a solution of (1.12) also represents a curve that does not change its shape and vanishes into a point. But because $\mu$ is smaller than the self-similar rate and equation (1.9) becomes stationary in the limit $t \to T$, we call this solution a quasi-stationary solution.

In [5] it is proven that there exists quasi-stationary blowup in the curve shortening flow. Consider the curve in Figure 1.1, which is the so-called Limaçon of Pascal. In [5] it is shown that the inner loop shrinks faster to a point than the outer loop. Hence, a singularity is created before the curve shrinks to a point, see Figure 1.1. The rate at which this singularity is created is faster than the self-similar rate in the case of disappearing circles. Hence, the blowup is quasi-stationary. The blowups we are concerned with in these investigations are all examples of singularities with quasi-stationary rates.
1.1 The equations

In this Section we discuss the two geometric flows studied in this thesis: the harmonic map heat flow and the Willmore flow.

The harmonic map heat flow

Consider two manifolds \( M \) and \( M' \) with their respective metrics given by \( g \) and \( g' \). For any map \( f: M \rightarrow M' \) we define its energy \( E \) as

\[
E[f] = \frac{1}{2} \int_M g^{ij} g'_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \mu_M,
\]

with \( \mu_M \) the volume element of \( M \). This energy represents the tension on \( M \) when restricted to \( M' \) through \( f \) and is a generalization of the energy of a curve. The minimizers of this energy, which are called harmonic maps, are of interest in matching geometric structures of different manifolds (see [44] and [112]). A natural question to ask is whether a particular map \( f \) from \( M \) to \( M' \) is homotopic to such a harmonic map.

Eells and Sampson suggested in [44] to study the \( L^2 \)-gradient flow of this energy given by

\[
\partial_t f^\alpha = \Delta_M f^\alpha + g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} \Gamma^\nu_{\alpha\beta} f^\nu,
\]

with \( \Delta_M \) the Laplace-Beltrami operator on \( M \) and \( \Gamma^\nu_{\alpha\beta} \) the Christoffel symbols on \( M' \) (see Subsections 2.2.1 and 4.1.1). The flow given by (1.14) is called the harmonic map heat flow and was one of the first geometric flows studied. To find a harmonic map homotopic to \( f_0 \), we use \( f_0 \) as an initial condition on (1.14). In [44] the authors proved the short-time existence of a solution to (1.14). If the solution is global in time and converges to some map \( \bar{f} \), then \( \bar{f} \) is a harmonic map homotopic to \( f_0 \). If this is not the case, the flow eventually creates a singularity. The subject of Chapter 4 is about the behavior of such infinite time singularities.

There are many ways to construct singularities, finite or infinite time, in the harmonic map heat flow. In Chapter 4 we study the infinite time blowup of the harmonic map heat flow from the disk to the sphere in a specific scenario.

Consider a map from the disk to the sphere given by

\[
u_t : \left( r \cos \theta \right) \mapsto \left( \cos \theta \sin \phi(r, t) \sin \theta \sin \phi(r, t) \cos \phi(r, t) \cos \phi(r, t) \right).
\]

The harmonic map heat flow of this map can be given in terms of the angle \( \phi(r, t) \) by

\[
\phi_t = \phi_{rr} + \frac{1}{r} \phi_r - r^2 \sin(2\phi) \frac{\sin(2\phi)}{2r^2}.
\]

As boundary conditions we choose

\[
\phi(0, t) = 0 \quad \text{and} \quad \phi(1, t) = \phi_0.
\]

This equation comes up in liquid crystals and ferromagnetism (see for instance [68] and [60]). Equation (1.16) is a variation of the type of equations that were used to first
show finite time singularities for the harmonic map heat flow (see [37]). For equation (1.16), with \( n = 1 \), explicit finite and infinite time blowup scenarios for the harmonic map between the disk and the sphere were given (see [31] and [32]).

Consider the harmonic map heat flow given by (1.16). The stationary solutions (harmonic maps) of this equation are given by

\[
\begin{align*}
    f &= m\pi + 2\arctan(q^n) \\
    f &= (\frac{1}{2} + m)\pi, \quad \text{with} \quad m \in \mathbb{N}, q \in \mathbb{R}.
\end{align*}
\]

(1.18)

We see immediately that none of these harmonic maps can fulfill the boundary conditions given by (1.17), if \( \phi \geq \pi \). Hence, a singularity has to be created. In Figure 1.2 we show a typical blowup for the harmonic map heat flow (1.16), with \( n = 1 \) and \( \phi = \pi \). One sees that the first derivative blows up in this case. Using the formal matched asymptotics calculations from [15], it is shown in [7] that the inverse blowup rate \( R \) of this singularity is given by

\[
R(t) = e^{-(2+\omega(t))\sqrt{t}},
\]

(1.19)

where \( R \) is given by the unique function such that \( \phi(R(t), t) = \frac{\pi}{2} \). Using similar ideas as in [7] we prove in Chapter 4 the asymptotics of the blowup rate in the infinite time blowup case \( n \geq 2 \) and \( \phi \in [\pi, 2\pi) \).

The Willmore flow

The Willmore flow is an example of an extrinsic geometric flow, where the metric as well as the immersion of the space changes during the evolution. Consider the integral

\[
W = \int_M H^2 d\mu,
\]

(1.20)

where \( H \) is the mean curvature on the compact surface \( M \) immersed in \( \mathbb{R}^3 \). The corresponding \( L^2 \)-gradient flow of this integral is given by

\[
V = -\Delta H - 2H(H^2 - K),
\]

(1.21)

with \( V \) the displacement of the surface in the normal direction, \( \Delta \) the Laplace-Beltrami operator and \( K \) the Gaussian curvature on \( M \). We refer to Subsection 2.2.3 for the
1.1. The equations

![Equations Diagram]

Figure 1.3: Consider a surface given by the curve on the left rotated around the z axis. The typical evolution of this surface is represented on the right.

derivation of this flow from the energy \( (1.20) \). The integral of the Gaussian curvature over a compact surface is a constant, due to the theorem of Gauss-Bonnet. This means that the corresponding \( L^2 \)-gradient flow of

\[
W_0 = \int_M \left( H^2 - K \right) d\mu,
\]

is also given by \( (1.21) \). In the literature both \( W \) and \( W_0 \) are used as a starting point for the flow \( (1.21) \).

The integral \( (1.20) \), or versions thereof, were already discussed in the early 19th century as energies of elastic surfaces (see [24], [38] and references therein). In the early 20th century this integral was studied again by geometers, such as in [20]. In the 1960s the integral was reintroduced by T. J. Willmore who proposed to study the global geometry of a surface through \( (1.20) \). For instance, one can show (see Subsection 2.2.2) that the global minimum of \( (1.20) \) for compact surfaces is \( 4\pi \). This global minimum is attained if and only if the surface is a sphere. Willmore conjectured that, for surfaces with genus 1, the global minimum of \( (1.20) \) equals \( 2\pi^2 \). For more on this conjecture we refer to [110]. Both the integral and the corresponding flow are, by now, named after Willmore. The integral \( (1.20) \) is also used as the bending energy of bio-membranes and in surface restoration, see for instance [98] and [36].

After the introduction of the conjecture there has been a lot of attention for minimizers of the integral \( (1.20) \), which are called Willmore surfaces. The corresponding flow has gotten less attention and there is not much known about this flow. Short-time existence is proven (see [73], [108] and references therein) and long-time existence of the solutions is proven for initial data close to a stationary surface (see, for instance [99], [80] and the references mentioned in Subsection 2.2.5). There exists an example of blowup, which is shown in Figure 1.3. Just as in the case of the Limaçon under the curve shortening flow, the only way to evolve towards a stationary solution is by forming a singularity. The fact that this surface has to blow up is proven in [21]. It is however not known whether this singularity is created in finite or infinite time, although numerical computations ([90]) suggest the blowup occurs in finite time. The investigations of Chapter 2 and 3 are concerned with this question.

There are many similarities between the harmonic map heat flow on surfaces and the Willmore flow. Both flows are scale invariant. This causes the singularity formation to
be quasi-stationary. Using this scale invariance one shows that both in the Willmore flow and in the harmonic map heat flow a certain amount of energy is concentrated in the singularity. This means that the singularity creation, if any, is local and happens in a single point. This difference in scales in the solution makes it ideal to investigate these flows using formal matched asymptotics. In Chapter 2 we use these methods to study the Willmore flow and in Chapter 4 we use the formal results from [15] to prove the asymptotics of blowup in the harmonic map heat flow.

There are also noticeable differences between the harmonic map heat flow and the Willmore flow. The harmonic map heat flow is an evolution on maps between two manifolds. During this evolution the geometries of these manifolds are not changed. The Willmore flow, however, is an evolution of immersions of a manifold into an ambient space. During this evolution the immersion changes and therefore also the (intrinsic and extrinsic) geometry of the manifold changes.

Another noticeable difference is the order of the equations. The harmonic map heat flow is a second order differential equation and the Willmore flow is of fourth order. This means we can not use the standard parabolic theory, such as the maximum principle, on the Willmore flow.

1.2 The methods

In this Section we discuss briefly the methods we use in this thesis.

Matched asymptotics

Consider the algebraic equation

$$\epsilon x^2 + 2x - 1 = 0,$$  \hspace{1cm} (1.23)

with $\epsilon$ a small parameter. The explicit solution is given by

$$x_\pm = \frac{-1}{\epsilon} \pm \frac{1}{\epsilon} \sqrt{1 + \epsilon}.$$  \hspace{1cm} (1.24)

Imagine that we do not know how to explicitly solve a second order algebraic equation. One method to approximate a solution to (1.23) is to substitute in this equation a formal series

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots.$$  \hspace{1cm} (1.25)

Gathering the different powers of $\epsilon$ in the equation and solving at each order consecutively gives $x = \frac{1}{2} - \frac{1}{2} \epsilon + \frac{1}{8} \epsilon^2 + \cdots$, which is of course the expansion of $x_\pm$ around $\epsilon = 0$. To first approximation this solution is the solution of equation (1.23), with $\epsilon = 0$. This is a first order algebraic equation with only one solution. Hence, we can not expect to find both solutions (1.24) in this manner. To find the second solution we need to rescale the equation such that the quadratic term is of the same order as $2x - 1$. Consider the rescaling $x = \frac{X}{\epsilon}$. If we substitute this into equation (1.23) we find the rescaled equation

$$X^2 + 2X - \epsilon = 0.$$  \hspace{1cm} (1.26)

Using the expansion

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots,$$  \hspace{1cm} (1.27)
we find two solutions. One solution has $X_0 = 0$ and corresponds to $x_+$. The other solution is $X = -2 - \frac{1}{2} \epsilon + \frac{1}{2} \epsilon^2 + \cdots$. Hence, this last solution corresponds to $x = -\frac{2}{\epsilon} - \frac{1}{2} + \frac{1}{\epsilon^2}$, which is the expansion of $x_+$ around $\epsilon = 0$. We have found that the solution of an algebraic equation with a small parameter can be approximated by a formal expansion. If the small parameter, however, is the coefficient of the highest order, we need to rescale the problem to find all solutions. We now give an example of an ODE with a small parameter and show that the ideas of approximating a solution stay the same and rescaling is the key notion.

Consider the following toy model
\[ \epsilon u'' + 2u' - u = 0, \] (1.28)
with $\epsilon$ a small parameter. The explicit solution of (1.28) is given in terms of the roots of (1.23). Namely,
\[ u(y) = Ae^{\frac{1 + \epsilon y}{\epsilon}} + Be^{-\frac{1 + \epsilon y}{\epsilon}}. \] (1.29)
If the boundary conditions are given by $u(0) = \alpha$ and $u(1) = \beta$, the solution can be rewritten as
\[ u(y) = \frac{\beta - \alpha e^{-\frac{1 + \epsilon y}{\epsilon}}}{\sinh \left( \frac{1 + \epsilon y}{\epsilon} \right)} e^{-\frac{1}{2} \epsilon^{1/2} \sinh \left( \frac{1 + \epsilon y}{\epsilon} \right)} + \alpha e^{-\frac{1 + \epsilon y}{\epsilon}}. \] (1.30)
Imagine again the situation that we do not know this explicit solution and need to find a way to approximate it. We can try and approximate a solution by the formal series
\[ u(y) = u_0(y) + \epsilon u_1(y) + \epsilon^2 u_2(y) + \cdots. \] (1.31)
If we substitute this expansion into equation (1.28) we find for $u_0$ a first order ODE with solution
\[ u_0(y) = Ce^{\frac{1}{\epsilon} y}. \] (1.32)
As the equation of $u_0$ is a first order ODE, it only admits one boundary condition, while the problem (1.28) has two boundary conditions. Solution (1.32) can only be a good approximation of the solution of (1.28) if the boundary values happen to obey
\[ \alpha = e^{-\frac{1}{2} \epsilon} \beta + O(\epsilon). \] (1.33)
Generally, this will not be the case and we need a different method to find an expansion for the solution.

Just as in the case of the algebraic equation we rescale the variables. Let $Y = \frac{y}{\epsilon}$ and $U(Y) = u(y)$. Equation (1.28) can then be rewritten as
\[ \epsilon U'' + 2U' - U = 0. \] (1.34)
If we substitute the series expansion
\[ U(Y) = U_0(Y) + \epsilon U_1(Y) + \epsilon^2 U_2(Y) + \cdots, \] (1.35)
we find for the zeroth order term
\[ U_0(Y) = A + Be^{-2Y}. \] (1.36)
1. Introduction

Figure 1.4: In the left figure the solid black line represents the solution (1.30), with $\alpha = 0, \beta = 1$ and $\epsilon = 0.1$. The dashed lines represent the first four approximations up to $U_0, U_1, U_2$ and $U_3$. The darker the gray the more terms we include and the more accurate the approximation. In the right figure we show the difference between the approximations $(U_1, U_2$ and $U_3)$ and the explicit solution. The dotted line represents the zeroth order term of the expansion found through matched asymptotics.

This solution does admit two boundary conditions and we can approximate $u(y)$ by $U_0(y)$. Gathering the different powers of $\epsilon$ gives the other terms of the expansion. The more terms we include the better the approximation. See Figure 1.4.

The approach of the matched asymptotics method is slightly different from the direct calculations of the expansion we just did. Note that both functions $u_0$ and $U_0$ are good local approximations, but in different areas of the domain. The main idea is that the global approximation of the solution can then be given by a combination of these local approximations, through a matching procedure. Consider the function $u_0$. Inspection of (1.28) shows that $u_0$ is a good approximation if $y$ and its derivatives are of order one. From Figure 1.4 we see this is the case when $y$ is of order one. Hence, we approximate $u$ around $y = 1$ by $u_0$ and we need to set $C = e^{-\frac{1}{2}}$ in (1.32) to fulfill the boundary condition. This function is called the outer solution and represents the regular part of the solution. The function $U_0$ is a good approximation of the solution $u$ if $y$ is small and the derivatives are large. This function is called the inner solution and represents the so-called boundary layer of the solution. To obey the boundary condition at $y = 0$ we need to set $B = -A$ in (1.36). The constant $A$ is still undetermined and will be found by matching. We now have two functions which are good approximations of the solution in two different, specific, regions. For the global approximation we would like that the solution $u_0$ gradually transforms into $U_0$ in an intermediate region. The matching procedure is exactly this condition. The solution $u_0$ has to have the same asymptotic behavior in the intermediate region as $U_0$.

The intermediate region, in this case, is the region where $Y$ is large and $y$ is small. The matching condition then states

$$\lim_{Y \to \infty} U_0 \sim \lim_{y \to 0} u_0.$$  \hspace{1cm} (1.37)

This gives $A = e^{-\frac{1}{2}}$. The global approximation of the solution is given by summing the inner and outer solution and subtracting their common part, that is,

$$u(y) \approx e^{-\frac{1}{2}} \left( e^{\frac{1}{2}y} - e^{-\frac{1}{2}y} \right).$$  \hspace{1cm} (1.38)
1.2. The methods

From Figure 1.4 we see that this function approximates the solution quite well, certainly when compared to the other method.

There are a lot of matters concerning matched asymptotics we did not mention in this example. For instance, it is not always obvious where the boundary layers lie and it is certainly not always possible to find a approximation by direct expansion in the right variables. Often one really needs to examine the equation in different variables and match the different solutions. Matched asymptotics can also be used in PDEs, as is done in Chapter 2. For more on matched asymptotics we refer to [74] and [67].

In Chapter 2 we use the matched asymptotics method to get information on the blowup behavior of the Willmore flow. In this PDE it is not a small parameter but the creation of a singularity that forces us to look at different scales at the problem. We refer to [9], [10], [65] and [15] for some similar investigations of a geometric flow through formal matched asymptotics.

Moving mesh methods

In Chapter 3 we investigate a blowup scenario of the Willmore flow numerically, using moving mesh methods. In this Subsection we give a quick introduction to this method by example. For more on these methods we refer to [72] and [71]. One could also consult the more general [26] and references therein.

Consider the harmonic map heat flow (1.16) in the case \( n = 2 \) and \( \bar{\phi} = \pi \) such that the initial condition \(|\phi_0| < \pi|\) on the interval \([0,1]\). From theory one knows that, due to the boundary conditions, the flow has to create a singularity and that this singularity is created in infinite time (see Subsection 4.1.3). It is further known that the singularity consists of the second derivative blowing up at \( r = 0 \). In Chapter 4 we prove that the asymptotics of the inverse blowup rate \( R \) is given by

\[
R \sim \left(\frac{32}{\pi^2}\right)^{-\frac{1}{2}}, \quad \text{for large } t, \tag{1.39}
\]

where \( R(t) \) is defined as the function such that \( \phi(R(t), t) = \frac{\pi}{2} \) holds, for the solution \( \phi \).

In Figure 1.5 we show the numerics (with a rigid mesh) of this evolution, up to time \( t = 50 \). We see that after a certain time the inverse blowup rate, given in Figure 1.5, drops suddenly and continues as a constant. This is obviously not what we want. The reason for the sudden failure of the numerics is that there are not enough points near \( r = 0 \), where the singularity is created. With this lack of points near the origin, where the second derivative increases, the accuracy of the method is lost. One method to attack this problem of blowup in numerics is by adapting the mesh. One could for instance add points to the mesh where the singularity is created. One can also try, instead of adding points, to move the grid points towards the regions where they are needed most.

A combination of these methods is also possible.

In this thesis we have chosen to use an adaptive mesh method that moves the grid points in time along with the solution, letting the number of points be constant. The method we use is called a moving mesh method (see [72] and [71]). The main idea of this method is to equidistribute a well-chosen, so-called monitor function, along the solution. We refer to the aforementioned articles and to Subsection 3.1.1 for more on the moving mesh method. For now, we try to explain the ideas of this method using the harmonic map heat flow as an example.
1. Introduction

Figure 1.5: Left: the evolution of the harmonic map heat flow for $n = 2$, calculated with a fixed mesh. The initial condition is given by the dashed curve and the evolution is from the lightest gray to the darker grays. Right: the inverse blowup rate of this evolution.

We saw that the numerics fail because of a lack of points in the singular part of the solution. The idea is to move some points of the grid towards the region that blows up. Consider the second derivative along the interval $[r, s]$ in the evolution of Figure 1.5. We see that, at the origin, the curvature of the function increases rapidly, while the curvature is small at the other end of the interval, near $r = 1$. If we could manage that the points on the grid move in such a way that the amount of curvature between two consecutive points stays the same, the points should move towards the singularity. Hence, the curvature could serve as a monitor function. Several differential equations have been introduced which equidistribute a function along a solution (see [72] and [71]). The moving mesh method consists of solving the original system of differential equations together with the differential equation moving the grid points.

Using a moving mesh method with a monitor function $M_n$ such that $M_n$ equals the second derivative near $r = 0$ and decreases monotonically on the interval $[0, 1]$, we find the results presented in Figure 1.6. Eventually this method also breaks down, but we are able to evaluate the evolution further, until at least $t = 200$.

Figure 1.7 illustrates the main idea of the moving mesh method. In this figure we show the $r$-coordinate of the first 50 grid points (counting from $r = 0$), divided by the inverse blowup rate $R_n$ against time. In the case of the rigid mesh we see that these values blow up. For the moving mesh method we see that the values stay constant. This means that, in the case of the moving mesh method, the grid points move towards the singularity exactly at the rate of the blowup. This ensures that we can continue the computations for a longer time.

The maximum principle

In Chapter 4 we make use of the maximum principle. This method pertains to the standard theory of parabolic second order PDEs and we refer to [95] for a thorough introduction to this subject. In [19] and in [57] a maximum principle is explicitly proved for the harmonic map heat flow (1.16) of degree $n$. In Chapter 4 we use this maximum principle to prove the asymptotics of singularities in this flow.
1.2. The methods

Figure 1.6: Left: the evolution of the harmonic map heat flow for $n = 2$, with a moving mesh procedure. The initial condition is given by the dashed curve and the evolution is from the lightest gray to the darker grays. Right: the inverse blowup rate of this evolution.

Figure 1.7: In these figures we show the $r$-coordinate of the first 50 grid points, divided by the inverse blowup rate $R$, against time. Left for the calculations with a rigid mesh and right for the moving mesh method.
1. Introduction

As an example of the maximum principle for elliptic equations we consider the Laplace equation on a two-dimensional (open and connected) domain \( D \), given by

\[ \Delta u = 0. \]  

(1.40)

Let \( K \) be the disk of radius \( r \) centered at the point \((x_0, y_0)\). Then

\[ \int_{K_r} \Delta u \, d\mu = r \int_0^{2\pi} \partial_r \bar{u} \, d\theta, \]

(1.41)

with \( r \) and \( \theta \) the polar coordinates on the disk \( K_r \) and \( u(x, y) = \bar{u}(r, \theta) \). If \( u \) obeys the Laplace equation (1.40) we have that equation (1.41) equals zero and therefore

\[ \int_0^R \int_0^{2\pi} \partial_r \bar{u} \, d\theta \, dr = \int_0^{2\pi} \bar{u}(R, \theta) \, d\theta - 2\pi u(x_0, y_0) = 0. \]

(1.42)

Hence, the value of \( u \) at a point \((x_0, y_0)\) in the domain \( D \) equals the mean value of \( u \) on a circle in \( D \) centered at that point \((x_0, y_0)\). Using this mean value theorem we can show a maximum principle for the Laplacian. This maximum principle does not only hold for harmonic functions obeying (1.40), but also for subsolutions of the Laplace equation. We say that a function \( u \) from \( D \) to \( \mathbb{R} \) is a subsolution of (1.40) if the function obeys

\[ -\Delta u \leq 0. \]

(1.43)

Let \( u \) be a subsolution of the Laplace equation and let \( u \) have a maximum \( M \) in a point \((x_0, y_0)\) in the domain \( D \). By similar arguments as above one can show that

\[ u(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \bar{u}(R, \theta) \, d\theta. \]

(1.44)

Hence, the value of \( u \) in the point \((x_0, y_0)\) is lower or equal to the mean value of \( u \) on a circle in \( D \) around \((x_0, y_0)\). We can therefore conclude that \( u \) has to be \( M \) on every such circle. By continuity of \( u \) in \( D \) we conclude that \( u \) has to be constant \( M \) in \( D \). We have shown the maximum principle for subsolutions of the Laplace equation which states that:

If \(-\Delta u \leq 0 \) and \( u \) attains its maximum \( M \) in \( D \), then \( u \equiv M \) on the whole of \( D \).

This result can be generalized to any second order linear elliptic differential equation.

Consider now the heat equation on a 2-dimensional domain \( D \), given by

\[ u_t = \Delta u, \]

(1.45)

with given initial condition on \( t = 0 \) and boundary conditions for \( t \geq 0 \). A solution \( u \) of the heat flow (1.45) is defined on the space \( E := D \times (0, T) \). We define the sides \( S \) of \( E \) as

\[ S := \partial D \times (0, 1), \]

(1.46)

with \( \partial D \) the boundary of the set \( D \). Let us further define the subsets \( E_s \) as

\[ E_s := \{ (x, t) \in E | t = s \}. \]

(1.47)

For the heat equation we have a maximum principle similar to the maximum principle for the Laplace equation.
1.3. Results

Theorem 1.2.1. For every function $u$ that obeys
\[ \partial_t u \leq \Delta u, \quad \text{on } E, \tag{1.48} \]
we have that its maximum on the closure $\overline{E}$ must be attained on $E_0$ or on $S$.

A function obeying (1.48) is called a subsolution of the heat equation. A function obeying the opposite inequality is called a supersolution. Consider a subsolution $u$ and a supersolution $v$ of the heat equation. Let $u \leq v$ on $E_0 \cup S$. The maximum principle as stated in Theorem 1.2.1 then tells us that $u \leq v$ on the interior of $E$. This is called the comparison principle. The maximum principle and comparison principle can be generalized to other linear (and even certain nonlinear) parabolic equations. We refer to [yu] for more on this.

The idea of Chapter 4 is to construct sub- and supersolutions that determine the qualitative behavior of the harmonic map heat flow. If we can find a subsolution which lies under the solution $u$ of the flow and blows up, we can immediately conclude that the solution has to blow up as well. If the subsolution blows up in finite time, the solution has to blow up in finite time as well. If the subsolution blows up in infinite time (as is the case in Chapter 4), we do not gain extra information. If we can construct a global (in time) supersolution which lies above the same $u$, we can conclude that the solution is global and must therefore blow up in infinite time. This principle is used to determine the behavior of the harmonic map heat flow in many specific blowup scenarios (see, for instance, [37], [31], [32], [55] and [56] and the later works [106], [17], [69], [7], [19] and [57]).

In Chapter 4 we prove the precise asymptotics of the blowup behavior of the harmonic map heat flow (1.16) in the case that $n \geq 2$ and $\phi \in [\pi, 2\pi)$. To do that we construct sub- and supersolutions that blow up with some specific rates, formally computed in [15]. Just as in [7], we need to rescale the sub- and supersolutions to actually put them under and above the solution, respectively. This rescaling makes use of the fact that the harmonic map heat flow is scale invariant.

In Chapter 4 we make also use of Sturmian theory, which follows from the maximum principle. We use this theory to make sure that the radius of the singularity (see Subsection 4.1.1) is well-defined. For Sturmian theory we refer to [6], [31] and references therein.

1.3 Results

We give a brief overview of our investigations and results in this thesis.

Chapter 2

In Chapter 2 we consider a Limaçon-like curve rotated around the $z$-axis, see Figure 1.3. The Willmore flow on this surface will create a singularity (see [21]). It is, however, unclear whether this surface blows up in finite or infinite time. Numerical calculations (see [90]) suggest that this happens in finite time. Besides this numerical suggestion there is no other result on finite time blowup of the Willmore flow.

In this Chapter we investigate the blowup behavior of this Limaçon using formal matched asymptotics. The idea is that these calculations, although they are merely
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formal, can add weight to the numerical evidence. These calculations give the asymptotic blowup rate of the blowup and can, in this way, contribute to the definitive answer on whether the Willmore flow can create a finite time singularity or not.

The idea of formal matched asymptotics is to divide the problem into different regions where different scales are predominant. As in these regions the scales are different, the approximations of the solutions are also different. The next step is to match the different approximations to each other. Through this matching we find the blowup rate. Our hypothesis was that the matching conditions would give a clear distinction between blowup in the self-intersecting (Limaçon) case and no blowup in the non self-intersecting case. The conditions we find, however, do not seem to agree with this hypothesis. We find a blowup scenario without self-intersection.

Assuming finite time blowup, we find that the curve in the neighborhood of the singularity looks like

\[ r \sim \lambda \cosh\left(\frac{z}{\lambda}\right), \quad \text{with} \quad \lambda \sim a_c(T-t)^{\frac{1}{2}} \ln\left(\frac{T-t}{T}\right)^{-\frac{1}{4}}, \quad \text{for} \quad t \to T, \quad (1.49) \]

for some positive constant \(a_c\) and with \(T\) the time of blowup. Hence, we find quasistationary finite time blowup. Matching gives us, however, also that the curve does not intersect itself. Hence, this is not the blowup scenario of the Limaçon. In Subsection 1.4 we give a possible way to interpret these results and we give a possible strategy to find the asymptotics of the blowup of the Limaçon.

We refer to [15] for a similar investigation of the blowup behavior, using formal matched asymptotics, of the harmonic map heat flow. In both equations we need three scales to catch the precise asymptotics of the problem and in both equations we use the scale invariance of the problem. A different feature of the Willmore flow in comparison to the harmonic map heat flow, however, is that the rescaling is necessary not only in the domain but also in the target.

Chapter 3

In Chapter 3 we calculate the Willmore flow on different initial surfaces, using a moving mesh method. These investigations were started to validate the finite time blowup results in the Willmore flow of [90] and to extract more information that could be useful for the matched asymptotics of Chapter 2. We find similar results as in [90]. That is, finite time blowup in the case of a Limaçon-like figure. We further see that all dumbbells seem to evolve to spheres, but we believe more experiments are appropriate for these initial conditions.

In Chapter 2 we make some assumptions on the evolution of the Willmore flow, which help us to approximate the solution. In Chapter 3 we give numerical results that strengthen these assumptions. Besides this numerical evidence we also find a possible aid in solving the problem of the blowup behavior of the Limaçon. We find that the inner loop of the Limaçon vanishes on a quasi-stationary scale. The numerics further show that every Limaçon-like surface, no matter how far from Figure 1.3, eventually flows in a similar way towards the singularity. That is, every such Limaçon-like surface looks, asymptotically, like two spheres.
1.4. Final considerations

Chapter 4

In this Chapter we discuss the blowup behavior of the harmonic map heat flow of a function from the disk to a sphere, in a specific scenario. Let this map, from the disk to the sphere, be such that it is equivariant under rotations of the disk and spherically symmetric. The resulting equation is \( (1.16) \) and the boundary conditions are given by \( (1.17) \).

Using the results from \([15]\) we are able to construct super- and subsolutions of the equation that blow up with a specific rate given by

\[
\ln(R) \sim -\frac{8}{9} (\alpha_0 t)^{\frac{1}{2}}, \quad \text{for} \quad n = 2, \quad \phi \in (\pi, 2\pi) \quad \text{and large} \quad t,
\]

\[
R \sim \left( \frac{(n-2)\alpha_0}{E_n} \right)^{\frac{1}{n-2}}, \quad \text{for} \quad n > 2, \quad \phi \in (\pi, 2\pi) \quad \text{and large} \quad t,
\]

\[
R \sim \left( \frac{4(n-1)\alpha_0}{E_n} t \right)^{\frac{1}{n-2}}, \quad \text{for} \quad n \geq 2, \quad \phi = \pi \quad \text{and large} \quad t,
\]

where \( E_n = \frac{n}{n-2} \) and \( \alpha_0 = 2 \tan(\frac{\pi}{2n}) \).

With these sub- and supersolutions we prove that any solution, with initial condition \( \phi_0 := \phi(\cdot, 0) \) obeying

\[
i. \quad C_1 \leq \lim_{t \to 0^+} \frac{\phi_0}{r^n} \leq C_2, \quad \text{for some} \quad C_1, C_2 \in \mathbb{R},
\]

\[
ii. \quad \phi_0^{-1} \left( \frac{n}{2} \right) \text{ is unique},
\]

\[
iii. \quad |\phi_0| < 2\pi,
\]

has to blow up with a blowup rate given by \( (1.50) \). This result proves the formal results of \([15]\).

We refer to \([7]\) where similar results have been given for the infinite time blowup rate of the harmonic map heat flow \( (1.16) \) with boundary conditions \( \phi(0, t) = 0 \) and \( \phi(1, t) = \pi \) and with \( n = 1 \). In this reference the authors also use the formal results of \([15]\) to build sub- and supersolutions. The main difference between our work and \([7]\) is the amount of scales needed to build the sub- and supersolutions. In \([7]\) it was sufficient to use only the inner scale. In our case we need to include the first term of the outer solution, which means we construct sub- and supersolutions with two scales.

1.4 Final considerations

In this Section we mention and discuss some loose ends of the investigations done in the main part of this thesis.

Chapter 2

The results of Chapter 2 suggest that we need a different strategy, when using matched asymptotics, to describe the singularity formation of a Limaçon under the Willmore flow. From theory and numerical simulations we know that the inner solution, given in \((2.146)\) and \((2.149)\), is a good approximation of the Limaçon in the tip. In fact, \((2.146)\) and \((2.149)\)
hold no more information than that the mean curvature goes to zero as the singularity is approached. The rate $\lambda$ can still be anything, although theory states that it has to be quasi-stationary. Matching with the right approximation of the solution in the outer region should give the blowup rate $\lambda$.

The numerics in Chapter 3 suggest that the self-intersection of the surface is quasi-stationary. Hence, if this is true, the whole loop vanishes on a quasi-stationary rate and could be considered as the inner solution. As the inner solution is a linearization around $H=0$, this would mean that the mean curvature on the loop vanishes. On the other hand, the numerics also suggest that the curvatures, near the intersection, are of order one and

$$\kappa_1 - \kappa_2 \to 0, \quad \text{as} \quad t \to T. \quad (1.52)$$

This would mean that the mean curvature does not vanish on the whole loop. It seems there is a possibility we did not take into account in Chapter 2, that the region near the intersection behaves quasi-stationary but is not given by the inner solution. This would mean we have to introduce a second quasi-stationary state in the matched asymptotics. One option to do this is to perturb the self-similar scale of the outer solution with a logarithmic term such that

$$r \sim c(T-t)^{\frac{1}{4}} \ln \left( \frac{1}{T-t} \right) \gamma \quad \text{near the intersection}, \quad (1.53)$$

for some positive $c$ and $\gamma$. This gives an outer solution with a quasi-stationary rate. This outer solution could replace the outer solution of Chapter 2. We leave the details for future research.

We could also try to match the inner solution immediately to the remote solution. To do this we need an approximation of the remote solution. Hence, we need a way to simplify the Willmore flow in the regular part of the evolution. The numerics of Chapter 3 suggest that the remote solution is an perturbation of the sphere. Future investigations on the finite time blowup in the Willmore flow could be directed to finding an expression of this remote solution, starting from initial data close to a double sphere.

Chapter 3

In Chapter 3 we study the Willmore flow through a moving mesh method. We find that, in the case of a self-intersection, the surface blows up in finite time, as is already shown in [90].

We further show that the inner loop of the Limaçon-like curve vanishes on a quasi-stationary scale. It is known that finding the precise asymptotic rates of an evolution, through a moving mesh method, is a subtle venture. There are cases known where the outcome is not correct (see, for instance, [26]). As we think it to be important to know the asymptotics of the loop, we suggest to investigate the rate at which the loop vanishes further. This can be done using more refined moving mesh methods. One can think of other discretization methods such as spline collocation or other ways of time integration. Another way to handle the loss of precision, when approaching a finite time singularity, is through a Sundman transform. We refer to [27] and [26] for discussions on these various methods in the moving mesh method.

Another interesting direction to proceed investigations, is trying to find the blowup solution we constructed in Chapter 2. In this Chapter we find a blowup solution that
1.4. Final considerations

is not the Limaçon, because the solution does not self-intersect (in the vanishing parts). Can the blowup solution, found in Chapter 2, be a dumbbell? The numerics of Chapter 3 suggest it can not or, if it does, it is not a stable blowup problem. It seems worthwhile to investigate this further, with more accurate moving mesh methods.

Chapter 4

To prove the theorem on the blowup rate in Chapter 4 we put some assumptions on the initial condition of the solution (see (1.51)). It is our believe that these assumptions can be relaxed and that Theorem 4.1.1, i.e. infinite time blowup with the stated blowup rates, holds for all initial conditions, provided $\phi \in [\pi, 2\pi)$ and $n \geq 2$.

Consider the assumption (1.51a). This assumption is quite natural if one considers that the original equation is the harmonic map heat flow on a map $u$ from the disk to the sphere. Smooth $u$ implies smooth $\phi$ such that $\phi \sim r^n$ as $r \to 0$, for $n$ integer. We believe this also to be true for non-integer $n$.

To define the (inverse) blowup rate of the singularity we make use of the fact that $R\phi$ is unique, through the assumption (1.51b). For the blowup rate to be well-defined it is, however, enough that $R\phi$ eventually becomes unique in a region around $r = 0$ where the singularity is created. If one can prove this for every solution $\phi$, one can drop the assumption (1.51b). This question is closely related to the question of convergence. In Subsection 4.3.3 we show that the solution evolves uniformly towards $\pi + 2 \arctan(\frac{\pi}{2} r^n)$, away from zero. From the parabolicity of (1.16) we know that the derivatives of the solution evolve uniformly, as well, to the derivatives of the stationary solution, away from zero. One would like to have a result similar to [sv] that states that the solution is monotone in the region near $r = 0$.

Assume that the solution is global in time, independent of assumption (1.51c). We show that, in this case, the solution eventually obeys all assumptions (1.51) and Theorem 4.1.1 holds, see Figure 1.8. Consider a solution $\phi$ with maximum $M \in (k\pi, (k+1)\pi)$, with $k \geq 2$ an integer. We can construct a function $\psi$ such that $\psi \geq \phi$ with $\psi_{\mid r=0} = k\pi$ and $\psi_{\mid r=1} = (k-2)\pi + \phi$. This function evolves to $k\pi - 2 \arctan(\frac{\pi}{2} r^n)$ for $t \to \infty$, with $\beta_0 = 2 \tan(\frac{\pi}{2})$. Hence, for $t$ big enough the solution $\psi$ with initial condition $\psi_0$ is smaller than $k\pi$ on $r \in (0, 1]$. By the comparison principle this means that for $t$ big enough $\phi < k\pi$. As we assumed that the solution does not blow up in finite time, we can continue this procedure until we have found that for $t$ big enough $\phi < 2\pi$. Hence, we can drop assumption (1.51c), if the solution is global in time.

In [19] the authors remark that they can prove that the solution of (1.16) and (1.17) is global in time if assumption (1.51a) holds, without any assumption on the maximum or minimum value of $\phi$. This would mean that we could relax assumption (1.51c). We think this can be shown as follows. In [69] it is shown, in the case $n = 1$, that for finite time blowup the solution at $r = 0$ jumps from 0 to $\pi$ and there is no bubble tree. This means there is a small interval $[0, \delta]$ where the solution of (1.16) and (1.17), in the case $n = 1$, is monotone and smaller than $2\pi$. As the author of [69] mainly uses intersection theorems which also hold for larger $n$, we believe one can prove, along similar lines, the same for the solution of (1.16) and (1.17), in the case $n \geq 2$. If the result of [69] could be generalized to our case this would mean that, if we assume finite time blowup, the solution is smaller than $2\pi$ and monotone on a small interval containing zero. This in turn would mean that we are able to bound the solution from above with a supersolution.
Figure 1.8: In this figure we show the evolution of several solutions to the harmonic map heat flow $(n = 2$ and $\bar{\phi} = \pi + 1)$. The initial condition is given by the dashed curves and the evolution is from the lighter grays to the darker grays. Assume that the solution is global. We can bound any solution with maximum between $3\pi$ and $4\pi$ by a solution as in the left figure. This means, by the comparison principle, we can eventually bound the solution with a solution as in the middle figure. Hence, after a while the solution lies well below the value of $2\pi$, as in the figure right.

as constructed in Chapter 4, with a well-chosen $\alpha_0$. As such a supersolution blows up in infinite time, this contradicts the assumption that the solution blows up in finite time. Hence, if one can generalize the results of [69] to $n \geq 2$, we can drop assumption (1.51c).

One could also try and relax the assumption on the boundary condition $\bar{\phi}$ and take, for instance, $\bar{\phi} \geq 2\pi$. As the stationary solutions can only span an interval smaller than $\pi$, we know that at least two harmonic maps have to bubble off. If all blowup happens at infinite time (in the $n \geq 2$ case), we have multiple blowup. The formal results in [15] give different blowup rates in this case. We believe one can generalize the theorems of Chapter 4 to the multiple blowup case with the corresponding rates of [15]. We refer to [105] for an explicit example of multiple blowup in the harmonic map heat flow.

One would hope that the techniques, used in Chapter 4, can also be applied to the case of finite time blowup. The difficulty with the finite time blowup, however, is that for every perturbation of a solution (or sub- or supersolution) the blowup time $T$ also changes. This means that the subsolutions and supersolutions of the finite time blowup case can not lie ordered. One needs other methods to prove the precise asymptotics for the finite time blowup case. For some results in this direction we refer to [69], [8] and [18]. Recently the precise blowup rates for some solutions in the finite time blowup case (with domain $\mathbb{R}^2$) have been proved in [96]. In [16] it is shown that the finite time singularities in the harmonic map heat flow are unstable. This could give a suggestion how to continue the flow after blowup.
Chapter 2
Matched asymptotics for finite time blowup in the Willmore flow

2.1 Introduction
In this Chapter we consider the Willmore flow on surfaces given by
\[ V = -\Delta H - 2H(H^2 - K), \]
with \( V \) the displacement of the surface in the normal direction, \( H \) and \( K \) the mean and Gaussian curvatures and \( \Delta \) the Laplace-Beltrami operator (see Section 2.2). This flow is the \( L^2 \)-gradient flow of the Willmore functional (see Subsection 2.2.3)
\[ W[f] = \int_M H^2 d\mu. \]

The stationary solutions of this flow are called Willmore surfaces (see, for instance, Chapter 7 of [109]). These surfaces play a role in conformal geometry (see [110] and Chapter 3 of [66]). Besides a geometric relevance, the integral (2.2) is also used in biology where it models the classical bending energy of biomembranes. See [28] and [63] or the overview article [98].

In [70] the flow was used to find, numerically, Willmore surfaces for different genus. In [81] and [99] the research on the analysis of the Willmore flow was initiated. The question was raised whether the Willmore flow can create finite time singularities, starting from a smooth surface. This is the subject of this Chapter.

In [90] a numerical example of finite time blowup was given. In this Chapter we study this case using formal matched asymptotics. Besides giving more evidence for the finite time blowup, the formal matched asymptotics can also give specific blowup rates.

2.2 Preliminaries
In this Section we give some preliminaries on differential geometry. This is by no means intended as a full introduction to the subject but is merely written as a reminder for the reader and with the purpose to introduce the Willmore flow on surfaces of revolution. We refer to [100] for a thorough introduction to differential geometry.

In the last part of this Section we give a short overview of the work, known to us, on the Willmore flow.

2.2.1 Geometry on 2-dimensional hypersurfaces
In this Subsection we consider the geometry of hypersurfaces in \( \mathbb{R}^3 \). We distinguish between the intrinsic geometry of a surface and the extrinsic geometry. By doing so, we introduce some notions and notations in differential geometry.
Let $M$ be a two dimensional manifold that can be immersed in $\mathbb{R}^3$. Hence, there exists an $f : M \to \mathbb{R}^3$ such that the rank of $f$ is 2 for every point $p \in M$. This implies that for every local coordinate system $(\phi, U)$ of $p \in M$ we can find a local coordinate system $(\psi, V)$ of $f(p)$ such that
\[
\psi \circ f \circ \phi^{-1}(q^1, q^2) = (q^1, q^2, 0),
\] (2.3)
with $\phi(q) = (q^1, q^2)$ for $q \in M$. Hence, we can consider $f$, locally, as an embedding of $M$ in $\mathbb{R}^3$. With this in mind one can show that every smooth function $F$ on $M$ can locally be extended to a smooth function $\hat{F}$ on $\mathbb{R}^3$ and that, likewise, every smooth vector field $X$ over $M$ can be locally extended to a smooth vector field $\hat{X}$ over $\mathbb{R}^3$. That is, for every neighborhood $U \subset M$ of $p$ we can find a neighborhood $V \subset \mathbb{R}^3$ of $f(p)$ such that $f(U) \subset V$ and we can find smooth $\hat{X}$ and $X$ on $V$ such that $\hat{X}|_{f(U)} = X$ and $X|_{f(U)} = X$. For notational simplicity we do not use explicit coordinate functions but assume them implicitly.

Let us put a geometry on $M$ by taking the induced metric $g = f^*\hat{g}$ coming from the Euclidean metric $\hat{g} = \langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$. For every point $p \in M$, the tangent space $T_pfM$ can be seen as a subspace of $T_{f(p)}\mathbb{R}^3$ via the push forward $f_*$ and the local extension we discussed above. Choosing local coordinates $(s, t)$ on $M$ and $(u, v, w)$ on $\mathbb{R}^3$ such that $f(s, t) = (u(s, t), v(s, t), w(s, t))$, we define
\[
f_1 := f_*\partial_s = \begin{pmatrix} u_s \\ v_s \\ w_s \end{pmatrix} \quad \text{and} \quad f_2 := f_*\partial_t = \begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix},
\] (2.4)
where we consider $f_1$ and $f_2$ as vector fields in a neighborhood of a point $f(s, t)$ in $\mathbb{R}^3$. The set $\{f_1, f_2, N\}|_{(s, t)}$ then forms a basis for $T_{f(s,t)}\mathbb{R}^3$ with $N$ the normal on the surface $f(M)$ given by
\[
N(s, t) = \frac{f_1 \times f_2}{\|f_1 \times f_2\|}(s, t).
\] (2.5)
Note that there are two choices for the direction of the normal, plus or minus $f_1 \times f_2$. Choosing a direction fixes the orientation on the neighborhood under consideration.

With the notation $\partial_1 := \partial_s$ and $\partial_2 := \partial_t$ (the same correspondence as above) the components of the induced metric on $M$ are given by
\[
g_{ij} = \langle f_1 \partial_i, f_1 \partial_j \rangle.
\] (2.6)
This induced metric determines the intrinsic geometry of the surface $M$, that is, the geometry of the surface without taking into account how it is immersed into the ambient space $\mathbb{R}^3$. This means that, given a metric on the space $M$, we can find the corresponding immersion $f$, up to local isometries. Since, for instance, the cylinder is locally isometric to the flat plane, the intrinsic geometry of these two spaces cannot distinguish them. If we want to take into account how the surface is immersed into $\mathbb{R}^3$ we need the so-called extrinsic geometry. This extrinsic geometry is an invariant structure under translations and rotations.

To study the extrinsic geometry of $M$ we need more information on how the surface is immersed into $\mathbb{R}^3$. This information is given by the so-called scalar second fundamental form $h$ of $M$. This object gives the derivatives of the normal and its components are given by
\[
h_{ij} = \langle -\partial_i N, f_j \rangle.
\] (2.7)
2.2. Preliminaries

The (scalar) second fundamental form \( h \), together with the metric \( g \) (also called the first fundamental form) give the local extrinsic geometry of the surface \( M \) immersed in \( \mathbb{R}^3 \). This is expressed by the fundamental theorem of surface theory stated by Bonnet: if for two immersions \( f \) and \( f' \) the corresponding \( g, g' \) and \( h, h' \) are equal on a neighborhood \( U \), then \( f(U) \) and \( f'(U) \) are equal up to rotation and translation. The theorem states further that if we have an \( h \) and a positive definite \( g \) which are symmetric in their components, together with some compatibility conditions on \( h \) and \( g \), then we can reconstruct an immersion \( f \) (up to rotation and translation in \( \mathbb{R}^3 \)), such that \( h \) and \( g \) are the corresponding second fundamental form and metric (all locally). These compatibility conditions, the so-called Gauss equations and Codazzi-Mainardi equations, ensure that we can find such an immersion. In this Subsection the focus lies on these conditions and we show the necessity of these equations. By doing so we also introduce some geometric constructions and notation for later use.

The necessity of the Gauss equation and the Codazzi-Mainardi equations follows from equating the third derivatives \( \partial_{ij} g \) and \( \partial_{ij} f \). Before we do this, we need some specific expressions for the second derivatives of the immersion \( f \). Consider \( \partial_i f \), which is a vector in \( \mathbb{R}^3 \) and can be expressed in terms of the vectors \( f_1, f_2 \) and \( N \). Hence,

\[
\partial_i f = (\partial_{i3} f, f_3) g^{13} f_1 + (\partial_{i3} f, N) N,
\]

with \( g^{ij} \) the components corresponding to the inverse of the metric \( g \). We are able to express \( \partial_i f \) in terms of the metric \( g \) in the following manner:

\[
\langle \partial_i f, f_k \rangle = \frac{1}{2} \langle \partial_i f, f_k \rangle + \frac{1}{2} \langle \partial_i f, f_k \rangle
\]

\[
= \frac{1}{2} \partial_i (f_j, f_k) - \frac{1}{2} (f_i, \partial_j f_k) + \frac{1}{2} (f_i, \partial_k f_j) - \frac{1}{2} (f_i, \partial_j f_k)
\]

\[
= \frac{1}{2} \langle \partial_i (f_j, f_k) \rangle + \frac{1}{2} \langle (f_i, \partial_j f_k) \rangle - \frac{1}{2} \langle (f_i, \partial_k f_j) \rangle
\]

\[
= \frac{1}{2} \langle \partial_i g_{jk} \rangle + \frac{1}{2} \langle \partial_j g_{ik} \rangle - \frac{1}{2} \langle \partial_k g_{ij} \rangle.
\]

We see that the components of \( \partial_i f \) in the directions tangent to \( f(M) \) can be given in terms of the metric \( g \) of \( M \). These factors are, in fact, the Christoffel symbols defined as

\[
\Gamma^i_{ij} = \frac{1}{2} g^{ik} (\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}).
\]

**Remark 2.2.1.** These Christoffel symbols determine the Levi-Civita connection on \( M \). The Levi-Civita connection is the unique linear connection \( \nabla \) that is symmetric and compatible with \( g \). The relation between the Christoffel symbols and the Levi-Civita connection is given by \( \nabla_i \partial_j = \Gamma^k_{ij} \partial_k \). This connection helps to compare tangent spaces in different points of \( M \) and measures how much a vector field changes along a certain direction in \( M \). This connection can be extended to a connection, also denoted by \( \nabla \), on every tensor bundle on \( M \). The Laplace-Beltrami operator \( \Delta \) is defined as

\[
\Delta := g^{ij} \nabla_i \nabla_j.
\]

Since \( g_{ij} g^{ij} = \delta^i_i \) and by the compatibility of the connection,

\[
\partial_i g_{jk} = \Gamma^m_{ij} g_{mk} + \Gamma^m_{ik} g_{jm},
\]

with \( \partial_i g_{jk} = \delta^i_i \delta^j_j \delta^k_k \).
we have that
\[ \partial_i g^{ij} = -g^{ik} \Gamma^j_{ik} - g^{jk} \Gamma^i_{jk}. \]  
(2.13)

Jacobi’s formula for the derivative of the determinant of a matrix states that
\[ \partial_i \det g = \partial_i g_{mn} \text{adj}(g)_{nm}, \]  
(2.14)

with \( \text{adj}(g) \) the adjugate of the matrix \( g \). Using equation (2.12) and expressing \( \text{adj}(g) \) in terms of \( \det g \) and the inverse of \( g \) gives
\[ \partial_i \det g = 2 \Gamma^m_{in} \det g. \]  
(2.15)

With equations (2.13) and (2.15) one can rewrite the Laplace–Beltrami operator in the form
\[ \Delta a = \frac{1}{\sqrt{\det g}} \partial_i \left( g^{ij} \sqrt{\det g} \partial_j a \right). \]  
(2.16)

Hence, \( \Delta \) obeys the symmetry relation
\[ \int_M a \Delta b \, d\mu = \int_M \Delta a b \, d\mu, \]  
(2.17)

for \( a, b \) functions on \( M \), \( M \) without boundary and \( d\mu \) the volume form on \( M \) given by the measure \( \sqrt{\det g} \, dx \) with \( dx \) the Lebesgue measure on \( M \).

The components of \( \partial_i f \) in the normal direction are given by
\[ \langle \partial_i f, N \rangle = -\langle f, \partial_i N \rangle = h_{ji}. \]  
(2.18)

Note that, since
\[ g_{ij} = \langle f_i, f_j \rangle = \langle f_j, f_i \rangle = g_{ji}, \]  
(2.19)

the Christoffel symbols are invariant under interchanging the lower indices:
\[ \Gamma^i_{ij} = \Gamma^j_{ji}. \]  
(2.20)

Further note that the scalar second fundamental form is symmetric as well, since
\[ h_{ij} = \langle -\partial_i N, \partial_j f \rangle = \langle N, \partial_i f \rangle = \langle N, \partial_j f \rangle = \langle -\partial_j N, \partial_i f \rangle = h_{ji}. \]  
(2.21)

The equation
\[ \partial_i f = \Gamma^j_{ij} f_j + h_{ij} N \]  
(2.22)

is known as the Gauss formula. This formula shows that the covariant derivative of \( M \) with the induced metric \( g \) is exactly the covariant derivative of the ambient space \( \mathbb{R}^3 \) in the direction tangent to \( f(M) \). The covariant derivative of the ambient space orthogonal to \( f(M) \) is given by \( h \) and determines, together with the metric, the extrinsic geometry of \( M \).

After expressing the derivatives of the tangent vectors \( f_i \) in terms of the scalar second fundamental form \( h \) and the metric \( g \), we want to do the same with the derivatives of \( N \). Since the length of the normal is always 1, a derivative of \( N \) is always in the tangent direction. Hence,
\[ \partial_i N = (\partial_i N, f_j) g^{ij} f_l = -h_{ij} g^{ij} f_l, \]  
(2.23)
2.2. Preliminaries

by definition. This equation is called the Weingarten equation. The derivative of a tangent vector $T_f(x)$ can be given in terms of the metric $g$ and the scalar second fundamental form $h$ through the following equations:

$$
\partial_j f_i = \Gamma^l_{ij} f_l + h_{ij} N_i, \\
\partial_i N = -h_{ij} g^{il} f_l.
$$

(2.24)

These two equations represent in fact 15 partial differential equations for the 9 functions $f_i$ and $N^k$ (due to the equality $\partial_j f_i = \partial_i f_j$). If we can find a solution to this equation we have found a local expression for the immersion $f$ and reconstructed our surface. For the differential equations to have a solution, they must fulfill certain compatibility conditions. These compatibility conditions are given by equating the higher order derivatives $\partial_{ij} f$ and $\partial_{jik} f$. The resulting equations are called the Gauss equations and the Codazzi-Mainardi equations.

Consider the third derivative $\partial_{ij} f$. Then

$$
\partial_{ijk} f = \partial_l \Gamma^l_{ij} f_l + \Gamma^l_{ij} \partial_l f_l + \partial_l h_{ij} N + h_{lij} \partial_l N \\
= \partial_l \Gamma^l_{ij} f_l + \Gamma^l_{ij} (\Gamma^m_{kl} f_m + h_k N) + \partial_l h_{ij} N - h_{lij} h_{km} g^{il} f_m
$$

(2.25)

$$
= \left( \partial_l \Gamma^l_{ij} + \Gamma^m_{kl} \Gamma^l_{ij} - h_{lij} h_{km} g^{il} \right) f_l + \left( \partial_l h_{ij} + \Gamma^l_{ij} h_{kl} \right) N.
$$

Hence, equating $\partial_{ijk} f$ and $f_{jik}$ and collecting the linearly independent components, corresponding to $f_i$ and $N$, gives the Gauss equations

$$
\partial_l \Gamma^l_{ij} - \partial_j \Gamma^j_{il} + \Gamma^m_{kl} \Gamma^l_{ij} - \Gamma^m_{kl} \Gamma^j_{im} = h_{ijk} g^{ml} - h_{ikl} h_{jm} g^{ml},
$$

(2.26)

and the Codazzi-Mainardi equations

$$
\Gamma^l_{ijkl} - \Gamma^l_{ikjl} = \partial_k h_{ij} - \partial_j h_{ik} = 0.
$$

(2.27)

The terms on the left hand side of equation (2.26) are exactly the components of the Riemannian curvature tensor $R^i_{jkl}$ which measures how far the surface deviates from being flat. That is,

$$
R^i_{jkl} := \partial_l \Gamma^l_{ij} - \partial_j \Gamma^j_{il} + \Gamma^m_{kl} \Gamma^l_{ij} - \Gamma^m_{kl} \Gamma^j_{im}.
$$

(2.28)

**Remark 2.2.2.** This expression was found by Riemann as the term that needs to be zero if a surface is locally isometric to the plane. Let the metric $g$ of a surface be isometric to the Euclidean metric on $\mathbb{R}^3$. Hence,

$$
g_{ij} = f^* (\partial_i, \partial_j) = (f_i \partial_i, f_j \partial_j) = (f_i \cdot, f_j \cdot) = \partial f^i / \partial x^j.
$$

(2.29)

Differentiating this equation results in a differential equation comparable to (2.22), namely

$$
\partial^2 f^k / \partial x^i \partial x^j = \Gamma^l_{ij} \partial f^k / \partial x^l.
$$

(2.30)

Equating the third derivatives of $f^k$ gives the integrability condition $R^i_{jkl} = 0$, with $R^i_{jkl}$ defined as in (2.28). Hence $R^i_{jkl}$ is an invariant of the intrinsic geometry and measures the deviation of a surface from being flat.
The Gauss equation is obtained by lowering an index. Hence

\[ R_{mjk} = g_{mk} R_{jk} = h_j h_k - h_k h_j. \]  

(2.31)

Due to the symmetries of the Riemannian curvature,

\[ R_{2112} = -R_{2121} = -R_{1212} = R_{1221}, \]

(2.32)

we see that, in this context, the Gauss equation is only one equation. That is,

\[ R_{2112} = h_{11} h_{22} - h_{12} h_{21}. \]  

(2.33)

**Remark 2.2.3.** Equation (2.33) gives in fact Gauss’ Theorema Egregium. Namely

\[ K := \frac{h_{11} h_{22} - h_{12} h_{21}}{g_{11} g_{22} - g_{12} g_{21}} = \frac{R_{2112}}{g_{11} g_{22} - g_{12} g_{21}}, \]

(2.34)

where \( K \) is the Gaussian curvature, which we introduce in Subsection 2.2.2. This equation is called remarkable (egregious) because the Gaussian curvature is defined in terms of the extrinsic structure \( h \), but in fact, only depends on the intrinsic structure \( g \).

Consider now equation (2.27). This equation is invariant under interchanging \( j \) and \( k \). Hence, the remaining meaningful equations are

\[
\begin{align*}
\Gamma^j_{11} h_{22} - \Gamma^j_{12} h_{21} + \partial_2 h_{11} - \partial_1 h_{12} &= 0, \\
\Gamma^j_{21} h_{12} - \Gamma^j_{22} h_{11} + \partial_1 h_{21} - \partial_2 h_{22} &= 0,
\end{align*}
\]

(2.35)

which give the derivative of \( h \) in terms of \( h \) and \( g \). These are the so-called Codazzi-Mainardi equations.

The Gauss equation and the Codazzi-Mainardi equations, (2.26) and (2.27), are the integrability conditions of the system of differential equations given by (2.24). They follow directly from the immersion, but also define the surface (given \( g \) and \( h \)) under consideration, up to rotations and translations. This gives the extrinsic geometry of a surface.

In Subsection 2.2.3 we use (2.24) to derive the first variation of certain geometric structures and, by that, the Willmore flow. Before we do that we introduce different curvatures on 2-dimensional hypersurfaces.

**2.2.2 Curvature on 2-dimensional hypersurfaces**

In this Subsection we introduce the principal curvatures, Gaussian curvature and mean curvature on 2-dimensional hypersurfaces. Both the Gaussian and the mean curvature can be given in terms of the principal curvatures. The Willmore flow, discussed in Subsection 2.2.3, is given in terms of these curvatures.

Just as the curvature at a point of a curve is given by the radius of a circle that fits best on the curve at this point, one can define the curvature of a point on a surface by comparing it to a sphere. This is the approach Gauss took to define the curvature of a surface and this is the first curvature we discuss.
2.2. Preliminaries

Consider an immersion $f: M \to \mathbb{R}^3$. We can choose local coordinates such that $f(M)$ can locally be given as the graph of a function $w(s,t)$. Hence, $f(s,t) = (s,t,w(s,t)) \in \mathbb{R}^3$. The tangent vectors $f_1, f_2$ and the normal $N$ on $f(M)$ are, in these coordinates, given by

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad N = \frac{1}{\sqrt{1 + w_s^2 + w_t^2}} \begin{pmatrix} -w_s \\ -w_t \\ 1 \end{pmatrix}. \quad (2.36)$$

With this choice of coordinates the components of the induced metric $g$ on $M$ are

$$g_{11} = 1 + w_s^2, \quad g_{12} = w_s w_t = g_{21}, \quad g_{22} = 1 + w_t^2, \quad (2.37)$$

and $\det g = 1 + w_s^2 + w_t^2$. Denote the map that attaches to every point $f(s,t) \in M$ the normal vector $N$ as $\nu : f(M) \to \mathbb{R}^3$. The vector $N \in \mathbb{R}^3$ has by definition norm 1 and can therefore be seen as a point on the sphere $S^2$ embedded in $\mathbb{R}^3$. Denote the embedding of the sphere as $j : S^2 \to \mathbb{R}^3$. Locally we choose coordinates such that the embedding of the sphere is given by a graph. Namely, $i_3(x,y) = (x, y, \sqrt{1 - x^2 - y^2})$. The tangent vectors $i_{31}, i_{32}$ and the normal $N_{i_3}$ on $i_3(S^2)$ are in these coordinates given by

$$i_{31} = \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{\sqrt{1-x^2-y^2}} \end{pmatrix}, \quad i_{32} = \begin{pmatrix} 0 \\ 1 \\ -\frac{x}{\sqrt{1-x^2-y^2}} \end{pmatrix}, \quad N_{i_3} = \begin{pmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{pmatrix}. \quad (2.38)$$

The induced metric $\sigma$ on $S^2$ is given by

$$\sigma_{xx} = \frac{1 - y^2}{1 - x^2 - y^2}, \quad \sigma_{xy} = \frac{xy}{1 - x^2 - y^2} = \sigma_{yx}, \quad \sigma_{yy} = \frac{1 - x^2}{1 - x^2 - y^2}, \quad (2.39)$$

with $\det \sigma = \frac{1}{1-x^2-y^2}$. Note that we can identify the normal $N_{i_3}$ on a point $i_3(x,y)$ with the point itself.

Consider now figure 2.1. We see that for every $(s,t) \in M$ we can define a point on $S^2$ with the map $n : M \to S^2$ given by

$$n = i_3^{-1} \circ \nu \circ f. \quad (2.40)$$

Here we explicitly used the fact that we can identify a normal vector on $f(M)$ with a point in $j(S^2)$. This map $n$ is called the Gauss map and is, in local coordinates, given by $n(s,t) = (x,y)$, with the identification

$$x = \frac{-w_s}{\sqrt{1 + w_s^2 + w_t^2}}, \quad y = \frac{-w_t}{\sqrt{1 + w_s^2 + w_t^2}}. \quad (2.41)$$

Consider the metric $\bar{g}$ on $M$ induced by the Gauss map. That is

$$\bar{g} = (i_3^{-1} \circ \nu \circ f)^* \sigma. \quad (2.42)$$

This metric gives a volume form $d\mu_{\bar{g}}$ on $M$, given by $\sqrt{\det \bar{g}} dx$. The Gaussian curvature $K$ on a point $(s,t) \in M$ is given as the ratio between $d\mu_{\bar{g}}$ and the volume form $d\mu_g$ on $M$ coming from the metric $g$. This quantity measures the local deviation of the normal
on the surface compared to its deviation on the unit sphere. To compute the Gaussian curvature we explicitly compute det $\bar{g}$. Using equations (2.41) we have

$$\bar{g}_{ij} = \partial_i x \partial_j x \sigma_{xx} + \partial_i x \partial_j y \sigma_{xy} + \partial_i y \partial_j x \sigma_{yx} + \partial_i y \partial_j y \sigma_{yy}$$

and

$$\det \bar{g} = \det g (\partial_i x \partial_j y - \partial_i y \partial_j x)^2.$$  (2.44)

A few more computations now show that

$$\det \bar{g} = \frac{(w_{xw} w_{yi} - w_{yw} w_{xi})^2}{(\det g)^2},$$

and the Gaussian curvature is, in these coordinates, given by

$$K(s, t) = \frac{dy_k(s, t)}{d\mu_p} = \frac{w_{xw} w_{yi} - w_{yw} w_{xi}}{(\det g)^2}(s, t).$$  (2.46)

We now express the Gaussian curvature in terms of the metric $g$ and the second fundamental form $h$. Consider the metric $\tilde{g}$. Since $\sigma$ is the induced metric coming from the Euclidean space $\mathbb{R}^3$, with metric $\hat{g}$, one can write $\tilde{g}_{ij} = \hat{g}(\nu, f_1, \nu, f_2)$, where $\nu$ is the push forward of $\nu : f(M) \to j(S^2),$$  (2.47)

with the identification $N \in \iota_2(S^2)$ as we did for equation (2.40). In the same manner we can identify any vector in $T_{\iota_2(S^2)}f(M)$ with a vector in $\mathbb{R}^3$ and a vector in $T_{f(M)}f(M)$. The reason we can do this is that the two spaces $T_{\iota_2(S^2)}f(M)$ and $T_{f(M)}f(M)$ both are perpendicular to the vector $N$ and therefore parallel. Hence, we can consider the map $\nu$, as a map from $(T_{f(M)}f(M))^\top$ to itself, where the space $(T_{f(M)}f(M))^\top$ is the space...
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spanned by the vectors tangent to \( f(M) \) in \( T_{f(s,t)}f(M) \). In this case the map \( \nu \) is called the shape operator or the Weingarten map and we can write (using (2.23))

\[
\nu_s f_i = \partial_i (\nu \circ f) = \partial_i N = -h_{ij}g^{jl}f_i.
\]

Hence,

\[
\det \bar{g} = \det h^2 \det g^{-1},
\]

and

\[
K = \frac{\det h}{\det g} = \det(-\nu).
\]

Being defined as a combination of the second fundamental form and the metric makes the Gaussian curvature an extrinsic geometric invariant. As we already mentioned in Remark 2.2.3, however, this specific combination of the components of the second fundamental form can be given in terms of the metric. Hence, the Gaussian curvature is, in fact, an intrinsic invariant of the geometry of the surface.

Consider minus the Weingarten map \(-\nu\). Since the components \(\nu_{ij}\) are symmetric, the map \(-\nu\) is a self-adjoint operator and we can find an orthonormal basis \(\{f_1, f_2\}\) of \((Tf(M))^\top\) such that \(-\nu_{ij}f_i = \kappa_1 f_j\) and \(-\nu_{ij}f_2 = \kappa_2 f_j\), with \(\kappa_1, \kappa_2\) real eigenvalues. These eigenvalues \(\kappa_1\) and \(\kappa_2\) are the so-called principal curvatures of the surface and their product equals the Gaussian curvature. This product contains the following geometric information. Consider the plane \(T_v\) spanned by a vector \(v \in T_{f(s,t)}f(M)\) and the normal vector \(N\). The curvature \(\kappa_v(s,t)\) is defined as the curvature of the curve obtained by taking the intersection of \(T_v\) with \(f(M)\) at a point \((s,t)\) (see Figure 2.2). Euler showed that if the \(\kappa_v\) are not all equal for different \(v \in T_{f(s,t)}f(M)\), then there exist two orthogonal directions, given by \(v_1\) and \(v_2\), such that

\[
\kappa_{v_1} = \max_v(\kappa_v), \quad \text{and} \quad \kappa_{v_2} = \min_v(\kappa_v),
\]

Figure 2.2: The curvature \(\kappa_v\) depends on the choice of orientation.
for every point on $M$. Gauss showed that these are exactly the principal curvatures $\kappa_1$ and $\kappa_2$. Hence, the Gaussian curvature can also be given as the product of the minimal and the maximal directional curvature of a surface. Although the principal curvatures themselves depend on the choice of the orientation, the Gaussian curvature does not. This is in accordance with the fact that the Gaussian curvature is an intrinsic object.

The Gaussian curvature distinguishes on a surface three kinds of points: the points that have positive, negative or zero Gaussian curvature. This characterizes a surface locally in the following way. Consider local coordinates on $M$ such that $f(M)$ is given by the graph $f(s,t) = (s,t,w(s,t))$ and the point we are considering is the origin. Moreover, we can choose the coordinates in such a way that the first derivatives of $w$ at the origin are zero. If the Gaussian curvature is negative there is a positive and a negative principal curvature, which means the surface at that point looks locally like a saddle. If the Gaussian curvature is positive the surface at that point looks locally like an extremum point and the surface bends away from the $(s,t)$-plane (positively or negatively). If the Gaussian curvature is zero, at least one of the principal curvatures is zero. If the Gaussian curvature is zero everywhere (like in the case of a cylinder) one can wrap the plane completely onto the surface.

The intrinsic geometry of a surface has consequences for the geometry of objects defined on it. For instance, a corollary of the local Gauss-Bonnet Theorem states that the sum of the angles of a geodesic triangle (the sides are geodesic on the surface) is given by $\pi$ plus the integral of the Gaussian curvature over the area enclosed by the triangle. This means that angles of a triangle on a sphere add up to a number bigger than $\pi$, while the angles of a triangle on a saddle add up to less than $\pi$. The angles of a triangle on a cylinder, like in the case of the plane, add up to exactly $\pi$. One can also give global statements concerning the curvatures. The global Gauss-Bonnet Theorem states that, for closed surfaces $M$,

$$\int_M Kd\mu = 2\pi \chi(M),$$

where $\chi(M)$ is the Euler number of the surface $M$ given by $2 - 2G$, with $G$ the genus of $M$.

We discuss two other results for the Gaussian curvature which are related to the Willmore conjecture we discuss at the end of Subsection 2.2.3. Consider a compact surface $M \subset \mathbb{R}^3$. Because $M$ is compact we can fit it inside a region bounded by some sphere $S_R$ with radius $R$. By decreasing the radius we shrink the sphere gradually until it touches the surface $M$ for the first time. In this point(s) the surface has to have positive Gaussian curvature. Because, if not, the surface would intersect with the sphere. We may conclude that any compact surface has at least one point $p \in M$ such that $K(p) > 0$. We can even say more about compact surfaces. Fix the surface in space. Put a plane against the surface such that it touches $M$. Any point that touches $M$ in this situation has a normal vector perpendicular to the plane and has non-negative Gaussian curvature by the same reasoning as above. Since this hold for every plane placed against the surface from any direction we can conclude that for a compact surface $M$

$$\int_{M^+} Kd\mu \geq 4\pi,$$

where $M^+ = \{ p \in M | K(p) \geq 0 \}$ and $4\pi$ the surface area of the unit sphere. This holds
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because the Gaussian curvature measures the area on the sphere covered by the normal vector of the area on the manifold $M$ through the Gauss map $\kappa : M \to S^2$.

There is another geometric invariant given in terms of the principal curvatures. This is the mean curvature $H$ given by half the trace of $-\nu$, which gives the mean of the principal curvatures. Hence,

$$H = \frac{1}{2} \text{Trace}(\nu^{-1}h) = \frac{1}{2}(\kappa_1 + \kappa_2).$$

The mean curvature is an extrinsic invariant of the geometry. The fact that it is not intrinsic can be seen by the difference between the mean curvature of the plane and a cylinder. Hence, the mean curvature does differentiate between surfaces that are isometric but are differently immersed into $\mathbb{R}^3$. Note that the mean curvature is defined in different ways throughout the literature. While some people define it as we do here, others prefer to define the mean curvature as the trace of the map $-\nu$, without the factor $\frac{1}{2}$.

Due to Gauss-Bonnet, (2.32) the total Gaussian curvature $K$ of a surface $M$ is a topological invariant. What can now be said about the integral over the squared mean curvature? For closed orientable surfaces, smoothly immersed in $M$, we have that

$$\int_M H^2 \, d\mu \geq 4\pi,$$

with equality for the spheres. Indeed, consider equation (2.53) and the definition for $M^+$ right underneath it. Then we have

$$\int_\partial H^2 \, d\mu \geq \int_\partial H^2 \, d\mu \geq \int_\partial K \, d\mu \geq 4\pi,$$

where we used the inequality

$$H^2 - K = \frac{1}{4}(\kappa_1 - \kappa_2)^2 \geq 0.$$

Equality follows whenever $\kappa_1 = \kappa_2$ on $M$, that is, whenever $M$ is embedded as a sphere in $\mathbb{R}^3$. Hence, the sphere minimizes, globally, the Willmore integral. The Willmore conjecture states that

$$\int_{M^1} H^2 \, d\mu \geq 2\pi^2,$$

for all closed orientable surfaces $M_1$ of genus one. Consider a torus such that the ratio between the two radii is $\sqrt{2}$. This torus is a local minimizer (see Subsection 2.2.3) with total square mean curvature equal to $2\pi^2$. Hence, if the conjecture were true, this torus would be a global minimizer of the Willmore integral (for surfaces of genus 1). In several different cases the Willmore conjecture has been confirmed (see for instance [110] and [4]). A complete proof of the Willmore conjecture is given in [97]. For more on Willmore surfaces we refer to [110] and Chapter 7 of [109] (and references therein).

One method of considering the minimizers of the total squared mean curvature is by the calculus of variations. This gives an Euler-Lagrange equation and a corresponding $L^2$-gradient flow, discussed in Subsection 2.2.3. This flow, called the Willmore flow, is the evolution under consideration in this Chapter.

For more on curvature and the local and global theory of surfaces we refer to [100], [77] and [85].
2.2.3 Evolution of geometric structures and the Willmore flow

Consider a moving smooth surface in $\mathbb{R}^3$. We can describe this surface as a family of immersions $\varphi_t : M \to \mathbb{R}^3$, which are parameterized by time $t$. For each point at time $t_0$ there is a corresponding induced metric given by the immersion $\varphi_{t_0}$. The dependence of the metric on $t$ is the subject of this Subsection. That is, how does the corresponding metric of a moving surface change in time? This, together with the time-dependence of the normal, gives us the evolution of the geometry of the surface. With this we can show that the $L^2$-gradient flow

$$\partial_t \phi_t$$

obeys the equation

$$\langle \partial_t \phi_t, N \rangle = -\Delta H - 2H (H^2 - K),$$

with $\Delta$ the Laplace-Beltrami operator defined in Remark 2.2.1, $H$ the mean curvature and $K$ the Gaussian curvature (both defined in Subsection 2.2.2). The term $d\mu$ in (2.59) is the volume form of $M$ given by $\sqrt{det g} \, dx$, with $dx$ the Lebesgue measure on $M$.

Consider a family of smooth immersions $\phi_t$ parameterized by $t$. Any displacement of a point $\phi_t(p)$ that is not in the normal direction $N$ but lies strictly in the tangent space $\phi_t^*(T_p M)$ is, in fact, a reparameterization, in $M$, of the immersion. Since reparameterizations of the immersion are local isometries they do not change any geometric structures. Hence, the integral of equation (2.59) is invariant under a displacement in the tangent direction. Therefore, to study the first variation of the Willmore integral, it is sufficient to consider the displacement of a point in a family of immersions in the normal direction. To do this we choose the parameterization in such a way that the displacement is always and everywhere in the normal direction. This family of immersions, still parameterized by $t$ is denoted by $f_t$, with displacement given by $\partial_t f = VN$.

We start with a Lemma that gives us the first variation of the metric and the normal.

Lemma 2.2.4. Let $f_t$ be a family of immersions as above. Then the following holds:

1. $\partial_t g_{ij} = -2h_{ij} V$,
2. $\partial_t N = -\nabla V$,

where

$$\nabla a = \partial_i ag^{ij} f_j,$$

for all smooth functions $a$ on $M$.

Proof.

1. Since the partial derivatives commute and the displacement of the immersion is in the normal direction, we have

$$\partial_t g_{ij} = \partial_i (\partial_t f, \partial_j f) + (\partial_t f, \partial_i \partial_j f)$$

$$= \partial_i (\partial_t f, \partial_j f) + \partial_j (\partial_t f, \partial_i f) - 2(\partial_t f, \partial_i \partial_j f)$$

$$= -2(\partial_t f, \partial_i \partial_j f)$$

$$= -2h_{ij} V,$$

by the Gauss formula (2.22).
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Since the normal \( N \) has norm one we know that the variation of \( N \), denoted as \( \partial_t N \), cannot have a component in the normal direction and must lie in the tangent plane of the surface. In other words, \( \langle \partial_t N, N \rangle = 0 \) and we can write

\[
\partial_t N = \langle \partial_t N, \partial_i f \rangle g^{ij} \partial_j f.
\]

Therefore,

\[
\partial_t N = -(N, \partial_t f) g^{ij} \partial_j f = -\partial_t (N, \partial_i f) g^{ij} \partial_j f + \langle \partial_t N, \partial_i f \rangle g^{ij} \partial_j f = -\nabla V.
\]

since \( \langle \partial_t N, \partial_i f \rangle = 0 \) by the same reasoning as above.

The following Lemma gives the variation of the second fundamental form.

**Lemma 2.2.5.** Let \( f_t \) be a family of immersions as above. The variation of the second fundamental form is given by

\[
\partial_t h_{ij} = \nabla_i \nabla_j V - \bar{V} h_{im} h_{jk} g^{mk},
\]

where \( \nabla_i \nabla_j a \) on a function \( a \) is given by

\[
\nabla_i \nabla_j a = \partial_i \partial_j a - \Gamma^k_{ij} \partial_k a.
\]

**Proof.** Using Lemma 2.2.4 and the construction of \( \partial_t f \) we obtain

\[
\partial_t h_{ij} = \langle \partial_i \partial_j \partial_t f, N \rangle + \langle \partial_i \partial_j f, \partial_t N \rangle = \partial_t \partial_i \partial_j V + \bar{V} (\partial_i \partial_j N, N) + \langle \partial_i \partial_j f, -\nabla V \rangle.
\]

Using the Gauss formula and the Weingarten equation (2.24) for the derivatives of \( f \) and \( N \) we have

\[
\partial_t h_{ij} = \partial_t \partial_i V - \bar{V} (\partial_i \partial_j N, N) + \langle \Gamma^k_{ij} \partial_k f + h_{ij} N, -\nabla V \rangle = \partial_t \partial_i V - \bar{V} h_{ik} g^{kl} h_{jm} g^{mn} g^{mn} - \langle \Gamma^k_{ij} \partial_k f, -\nabla V \rangle = \nabla_i \nabla_j V - \bar{V} h_{im} h_{jk} g^{mk}.
\]

Lemmas 2.2.4 and 2.2.5 give us all the information we need to calculate the evolution of any geometric object. Let us calculate the evolution of \( d\mu \) and \( H \). The evolution of \( d\mu \) is now straightforward:

\[
\partial_t (d\mu) = \frac{1}{2} \sqrt{\det g} \partial_t (g_{11} g_{22} - g_{12} g_{21}) d\mu = -V \sqrt{\det g} (h_{ij} g^{ij}) d\mu = -2HV d\mu,
\]

by the definition of \( H \) (see (2.54)). For the evolution of the mean curvature we observe that

\[
0 = \partial_t (g^{ij} g_{ij}) = \partial_t g^{ij} g_{ij} + g^{ij} \partial_t g_{ij}.
\]
and therefore, using Lemma 2.2.4,
\[ \partial_t g^{ij} = 2g^{im}g^{lj}V h_{mi}. \]  
(2.71)

Hence,
\[ \partial_t H = \frac{1}{2} \partial_t g^{ij}h_{ij} + \frac{1}{2} g^{ij} \partial_t h_{ij} \]
\[ = g^{im}g^{lj}h_{mi} + \frac{1}{2} g^{ij}(\nabla_i \nabla_j V - V h_{im}h_{jk} g^{mk}) \]
\[ = \frac{1}{2} \Delta V + \frac{1}{2} V |h|^2 , \]  
(2.72)

where the square of the norm \( |h| \) of \( h \) is given by
\[ g^{ij}g^{mk}h_{im}h_{jk} = \text{Tr} \left( (g^{-1}h)^2 \right) = \kappa_1^2 + \kappa_2^2 . \]  
(2.73)

We are now able to give the first variation of the Willmore functional \( W(f_t) \):
\[ \partial_t W(f_t) = \frac{\partial}{\partial t} \int_M H^2 d\mu \]
\[ = 2 \int_M H \frac{\partial}{\partial t} H d\mu + \int_M H^2 \frac{\partial}{\partial t} d\mu \]
\[ = 2 \int_M H \left( \frac{1}{2} \Delta V + \frac{1}{2} V |h|^2 \right) d\mu - 2 \int_M H^2 V d\mu \]
\[ = \int_M \left( \Delta H + H (2H^2 - 2K) \right) V d\mu , \]  
(2.74)

for every variation \( \partial_t f = V N \) of \( f_t \), where we used equation (2.17) and
\[ 4H^2 - 2K = (\text{Trace}(g^{-1}h))^2 - 2 \det(g^{-1}h) = \kappa_1^2 + \kappa_2^2 = |h|^2 . \]  
(2.75)

The Willmore flow is defined as the \( L^2 \) gradient flow corresponding to the Willmore functional \( W(f) \) of equation (2.59) and is therefore given by
\[ V = -\Delta H - 2H(H^2 - K) . \]  
(2.76)

Hence, the time derivative of the integral is given by
\[ \partial_t W(f_t) = - \int_M V^2 d\mu \]
\[ = - \int_M \left( \Delta H + 2H(H^2 - K) \right) d\mu . \]  
(2.77)

If the flow obeys (2.76), the Willmore functional is minimized in the quickest possible way in the \( L^2 \) norm.

Setting the first variation of \( W(f) \) equal to zero, hence looking at the critical points of \( W(f) \), gives
\[ \Delta H + 2H(H^2 - K) = 0 . \]  
(2.78)

Every surface that satisfies equation (2.78), is called a Willmore surface.
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2.2.4 The axisymmetric Willmore flow

In this Section we give the Willmore flow for a surface of revolution given by an immersion

\[ f : \left( s, \theta \right) \mapsto \begin{pmatrix} r(s) \cos(\theta) \\ r(s) \sin(\theta) \\ z(s) \end{pmatrix}, \quad (2.79) \]

The resulting differential equation is only dependent on one spatial parameter \( s \). We choose two different specific parameterizations for \( s : s = z \) and \( s = r \). To describe the Limaçon in Section 2.3, we need both points of view.

Consider the immersion given by equation (2.79). The corresponding tangent vectors \( f_1, f_2 \) and normal vector \( N \) are

\[ f_1 = \begin{pmatrix} r' \cos \theta \\ r' \sin \theta \\ z' \end{pmatrix}, \quad f_2 = \begin{pmatrix} -r' \sin \theta \\ r' \cos \theta \\ z' \end{pmatrix}, \quad N = \frac{-1}{\sqrt{r'^2 + z'^2}} \begin{pmatrix} z' \cos \theta \\ z' \sin \theta \\ -r' \end{pmatrix}, \quad (2.80) \]

while the components of the metric are given by

\[ g_{11} = r'^2 + z'^2, \quad g_{12} = 0 = g_{21}, \quad g_{22} = r'^2, \quad (2.81) \]

so that \( \det g = r^2(r'^2 + z'^2) \). The orientation of the surface is chosen in the direction of \( f_1 \times f_2 \).

With this choice for the orientation the components of the scalar second fundamental form are given by

\[ h_{11} = -\frac{z''r' - r'z''}{\sqrt{r'^2 + z'^2}}, \quad h_{12} = 0 = h_{21}, \quad h_{22} = \frac{z'}{\sqrt{r'^2 + z'^2}}, \quad (2.82) \]

and the principal curvatures are

\[ \kappa_1 = -\frac{z''r'' + r'z''}{(r'^2 + z'^2)^{3/2}} \quad \text{and} \quad \kappa_2 = \frac{z'}{r\sqrt{r'^2 + z'^2}}. \quad (2.83) \]

Hence,

\[ H = \frac{1}{2} \frac{1}{r(r'^2 + z'^2)^{1/2}} \left[ r'^2z'' + z'^3 - r'z'z'' + r'z'^2 \right]. \quad (2.84) \]

and

\[ K = \frac{-r'^2z'' + r'z'z''}{r(r'^2 + z'^2)^{3/2}}. \quad (2.85) \]

Since \( H \) only depends on \( s \) (and not on \( \theta \)) the Laplace-Beltrami operator on \( H \) is, in these coordinates, given by (see equation 2.16)

\[ \Delta H = \frac{1}{\sqrt{\det g}} \partial_i (g^{ij} \sqrt{\det g} \partial_j H) = \frac{1}{\sqrt{\det g}} \partial_i \left( \frac{r^2}{\sqrt{\det g}} \partial_j H \right). \quad (2.86) \]
In Section 2.3 we describe the behavior of blowup for a certain surface using matched asymptotics. This surface, given in Figure 2.3, has a self intersection. The main idea of the matched asymptotics is that different spatial regions correspond to different temporal regions. For these various regions we try to find solutions to the Willmore flow with appropriate boundary conditions. Due to the form of the surface we see that different regions of the surface need different parameterizations. We give the Willmore flow in terms of both parameterizations: \( s = z \) and \( s = r \).

Consider the parameterization \( s = z \) such that we can describe the surface as the graph of \( r(z) \) rotated around the \( z \)-axis. In this case the mean and Gaussian curvature, \( H[r] \) and \( K[r] \) respectively, are given by

\[
H[r] = \frac{1}{2} \frac{r_z^2 + 1 - r^2_{zz}}{r(1 + r_z^2)^{3/2}}
\]

and

\[
K[r] = \frac{-r_{zz}}{r(1 + r_z^2)^2}.
\]

If we denote the Laplace-Beltrami operator in this parameterization as \( \Delta[r] \), this gives us, writing \( q = 1 + r_z^2 \),

\[
\left( \Delta H + H(H^2 - K) \right)[r] = \frac{1}{4} \frac{1}{q^{3/2}} \left[ 2r_{zzz}q^2 - 20r_z r_{zz} r_{zzz} q - 5r_{zz}^3 + 30r_z^2 r_{zz}^2 q \right.
\]

\[
\left. + \frac{1}{r} \left( 4r_z r_{zz} q^2 + 3r_z^2 q - 12r_z^2 r_{zz} q \right) \right.
\]

\[
\left. + \frac{1}{r} (r_z q^3 - 3r_z q r_{zz} q) \right]
\]

\[
\left. + \frac{1}{r} \left( -q^3 - r_z^2 q^2 \right) \right] .
\]

This is, up to sign, the Willmore flow for the graph of \( r(z) \) in the normal direction. To obtain the Willmore flow in the \( r \) direction we have to compensate with an extra factor \( 1/ \cos \alpha = \sqrt{1 + r_z^2} \), with \( \alpha \) the angle between \( N \) and the \( r \) direction. Hence, in the region around the tip \( \lambda \) of the Limaçon, the Willmore flow is given by (see Figure 2.3)

\[
r_t = \sqrt{1 + r_z^2} \left( \Delta H + H(H^2 - K) \right)[r] .
\]

Consider now the parameterization where the surface is given by the graph of \( z(r) \) rotated around the \( z \)-axis. Hence \( s = r \). In this case the mean and Gaussian curvature \( (H[z] \) and \( K[z] \) are given by

\[
H[z] = \frac{1}{2} \frac{1 + z_r^2 + r z_{rr}}{r(1 + z_r^2)^{3/2}}
\]

and

\[
K[z] = \frac{z_r z_{rr}}{r(1 + z_r^2)^2}.
\]
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Denoting the Laplace-Beltrami operator now as $\Delta[z]$ this gives us, writing $\tilde{q} = 1 + z^2$, 
\[
\left(\Delta H + 2H(H^2 - K)\right)[z]
\]
\[
= \frac{1}{4} \tilde{q}^{-\frac{3}{2}} \left[2z_{rrr}^2 - 20z_{rr}z_{rrr}^2 + 30z_{rr}^2 - 5z_{rr} \right]
\]
\[
+ \frac{1}{7} \left(4z_{rrr}^2 - 15z_{rr}^2 \tilde{q} \right)
\]
\[
+ \frac{1}{14} \left(-2z_{rr} \tilde{q} + 3z_{rr}^2 \tilde{q}^2 \right)
\]
\[
+ \frac{1}{14} \left(2z_{rr} \tilde{q} - z_{rr} \tilde{q}^3 \right),
\]
which is minus the Willmore flow in the direction normal to the surface. In this case, where we express the surface as a surface of revolution of the graph of $z(r)$, we are interested in the Willmore flow in the direction of $z$. Hence, in the neighborhood of the crossing of the Limaçon with the $r$-axis (see Figure 2.3), the Willmore flow is given by
\[
z_t = -\sqrt{1 + z_t^2} \left(\Delta H + 2H(H^2 - K)\right)[z].
\]

Although the axisymmetric Willmore flow is simpler than the Willmore flow on a general closed surface, it is still quite involved. In Section 2.3 we use matched asymptotics to analyze the blowup behavior of the Limaçon. Every different spatial region, corresponding to a particular scaling, gives rise to a specific simplification on the surface. Due to these simplifications we are able to study the Willmore flow.

2.2.5 Some results on the Willmore flow

Since their reintroduction in the 1960s there has been quite some attention to the study of Willmore surfaces, which solve equation (2.78), see, for instance, Chapter 7 of [109] and references therein for the history of and investigations on these surfaces. However, the related flow has been studied much less. In this Subsection we discuss the results, known to us, on the Willmore flow.

In [109] and [73] the local existence of the Willmore flow with smooth (enough) initial data is proved. In both references it is shown that this flow can be restated as a parabolic quasilinear fourth order differential equation. The short time existence is then proved by using general results on higher order parabolic quasilinear equations. We refer to [108] for a discussion on these proofs and another self-contained proof of the local existence.

Besides the local result in [109], the author also gives a global result. He proves that for any surface $f(M)$ sufficiently close to a sphere, and smooth enough, there exists a smooth unique solution $f_t$, for all $t > 0$ and $f_0(M) = f(M)$, which evolves to a sphere exponentially fast. In [80] a similar result is proved. This is done using the trace-free second fundamental form, given by $h_0 := h - gH$. This $h_0$ measures how far a point on a
surface deviates from having equal principal curvatures. Namely,

\[
|h|^2 = g^{ij}g^{km}(h_{ik} - g_{ik}H)(h_{jm} - g_{jm}H) \\
= |h|^2 - 2g^{km}h_{mk}H + 2H^2 = |h|^2 - 2H^2 = 2H^2 - 2K
\]

where we used equation (2.75) in the penultimate equation. The authors of [8] show that there exists an \( \epsilon_0 > 0 \) such that if for a closed immersed surface

\[
\int_M |h|^2 d\mu \leq \epsilon_0,
\]

then the Willmore flow exists smoothly for all time and converges exponentially to a sphere.

In [8], the authors give a lower bound on the lifespan of a smooth solution \( f_t \) dependent on the initial density of curvature of the surface \( f_0(M) \). They show that there exist constants \( \epsilon_0 \) and \( c \) such that if \( \rho \) is chosen such that

\[
\int_{\mathbb{R}^3} |h|^2 d\mu \leq \epsilon \leq \epsilon_0,
\]

then the maximal lifespan of the Willmore flow with initial data \( f_0 \) is given by

\[
T \geq \frac{1}{c} \rho^4,
\]

and one has the estimate

\[
\int_{\mathbb{R}^3} |h|^2 d\mu \leq c \epsilon, \quad \text{for } 0 \leq t \leq \frac{1}{c} \rho^4.
\]

This implies that, if the solution blows up, a bit of \( |h|^2 = \kappa_1^2 + \kappa_2^2 \) concentrates in a vanishing region of the surface. Hence, if the Willmore flow forms a singularity, at least one of the principal curvatures blows up in a single point. The authors further conclude that this disappearing part of the surface the Willmore flow has to be stationary. This is the reason why the flow exhibits different scalings if it blows up. This information is used to study the flow through matched asymptotics in Section 2.3 and onwards.

In [81], the authors generalize the result of [80] by dropping the smallness assumption in (2.96). They state that, for immersed spheres, if

\[
\int_M H^2 d\mu \leq 8\pi, \quad \text{or equivalently } \int_M |h|^2 d\mu \leq 8\pi,
\]

then the Willmore flow exists smoothly for all times and converges exponentially to a round sphere.

There are a lot of numerical investigations on the Willmore flow, which we discuss in Subsection 3.1.2. Here we only mention the numerics of [90]. In this article the authors investigate the geometric properties of the Willmore flow numerically. They show that there is evidence that the Willmore flow can drive embedded surfaces to non-embedded
2.3. Formal asymptotics for the Limaçon

It is still an open problem whether the Willmore flow can create a singularity in finite time, starting from a smooth surface. In reference [90] the authors show that numerical considerations suggest that the Limaçon, depicted in Figure 2.3 and rotated around the \( z \)-axis, develops a singularity in finite time. Besides [90] one can consult [89] for an animation of this evolution. In Chapter 3 we use a moving mesh method to investigate certain aspects of this evolution. The evolution is also given in Figure 2.3. We see that a singularity develops as the tip, denoted by \( \lambda \), drops to zero and the curvature (of the curve) in the tip tends to infinity. We further see that, together with the tip \( \lambda \), the crossing of the curve with the \( r \)-axis drops and reaches zero at the same time as the tip. Hence, the loop shrinks to a point and vanishes into the origin at time \( T < \infty \). In [80] and [21] it is shown that the blowup limit, that is, the solution in the vanishing region of the singularity in the limit \( t \to T \), must be stationary and consists of catenoids and/or planes. Both the plane and the catenoid have mean curvature zero. Hence, \( H \to 0 \) as \( t \to T \).

Due to [81], one knows that the blowup, in the the Willmore flow, has to occur in a small, disappearing region, where at least one of the principal curvatures blows up. This

---

Figure 2.3: Left: the Limaçon. We denote the tip of the Limaçon by \( \lambda \). Right: typical evolution of the Limaçon.

immersions and that the Willmore flow can create finite time singularities starting with a smooth immersion. The evidence they found for finite time blowup is with a Limaçon as initial condition. That is, the curve of Figure 2.3 rotated around the \( z \)-axis. In [91] the authors prove that the Willmore flow can indeed turn embedded surfaces into immersed surfaces with self intersections.

In [21], the author specifies the singularity results found in [80], [81] and [82] for the axisymmetric case. He proves the singularity formation (not specifically in finite time) in the Limaçon and shows that at blowup time the singularity has to look like a hyperbolic cosine. With this example he also shows that the estimate in (2.100) is sharp.

In [35] the global result of [99] is generalized. For every (smooth enough) initial surface close enough to a local minimizer of the Willmore integral, the Willmore flow exists globally and converges smoothly to a local minimizer. Finally, in [78] the authors find global existence of solutions for 2 dimensional graphs with some smallness assumptions.
means that there are at least two different scales in this problem. The first scale is related to the rate at which the curvature blows up and the blowup region disappears. We see in Subsection 2.3.1 that this rate, in turn, is related to $\lambda(t)$, the rate at which the tip of the loop drops to zero. This function $\lambda(t)$ is, in fact, the focus of the investigations in this Chapter. In [81] it is also shown that, in case of blowup, the rate $\lambda$ has to be smaller than the self-similar rate $(T - t)^{\frac{1}{4}}$, with $T$ the time of blowup. This corresponds to the fact that the blowup limit is quasi-stationary. We call the vanishing region corresponding to the blowup the inner region of the problem.

The second scale is the scale of those regions of the surface that evolve without being (or hardly being) influenced by the blowup. In these regions the surface does not disappear and the scale is of order 1. These regions form the remote region. We expect a third region between the inner and the remote region, that vanishes with the self-similar scale. This region is called the outer region.

The method we use to investigate the blowup behavior is the method of formal matched asymptotics. That is, after finding different solutions to the evolution problem in the different regions, we try to match them and hope to find information on the blowup rate.

The solutions we find are, in fact, expansions. In the inner region we expand around $H = 0$, suggested by the literature and the numerics. In the outer region we expand around $\partial_r z(r, t) = 0$, which is suggested by the numerics in Chapter 3 and can be seen in Figure 2.3. In this figure we see that the tangent to the point on the curve that crosses the $r$-axis evolves to a vertical position. This is the crucial assumption of this Chapter. The outer region is that part of the Limaçon that vanishes on a self-similar scale into the origin. Hence, in this region the $r$-coordinate and the $z$-coordinate drop to zero with some specific rates. The assumption that in the outer region the derivative $\partial_r z(r, t)$ drops to zero, means that the $z$-coordinate is much smaller than the $r$-coordinate. As we assume that there are only three different scales in this problem (order 1, self-similar, and quasi-stationary), we expect the first order approximations of the scales of the coordinates, in the limit $t \to T$, to be one of these three. Hence, if the scale of the $z$-coordinate in the outer solution is self-similar, the scale of the $r$-coordinate also has to be self-similar or of order 1. Since the scale of the $r$-coordinate in the outer region cannot be of order 1 (the region vanishes in the origin), we state that in the outer region the $r$-coordinate drops with a self-similar rate. The $z$-coordinate can drop with a similar rate or faster (quasi-stationary). For the remote region we do not need an explicit (or expansion of a) solution. For information on the blowup, it is sufficient to use the remote solution as a boundary condition for the outer solution. This is explained when we construct the outer solutions in Subsections 2.3.2 and 2.3.4. We do, however, remark that the numerics in Chapter 3 suggest that the blowup limit is close (energy-wise) to two spheres. This could mean that the remote solution can be approximated, in the limit $t \to T$, by a sphere.

In the remainder of this Chapter we construct inner and outer solutions and match them to each other. This gives a family of blowup rates. The biggest of these rates is given by

$$\lambda(t) \sim a_c \ln \left( \frac{1}{T - t} \right)^{-1} (T - t)^{\frac{1}{4}}, \quad \text{for} \quad t \to T,$$

for some positive constant $a_c$. We also conclude, however, that the blowup we find is not the blowup of a Limaçon. The approximation of the solution we find does not cross the $r$-axis in the relevant scales (disappearing region). In Subsection 2.4 we discuss and interpret the assumptions we make and the results we find.
2.3. Formal asymptotics for the Limaçon

2.3.1 The inner solution.

In this Subsection we consider the inner solution. As parameterization of the curve we choose $s = z$ and consider the surface as the graph of $r$ with respect to $z$, rotated around the $z$-axis. Hence, we are interested in the movement of the surface in the $r$-direction and we have to consider the Willmore flow given by equation (2.90). The inner region is the region where the singularity occurs and at least one of the principal curvatures blows up. In [um] and [on] it is shown that the blowup profile in this region is a Willmore surface and that $H \to 0$ as $t \to T$, with $T$ the blowup time. Hence, both principal curvatures blow up in the tip of the loop. We use these facts to construct the inner solution.

Consider the two principal curvatures $\kappa_1$, $\lambda$ and $\kappa_2$, $\lambda$ in the tip $\lambda$ of the Limaçon (see Figure 2.3). In the parameterization where $r$ is given as a function of $z$ the principal curvatures $\kappa_1$, $\lambda$ and $\kappa_2$, $\lambda$ are given by

$$\kappa_1(t) = -r_z(z = 0, t) \quad \text{and} \quad \kappa_2(t) = \frac{1}{r(z = 0, t)}.$$

As $t \to T$, we have that $\kappa_1$ tends to minus infinity and $\kappa_2$ tends to plus infinity. These principal curvatures determine the blowup rate of our problem and are the focus of these investigations. We study them through their inverses. Define $\bar{\lambda}$ and $\lambda$, both positive, and such that they control the principal curvatures in the following way. For $t \to T$, we define

$$\bar{\lambda}(t) := -\frac{1}{\kappa_2}, \quad \lambda(t) := \frac{1}{\kappa_2} = r(z = 0, t).$$

Since $\kappa_1 + \kappa_2 = 2H \to 0$ we also have that

$$\lambda - \bar{\lambda} \to 0,$$

and in the limit the blowup rate is in fact given by only one rate, namely $\lambda \sim \bar{\lambda}$. Consider the change of variables $\zeta = \frac{t}{\lambda}$ and the equality $r(z, t) = \lambda \rho(\zeta, t)$. Then

$$r_t = \lambda \rho_t + \lambda \left( \rho - \zeta \rho_{\zeta} \right),$$

and

$$r_z = \rho_{\zeta}, \quad r_{zz} = \frac{1}{\lambda} \rho_{\zeta\zeta}, \quad r_{zzz} = \frac{1}{\lambda^2} \rho_{\zeta\zeta\zeta}, \quad r_{zzzz} = \frac{1}{\lambda^3} \rho_{\zeta\zeta\zeta\zeta}.$$

Consider the mean and Gaussian curvature in the rescaled variables. We have

$$H[r] = \lambda \rho + \lambda \left( \rho - \zeta \rho_{\zeta} \right),$$

and

$$K[r] = \lambda^2 \rho(1 + \rho_{\zeta})^2 = \frac{1}{\lambda^2} H_{\rho r}[\rho].$$

Similarly we have

$$\Delta[r] = \frac{1}{\lambda^2} \rho \sqrt{1 + \rho^2} \rho_{\zeta} \left( \frac{\rho}{\sqrt{1 + \rho^2}} \right) = \frac{1}{\lambda^2} \Delta_{\rho r}[\rho].$$
2. Matched asymptotics for finite time blowup in the Willmore flow

with $\Delta_\nu[\rho]$ the rescaled Laplace-Beltrami operator. The rescaled Willmore flow is given by

$$
\lambda^4 \dot{\rho} + \lambda^3 \rho - \xi \rho = \sqrt{1 + \rho^2} \left[ \Delta_\nu \rho + 2H^2 \rho \right] \left[ \rho \right].
$$

(2.110)

Since in the limit $t \to T$, the solution to the Willmore flow is given by a (quasi) static solution, we know that $\lambda$ is smaller than the self-similar rate $(T - t)^{\frac{1}{3}}$ and therefore that $\lambda^3 \lambda \to 0$ as $t \to T$. This means that the rescaled Willmore flow to zeroth order in $\lambda^3$ and $\lambda^3 \lambda$ is given by the stationary Willmore equation. Let $\rho_0$ be the stationary solution corresponding to this equation. Hence,

$$
\left( \Delta_\nu \rho + 2H^2 \rho \right) \left[ \rho_0 \right] = 0,
$$

(2.111)

with boundary conditions

$$
\begin{align*}
\rho(z = 0) &= \lambda, \quad \Rightarrow \quad \rho_0(0) = 1, \\
\rho(z = 0) &= 0, \quad \Rightarrow \quad \rho_0(0) = 0, \\
\rho(z = 0) &= \frac{1}{\lambda}, \quad \Rightarrow \quad \rho_0(0) = \frac{\lambda}{\lambda}, \\
\rho(z = 0) &= 0, \quad \Rightarrow \quad \rho_0(0) = 0.
\end{align*}
$$

(2.112)

From [80] and [21] we know that the mean curvature in the inner region goes to zero as $t \to T$. Hence, $H_{\nu,0}[\rho_0] \to 0$. This means we can write the quasi-stationary solution $\rho_0$ as the expansion

$$
\rho_0(\zeta, t) = \phi_0(\zeta) + \epsilon \phi_1(\zeta) + \epsilon^2 \phi_2(\zeta) + \cdots ,
$$

(2.113)

with $\epsilon = \epsilon(t) \to 0$ as $t \to T$. The choice for $\phi_0$ is such that

$$
\phi_0(0) = \rho_0(\zeta = 0, t) = 1 \quad \text{and} \quad \phi_0(0) = \rho_0(\zeta = 0, t) = 0.
$$

(2.114)

Hence, $\phi_i(0) = \phi_i(0) = 0$ for all $i \geq 1$. With $\rho_0$ expanded as in (2.113) we can expand the rescaled mean and Gaussian curvature and the Laplace-Beltrami operator as

$$
\begin{align*}
H_{\nu,0}[\rho_0] &= H^2[\phi_0, \phi_1] + \epsilon H^2[\phi_0, \phi_1] + \epsilon^2 H^2[\phi_0, \phi_1, \phi_2] + O(\epsilon^3), \\
K_{\nu,0}[\rho_0] &= K^2[\phi_0, \phi_1] + \epsilon K^2[\phi_0, \phi_1] + \epsilon^2 K^2[\phi_0, \phi_1, \phi_2] + O(\epsilon^3),
\end{align*}
$$

(2.115)

and

$$
\begin{align*}
\Delta_\nu[\rho_0] &= \Delta^2[\phi_0] + \epsilon \Delta^2[\phi_0, \phi_1] + \epsilon^2 \Delta^2[\phi_0, \phi_1, \phi_2] + O(\epsilon^3).
\end{align*}
$$

(2.116)

Since $H \to 0$ as $t \to T$ we have $H^2[\phi_0] = 0$, which immediately fulfills the Willmore equation. This gives

$$
\phi_0^{\zeta} + 1 - \phi_0 \phi_0 = 0,
$$

(2.118)

with boundary conditions $\phi_0(0) = 1$ and $\phi_0(0) = 0$. Hence,

$$
\phi_0 = \cosh(\zeta).
$$

(2.119)

With equation (2.119) we find

$$
K_{\nu,0}[\phi_0] = -\frac{1}{\cosh^2(\zeta)} \quad \text{and} \quad \Delta_\nu[\phi_0] = \frac{1}{\cosh^2(\zeta)} p^2
$$

(2.120)
2.3. Formal asymptotics for the Limaçon

The Willmore equation (2.111) to first order in $\epsilon$ is now given by

$$\partial^2 H_1[\phi_0, \phi_1] + \frac{2}{\cosh^2(\zeta)} H_1[\phi_0, \phi_1] = 0,$$  \hspace{1cm} (2.121)

with

$$H_1[\phi_0, \phi_1] = \frac{1}{2} \frac{-1}{\cosh(\zeta)} (\cosh(\zeta) \phi_1 - 2 \sinh(\zeta) \phi_1 + \cosh(\zeta) \phi_{1CC}).$$  \hspace{1cm} (2.122)

Two solutions to the homogeneous equation (2.121), interpreted as an equation for $H_1$, are

$$A(\frac{\sinh(\zeta)}{\cosh(\zeta)} - 1) \text{ and } B\left(\frac{\sinh(\zeta)}{\cosh(\zeta)}\right),$$  \hspace{1cm} (2.123)

with $A, B \in \mathbb{R}$. Hence, to first order in $\epsilon$, the rescaled mean curvature (2.122) is

$$H_1[\phi_0, \phi_1] = A\left(\frac{\sinh(\zeta)}{\cosh(\zeta)} - 1\right) + B\frac{\sinh(\zeta)}{\cosh(\zeta)},$$  \hspace{1cm} (2.124)

and the corresponding $\phi_1$ solves

$$\phi_{1CC} - \frac{2}{\cosh(\zeta)} \phi_{1C} + \phi_1 = -2 \cosh^3(\zeta)\left(A\left(\frac{\sinh(\zeta)}{\cosh(\zeta)} - 1\right) + B\frac{\sinh(\zeta)}{\cosh(\zeta)}\right).$$  \hspace{1cm} (2.125)

Solutions to the homogeneous version of the equation (2.125) are

$$C\sinh(\zeta) \text{ and } D\left(\zeta \sinh(\zeta) - \cosh(\zeta)\right),$$  \hspace{1cm} (2.126)

with $C, D \in \mathbb{R}$. Hence, using variation of constants we find, denoting $\cosh(\zeta)$ and $\sinh(\zeta)$ with $ch$ and $sh$ respectively,

$$\phi_1(\zeta) = A\left(\frac{5}{2} ch + ch^3 + \frac{5}{2} sh - \frac{1}{2} \zeta ch ch^2 - \frac{1}{4} ch + \frac{1}{6} \zeta ch\right)$$
$$+ B\left(-\frac{1}{2} ch ch^2 - \zeta ch + \frac{1}{2} ch^2 + Csh + D(\cosh - ch)\right).$$  \hspace{1cm} (2.127)

Considering the boundary conditions (2.112) and the solution $\phi_0$ in (2.119) we see that $\phi_1$ satisfies $\phi_1(0) = \phi_{1C}(0) = 0$ and

$$\rho_{0CC}(\zeta) = 0 = \rho_{0C}(0) + \epsilon \phi_{1C}(0) = \frac{\lambda}{\bar{\lambda}},$$
$$\rho_{0},(\zeta) = 0 = \rho_{0CC}(0) + \epsilon \phi_{1CC}(0) = 0.$$  \hspace{1cm} (2.128)

Hence, with

$$\epsilon := \frac{\lambda}{\bar{\lambda}} - 1,$$  \hspace{1cm} (2.129)

we have $\phi_{1C}(0) = 1$ and $\phi_{1CC}(0) = 0$. Therefore, $A = \frac{1}{4}$, $B = 0$, $C = 0$, $D = -\frac{3}{4}$. This gives

$$\phi_1(\zeta) = \frac{1}{2} ch sh^2 + \frac{1}{4} (\zeta ch - \frac{1}{4} \zeta ch^2 - \frac{1}{4} \zeta^2 ch + \frac{1}{16} \zeta^3 sh),$$  \hspace{1cm} (2.130)
and \( H_1^\ell [\phi_0, \phi_1] \) is given by
\[
H_1^\ell [\phi_0, \phi_1] = \frac{1}{2} \phi_0 \phi_1 \sinh(\zeta) - \cosh(\zeta) \cosh(\zeta) \tag{2.131}
\]

The Willmore equation (2.111) to second order in \( \epsilon \) is given by the system of equations
\[
\frac{1}{\epsilon^2} \partial_t^2 H_1^\ell [\phi_0, \phi_1, \phi_2] + \frac{2}{\epsilon^2} \partial_t H_2^\ell [\phi_0, \phi_1, \phi_2] = \left( -\Delta [H_1^\ell] + 2H_1^\ell [K_1^\ell] \right) [\phi_0, \phi_1, \phi_2], \tag{2.132}
\]
and
\[
H_2^\ell [\phi_0, \phi_1, \phi_2] = a_s [\phi_0, \phi_1] \partial_x^2 \phi_2 + b_s [\phi_0, \phi_1] \partial_y^2 \phi_2 + c_s [\phi_0, \phi_1] \phi_2 + d_s [\phi_0, \phi_1], \tag{2.133}
\]
where \( a, b, c, d \) and \( H_1^\ell, K_1^\ell, \Delta^\ell \) can be explicitly expressed in terms of \( \phi_0 \) and \( \phi_1 \). This gives, using the boundary conditions \( \phi_2(0) = \phi_{2c}(0) = \phi_{2c}(0) = 0 \),
\[
\phi_2(\zeta) = \frac{37}{64} \phi_0 + \frac{69}{64} \phi_0 \zeta^3 + \frac{1}{2} \phi_0 \zeta^5 + \zeta \left( \frac{37}{64} \phi_0 \zeta + \frac{17}{16} \phi_0 \zeta^2 - \frac{9}{16} \phi_0 \zeta^3 \right)
+ \zeta^3 \left( \frac{9}{16} \phi_0 \zeta^2 + \frac{5}{32} \phi_0 \zeta^3 \right) + \zeta^4 \left( -\frac{7}{48} \phi_0 \zeta + \frac{3}{16} \phi_0 \zeta^2 \right) \tag{2.134}
+ \zeta^5 \left( \frac{1}{12} \phi_0 \zeta + \frac{1}{16} \phi_0 \zeta^2 \right) - \frac{1}{120} \phi_0 \zeta^3 + \frac{1}{288} \phi_0 \zeta^4.
\]

The function \( \rho_0 \) (see (2.131)) is the quasi-stationary part of the solution of the rescaled Willmore flow (2.110). Consider the left hand side of equation (2.110), where we substitute \( \rho \) for \( \rho_0 \). Then,
\[
\lambda^4 \rho_0 + \lambda^3 \rho_0 (\zeta - \zeta \phi_{0c}) = \lambda^4 (\zeta \phi_0 + 2 \epsilon \phi_2 + \cdots)
+ \lambda^3 (\phi_0 - \zeta \phi_{0c} + \epsilon \phi_1 - \epsilon \zeta \phi_{1c} + \cdots). \tag{2.135}
\]

To find a higher approximation of a solution to (2.110) we have to add terms of order \( \lambda^3 \), \( \lambda^2 \epsilon \), \( \lambda \epsilon^2 \), etc. We assume \( \lambda \ll \lambda \). Hence, we start by perturbing the quasi-stationary solution with \( \lambda \lambda \rho_0 (\zeta) \). Consider the asymptotic expansion
\[
\rho(\zeta, t) \sim \rho_0(\zeta, t) + \lambda \lambda \rho_0 (\zeta) + O((\lambda \lambda^3)^2), \quad \text{for } t \rightarrow T, \tag{2.136}
\]
and write
\[
H_{\lambda}[\rho] = H_0^\ell [\rho_0] + \lambda \lambda H_1^\ell [\rho_0, \rho_1] + O((\lambda \lambda^3)^2), \tag{2.137}
\]
\[
K_{\lambda}[\rho] = K_0^\ell [\rho_0] + \lambda \lambda K_1^\ell [\rho_0, \rho_1] + O((\lambda \lambda^3)^2), \tag{2.138}
\]
and
\[
\Delta_{\lambda}[\rho] = \Delta_0^\ell [\rho_0] + \lambda \lambda \Delta_1^\ell [\rho_0, \rho_1] + O((\lambda \lambda^3)^2). \tag{2.139}
\]

The rescaled Willmore flow (2.110) up to first order in \( \lambda \lambda \) (and zeroth order in \( \epsilon \)) is, in this notation, given by
\[
\sqrt{1 + \phi_{0c}^2} \left( \frac{1}{\epsilon^2} \partial_t^2 H_1^\ell [\phi_0, \rho] + \frac{2}{\epsilon^2} \partial_t H_2^\ell [\phi_0, \rho] \right) = \rho_0 - \zeta \phi_{0c}, \tag{2.140}
\]
where \( H_1^\ell [\phi_0, \rho] \) is now given by
\[
H_1^\ell [\phi_0, \rho] = \frac{1}{2 \cosh^2(\zeta)} \left( \cosh(\zeta) \rho_1 - 2 \sinh(\zeta) \rho_{1c} + \cosh(\zeta) \rho_{1c} \right). \tag{2.141}
\]
2.3. Formal asymptotics for the Limaçon

Using the same methods as in the case of the \( \epsilon \) perturbations and using the boundary conditions

\[
\rho_i(0) = \rho_{\phi}(0) = \rho_{\psi}(0) = \rho_{\phi\psi}(0) = 0, \tag{2.142}
\]

we find

\[
\rho_1 = -\frac{5}{64} \cosh^4 + \frac{1}{128} \sinh \cosh^3 + \frac{81}{128} \cosh^2 + \frac{21}{32} \sinh \cosh^3 + \frac{1}{4} \cosh^3 \sinh
- \frac{1}{24} \sinh \cosh^2 + \frac{71}{128} \cosh - \frac{21}{32} \sinh \cosh - \frac{7}{32} \sinh
- \frac{1}{48} \sinh \cosh + \frac{1}{240} \sinh. \tag{2.143}
\]

Hence, the inner solution \( r(z) \) for \( t \to T \) is given by

\[
r = \lambda \rho \sim \lambda \cosh(\zeta) + \epsilon \lambda \phi_1(\zeta) + \epsilon^2 \lambda \phi_2(\zeta) + \lambda \lambda^3 \rho_1(\zeta). \tag{2.144}
\]

As \( \phi_1, \phi_2 \) and \( \rho_1 \) are perturbations of \( \cosh(\zeta) \), this expression of the inner solution is valid for all \( \zeta \) such that

\[
\epsilon \phi_1(\zeta) \ll \cosh(\zeta), \quad \epsilon^2 \phi_2(\zeta) \ll \cosh(\zeta) \quad \text{and} \quad \lambda \lambda^3 \rho_1(\zeta) \ll \cosh(\zeta). \tag{2.145}
\]

The idea of matched asymptotics is to equate the inner solution to the outer solution of Subsection 2.3.4, in an intermediate region. The inner solution \( \rho^+ \) in this intermediate region is given by large \( \rho \) and \( \zeta \). Hence, for \( t \to T \),

\[
\rho^+(\zeta, t) \sim \frac{1}{2} \zeta^2 + \epsilon \left( \frac{1}{16} - \frac{1}{32} \zeta \right) \cosh + \epsilon^2 \left( \frac{1}{512} + \frac{9}{512} \zeta^2 + \frac{1024}{512} \zeta \right) \cosh
+ \lambda \lambda^3 \left( \frac{5}{2048} + \frac{1}{1024} \zeta \right) \cosh \tag{2.146}
\]

\[
= \phi^+_0(\zeta) + \epsilon \phi^+_1(\zeta) + \epsilon^2 \phi^+_2(\zeta) + \lambda \lambda^3 \rho^+_1(\zeta),
\]

with \( \epsilon \phi^+_0 \ll \phi^+_0, \epsilon^2 \phi^+_2 \ll \phi^+_0 \) and \( \lambda \lambda^3 \rho^+_1 \ll \phi^+_0 \).

Since the outer solution (see Subsection 2.3.4) is expressed as a function \( z \) of \( r \), we invert the inner solution \( \rho^+(\zeta, t) \) so that we are able to match them. Let the inversion \( \zeta^+(\rho, t) \) of the inner solution (2.146), for large \( \rho \), be given by

\[
\zeta^+(\rho, t) \sim \varphi^+_0(\rho) + \epsilon \varphi^+_1(\rho) + \epsilon^2 \varphi^+_2(\rho) + \lambda \lambda^3 \varphi^+_1(\rho), \quad \text{for} \ t \to T, \tag{2.147}
\]

with \( \varphi^+_1 \ll \varphi_0, \epsilon \varphi^+_2 \ll \varphi_0 \) and \( \lambda \lambda^3 \varphi^+_1 \ll \varphi_0 \), such that \( \rho^+(\zeta^+(\rho, t), t) \sim \rho \). Then,

\[
\varphi^+_0 = \ln(2\rho)
\varphi^+_1 = \left( \frac{1}{2} + \frac{1}{4} \ln(2\rho) \right) \rho^2
\varphi^+_2 = 0 \tag{2.148}
\]

\[
\varphi^+_1 = \left( \frac{5}{64} - \frac{1}{32} \ln(2\rho) \right) \rho^4.
\]

Hence, the inner solution near the outer region, denoted by \( \zeta^+ \) is given by

\[
\zeta^+(\rho, t) = \ln(2\rho) + \epsilon \left( \frac{1}{2} + \frac{1}{4} \ln(2\rho) \right) \rho^2
+ \lambda \lambda^3 \left( \frac{5}{64} - \frac{1}{32} \ln(2\rho) \right) \rho^4 + O(\epsilon^3, \lambda^2 \rho^4), \tag{2.149}
\]

for \( t \to T \).
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2.3.2 One specific outer solution

In this Subsection we consider one specific outer solution. As explained in the introduction of this Section, the typical scale of the outer region is the self-similar scale. In this Chapter we assume that this scale is given by the rate by which the $r$-coordinate of the Limaçon drops towards the $z$-axis. The scale of the $z$-coordinate does not have to be self-similar. We do expect it to be smaller than order 1, as the vanishing region disappears into the origin. In fact, we assume the scale of the $z$-coordinate to be smaller than the scale of the $r$-coordinate. See the introduction of this Section for a discussion on these assumptions.

In this Subsection we construct a specific outer solution after choosing a particular scale for the $z$-coordinate. In Subsection 2.3.4 we see that this choice of the outer solution is the slowest disappearing solution in a family of possible outer solutions. To simplify calculations we first construct this specific outer solution and match it to the inner and remote solution. In Subsections 2.3.4 and 2.3.5 we see that the matching for the other, smaller solutions, is similar. Since we assume that the $z$-scale drops faster to zero than the $r$-scale, we express $z$ in terms of $r$ such that $\partial_r z \to 0$ as $t \to T$. Therefore we use equation (2.94) combined with equation (2.93).

Define the functions

$$
\mu(t) := |T - t|^{\frac{1}{4}}, \quad \text{and} \quad \nu(t) := |T - t|^{\frac{1}{2}},
$$

for $T$ the blowup time of the problem. Consider the following rescaling for $z$.

$$
z(r, t) = \nu Z(P, t), \quad \text{with} \quad P = \frac{r}{\mu},
$$

where we assume that the asymptotic behavior of $Z$, for $t \to T$, is given by

$$
Z = O\left(\ln\left(\frac{T}{T - t}\right)^{\gamma}\right),
$$

for a certain $\gamma$. Hence, we assume that the scale of $z$, corresponding to the self-similar scale of $r$, is given by

$$
z = O\left(|T - t|^{\frac{1}{4}} \ln\left(|T - t|\right)^{\gamma}\right), \quad \text{for} \quad t \to T,
$$

where $\gamma$ is determined by the matching. In Section 2.4 we discuss this choice of algebraic decay. Having defined the scales, we find

$$
z_t = \nu Z_t + \nu Z_i - \frac{\nu}{\mu} P Z_P,
$$

and

$$
z_r = \frac{\nu}{\mu} Z_P, \quad z_{rr} = \frac{\nu}{\mu^2} Z_{PP}.
$$

We can rescale the geometric structures $H$, $K$ and $\Delta$ and expand them around $\frac{r}{\mu} = (T - t)^{\frac{1}{4}} \to 0$. Rescaling gives,

$$
H[z] = \frac{1}{2} z_t + z^2 + r z_r = \frac{1}{\mu} \frac{\nu}{\mu} Z_P + \frac{\nu}{\mu^2} P Z_{PP} + \frac{\nu}{\mu^2} \bar{Z}_{PP} = \frac{\nu}{\mu^2} H_{RS}[Z],
$$

for
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\[ K[z] = \frac{z \, z''}{r(1 + z^2)^2} = \frac{\nu^2}{\mu^4} \frac{Z_{PP} P}{(1 + \frac{\nu^2}{\mu^4} Z_r^2)^2} = \frac{\nu^2}{\mu^4} K_{RS}[Z], \quad (2.157) \]

and

\[ \Delta[z] = \frac{1}{r \sqrt{1 + z^2}} \partial_r \left( \frac{r}{\sqrt{1 + z^2}} \partial_r \right) = \frac{1}{\mu^4} \frac{1}{P \sqrt{1 + \frac{\nu^2}{\mu^4} Z_r^2}} \partial_r \left( \frac{P}{\sqrt{1 + \frac{\nu^2}{\mu^4} Z_r^2}} \partial_r \right) = \frac{1}{\mu^4} \Delta_{RS}[Z]. \quad (2.158) \]

The rescaled Willmore flow is

\[ \frac{\dot{\nu}}{\nu^4} Z + \mu^4 Z_t - \dot{\mu}^3 P Z_P = -\sqrt{1 + \frac{\nu^2}{\mu^4} Z_r^2} \left[ \Delta_{RS} H_{RS} + \frac{\nu^4}{\mu^4} 2 H_{RS} \left( H_{RS}^2 - K_{RS} \right) \right]. \quad (2.159) \]

Since \( \mu \) is the self-similar scale \((T - t)^{1/4}\) we find that

\[ \frac{\dot{\nu}}{\nu^4} = -\frac{1}{2} \quad \text{and} \quad \dot{\mu}^3 = -\frac{1}{4}, \quad (2.160) \]

and, using assumption (2.152), \( \mu^4 Z_t \ll Z \) as \( t \to T \). Expanding the rescaled Willmore flow (2.159) around \( \mu^4 = 0 \) and \( \nu^4 = 0 \) and taking the zeroth order terms, we find a linear ordinary differential equation given by

\[ -\frac{1}{2} Z + \frac{1}{4} P Z_P = -\frac{1}{2} \partial_r \left( P \partial_r \left( \frac{1}{P} \partial_r (P \partial_r Z) \right) \right) \]

\[ = -\frac{1}{2} \left( Z_{PPP} + 2 \frac{Z_{PPP}}{P} - \frac{Z_{PP}}{P^2} + \frac{Z_P}{P^3} \right). \quad (2.161) \]

To solve this equation we use the Frobenius method. Consider a solution \( Z \) of (2.161) as a power series given by

\[ Z = P^m \sum_{i=0}^{\infty} a_i P^i, \quad (2.162) \]

with \( a_0 \neq 0 \). If this is a solution of equation (2.161) it has to obey

\[ \sum_{i=0}^{\infty} a_i E_{m,i} P^{m+i} + 2 \sum_{i=0}^{\infty} a_i \left( 1 - \frac{1}{2} (m + i) \right) P^{m+i} \]

\[ = \sum_{i=0}^{\infty} a_i \left( 1 - \frac{1}{2} (m + i) \right) P^{m+i}, \quad (2.163) \]

where

\[ E_{m,i} := (m + i)(m + i - 1)(m + i - 2)(m + i - 3) \]

\[ + 2(m + i)(m + i - 1)(m + i - 2) - (m + i)(m + i - 1) + (m + i) \]

\[ = (m + i)^2 \left( (m + i - 2) \right)^2. \quad (2.164) \]
This gives the indicial equation
\[ E_{0,m} = m^4 - 4m^3 + 4m^2 = 0, \tag{2.165} \]
since \( a_0 \neq 0 \) and
\[
\begin{align*}
0 &= a_1(m^4 - 2m^2 + 1), \\
0 &= a_2(m^4 + 4m^3 + 4m^2), \\
0 &= a_3(m^4 + 8m^3 + 22m^2 + 24m + 9),
\end{align*}
\tag{2.166}
\]
and
\[
a_i = -a_i - \frac{1}{2}(m + i - 4) - \frac{1}{E_{i,m}}, \quad \text{for} \ i \geq w.
\tag{2.167}
\]
The indicial equation (2.165) gives \( m = 0 \) or \( m = 2 \), both with multiplicity 2. In both cases we need to set \( a_1 = a_3 = 0 \). Consider the case where \( m = 2 \). In this case we also have to set \( a_2 = 0 \) and the solution is
\[
Z_1 := Z \bigg|_{m=2} = p^2, \tag{2.168}
\]
where we choose \( a_0 = 1 \). Define the operator \( L \) by
\[
L := \frac{1}{2} - \frac{1}{4} P \partial_p - \frac{1}{2} \left( \partial_p^2 + \frac{2}{P} \partial_p^3 - \frac{1}{P^2} \partial_p^4 + \frac{1}{P^3} \partial_p^5 \right).
\tag{2.169}
\]
Then \( L[Z] = 0 \). Since the derivatives \( \partial_p \) and \( \partial_p \) commute, we have (if the \( a_i \) are defined as in (2.167))
\[
L[\partial_p Z] = \partial_p L[Z] \bigg|_{m=2} = \partial_p \left( \frac{1}{2} p^{n-4} \left( E_{0,m} + \sum_{i=1}^{\infty} a_i E_{i,m} - a_{i-4} \left( 1 - \frac{1}{2}(m + i - 4) \right) \right) \right) \bigg|_{m=2}
\]
\[
= \partial_p \left( \frac{1}{2} p^{n-4} E_{0,m} \right) \bigg|_{m=2}
\]
\[
= \left( \frac{1}{2} p^{n-4} \ln(P) E_{0,m} - \frac{1}{2} p^{n-4} \partial_p E_{0,m} \right) \bigg|_{m=2}.
\tag{2.170}
\]
Because \( m = 2 \) is a root of the indicial equation (2.165) with multiplicity 2, we know that \( \partial_p Z \bigg|_{m=2} \) also has to be a solution of equation (2.161). Hence, we find a second solution \( Z_3 \) given by
\[
Z_3 := \partial_p Z \bigg|_{m=2} = \ln(P) Z_1 + \sum_{i=1}^{\infty} \left( \partial_p a_i \right) \bigg|_{m=2} p^{2+i} = p^2 \ln(P) - \frac{1}{1152} p^6 + \frac{1}{3686400} p^8 + \cdots. \tag{2.171}
\]
The other two solutions of (2.161) are given in a similar way by \( m = 0 \). Again \( a_1 = a_3 = 0 \). We can choose \( a_2 = 0 \), since its contribution just gives a multiple of \( Z_1 \). We find (again taking \( a_0 = 1 \)),
\[
Z_4 := Z \bigg|_{m=0} = 1 + \frac{1}{64} p^4 - \frac{1}{73728} p^8 + \cdots, \tag{2.172}
\]
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and

\[ Z_4 = \ln(P) Z_3 + \sum_{i=1}^{\infty} (\partial_n a_i) \bigg|_{n=0} P^n \]

\[ = \ln(P) - \frac{1}{32} P^4 + \frac{1}{64} P^4 \ln(P) + \cdots. \]  

Hence, the general solution \( Z_0 \) to (2.161) is given by a linear combination

\[ Z_0 = A_1 Z_1 + A_2 Z_2 + A_3 Z_3 + A_4 Z_4. \]  

The constants are determined by the boundary conditions. These conditions are in fact the matching conditions we acquire when matching the outer solution to the inner and remote solution. To match the outer solution to the remote solution we need to study the behavior of (2.174) for large \( P \). To do this we examine equation (2.161) for large \( P \) and use the WKB approach. Approximate a solution of (2.161) in the limit \( P \to \infty \) with

\[ e^{i\Theta(P)} \]

for some function \( \Theta \) of \( P \). Assume that \( \Theta' \) obeys \( \Theta'' \ll \Theta' \) for \( P \to \infty \). Hence, equation (2.161) in the limit \( P \to \infty \) is then given by

\[ \frac{1}{2} - \frac{1}{4} P \Theta' - \frac{1}{2} \left( \Theta'^2 + 2 \Theta' \frac{\Theta''}{P} - \frac{\Theta'^2}{P^2} + \frac{\Theta'}{P} \right) = 0. \]  

Assuming superlinear growth, \( \Theta' \to \infty \), this leads to

\[ \frac{1}{4} P \Theta' + \frac{1}{2} \Theta'^2 = 0. \]  

Hence,

\[ \Theta_0' \sim -2^{\frac{3}{4}} P^{\frac{1}{2}} \quad \text{and} \quad \Theta_1' \sim 2^{-\frac{1}{4}} \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{3} \right) P^{\frac{1}{2}}, \]  

and \( \Theta' = 0 \) which does not obey the assumption \( \Theta' \to \infty \). This gives

\[ \Theta_0 \sim 2^{\frac{3}{8}} P^{\frac{1}{2}} + C_0 \quad \text{and} \quad \Theta_1 \sim 2^{\frac{3}{16}} \left( \frac{3}{16} \pm \frac{3}{16} \sqrt{3} \right) P^{\frac{1}{2}} + C_\pm. \]  

The last solution we find by taking \( \Theta' \ll 1 \) as \( P \to \infty \). Then

\[ \Theta' \sim \frac{2}{P}. \]  

Hence,

\[ \Theta = 2 \ln(P) + C. \]  

Thus, in the limit \( P \to \infty \) we have

\[ Z_0 \sim \tilde{C} P^2 + \tilde{C}_0 e^{-\frac{1}{8} P^{\frac{1}{2}}} + \tilde{C}_- e^{\frac{1}{4} \pm \frac{1}{2} \sqrt{3} P^{\frac{1}{2}}} + \tilde{C}_- e^{\frac{1}{4} \pm \frac{1}{2} \sqrt{3} P^{\frac{1}{2}}}. \]  

This can be written as

\[ Z_0 \sim \tilde{C} P^2 + \tilde{C}_0 \exp\left( -\frac{3}{8} P^{\frac{1}{2}} \right) + \tilde{C}_- \exp\left( \frac{3}{16} P^{\frac{1}{2}} \right) \sin\left( 2^{\frac{3}{16}} \frac{3}{16} P^{\frac{1}{2}} \right) + \tilde{C}_+ \exp\left( 2^{\frac{3}{16}} \frac{3}{16} P^{\frac{1}{2}} \right) \cos\left( 2^{\frac{3}{16}} \frac{3}{16} P^{\frac{1}{2}} \right). \]
We note that these C’s can still depend "weakly" on P.

To improve the approximation of Z₀ for P → ∞, we approximate Z₀ now with

\[ Z₀ \sim c₁Z₁,∞ + c₂Z₂,∞ + c₃Z₃,∞ + c₄Z₄,∞, \]  

(2.183)

with

\[ Z₁,∞ := P^2 \]
\[ Z₂,∞ := P^{-3} \exp\left(-\frac{3}{8}P^⁺\right) \]
\[ Z₃,∞ := P^{-3} \exp\left(\frac{21}{16}P^⁺\right) \sin\left(\frac{21}{16}\sqrt{3}P^⁺\right) \]
\[ Z₄,∞ := P^{-3} \exp\left(\frac{21}{16}P^⁺\right) \cos\left(\frac{21}{16}\sqrt{3}P^⁺\right). \]

(2.184)

We see that two of the four solution grow exponentially fast as P → ∞. These functions give the asymptotic behavior of the solution in the remote region and have to be matched with the remote solution. But because the remote solution does not exhibit oscillatory behavior we disregard this option and conclude that c₃ and c₄ in (2.183) have to be zero. Hence, the boundary conditions on Z₀ are such that Z₀ does not grow exponentially fast as P → ∞.

Consider again the linear combination (2.174). For large P every Zᵢ can be written as

\[ Zᵢ \sim Γᵢ₁Z₁,∞ + Γᵢ₂Z₂,∞ + Γᵢ₃Z₃,∞ + Γᵢ₄Z₄,∞. \]

(2.185)

For example, the contribution to the exponential sine growth of the solution Z₁ is given by Γ₁, and the contribution to the exponential cosine growth of Z₃ is given by Γ₄. Since Z₁ = P², we immediately conclude that Γ₁ = Γ₄ = 0. The matching conditions, i.e., matching with the remote solution, determine two of the four constants of (2.174) by

\[ A₁Γ₂₃ + A₃Γ₃₃ + A₄Γ₄₃ = 0 \]
\[ A₄Γ₂₄ + A₃Γ₃₄ + A₄Γ₄₄ = 0. \]

(2.186)

The other two constants are determined by matching with the inner solution.

To find a solution Z to the rescaled Willmore flow (2.159) we use Z₀ as a first approximation and perturb it by multiplying with logarithmic terms in time and adding terms of order \( ν²µ² \). Consider the term \( µ^₄Z_t \) in the rescaled Willmore flow (2.159). Since the time scale of Z is given by (2.152) we find

\[ µ^₄Z_t \sim \gamma Z \ln\left(\frac{1}{t - T}\right), \]

(2.187)

which is much bigger than the perturbations of order \( ν²µ² \). Hence, we can write the asymptotic expansion of a solution to (2.159) as

\[ Z = F₁Z₀ + γF₂₋₁Z₁ + γ(γ - 1)F₁₋₂Z₂ + \cdots + O(\frac{ν²}{µ²}). \]

(2.188)

for \( t \to ∞ \), with

\[ F_γ(t) := \ln\left(\frac{1}{t - T}\right)^γ. \]

(2.189)
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For matching (2.188) with the inner solution we, most likely, need the explicit expressions for $Z_1$ and $Z_2$. These expressions are given in Subsection 2.3.3. The terms of order $\nu^2$ are so small that they do not play a role in matching. Substituting the expansion (2.188) into equation (2.159) we find

$$
\gamma F_{\gamma-1} \left( Z_0 - L[Z_1] \right) + \gamma(\gamma - 1) F_{\gamma-2} \left( Z_1 - L[Z_2] \right) + \cdots + O \left( \frac{\nu^2}{\rho^2} \right) = 0
$$

(2.190)

with $L$ as in (2.169). This gives

$$
L[Z_i] = Z_{i-1},
$$

(2.191)

with the the same boundary conditions as on $Z_0$, that is, no exponential growth for $P \to \infty$ and the matching conditions at $P \to 0$. Any solution to (2.191) can be written as

$$
Z_i = P^m \left( \sum_{n=0}^{\infty} a_n P^n + \sum_{n=0}^{\infty} b_n P^n \ln(P) \right) + B_1 Z_1 + B_2 Z_2 + B_3 Z_3 + B_4 Z_4,
$$

(2.192)

with $a_n$ and $b_n$ still to be determined and with $m \geq 4$, since for $0 \leq m < 4$ we just get the same indicial equation (2.165), which gives one of the homogeneous solutions. The matching conditions determine the constants $B_k$ of the homogeneous solutions.

Since we want to match this solution to the inner solution, we have to consider (2.188) in the limit of $P \to 0$. That is,

$$
Z \sim F_{\gamma}(t) \left[ A_1 P^2 + A_2 P^2 \ln(P) + A_3 + A_4 \ln(P) \right]
$$

$$
+ \gamma F_{\gamma-1}(t) \left[ B_1 P^2 + B_2 P^2 \ln(P) + B_3 + B_4 \ln(P) \right]
$$

$$
+ \gamma(\gamma - 1) F_{\gamma-2}(t) \left[ C_1 P^2 + C_2 P^2 \ln(P) + C_3 + C_4 \ln(P) \right],
$$

(2.193)

for $P \to 0$.

2.3.3 Matching the inner solution to the specific outer solution

In this Section we match the inner solution to the outer solution obtained in Subsection 2.3.2. This determines the constant $\gamma$ in (2.152) and the behavior of the blowup rate $\lambda$.

We note that matching only succeeds for curves that do not cross the $r$-axis.

Consider the inner solution in the limit of the outer scale given by equation (2.149). That is,

$$
\zeta^* = \ln(2\rho) + \rho \left( -\frac{1}{2} + \frac{1}{4} \ln(2\rho) \right) \rho^2 + \lambda \rho^3 \left( \frac{5}{64} - \frac{1}{32} \ln(2\rho) \right) \rho^4 + \cdots
$$

(2.194)

As we want to match this solution to the outer solution we have to consider $Z = \frac{1}{r} z = \frac{1}{\rho} \zeta$. 

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and express it in terms of the self-similar coordinate \( P = \frac{t}{\lambda} \). Hence,

\[
\begin{align*}
\mathcal{Z} &= \lambda^\nu \ln \left( \frac{\nu}{\lambda} P \right) + \epsilon \left( -\frac{1}{2} + \frac{1}{4} \ln \left( \frac{\nu}{\lambda} P \right) \right) \mu^2 \lambda^2 P^2 \\
&\quad + \lambda \lambda^\nu \left( \frac{5}{64} - \frac{1}{32} \ln \left( \frac{\nu}{\lambda} P \right) \right) \mu^2 \lambda^2 P^4 + \cdots \\
&= \lambda^\nu \left[ \ln(2) - \frac{1}{4} \ln \left( \frac{1}{T-t} \right) - \ln(\lambda) + \ln(P) \right] \\
&\quad + \epsilon \left( -\frac{1}{2} + \frac{1}{4} \ln(2) - \frac{1}{16} \ln \left( \frac{1}{T-t} \right) - \frac{1}{4} \ln(\lambda) + \frac{1}{4} \ln(P) \right) \mu^2 \lambda^2 P^2 \\
&\quad + \lambda \lambda^\nu \left( \frac{5}{64} + \frac{1}{32} \ln(2) + \frac{1}{128} \ln \left( \frac{1}{T-t} \right) + \frac{1}{32} \ln(\lambda) - \frac{1}{32} \ln(P) \right) \mu^2 \lambda^2 P^4 + \cdots \\
&= \lambda^\nu \ln(2) - \frac{1}{4} \ln \left( \frac{1}{T-t} \right) - \ln(\lambda) + \ln(P) \\
&\quad + \epsilon \left( -\frac{1}{2} + \frac{1}{4} \ln(2) - \frac{1}{16} \ln \left( \frac{1}{T-t} \right) - \frac{1}{4} \ln(\lambda) + \frac{1}{4} \ln(P) \right) \mu^2 \lambda^2 P^2 \\
&\quad + \lambda \lambda^\nu \left( \frac{5}{64} + \frac{1}{32} \ln(2) + \frac{1}{128} \ln \left( \frac{1}{T-t} \right) + \frac{1}{32} \ln(\lambda) - \frac{1}{32} \ln(P) \right) \mu^2 \lambda^2 P^4 + \cdots.
\end{align*}
\]

(2.195)

where the bar in \( \mathcal{Z} \) denotes the fact that this is the inner solution in the self-similar coordinates and scaling.

Consider now the outer solution (2.193), for small \( P \). Matching (2.195) with (2.193) we get, for the terms up to order two in \( P \),

\[
\begin{align*}
O(1) & : F_0 A_3 + \cdots \sim \lambda^\nu \ln(2) - \frac{1}{4} \ln \left( \frac{1}{T-t} \right) - \ln(\lambda), \\
O(\ln(P)) & : F_0 A_4 + \cdots \sim \lambda^\nu, \\
O(P^2) & : F_0 A_1 + \cdots \sim \epsilon \left( -\frac{1}{2} + \frac{1}{4} \ln(2) - \frac{1}{16} \ln \left( \frac{1}{T-t} \right) - \frac{1}{4} \ln(\lambda) + \frac{1}{4} \ln(P) \right) \mu^2 \lambda^2 P^2 \\
&\quad + \lambda \lambda^\nu \left( \frac{5}{64} + \frac{1}{32} \ln(2) + \frac{1}{128} \ln \left( \frac{1}{T-t} \right) + \frac{1}{32} \ln(\lambda) - \frac{1}{32} \ln(P) \right) \mu^2 \lambda^2 P^4 + \cdots.
\end{align*}
\]

(2.196)

Considering the \( O(\ln(P)) \)-terms and the \( O(\ln(P)P^2) \)-terms in (2.196) gives us

\[
\lambda^\nu = O \left( \ln \left( \frac{1}{T-t} \right)^{\gamma_a} \right) \quad \text{and} \quad \mu^2 \lambda^2 = O \left( \ln \left( \frac{1}{T-t} \right)^{\gamma_b} \right),
\]

(2.197)

with, for now, undetermined \( \gamma_a \) and \( \gamma_b \). Hence, for \( t \) close to \( T \),

\[
\lambda \sim a(t)(T-t)^{\frac{\gamma_a}{2}} \quad \text{and} \quad \epsilon \sim b(t)(T-t)^{\frac{\gamma_b}{2}},
\]

(2.198)

with

\[
a(t) = \alpha \ln \left( \frac{1}{T-t} \right)^{\gamma_a} \quad \text{and} \quad b(t) = \beta \ln \left( \frac{1}{T-t} \right)^{\gamma_a + \gamma_b}.
\]

(2.199)

Substituting this into the matching conditions (2.196) and taking the limit \( t \to T \), gives

\[
\begin{align*}
O(1) & : F_0 A_3 + \gamma F_0 - 1 B_3 + \gamma (\gamma - 1) F_0 - 2 C_3 + \cdots \sim \frac{a}{4} \ln \left( \frac{1}{T-t} \right), \\
O(\ln(P)) & : F_0 A_4 + \gamma F_0 - 1 B_4 + \gamma (\gamma - 1) F_0 - 2 C_4 + \cdots \sim \alpha, \\
O(P^2) & : F_0 A_1 + \gamma F_0 - 1 B_1 + \gamma (\gamma - 1) F_0 - 2 C_1 + \cdots \sim \frac{b}{a} \frac{1}{16} \ln \left( \frac{1}{T-t} \right), \\
O(P^2 \ln(P)) & : F_0 A_2 + \gamma F_0 - 1 B_2 + \gamma (\gamma - 1) F_0 - 2 C_2 + \cdots \sim \frac{b}{a} \frac{1}{4}.
\end{align*}
\]

(2.200)
Comparing the \(O(1)\)-terms with the \(O(\ln(P))\)-terms in (2.200) gives

\[
F, A_3 + \gamma F_{\gamma - 1} B_1 + \cdots = \frac{1}{4} \ln \left( \frac{1}{P - T} \right) \left[ F, A_4 + \gamma F_{\gamma - 1} B_4 + \cdots \right]
\]

\[
= \frac{1}{4} \left[ F, A_4 + \gamma F_{\gamma} B_4 + \cdots \right],
\]

where we used \(F, A_3\) as defined in (2.189). Hence,

\[
A_4 = 0, \quad B_4 = \frac{4}{\gamma} A_3, \quad C_4 = \frac{4}{\gamma - 1} B_3, \quad \text{etc.,}
\]

(2.202)

and similarly, by comparing the \(O(P)\) and the \(O(P^2 \ln(P))\) terms,

\[
A_2 = 0, \quad B_2 = \frac{4}{\gamma} A_1, \quad C_2 = \frac{4}{\gamma - 1} B_1, \quad \text{etc.}
\]

(2.203)

Consider again the boundary conditions on \(P \to \infty\), given by (2.186). Because \(A_2 = A_4 = 0\), the boundary conditions are reduced to

\[
A_3 \Gamma_{33} = t \quad \text{and} \quad A_3 \Gamma_{34} = 0.
\]

(2.204)

Since \(\Gamma_{33} \neq 0\) (see Subsection 2.3.6), we conclude that \(A_3\) is also zero. Hence, the zeroth order term of the expansion of the solution to the rescaled Willmore flow (2.161) is given by the monomial

\[
Z_0 = A_1 P^2,
\]

(2.205)

where \(A_1 \neq 0\), by definition. The first order term \(Z_1\) of the expansion of the solution of (2.161), corresponding to a logarithmic perturbation, is then given by a solution of

\[
L[Z_1] = A_1 P^2.
\]

(2.206)

A solution to this equation is

\[
Z_1 = -4A_1 P^2 \ln(P) + \mathcal{B}_1 Z_1 + \mathcal{B}_2 Z_2 + \mathcal{B}_3 Z_3 + \mathcal{B}_4 Z_4,
\]

(2.207)

The \(\mathcal{B}_i\)'s are fixed by the boundary conditions on \(Z_1\) at \(P \to \infty\) and the matching conditions (2.200). The boundary conditions on \(Z_1\) for \(P \to \infty\) are similar to (2.186) and given by

\[
\mathcal{B}_1 \Gamma_{23} + \mathcal{B}_2 \Gamma_{31} + \mathcal{B}_3 \Gamma_{43} = 0,
\]

\[
\mathcal{B}_2 \Gamma_{23} + \mathcal{B}_4 \Gamma_{44} = 0.
\]

(2.208)

The relations between the \(B_i\) and the \(\mathcal{B}_i\) are given by

\[
B_1 = \mathcal{B}_1, \quad B_2 = -4A_1 + \mathcal{B}_2, \quad B_3 = \mathcal{B}_3, \quad \text{and} \quad B_4 = \mathcal{B}_4.
\]

(2.209)

Since we have concluded that \(A_3 = 0\) we have, by (2.202), that \(B_4 = 0\). Hence, the boundary conditions (2.208) reduce to

\[
\mathcal{B}_1 \Gamma_{23} + \mathcal{B}_3 \Gamma_{33} = 0,
\]

\[
\mathcal{B}_2 \Gamma_{24} + \mathcal{B}_4 \Gamma_{44} = 0.
\]

(2.210)
In Subsection 2.3.6 we find that \( \Gamma_2 \Gamma_{33} - \Gamma_{31} \Gamma_{33} \neq 0 \). Hence, we conclude that \( \overline{\Gamma_2} = \overline{\Gamma_3} = 0 \).

Combining (2.203) with (2.209) and \( \overline{\Gamma_2} = 0 \) gives \(-4A_1 = \frac{2}{3}A_4 \). Hence, \( \gamma = -1 \). The second order term, \( Z_2 \), in the expansion of the solution of (2.161) is then given by

\[
L[Z_2] = Z_1 = -4A_1 P^2 \ln(P) + B_1 P^2.
\]

Consider the series

\[
Z_2 = \sum_{i=0}^{\infty} \left( e_i + f_i \ln(P) \right) P^{n+i}.
\]

Then

\[
L[Z_2] = -\frac{1}{2} \left[ \sum_{i=0}^{\infty} e_i E_{i,n} P^{n+i-4} - \sum_{i=0}^{\infty} e_{i-4} \left( \frac{1}{2} (n + i - 6) \right) P^{n+i-4} \right]
\]

\[
+ \frac{1}{2} \left[ \sum_{i=0}^{\infty} f_i E_{i,n} \ln(P) P^{n+i-4} - \sum_{i=4}^{\infty} f_{i-4} \left( \frac{1}{2} (n + i - 6) \right) \ln(P) P^{n+i-4} \right]
\]

\[
- \frac{1}{2} \left[ \sum_{i=0}^{\infty} f_i E_{i,n} P^{n+i-4} + \sum_{i=4}^{\infty} \left( \frac{1}{2} f_{i-4} P^{n+i-4} \right) \right]
\]

where

\[
E_{i,n} := 4(n+i)^3 - 12(n+i)^2 + 8(n+i),
\]

and \( E_{i,n} \) as in equation (2.164). Hence, if we define

\[
Z_2 := \sum_{i=0}^{\infty} \left( e_i + f_i \ln(P) \right) P^{n+i},
\]

with \( e_0 = -\frac{1}{127} \), \( f_0 = \frac{1}{2} \), \( e_1 = e_2 = e_3 = f_1 = f_2 = f_3 = 0 \), and

\[
f_i = -f_{i-4} \frac{i}{2E_{i,b}}.
\]

and

\[
e_i = -e_{i-4} \frac{i}{2E_{i,b}} - f_i \frac{F_{i,0}}{E_{i,b}} - \frac{1}{2} f_{i-4} \frac{i}{E_{i,b}}.
\]

we have

\[
L[Z_2] = -4P^2 \ln(P).
\]

Therefore, the general solution \( Z_2 \) to (2.211) is given by

\[
Z_2 = A_1 \overline{Z_2} - 4B_1 P^2 \ln(P) + \overline{c_1} Z_1 + c_2 \overline{Z_2} + \overline{c_3} Z_3 + \overline{c_4} Z_4,
\]

with \( \overline{Z_2} \) as in (2.218). The relations between the \( C_i \) and the \( \overline{C}_i \) are given by

\[
C_1 = \overline{C}_1, \quad C_2 = -4B_1 + \overline{C}_2, \quad C_3 = \overline{C}_3, \quad \text{and} \quad C_4 = \overline{C}_4.
\]

The boundary conditions on \( Z_2 \) are such that \( Z_2 \) does not grow exponentially fast, as \( P \to \infty \). This means that, if \( C_i \) is the contribution to \( \overline{Z_2} \) of the cosine exponential
growth and $C_s$ is the contribution to $Z_2$ of the sine exponential growth, the boundary conditions on $P \to \infty$ are given by

\begin{align}
C_2 \Gamma_{23} + C_3 \Gamma_{33} + C_4 \Gamma_{43} &= -A_1 C_s, \\
C_2 \Gamma_{24} + C_3 \Gamma_{34} + C_4 \Gamma_{44} &= -A_1 C_c.
\end{align}

Since $B_1 = 0$, we can conclude, by (2.202), that $C_3 = 0$. This reduces the boundary conditions (2.221) to

\begin{align}
C_2 \Gamma_{23} + C_3 \Gamma_{33} &= -A_1 C_s, \\
C_2 \Gamma_{24} + C_3 \Gamma_{34} &= -A_1 C_c.
\end{align}

Since, due to the computations in Subsection 2.3.6, we know that $\Gamma_{24} \neq 0$, we can rewrite the boundary conditions such that $\Gamma_3$ is given by

\begin{equation}
\Gamma_3 \left( \Gamma_{23} \Gamma_{33} - \Gamma_{24} \Gamma_{34} \right) = A_1 \left( C_3 \Gamma_{23} - C_4 \Gamma_{24} \right). \tag{2.223}
\end{equation}

Consider the $O(1)$-terms in (2.200). Because $A_1 = B_3 = 0$, $\gamma = -1$ and $\alpha > 0$, by positivity of $\lambda$, we know that $C_3 \geq 0$. We further know that $C_2 \Gamma_{23} - C_4 \Gamma_{24} > 0$, $A_1 \neq 0$ and $C_1 \Gamma_{23} - C_4 \Gamma_{24} \neq 0$ (see Subsection 2.3.6). Hence, $\Gamma_3 = C_3 > 0$ and therefore (see (2.202)) $D_4 \neq 0$.

The expansion (2.188) of the solution to the Willmore flow in the outer region can now be given by

\begin{align}
Z &= F_{-1} A_1 P^3 - F_{-2} \left( -4 A_1 P^2 \ln(P) + B_1 P^3 \right) \\
&+ 2 F_{-3} \left( A_1 \Gamma_2 - 4 B_1 P^2 \ln(P) + \Gamma_1 \Gamma_3 + \Gamma_2 \Gamma_4 + \Gamma_3 \Gamma_5 \right) \\
&- 6 F_{-4} \Gamma_3 + \cdots + O \left( \frac{1}{P^4} \right).
\end{align}

We remark that we can always add a homogeneous solution $V$ to (2.224), but that this $V$ does not affect $C_s$ or the sign of $A_1$. Even if the asymptotic behavior of $V$ is given by $V = O(P^4)$, with $-3 < \delta < -1$, we find that the boundary conditions reduce to (2.210) and $V$ does not influence $C_s$ or the sign of $A_1$. The matching conditions (2.200) then reduce to

\begin{align}
O(1) & : 2 F_{-3} \Gamma_3 + \cdots \sim \frac{a}{4} \ln \left( \frac{1}{T-t} \right), \\
O(\ln(P)) & : -6 F_{-4} D_4 + \cdots \sim a, \\
O(P^3) & : A_1 F_{-1} + \cdots \sim \frac{b}{10} \ln \left( \frac{1}{T-t} \right), \\
O(P^3 \ln(P)) & : 4 A_1 F_{-2} + \cdots \sim \frac{b}{10} \quad (2.225)
\end{align}

Hence, the exponents $\gamma_a$ and $\gamma_b$ in (2.197) are given by $\gamma_a = -4$ and $\gamma_b = -2$ such that

\begin{equation}
\lambda \sim a \ln \left( \frac{1}{T-t} \right) \, (T-t)^{\frac{3}{2}} \quad \text{and} \quad \epsilon \sim b \ln \left( \frac{1}{T-t} \right) \, (T-t)^{\frac{3}{2}}, \tag{2.226}
\end{equation}

for some constants $a_c$ and $b_c$. 

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Consider again equation (2.223). Since, by the computations in Subsection 2.3.6, we have that both $\Gamma_{23}\Gamma_{33} - \Gamma_{24}\Gamma_{34}$ and $C_1\Gamma_{23} - C_1\Gamma_{24}$ are positive, we find that the sign of $A_1$ is also positive. Consider the outer solution (2.188). For $t$ close to the blowup time $T$, the curve $z(r)$ in the outer region is given by

$$z \sim A_1 \ln\left(\frac{1}{T-t}\right)^{-1} (T-t)^{\frac{1}{2}} P^2.$$  \hspace{1cm} (2.227)

If we express this in the order 1 scale of the remote solution we have to include the second term of the expansion (2.224). We then find

$$z \sim A_1 \ln\left(\frac{1}{T-t}\right)^{-1} r^2 \left[2 + 4\ln\left(\frac{1}{T-t}\right)^{-1} \ln(r)\right].$$ \hspace{1cm} (2.228)

Hence, a positive value for $A_1$ means that, in the limit $t \to T$, the curve $z(r)$, in the inner and outer region (even in the remote limit), lies in the positive quadrant of the $(r, z)$-plane. This means that the curve in the vanishing regions does not cross the $r$-axis. This is in contradiction with the numerical computations we find in Chapter 3. In Chapter 3 we find that the Limaçon evolves towards two spheres and that the loop vanishes completely. This means we should detect the crossing of the $r$-axis within the vanishing scales. Hence, we have not found the blowup behavior for the Limaçon. In Subsection 2.4 we discuss how we can interpret this result. In the next Subsections we construct a more general outer solution and match this to the inner solution. We find similar results as in this case.

### 2.3.4 The general outer solution

In this Subsection we consider the general outer solution. Again, we assume that the scale of the $r$-coordinate is self-similar. The difference with Subsection 2.3.2 is that we do not predetermine the scale of $z$. The only assumptions we make on the scale of $z$ is that it is algebraic in $t$ and that it is smaller than the scale of $r$, so that \( \frac{\partial}{\partial t} z(r, t) \) goes to zero. We find a family of outer solutions similar to the one from Subsection 2.3.2. None of these approximations crosses the $r$-axis.

The outer solution is the solution in the region where $r$ is of the order of the self-similar scale. Consider the following rescaling for $z$.

$$z(r, t) = \nu Z(P, t), \quad \text{with} \quad P = \frac{r}{R},$$ \hspace{1cm} (2.229)

and

$$\mu(t) := (T-t)^{\frac{1}{2}} \quad \text{and} \quad \nu(t) := (T-t)^{\beta},$$ \hspace{1cm} (2.230)

with $T$ the blowup time. The value for $\beta$ is such that the asymptotic behavior of $Z$, for $t \to T$, is given by

$$Z = O\left((T-t)^{\gamma} \ln\left(\frac{1}{T-t}\right)^{\gamma}\right).$$ \hspace{1cm} (2.231)

for a certain $\gamma$. Again we mean, by this, that the scale of $z$ corresponding to the self-similar scale of $r$ is given by

$$z = O\left((T-t)^{\beta} \ln\left(\frac{1}{T-t}\right)^{\gamma}\right), \quad \text{for} \quad t \to T,$$ \hspace{1cm} (2.232)
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where $\beta$ and $\gamma$ are determined through the matching. It is assumed that $\beta \geq \frac{1}{4}$, so that $\nu \ll \mu$. Then

$$z_t = \dot{\nu} Z + \nu Z_t - \frac{\nu}{\mu} P Z_P,$$

(2.233)

and

$$z_r = \frac{\nu}{\mu} Z_P, \quad z_{rr} = \frac{\nu}{\mu^2} Z_{PP}.$$

(2.234)

We assume that the scale of $z_r$ goes to zero as $t \to T$. Hence, by assumption, we have that $\frac{\nu}{\mu} \to 0$. Just as in Subsection 2.3.2 we can rescale the Willmore flow and we find

$$\dot{\nu} Z + \mu Z_t - \mu^3 P Z_P = -\left(1 + \frac{\nu^2}{\mu^2} Z_t^2 \right) \Delta_{RS} H_{RS} + \frac{\nu^2}{\mu^2} 2 H_{RS} \left[H_{RS} - K_{RS}\right].$$

(2.235)

This gives, in zeroth order, the ordinary differential equation

$$-\beta Z + \frac{1}{4} P Z_P = -\frac{1}{2} P \partial_P \left(P \partial_P \left(\frac{1}{P} \partial_P (P \partial_P Z)\right)\right)$$

$$= -\frac{1}{2} \left(Z_{PPP} + 2 Z_{PP} \frac{Z_{PP}}{P^2} - \frac{Z_{P}}{P^2}\right).$$

(2.236)

Using the same techniques as in Subsection 2.3.2 we find the solutions

$$Z_1 := \sum_{i=0}^{\infty} a_i P^{2+i} \mid_{m=2},$$

$$Z_2 := \ln(P) Z_1 + \sum_{i=1}^{\infty} \left(\partial_m a_i\right) \mid_{m=2} P^{2+i},$$

$$Z_3 := \sum_{i=0}^{\infty} a_i P^i \mid_{m=0},$$

$$Z_4 := \ln(P) Z_3 + \sum_{i=1}^{\infty} \left(\partial_m a_i\right) \mid_{m=0} P^i,$$

(2.237)

with

$$a_i = -a_{i-1} \frac{1}{2} (m + i - 4) - \frac{2 \beta}{E_{ij,m}}, \quad \text{for } i \geq 4,$$

(2.238)

and $E_{ij,m}$ as in (2.164). Again, we choose $a_0$ to be 1 and $a_1 = a_2 = a_3 = 0$.

Hence, a general solution $Z_0$ to (2.236) is given by a linear combination

$$Z_0 = A_1 Z_1 + A_2 Z_2 + A_3 Z_3 + A_4 Z_4.$$

(2.239)

The constants are, just as in Subsection 2.3.2, determined by matching this to the inner solution and the remote solution. A similar calculation as in Subsection 2.3.2 shows that, for $P \to \infty$,

$$Z_0 \sim c_1 Z_{1,\infty} + c_2 Z_{2,\infty} + c_3 Z_{3,\infty} + c_4 Z_{4,\infty}.$$

(2.240)
with
\[
Z_{1,\infty} = P^{t^3}
\]
\[
Z_{2,\infty} = P^{t^4} \exp\left(\frac{3}{8} t^4 P_t^3\right)
\]
\[
Z_{3,\infty} = P^{t^4} \exp\left(2 t^4 \frac{3}{16} P^2_t\right) \sin\left(2 t^4 \frac{3}{16} \sqrt{3} P^2_t\right)
\]
\[
Z_{4,\infty} = P^{t^4} \exp\left(2 t^4 \frac{3}{16} P^2_t\right) \cos\left(2 t^4 \frac{3}{16} \sqrt{3} P^2_t\right).
\]

Again, we do not expect that the outer solution in the remote region exhibits oscillatory behavior with exponential growth. Hence, the boundary conditions on \(Z_0\) are such that \(Z_0\) does not grow exponentially fast as \(P \to \infty\). Defining the \(\Gamma\)’s similarly to the \(\Gamma\)’s in Subsection 2.3.2, but now dependent on \(\beta\), gives for the boundary conditions as \(P \to \infty\),
\[
A_1 \Gamma_{13} + A_2 \Gamma_{23} + A_3 \Gamma_{33} + A_4 \Gamma_{43} = 0
\]
\[
A_1 \Gamma_{14} + A_2 \Gamma_{24} + A_3 \Gamma_{34} + A_4 \Gamma_{44} = 0.
\]

Inspection of the inner solution in the outer limit shows that the \(O(1)\)-term is bigger than the \(\ln(P)\)-term. This means, after matching, that \(A_1 = 0\) (see Subsection 2.3.5). Similarly we find \(A_2 = 0\). Using numerical computations (see Subsection 2.3.6) we further show that, in this case, equations (2.242) are only satisfied if \(\beta = \frac{1}{k}\), with \(k \in \mathbb{N}\). Since we assumed \(\beta \geq \frac{1}{4}\), we have that \(\beta > \frac{1}{4}\) by the above arguments.

Since \(\beta > \frac{1}{4}\), we see that \(\frac{1}{P^3} \ll \ln\left(\frac{1}{P^3}\right)^{-1}\). Therefore, in the expansion of the solution to (2.235) the following terms are given by the logarithmic perturbations. Hence, we can write, as in Subsection 2.3.2, the asymptotic expansion of a solution to (2.235) as
\[
Z = F_\gamma Z_0 + \gamma F_{\gamma-1} Z_1 + \gamma (\gamma - 1) F_{\gamma-2} Z_2 + \cdots + O\left(\frac{1}{P^{\beta}}\right),
\]

with
\[
F_\gamma(t) := \ln\left(\frac{1}{T-t}\right)^\gamma.
\]

If we substitute (2.243) into the rescaled Willmore flow (2.235) and define the operator
\[
L := \beta - \frac{1}{4} P \partial_P - \frac{1}{2} \left( \frac{\partial^2_P}{P^2} + \frac{2}{P^3} \frac{\partial^3_P}{P^4} - \frac{1}{P^4} \frac{\partial^4_P}{P^5} + \frac{1}{P^5} \right)
\]
we find, for the \(Z_i\) in (2.243),
\[
L[Z_i] = Z_{i-1},
\]

with \(Z_0\) as in (2.239). The boundary conditions on the \(Z_i\) are such that there is no exponential growth for \(P \to \infty\) and such that the outer solution can be matched to the inner solution. Any solution to (2.246) can be written as
\[
Z_i = P^m \left( \sum_{n=0}^{\infty} a_n P^n + \sum_{n=0}^{\infty} b_n P^n \ln(P) \right) + B_1 Z_1 + B_2 Z_2 + B_3 Z_1 + B_4 Z_2,
\]

with \(m \geq 4\), since for \(0 \leq m < 4\) we just get the homogeneous solutions.
As we want to match this solution to the inner solution, we have to consider \( P \to \infty \). That is,

\[
Z \sim F_r(t) \left[ A_1 P^2 + A_2 P^2 \ln(P) + A_3 + A_4 \ln(P) \right] + \gamma F_{r-1}(t) \left[ B_1 P^2 + B_2 P^2 \ln(P) + B_3 + B_4 \ln(P) \right] + \gamma (\gamma - 1) F_{r-2}(t) \left[ C_1 P^2 + C_2 P^2 \ln(P) + C_3 + C_4 \ln(P) \right],
\]

for \( P \to 0 \).

### 2.3.5 Matching the inner solution to the general outer solution

In this section we match the inner solution to the general outer solution \((2.243)\). This determines the constants \( \beta, \gamma \) in \((2.230)\) and \((2.231)\) and the behavior of the blowup rate \( \lambda \). Just as in the case of \( \beta = \frac{1}{2} \), we find that the matching only succeeds for curves that do not cross the \( r \)-axis.

The matching conditions \((2.196)\), found in Subsection 2.3.3, hold for all \( \beta \) in \( \nu \). Hence, the matching conditions for the general outer solution are also given by

\[
\begin{align*}
O(1) & : F_r A_3 + \cdots \sim \frac{\lambda}{\nu} \left[ \ln(2) - \frac{1}{4} \ln\left( \frac{1}{T-t} \right) - \ln(\lambda) \right], \\
O(\ln(P)) & : F_r A_4 + \cdots \sim \frac{\lambda}{\nu}, \\
O(P^2) & : F_r A_1 + \cdots \sim \epsilon \left( \frac{1}{2} + \frac{1}{4} \ln(2) - \frac{1}{16} \ln\left( \frac{1}{T-t} \right) - \frac{1}{4} \ln(\lambda) \right) \frac{\mu^2}{\lambda^\nu}, \\
O(P^2 \ln(P)) & : F_r A_2 + \cdots \sim \frac{\mu^2}{\lambda^\nu} \frac{1}{4}.
\end{align*}
\]

As \( \lambda \) goes to zero with a quasistationary rate, we can write

\[
\lambda(t) = (T - t)^{\frac{1}{2}} f(t),
\]

with \( f(t) \to 0 \) as \( t \to T \). Hence,

\[
\frac{1}{4} \ln\left( \frac{1}{T-t} \right) - \ln(\lambda) = - \ln(f) \to \infty,
\]

for \( t \to T \). We can therefore conclude that the \( O(\ln(P))\)-terms and the \( O(P^2 \ln(P))\)-terms of the outer solution in the inner region have to be smaller than the \( O(1)\)-terms and the \( O(P^2)\)-terms. This holds in any case (\( \beta = \frac{1}{2} \) or \( \beta > \frac{1}{2} \)).

Hence, the matching \((2.249)\) gives \( A_2 = A_4 = 0 \) and the boundary conditions \((2.242)\) on \( P \to \infty \) reduce to

\[
A_1 \Gamma_{31} + A_3 \Gamma_{33} = 0, \\
A_1 \Gamma_{14} + A_3 \Gamma_{34} = 0.
\]

We can assume that at least one of \( A_1 \) and \( A_3 \) is not zero. Because, if not, all \( A_i \) are zero and the \( B_i \) play the role of the \( A_i \).

Assume that both \( A_1 \) as \( A_3 \) are not zero. Then \( \Gamma_{13} \Gamma_{34} - \Gamma_{14} \Gamma_{33} \) is zero. In Subsection 2.3.6 we show that (see \((2.302)\))

\[
\Gamma_{13} \Gamma_{34} - \Gamma_{14} \Gamma_{33} = 0 \iff \beta = \frac{1}{2} k,
\]

for \( P \to 0 \).
and

\[ \Gamma_{13} = \Gamma_{14} = 0 \iff \beta = \frac{1}{2} + k, \]
\[ \Gamma_{33} = \Gamma_{34} = 0 \iff \beta = k, \]

(2.254)

with \( k \) an integer. Hence, their does not exist a \( \beta \) such that (2.252) holds with \( A_1 \) and \( A_3 \) both non-zero. We, therefore, conclude that either \( A_1 = 0 \) or \( A_3 = 0 \).

Let \( A_1 = 0 \). Then \( Z_0 = A_3 Z_3 \) and in the limit \( t \to T \) we find

\[ Z = F_A Z_3, \]

(2.255)

with boundary conditions

\[ A_1 \Gamma_{13} = A_3 \Gamma_{34} = 0. \]

(2.256)

Hence \( \Gamma_{13} = \Gamma_{34} = 0 \), which only occurs for integer \( \beta \) as shown in Subsection 2.3.6. Let \( \beta \) be an integer. Since

\[ Z_3 = 1 + \frac{3}{2} \beta p^2 - \frac{3}{50} \beta (3 - 2\beta) p^4 + \frac{\beta(2 - 2\beta)(4 - 2\beta)}{1001653200} p^{12} + \ldots \]

(2.257)

is a polynomial of degree 4\( \beta \) with positive coefficients we conclude that \( Z_3 \) is positive for all values \( P \). Therefore, \( Z_3 \) is also positive for \( P \to \infty \). Since \( \lambda \) and \( \nu \) are positive by definition and (2.251) holds, we have that \( A_3 > 0 \). Hence, in this case, the outer solution (2.255) in the remote region, for \( t \to T \), has positive \( z \). The curve we describe in this way does not cross the \( r \)-axis, at least not within the scaling we considered, that is, the quasi-stationary scaling up to the self-similar scaling. At the end of this Subsection we show that the solution stays positive in the remote limit of the outer solution.

Let \( A_3 = 0 \), with \( A_1 \neq 0 \). The boundary conditions (2.252) reduce to

\[ A_1 \Gamma_{13} = A_1 \Gamma_{14} = 0. \]

(2.258)

Hence \( \Gamma_{13} = \Gamma_{14} = 0 \). This happens if and only if \( \beta = \frac{1}{2} + k \), with \( k \) an integer, by (2.302) in Subsection 2.3.6. In this case, the biggest term of the outer solution in the limit \( t \to T \) is given by the polynomial \( F_A Z_1 \), with

\[ Z_1 = p^2 - \frac{1 - 2\beta}{50} p^4 + \frac{(1 - 2\beta)(3 - 2\beta)}{1001653200} p^{12} + \ldots \]

(2.259)

Since \( Z_1 \) is positive for all \( P \) and all \( \beta = \frac{1}{2} + k \), the sign of \( A_1 \) determines whether the solution crosses the \( r \)-axis, or not. By matching we find the constant \( \gamma \) and the sign of \( A_1 \).

Let \( \beta = \frac{1}{2} + k \). Inspection of the \( O(\ln(P)) \)-terms and the \( O(P^2 \ln(P)) \)-terms in (2.249) gives \( \lambda = O(\nu) \) and \( \epsilon = (\frac{\nu}{\mu}) \). Hence,

\[ \lambda \sim a(t)(T - t)^{\beta} \quad \text{and} \quad \epsilon \sim b(t)(T - t)^{2\beta - \frac{1}{2}}, \]

(2.260)

with \( a, b \) given by

\[ a(t) = a_0 \ln\left( \frac{1}{T - t} \right)^{\gamma} \quad \text{and} \quad b(t) = b_0 \ln\left( \frac{1}{T - t} \right)^{\gamma}, \]

(2.261)
2.3. Formal asymptotics for the Limaçon

From this we see that
\[ \epsilon^2 = O(\lambda^3). \] (2.262)

If we substitute this into (2.249) and taking the limit \( t \to T \) we get
\[
O(1) : F_1 A_1 + \gamma F_{\alpha - 1} B_1 + \cdots \sim a \left( -\frac{1}{4} + \beta \right) \ln \left( \frac{1}{T-t} \right),
\]
\[
O(\ln(P)) : F_1 A_1 + \gamma F_{\alpha - 1} B_1 + \cdots \sim a,
\]
\[
O(P^2) : F_1 A_1 + \gamma F_{\alpha - 1} B_1 + \cdots \sim \frac{b}{\alpha} \left( \frac{1}{16} + \frac{\beta}{4} \right) \ln \left( \frac{1}{T-t} \right),
\]
\[
O(P^2 \ln(P)) : F_1 A_2 + \gamma F_{\alpha - 1} B_2 + \cdots \sim \frac{b}{\alpha} \frac{1}{4}.
\]

Hence,
\[
A_4 = 0, \quad B_4 = \frac{4}{4\beta - 1\gamma} A_3, \quad C_4 = \frac{4}{4\beta - 1\gamma} B_3, \quad \text{etc.}
\] (2.264)

and similarly
\[
A_2 = 0, \quad B_2 = \frac{4}{4\beta - 1\gamma} A_1, \quad C_2 = \frac{4}{4\beta - 1\gamma} B_1, \quad \text{etc.}
\] (2.265)

Since, by construction, we have that \( A_1 = 0 \), we can immediately conclude that \( B_4 = 0 \).

Consider the recursive equation (2.238) for \( a_i \). Then, for \( \beta = \frac{1}{4} + k \),
\[
a_{4k+4} \mid_{m=2} = -a_{4k} \mid_{m=2} \frac{\frac{1}{4} (2 + 4k) - 2 \left( \frac{1}{4} + k \right)}{E_{4k+4}} = 0,
\] (2.266)

and we can conclude that \( Z_1 \) in (2.237) is a finite series of degree \( 4k + 2 \) and \( \Gamma_{13} = \Gamma_{14} = 0 \), as anticipated.

For matching we need to know the explicit expression of \( Z_1 \), given by,
\[ L[Z_1] = Z_0 = A_1 Z_1. \] (2.267)

Consider the finite series
\[ Z_1 = -4 \ln(P) Z_1 + P^6 \sum_{i=0}^{4k+3} b_i P^i. \] (2.268)

Since,
\[ L[\ln(P) \Phi] = \ln(P) L[\Phi] - \frac{1}{4} \Phi - \frac{2}{P^2} \Phi, \] (2.269)

for every function \( \Phi \), we have
\[ L[Z_1] = L[-4 \ln(P) Z_1] + L[P^6 \sum_{i=0}^{4k+3} b_i P^i] \]
\[ = -4 \ln(P) L[Z_1] + Z_1 + \frac{8}{P^2} \Phi_1 Z_1 - \frac{1}{2} \left[ 2 \sum_{i=0}^{4k+3} b_i E_{i} P^{2i+1} - \sum_{i=3}^{4k+7} b_{i} (2k - \frac{1}{2}) P^{2i+1} \right] \]
\[ = Z_1 + \frac{4i(1+2i)}{(6-1)} a_{i} |_{m=2} P^{i-2} - \frac{1}{2} \left[ 2 \sum_{i=0}^{4k+3} b_i E_{i} P^{2i+1} - \sum_{i=3}^{4k+7} b_{i} (2k - \frac{1}{2}) P^{2i+1} \right], \] (2.270)
where we used (2.266) to write the third derivative of \( Z_1 \) as a finite sum. Hence, if
\[
b_0 = \frac{6}{r^6}, \quad b_1 = b_2 = b_3 = 0, \quad \text{and}
\]
\[
b_i = -b_{i-1} \frac{26}{E_{r,0}} + \frac{(i + 6)!(a_{i+4})_{m=2}}{(i + 3)!} E_{r,0}, \quad \text{for } i \geq 4, \tag{2.271}
\]
then \( Z_1 \), as given by (2.268), is a solution to
\[
L[Z_1] = Z_1. \tag{2.272}
\]
The general solution to equation (2.267) is then given by
\[
Z_1 = A_1 Z_1 + \mathcal{B}_1 Z_1 + \mathcal{B}_2 Z_2 + \mathcal{B}_3 Z_3 + \mathcal{B}_4 Z_4. \tag{2.273}
\]
The relations between \( B_i \) and \( \mathcal{B}_i \) are, again, given by (2.209).

Since \( B_4 = 0 \), we conclude that \( \mathcal{B}_4 = 0 \). Since \( Z_1 \) is a finite series, its contribution to the exponential growth of \( Z_1 \) is zero. Hence, the boundary conditions for \( Z_1 \) at \( P \to \infty \) are given by
\[
\mathcal{B}_2 \Gamma_{23} + \mathcal{B}_3 \Gamma_{34} = 0,
\]
\[
\mathcal{B}_2 \Gamma_{24} + \mathcal{B}_4 \Gamma_{34} = 0. \tag{2.274}
\]
These boundary conditions equal the boundary conditions (see 2.210) for \( Z_1 \) in the \( \beta = \frac{1}{2} \) case. By the same arguments used in the case \( \beta = \frac{1}{2} \), we can conclude that \( \mathcal{B}_2 = \mathcal{B}_4 = 0 \). This gives, using (2.209) and (2.205), \( \gamma = \frac{1}{2} \).

The next term \( Z_2 \) in the expansion of the solution of the rescaled Willmore flow (2.159) is given by,
\[
L[Z_2] = Z_1 = A_1 Z_1 + B_1 Z_1. \tag{2.275}
\]
Consider the series
\[
Z_2 = -4 \ln(P)(Z_1 + 4Z_2) + P^d \sum_{i=0}^{\infty} i! P^i. \tag{2.276}
\]
Since, by definition of \( Z_1 \) and \( Z_2 \),
\[
Z_1 + 4Z_2 = P^d \sum_{i=0}^{\infty} b_i P^i + 4 \sum_{i=1}^{\infty} (\partial_{\gamma} \alpha_i) \bigg|_{m=2} P^{2+i}, \tag{2.277}
\]
we find, using equation (2.269),

\[ L[Z_2] = -4 \ln(P) Z_1 + 4 Z_2 + 8 \frac{d}{P^3} (Z_1 + 4 Z_2) + L[P^2 \sum_{i=0}^{\infty} d_i P^i] \]
\[ = -4 \ln(P) Z_1 + 4 Z_2 + \sum_{i=0}^{4k+4} \frac{8 (i + 6)!}{(i + 3)!} b_i P^{2i+3} + \sum_{i=1}^{4} \frac{32 (i + 2)!}{(i - 1)!} (\partial_m a_i) \bigg|_{m=2} P^{i-2} \]
\[ - \frac{1}{2} \sum_{i=0}^{\infty} d_i E_i P^{2i+1} - \sum_{i=4}^{\infty} d_{i-4} (2k - \frac{1}{2}) P^{2i+2} \bigg] \]
\[ = Z_1 + 4 \sum_{i=0}^{\infty} (\partial_m a_i) \bigg|_{m=2} P^{2i+3} + \sum_{i=0}^{4k+4} \frac{8 (i + 6)!}{(i + 3)!} b_i P^{2i+1} + \sum_{i=1}^{4} \frac{32 (i + 2)!}{(i - 1)!} (\partial_m a_i) \bigg|_{m=2} P^{i-2} \]
\[ - \frac{1}{2} \sum_{i=0}^{\infty} d_i E_i P^{2i+1} - \sum_{i=4}^{\infty} d_{i-4} (2k - \frac{1}{2}) P^{2i+1} \bigg]. \quad (2.278) \]

Hence, if \( d_0 = -\frac{1}{2} \pi, d_1 = d_2 = d_3 = 0, \) and

\[ d_i = \frac{1}{E_i} \left( -d_{i-4} \left( \frac{1}{2} - 2k \right) + 8(\partial_m a_i) \bigg|_{m=2} + 16 \frac{(i + 6)!}{(i + 3)!} b_i + 64 \frac{(i + 6)!}{(i + 3)!} (\partial_m a_i) \bigg|_{m=2} \right), \quad (2.279) \]

for \( i \geq 4 \) and \( Z_2 \) is given by (2.276), then

\[ L[Z_2] = Z_1. \quad (2.280) \]

Therefore, the general solution to (2.275) is given by

\[ Z_2 = A_1 Z_2 + B_1 Z_1 + C_1 Z_1 + C_2 Z_2 + C_3 Z_2 + C_4 Z_4 \]
\[ = -4 A_1 \ln(P) \left[ P^6 \sum_{i=0}^{4k+3} b_i P^i + 4 \sum_{i=1}^{4} (\partial_m a_i) \bigg|_{m=2} P^{2i+1} \right] \quad (2.281) \]
\[ -4 B_1 \ln(P) Z_1 + B_1 P^6 \sum_{i=0}^{4k+3} b_i P^i + C_1 Z_1 + C_2 Z_2 + C_3 Z_2 + C_4 Z_4. \]

The relations between \( C_i \) and \( \overline{C}_i \) are given by (2.220). Since \( B_1 = 0 \) we have \( C_4 = 0 \). Let \( C_1 \) and \( C_2 \) be the contributions of \( Z_2 \) to the exponential cosine ans sine growth, respectively. The boundary conditions for \( Z_2 \) at \( P \to \infty \) are equal to the ones in the case \( \beta = \frac{1}{2} \) given by equation (2.222). The computations in Subsection 2.3.6 (see (2.302)) show that \( \Gamma_{33} \) and \( \Gamma_{34} \) are not both zero for the same value of \( \beta = \frac{1}{2} + k \), with \( k \) an integer. Hence, just as in Subsection 2.3.3 we conclude that the boundary conditions reduce to

\[ C_4 \left( \Gamma_{33} \Gamma_{24} - \Gamma_{34} \Gamma_{23} \right) = A_1 \left( C_1 \Gamma_{21} - C_4 \Gamma_{24} \right). \quad (2.282) \]

Because \( A_1 = B_1 = 0 \) and \( \gamma = \frac{1}{2} \frac{1}{2} \), we find \( C_3 = C_5 \geq 0 \), by positivity of \( \lambda \). For \( \beta = \frac{1}{2} + k \) we have (see Subsection 2.3.6) that \( \Gamma_{33} \Gamma_{24} - \Gamma_{34} \Gamma_{23} > 0 \) and \( C_1 \Gamma_{21} - C_4 \Gamma_{24} \neq 0 \). This means that the sign of \( A_1 \) is given by the sign of \( C_1 \Gamma_{21} - C_4 \Gamma_{24} \), which is (see Subsection 2.3.6) positive.
The matching conditions then reduce to
\[
\lambda \sim a_r \ln \left( \frac{1}{T-t} \right)^{\gamma_r} \frac{4k}{\alpha} (T-t)^{\gamma_r + k} \quad \text{and} \quad \epsilon \sim b_i \ln \left( \frac{1}{T-t} \right)^{\gamma_i} \frac{4k}{\beta} (T-t)^{\gamma_i + k},
\]
for some constants \(a_r\) and \(b_i\), and for \(k\) an integer.

Consider the outer solution (2.243) for \(t\) close to \(T\). We see that the curve \(z(r)\) of our problem is given, in the outer region, by
\[
z \sim A_1 \ln \left( \frac{1}{T-t} \right)^{\gamma_r} \frac{4k}{\alpha} (T-t)^{\gamma_r + k} \mathcal{Z}_t,
\]
where \(\mathcal{Z}_t\) is a polynomial of degree \(2 + 4k\) in \(P\). Again, if we want to express this in the order 1 scale of the remote solution, we have to include the terms of order \(F_{r-1}\) in (2.283). Then
\[
z \sim A_1 \ln \left( \frac{1}{T-t} \right)^{\gamma_r} \frac{4k}{\alpha} \left|_{\gamma \to \gamma_r} \right. \left[ 2 - \frac{4k}{4k+1} \right] \ln \left( \frac{1}{T-t} \right)^{-1} \ln(r).
\]
Just as in Subsection 2.3.3, we have found an outer solution which is completely positive, in the limit \(t \to T\). In the case of singularity formation in the Limaçon we expect that the solution crosses the \(r\)-axis. This does not happen. This means that we did not describe the blowup behavior of the Limaçon under the Willmore flow. In Subsection 2.4 we discuss the results we obtained in this Subsection and Subsection 2.3.3.

In the remote limit of the outer solution, the second term in the expansion is of the same order as the first term. We now investigate the second term of the expansion of the
outer solution in the case that $\beta$ is integer. Although the first term is positive it could happen that the second term turns the outer solution negative in the remote region.

Let $\beta$ be an integer $k$, such that $\Gamma_{13} = \Gamma_{44} = 0$. Matching gives immediately $A_2 = A_4 = 0$ and therefore also $A_1 = 0$. The term $Z_1$ of equation (2.243) is then given by

$$Z_1 = -4A_3 \ln(P)Z_3 + A_3P^4\Phi + \Gamma_1 Z_1 + \Gamma_2 Z_2 + \Gamma_3 Z_3 + \Gamma_4 Z_4,$$  \hspace{1cm} (2.289)

for some polynomial $\Phi$ of degree, at most, $4k - 4$ (see (2.241)). The relations between $B_i$ and $\bar{B}_i$ are given by

$$B_1 = \bar{B}_1, \quad B_2 = \bar{B}_2, \quad B_3 = \bar{B}_3, \quad B_4 = -4A_3 + \bar{B}_4.$$  \hspace{1cm} (2.290)

Since $A_1 = 0$, (2.265) gives us $B_2 = 0$ and therefore $\bar{B}_2 = 0$. The boundary conditions on $P \to \infty$ then give

$$\bar{B}_1 = -B_4 \Gamma_{13},$$  \hspace{1cm} (2.291)

Since, from the numerical computations in Subsection 2.3.6, we know that $\Gamma_{44} \Gamma_{13} - \Gamma_{34} \Gamma_{14} \neq 0$ for integer $\beta$, we find $\bar{B}_4 = \bar{B}_1 = 0$ and

$$Z = F_3 A_3 Z_3 + \gamma F_{-1} \left(-4A_3 \ln(P)Z_3 + A_3 P^4 \Phi + \bar{B}_3 Z_3\right) + \cdots,$$  \hspace{1cm} (2.292)

with

$$\gamma = \frac{-1}{4j - 1}, \quad \text{with} \quad \beta = k \in \mathbb{N}.$$  \hspace{1cm} (2.293)

Since $\Phi$ is of degree, at most, $4k - 4$ and

$$Z_3 \sim a_{4k} P^{4k}, \quad \text{for large} \quad P,$$  \hspace{1cm} (2.294)

we find

$$Z \sim F_3 a_{4k} \left|_{m=0}^{P^{4k}} \right| \left[ 1 - \gamma - 4\gamma F_{-1} \right]$$

$$\sim F_3 a_{4k} \left|_{m=0}^{P^{4k}} \right| \frac{4\beta}{4\beta - 1},$$  \hspace{1cm} (2.295)

for large $P$ and $t \to T$. From the recurrence relation (2.238) and $a_0 = 1$ one shows that $a_{4k} \left|_{m=0}^{P^{4k}} \right. > 0$ for $\beta = k$. Hence, we see that the outer solution stays positive, also in the remote limit.

### 2.3.6 Numerical computations of the $\Gamma$'s

In this Subsection we investigate the $\Gamma$'s for different values of $\beta$. This helps us to match the inner with the outer solution.

The constant $\Gamma_{13}$ is the contribution of the homogeneous solution $Z_3$ to the sine exponential growth at $P \to \infty$, while $\Gamma_4$ its contribution to the cosine exponential growth. Hence, for $P \to \infty$ we can approximate $\Gamma_{13}$ and $\Gamma_4$ by (see (2.184))

$$\Gamma_{13} = \frac{2}{\pi} \int \left(\frac{d\varphi}{\varphi}\right)^{3/4} \frac{d\varphi}{\varphi} e^{-\varphi} Z_3 \sin \left(t P^4 \frac{4}{3} P^2\right) dP,$$  \hspace{1cm} (2.296)
2. Matched asymptotics for finite time blowup in the Willmore flow

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
 & $m = 51$ & $m = 224$ & $m = 443$ & $m = 694$ & $m = 971$ \\
\hline
$N = 3000$ & -1.8574 & -1.8461 & - & - & - \\
\hline
$N = 6000$ & -1.8574 & -1.8461 & -1.8444 & - & - \\
\hline
$N = 9000$ & -1.8574 & -1.8461 & -1.8444 & -1.8438 & - \\
\hline
$N = 12000$ & -1.8574 & -1.8461 & -1.8444 & -1.8438 & -1.8434 \\
\hline
\end{tabular}
\end{center}

Table 2.1: The value for $\Gamma_{33}$ for different numbers of terms $N$ and different values of $m$.

and

$$
\Gamma_{34} = \frac{2}{\pi} \left(\frac{m+1}{m}\right)^{3/4} P \frac{\pi}{\sqrt{m}} \cos\left(P^4\right) \frac{4}{3} \int P^2 \, dP, 
$$

(2.297)

for some large integer value of $m$ and $l = \frac{16}{3} \sqrt{m}$. In this Subsection we give an approximation of this $\Gamma$’s, for different values of $\beta$, using the computer software Maple.

**The case $\beta = \frac{1}{2}$**

We consider the case $\beta = \frac{1}{2}$. Approximate the functions $Z_2$, $Z_3$ and $Z_4$, as given in (2.171), (2.172) and (2.215) in Subsections 2.3.2 and 2.3.3, by a varying number of terms $N$. In Table 2.1 we denote the results for $\Gamma_{33}$ using a finite approximation of the series $Z_3$ in equation (2.296). It is clear that, the more terms we take to approximate $Z_3$, the larger we can take $m$. It seems that the value for $\Gamma_{33}$ converges as $m$ becomes larger and that the limit is $-1.84$ with an error of the order $10^{-3}$. Hence, the value for $\Gamma_{33}$ is given, up to order $10^{-3}$, by the approximation with $N = 3000$ and $m = 224$. Doing a similar analysis we find

$$
\Gamma_{34} = O(10^{-3}), \\
\Gamma_{23} = O(10^{-3}), \\
\Gamma_{24} = -4.62 + O(10^{-3}).
$$

(2.298)

We further find

$$
C_s = 58 + O(10^{-1}), \\
C_t = 80 + O(10^{-1}).
$$

(2.299)

We see that $\Gamma_{34}$ and $\Gamma_{23}$ could be zero, and the numerical evidence for higher $N$ and $m$ certainly suggests so. This is remarkable, as it would mean that the exponential contribution of $Z_3$, in the limit $P \to \infty$, is exactly given by a constant times $Z_{3,\infty}$ in (2.184) and that the exponential contribution of $Z_2$, in the limit $P \to \infty$, is exactly given by a constant times $Z_{4,\infty}$ in (2.184). For the matching in Subsection 2.3.3 we do not need, however, the precise value for the $\Gamma$’s. It suffices to observe that

$$
\Gamma_{33}\Gamma_{24} - \Gamma_{23}\Gamma_{34} > 0, 
$$

(2.300)

and

$$
C_s\Gamma_{23} - C_t\Gamma_{24} > 0.
$$

(2.301)
2.3. Formal asymptotics for the Limaçon

\[ \Gamma^1_{13}e^{-2\beta^{1/3}} \]

\[ \Gamma^1_{14}e^{-2\beta^{1/3}} \]

Figure 2.4: This figure shows \( \Gamma^1_{13}e^{-2\beta^{1/3}} \) (left) and \( \Gamma^1_{14}e^{-2\beta^{1/3}} \) (right) as a function of \( \beta \). Since the \( \Gamma^1 \)’s grow exponentially and we are interested in their zeros, we multiply them with \( e^{-2\beta^{1/3}} \). The exponent 1.21 is found by trial and error.

\[ \Gamma^3_{33}e^{-2\beta^{1/3}} \]

\[ \Gamma^3_{34}e^{-2\beta^{1/3}} \]

Figure 2.5: This figure shows \( \Gamma^3_{33}e^{-2\beta^{1/3}} \) (left) and \( \Gamma^3_{34}e^{-2\beta^{1/3}} \) (right) as a function of \( \beta \). Since the \( \Gamma^3 \)’s grow exponentially and we are interested in their zeros, we multiply them with \( e^{-2\beta^{1/3}} \). The exponent 1.21 is found by trial and error.

The case for general \( \beta \)

In Figure 2.4 and 2.5 we have plotted the approximations of \( \Gamma^1_{13}, \Gamma^1_{14}, \Gamma^3_{33} \) and \( \Gamma^3_{34} \) as functions of \( \beta \). We used \( N = 3000 \) and \( m = 224 \), which gave us in the case of \( \beta = \frac{1}{2} \) a good approximation up to order \( 10^{-3} \). For higher \( \beta \) the approximations are less accurate. Examining some values of \( \beta \) with higher \( N \), however, shows that the qualitative behavior of the figures stays the same but that the zeros (of the \( \Gamma^1 \)’s) are slightly shifted. Computations with higher accuracy suggest that, for instance, \( \Gamma^1_{34} = 0 \), for \( \beta = \frac{1}{2} + 3k \) (\( k \) an integer). On the other hand it is clear that \( \Gamma^3_{33} \not= 0 \) for \( \beta = \frac{1}{2} + 3k \) (\( k \) an integer). Combining Figures 2.4, 2.5 and 2.6 with the explicit recurrence relation (2.238) of the coefficients of the series \( Z_j \) and \( Z_k \) we see that

\[ \Gamma_{13} = \Gamma_{14} = 0 \iff \beta = \frac{1}{2} + k, \]

\[ \Gamma_{33} = \Gamma_{34} = 0 \iff \beta = k, \]

\[ \Gamma_{13}\Gamma_{34} - \Gamma_{14}\Gamma_{33} = 0 \iff \beta = \frac{1}{2} k, \]  \hspace{1cm} (2.302)

with \( k \) an integer. Also in the case of \( \beta = \frac{1}{2} + k \), we do not need the precise values for the \( \Gamma^1 \)’s. To match the inner solution to the outer solution in Subsection 2.3.5 it is sufficient to know the sign of \( \Gamma_{31}\Gamma_{24} - \Gamma_{23}\Gamma_{14} \) and \( C^1_j\Gamma_{23} - C^3_j\Gamma_{24} \), for \( \beta = \frac{1}{2} + k \). From Figures 2.6
and 2.7 we see that

\[ \Gamma_{33}\Gamma_{24} - \Gamma_{23}\Gamma_{34} > 0, \quad \text{and} \quad C_4\Gamma_{23} - C_3\Gamma_{24} > 0, \quad \text{for} \quad \beta = \frac{1}{2} + k. \quad (2.303) \]

From Figure 2.7 we see that \( \Gamma_{13}\Gamma_{43} - \Gamma_{43}\Gamma_{14} \) has two zeros on any interval \([k, k + 1]\), with \( k \in \mathbb{N} \). The recurrence relation (2.238) gives one zero for \( \beta = k + \frac{1}{2} \). The other zero seems to be close to \( \beta = k + \frac{1}{2} \) and certainly not at \( \beta = k \), for \( k \) an integer. Computations with higher accuracy at \( \beta = 1 \) support this observation. Hence, we conclude that

\[ \Gamma_{14}\Gamma_{43} - \Gamma_{43}\Gamma_{14} \neq 0, \quad \text{at} \quad \beta = k + \frac{1}{2} \quad (2.304) \]

with \( k \) an integer.
2.4 Conclusions and outlook

In Section 2.3 we have tried to examine the blowup behavior of a surface under the Willmore flow, using formal matched asymptotics. We have found that matching only occurs for inner and outer solutions that do not cross the $r$-axis. This means that we have not found the blowup behavior for the Limaçon. There are several possibilities for the reason why we have not found this solution. In this Section we discuss these possibilities.

Consider the inner solution we constructed in Subsection 2.3.1. The only assumption we made in this Subsection was that $\lambda \epsilon \ll \lambda$. This assumption holds for $\epsilon \to 0$ and $\epsilon$ and $\lambda$ differentiable up to time $T$. This is easily seen whenever $\frac{\epsilon}{\lambda}$ exists in the limit $t \to T$ using l'Hopital.

We disregard the case that $\epsilon$ and $\lambda$ are not differentiable such that $\lambda \epsilon \ll \lambda$ would not hold, which is supported by the numerical computations in Chapter 3. The inner solution (2.144) is valid for $z$ such that (2.145) holds. Matching gives us the rates of $\epsilon$ and $\lambda$ and we can conclude that the validity still holds in the matching region. Note that for the inner solution we have not made any assumptions on the blowup time or the blowup behavior, besides $\lambda \epsilon \ll \lambda$.

Consider the outer solutions we constructed in Subsections 2.3.2 and 2.3.4. To find these solutions we made several assumptions. The first assumption is that the blowup time is finite. If the blowup time is not finite the self-similar rate is given by $t^\frac{1}{4}$ and goes to infinity. Hence, for infinite time blowup we only have two scales: a quasi-stationary scale and a scale of order one. This means that we ought to match the inner solution immediately to the remote solution. We remark, however, that if the matched asymptotics would show infinite blowup time, this would contradict the results of Chapter 3 and of reference [vm]. Motivated by this possibility, one could wonder if it is possible to match the inner solution immediately to the remote solution in the finite time case. This would mean we ignore the outer solution. We discuss this possibility in the Introduction of this thesis.

A second assumption we made is that the vanishing rate of the $z$-coordinate in the outer solution is algebraic in $t$. This made the $z^\mu$ term and the $\mu^4$ term in the rescaled Willmore flow (2.159) of equal order. For a different scale of the $z$-coordinate this does not hold anymore. Let the scale of $z$ go faster to zero than with an algebraic rate. This means that $z^\mu$ is of order one for $\mu$ smaller than the self-similar rate. Hence, we have found another scale in between the inner and outer solution that could give us more information.

Taking these scales for the $z$ and the $r$ coordinates gives us, instead of equations (2.161) or (2.236), the equation

$$-\beta \frac{1}{2} Z = -\frac{1}{2} \left( Z_{PPP} + \frac{Z_{PPP}^2}{P} - Z_{PP} \frac{Z_P}{P} + Z_P \frac{P}{P} \right),$$

(2.305)

where $Z_P = -\beta$ in the limit $t \to T$. Doing a calculation similar to the one we did in Subsection 2.3.2, gives for the zeroth order term $Z_0$ of the expansion of a solution of the rescaled Willmore flow (2.305)

$$Z_0 \sim c_1 P^{-\frac{1}{2}} \exp \left( -(2\beta) \frac{1}{2} P \right) + c_2 P^{-\frac{1}{2}} \exp \left( (2\beta) \frac{1}{2} P \right) + c_3 P^{-\frac{1}{2}} \sin \left( (2\beta) \frac{1}{2} P \right) + c_4 P^{-\frac{1}{2}} \cos \left( (2\beta) \frac{1}{2} P \right),$$

(2.306)
for $P \to \infty$. This has to be matched to the outer solution of Subsection 2.3.2 or 2.3.4, which is not possible. Hence, for $\nu$ smaller than algebraic, we can not construct a global approximation of the Limaçon.

A third assumption is that the asymptotic behavior of the rescaled outer solution $Z$ can be given by (2.231). Consider the very particular case that $Z$ is not dependent on time $t$. By this we mean that the scale of $z$, corresponding to the self-similar scale of $r$, is exactly given by $(T - t)^{\beta}$, without any logarithmic (or other) perturbations. This means that the expansion of the solution to the rescaled Willmore flow (2.235) is given by

$$Z = Z_0 + \frac{\nu^2}{\mu^2} Z_1 + \frac{\nu^4}{\mu^4} Z_2 + O(\nu^6).$$

The zeroth order term is still given by $Z_0 = A_1 Z_1$, by the same reasoning as in Subsection 2.3.5. This means that $\beta = \frac{1}{2} + k$, with $k$ integer. Hence,

$$\frac{\nu^2}{\mu^2} \sim (T - t)^{2k + 4}.$$  (2.308)

Inspection of the inner solution in the outer region (2.195), shows that the $P^2 \ln(P)$-terms have to differ a logarithmic factor of $T - t$ with the $P^2$-terms. Hence, we see immediately that we are not able to match the solutions. This means that the rescaled outer solution $Z$ has to be dependent on time.

The fourth assumption we made, constructing the outer solution, is that $z_\tau \to 0$ actually implies $\frac{z}{r} \to 0$, where $\nu$ and $\mu$ are the vanishing scales of the $z$- and $r$-coordinate, respectively. Although the numerics in Chapter 3 confirms this, it could be possible that this assumption does not hold. If that would be the case, we need another assumption to simplify the Willmore flow in the outer region.

In Section 2.3 we did not succeed in approximating the Limaçon near blowup. We did find, however, self-consistent blowup behavior, although it is unclear which scenario it is. In the Introduction of this thesis we discuss this further and we give some suggestions on other possibilities to do the formal asymptotics for the Limaçon.
Chapter 3
Moving mesh method for the Willmore flow on axisymmetric surfaces

In this Chapter we investigate the Willmore flow numerically, using a moving mesh method. The reason for this project is to validate the results obtained on the Limaçon in [90]. We further give results that justify the assumptions we make in Chapter 2 and give suggestions on the blowup behavior of a Limaçon. Motivated by the results of Chapter 2 we also investigate the evolution of several dumbbells. Just as in [90] we find that dumbbells evolve into spheres.

3.1 Introduction

3.1.1 Moving mesh methods

In this Subsection we give a short introduction to the moving mesh method. For more on the subject we refer to [ys] and [yr]. One could also consult the more general [sw] and references therein. We further refer to [25] and [27], where the moving mesh method is investigated in specific blowup problems.

Let \( N+1 \) be the number of grid points we use. The idea of a moving mesh is to move the grid points in time along with the solution towards places where the solution varies a lot. To do this, we distinguish between the rigid computational coordinates \( \xi_i \) and the physical coordinates \( s_i \). These coordinates are given by

\[
\xi_i := \frac{i}{N},
\]

for \( i \) an integer from 0 to \( N-1 \), and

\[
s_i := s(\xi_i, t),
\]

with \( s_0 \) and \( s_N \) constants, independent of time. The main idea of the moving mesh method is the equidistribution of a certain positive measure \( M \) of a yet to be chosen error function. This means that the grid points are arranged in such a manner that \( M \), which is called the monitor function, is equally distributed between the points. This is the case if

\[
\int_{s(i, t)}^{s(i+1, t)} M(x, t) dx = \frac{1}{N} \int_{s(0, t)}^{s(1, t)} M(x, t) dx,
\]

for all \( i \) from 0 to \( N-1 \).
The equidistribution formula (3.3) is given in the continuous form. The equations that govern the speed of the grid points along the solution are given by partial differential equations following from (3.3). These equations are called moving mesh partial differential equations and abbreviated as MMPDE’s. The solution to the problem is then given by discretizing the MMPDE together with the Willmore flow and solving the resulting system of differential equations.

By differentiating (3.3) we get two different versions of the equidistribution formula,

\[ M(s, t) \frac{\partial}{\partial \xi} \xi = \int_{s(t)}^{s(t+1)} M(x, t) dx, \]  

(3.5)

and

\[ \frac{\partial}{\partial \xi} \left( M(s, t) \frac{\partial}{\partial \xi} \xi \right) = 0. \]  

(3.6)

Several MMPDE’s are constructed by differentiating (3.5) or (3.6) with respect to time (see [wr]). The MMPDE we use is constructed slightly different. One can choose the velocity of a grid point in such a manner that it lowers the term \( \frac{\partial}{\partial \xi} f M \frac{\partial}{\partial \xi} s \), which measures the deviation from being equidistributed. One way to move the grid points is by the equation (already in discretized form)

\[ \dot{s}_{i+1} - \dot{s}_i = -\frac{1}{\tau} \left( \int_{s_i}^{s_{i+1}} M dx - \frac{1}{N} \int_{s_0}^{s_N} M dx \right), \]  

(3.7)

introduced in [2] and [3]. Here, \( \tau \) is some (small) positive constant. When the contribution of the error between two points \( s_i \) and \( s_{i+1} \) is greater than the mean contribution, the right hand side of equation (3.7) is negative and the points \( s_i \) and \( s_{i+1} \) flow towards each other. Subtracting the movement of two consecutive intervals, and denoting \( M_i = M(s_i) \), gives

\[ \dot{s}_{i+1} - 2\dot{s}_i + \dot{s}_{i-1} = -\frac{1}{\tau} \left( \int_{s_i}^{s_{i+1}} M dx - \int_{s_{i-1}}^{s_i} M dx \right) \]

\[ \approx -\frac{1}{\tau} \frac{1}{2} \left( (M_{i+1} + M_i)(s_{i+1} - s_i) - (M_i + M_{i-1})(s_i - s_{i-1}) \right), \]  

(3.8)

which is the central difference discretization (see Subsection 3.1.3) of the differential equation

\[ \frac{\partial^2}{\partial t^2} \xi = -\frac{1}{\tau} \frac{\partial}{\partial \xi} \left( M \frac{\partial}{\partial \xi} \xi \right). \]  

(3.9)

This equation is called MMPDE6 (see [72]) and is the equation we use to move the grid points.

The moving mesh method consists of solving the original PDE together with MMPDE6. Let the PDE be given by

\[ f_t = L[f], \]  

(3.10)

for some function \( f = f(s, t) \) and some differential operator \( L \), with only spatial derivatives. If the parameter \( s \) is time-dependent, the time derivative of \( f \) is given by

\[ \frac{df}{dt} = f_t + f \delta. \]  

(3.11)
Discretization gives, with $f_i = f(s_i(t), t)$ for every function $f$,
\[ \frac{d}{dt} f_i = (L_{1i}) + \left( \frac{\partial f}{\partial s} \right) \dot{s}_i, \]
where $\frac{d}{dt} f_i$ is given for fixed $i$, while $(f_i)$, is given for fixed $s_i$. Hence, we can substitute $(f_i)$ by $L[f]$, and the PDE in discretized form can be given by
\[ \frac{d}{dt} f_i = L[f] + \left( \frac{\partial f}{\partial s} \right) \dot{s}_i. \]

The moving mesh method for the Willmore flow thus consists of solving the system
\begin{align*}
\frac{dr_i}{dt} - (r_i) \dot{s}_i &= \left[ -\frac{z_i}{\sqrt{r_i^2 + z_i^2}} \left( \Delta H + 2H(H^2 - K) \right) \right], \\
\frac{dz_i}{dt} - (z_i) \dot{s}_i &= \left[ -\frac{r_i}{\sqrt{r_i^2 + z_i^2}} \left( \Delta H + 2H(H^2 - K) \right) \right], \\
\partial_\xi^2 \dot{s}_i &= \left[ \frac{1}{\tau} \partial_\xi \left( M \partial_\xi s \right) \right].
\end{align*}

In moving mesh methods the choice of the monitor function is crucial. Choosing the right monitor function can enable one to accurately and efficiently compute the blowup. All information we know beforehand can be useful in choosing the right monitor function, whether it be scale invariance or symmetry of the PDE or the actual blowup rate of the singularity, see [qv] and [qt] for some examples. The smoothness of the monitor function is also of importance. It has been shown (see [26] and references therein) that the smoothness of the monitor function is directly related to the smoothness of the node concentration and therefore to the accuracy of the discretization. For this reason one often smoothens the monitor function. We choose not to smooth the monitor function as the diffusive term in MMPDE6 (3.9) already takes care of the smoothing of the node concentration.

As the computations approach a singularity, the mesh points flow towards the region where the monitor function is highest. This could cause difficulties in the regular part of the solution because of a lack of points. To overcome this problem we add a correction term to $M$ and use this as the new monitor function $M_c$. In [14] the authors introduce a constant term. This makes sure the monitor function is nowhere zero and some points are left in the regular part of the solution. To control the amount of points that stay in the regular part, we can modify the monitor function as follows (see, again, [26] and references therein). Consider the correction term
\[ M_c = M + \beta \int_{x_0}^{x_f} M(x, t) dx. \]

If $M_c$ is equally distributed we have
\[ \int_{x_0}^{e^{\xi_0}} M_c(x, t) dx = \xi^* \int_{x_0}^{e^{\xi_0}} M_c(x, t) dx, \]
for $\xi^*$ the ratio of points, that are allowed to move into the singularity. Note that if a singularity occurs in our problem, it occurs at one of the boundaries. In fact, we have
chosen the parameterization in such a way that it occurs at $s_0$ in the origin. This allows us to determine how many points flow into the singularity and how many points we keep to describe the regular part of the solution. Let $M_t$ be given by (3.15), then

$$\xi^* \int_{s_0}^{s_N} M_t(x,t)dx = \xi^* \left( 1 + \beta (s_N - s_0) \right) \int_{s_0}^{s_N} M(x,t)dx. \quad (3.17)$$

If (3.16) holds and $\xi^*$ is the ratio of points which are allowed into the singularity, we have that $(s(\xi^*) - s_0) \to 0$ as the singularity is approached and

$$\int_{s_0}^{s(\xi^*,t)} M_t(x,t)dx = \int_{s_0}^{s(\xi^*,t)} M(x,t)dx + \beta (s(\xi^*,t) - s_0) \int_{s_0}^{s_N} M(x,t)dx \approx \int_{s_0}^{s(\xi^*,t)} M(x,t)dx. \quad (3.18)$$

Hence,

$$\xi^* \left( 1 + \beta (s_N - s_0) \right) \to 1. \quad (3.19)$$

as the singularity is approached. Therefore, by choosing $\beta$, we can set the ratio $\xi^*$. When we want $N_s$ of the $N + 1$ points to flow into the boundary layer, we need

$$\beta = \frac{N + 1 - N_s}{N_s (s_N - s_0)}. \quad (3.20)$$

### 3.1.2 Previous numerical results on the Willmore flow

In this Subsection we give a very short overview of the literature about the numerical aspects of the Willmore flow.

One of the earliest papers on this subject is [70]. The authors of this paper look for minimizers of the Willmore integral, using Brakke’s surface evolver. Starting with different shapes of genus one they find the $\sqrt{2}$-torus. Starting with different shapes of genus two and three they find certain smooth minimizers. This supports the conjecture, proposed in [79], that one can find a minimizer for every genus and that this minimizer is the stereographic projection of certain minimal surfaces in the three sphere. In [13] the first statement is proved. The second statement, which generalizes the Willmore conjecture, remains unsolved.

Most papers on the numerics of the Willmore flow concentrate on finding minimizers of the Willmore functional. In [90], however, the authors study the geometric evolution of the flow and its properties. In this article they study the evolution of certain axisymmetric surfaces and find, for instance, that an embedded surface can become immersed and create self-intersections. This was proven later in [91]. The authors further show in [90] a numerical example of finite time blowup in the Willmore flow. This observation is the principle idea in the investigations of Chapter 2 and this Chapter. In this Chapter we try to verify their results and extract quantitative information, such as the blowup rate, from the calculations. In the same article the authors also find that several dumbbells flow towards a sphere. This in contrast with the surface diffusion flow, which can (at least numerically) create a singularity starting from a dumbbell by pinching the neck (see [46]). As these dumbbells all evolve towards spheres, the authors of [90] conjecture that
3.1. Introduction

this happens for all dumbbells. With the results of Chapter 2 in mind, we also investigate this claim numerically in this Chapter. We also find that dumbbells evolve into spheres.

We further mention [40], [36], [58], [22], [39], [12], [41] and [94]. They all give a different formulation of the Willmore flow or study a different aspect of the numerical computations. Most of the surfaces they study are typically not axisymmetric and can have every shape possible. This is different from our investigations, which are only on axisymmetric surfaces. In most of these references blowup is not discussed.

3.1.3 Discretization

In this Section we give two centered difference approximations for a non-uniform mesh: one with 3 points and one with 7 points.

Let the grid points \( s_i \) be given as in Subsection 3.1.1. Then for a given function \( f \) dependent on \( s \) we have the following Taylor expansions for the discretized function,

\[
\begin{align*}
    f_{i+1} &:= f(s_{i+1}) = f(s_i + \Delta_1(i)) = f(s_i) + f'(s_i)\Delta_1(i) + \frac{1}{2} f''(s_i)\Delta_1(i)^2 + \frac{1}{6} f'''(s_i)\Delta_1(i)^3 + O(\Delta_1(i)^4) \\
    f_{i-1} &:= f(s_{i-1}) = f(s_i - \delta_1(i)) = f(s_i) - f'(s_i)\delta_1(i) + \frac{1}{2} f''(s_i)\delta_1(i)^2 - \frac{1}{6} f'''(s_i)\delta_1(i)^3 + O(\delta_1(i)^4),
\end{align*}
\]

with \( \Delta_1(i) \) and \( \delta_1(i) \) given by

\[
\begin{align*}
    \Delta_1(i) &:= s_{i+1} - s_i \\
    \delta_1(i) &:= \Delta_1(i-1).
\end{align*}
\]

Similarly one has

\[
\begin{align*}
    f_{i+2} &= f(s_i) + f'(s_i)\Delta_{12}(i) + \frac{1}{2} f''(s_i)\Delta_{12}(i)^2 + \frac{1}{6} f'''(s_i)\Delta_{12}(i)^3 + O(\Delta_{12}(i)^4), \\
    f_{i-2} &= f(s_i) - f'(s_i)\delta_{12}(i) + \frac{1}{2} f''(s_i)\delta_{12}(i)^2 - \frac{1}{6} f'''(s_i)\delta_{12}(i)^3 + O(\delta_{12}(i)^4), \\
    f_{i+3} &= f(s_i) + f'(s_i)\Delta_{123}(i) + \frac{1}{2} f''(s_i)\Delta_{123}(i)^2 + \frac{1}{6} f'''(s_i)\Delta_{123}(i)^3 + O(\Delta_{123}(i)^4), \\
    f_{i-3} &= f(s_i) - f'(s_i)\delta_{123}(i) + \frac{1}{2} f''(s_i)\delta_{123}(i)^2 - \frac{1}{6} f'''(s_i)\delta_{123}(i)^3 + O(\delta_{123}(i)^4),
\end{align*}
\]

with

\[
\begin{align*}
    \Delta_{12}(i) &:= \Delta_1(i) + \Delta_1(i+1), \\
    \Delta_{123}(i) &:= \Delta_1(i) + \Delta_1(i+1) + \Delta_1(i+2), \\
    \delta_{12}(i) &:= \delta_{12}(i-2), \\
    \delta_{123}(i) &:= \delta_{123}(i-3).
\end{align*}
\]

The finite differences approximations are now determined as follows. Let the second derivative of \( f \) be given by finite differences. Then we can write

\[
f''(s_i) = Af_{i-3} + Bf_{i-2} + Cf_{i-1} + Df_i + Ef_{i+1} + Ff_{i+2} + Gf_{i+3},
\]
where we want to determine the constants $A$ to $G$. These constants are determined by substituting equations (3.21) and (3.23) into equation (3.28). We need at least the contributions to the terms $f(s)$ and $f'(s)$ in (3.28) to be zero and the contribution to $f''(s)$ to be one. These are three requirements which can be solved by three constants in (3.28). Hence, we can calculate the second derivative of $f$ using only three points. We choose the centered finite differences approximation and we may set $A = B = F = G = 0$. We are left with the equations

$$
C + D + E = 0,
$$
$$
-C\delta_1(i) + E\Delta_1(i) = 0,
$$
$$
\frac{1}{2}C\delta_1(i)^2 + \frac{1}{2}E\Delta_1(i)^2 = 1,
$$

which are solved by

$$
C = \frac{2}{\Delta_1(i)(\Delta_1(i) + \delta_1(i))}, \quad D = -\frac{2}{\Delta_1(i)\delta_1(i)}, \quad E = \frac{2}{\delta_1(i)(\Delta_1(i) + \delta_1(i))}.
$$

Hence,

$$
f''(s_i) = \frac{2}{\Delta_1(i) + \delta_1(i)} \left( \frac{f_{i+1} - f_i}{\Delta_1(i)} - \frac{f_i - f_{i-1}}{\delta_1(i)} \right) - \frac{1}{3}f''(s_i)(\Delta_1(i) - \delta_1(i)) - O(\Delta_1^2).
$$

The smoother the node concentration the smaller the value of $\Delta_1(i) - \delta_1(i) = \Delta_1(i) - \Delta_1(i - 1)$. Hence, with a smooth enough node concentration, the centered differences approximation (3.31) is up to order 2 in $\Delta_1$. That is why we have chosen MMPDE6 in Subsection 3.1.1, where the diffusive term smoothens the monitor function. We see that, up to order 2 in $\frac{1}{\Delta_1}$, the centered difference approximation of $2 \partial_t (M\partial_s s)$ can be given by

$$
2\partial_t (M\partial_s s) = \partial_s^2(Ms_i) - \partial_s^2Ms_i + M\partial_s^2s_i
$$

$$
= N^2(M_{s_{i+1}}s_{i+1} - 2Ms_s + M_{s_{i-1}}s_{i-1})
$$

$$
- N^2(M_{s_{i+1}} - 2Ms + M_{s_{i-1}}) + N^2Ms(s_{i+1} - 2s_s + s_{i-1})
$$

$$
= N^2\left( \left( M_{s_{i+1}} + M_s \right)(s_{i+1} - s_i) - \left( M_s + M_{s_{i-1}} \right)(s_i - s_{i-1}) \right),
$$

and we note that (3.9) is indeed the continuous version of equation (3.8).

If we want to approximate the second derivative up to higher order in $\Delta_1$, we need to use more points. For instance, an approximation up to order 5 in $\Delta_1$ can be achieved by solving the system

$$
\begin{pmatrix}
-\delta_{22}(i) & -\delta_{21}(i) & -\delta_1(i) & 0 & \Delta_1(i) & \Delta_{12}(i) & \Delta_{122}(i) \\
\frac{1}{2}\delta_{12}(i)^2 & \frac{1}{2}\delta_{12}(i)^2 & \frac{1}{2}\delta_1(i)^2 & 0 & \frac{1}{2}\Delta_1(i)^2 & \frac{1}{2}\Delta_{122}(i)^2 & \frac{1}{2}\Delta_{1222}(i)^2 \\
-\delta_{21}(i)^3 & -\delta_{21}(i)^3 & -\delta_1(i)^3 & 0 & \Delta_1(i)^3 & \Delta_{12}(i)^3 & \Delta_{122}(i)^3 \\
\delta_{12}(i)^4 & \delta_{12}(i)^4 & \delta_1(i)^4 & 0 & \Delta_1(i)^4 & \Delta_{12}(i)^4 & \Delta_{122}(i)^4 \\
-\delta_{21}(i)^5 & -\delta_{21}(i)^5 & -\delta_1(i)^5 & 0 & \Delta_1(i)^5 & \Delta_{12}(i)^5 & \Delta_{122}(i)^5 \\
\delta_{12}(i)^6 & \delta_{12}(i)^6 & \delta_1(i)^6 & 0 & \Delta_1(i)^6 & \Delta_{12}(i)^6 & \Delta_{122}(i)^6
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C \\
D \\
E \\
F \\
G
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}
$$

(3.33)

Solving this system gives constants of order $\frac{1}{\Delta_1^5}$. For explicit expressions of the constants we refer to the code given in Section 3.4. Hence, the centered differences approximation (3.28) is accurate up to, at least, order $7 - 2 = 5$. For a smooth node concentration we find that, by the same reasoning as above, we can get the approximation up to order 6.
3.2. Computational results on the Willmore flow

3.1.4 Boundary conditions

In this Chapter we compute the evolution of axisymmetric surfaces. These surfaces can be characterized by a curve in the \((z, r)\) plane rotated around the \(z\)-axis. Hence, we need to solve the system of equations (3.14). To make the calculations easier we consider surfaces which are mirrored in the \(r\)-axis. To compute the derivatives at the boundaries of the curve we make use of the geometry of the problem. Let the curve, corresponding to the surface, start on the \(r\)-axis. By the symmetry we have that

\[
r'(0) = 0, \quad z''(0) = 0.
\]

If the curve ends on the \(z\)-axis, as is the case for the Limaçon or a dumbbell, we have, due to the rotational symmetry, that

\[
z'(N) = 0, \quad r''(N) = 0.
\]

If the curve ends on the \(r\)-axis, as is the case for a torus, we have

\[
r'(N) = 0, \quad z''(N) = 0.
\]

To calculate the first and second derivatives of the principal curvatures \(\kappa_1\) and \(\kappa_2\), we need to compute the third and fourth derivatives of the coordinates \(r\) and \(z\). But we can also discretize the curvatures themselves and compute their derivatives directly. We choose the latter option. Besides that this shortens the computation time considerably, it also captures the geometric nature of the problem. Hence, for the derivatives of the curvatures we use the centered differences approximation with boundary conditions

\[
k'_1(0) = k'_1(N) = k'_2(N) = k'_2(0) = 0.
\]

In Subsection 3.2.1 we justify this choice for the computation of the curvatures.

To compute the evolution of the Willmore flow we solve the system of ordinary differential equations (3.14) with the implicit ode-solver ode15s in Matlab. We refer to Subsection 3.4 for an example of the code we use to calculate the Willmore flow.

3.2 Computational results on the Willmore flow

In this Section we give the results of our numerical computations. We compare them with the results of reference [90].

3.2.1 A torus

In this Subsection we give the numerical results, as a test case, for the Willmore flow on a torus. The Willmore conjecture in the case of radially symmetric tori is proved in [84]. Hence, we know that the torus has to evolve, in infinite time, towards a \(\sqrt{2}\)-torus with Willmore energy \(2\pi^2\). Since the evolution does not exhibit singular behavior we do not need a moving mesh method. To test our calculations we compare computations with an adaptive mesh with computations with a rigid mesh. We further compare several runs with different discretization methods to justify the choices we make in Subsection 3.2.2.
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Figure 3.1: Every torus evolves towards a minimizing $\sqrt{2}$-torus. On the left one sees the generating curve of the surface at different points in time. On the right the Willmore energy is plotted against time, where the theoretical value $2\pi^2$ is represented by the dashed line.

We compare our results with the results obtained in reference [90]. To do this we take as the generating curve for the initial surface a torus given by

$$r(s) = 5 + 2\sin(s), \quad z(s) = 2\cos(s),$$

with $s$ from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$.

Numerical evaluation of the Willmore flow gives the evolution given in Figure 3.1. We see that the torus flows indeed towards a torus with energy $2\pi^2$, which is the theoretical value for the Willmore energy of a $\sqrt{2}$-torus.

As mentioned before, we want to compare computations without moving mesh with computations with a moving mesh. Let the monitor function $M$ be given by

$$M = \sqrt{r^2 + z^2}.$$

Hence, the mesh is moved in such a way that the points are equally distributed, lengthwise, along the curve. The differences between both methods are not noticeable in Figure 3.1. In Figure 3.2 (left), however, we plot the integral over the Gaussian curvature against time and we do see a difference. Due to the Theorem of Gauss-Bonnet the integral over the Gaussian curvature is a constant with value 0. We use the computed total Gaussian curvature of the surface as a measure for the accuracy of the numerics. We see that the computations with the moving mesh are closer to the theoretical value. For both computations we have used a 7-point discretization for the derivatives of the coordinates and the curvatures.

We consider two ways to discretize the derivatives of the curvatures. One method, which we refer to as METHOD1, is by discretizing all derivatives of the coordinates (up to the fourth) and compute the derivatives of the curvatures with these. For this method we consider a 3-point discretization for the first and second derivative and a 7-point discretization for the third and fourth derivative or a 7-point discretization for all derivatives. Another method to compute the derivatives of the curvatures, which we call METHOD2, is by direct discretization of the curvatures. For this method we can use a 3-point or a 7-point discretization. In Figure 3.2 we plotted the total Gaussian curvature and Willmore energy for several runs with these different methods. We find that the total Gaussian curvature is almost identical for METHOD1, with both types of discretization,
and METHOD2 with a 3-point discretization (dashed curves, indistinguishable from the black uninterrupted curve in Figure 3.2). METHOD2 with a 7-point discretization clearly shows a different total Gaussian curvature of the same order (gray uninterrupted curve in Figure 3.2). The Willmore energy is almost identical for METHOD2 and METHOD1, both with a 7-point discretization (uninterrupted black and gray curves). The methods that use a 3-point discretization clearly have a different Willmore energy (dashed curves). In this case the energy of METHOD1 lies further away from the others than METHOD2.

Figure 3.2 suggests that we should use a 7-point discretization for METHOD1 and METHOD2. Since the computation time of METHOD1 is considerably larger than the computation time of METHOD2 we would like to use METHOD2. To justify the accuracy of this choice we refer to Figure 3.3. In this figure we plot the total Gaussian curvature for several runs with different number of grid points. We see that the more grid points we use, the more accurate the calculations are, as we would expect. We further see that if we use more grid points the energies of both methods come closer to each other, until they are almost indistinguishable. This justifies our choice for METHOD2 with a 7-point discretization of the derivatives.

In Figure 3.4 we show the results of computations with moving mesh, 7-point discretization, METHOD2 and a varying amount of points $N + 1$. We also included the results for a run with METHOD2 and a 3-point discretization, which immediately shows the failure at large times. We see that the total energy comes closer to the theoretical value if we take more points. On the other hand we see that around $t = 300$, and onwards, the noise of numerical inaccuracies is visible. In [90] the authors were able to compute up to time $t = 1000$, which we can not because of the prominent inaccuracies. Another difference of our computations with theirs is that they find that the surface area drops after $t = 50$ and continues dropping. We do not see that behavior as becomes clear in Figure 3.4. In this figure we see that in our calculations the surface area keeps increasing.

We conclude that the moving mesh computations are, for this test case, in agreement
Figure 3.3: On the left we have plotted the total Gaussian curvature $K$ over the surface against time, for different runs. On the right we show the energy for these runs. The black curve corresponds to $N + 1 = 200$. The darker gray curves corresponds to $N + 1 = 150$, while the lighter gray corresponds to $N + 1 = 100$. The dash-dot curves correspond to method 1 and the uninterrupted curves correspond to method 2, both with a 7-point discretization.

Figure 3.4: The energy (left) and the surface area (right) of several runs with moving mesh, method 2 and different amount of points. The lighter gray has $N + 1 = 150$ points, the darker gray has 250 points, while the black curve is the evolution with 500 points. The dashed curve corresponds to method 2 with a 3-point discretization and $N + 1 = 250$ points.
with theory and (for most parts) with [90]. The main difference of our computations with [90] is the behavior of the surface area for large \( t \). In the rest of this Chapter we keep using the total Gaussian curvature as a measure for the accuracy of the computations.

### 3.2.2 The Limaçon

In this Subsection we discuss the numerical computations of the Willmore flow of a Limaçon-like surface. Just as in [90] we find that the Limaçon evolves towards a singularity in finite time. In this Subsection we also investigate the different scales in the evolution of the Limaçon and we test some assumptions we make in Chapter 2.

As initial surface we start with the curve

\[
  r(s) = \cos(s) - \frac{4}{5} \cos(3s), \quad z(s) = -\sin(s) + \frac{4}{5} \sin(3s),
\]

with \( s \) from \(-\frac{\pi}{2}\) to \( \frac{\pi}{2}\), rotated around the \( z \)-axis, see Figure 3.5. This curve is actually not a Limaçon of Pascal, but another curve with an inner loop. Since all curves with an inner loop, similar to (3.40), behave identically near blowup (see also Subsection 3.2.3) we call every such curve (and its corresponding surface) a Limaçon. The Willmore integral over this surface is about \( 28.44 \), which is relatively close to the theoretical value of two spheres, given by \( W = 8\pi \). Therefore, the surface given by (3.40) is, energy-wise, close to blowup, which is the reason why we choose this as the initial surface. In Subsection 3.2.3 we study other Limaçon-like surfaces, whose energies lie far from \( 8\pi \), and conclude that these surfaces behave similarly under the Willmore flow.

The monitor function we use is

\[
  M = |\kappa_1 - \kappa_2|^\alpha,
\]

where \( \alpha \) is still to be determined. This can be justified by considering the monitor function in the region of the tip (minimal \( r \)-coordinate in the inner loop). Near the tip, where the curve is given in first order by \( r = \lambda \cosh(\frac{\pi}{4}) \), the principal curvatures are equal in magnitude and differ a sign. In the outer region, where the curve resembles a circle, the principal curvatures are approximately equal. Hence, the monitor function has a large contribution of order \( (\frac{\pi}{4})^\alpha \) near the tip and has a small contribution in the outer region. This makes the grid points flow towards the tip, with the right scale (if \( \alpha \) is chosen correctly).

The Willmore flow is scale invariant under self-similar transformations. It has been shown in several cases to be effective to choose the monitor function in such a way that the resulting system of equations stays scale invariant (see for instance [27]). This would mean in our case (see (3.14)) we have to choose \( \alpha = 4 \) in (3.41).

The complication for such high \( \alpha \), however, is that after a short time almost all points are driven into the singularity and there are no points left for the regular part of the solution. Even for the corrected monitor function \( M_c \) (see (3.15)) we have that after a short time all permissible points are in the singularity and the accuracy breaks down. We find that \( \alpha = 1 \) is a relatively good candidate. Although the points do not go fast enough into the singularity, we are still able to approach the singularity and keep a relatively good accuracy.

For the following calculations we use \( \alpha = 1 \) and METHOD2 with a 7-point discretization (see Subsection 3.2.1). For \( N = 150 \), with tolerances of \( 10^{-8} \), we find the evolution...
depicted in Figure 3.5. Denote the crossing of curve with the $r$-axis by $c(t)$ and denote the point on the inner loop where the $z$-coordinate is maximal by $b(t)$. We then denote the $z$-coordinate of $b$ by $b'$ and the $r$-coordinate of $b$ by $b''$. The tip (minimal $r$-coordinate) of the inner loop is denoted, as in Chapter 2, by $\lambda(t)$. We see that a singularity is created in finite time $T$. We further see that the crossing $c(t)$ of the curve drops towards the $z$-axis and the whole loop vanishes in the origin at time $T$. See also Figure 3.6. In this figure we see, on the left, the evolution of the points $c(t)$, $b(t)$ and $\lambda(t)$. On the right of Figure 3.6 we see the evolution of the Willmore energy, where the value of $8\pi$ is denoted by a dashed line. At time $t = 1.5$ the energy of the Limaçon differs 0.4 percent from the energy of two spheres and the tip of the Limaçon has dropped from 0.2 to the value 0.0016. This seems to indicate that the computations are close to the singularity.

In Figure 3.7 we compare different runs for different numbers of points. Superficially the runs seem to be the same, but if we zoom in on the tip we do see a difference. In this figure the total Gaussian curvature over the surface is also given. Theoretically this should be a constant given by $4\pi$ (Gauss-Bonnet). We see that the accuracy is lost dramatically as the blowup time is approached. This loss of accuracy is due to a lack of points in the tip of the loop. In Figure 3.8 we show the $r$-coordinate of the first 25 grid points divided
3.2. Computational results on the Willmore flow

Figure 3.7: Three different runs for different number of grid points. The black curve corresponds to $N + 1 = 100$ points, the darker gray corresponds to $N + 1 = 125$ points and the lighter gray corresponds to $N + 1 = 150$ gray.

Figure 3.8: The rescaled $r$-coordinate of the first 25 grid points with $\alpha = 1$ for $N + 1 = 150$, on the left. The total Gaussian curvature minus $4\pi$ on the right.

by $\lambda(t)$, against time. If the rate, by which the grid points move, was chosen rightly we would expect to see vertical lines in this picture (see also the example in Section 1.2). Now, we see that the grid points, divided by $\lambda$, move outwards, which is to be expected as we have chosen $\alpha$ equals 1, instead of 4. To improve the calculations we need to include more points. Consider again the total Gaussian curvature, minus $4\pi$, in Figure 3.8. If we decide that a deviation of $10^{-3}$ is still acceptable, we see that we can use the computations with $N + 1 = 150$ grid points up to time $t = 1.35$. At this time the energy differs 0.7 percent from the theoretical value $8\pi$ and the tip $\lambda$ has dropped from 0.2 to 0.005. If one fits the functions $\lambda$, $b$, $b'$ and $c$ on $t \in [0.75, 1.35]$ to an algebraic expression, one finds

$$
\lambda \sim 0.04(1.69 - t)^{2.01},
$$
$$
b \sim 0.32(1.72 - t)^{1.09},
$$
$$
b' \sim 0.11(1.67 - t)^{1.08},
$$
$$
c \sim 0.78(1.66 - t)^{0.81}.
$$

For computations with different initial values we find different scales (see also Subsection 3.2.3). The exponent of $\lambda$ varies, in these calculations, between 1.0 and 3.8. The exponent we find for $c$ varies between 0.4 and 1.2. This shows we can not draw conclusions about the exponents at this level of accuracy. It does suggests, however, that the scale of the curve up to the crossing $c$ is quasi-stationary.
For the matched asymptotics of Chapter 2 it would be convenient to know where the self-similar region of the solution lies. However, since we do not know the precise blowup time $T$, it is very hard to find this self-similar region numerically.

In Figure 3.9, on the right, we show the evolution of a part of the Limaçon together with the evolution of the inner solution we found in Chapter 2. One sees that they agree quite well, up to a point on the left side of the crossing. This, again, suggests that the part of the curve up to the crossing has a quasi-stationary scale. This could mean that the inner solution should be matched immediately to the remote solution. See Chapter 1 for more on this. In Figure 3.9 we further show the evolution of $\epsilon, \frac{\lambda}{\epsilon}$ and the derivative of $z$ with respect to $r$ at the grid point nearest to the crossing $c(t)$. The function $\epsilon$ is defined in Chapter 2 as $-\frac{\kappa_2}{\kappa_1}$ at the tip minimal $r$-coordinate of the inner loop. The matching in Chapter 2 shows that a negative $\epsilon$ corresponds to a curve that crosses the $r$-axis. This is consistent with the numerics. The assumption we make in Chapter 2 that $\lambda \epsilon \ll \lambda$ is supported by the fact that $\frac{\lambda}{\epsilon} \rightarrow 0$ in the numerics. The main assumption we make in Chapter 2 is that the derivative of $z$ with respect to $r$ in the outer region has to go to zero. This holds certainly for the point $c(t)$ on the curve. Hence, if the outer solution lies somewhere between the point $b(t)$ on the curve and the point on the curve where the $r$-coordinate is maximal, then $z$ certainly goes to zero.

In Figure 3.9 we have plotted the principal curvature $\kappa_2$ divided by the other principal curvature $\kappa_1$ at different points in time. We see that around $s = 0, \kappa_2$ evolves to $-\kappa_2$ and therefore $H = 0$. This agrees with theory. Around the point $s = \frac{\pi}{2}, \kappa_1$ seems to evolve towards $\kappa_2$, that is, towards a sphere. The point $c(t)$, where the curve crosses the $r$-axis, is denoted with squares. Also this point seems to evolve towards $\kappa_1 = \kappa_2$, although it is hard to say as the computations do not go far enough into the singularity. A better understanding of the evolution of the curvatures at the crossing point of the Limaçon would be very helpful for building the outer solution in the matched asymptotics. In the same Figure 3.9 we plot the evolution of $\kappa_2$ along the curve. We see that $\kappa_2$ at the crossing, seems to be of order 1 at all times, up to blowup time. In Chapter 1 we give a sketch of the matched asymptotics assuming that the Limaçon tends to evolve towards two spheres as it blows up in finite time. In the next Subsection we show that this assumption could hold for all Limaçon-like initial data.
3.2. Computational results on the Willmore flow

Figure 3.10: On the left the function $\frac{\kappa_2}{\kappa_1}$ with respect to the parameter $s$ of the curve. On the right the function $\kappa_2$. The squares denote the points where the curve crosses the $r$-axis.

Figure 3.11: Evolution of a Limaçon-like surface with initial energy $W = 135.4$.

3.2.3 Variants of the Limaçon

In Subsection 3.2.2 we compute the Willmore flow with an initial surface that already lies close to two spheres. Consider a curve in the upper half plane of $(z, r)$, such that the normal along this curve turns one and a half times. If we rotate this curve around the $z$-axis we call the resulting surface Limaçon-like. In this Subsection we study the evolution of Limaçon-like surfaces, whose energies lie far from the energy of two spheres. The evolutions of these surfaces suggest that any Limaçon-like surface creates a singularity in finite time with a blowup profile close to two spheres.

Consider the surfaces corresponding to the dashed curves in Figures 3.11 to 3.15. All these surfaces lie far, energy-wise, from two spheres. In the same figures we also show the evolution of the surface and we see that the surfaces blow up in finite time. Throughout the computations the total Gaussian curvature minus $4\pi$ stays below $10^{-3}$. From the evolution of the energies we see that the blowup occurs near $8\pi$, which is the Willmore energy of two spheres. In Figures 3.11 to 3.15 we also plot the curvature $\kappa_2$ divided by the curvature $\kappa_1$ at the end of the evolution (from $W = 35$ onwards). We see that the surfaces eventually blow up in the same manner as the surface discussed in Subsection 3.2.2.
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Figure 3.12: Evolution of a Limaçon-like surface with initial energy $W = 58.0$.

Figure 3.13: Evolution of a Limaçon-like surface with initial energy $W = 40.9$.

Figure 3.14: Evolution of a Limaçon-like surface with initial energy $W = 100.3$.

Figure 3.15: Evolution of a Limaçon-like surface with initial energy $W = 85.2$. 
3.2. Computational results on the Willmore flow

3.2.4 Different dumbbells

In this Subsection we discuss the evolution of dumbbells. In [90] it is claimed that every dumbbell evolves towards a sphere in infinite time. Because of the result of Chapter 2 we felt the necessity to verify this claim. Since, every initial surface with energy smaller than $8\pi$ flows towards a sphere (see Subsection 2.2.5), the dumbbells under considerations in this Subsection should have energy higher than $8\pi$. We find, with our calculations, that every dumbbell evolves to a sphere.

Consider the initial surface given by the dashed curve in Figure 3.16, rotated around the $z$-axis. The energy of this initial surface is about $7.5\pi$. In Figure 3.16 we also plotted the evolution of the energy under the Willmore flow and we see that, after a short time, the energy is well below $8\pi$, the theoretical value for two spheres. Theory then states that the surface has to evolve towards a sphere in infinite time. Throughout the computations the total Gaussian curvature, minus $4\pi$ stays order $10^{-5}$. Consider now a dumbbell that lies even further away (energy-wise) from a sphere. The evolution is given in Figure 3.17. The point at the $r$-axis drops towards zero, as if creating a singularity. At a certain point the energy drops below $8\pi$, however, and the tip starts rising. This would imply that the dumbbell evolves to a sphere. At this point the accuracy of the computations is still reasonable.
3.3 Conclusions

In this Section we computed the evolution of several surfaces under the Willmore flow, using moving mesh methods. We find the same results as in [90]: finite time blowup in the case of a Limaçon and no blowup in the case of a dumbbell. We further show that any Limaçon-like surface blows up in finite time with energy close to the energy of two spheres. The computed evolution suggests that the inner loop of the Limaçon vanishes on a quasi-stationary rate.

There is some room for improvement of the code. In the moving mesh method we flow our points towards the singularity on a slower scale than the blowup scale. We do this because we notice a dramatic loss of accuracy otherwise. A better understanding of this behavior could help to improve the calculations. We further refer to Section 1.4 for possible enhancements of the code.

3.4 The code

```matlab
function [] = WillmoreFlow(n,N)

% Choose initial curve
% --- a Limacon
p = inline ('cos(x) - 4/5*cos(3*x)''x');
qu = inline ('-sin(x) + 4/5*sin(3*x)''x');
s0 = 0;
s1 = (pi/2);

% Number of points
N = 100;

% Choose type of mesh
% --- linear
P = linspace (s0,s1,N);

% Initial curve
F(1:N) = feval (p,P);
F(N+1:2*N) = feval (q,P);
F(2*N+1:3*N) = P;

% Initial guess for velocity
G = zeros (1,3*N);

% Make initial velocity consistent
[FF,GG] = decic (WWillmoreEQN,0,F,1,G,[],N);

% set tolerances
options = odeset ('RelTol',1e-8,'AbsTol',1e-8);

% set time
n = 0.001;

% RUN
[t,x] = ode15i (WWillmoreEQN,linspace(0,n,1000),FF,GG,options,N);
save ('EvolutionData',t,x);
```
function Y = WillmoreEQN (t,F,Ft,N)

F=F';
Ft=Ft';

% Set the parameters:
% The relaxation parameter tau is fixed
tau = si(4)^-uh;
% Parameter alpha determines the shape
alpha = s;
% Number of points in the regular part
nn = tx;

beta = 1/(F(3+N)-F(2+N+1)) * nn/(N-nn);

rename function for convenience
r = F(1:N);
z = F(N+1:2+N);
s = F(2+N+1:3+N);

% Set the geometry of the boundaries
R(1:3) = [r(4),r(3),r(2)];
R(4:N+3) = t;
R(N+4:N+6) = [-r(N-1),r(N-2),r(N-3)];
Z(1:3) = [z(4),z(3),z(2)];
Z(4:N+3) = z;
Z(N+4:N+6) = [z(N-1),z(N-2),z(N-3)];

% Extend the running parameter
S(1:3) = 2*s(1) - [s(4),s(3),s(2)];
S(4:N+3) = s;
S(N+4:N+6) = 2*s(N) - [s(N-1),s(N-2),s(N-3)];

% Define
EE1(i+3) = Delta1(i) = s(i+1)-s(i) = S(i+4)-S(i+3)
and EE2(i+3) = Delta2(i) = s(i+2)-s(i) = S(i+5)-S(i+3)
and EE123(i+3) = Delta123(i) = s(i+3)-s(i) = S(i+6)-S(i+3)

EE1 = S(2:N+6)-S(1:N+5);
EE12 = S(3:N+6)-S(1:N+4);
EE123 = S(4:N+6)-S(1:N+3);

% Rename the Delta’s for convenience
E1 = EE1(3:N+2);
E2 = EE1(2:N+1);
E3 = EE1(1:N);
E12 = EE12(2:N+1);
E123 = EE123(1:N);

D1 = EE1(4:N+3);
D2 = EE1(5:N+4);
D3 = EE1(6:N+5);
D12 = EE12(4:N+3);
D123 = EE12(5:N+4);
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D123 = EE123(4·N+3);

% Some reoccurring combinations
DD = D1 +D12 + D1 ·D123 + D12 ·D123;
Drond = D1 ·D12 ·D123;

EE = E1 ·E12 + E1 ·E123 + E12 ·E123;

Emaal = Dur ·Duv ·Duvw;

FD = Euvw ·Duvw;

Fu = Euvw ·Euv ·Evw ·k Eun ·Duvw ·Duvw;

Fv = Euv ·Ew ·Ev ·k Euv ·Duvw ·Duvw;

Fw = Eu ·Evw ·Ew ·k Eun ·Duvw ·Duvw;

Fz = Eu ·Euv ·Euvw ·Dur ·Duv ·Duvw;

% The Ak = A(k) are the constants belonging to f’
% with f’(i) = sum (A(k·i+4) f(k) k=i-3..i+3)
A1 = -E1 ·E12 ·Dmaal .'/F1;
A2 = E1 ·E123 ·Dmaal .'/F2;
A3 = -E12 ·E123 ·Dmaal .'/F3;
A4 = (-Emaal ·DD + Dmaal ·EE).'/F4;
A5 = D12 ·D123 ·Emaal .'/F5;
A6 = -D1 ·D123 ·Emaal .'/F6;
A7 = D1 ·D12 ·Emaal .'/F7;

% The Bk = B(k) are the constants belonging to f”
% with f”(i) = sum (B(k·i+4) f(k) k=i-3..i+3)
B1 = 2* (E1 ·E12 ·D123 ).'/F1;
B2 = -2* (E1 ·E123 ·DD - (E1 +E123) ·Dmaal ).'/F2;
B3 = 2* (E12 ·E123 ·DD - (E12 +E123) ·Dmaal ).'/F3;
B4 = 2* (EE ·D123 ·Emaal - Dmaal ·Drond + Emaal ).'/F4;
B5 = 2* (D12 ·D123 ·EE - (D12 +D123) ·Dmaal ).'/F5;
B6 = -2* (D1 ·D123 ·EE - (D1 +D123) ·Emaal ).'/F6;
B7 = 2* (D1 ·D12 ·EE - (D1 +D12) ·Emaal ).'/F7;

% First derivatives of r and z
r = A1 ·R(1·N) + A2 ·R(2·N+1) + A3 ·R(3·N+2) + A4 ·R(4·N+3) .
+ A5 ·Z(5·N+4) + A6 ·Z(6·N+5) + A7 ·Z(7·N+6);

% Second derivatives of r and z
x = B1 ·R(1·N) + B2 ·R(2·N+1) + B3 ·R(3·N+2) + B4 ·R(4·N+3) .
+ B5 ·Z(5·N+4) + B6 ·Z(6·N+5) + B7 ·Z(7·N+6);

% Define q
Q = r1 ·2+z1 ·2;
3.4. The code

% The principle curvatures
kappa1 = (-z1.*r2+r1.*s2)./Q.*((3/2);
kappa2 = z1./r./sqrt(Q);

% kappa2 at the z-axis
kappa2(N)=(s(N)-s(N-2))^2/(s(N)-s(N-1))^2+kappa2(N-1) ... 
- (s(N)-s(N-1))^2/(s(N)-s(N-2))^2-kappa2(N-2);

% The mean and Gaussian curvatures
h = 1/2.*(kappa1+kappa2);
k = kappa1.*kappa2;

% The geometry at the boundaries imply
Kappa1(1:3) = [kappa1(4),kappa1(1),kappa1(2)];
Kappa1(N+4:N+6) = [kappa1(N-1),kappa1(N-2),kappa1(N-3)];
Kappa2(1:3) = [kappa2(4),kappa2(1),kappa2(2)];
Kappa2(N+4:N+6) = [kappa2(N-1),kappa2(N-2),kappa2(N-3)];

% First derivatives of kappa1 and kappa2
KK11 = A1.*Kappa1(1:N)+A2.*Kappa1(2:N+1)+A3.*Kappa1(3:N+2) +... 
A4.*Kappa1(4:N+3)+A5.*Kappa1(5:N+4)+A6.*Kappa1(6:N+5) +... 
A7.*Kappa1(7:N+6);
KK12 = A1.*Kappa2(1:N)+A2.*Kappa2(2:N+1)+A3.*Kappa2(3:N+2) +... 
A4.*Kappa2(4:N+3)+A5.*Kappa2(5:N+4)+A6.*Kappa2(6:N+5) +... 
A7.*Kappa2(7:N+6);

% Second derivatives of kappa1 and kappa2
KK21 = B1.*Kappa1(1:N)+B2.*Kappa1(2:N+1)+B3.*Kappa1(3:N+2) +... 
B4.*Kappa1(4:N+3)+B5.*Kappa1(5:N+4)+B6.*Kappa1(6:N+5) +... 
B7.*Kappa1(7:N+6);
KK22 = B1.*Kappa2(1:N)+B2.*Kappa2(2:N+1)+B3.*Kappa2(3:N+2) +... 
B4.*Kappa2(4:N+3)+B5.*Kappa2(5:N+4)+B6.*Kappa2(6:N+5) +... 
B7.*Kappa2(7:N+6);

% First and second derivatives of the mean curvature
h1 = 1/2.*(KK1+KK2);
h2 = 1/2.*(KK12+KK22);

% Laplace-Beltrami on the mean curvature
d = (r1./r./Q-(r1.*r2+z1.*z2)./Q.*2).*h1 + 1./Q.*h2;

% Choose the monitor function
% — sphere vs cosh
M = (abs(kappa1-kappa2)).*alpha;

% The monitor function with correction term
MC = M + beta*sum([s(2:N)-s(1:N-1)];2*(M(2:N)+M(1:N-1)));

% diff(M*diff(s))
x1=zeros(1,N);
xt(1) = 1/tau*(1/2*(MC(2)+s(3)-s(1)) - MC(1)+s(2)-s(1))) ;
x(2:N-1) = 1/tau/2*(MC(3)+MC(1)/2*(MC(3)+MC(2)-s(3)-s(2)-s(1))) ... 
- (MC(2)-s(2)-s(1));
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\[ x_t(N) = \frac{1}{\tau} \left( \sum \left( s(N) - s(N-1) \right) - \frac{1}{2} \sum \left( s(N) - s(N-2) \right) \right); \]

\% The Willmore flow

\[ \text{eqn1}(1:N) = r1 \cdot F \left( 2+N+1:3+N \right) + z1 \cdot Q \cdot (1/2) \cdot (d+2+h) \cdot (h' \cdot 2-k); \]

\[ \text{eqn1}(N+1:2+N) = z1 \cdot F \left( 2+N+1:3+N \right) - r1 \cdot Q \cdot (1/2) \cdot (d+2+h) \cdot (h' \cdot 2-k); \]

\[ y(1:2+N) = F \left( 1:2+N \right) - \text{eqn1}; \]

\% MMTREE

\[ y(2+N+1) = F \left( 2+N+1 \right); \]

\[ y(2+N+2:3+N-1) = \ldots \]

\[ F \left( 2+N+3:3+N \right) = 2 \cdot F \left( 2+N+2:3+N-1 \right) + F \left( 2+N+1:3+N-2 \right) + x_t(2:N-1); \]

\[ y(3+N) = F \left( 3+N \right); \]

\[ y = y'; \]
Chapter 4
Singularities in the 2-dimensional radially symmetric equivariant harmonic map heat flow

4.1 Introduction and some preliminaries
In this section we give a brief introduction to the harmonic map heat flow and the 2-dimensional radially symmetric version. For further references on the harmonic map heat flow we refer to part 1 of [103], [42] and [93]. In the final part of this Section we state the main Theorem of this Chapter.

4.1.1 The harmonic map heat flow
Consider two complete Riemannian manifolds $M$ and $M'$, with $g$ and $g'$ their metrics, respectively. Assume both $M$ and $M'$ to be compact. Let $f : M \to M'$ be a map between the two spaces. We call $\varepsilon(f)$ the energy density of $f$ defined by

$$
\varepsilon(f) := \frac{1}{2} \sum_{i,j} g^{ij} g'_{\alpha \beta} f^\alpha_i f^\beta_j
$$

where

$$
f^\alpha_i = \frac{\partial f^\alpha}{\partial x^i},
$$

with local coordinates in $M$ and $M'$ given by $x^i$ and $f^\alpha$, respectively. We define the total energy of the map $f$ as

$$
E[f] := \int_M \varepsilon(f) \, d\mu_M,
$$

with $d\mu_M$ the volume element of $M$.

Consider the differential $df$ from the tangent space $TM$ to the tangent space $TM'$. That is,

$$
df_i(\partial \cdot) = f^\alpha_i \partial \alpha \in T_{f(x)} M', \quad \text{for} \quad \partial \alpha \in T_x M.
$$

The norm squared of this differential is given by

$$
|df|^2 = \sum_{i,\alpha} g^{ij} g'_{\alpha \beta} f^\alpha_i f^\beta_j,
$$

and equals two times the energy density (4.1). Let $\partial \alpha$ be an orthonormal base for $TM$ and let $\partial \alpha$ be an orthonormal base for $TM'$. Then

$$
|df|^2 = \sum_{i,\alpha} (f^\alpha_i)^2.
$$

Hence, the energy density measures the sum of the squares of the expansion of the vector fields from $TM$ to $TM'$ under the map $f$, in the different directions. The total energy (4.3) of $f$ represents the so-called tension when bending the space $M$ onto the space $M'$. 

Consider a family of maps $f_t : M \to M'$ parameterized by $t$. If $M$ has a boundary we consider a family of maps $f_t$ such that the restriction on the boundary stays the same for all $t$. Hence, $\partial_t f_t = 0$ on the boundary of $M$. Then

$$\partial_t E[f_t] = \frac{1}{2} \int_M \partial_t \left( g^{ij} \partial_i f_t \partial_j f_t \right) d\mu_M = \frac{1}{2} \int_M g^{ij} \left( 2 f_t^i \partial_i f_t + f_t^i \partial_i g_{\alpha \beta} \frac{\partial g^{\alpha \beta}}{\partial f^j} \right) d\mu_M. \quad (4.7)$$

The first term in the last line of equation (4.7) can be rewritten, using integration by parts and equation (2.15), as

$$\int_M g^{ij} f_t^i \partial_j f_t \partial g_{\alpha \beta} d\mu_M = - \int_M \partial_t \left( g^{ij} f_t^i \partial_j f_t g_{\alpha \beta} + g^{ij} \partial_i f_t^i \partial_j f_t g_{\alpha \beta} \right. \left. + g^{ij} f_t^i \partial_j g_{\alpha \beta} \frac{\partial f_t}{\partial f^j} \right) d\mu_M$$

$$= - \int_M \left( g^{ij} \partial_t f_t^i \partial_j f_t g_{\alpha \beta} - g^{ij} \Gamma^\nu_{\alpha \beta} f_t^i \partial_t f_t^\nu \right) d\mu_M$$

$$= - \int_M \left( \Delta_M f_t^\nu + g^{ij} f_t^i \partial_j g_{\alpha \beta} \frac{\partial f_t}{\partial f^j} \right) \partial_t f_t^{\alpha \beta} d\mu_M. \quad (4.8)$$

with $\Delta_M$ the Laplace-Beltrami operator on $M$ (see Subsection 2.2). Hence, the first variation of the total energy of the map $f$ is

$$\partial_t E[f_t] = \frac{1}{2} \int_M \left( -2 \Delta_M f_t^\nu g_{\alpha \beta} + 2 g^{ij} f_t^i \partial_i f_t \frac{\partial g_{\alpha \beta}}{\partial f^j} + g^{ij} f_t^i \partial_i g_{\alpha \beta} \frac{\partial f_t}{\partial f^j} \right) \partial_t f_t^{\alpha \beta} d\mu_M$$

$$= - \int_M \left( \Delta_M f_t^\nu + g^{ij} f_t^i \partial_j g_{\alpha \beta} \frac{\partial f_t}{\partial f^j} \right) \partial_t f_t^{\alpha \beta} d\mu_M. \quad (4.9)$$

Therefore, a critical point of the total energy $E[f]$ is equivalent with a solution to the equation

$$\Delta_M f_t^\nu + g^{ij} f_t^i \partial_j g_{\alpha \beta} \frac{\partial f_t}{\partial f^j} = 0. \quad (4.10)$$

The terms on the left hand side of (4.10) form the $\alpha$ component of the tension field $\tau$. This tension field is defined as

$$\tau(f) := \text{Trace} \nabla df,$$  

where $\nabla$ is the induced connection on the tensor bundle $T^* M \otimes f^{-1} TM'$ (see, for instance, [42] or [93]). The equation for the first variation, (4.9), can then be written as

$$\partial_t E[f_t] = - \int_M \langle \tau(f), \partial_t f \rangle d\mu_M. \quad (4.12)$$

where $\langle \cdot, \cdot \rangle$ is the inner product coming from the metric $g'$ of $M'$. We call $f$ a harmonic map if it is a critical point of the total energy (4.3). Hence, the map $f$ is harmonic if its tension field, $\tau(f)$, equals zero.
4.1. Introduction and some preliminaries

One can rewrite (4.10) in an extrinsic, more illuminating, way. Due to Nash’s embedding theorem we may assume that there exists a map \( \varphi' : M' \to \mathbb{R}^k \), such that the metric \( g' \) is isometric to the Euclidean metric on \( \mathbb{R}^k \). Then,

\[
\Delta_M \left( \varphi' \circ f \right) = \left( \Delta_M f^{\alpha} + g^{ij} f_\alpha f_\beta \Gamma^{\alpha}_{\beta\gamma} \right) \partial_\alpha \varphi' + g^{ij} f_\alpha f_\beta h_{\alpha \beta}, \tag{4.13}
\]

where \( h_{\alpha \beta} \) is the second fundamental form of \( M' \). This second fundamental form \( h_{\alpha \beta} \) is that part of \( \partial_\alpha \partial_\beta \varphi' \) that is orthogonal to \( T_y M' \). Hence, it is the generalization of the second fundamental form defined in Section 2.2, for spaces with codimension greater than 1. Equation (4.10) can now be rewritten to the extrinsic form

\[
\Delta_M \left( \varphi' \circ f \right) - g^{ij} f_\alpha f_\beta h_{\alpha \beta} = 0. \tag{4.14}
\]

Hence, the map \( f \) is harmonic if the tangent part of the Laplace-Beltrami operator of \( M \) on \( \varphi' \circ f \) vanishes.

Harmonic maps appear in many different contexts.

Constant maps and the identity map

For a constant map \( f \) from \( M \) to \( M' \) we have \( df \equiv 0 \) and therefore \( \tau(f) = 0 \). If \( f \) is the identity map then \( df \) is also the identity map and \( \tau(f) \equiv 0 \). Hence, every constant map and the identity map are harmonic.

Geodesics

If \( M = S^1 \), the circle, we have that the Christoffel symbols \( \Gamma^{\alpha}_{\beta\gamma} \) are zero and equation (4.10) turns into

\[
\partial^2 f^{\alpha} + \partial_\alpha f_\beta \partial_\gamma f \Gamma^{\alpha}_{\beta\gamma} = 0, \tag{4.15}
\]

which is the geodesic equation for the curve \( f : S^1 \to M' \). Harmonic maps thus generalize geodesics.

Harmonic functions

If \( M' = \mathbb{R} \) we have that \( \Gamma^{\alpha}_{\beta\gamma} = 0 \). Then equation (4.10) is equivalent to

\[
\Delta f = 0. \tag{4.16}
\]

Hence, the function is harmonic if it is a harmonic map.

Minimal submanifolds

Let \( f : M \to M' \) be an isometric immersion. The second fundamental form \( \nabla df \) of the map \( f \) now equals the ordinary second fundamental form \( h \) of the space \( M \) as a submanifold in \( M' \) (see, for instance, [93]). Hence, the tension field, which is the trace over the second fundamental form of \( f \), is in this case given by \( m \) times the mean curvature of \( M \), with \( m \) the dimension of \( M \). Therefore, if \( M \) is a minimal submanifold (\( H \equiv 0 \)) in \( M' \), with \( f \) the immersion, then \( f \) is harmonic.

See [44], [42] and [93] for more details and other examples.

One of the main questions concerning harmonic maps, is the question of existence, which can be stated as follows. Can every map \( f_0 \) between Riemannian manifolds \( M \) and \( M' \) be deformed into a harmonic map \( f : M \to M' \)? This question has been answered for many specific cases. We refer to [44] and [43] for an overview of results.

To study the existence question for the harmonic maps, Eells and Sampson suggested in [44] to investigate the flow

\[
\partial_t f = \tau(f), \tag{4.17}
\]
which is, by now, called the harmonic map heat flow. This flow is the $L^2$-gradient flow for the total energy of the map $f$, which can immediately be seen from equation (4.12). In [44] the authors proved the small time existence of the flow and proved global existence in the case of $M$ without boundary and non-positive sectional curvature on $M'$. In [61] Hamilton extended this result to the case where $M$ has a boundary.

To prove global existence for more general targets, a study of the possible singularities in the flow is needed. The formation of finite time singularities for the harmonic map heat flow was first shown in [37]. The authors showed the blowup in a spherically symmetric setting from $\mathbb{R}^n$ to $S^n$, such that they could use the comparison principle. The examples that are given in [37] hold for $\dim M = \dim M' \geq 3$. In [33] it is shown why these examples have to blow up in finite time. This is restated in [103] as follows. Let $M$ and $M'$ be compact manifolds, with $\dim M \geq 3$. For any $T$ there exists a constant $\epsilon$ such that the harmonic map heat flow with initial condition $f(r, 0) = f_0$, which is not homotopic to a constant, blows up before time $2T$ if $E[f_0] < \epsilon$.

The result of [33] is specific for higher dimensions and can not be extended to $\dim M = 2$. Even if one explicitly takes an initial map that does not lie in a homotopy class with a harmonic map, the solution to the harmonic map heat flow can be global. In this case the solution blows up in infinite time. An example of such behavior is shown in [31]. After [33] and [31] it was believed that finite time singularities did not exist for dimension 2. In [32], however, the authors show that the harmonic map heat flow can create finite time singularities on surfaces. We discuss the precise statements in Subsection 4.1.3. Both in [31] and in [32] the authors study maps from the disk to the sphere. And, just as in [37], they restrict the maps to a specific spherically symmetric setting such that they are able to use a comparison principle. In this Chapter we study similar maps from the disk to the sphere and give the explicit blowup rates.

In [101] it is proven that there exists a unique global weak solution to the harmonic map heat flow, in the case $M$ has no boundary and $\dim M = 2$, which is regular in all but finitely many points $(x_k, t_k)$. In [30] the existence of such a weak solution is extended to the case where $M$ does have a boundary. The uniqueness of weak solutions on manifolds with boundary is then guaranteed, by [47], as long as the energy is non-increasing along the flow. In [101] the singular points are described as follows. The singularity occurs in a vanishing region where the energy is non-zero and the map in this region can be described as a harmonic map between the sphere $S^2$ and $N$. When such a harmonic map vanishes, one says that a sphere “bubbles off” and this process is called bubbling. We refer to [101], [107] and references therein for the precise statements of bubbling.

In this Chapter we investigate the rate of blowup of the harmonic map heat flow in some specific cases. The rate of blowup is measured as the inverse of the radius of the vanishing bubble. See, for instance, [18] for a discussion on the different definitions of the blowup rate. The fact that, in the case of surfaces, in the limit the blowup profile is harmonic, means that the blowup rate is quasi-stationary. By this we mean that the radius $R$ of the bubble disappears faster than the self-similar rate. That is,

$$R(t) = o(\sqrt{T-t}),$$

in the case of finite time blow up and $R \to 0$, in the case of infinite time blow up. See, for instance [107]. In this Chapter we use the term “blowup rate” for the inverse $R(t)^{-1}$ of the radius, but also for the radius $R(t)$ itself.
4.1. Introduction and some preliminaries

Existence of weak solutions in higher dimensions is proved in [tu]. For more on the harmonic map heat flow in higher dimensions we refer to [rqs] and [rqt].

4.1.2 The 2-d radially symmetric equivariant harmonic map heat flow

Consider a family of spherically symmetric maps $f_t : D^2 \to S^2$ from the disk $D^2$ to the sphere $S^2$ given by

$$f_t : \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} \mapsto \begin{pmatrix} \cos n \theta \sin \phi(r,t) \\ \sin n \theta \sin \phi(r,t) \\ \cos \phi(r,t) \end{pmatrix}, \quad \text{with } n \in \mathbb{N}, \quad (4.19)$$

where the restriction of $f_t$ on the boundary of the disk is independent of time and $\phi(0, t) = 0$. We consider the disk and the sphere here in the extrinsic geometry. Intrinsically the map $F_t$ is given by

$$F_t : (r, \theta) \mapsto (\Theta, \phi) = (n\theta, \phi(r, t)). \quad (4.20)$$

Since $\Theta$ is independent of $r$ and linear in $\theta$, the map $F_t$ is equivariant under rotations of the disk and we call the map $S^1$-equivariant. Define

$$\tilde{\phi} := \phi(r=1, t), \quad \text{for all } t \geq 0. \quad (4.21)$$

While the origin of the disk is mapped onto the north pole of the sphere, the boundary of the disk is mapped $n$ times on the circle on the sphere, with height $\cos \tilde{\phi}$.

The energy of this map is given by

$$E = \pi \int_0^1 \left( \tilde{\phi}_r^2 + \frac{n^2 \sin^2(\phi)}{r^2} \right) r dr, \quad (4.22)$$

and the corresponding harmonic map heat flow is

$$\phi_t = \phi_{rr} + \frac{1}{r} \phi_r - \frac{n^2 \sin(2\phi)}{2r^2}. \quad (4.23)$$

This equation is a generalization of the equation studied in [31] and [32]. In this Chapter we study cases of infinite time blowup for $n \geq 2$, where $n$ can now be a real number.

Standard quasilinear parabolic theory (see, for instance, [83]) implies that the solution to (4.23) stays smooth on $(0,1]$. Hence, the blowup, if any, must occur at $r = 0$. Consider the stationary solutions $f$ of (4.23). These are given by

$$f = m\pi + 2 \arctan(qr^n), \quad \text{with } m \in \mathbb{N} \quad \text{and} \quad q \in \mathbb{R}, \quad (4.24)$$

and $f = (\frac{1}{2} + m)\pi$, with $m \in \mathbb{N}$. The harmonic maps $f = (\frac{1}{2} + m)\pi$, with $m \in \mathbb{N}$, have infinite energy (4.22) and are disregarded. Because of the boundary condition on $r = 0$ we have that $m = 0$ in (4.24), and the stationary solutions under consideration in this Chapter are given by $f = 2 \arctan(qr^n)$, with $q \in \mathbb{R}$.

Consider the function

$$f_3 = 2 \arctan\left(\frac{r^n}{\lambda^2}\right), \quad (4.25)$$
where \( \lambda = \lambda(t) \) is a function dependent on \( t \). If we substitute this into the harmonic map heat flow it makes the right-hand side of equation (4.23) zero. The left-hand side of (4.23) goes to zero if \( \lambda \) goes to zero with a quasi-stationary rate, comparable with the example of Subsection 1.0.3. As \( \lambda \) goes to zero, the function \( f_\lambda \) blows up in \( r = 0 \). This is the only possible blowup scenario of (4.23). The bubbles, discussed in Subsection 4.1.1, are exactly these functions (4.25), where \( f_\lambda \) is the inverse stereographic projection from \( M \) to the sphere \( S^2 \).

Equation (4.23), with \( n \in \mathbb{N} \), also has a physical origin. See, for instance, [68] and [60], where the harmonic map heat flow is used to describe liquid crystals and ferromagnetic spin chains.

4.1.3 Previous work

We give a very short (and incomplete) overview of work done on the 2-dimensional radially symmetric equivariant harmonic map heat flow. In all the cases we discuss, the boundary conditions of the solution \( \phi \) are \( \phi(r = 0, t) = 0 \) and \( \phi := \phi(r = 1, t) \) and the initial condition is denoted by \( \phi_0(r) := \phi(r, t = 0) \).

The case \( n = 1 \)

Consider the harmonic map heat flow (4.23), in the case \( n = 1 \). This is the equation studied in [31] and [32]. In [31] it is shown that if \( |\phi_0| \leq \pi \), then the solution \( \phi \) is global. In fact, if \( |\phi_0| < \pi \), then the solution flows towards a harmonic map given by \( 2 \arctan(\beta r^\alpha) \), with \( \beta = \tan(\phi/2) \). Consider now a solution, which is monotone and has \( \phi = \pi \) and is therefore global. This solution is a family of maps from the disk to the sphere, where the boundary of the disk is mapped to one point on the sphere, namely the south pole. According to [86] all harmonic maps from the disk to the sphere, which map the boundary of the disk to one point, are constant. Hence, the solution has to blow up in infinite time. In [32] it is proved that any solution, with \( \phi > \pi \), blows up in finite time. In both references [31] and [32] the comparison principle is used. In [31] a supersolution that exists for all time and lies above the solution is constructed. In [32] a subsolution that blows up in finite time and lies under the solution is constructed. The comparison principle, used in [31] and [32], follows from the fact that we can use the maximum principle for the equation

\[
 f_t = f_{rr} + \frac{1}{r^2} f_x + p(r, t) f ,
\]

if the function \( p \) can be bounded from above. See [31] and [95] for the precise statements.

In [17] a variation of finite time blowup is proven. The authors give an example of finite time blowup in the case that \( \phi \leq \pi \), but \( \phi_0(r) > \pi \) somewhere on \( r \in (0, 1) \). They further show that the flow can become non-unique if one drops the monotonicity condition of [47] on the energy (see the Subsection 4.1.1). Also in this case, a comparison principle is used. This comparison principle holds for weak sub- and supersolutions, which may jump in the origin. This means that the authors are able to compare solutions even after blowup. A similar result was found in [106].

After one has settled the question whether blowup occurs and whether it occurs in finite or infinite time, the next step is to study the blowup rate and other properties of
the bubbles. In [69] it is shown that in finite time blowup only one bubble vanishes. This prevents the possibility of multiple bubbles bubbling off at the same time. See also the discussions on bubble-trees in [18] and [107] and references therein. As we already discussed in Subsection 4.1.1, any bubble that vanishes in finite time vanishes with a rate $R$ that obeys (4.18). This holds actually for all maps from the disk to a general target manifold.

In [107] it is proven that the blowup rate $R$ of finite time singularities from the disk, is given by

$$R(t) = o\left(\left[\frac{T-t}{\ln(T-t)}\right]^{\frac{1}{2}}\right),$$

(4.27)

for $T$ the blowup time, which is a slight improvement of the upper bound in (4.18).

In [15] an attempt is made to retrieve the precise asymptotics of the blowup rates $R$ of singularity formation in (4.28) for different scenarios. In this reference the authors find different blowup rates, for several different boundary conditions, using formal matched asymptotics. In the finite time blowup case of [32], that is $\phi > \pi$, they find

$$R \sim \kappa \frac{T-t}{\ln(T-t)^{\frac{1}{2}}}, \quad \text{as} \quad t \uparrow T,$$

(4.28)

for $T$ the blowup time and $\kappa$ some non-determined constant. This suggests that the upper bound in (4.27) can be improved. In fact, this has be done in [8], where the authors show that in a relatively general case the finite time blowup must obey

$$R = o(T-t), \quad \text{as} \quad t \uparrow T.$$  

(4.29)

This partly confirms the asymptotics of [15] in the finite time blowup case.

In the case of $\tilde{\phi} = \pi$ the authors of [15] find two blowup scenarios: one finite (as in [17] and [106]) and one infinite (as in [31]). For the infinite time blowup case they find

$$R \sim e^{-2\pi T^{-1/4}}, \quad \text{as} \quad t \to \infty.$$  

(4.30)

In [7] the asymptotics of the blowup rate in this case is proved. Under some mild assumptions on the initial condition $\phi_0$, the authors show that

$$R \sim e^{(-2+\alpha(1))\sqrt{T}}, \quad \text{as} \quad t \to \infty.$$  

(4.31)

This is done by comparing the solution $\phi$ with sub- and supersolutions that blow up with this specific rate $R$.

Besides the cases already mentioned, the authors in [15] also discuss less generic variants of blowup.

**The case $n \geq 2$**

In [15] the authors also use formal matched asymptotics to study the blowup rate of the harmonic map heat flow for all values of $n \geq 2$. In this case they find that all singularities are created in infinite time. For $\tilde{\phi} \in (\pi, 2\pi)$ they find

$$R \sim \kappa e^{-\frac{t}{\ln t}}, \quad \text{for} \quad n = 2 \quad \text{and large} \quad t,$$

$$R \sim \left(\frac{(n-2)\ln t}{E_n}\right)^{\frac{1}{n-2}}, \quad \text{for} \quad n > 2 \quad \text{and large} \quad t,$$

(4.32)
where $n > 0$ is arbitrary, $E_n = \frac{n}{2n - 1}$ and $\alpha_0 = 2 \tan \left( \frac{\pi}{2n} \right)$. If $\phi = \pi$ the case $n = 2$ equals the case $n > 2$ (in fact, $n > 1$) and it is shown in [15] that the blowup rate is
\[ R \sim \left( \frac{4(n - 1)}{E_n} \right)^{\frac{1}{n-1}} , \quad \text{for large } t. \quad (4.33) \]

The formal matched asymptotics of [15] further suggest that the case $n = 2$ is critical, for $\phi \in (\pi, 2\pi)$, in the sense that finite time blowup can only occur for $n < 2$. This is proven in [19], where they left the case $n = 2$ undecided. In [57] it is proved that for $|\phi_0| < 2\pi$ and $n \geq 2$, the solution $\phi$ has to be global in time. In both articles [19] and [57] the authors prove a maximum principle for the harmonic map heat flow in the case $n \geq 2$ and construct a supersolution that does not blow up in finite time. To prove that the solution $\phi$ lies under the supersolution the assumption that $\phi \sim r^q$, for small $r$, is made. This is an assumption we also need and which we discuss in Subsection 4.1.4.

In this Chapter we make the formal results on the blowup rate for $n \geq 2$ rigorous, using ideas proposed in [7]. We first construct subsolutions and supersolutions that blow up with the explicit blowup rates of equations (4.32) and (4.33). We then show that the solution lies in between the subsolution and supersolution and that the blowup rate $R_n$ of the solution $\phi$ is of the same order as $R$. The precise statement and the definition of $R_n$ are given in Subsection 4.1.4.

To conclude this Subsection we mention [55], [56] and [59], where blowup behavior in a $S^1$-equivariant setting from the ball and the plane to the sphere is discussed.

### 4.1.4 Some preliminaries and the main Theorem

Let $\phi$ be a solution of the harmonic map heat flow (4.23) with $n \geq 2$ and $\bar{\phi} \in (\pi, 2\pi)$. In [15] the blowup rates in this case where given using formal matched asymptotics. In this Chapter we prove the asymptotics of these blowup rates rigorously.

The fact that the solutions blow up in infinite time is already proven in [57] and [19], where the authors had to make the assumption that the solution is order $r^n$ as $r \to 0$. We have to make the same assumption. This assumption is not too strict as it can be shown that in the case of integer $n$ this is just a smoothness condition for the solution (4.19) from the disk to the sphere.

As in most results on the blowup behavior of (4.23) we make use of the comparison principle. Following the ideas of [7] we construct sub- and supersolution with the specific blowup rates (4.32). To make sure that the subsolution actually lies under the solution and the supersolution lies above it, we use the invariance of the flow under self-similarity transformations. By this we mean that if $\phi(r, t)$ is a solution to the harmonic map heat flow (4.23), then the rescaled function
\[ \phi(\lambda r, t_0 + \lambda^2 t), \quad \text{with } \lambda, t_0 \in \mathbb{R}, \quad (4.34) \]
is also a solution. Namely, let $y := \lambda r, \tau := t_0 + \lambda^2 t$ and $\hat{\phi}(y, \tau) = \phi(\lambda r, t_0 + \lambda^2 t)$. Then
\[ \partial_y \hat{\phi} - \partial_y^2 \hat{\phi} - \frac{1}{y} \partial_y \hat{\phi} + \frac{\lambda^2}{2y^2} \sin(2\hat{\phi}) = \lambda^{-2} \left( \partial_r \phi - \partial_r^2 \phi - \frac{1}{r} \partial_r \phi + \frac{\alpha^2}{2r^2} \sin(2\phi) \right) = 0. \quad (4.35) \]

By the same arguments we have that $\phi(\lambda r, t_0 + \lambda^2 t)$, with $\lambda \in \mathbb{R}$, is a sub- or supersolution if and only if $\phi(r, t)$ is a sub- or supersolution, respectively.
4.1. Introduction and some preliminaries

The blowup scale of the problem is defined in the following way (see also [18]). Consider a solution of the harmonic map heat flow \((4.23)\) with \(n \geq 2\) and \(\tilde{\phi} \in (\pi, 2\pi)\). We define the radius \(R_\phi(t)\) of a solution such that

\[
\phi(R_\phi(t), t) = \frac{\pi}{2}.
\]

(4.36)

If this radius is unique, it is well defined, and we consider this radius as the inverse blowup rate of our problem. Using Sturmian theory one can show that if \(R_\phi_0\) is unique for the initial condition \(\phi_0\), the radius \(R_\phi\) stays unique for the solution (see [7]).

We prove the blowup rates \((4.32)\) for a family of solutions obeying some technical assumptions on the initial conditions. Consider a smooth function \(\phi_0\) such that

\[
\begin{align*}
&i. \quad C_1 \leq \lim_{r \to 0} \frac{\phi_0}{r} \leq C_2, \quad \text{for some } C_1, C_2 \in \mathbb{R}, \\
&ii. \quad R_{\phi_0} \big|_{t=0} \text{ is unique}, \\
&iii. \quad |\phi_0| < 2\pi,
\end{align*}
\]

(4.37)

We are now able to give the main Theorem of this Chapter.

**Theorem 4.1.1.** Let \(\phi\) be the solution of the harmonic map heat flow \((4.23)\) with \(n \geq 2\), \(\phi(0, t) = 0\) and \(\phi(1, t) = \tilde{\phi} \in (\pi, 2\pi)\) and with an initial condition obeying \((4.37)\). Denote the unique radius given by \((4.36)\) as \(R_\phi\). Then \(R_\phi \to 0\) as \(t \to \infty\) and the asymptotics of \(R_\phi\) are given by

\[
\begin{align*}
\ln(R_\phi) &\sim -\frac{8}{\pi} \phi_0 t, \quad \text{for } n = 2 \text{ and large } t, \\
R_\phi &\sim \left(\frac{(n-2)\phi_0}{E_n}\right)^{1/n}, \quad \text{for } n > 2 \text{ and large } t,
\end{align*}
\]

(4.38)

with \(E_n = \frac{\pi}{2\pi - \sin(\pi/n)}\) and \(\phi_0 = 2\tan\left(\frac{\phi_0}{2}\right)\).

This is proven by constructing subsolutions and supersolutions that blow up with rates given by \((4.32)\). Using the invariance of the flow under self-similar transformations we are able to put any solution in between these subsolutions and supersolutions. After relating the blowup rates of the sub- and supersolution with their blowup radius we are able to draw the conclusion using the comparison principle.

**Remark 4.1.2.** In a similar manner one can prove that in the case of \(\tilde{\phi} = \pi\), the asymptotic behavior of the blowup rate \(R_\phi\) is given by \((4.33)\). This is shown at the end of Subsection 4.3.3

**Remark 4.1.3.** For the theorem to hold its crucial that \(\tilde{\phi} < 2\pi\). Namely, if \(\tilde{\phi} \geq 2\pi\), multiple blowup occurs and the blowup rates differ from \((4.32)\) and \((4.33)\), see [15]. The assumptions \((4.37)\) seem to be technical and not necessary for the result to hold. For more on this we refer to Section 1.4.
4.2 Formal solution of the harmonic map heat flow for $n \geq 2$

In this Section we give a part of the formal solution of the 2-dimensional radially symmetric equivariant harmonic map heat flow (4.23) in the case that $n \geq 2$ and the boundary conditions are such that a singularity occurs. We refer to [15] for a full account (including this case) of formal asymptotics for the 2-dimensional radially symmetric equivariant harmonic map heat flow.

Consider the harmonic map heat flow

$$\phi_t = \phi_{rr} + \frac{1}{r} \phi_r - \frac{n^2}{2y^2} \sin(2\phi),$$

(4.39)

with $n \geq 2$ and boundary conditions

$$\phi(r = 0, t) = 0, \quad \phi(r = 1, t) = \bar{\phi}, \quad \text{with} \quad \pi < \bar{\phi} < 2\pi.$$

(4.40)

In this Section we construct formal solutions near blowup. At blowup, the solution is stationary. Since the stationary solutions are given by

$$\phi_0 = m\pi + 2\arctan(qy^n), \quad \text{with} \quad m \in \mathbb{Z}, q \in \mathbb{R},$$

(4.41)

they do not obey both boundary conditions. Hence, we need a boundary layer to solve the problem. From the literature (see Subsection 4.1.3) it is known that the vanishing region is concentrated at $r = 0$. Hence, the boundary layer, given by the inner solution, lies at $r = 0$.

Consider the inner solution. Let $\phi(r, t) = F(y, t)$ with $y = \frac{r}{R}$ and $R(t)$ the scaling of the problem with $R(t) \to 0$ as $t \to T$, with $T$ the blowup time (possibly infinite). Define $R(t)$ such that it obeys

$$R^n \lim_{r \to 0^+} \frac{\phi(r, t)}{r^n} = 2,$$

(4.42)

for all $t$. In this notation the rescaled equation of (4.39) is given by

$$R^2 F_t - RR_y F_y = F_{yy} + \frac{1}{y} F_y - \frac{n^2}{2y} \sin(2F).$$

(4.43)

Since the scaling is quasi-stationary (blowup limit is stationary) we have that $R^2 \to 0$ and $RR \to 0$, as $t \to T$. Express the inner solution as the formal series

$$F(y, t) = f_0(y) + RR f_1(y) + O\left((RR)^2\right).$$

(4.44)

Plugging this into the rescaled harmonic map heat flow (4.43) gives the equations

$$f_{0yy} + \frac{1}{y} f_{0y} - \frac{n^2}{2y^2} \sin(2f_0) = 0,$$

(4.45)

and

$$f_{1yy} + \frac{1}{y} f_{1y} - \frac{n^2}{2y^2} \cos(2f_0) f_1 = -y f_{0y}.$$

(4.46)
4.2. Formal solution of the HMHF for $n \geq 2$

This gives

$$f_0 = m\pi + 2 \arctan(qy^n), \quad \text{with } m \in \mathbb{Z}, q \in \mathbb{R},$$

and, using variation of parameters,

$$f_1(y) = -\frac{1 + 4ny^n \ln(y) + y^{2n}}{y^n(1 + y^{2n})} \int_0^{\zeta} \frac{\zeta^{2n+1}}{(1 + \zeta^{2n})^2} \, d\zeta$$

$$+ \frac{y^n}{1 + y^{2n}} \int_0^{\zeta} \frac{1 + 4n\zeta^n \ln(\zeta) + \zeta^{2n}}{(1 + \zeta^{2n})^2} \, d\zeta$$

$$+ a_1 \frac{y^n}{1 + y^{2n}} + b_1 \frac{1 + 4ny^n \ln(y) + y^{2n}}{y^n(1 + y^{2n})}$$

(4.48)

Because of the boundary conditions $f_0(y) = f_1(y) = 0$ at $y = 0$, we take $m = b_1 = 0$. The definition of $R$ gives, when approximating $\phi$ with $F$ for small $r$ that $q = 1$ while the constant $a_1$ is yet to be determined.

The asymptotic behavior of $f_1$, at $y \to \infty$, is governed by $-E_n y^n$, with

$$E_n := \int_0^{\infty} \frac{\zeta^{2n+1}}{(1 + \zeta^{2n})^2} \, d\zeta = \frac{\pi}{2n^2 \sin(\frac{\pi}{n})}$$

(4.49)

For convenience we rewrite $f_1$ as

$$f_1 = -E_n y^n + I + (a_1 + E_n) \frac{y^n}{1 + y^{2n}},$$

(4.50)

with

$$I := \frac{y^n}{1 + y^{2n}} \int_y^{\infty} \frac{\zeta^{2n+1}}{(1 + \zeta^{2n})^2} \, d\zeta - \frac{1 + 4ny^n \ln(y)}{y^n(1 + y^{2n})} \int_0^y \frac{\zeta^{2n+1}}{(1 + \zeta^{2n})^2} \, d\zeta$$

$$+ \frac{y^n}{1 + y^{2n}} \int_0^y \frac{1 + 4n\zeta^n \ln(\zeta) + \zeta^{2n}}{(1 + \zeta^{2n})^2} \, d\zeta$$

(4.51)

where the asymptotic behavior of $I$ is given by

$$I \sim -\frac{n}{2n + 2} y^{n+2}, \quad \text{for small } y,$$

(4.52)

and

$$I \sim \frac{n}{2n - 2} y^{-n+2}, \quad \text{for large } y.$$

(4.53)

We are now able to determine the constant $a_1$ in $f_1$ by approximating $\phi$ with the inner solution $F$ in equation (4.42). The inner solution (4.44) is given by

$$F(y, t) = 2 \arctan(y^n) + RRf_1 + \cdots, \quad \text{for large } t.$$

Therefore, using (4.50) and (4.52),

$$R^n \lim_{r \to 0} \frac{\phi(r, t)}{r^n} \sim R^n \lim_{r \to 0} \frac{F(y, t)}{r^n}$$

$$= R^n \lim_{r \to 0} \left[ \frac{1}{r^n} \left( 2y^n + RR \left[ a_1 y^n - \frac{n}{2n + 2} y^{n+2} + \cdots \right] \right) \right]$$

$$= 2 + RRa_1 + \cdots,$$

(4.55)
4. Singularities in the 2-d radially symmetric equivariant HMFH

for large \( t \). Hence, \( a_1 = 0 \) and equation (4.50) simplifies to \( f_1 = -E_n y^n + I + E_n \frac{n}{2n-2} \).

The inner solution in the outer scale, denoted by \( \overline{r} \), is given by \( F \) for large \( y \). Hence,

\[
\overline{r}(y, t) \sim \pi - 2y^n + R R \left( -E_n y^n + \frac{n}{2n-2} y^{-n+2} \right).
\] (4.56)

The inner solution is the approximation of the solution in the vanishing region near \( r = 0 \). To get a global approximation of the solution we also need the outer solution. Matching the inner with the outer solution gives a relation for the blowup rate \( R \).

The first two terms in the expansion of the outer solution \( G \) are given by the stationary solution \( G \) for large \( y \), hence

\[
G(r, t) \sim \pi + 2 \arctan \left( \frac{1}{2} \alpha_0 y^n \right), \quad \text{with} \quad \alpha_0 := 2 \tan \left( \frac{\phi - \pi}{2} \right).
\] (4.57)

Since, the inner scale of the outer solution (4.57) is given by small \( r \), and therefore

\[
G(r, t) \sim \pi + \alpha_0 R^n + \cdots = \pi + \alpha_0 R^n y^n + \cdots,
\] (4.58)

we find, by matching the \( y^n \) terms of (4.56) and (4.58),

\[
-R R E_n = \alpha_0 R^n.
\] (4.59)

Hence, we find infinite blowup time and for \( t \to \infty \) we find,

\[
R(t) = \kappa e^{-\frac{\pi}{2} R_n}, \quad \text{with} \quad \kappa \in \mathbb{R}, \quad \text{for} \quad n = 2,
\]

\[
R(t) = \left( \frac{(n-2) \alpha_0}{E_n} \right)^{\frac{1}{n}}, \quad \text{for} \quad n > 2
\] (4.60)

These are the blowup rates found in [15]. In the remainder of this Chapter we use the solutions (4.60) of the differential equation (4.59) as the definition of \( R \). In the case of \( n = 2 \), we choose the solution with \( \kappa = 1 \).

In matched asymptotics, a global approximation of the solution is usually obtained by adding the inner and outer solution and subtracting their common parts in the matching region. We construct a global solution in a similar way. As the scale of \( R \) is determined by matching the first two terms of the inner solution (4.44) with the zeroth order terms (in \( R \)) of the outer solution (4.57) we only use these terms to construct a global approximation. Adding these terms and subtracting the first two terms of the outer solution in the inner scale (4.58), gives

\[
\overline{r} = 2 \arctan(y^n) - \frac{\alpha_0}{E_n} R^n f_1 + \pi + 2 \arctan \left( \frac{1}{2} \alpha_0 y^n \right)
\]

\[
- \pi - \alpha_0 R^n y^n
\]

\[
= f_0 + g_0 - \frac{\alpha_0}{E_n} R^n (f_1 + E_0 y^n),
\] (4.61)

with

\[
g_0 := 2 \arctan(s^n) \quad \text{and} \quad s = \left( \frac{\alpha_0}{2} \right) \frac{1}{r}.
\] (4.62)

In the next Section we modify this global approximation \( \overline{r} \), by adding a yet to be determined, function \( \epsilon \). By modifying this global approximation we are able to build sub- and supersolution to the harmonic map heat flow.
4.3 Sub- and supersolutions for the HMHF with \( n \geq 2 \)

**Remark 4.2.1.** In the case of \( \hat{\phi} = \pi \) one can make a similar analysis. Due to the boundary condition, the outer solution is just a perturbation of \( \pi \). A global approximation can then be given by

\[
\Phi = 2\arctan(y^n) - \frac{2}{E_n} R^{2n} f_i,
\]

with

\[
R \sim \left( \frac{4(n-1)}{E_n} t \right)^{\frac{1}{2n}}.
\]

See [15] for more on these asymptotics.

### 4.3 Sub- and supersolutions for the harmonic map heat flow with \( n \geq 2 \)

In this Section we construct sub- and supersolutions, \( \phi_{\text{sub}} \) and \( \phi_{\text{sup}} \), for the harmonic map heat flow (4.23) in the case \( \pi < \hat{\phi} < 2\pi \), which blow up in infinite time with a blowup rate given by equation (4.59). We show that the supersolution \( \phi_{\text{sup}} \) lies above the solution of the harmonic map heat flow if the initial condition obeys the assumptions (4.37), given in Subsection 4.1.4. This proves that the solution cannot create a singularity in finite time. This result has already been proved in [19] for \( n > 2 \) and in [57] for the case \( n \geq 2 \). Our result improves the estimate for the possible blowup rate. We further show that the subsolution \( \phi_{\text{sub}} \) lies under the solution. Hence, we can conclude that the solution blows up in infinite time with the rate given by equation (4.59). At the end of this Section we show how to prove that the blowup rate for the case \( \hat{\phi} = \pi \) is given by (4.33).

Assume that the function \( R(t) \) obeys the differential equation (4.59). Define, as in Section 4.2, \( y = \hat{y} \hat{f} \), for \( r \in [0, 1] \). Consider the function

\[
\Phi = f_0 + g_0 - \frac{\alpha_0}{E_n} R^n \left( f_1 + E_n y^n \right) + \epsilon(y, t),
\]

with \( f_0, g_0 \) and \( f_1 \) as in Section 4.2 and where we assume that the unknown function \( \epsilon \) is small compared to \( f_0 \) and \( g_0 \). This function is the global approximation of the solution, given in (4.61), with a small perturbation \( \epsilon(y, t) \). We choose the function \( \epsilon \) in such a way that it makes \( \Phi \) a super- or subsolution.

For convenience, we define

\[
\delta(y, t) := \frac{\alpha_0}{E_n} R^n \left( f_1 + E_n y^n \right),
\]

and write (4.65) as

\[
\Phi = f_0 + g_0 + \delta + \epsilon.
\]

Since both \( \delta \) and \( \epsilon \) are smaller than \( f_0 \) and \( g_0 \), on \( (0, 1] \), we can write

\[
\sin(2\Phi) = \sin(2f_0 + 2g_0) + 2 \cos(2f_0 + 2g_0)(\delta + \epsilon) + h^2[\Phi],
\]

where \( h[\Phi] \) is the second-order term of the approximation. This completes the proof of the existence of sub- and supersolutions.
with $h^{(2)}[\Phi]$ the remainder term. Then,
\[
\partial_t^2 \Phi + \frac{1}{r} \partial_r \Phi - \frac{n^2}{2r^2} \sin(2\Phi)
= \partial_t^2 (f_0 + \delta_0) + \frac{1}{r} \partial_r (f_0 + \delta_0) - \frac{n^2}{2r^2} \sin(2f_0 + 2\delta_0)
- \frac{n^2}{r^2} \cos(2f_0 + 2\delta_0)(\delta + \epsilon) - \frac{n^2}{2r^2} h^{(3)}[\Phi] + \partial_t^2 (\delta + \epsilon) + \frac{1}{r} \partial_r (\delta + \epsilon)
= \partial_t^2 \epsilon + \frac{1}{r} \partial_r \epsilon - \frac{n^2}{r^2} \cos(2f_0 + 2\delta_0) + T_1 + 2n \frac{\alpha_0}{E_n} R^{n-2} \frac{y^\alpha}{1 + y^\alpha} - \frac{n^2}{2r^2} h^{(2)}[\Phi],
\]
(4.69)
with
\[
T_1 := \frac{n^2}{2r^2} \left[ \sin(2f_0) + \sin(2\delta_0) - \sin(2f_0 + 2\delta_0) \right] - \frac{n^2}{r^2} \cos(2f_0 + 2\delta_0)\delta
+ \frac{n^2}{r^2} \cos(2f_0) \left( \frac{\alpha_0}{E_n} f_1 - \alpha_0 \frac{n^2}{r^2} R^n y^\alpha. \right)
(4.70)
\]

The time derivative of $\Phi$ is given by
\[
\partial_t \Phi = nR^{n-1} \frac{\alpha_0}{E_n} \left( f_1 + E_n y^\alpha \right) - \frac{R}{y} y^\alpha \left[ f_0 - \frac{\alpha_0}{E_n} R^n \left( f_1 + E_n y^\alpha \right) \right] + \partial_r \epsilon.
(4.71)
\]
with
\[
T_2 := \frac{\alpha_0}{E_n} R^{2n-2} \left[ n \left( f_1 + E_n y^\alpha \right) - y^\alpha f_1 - f_1 + E_n y^\alpha \right].
(4.72)
\]
Equating (4.69) with (4.71) we see that the terms of order $R^{n-2}$ cancel and that $\Phi$ is a supersolution if
\[
M[\Phi] := \partial_t \epsilon - \partial_t^2 \epsilon - \frac{1}{r} \partial_r \epsilon + \frac{n^2}{r^2} \cos(2f_0 + 2\delta_0) + T_2 - T_1 + \frac{n^2}{2r^2} h^{(2)}[\Phi] \geq 0.
(4.73)
\]
and $\Phi$ is a subsolution if
\[
M[\Phi] \leq 0.
(4.74)
\]
Hence, to construct a supersolution we need to find an $\epsilon$ such that equation (4.73) holds. And to construct a subsolution we need to find an $\epsilon$ such that (4.74) holds. To do this we need to know the behavior of $T_1$ and $T_2$, which are independent of the choice of $\epsilon$, and we need some control on $h^{(2)}[\Phi]$, which does depend on $\epsilon$.

### 4.3.1 Estimation of $T_1$ and $T_2$

In this subsection we estimate $T_1$ and $T_2$.

Consider $T_1$, defined by (4.70), rewritten as
\[
T_1 = \frac{n^2}{r^2} \left[ \left( \delta + \frac{1}{2} \sin(2f_0) \right) \left( 1 - \cos(2\delta_0) \right) + \left( -\alpha_0 R^n y^\alpha + \frac{1}{2} \sin(2\delta_0) \right) \left( 1 - \cos(2f_0) \right) - S8 \right],
(4.75)
\]
4.3. Sub- and supersolutions for the HMHF with \( n \geq s \geq 10^9 \)

with

\[
S := \left(1 - \cos(2f_0)\right)\left(1 - \cos(2g_0)\right) - \sin(2f_0) \sin(2g_0),
\]

which is necessarily small since \( y = \frac{r}{\varepsilon} \) and \( s \) are of different order (see (4.62)).

Since \( I \) is bounded and its asymptotic behavior is given by (4.52) and (4.53) we may conclude that

\[
|f_1 + E_n y^n| = \left| I + E_n \frac{y^n}{1 + y^{2n}} \right| \leq M y^n + y^{n+2} \frac{1}{1 + y^{2n}}.
\]

for a certain non-specified positive constant \( M \) which, from now on, can differ from line to line.

Rewriting the sine and cosine of \( 2f_0 \) and \( 2g_0 \) in terms of rational functions gives the estimate

\[
|S| \leq R^n \frac{y^{2n}}{1 + y^{2n}}.
\]

This gives, where \( s \) is given by (4.62),

\[
|T_1| \leq \frac{n^2}{r^2} \left( M R^n y^n + y^{n+2} + 2 \frac{y^n}{1 + y^{2n}} \right) \frac{y}{(1 + y^{2n})}
\]

\[
+ 6s^n + 2s^n \frac{y^{2n}}{(1 + y^{2n})^2} + M R^n y^n + y^{n+2} \frac{1}{1 + y^{2n}} R^n \frac{y^{2n}}{1 + y^{2n}}
\]

\[
\leq R^n M \frac{y^{2n} + y^{2n+2}}{1 + y^{2n}} + s^n + \frac{y^{2n}}{1 + y^{2n}} + \frac{y^{2n}}{(1 + y^{2n})^2} s^n + \frac{y^{2n} + y^{2n+2}}{(1 + y^{2n})^2}
\]

\[
\leq R^{2n-2} M y^{2n-2} + y^{2n-2}
\]

for all \( n \geq 2, \) as \( r \) and \( s \) are bounded.

To get an estimate for \( T_2 \) we need a bound on \( y \partial_s (f_1 + E_n y^n) \). Direct differentiation gives a bounded expression with the same integrals as in \( I \) and

\[
|y \partial_s (f_1 + E_n y^n)| \leq M \frac{y^n + y^{n+2}}{1 + y^{2n}}
\]

Hence,

\[
|T_2| \leq M R^{2n-2} \frac{y^n + y^{n+2}}{1 + y^{2n}}.
\]

4.3.2 Choice of \( \varepsilon \)

In this subsection we give, and justify, our choice for \( \varepsilon \). Consider the function \( \Phi \) given by (4.65). To make \( \Phi \) a super- or subsolution of the harmonic map heat flow we need to find an \( \varepsilon \) such that \( \Phi \) obeys inequality (4.73) or (4.74). Define

\[
N[\varepsilon] := \partial_\varepsilon^2 \varepsilon + \frac{1}{r^2} \partial_\varepsilon \varepsilon - \frac{n^2}{r^2} \cos(2f_0 + 2g_0) \varepsilon.
\]

Then,

\[
M[\Phi] = \partial_\varepsilon \varepsilon - N[\varepsilon] + T_2 - T_1 + \frac{n^2}{2v^2} \beta^{(2)}[\Phi].
\]
We choose $\epsilon$ such that
\[ |N[\epsilon]| \geq |T_1| + |T_2|. \] (4.84)
If for this $\epsilon$ also holds that
\[ |N[\epsilon]| \geq \frac{n^2}{2\pi^2} h^{(2)}[\Phi], \] (4.85)
then $N[\epsilon]$ determines, together with $\partial \epsilon$, the sign of $M[\Phi]$.

Consider $N[\epsilon]$. If $\epsilon$ equals $y^n$, for any integer $m \in \mathbb{N}$, we find
\[ N[y^n] = R^{-\epsilon} \left[ (m^2 - u^2) + n^2 \left( 1 - \cos(2f_0 + 2\varphi) \right) \right] y^{n-2}. \] (4.86)
where the term, with $1 - \cos(2f_0 + 2\varphi)$ is positive. We choose $\epsilon = \kappa_1 R^{2n} y^n$, with non-zero $\kappa_1 \in \mathbb{R}$ to be fixed later. The idea of constructing the sub- and supersolutions is to find an integer $m$ such that (4.84) holds. Hence, we want $\epsilon$ to obey (see Subsection 4.3.1).

To determine the sign of $M[\Phi]$ we still need to estimate the terms $\partial \epsilon$ and $\frac{n^2}{2\pi^2} h^{(2)}[\Phi]$. Consider $h^{(2)}[\Phi]$. This remainder term of the expansion of $\sin(2v + 2w)$, with $w \ll v$, can be written as
\[ h^{(2)}[2v + 2w] := \sin(2v + 2w) - \sin(2v) - \cos(2v)2w 
= 4w^3 \int_0^1 (\tau - 1) \sin(2v + 2\tau w) d\tau 
= \sin(2v + 2\theta u) 4w^3 \int_0^1 (\tau - 1) d\tau 
= -2 \sin(2v + 2\theta u) w^2, \] (4.89)
for certain $\theta \in (0, 1)$, by the Mean Value Theorem. Hence, in the notation of (4.67),
\[ \left| h^{(2)}[\Phi] \right| \leq 2(\delta + \epsilon)^3. \] (4.90)

Consider the square of $\epsilon$. For $t$ large enough, such that $\kappa_2^2 R(t) \leq 1$, we have
\[ \epsilon^2 = \kappa_2^2 R^{4n} y^{2n} \leq R^{4n-1} y^{2n} \leq R^{4n-m-1} y^m \leq R^{4n} y^m, \] (4.91)
if $m \leq 2n - 1$. Since, by (4.77),
\[ \delta^2 \leq M R^{2n} y^{2n} + y^{2n+4} \leq M R^{2n} y^m, \] (4.92)
if $-2n + 4 \leq m \leq 2n$, we conclude
\[ \left| \frac{n^2}{2\pi^2} h^{(2)}[\Phi] \right| \leq \frac{n^2}{2\pi^2} (\delta + \epsilon)^3 \leq \frac{2n^2}{\pi^2} (\delta^2 + \epsilon^3) \leq M R^{2n-2} y^{m-2}, \] (4.93)
4.3. Sub- and supersolutions for the HMHF with \( n \geq t \)

If \(-2n + 4 \leq m \leq 2n - 1\) and for \( t \) large enough such that \( \kappa_1^2 R(t) \leq 1 \). As we want this to hold for all \( n \geq 2 \), and we also have \( n < m \leq n + 2 \), we choose \( m = n + 1 \). Hence, \( \epsilon = \kappa_1 R^{2n-2} y^{n-1} \) so that

\[
N[\epsilon] = \kappa_1 R^{2n-2} y^{n-1} \left[ (2n + 1) + n^2 \left( 1 - (2f_0 + 2g_0) \right) \right].
\] (4.94)

The time derivative of \( \epsilon \) is, in this case, given by

\[
\partial_t \epsilon = - \frac{\alpha_0}{E_n} (n-1) \kappa_1 R^{2n-2} y^{n-1},
\] (4.95)

and gives, for \( \kappa_1 > 0 \), a negative contribution to equation (4.83), which determines whether \( \Phi \) is a sub- or supersolution. Let \( \kappa_1 \) be positive. Since \( 1 - (2f_0 + 2g_0) \) is positive, we have

\[
N[\epsilon] \geq \kappa_1 R^{2n-2} y^{n-1},
\] (4.96)

and therefore

\[
M[\Phi] \leq - \frac{\alpha_0}{E_n} 2\kappa_1 R^{2n-2} y^{n-1} - \kappa_1 R^{2n-2} y^{n-1} + T_2 - T_1 + \frac{n^2}{2r^2} h^2[\Phi]
\]

\[
\leq - \kappa_1 R^{2n-2} y^{n-1} + T_2 - T_1 + \frac{n^2}{2r^2} h^2[\Phi],
\] (4.97)

with \( \epsilon = \kappa_1 R^{2n-2} y^{n-1} \). One sees that for \( t \) and \( \kappa_1 > 0 \) large enough \( M[\Phi] \) becomes negative and we have constructed a subsolution. A supersolution is constructed similarly with \( \kappa_1 \) negative. As is shown in Subsection 4.3.3, the subsolution does not lie under the solution with boundary conditions (4.49). Hence, we can not use the Comparison Principle on the subsolution we just created. The same holds for the supersolution, for which it is shown that it does not lie above the solution. In the next Subsection we modify the sub- and supersolutions \( \Phi \) we have just found, such that they do lie under and above the solution, respectively. We are then able to use the Comparison Principle and to prove Theorem 4.1.1

### 4.3.3 The sub- and supersolution

Consider the functions \( \phi_+ \) and \( \phi_- \) given by

\[
\phi_+(t, r) = f_0 + g_0 - \frac{\alpha_0}{E_n} R^n \left( f_1 + E_n y \right) \pm \epsilon.
\] (4.98)

with

\[
\epsilon = \kappa_1 R^{2n-2} y^{n-1}, \quad \text{and} \quad \kappa_1 > 0.
\] (4.99)

Using the estimates of Subsections 4.3.1 and 4.3.2 we have

\[
\partial_\phi \phi_+ - \partial_\phi^2 \phi_+ - \frac{1}{r} \partial_\phi \phi_+ + \frac{n^2}{2r^2} \sin(2\phi_+) = \partial_\phi \epsilon - N[\epsilon] + T_2 - T_1 - h^{[2]}(\phi_+)
\]

\[
\leq - \frac{\alpha_0}{E_n} 2\kappa_1 R^{2n-2} y^{n-1} - \kappa_1 R^{2n-2} y^{n-1} + M R^{2n-2} y^{n-1},
\] (4.100)

\[
\leq (M - \kappa_1) R^{2n-2} y^{n-1}.
\]
for $t$ large enough such that $\kappa^2 I(t) \leq 1$. Similarly we find
\[
\partial_t \phi_- - \frac{\partial^2 \phi_-}{\partial r^2} - \frac{1}{r} \partial_r \phi_- + \frac{n}{2r^2} \sin(2\phi_-) \geq (-M + \kappa_1) R^{n-2} y^{n-1},
\]
for $t$ large enough such that $\kappa^2 I(t) \leq 1$. Hence, choosing $\kappa_1 > M$ and taking $t$ large enough we have that $\phi_+$ is a subsolution and that $\phi_-$ is a supersolution.

Consider $\phi_+(r, t)\big|_{t=1}$. Then, using (4.53),
\[
\phi_+(r, t)\big|_{t=1} = 2 \arctan\left( \frac{1}{R} \right) + 2 \arctan\left( \frac{\alpha_0}{\sqrt{2}} \right) - \frac{\alpha_0}{\sqrt{2}} R^n \left( f_1 + E_n y^n \right) \big|_{t=1} \pm \kappa_1 R^{n-1} \frac{1}{R^{n+1}}
\]
\[
= \pi + O(R^n) + \tilde{\phi} - \pi - \frac{\alpha_0}{\sqrt{2}} R^n O(R^{n-2}) \pm \kappa_1 R^{n-1}
\]
\[
= \tilde{\phi} \pm \kappa_1 R^{n-1} + O(R^n).
\]

Hence, the supersolution $\phi_+$ does not lie above the solution and the subsolution $\phi_-$ does not lie under the solution. In this section we use the fact that the harmonic map heat flow is invariant under similarity transformations to build sub- and supersolutions that lie respectively under and above the solution.

By a similar calculation as in (4.102) we show that
\[
\partial_r \phi_+(r, t)\big|_{t=1} = 2n \frac{\alpha_0}{1 + \frac{n}{2}} \pm \kappa_1 (n+1) R^{n-1} + O(R^n).
\]

Hence, for a function $f(t) \ll 1$,
\[
\phi_+(r, t)\big|_{t=1+f} = \phi_+(r, t)\big|_{t=1} + \partial_r \phi_+(r, t) f(t)\big|_{t=1} + \cdots
\]
\[
= \tilde{\phi} \pm \kappa_1 R^{n-1} + 2n \frac{\alpha_0}{1 + \frac{n}{2}} f(t) + O(R^n, R^{n-1} f)
\]
\[
= \tilde{\phi}.
\]

if
\[
f(t) = \frac{n+1}{n} \frac{\alpha_0}{\sqrt{2}} R^{n-1} + O(R^n).
\]

Inspection of $\partial_r \phi_+$ gives, using (4.80),
\[
\partial_r \phi_+ = \partial_r \left[ 2 \arctan(y^n) + 2 \arctan(s^n) - \frac{\alpha_0}{\sqrt{2}} R^n \left( f_1 + E_n y^n \right) \pm \kappa_1 R^{n-1} y^{n-1} \right]
\]
\[
\geq \left( \frac{2n y^{n-1}}{R^{n+1} + g^{2n}} + \frac{\alpha_0}{\sqrt{2}} \right) \left( 2n s^{n-1} + 2n y^{n-1} + y^{n+1} \right) - \kappa_1 R^{n-1} (n+1) r^n
\]
\[
> 0,
\]
on $r \in (0, \infty)$, for $t$ big enough. Let us define $r_+(t)$ as the unique value such that $\phi_+(r_+(t), t) = \tilde{\phi}$. Hence,
\[
r_+(t) = 1 \mp \frac{\alpha_0}{n} \frac{1 + \frac{n}{2}}{\alpha_0} R^{n-1} + O(R^n).
\]
4.3. Sub- and supersolutions for the HMHF with \( n \geq 2 \)

We see that \( r_+ \to 1 \) as \( t \to \infty \).

In the upcoming Lemmas we construct a subsolution that lies under the solution \( \phi \) and a supersolution that lies above the solution \( \phi \). This is done using the invariance of the flow under self-similarity transformations. In the proofs of the Lemmas we use repeatedly the identities

\[
\arctan \left( \frac{1}{a} \right) = \frac{\pi}{2} - \arctan(a), \quad \text{for all } a > 0,
\]

and

\[
\arctan(a) - \arctan(b) = \arctan \left( \frac{a - b}{1 + ab} \right), \quad \text{for all } a, b > 0.
\]

We further use the inequality

\[
a - b \leq \frac{\arctan(a)}{\arctan(b)} \quad \text{for all } a, b \text{ such that } 0 \leq a < b \quad \text{and} \quad b > 0.
\]

In the following Lemma we construct a supersolution, which lies above the solution to (4.39) with boundary conditions (4.40). As the constructed supersolution does not blow up in finite time, we can immediately conclude that the solution is global.

**Lemma 4.3.1.** Let \( \phi \) be a solution of (4.39) and (4.40), with initial condition obeying the assumptions (4.37) in Subsection 4.1.4. Let further \( t_1 \) be such that \( r_-(t) < \frac{1}{2} \) for all \( t \geq t_1 \). There exists a number \( N \) and a time \( t_2 > t_1 \) such that

\[
\phi(r, t) \leq \phi_-(Nr, t_2 + N^2 t),
\]

for all \( t \geq 0 \).

**Proof.** Let \( \phi_{sup}(r, t) := \phi_-(Nr, t_2 + N^2 t) \), which is a supersolution to the harmonic map heat flow. To prove this Lemma we use the Comparison Principle. By definition and by construction we have \( \phi(0, t) = \phi_{sup}(0, t) = 0 \) and \( \phi(1, t) = \phi < \phi_{sup}(1, t) \) for all \( N \geq \frac{1}{2} \), where we used (4.107) and the fact that \( \partial_r \phi > 0 \) on \( r \in (0, \infty) \), see (4.106). To prove the Lemma it remains to show that \( \phi(r, 0) \leq \phi_{sup}(r, 0) = \phi_-(Nr, t_2) \). We prove this in two steps.

By assumption (see Subsection 4.1.4) we have \( \phi(r, 0) \leq Dr^n \), for some large positive \( D \in \mathbb{R} \). Choose \( \delta_1 > 0 \) such that \( D\delta_1^n < \pi \). Then we find on the interval \([0, \delta_1]\), using (4.110),

\[
Dr^n \leq \frac{\text{C}R^n}{2 \arctan(\frac{\pi}{2^n})} \frac{2 \arctan\left( \frac{
}{\text{C}R} \right)}{\arctan(\phi^n)},
\]

\[
\leq 2 \arctan(\phi^n),
\]

for \( t \) large enough such that \( D\delta_1^n \leq 2 \arctan(\frac{\pi}{2^n}) \).

Using the bound (4.77) and the identity (4.109) we have

\[
\phi_-(Nr, t) \geq 2 \arctan(\text{N}^n y^n) + 2 \arctan(\text{N}^n s^n) - MR^n \frac{\text{N}^n y^n + \text{N}^n s^n + \text{N}^n s^n}{1 + \text{N}^n y^n} - \kappa I R_{2n} \text{N}^{n+1} y^{n+1}
\]

\[
\geq 2 \arctan(y^n) + 2 \arctan\left( \frac{\text{N}^n y^n - y^n}{1 + \text{N}^n y^n} \right) + 2 \arctan(\text{N}^n s^n) - MR^n \frac{\text{N}^n y^n + \text{N}^n s^n + \text{N}^n s^n}{1 + \text{N}^n y^n} - \kappa I R_{2n} \text{N}^{n+1} y^{n+1}
\]

\[
\geq 2 \arctan(y^n),
\]

for all \( t \geq 0 \).
for \( t \) large enough. Hence, on the interval \([0, \delta_1]\) we have \( \phi(r, 0) \leq \phi_-(Nr, t) \), for \( t \) large enough.

For any small \( \delta_1 > 0 \) we have, using the bound (4.77) and (4.108),

\[
\phi_-(Nr, t) \geq 2 \arctan(N^n y^n) + 2 \arctan(N^n s^n) - MR^n \frac{N^n y^n + N^{n+2} y^{n+2}}{1 + N^{n+2} y^{n+2}} \\
- \kappa_1 R^{2n} N^{n+1} y^{n+1} \\
\geq 2\pi - 2 \arctan\left(\frac{R^n}{N^n y^n}\right) - 2 \arctan\left(\frac{1}{N^n y^n}\right) - MR^n - \kappa_1 R^{2n} N^{n+1} \\
= 2\pi + O\left(\frac{1}{N}, R^n, R^{2n} N^{n+1}\right),
\]

(4.114)
on \( r \in [\delta_1, 1] \). For every \( N \) we can make \( \kappa_1 N^{n+1} R^{2n} \) as small as we want by increasing \( t \). Hence, as \( \phi(r, 0) < 2\pi \), we can find an \( N \) large enough and a corresponding \( t_2 \) large enough such that \( \phi(r, 0) < \phi_-(Nr, t_2) \) on \([\delta_1, 1]\). Taking \( t_2 \) such that equation (4.112) also holds, finishes the proof.

Comparing the solution with the subsolution \( -\phi_{(s)}(r, t) \), gives us a lower bound for the solution. We can conclude that any solution with the smoothness assumption (4.37a) on the initial condition and \( |\phi(r, 0)| < 2\pi \) is global. This has already been proven along similar lines in [19] and [57]. The difference with this Lemma is the choice of the blowup rate of the supersolution \( \phi_{(s)} \). In [19] and [57] the authors compare the solution with a supersolution that blows up exponentially fast. The blowup rate for our supersolution, however, is (in the case of \( n > 2 \)) algebraic in \( t \). Hence, we have found a better approximation of the blowup rate. We make this more precise after Lemma 4.3.2.

In the following Lemma we show that we are able to put a subsolution under the solution. As this subsolution blows up in infinite time, this also has to hold for the solution \( \phi \).

**Lemma 4.3.2.** Let \( \phi \) be a solution of (4.39) and (4.40), with initial condition obeying the assumptions (4.37) in Subsection 4.1.4. Let \( t_1 \) be such that \( r_+(t) > \frac{\delta}{2} \) for all \( t \geq t_1 \).

There exists an \( \epsilon_1 > 0 \) and a \( t_2 \) such that

\[
\phi_+(r, t, t_1 + \epsilon_1^2 t) \leq \phi(r, t_0 + t),
\]

(4.115)

for all \( t \geq 0 \).

**Proof.** Consider \( \phi_{(s)}(r, t) := \phi_+(r, t, t_1 + \epsilon_1^2 t) \) which is a subsolution of the harmonic map heat flow. We, again, use the Comparison Principle. Since, by definition and construction we have \( \phi_{(s)}(0, t) = 0 \) and \( \phi(1, t) = \phi \geq \phi_{(s)}(1, t) \), for \( \epsilon_1 \leq \frac{\delta}{2} \), it remains to show that \( \phi(r, t_0) \geq \phi_{(s)}(r, 0) \).

Compare the solution \( \phi \) with a solution \( \psi \) (see Figure 4.1), which has an initial condition \( \psi_0 \) obeying \(-2\pi < \psi_0 < 0 \) and \( \psi_0 \leq \phi_0 \) and boundary conditions \( \psi(0, t) = -\pi \) and \( \psi(1, t) = -\pi + 2\pi \). The solution \( \psi \) is global and goes to \(-\pi + 2 \arctan\left(\frac{\epsilon_0}{\epsilon_0}\right) \), with \( \epsilon_0 \) as in (4.57). From the parabolicity of (4.39) we know that the derivatives of \( \psi \) converge as well to the derivatives of \(-\pi + 2 \arctan\left(\frac{\epsilon_0}{\epsilon_0}\right) \). Hence, after a certain time \( t_0 \) the solution \( \psi \), and therefore also \( \phi \), lies above \(-\pi \) on the interval \([0, 1]\). We are now able to compare the solution \( \phi \) with a solution \( \chi \), which has an initial condition \( \chi_0 \) obeying \( |\chi_0| < \pi \) and
4.3. Sub- and supersolutions for the HMHF with $n \geq 2$

Figure 4.1: In this figure we see the evolution of several solutions to the harmonic map heat flow with $n = 2$, and $\bar{\phi} = \pi + 1$. The dashed lines represent the initial conditions and the evolution is from the lighter grays towards the darker grays. In the left figure we see the function $\psi$ that bounds the solution $\phi$ from below. This means we can bound the solution $\phi$ after a while with the solution $\chi$ in the middle figure. Hence, any solution, with initial condition $|\phi_0| < 2\pi$, eventually becomes positive on the interval $(0, 1]$.

$\chi_0 \leq \phi_0$. This function $\chi$ is global and becomes positive if we take boundary conditions $\chi(0, t) = 0$ and $0 < \chi(1, t) < \pi$. Hence, after a certain time, $\phi > 0$ on the interval $(0, 1]$. We may conclude, using the assumption (4.37a) of Subsection 4.1.4 that $\phi(r, t_0) \geq Dr^n$, for some small positive $D$.

By construction and boundedness of the subsolution $\phi_-$, we can bound it with $C(t)r^n$. Hence,

$$\phi_-(r, 0) = \phi_-(\epsilon_1 r, t_0) \leq C(t_0)\epsilon_1^n r^n \leq Dr^n \leq \phi(r, t_0),$$

for $\epsilon_1 \leq \left(\frac{D}{C(t_0)}\right)^{\frac{1}{n}}$. Hence, equation (4.115) holds for every $\epsilon_1 \leq \min\left(\left(\frac{D}{C(t_0)}\right)^{\frac{1}{n}}, \frac{1}{2}\right)$.

This shows that any solution of the harmonic map heat flow (4.39), with initial condition obeying (4.37) and boundary conditions as in (4.40), blows up in infinite time. We can make the assumptions on the initial condition less restrictive. If we can show that the solution is global, also in the case $\max_{|\theta|} |\theta_0| \geq 2\pi$, we can use the same arguments as in the proof of Lemma 4.3.2 to show that the solution has to blow up in infinite time. We refer to Section 1.4 for a thorough discussion on the assumptions of the main Theorem.

Combining Lemmas 4.3.1 and 4.3.2 we find that any solution, obeying the assumptions (4.37), blows up with a blowup rate that goes to zero faster than $R(t_2 + N^2)$ but slower than $R(t_2 + \epsilon_1 t)$. Let us make this more precise.

Consider the blowup rate of the solution $\phi$, defined in Subsection 4.1.4 as the unique function $R_\phi(t)$ such that

$$\phi(R_\phi(t), t) = \frac{\pi}{2}.$$  

For the subsolution and supersolution we define similar rates $R_\pm$ given by

$$\phi_\pm(R_\pm(t), t) = \frac{\pi}{2}.$$  

As the functions $\phi_\pm$ are monotone, these rates are unique. Let $r = \alpha R$. Then

$$\phi_\pm(r, t) = 2 \arctan(\alpha^n) + O(R^n) = \frac{\pi}{2},$$

if $\alpha^n = 1 + O(R^n)$. Hence,

$$R_\pm(t) = \left(1 + o(1)\right)R(t),$$

if $\alpha^n = 1 + O(R^n)$. Hence,
and $R_\phi$ has the same asymptotic expansion as $R$. Using Lemma 4.3.1 and the definition of $R_\phi$, we conclude that

$$\frac{\pi}{2} = o(R_\phi(t), t) \leq \phi_\ast(NR_\phi(t), t_2 + N^2t).$$

(4.121)

and therefore,

$$NR_\phi(t) \geq R_\ast(t_2 + N^2t) = \left(1 + o(1)\right) R(t_2 + N^2t).$$

(4.122)

Using Lemma 4.3.2 we have

$$\frac{\pi}{2} = \phi_\ast(R_\phi(t_0 + t), t_0 + t) \geq \phi_\ast \left( \epsilon_1 R_\phi(t_0 + t), t_3 + \epsilon^2 t \right).$$

(4.123)

and therefore,

$$\epsilon_1 R_\phi(t_0 + t) \leq R_\ast(t_3 + \epsilon^2 t) = \left(1 + o(1)\right) R(t_3 + \epsilon^2 t).$$

(4.124)

Consider the case $n > 2$, where $R$ from (4.59) is given by

$$R(t) = \left(\frac{(n - 2)\alpha_0}{E_n}\right)^{\frac{1}{n - 2}}, \text{ for large } t.$$

(4.125)

Hence, for large $t$, we have

$$R_\phi(t) \geq \frac{1}{N} \left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0}{E_n}\right)^{\frac{1}{n - 2}} \left(\frac{(n - 2)\alpha_0 (t_2 + N^2t)}{E_n}\right)^{\frac{1}{n - 2}}$$

$$= \frac{1}{N} \left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0 N^2t}{E_n}\right)^{\frac{1}{n - 2}}$$

$$= \left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0 N^2t}{E_n}\right)^{\frac{1}{n - 2}}.$$

(4.126)

With a similar calculation we find, for large $t$,

$$R_\phi(t) \leq \left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0}{E_n}\right)^{\frac{1}{n - 2}}.$$

(4.127)

Hence, we have found solutions $\phi$ that blow up with an inverse blowup rate $R_\phi$, given by

$$\left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0}{E_n}\right)^{\frac{1}{n - 2}} \leq R_\phi(t) \leq \left(1 + o(1)\right) \left(\frac{(n - 2)\alpha_0}{E_n}\right)^{\frac{1}{n - 2}}.$$

(4.128)

This statement can be improved and it can be shown that the blowup rate of the solution is in fact precisely as in Theorem 4.1.1. To do this we first show that, away from the singularity in $r = 0$, the solution evolves uniformly to

$$\phi(r, t) \to \pi + 2 \arctan \left(\frac{\alpha_0}{2} \frac{n}{\sqrt{n}}\right), \text{ as } t \to \infty.$$

(4.129)

With this information on can prove Lemmas similar to 4.3.1 and 4.3.2, with $N$ and $\epsilon_1$ replaced by values as close to 1 as we want.
4.3. Sub- and supersolutions for the HMHF with \( n \geq 2 \)

We show that the limit (4.129) holds by constructing explicit super- and subsolutions that evolve towards the limit profile, away from \( r = 0 \). Consider a solution \( \psi \) to the harmonic map heat flow with boundary conditions

\[
\psi(0, t) = \pi \quad \text{and} \quad \psi(1, t) = \tilde{\phi},
\]  

(4.130)

and initial condition \( \psi_0 = \psi(r, 0) \) such that \( \pi < \psi_0 < 2\pi \) on \( (0, 1) \). This solution does not blow up and evolves to the harmonic map given by (4.129). As long as the initial condition of \( \phi \) obeys \( \phi(0, 0) < 2\pi \) on \( (0, 1) \), we can put a \( \psi \) above it. The solution \( \phi \) then has to stay below \( \psi \) and we can conclude that for any \( \epsilon > 0 \), we can find a \( t_\epsilon \) such that

\[
\phi(r, t) \leq \pi + 2 \arctan \left( \frac{\alpha_0}{2} \right) + \epsilon, \quad \text{for all} \quad t \geq t_\epsilon.
\]  

(4.131)

To show the other inequality, away from \( r = 0 \), we need to do a bit more work.

Consider the function \( \phi^{\alpha}_l(r, t) \), which is defined as \( \phi^{\alpha}_l(r, t) \) from equation (4.98) but with \( \alpha_0 \) replaced by a parameter \( \alpha \in (0, \infty) \). Just as \( \phi^{\alpha}_l \), the function \( \phi^{\alpha}_l \) is a subsolution of the harmonic map heat flow. Instead of \( \phi^{\alpha}_l \), \( (\epsilon r, t_4 + \epsilon^2 t_5) \) from Lemma 4.3.2, we now use the subsolution \( \phi^{\alpha}_l(\epsilon r, t_4 + \epsilon^2 t_5) \), with

\[
\alpha_1 := \frac{\alpha_0}{\epsilon^2} \quad \text{and} \quad \theta \in (0, 1),
\]  

(4.132)

to bound the solution from beneath. This is done in Lemma 4.3.3. Considering the function \( \phi^{\alpha}_l(\epsilon r, t_4 + \epsilon^2 t_5) \) for \( \epsilon \) small and \( t_4 \) large and using the Comparison Principle we conclude that for every \( \theta \in (0, 1) \), arbitrary close to 1, and \( \epsilon, \delta > 0 \) we can find a \( t_\epsilon \) such that

\[
\pi + 2 \arctan \left( \frac{\alpha_0}{2} \theta \right) - \epsilon \leq \phi(r, t), \quad \text{for all} \quad t \geq t_\epsilon \quad \text{on} \quad [\delta, 1].
\]  

(4.133)

Let us denote for convenience the function \( R \) and \( g \) on time \( t_\epsilon \), as \( R_\epsilon \) and \( g_\epsilon \).

**Lemma 4.3.3.** Let \( \alpha_1 \) as in (4.132) and \( \theta \in (0, 1) \) arbitrary. There exist an \( \epsilon_2 > 0 \) and \( t_4, t_5 \) such that

\[
\phi^{\alpha}_l(\epsilon r, t_4 + \epsilon^2 t_5) \leq \phi(r, t_5 + t),
\]  

(4.134)

for all \( t \geq 0 \).

**Proof.** Since, \( \phi^{\alpha}_l(\epsilon r, t_4 + \epsilon^2 t_5) \) is a subsolution for the harmonic map heat flow we can use the Comparison Principle again. Consider \( \phi^{\alpha}_l(\epsilon r, t) \) at \( r = 1 \). We have, using the bound (4.77) and (4.108) and (4.109),

\[
\phi^{\alpha}_l(\epsilon r, t) \leq 2 \arctan \left( \frac{\epsilon}{R} \right) + 2 \arctan \left( \frac{\alpha_0}{2} \right) + MR^\alpha + \epsilon_1 R^{\alpha-1}\epsilon_2^{n+1}
\]

\[
= \tilde{\phi} - 2 \arctan \left( \frac{\epsilon}{R} \right) - 2 \arctan \left( \frac{\alpha_0 (1 - \theta)}{1 + \theta} \right) + MR^\alpha + \epsilon_1 R^{\alpha-1}\epsilon_2^{n+1}
\]  

(4.135)

\[
\leq \tilde{\phi} = \phi(1, t_\epsilon),
\]

for large \( t \). Let \( t_\epsilon \) be such that (4.135) holds for all \( t \geq t_\epsilon \). By construction we have \( \phi^{\alpha}_l(0, t) = \phi(0, t) = 0 \). Hence, it remains to show that there exist \( \epsilon_2 \) and \( t_\epsilon \) such that \( \phi^{\alpha}_l(\epsilon r, t_4) \leq \phi(r, t_5) \). We do this by comparing \( \phi^{\alpha}_l(\epsilon r, t_4) \) with \( \phi^{\alpha}_l(\epsilon r, t) \) and then using Lemma 4.3.2.
Using the bound (4.77) we have
\[
\phi_{\text{in}}^n(\epsilon r, t_4) \leq 2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) + 2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) + M_t R_4 \frac{\epsilon y^*_4 + \epsilon y^*_4 y^*_4 + 2 + \epsilon y^*_4 y^*_4 + 2}{1 + \epsilon y^*_4 y^*_4} + \kappa_t R_4 \epsilon y^*_4 y^*_4 + 2.
\]
and
\[
\phi_{\text{out}}(\epsilon r, t_4) \geq 2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) + 2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) - M_t R_4 \frac{\epsilon y^*_4 + \epsilon y^*_4 y^*_4 + 2}{1 + \epsilon y^*_4 y^*_4} \tag{4.137}
\]
Let \( \epsilon_1 \) be as in Lemma 4.3.2 and \( t_4 \) such that (4.135) holds. We show that there exist \( \epsilon_2 \) and \( t_6 \) such that
\[
\phi_{\text{in}}^n(\epsilon r, t_4) \leq \phi_{\text{out}}(\epsilon r, t_6),
\]
by comparing the inner part of (4.137) with the outer part of (4.136) and the other way around, see also Figure 4.2.

If
\[
\epsilon_2 \leq \left( \frac{\epsilon_1}{2} \right),
\]
we have
\[
\arctan \left( \frac{\epsilon y^*_4}{2} \right) \leq \arctan \left( \frac{\epsilon y^*_4}{2} \right). \tag{4.140}
\]
Hence, for (4.138) to hold, we need to show that
\[
2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) + M_t R_4 \frac{\epsilon y^*_4 + \epsilon y^*_4 y^*_4 + 2}{1 + \epsilon y^*_4 y^*_4} + \kappa_t R_4 \epsilon y^*_4 y^*_4 + 2 \leq 2 \arctan \left( \frac{\epsilon y^*_4}{2} \right) - M_t R_4 \frac{\epsilon y^*_4 + \epsilon y^*_4 y^*_4 + 2}{1 + \epsilon y^*_4 y^*_4} \tag{4.141}
\]
The inequality (4.141) holds for \( \epsilon_2 \) small enough, such that the left-hand side of the equation is well below \( \pi \), and for \( t_6 \) large enough such that the right-hand side is close enough to \( \pi \), on an interval away from \( 0 \). Hence, we can indeed find \( t_6 \) and \( \epsilon_2 \) such that (4.138) holds. Choose \( t_6 \) to be larger than \( t_3 \) from Lemma 4.3.2. Then, using Lemma 4.3.2 we find
\[
\phi_{\text{in}}^n(\epsilon r, t_4) \leq \phi_{\text{out}}(\epsilon r, t_6) \tag{4.142}
\]
with \( t_5 := t_6 + \frac{\epsilon_1}{2} t_4 \), where \( t_6 \) as in Lemma 4.3.5. This finishes the proof.

Lemma 4.3.3 implies that (4.133) indeed holds. Combining equations (4.131) and (4.133) we conclude that, away from \( r = 0 \), the solution \( \phi \) evolves towards the harmonic map given in (4.129). This information enables us to prove that the blowup rate of a solution is exactly as given in Theorem 4.1.1.

In the next Lemma we show that one can relax the condition \( \epsilon_1 \leq \min \left( \frac{2 \epsilon_1}{\epsilon_4} \right) \) in Lemma 4.3.2 and take \( \epsilon_1 \) arbitrarily close to \( 1 \), at the cost of having to take a larger \( t \).

**Lemma 4.3.4.** Let \( \theta \in (0, 1) \) be given and choose \( t_7 \) such that \( r_\theta(t) > 0 \) for all \( t \geq t_7 \). There exist \( t_8, t_9 \geq t_7 \) such that
\[
\phi_{\text{in}}(\theta r, t_8 + \theta^2 t) \leq \phi(r, t_9 + t) \tag{4.143}
\]
for all \( t \geq 0 \).
4.3. Sub- and supersolutions for the HMHF with $n \geq 2$

Figure 4.2: Sketch of the proof of equation (4.138). Consider the function $\phi_{e_1}(x, t_4)$. The time $t_4$ is chosen such that the function lies beneath $\phi$. In the left figure we show the evolution (from the dashed curve to the darker curves) of the first two terms of the right hand side of (4.136) when decreasing $e_2$. In the right figure we see the first two terms of the right hand side of (4.137), for fixed $e_1$, when increasing $t$. Also in this figure the evolution is from the dashed line to the darker curves. Hence, for fixed $t_4$ and $e_1$, it is possible to bound $\phi_{e_1}(x, t_4)$, from above, with $\phi_1(x, t_4)$, by choosing small enough $e_2$ and large enough $t_4$.

Proof. The function $\phi_1(\theta, t_4 + \theta^2t)$ is a subsolution and we use the Comparison Principle again. By construction and by assumption we have that $0 = \phi(0, t_4 + t) = \phi_1(0, t_4 + t)$ and $\phi_1(\theta, t_4 + \theta^2t) < \phi = \phi(1, t_4 + t)$. Hence, it remains to show that $\phi(\theta, t_4) \geq \phi_1(\theta, t_4)$.

Consider the subsolution $\phi_1$. Using the bound (4.77) and the equality (4.109) we can find a $t_4$, large enough, such that

$$
\phi_1(\theta, t_4) \leq 2 \arctan(\theta y_0^\theta) + 2 \arctan(\theta s) + M R_{n}^{\theta y_0^\theta + \theta s^2 y_0^\theta + \theta s^2 y_0^\theta + 1} + \kappa_{1} R_{n}^{\theta y_0^\theta + \theta s^2 y_0^\theta + 1}
$$

$$
= 2 \arctan(\theta y_0^\theta) + 2 \arctan(s) + M R_{n}^{\theta y_0^\theta + \theta s^2 y_0^\theta + \theta s^2 y_0^\theta + 1} + \kappa_{1} R_{n}^{\theta y_0^\theta + \theta s^2 y_0^\theta + 1}
$$

$$
\leq 2 \arctan(\theta y_0^\theta) + 2 \arctan(s)
$$

$$
< \pi + 2 \arctan(s),
$$

(4.144)

on the interval $r \in [0, 1]$.

The function $\phi$ blows up in infinite time, due to Lemmas 4.3.1 and 4.3.2, with blowup profile given by (4.129), away from $r = 0$. Hence, for any small $\delta_1 > 0$ we have that $\lim_{r \to 0} \phi(r, t) = \pi + 2 \arctan(s)$ uniformly on the interval $[\delta_1, 1]$. This means that we can find a $t_4$ such that $\phi_1(\theta, t_4) \leq \phi(r, t_4)$ on any interval $[\delta_1, 1]$.

Choose $\delta_1 = (4\pi)^2 R_{n}^{\theta y_0^\theta}$ such that on the interval $[0, \delta_1]$ we have $y_0^\theta s \leq 1$ and

$$
2 \arctan(y_0^\theta) + 2 \arctan(s) \leq \pi.
$$

(4.145)

Consider the inequality (4.144) that states

$$
\phi_1(\theta, t_4) \leq 2 \arctan(\theta y_0^\theta) + 2 \arctan(s),
$$

(4.146)
on the whole interval $[0,1]$, for $t_3$ large enough. Due to (4.145) we can sum the two arctangents of (4.146) on the interval $[0, \delta_1]$ and we can find a time $t_9 > t_3$, with $t_3$ as in Lemma 4.3.2, such that

$$\phi_+ (\theta r, t_3) \leq 2 \arctan \left( \frac{\theta^n + \theta^{2n}}{1 - \theta^n \theta^{2n}} \right)$$

$$\leq 2 \arctan \left( \frac{\theta^n + \theta^{2n}}{1 - \theta^n} \right)$$

$$= 2 \arctan \left( \frac{\theta^n}{R^0_3} \right),$$

with $\frac{1}{R^0_3} = \left( \frac{\frac{\theta^n}{1 - \theta^n}}{1 - \theta^n \theta^{2n}} \right)^\frac{1}{2}$. Consider a $t_\beta$ such that $t_\beta > t_9$. Then, using the bound (4.77) and (4.109), we have

$$\phi_+ (\epsilon r, t_\beta) \geq 2 \arctan \left( \epsilon^n y^n_\beta \right) + 2 \arctan \left( \epsilon^n \theta^n \right) - M R^0_3 \frac{\epsilon^{n+1} y^{n+2} - \epsilon^{n+2} y^{n+2}}{1 + \epsilon^n \theta^n}$$

$$= 2 \arctan \left( \epsilon^n y^n_\beta \right) + 2 \arctan \left( \frac{y^n_\beta - y^n_0}{1 + \epsilon^n \theta^n} \right) + 2 \arctan \left( \epsilon^n \theta^n \right)$$

$$\geq 2 \arctan \left( \epsilon^n y^n_\beta \right),$$

for $t_\beta$ large enough. Hence, using Lemma 4.3.2, we have on $r \in (0, \delta_1]$}

$$\phi_+ (\theta r, t_3) \leq \phi_+ (\epsilon r, t_\beta) \leq \phi (r, t_3),$$

with $t_\beta := t_9 + \frac{t_9 - t_3}{t_3}$, with $t_0$ as in Lemma 4.3.2.

Using the limit profile of the solution $\phi$, away from $r = 0$, one can also relax the condition for large $N$ in Lemma 4.3.1. With this information it is readily shown that $N$ can take any value bigger than 1. Hence, equation (4.128), with arbitrary $N > 1$ and $0 < \epsilon_1 < 1$ now gives

$$R_n(t) = \left( 1 + o(1) \right) \left( \frac{(n - 2) \alpha_0}{E_n} \right)^{\frac{1}{n-2}}, \text{ for } n > 2 \text{ and large } t. \quad (4.150)$$

A similar calculation for the case $n = 2$, gives

$$\ln \left( R_n(t) \right) = -\frac{8}{\pi} \alpha_0 t, \text{ for } n = 2 \text{ and large } t. \quad (4.151)$$

This finishes the proof of Theorem 4.1.1.

**Remark 4.3.5.** In the case that the boundary condition on $r = 1$ is given by $\varphi = \pi$, one can prove along similar lines that the blowup rate is given by (4.33). One takes for the sub- and supersolutions, $\phi_-$ and $\phi_-$,

$$\varphi_\pm = 2 \arctan (y^n) - \frac{2}{E_n} R^{2n} f_1 \pm \kappa_1 R^{4n} y^{2n}, \quad (4.152)$$
4.3. Sub- and supersolutions for the HMHF with $n \geq 2$

which are variations of the global approximation, given in Remark 4.2.1. To construct sub- and supersolutions that actually lie beneath and above the solution $\phi$ we use Lemmas similar to 4.3.1 and 4.3.2. Since we explicitly know the limit profile of the solution, it is straightforward to show that we can take $\epsilon_1$ and $N$ as close to 1, as we want. Relating $\Re_\delta$ to $R$ then gives the asymptotics of the blowup rate.
4. Singularities in the 2-d radially symmetric equivariant HMHF
Bibliography


Bibliography


Samenvatting

Het opblazen van twee meetkundige stromingen

Wiskunde is de taal waarin de natuurwetten zijn geformuleerd. Dit stelt ons in staat de natuur niet alleen te begrijpen maar ook te beheersen. De moderne wereld met alle technologie is hierop gebaseerd.

Vaak zal het voorkomen dat we iets willen beschrijven dat verandert in de tijd. Men zou kunnen denken aan de positie van een geworpen bal, de temperatuur na het aanzetten van de kachel of het aantal konijnen in een duingebied. Een manier om een systeem waarin veranderingen in de tijd plaatsvinden te beschrijven is via differentiaalvergelijkingen. Een bekend voorbeeld is de tweede wet van Newton die de versnelling van een object relateert aan de kracht op dat object. Als we nu precies weten van welk punt en met wat voor snelheid en in welke richting een bal wordt weggegooid, dan kunnen we precies voorspellen wat de baan van die bal is door de differentiaalvergelijking van Newton op te lossen. De oplossing van een differentiaalvergelijking die een tijdsevolutie beschrijft noemen we een *stroming*. We verwachten in dit geval dat we één oplossing (en slechts één) vinden. Per tijdstip is er immers altijd één positie ten slechts één positie waar de bal zich bevindt.

Deze eigenschap van de tweede wet van Newton, dat er precies één oplossing is, is een eigenschap die we verwachten bij meer differentiaalvergelijkingen die natuurwetten beschrijven. Het aantal oplossingen voor een willekeurige differentiaalvergelijking is echter zeer divers en kan veranderen in de tijd. Er zijn zeer simpele voorbeelden van tijdsevoluties waar op het begin tijdstip er een unieke oplossing bestaat, maar na een bepaalde tijd de oplossing niet goed gedefinieerd is (bijvoorbeeld omdat de oplossing oneindig is geworden). Dit noemen we het *opblazen* van een oplossing. Dit proefschrift bespreekt dit fenomeen in twee specifieke tijdsevoluties uit de natuurkunde en de biologie.

In dit proefschrift bespreken we twee stromingen van oppervlakken. We kijken dus naar bewegende oppervlakken. De ene stroming is de zogenaamde Willmore Stroming en de andere stroming is de Warmte Stroming naar Harmonische Afbeeldingen (WSHA). Deze vergelijkingen komen voor in de biologie en natuurkunde als beschrijvingen van, bijvoorbeeld, rode bloedcellen (Willmore Stroming) en vloeibare kristallen (WSHA). Beide stromingen zijn voorbeelden van zogenaamde *meetkundige stromingen*. Een stroming van een oppervlak wordt meetkundig genoemd zodra de evolutie van het oppervlak afhankelijk is van de vorm van het oppervlak zelf. Als voorbeeld zou men de beweging van een ballon kunnen nemen. De vorm van een ballon zal niet zomaar veranderen. Maar als men een ballon van vorm verandert, bijvoorbeeld door hem in te drukken, zal hij na het loslaten weer willen uitduiken. Er zal verder gelden dat hoe meer de ballon is ingedrukt hoe harder hij weer zal willen uitduiken. Net zoals de ballon altijd weer naar zijn oorspronkelijke vorm terug verandert, geldt dat in de Willmore Stroming en in de WSHA de oplossing naar een “natuurlijke” situatie stroomt. We zullen nu in het kort beide stromingen en hun opblaasgedrag bespreken.
De Willmore Stroming

De Willmore Stroming is de evolutievergelijking die we behandelen in hoofdstukken 2 en 3. Voor deze stroming kiezen we als beginsituatie een oppervlak dat niet naar zijn "natuurlijke" vorm kan veranderen zonder op te blazen. We zullen dit proberen te verduidelijken met figuur 4.3. Beschouw de kromme in het linkervlakje van figuur 4.3 en draai deze om de horizontale as. De kromme laat tijdens het $360^\circ$ draaien van de as een spoor achter zo dat er een ruimtelijke vorm ontstaat. Dit vormt een oppervlak dat lijkt op een bal (sfeer) waarin nog een rond gedraaide koker is verwerkt. De Willmore Stroming wil van dit oppervlak één of meer sferen maken en zal dat doen volgens de evolutie rechts. Men ziet dat de lus wordt dichtgetrokken en daalt om op de $z(s)$-as in één punt over te gaan. Het eindresultaat is twee halve sfeerige oefecten die één punt gemeen hebben. Het feit dat er nu een knik in het oppervlak zit, maakt dat de oplossing (het oppervlak) niet goed gedefinieerd kan worden en de vergelijking dus opblaast. Het is nu niet de oplossing die naar oneindig gaat, maar de kromming in de lus.

Figuur 4.3: Laat een oppervlak gegeven zijn door de kromme links die men draait om $z$-as. De evolutie van het oppervlak wordt geportrayeerd in de afbeelding rechts.

Ons onderzoek stelt de vraag: "Wanneer verdwijnt de lus en hoe snel?". Om hier een antwoord op te geven maken wij gebruik van twee verschillende methoden: computersimulaties (Hoofdstuk 3) en de methode van consistentie asymptotiek (Hoofdstuk 2). Beide methoden maken gebruik van simplificaties van de Willmore Stroming. In het geval van de computersimulaties benaderen we de tijd en de ruimte met een eindig aantal punten. In het geval van de consistentie asymptotiek maken we gebruik van een andere benadering, waar we nu iets specifieker op in gaan.

Uit theorie van de Willmore Stroming weten we dat verschillende plekken op de kromme in figuur 4.3 (links) zich verschillend gedragen. Dit zien we terug in figuur 4.3 (rechts). We kunnen zien dat, terwijl de lus verdwijnt (en een grote verandering ondergaat), de bal slechts een klein beetje beweegt en niet veel groter of kleiner wordt. Dit onderscheid in gedrag op verschillende plekken op de kromme zorgt ervoor dat het mogelijk is op deze verschillende plekken een andere benadering van de Willmore Stroming te nemen. Dit geeft andere oplossingen op verschillende gebieden op de kromme. Het idee achter de consistentie asymptotiek is dat in de gebieden waar deze verschillende oplossingen elkaar overlappen de oplossingen op elkaar lijken (consistentie asymptotiek). Het feit dat de oplossingen op elkaar moeten lijken in de overlappende gebieden geeft ons
enkele restricties. Deze restricties geven ons informatie over het opblaasgedrag van de vergelijking.

In de computersimulaties van Hoofdstuk 3 verdwijnt de lus in eindige tijd. Dit is consistent met eerdere computersimulaties en geeft meer gewicht aan het vermoeden dat de Willmore stroming in eindige tijd kan opblazen. De conclusies die we uit de consistente asymptotiek kunnen trekken zijn iets anders. Wat wij vinden met de consistente asymptotiek is een kromme die opblaast, maar niet met een verdwijnende lus. Of een kromme zonder lus werkelijk kan opblazen is nog maar de vraag en verdient meer onderzoek. De vraag of de kromme met de lus in eindige tijd opblaast hebben we niet kunnen beantwoorden met de consistente asymptotiek. Dit betekent niet dat we hiermee hebben laten zien dat de kromme niet in eindige tijd opblaast, maar wel dat dat in ieder geval niet volgens de meest "intuïtieve" manier gebeurt.

De Warmte Stroming naar Harmonische Afbeeldingen

In Hoofdstuk 4 behandelde we de Warmte Stroming naar Harmonische Afbeeldingen (WSHA). De WSHA kan gebruikt worden in de studie van vloeibare kristallen. Vloeibare kristallen bestaan uit langwerpige moleculen en kunnen in een erg versimpelde vorm beschreven worden als korte staafjes (allen van dezelfde lengte) in een vloeibare oplossing. Een volgende vereenvoudiging is om de kristallen niet in drie dimensies, maar in twee dimensies te beschouwen. Dit is ongeveer de configuratie die we bespreken in Hoofdstuk 3. De manier waarop de staafjes zich ordenen wordt beschreven door de WSHA. Deze stroming kan ook opblaasgedrag vertonen.

Beschouw vloeibare kristallen (staafjes) op een schijf. Laat de ordening op de schijf rotatie symmetrisch zijn. Dit betekent dat de ordening van de staafjes gelijk blijft als we de schijf onder een willekeurige hoek draaien. Het is dan voldoende om de staafjes op één "spaak" van de schijf te bekijken. De staafjes op de spaak zijn zo verdeeld dat het staafje op \( r = 0 \) omhoog staat en het staafje op \( r = 1 \) omlaag (zie het linkerplaatje in Figuur 4.4).

![Figuur 4.4: De evolutie van de staafjes op verschillende tijdstippen. Rond \( r = 0 \) wijzen de staafjes alle kanten op. De oplossing is hier niet goed gedefinieerd en de oplossing blaast op.](image)

Net zoals in het geval van de ballon en van de Willmore stroming zal de WSHA de ordening van de staafjes veranderen naar een meer "natuurlijke" ordening. Deze ordening is zo dat alle staafjes dezelfde kant op wijzen. Als we er voor zorgen dat de staafjes aan de uiteinden niet kunnen bewegen, zal deze ordening niet bereikt worden tenzij de oplossing opblaast. Het opblazen wordt geschetst in Figuur 4.4. De staafjes zullen van rechts naar links omlaag gaan staan. Er onstaat dan een gebied bij \( r = 0 \) waar een opeenhoping van
staafjes is die allen een andere kant opwijzen. Dit betekent dat de eerste afgeleide van de hoek die de staafjes maken naar oneindig gaat. De ordening is dan niet goed meer gedefinieerd en de WSHAblaast dus op.

De zojuist beschreven herordening van de staafjes op een schijf kunnen we zien als een bewegend oppervlak. Dit doen we door de orientatie van een staafje te identificeren met een punt op de bol. De herordening van de staafjes op een schijf is dan te vergelijken met een bewegende schijf over een bol. De situatie die wij bekijken in Hoofdstuk 4 is iets anders. We eisen geen rotatie symmetrie op de schijf, maar sferische symmetrie op de bol. Het opblaasgedrag zoals net onschreven blijft echter hetzelfde.

In Hoofdstuk 4 bewijzen we de snelheid waarmee verschillende oplossingen opblazen. De methode die we gebruiken in dit hoofdstuk is een zogenaamde vergelijkingsmethode. Met deze methode kunnen we, ondanks dat we niet weten hoe de oplossing eruitziet, de oplossing van onder en van boven begrenzen met functies die we wel kennen. Omdat we weten hoe deze functies opblazen, kunnen we er uiteindelijk ook achter komen of en hoe de oplossing precies opblaast.

Behalve voor de toepassingen is er ook een wiskundige reden om het opblaasgedrag van specifiek de Willmore Stroming en de WSHA te onderzoeken. Beide vergelijkingen zijn standaardvoorbeelden van meetkundige stromingen. De afgelopen decennia zijn er veel successen geboekt in de wiskunde door gebruik te maken van meetkundige stromingen, met als wellicht het bekendste voorbeeld het bewijs van het vermoeden van Poincaré met behulp van de zogenaamde Ricci stroming. Om zulke stromingen te kunnen gebruiken, is kennis over het opblaasgedrag van belang. De wetenschap hoe de Willmore Stroming en WSHA zich (kunnen) gedragen, maar ook de kennis van de methoden die dit gedrag blootleggen, kan nieuw inzicht geven in de studie van opblaasgedrag in andere meetkundige stromingen.
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