Random fractals and scaling limits in percolation

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Het figuur op de omslag is een simulatie van Mandelbrot fractalpercolatie in drie dimensies. De simulatie-software is geschreven door Wouter Kager.

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Random fractals and scaling limits in percolation

ACADEMISCH PROEFSCHRIFT

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door

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geboren te Amsterdam

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1 Introduction

1.1 Percolation

Percolation theory is a fascinating branch in mathematics. It is concerned with important questions, arising from different scientific areas such as statistical physics, epidemiology and complex networks. Percolation theory was started by the engineer Simon Broadbent and mathematician John Hammersley in the paper [15], in which they introduced the following model to study the behaviour of fluids through random media. Consider the square lattice $\mathbb{Z}^2$ and declare each edge open with probability $p \in [0,1]$ and closed with probability $1-p$, independently of other edges. See Figure 1.1 for an illustration. This model is called Bernoulli bond percolation with parameter p and the probability measure associated to it is denoted by $\mathbb{P}_p$. Performing Bernoulli bond percolation results in a random graph, where the connectivity properties of open clusters are natural objects of interest. To make the connection with the original real-world problem of fluid flowing through a porous medium one can interpret an open edge as an edge which is permissable for the flow of fluid. So open clusters are channels which can be used for the flow of fluids.

The first question that arises (among mathematicians) is what the probability is of the event that the origin is contained in an infinite open cluster. Denoting this event by $\{O \leftrightarrow \infty\}$ this means we are interested in the function

$$\vartheta(p) = \mathbb{P}_p(O \leftrightarrow \infty).$$

Clearly, ϑ is increasing in p and $\vartheta(1) = 1$, so we can ask when ϑ is positive. To this

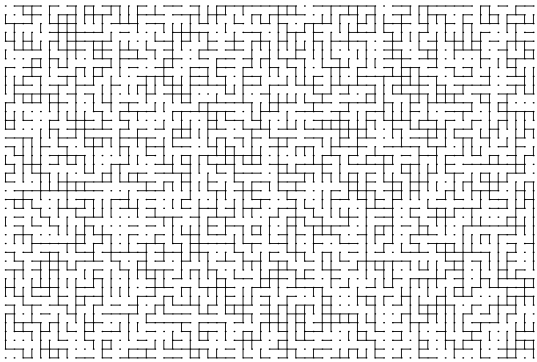


Figure 1.1: Typical realization of the Bernoulli bond percolation model with parameter $p = 1/2$.

end, define the critical probability

$$p_c = \inf\{p : \vartheta(p) > 0\}.$$

It is not hard to show that $0 < p_c < 1$, see for instance [36]. It was shown by Harris already in 1960 [37] that $\vartheta(1/2) = 0$, implying that $p_c \geq 1/2$, but it took another 20 years before Kesten proved the inequality in the other direction, yielding that $p_c = 1/2$ [38]. This particular problem is examplary for the field of percolation: it is typically easy to pose a natural and interesting question, but providing an answer requires the development of exciting new mathematics.

The percolation model has a phase transition at $p_c = 1/2$: below p_c there exists no infinite open cluster, while above p_c there exists a gigantic infinite open cluster and at p_c neither an open nor a closed infinite cluster exists. Phase transitions are of great interest for physicists and percolation is probably the easiest example of a model with a non-trivial phase transition. Examples of phase transitions in every day life include boiling an egg (the egg white and yolk solidify after sufficient heating) and adding water to Pernod (the liquid becomes opaque after the “critical” drop of water).

Mathematicians have a deep and passionate love for generalization and this love was equally returned by percolation. One possible generalization is to declear vertices instead of edges open or closed, this is called site percolation. Although this looks like an almost trivial generalization, interestingly enough, the critical probability for site percolation on $\mathbb{Z}^2$ is not equal to the critical probability for bond percolation on $\mathbb{Z}^2$. Moreover, until now the critical value for site percolation on the square lattice is unknown. Also, other lattices than the square lattice have been studied. Consider the triangular lattice $\mathcal{T}$ and perform site percolation on $\mathcal{T}$. It has been shown (using Kesten's method) that the critical probability for site percolation on the triangular lattice is also 1/2.

Instead of restricting ourselves to lattices we can also perform percolation on random point processes. A natural candidate for this is the Poisson process. Let $\mathcal{P}$ be a Poisson process with intensity λ and consider the associated Voronoi tiling: let x be a Poisson point then the corresponding Voronoi cell $V(x)$ is the closure of the set of points which are closer to x than to any other Poisson point. In this way, we obtain a random tessellation of the plane. Next, declare every point of $\mathcal{P}$ open with probability p, and closed otherwise. Let every point in the Voronoi cell $V(x)$ have the same state as the Poisson point x. This yields a random coloring of the tessellation, see Figure 1.2 for an illustration. This model is called random Voronoi percolation and it was proven by Bollobas and Riordan in [12] that the critical probability for random Voronoi percolation also equals 1/2. We denote the probability measure of the random Voronoi percolation model with $\mathbb{P}_{\lambda,p}$.

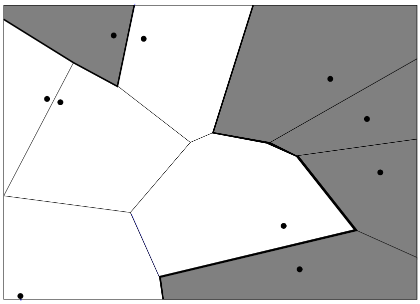


Figure 1.2: A typical Voronoi configuration.

The macroscopic behavior of systems with microscopic random inputs is of great interest to both mathematicians and physicists. A scaling limit is the object which arises as the limit of an "observant" of the random system when the lattice spacing is sent to zero. So by studying the scaling limit we gain information, depending on the observant, about the macroscopic behavior of a random system. We introduce our choice for the observant of percolation and point out our main results in Section 1.2.

Instead of considering lattice models and sending the lattice space to zero we can also study models which are already defined at all smaller scales. Such objects which also exhibit statistical self-similarity are called random fractals. In Section 1.3 the percolation fractals which are studied in this thesis are introduced.

1.2 Scaling limits in percolation

Since 1999 tremendous progress has been made in understanding the macroscopic behavior of site percolation on the triangular lattice. A lot of this progress is based on the celebrated result of Smirnov that critical site percolation on the triangular lattice is conformally invariant [53]. This, loosely speaking, means that crossing probabilities converge to conformally invariant functions as the lattice spacing is sent to zero. However, Smirnov's result depends heavily on the symmetries of the triangular lattice and could therefore not been extended yet to other lattices as well. Therefore, we take site percolation on the triangular lattice as our starting point.

Recall the definition of site percolation on the triangular lattice: each site is declared open with probability p and closed otherwise, independently of other sites. Each site of the triangular lattice can be identified with a face of the hexagonal lattice, the dual lattice of the triangular lattice. An interface is an edge between an open and closed hexagon. A concatination of interfaces is called an interface curve, see Figure 1.3. Note that open and closed clusters are seperated by an interface curve.

Let δ denote the mesh size of the lattice, e.g. the length of the edges of the hexagonal lattice. The probability measure $\mathbb{P}_p$ naturally induces a probability measure $\mu_{\delta,p}$, which denotes the distribution of interface curves, on a suitable space of curves (see Chapter 2 for the precise definition of this space). In this thesis we focus on the distribution of interface curves and therefore take the probability measure $\mu_{\delta,p}$ as our "observant" of the percolation model. A celebrated result by Aizenman and Burchard [2] tells us that for any sequence $\delta_n \to 0$ there exists a subsequence δ_{n_k} along which $\mu_{\delta_{n_k},p}$ converges weakly to a probability measure μ. This probability measure μ is the scaling limit we are interested in.

In Chapter 3 we study the scaling limit when we let p depend on δ in such a way that $p_\delta \to 1/2$ as $\delta \to 0$. We show that, depending on the choice of p_δ, there are

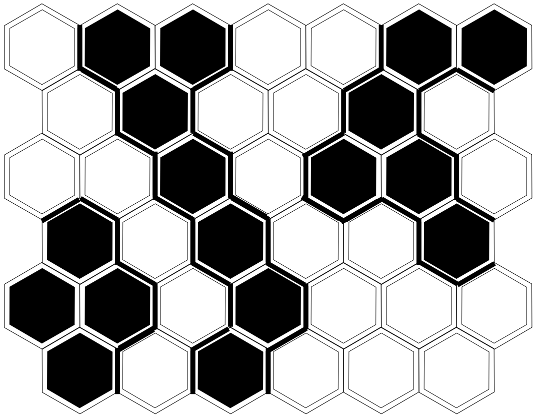


Figure 1.3: Realization of percolation on the hexagonal lattice, where open hexagons are colored black and closed hexagons are colored white. Interfaces are denoted by heavy lines.

three possible scaling limits and we give a qualitative description of these scaling limits.

Main Result 1. *Suppose that μ is the weak limit of a sequence $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$, with $\delta_j \to 0$ as $j \to \infty$ and $p_j \geq 1/2$ for all j. Then one of the following non-void scenarios holds.*

(1) Trivial scaling limit: μ-a.s. there are no loops of diameter larger than zero.
(2) Critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a countably infinite family of nested loops with radii going to zero. Moreover, every point is surrounded by a countably infinite family of nested loops with radii going to infinity.
(3) Near-critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a largest loop and by a countably infinite family of nested loops with radii going to zero.

For the scaling limit of Voronoi percolation we have to employ a somewhat different strategy. Since we do not have a lattice in Voronoi percolation, we take the intensity of the Poisson process as the scaling parameter. Note that the intensity λ is inversely related to the size of Voronoi cells, so sending $\lambda \to \infty$ corresponds with sending the size of Voronoi cells to zero. We denote by μ_λ the distribution of interface curves, which are also defined as the boundaries between black and white clusters, in random Voronoi percolation. Let $H(\rho, 1)$ denote the event that the rectangle $[0,\rho]\times[0,1]$ contains a black path which joins $\{0\}\times[0,1]$ with $\{\rho\}\times[0,1]$. We make the following assumption on critical Voronoi percolation.

$$\text{If } \limsup_{\lambda\to\infty} \mathbb{P}_{\lambda,1/2}(H(\rho,1)) > 0 \text{ for some } \rho > 0,$$
$$\text{then } \liminf_{\lambda\to\infty} \mathbb{P}_{\lambda,1/2}(H(\rho,1)) > 0 \text{ for all } \rho > 0. \quad (\star)$$

Based on this natural assumption we derive the following result in Chapter 4.

Main Result 2. *Assume Condition* $(\star)$ *holds for random Voronoi percolation, then for every sequence* $\lambda_n \to \infty$ *there exists a subsequence* $\lambda_{n_k} \to \infty$ *such that* $\mu := \lim_{k\to\infty} \mu_{\lambda_{n_k}}$ *exists.*

1.3 Random fractals

The following fractal percolation model was introduced by Mandelbrot in [44] and is therefore called Mandelbrot fractal percolation. Consider the unit cube $[0,1]^d$, for $d \geq 2$. Let $N \geq 2$ be an integer and partition the unit cube into N^d subcubes of equal size. Each subcube is retained with probability p and discarded otherwise, independently of other subcubes. This yields a random subset $D_p^1 \subset [0,1]^d$. We repeat the previous procedure in every retained subcube at all smaller scales. This yields an infinite decreasing sequence $(D_p^n)_{n\geq 1}$ of (compact) random subsets of $[0,1]^d$. We are interested in its limit set $D_p = \bigcap_{n=1}^{\infty} D_p^n$. See Figure 1.4 for an illustration of Mandelbrot fractal percolation.

For $p \leq N^{-d}$ it is easy to show [23] that D_p is empty with probability 1. For $p > N^{-d}$ the limit set D_p contains "dust", i.e. totally disconnected points, with positive probability [23]. Denote the set of dust points with D_p^d. It was shown [23, 16] that there exists some $p_c(N,d) < 1$ such that for $p \geq p_c(N,d)$ the random set D_p contains connected components with positive probability. Denote the latter set with D_p^c. It is not hard to show that the Hausdorff dimension of D_p equals $d + \log p/\log N$ with probability 1. Note that D_p is the disjoint union of D_p^c and D_p^d. Let $\dim_B(S)$ denote the Box dimension of a set S and $\dim_{\mathcal{H}}(S)$ the Hausdorff dimension of S. We prove the following in Chapter 5.

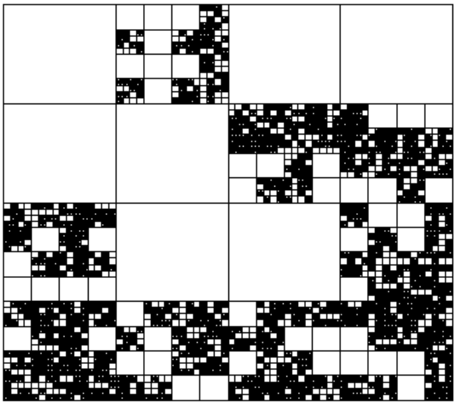


Figure 1.4: Realization of the Mandelbrot fractal percolation model with $N = 4$ until iteration step 3. Retained cubes are colored black and discarded ones white.

Main Result 3. *For* $p_c(N,d) \le p < 1$ *we have* $\dim_B(D_p^c) = \dim_B(D_p)$ *a.s., but* $\dim_{\mathcal{H}}(D_p^c) < \dim_{\mathcal{H}}(D_p)$ *a.s. if* $D_p \neq \emptyset$.

Since Hausdorff dimension can be interpreted as a measure for "fractality" this result implies that the dust set is more fractal than the set of connected components. In addition, we show in dimension $d = 2$ with a scaling limit approach that the set of connected components is a union of non-trivial Hölder continuous curves. To provide a mathematically rigorous formulation of the previous we need quite some technical concepts, but we can also focuse on a particular connected component, e.g. the lowest horizontal crossing of the unit sqare. The result is then easier to formulate.

Main Result 4. *If* D_p *contains a horizontal crossing of the unit square, then the lowest horizontal crossing in* D_p *is a Hölder continuous curve.*

In Chapter 6 we study two modified versions of Mandelbrot's fractal percolation model. An equivalent construction of the Mandelbrot fractal percolation model with parameter p is as follows. We start again by dividing the unit cube $[0,1]^d$ in N^d equal cubes with side length $1/N$. Draw a number R from a binomially distributed random variable with parameters N^d and p. Next, choose R subcubes (out of N^d possible subcubes) in a uniform way and call these cubes retained. Repeat this procedure in every retained cube at every smaller scale. It follows from standard probabilistic arguments that the limit set obtained in this way has the same distribution as D_p. Now, we modify this construction by taking R deterministic, e.g. $R = k$ for an integer k, instead of random. The other parts in the construction remain unchanged: we choose k retained subcubes in a uniform way and repeat this procedure in every retained subcube. We call this k-fractal percolation and denote its limit set with D_k. To the author's knowledge, the first reference to this model is in [25]. Let $k_c(N,d)$ be the critical value, such that for $k \geq k_c(N)$ the unit cube is crossed with positive probability by D_k. Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $x \sim y$ if and only if $|x_i - y_i| \leq 1$ for all i and $x_i = y_i$ for at least one value of i. For $d = 2$, $\mathbb{L}^d$ equals just the square lattice. We prove the following.

Main Result 5. *Let $d \geq 2$. Then*

$$\lim_{N\to\infty} \frac{k_c(N,d)}{N^d} = p_c(d),$$

where $p_c(d)$ is the critical probability for site percolation on $\mathbb{L}^d$.

The second model is called fat fractal percolation, because of the analogy with the fat Cantor set. The fat Cantor set is reminiscent of the ordinary Cantor set in the sense that it is a nowhere dense set, yet has positive Lebesgue measure, as opposed to the Cantor set. The fat fractal percolation model is obtained as follows from Mandelbrot fractal percolation. Instead of a fixed percolation parameter p we retain at iteration step n a cube with probability p_n, where the sequence $\{p_n\}_{n\geq 1}$ is non-decreasing and satisfies $\prod_{n=1}^{\infty} p_n > 0$. Denote the limit set of fat fractal percolation with D_{fat}. Again, we can seperate D_{fat} in two disjoint sets: the set of totally disconnected points D^d_{fat} and the set of connected components D^c_{fat}. Let λ denote d-dimensional Lebesgue measure. We prove the following.

Main Result 6. *Given $D_{fat} \neq 0$, we have that $\lambda(D_{fat}) > 0$ a.s. Moreover, given $D_{fat} \neq \emptyset$, either $\lambda(D^d_{fat}) = 0$ and $\lambda(D^c_{fat}) > 0$ a.s. or $\lambda(D^d_{fat}) > 0$ and $\lambda(D^c_{fat}) = 0$ a.s.*

1.4 List of publications

This thesis is based on the following articles.

- F. Camia, M. Joosten and R. Meester. Trivial, critical and near-critical scaling limits of two-dimensional percolation. *J. Stat. Phys.* 137(1):57–69, 2009.
- E. Broman, F. Camia, M. Joosten and R. Meester. Dimension (in)equalities and Hölder continuous curves in fractal percolation. Submitted, 2010.
- T v.d. Brug, E. Broman, F. Camia, M. Joosten and R. Meester. Fat fractal percolation and k-fractal percolation. Submitted, 2011.
- F. Camia, M. Joosten and R. Meester. Conditional existence of scaling limits in random Voronoi percolation. Work in progress, 2011.

2 Scaling limits: general setup

In this chapter we will provide a detailed account of the Aizenman-Burchard concept [2] of a scaling limit for systems of random curves. We will apply the developed techniques in several other chapters to interface curves in percolation models. There are several options for the definition of a scaling limit of percolation models, see [52] for an extensive list of possibilities. Since we are interested in the behaviour of the collection of interface curves, we follow Aizenman and Burchard [2] and the survey paper by Sun [57] in defining the scaling limit.

2.1 The space of curves

Since we will consider curves in planar percolation models, our setup will be in dimension $d = 2$. However, everything stated and proved in this chapter can easily be generalized to higher dimensions. Let us start by giving a precise definition of a curve. Let $f_1, f_2 : [0,1] \to \mathbb{R}^2$ be two continuous functions and we say that f_1 and f_2 are *the same up to reparametrization*, which we denote by $f_1 \sim f_2$, if there exist increasing homeomorphisms $\varphi, \tilde{\varphi} : [0,1] \to [0,1]$ such that

$$f_2 = f_1 \circ \varphi, \quad f_1 = f_2 \circ \tilde{\varphi}.$$

A *(directed) curve* C is then defined to be an equivalence class modulo $\sim$. The space of curves in a closed subset $D \subset \mathbb{R}^2$ will be denoted $\mathcal{C}_D$. When we study scaling limits of fractal percolation the space $\mathcal{C}_D$ will suffice, since we only consider the unit square in fractal percolation. However, when considering scaling limits of percolation models in $\mathbb{R}^2$ we will have to use a metric on $\mathbb{R}^2$ to make it precompact, as remarked by Aizenman et al. [3] (see also [21]). Define the distance function $\Delta(\cdot,\cdot)$ by

$$\Delta(u,v) := \inf_{\varphi} \int (1+|\varphi(s)|^2)^{-1} ds,$$

where the infimum is over all smooth paths joining u with v, parametrized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded domains, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere. In order to be notationally consistent with Aizenman and Burchard [2] and Sun [57] we focus hereafther on a fixed finite domain D endowed with the Euclidean metric. However, by the previous comments, replacing the Euclidean norm $|\cdot|$ with $\Delta(\cdot,\cdot)$ and D by $\dot{\mathbb{R}}^2$ yields the same results for curves in $\mathbb{R}^2$.

We define a distance function on the space of curves by

$$d_C(C_1,C_2) := \inf_{\varphi}\left(\sup_{0\le t\le 1} |f_1\circ\varphi(t) - f_2(t)|\right),$$

where f_1, resp. f_2, are functions in the equivalence class C_1, resp. C_2, and the infimum is taken over all reparametrizations φ which are increasing homeomorphisms on $[0,1]$.

Lemma 2.1. *The distance function d_C is well-defined and gives a metric on the space of curves.*

Proof. It is obvious that d_C satisfies non-negativity, symmetry and the triangle-inequality. For proofs of $d_C(C,C) = 0$ and $C_1 \neq C_2$ implies $d_C(C_1,C_2) > 0$ we refer the interested reader to [2, Lemma 2.1] or [57, Lemma 5.1]. □

We will refer to d_C as the *uniform metric*, and to the topology it generates as the *uniform topology*. C_D is complete and separable [50], but generally not compact, even for compact D.

For each $\delta > 0$ we are interested in simple polygonal curves of step size δ, so define C_D^δ to be the subspace of such curves in C_D. We refer to δ as the *mesh size* or *short-distance cutoff*. Let Ω_D be the space of closed subsets of C_D and let $\Omega_D^\delta \subset \Omega_D$ denote the subspace of closed sets whose elements are curves in C_D^δ. We define a *δ-configuration of curves* to be an element F_δ of Ω_D^δ. A *system of configurations F* is an assignment $\delta \mapsto F_\delta$ for $0 < \delta \le \delta_0$, for some $\delta_0 < \infty$; we write $F = (F_\delta)_{0<\delta\le\delta_0}$.

Our main objects of interest are *random systems of configurations*, given by an assignment of probability measures $(\mathbb{P}_\delta)_{0<\delta\leq\delta_0}$ on Ω_D, with each $\mathbb{P}_\delta$ supported on Ω_D^δ. We will formalize the notion of the scaling limit as the convergence of the measures $\mathbb{P}_\delta$ to a probability measure on Ω_D as $\delta \to 0$. To this end, we need a metric on Ω_D, and we will use the Hausdorff metric naturally induced by the metric d_C on C_D. Let $\mathcal{K}, \mathcal{K}' \in \Omega_D$ and define $d_H(\mathcal{K}, \mathcal{K}')$ by

$$d_H(\mathcal{K}, \mathcal{K}') \leq \varepsilon \quad \text{iff} \quad \forall C \in \mathcal{K}\, \exists C' \in \mathcal{K}' \text{ such that } d_C(C, C') \leq \varepsilon \text{ and vice versa.}$$

Completeness and separability of Ω_D with respect to the metric d_H follow easily from completeness and separability of C_D.

2.2 Weak convergence

Let $\mathcal{S}$ be a general metric space and let $\mathcal{B} = \mathcal{B}_\mathcal{S}$ be its Borel σ-field. A sequence $\{\mathbb{P}_n\}_{n\geq 1}$ of probability measures on $(\mathcal{S}, \mathcal{B})$ *converges weakly* to the probability measure $\mathbb{P}$ on $(\mathcal{S}, \mathcal{B})$, denoted by $\mathbb{P}_n \Rightarrow \mathbb{P}$, if $\int_\mathcal{S} f d\mathbb{P}_n \to \int_\mathcal{S} f d\mathbb{P}$ for every bounded continuous function $f : \mathcal{S} \to \mathbb{R}$. We refer the interested reader to Billingsley [11] for an excellent account on the theory of weak convergence of probability measures.

A family Π of probability measures is said to be *relatively compact* if every sequence in Π has a weakly convergent subsequence. Our goal is to obtain relative compactness of the family $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$ since this leads directly to our original goal: existence of a subsequential scaling limit. We use Prohorov's theorem to prove relative compactness and therefore we need the following definition. A family Π of probability measures (parametrized by some directed set A) is *uniformly tight* if, for every $\varepsilon > 0$, there is a compact set $K = K(\varepsilon)$ such that $\mu(K) > 1 - \varepsilon$ for all $\mu \in \Pi$. We say that Π is *asymptotically tight* if, for every $\varepsilon > 0$, there is a compact set $K = K(\varepsilon)$ such that $\liminf_{i\in A} \mu_i(K) > 1 - \varepsilon$. Note that when $A = \mathbb{N}$, asymptotically tight and uniformly tight are equivalent. Prohorov's result is the following.

Theorem 2.2. *If a family Π of probability measures is asymptotically tight, then it is relatively compact.*

A proof of this particular version of Prohorov's theorem can be found in [58]. Billingsley's book [11] also provides a nice proof of Prohorov's theorem, but uniform tightness is taken as the condition. And the latter version of Prohorov's theorem is used by Aizenman and Burchard in [2]. However, as we will see later, by using asymptotical tightness as condition in Prohorov's theorem, we can relax the condition for crossing probabilities of random curves. We will now provide the framework to prove asymptotic tightness of the family $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$, under a condition for crossing probabilities of random curves.

The Arzelà-Ascoli theorem (see e.g. [50]) states that if a sequence $\{f_n\}_{n\geq 1}$ of functions from $[0,1]$ to $\mathbb{R}^2$ is uniformly bounded and equicontinuous, then $\{f_n\}$ contains a uniformly convergent subsequence. The Arzelà-Ascoli theorem allows us to characterize compactness in C_D.

Lemma 2.3. *Let K be a closed subset of C_D, where $D \subset \mathbb{R}^2$ is compact. K is compact if the curves in K can be simultaneously parametrized in such a way that the parametrizations are equicontinuous.*

Proof. Direct consequence of the Arzelà-Ascoli theorem. □

The following lemma is a standard exercise (see e.g. Munkres [46, Exercise 45.7]) and its proof is therefore omitted.

Lemma 2.4. *Let (S,d) be a metric space. Let $\mathcal{H}(S)$ denote the collection of all (non-empty) closed, bounded, subsets of S, under the Hausdorff metric d_H induced by d. If (S,d) is compact then $(\mathcal{H}(S), d_H)$ is compact.*

This lemma characterizes compactness in Ω_D.

Corollary 2.5. *Let $\mathcal{K}$ be a closed subset of Ω_D, where $D \subset \mathbb{R}^2$ is compact. $\mathcal{K}$ is compact if the union of all $K \in \mathcal{K}$ is contained in a compact subspace K_0 of C_D.*

Proof. By the hypothesis, $\mathcal{K}$ is a closed subset of the compact space $(\mathcal{H}(K_0), d_H)$, hence compact. □

2.3 Asymptotic tightness of $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$

Our goal is to show that the family of measures $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$ is asymptotically tight under the assumption of a suitable hypothesis. That is, for every $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset \Omega_D$ such that $\mathbb{P}_\delta(\mathcal{K}) > 1 - \varepsilon$ for δ sufficiently small. Suppose that we have, for δ sufficiently small, an equicontinuity bound which holds with probability asymptotically larger than $1-\varepsilon$. By Lemma 2.3 it follows that a closed collection of curves F_δ, whose curves satisfy the equicontinuity bound, is compact for δ sufficiently small. Corolloray 2.5 then yields that the set $\mathcal{K}$ which contains all such F_δ for δ sufficiently small is compact. It follows that $\mathbb{P}_\delta(\mathcal{K}) > 1-\varepsilon$ for δ sufficiently small. Hence, to prove asymptotic tightness an equicontuinty bound which holds with probability asymptotically larger than $1-\varepsilon$ suffices.

This leads to the following definition. A system of random variables $X = (X_\delta)_{0<\delta\leq\delta_0}$, where each X_δ is a random variable with respect to the probability triple $(\Omega, \mathcal{F}_\delta, \mathbb{P}_\delta)$, is called *stochastically bounded* if for every $\varepsilon > 0$ there exists $M < \infty$ such that $\limsup_{\delta\to 0} \mathbb{P}_\delta(|X_\delta| \geq M) < \varepsilon$. A system X is *stochastically bounded away from zero* if X^{-1} is stochastically bounded.

Let X, Y be two metric spaces with respective metric d_X, d_Y. A function $f : X \to Y$ is *Hölder continuous with exponent* $\alpha > 0$

$$d_Y(f(x_1), f(x_2)) \leq c \cdot d_X(x_1, x_2)^\alpha,$$

for all $x_1, x_2 \in X$ and for some $c_\alpha < \infty$. We will prove that under an appropriate hypothesis, for δ sufficiently small the curves $C \in F_\delta$ have simultaneous parametrizations $\gamma : [0,1] \to D$ such that for all $0 \leq t_1, t_2 \leq 1$

$$|\gamma(t_1) - \gamma(t_2)| \leq c_\alpha^\delta |t_1 - t_2|^\alpha, \tag{2.1}$$

where $|\cdot|$ denotes the Euclidean distance and the c_α^δ are stochastically bounded. Our hypothesis will be a bound on crossing probabilities of annuli. Let $x \in D$ and $0 < r \leq R \leq 1$, we define the following annulus:

$$D(x; r, R) := \{y \in D \subset \mathbb{R}^2 : r < |y - x| < R\}.$$

Let $F \subset D$ be a connected component. An annulus $D(x; r, R)$ *is crossed by* F (or *F crosses* $D(x; r, R)$) if F joins the outer boundary of $D(x; r, R)$ with its inner boundary. A *segment* of a curve C is an image $\gamma[a, b]$, where $[a, b] \subset [0, 1]$ and γ is in the equivalence class of C. Two segments are *disjoint* if their intersection is empty.

Hypothesis 2.6. *Let $x \in D$ and let $r, R > 0$ be such that $R/r \geq 2$. There exists a sequence $\{\lambda(k)\}_{k\geq 1}$, with $\lambda(k) \to \infty$ as $k \to \infty$ and each $\lambda(k)$ independent of r and R, such that*

$$\limsup_{\delta\to 0} \mathbb{P}_\delta(D(x; r, R) \textit{ is crossed by } k \textit{ disjoint segments of curves in } F_\delta) \leq c_k \left(\frac{r}{R}\right)^{\lambda(k)}, \tag{2.2}$$

where c_k is a constant depending only on k.

We will show later that Hypothesis 2.6 allows us to prove a bound of the form (2.1). The latter bound gives us the desired tightness result.

Theorem 2.7. *Suppose we have a system $(F_\delta)_{0<\delta\leq\delta_0}$ of random curves given by an assignment $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$ of probability measures on Ω_D, where D is compact. Assume that, for δ sufficiently small, the curves $C \in F_\delta$ have simultaneous parametrizations $\gamma : [0,1] \to D$ such that for all $0 \leq t_1, t_2 \leq 1$*

$$|\gamma(t_1) - \gamma(t_2)| \leq c_\alpha^\delta |t_1 - t_2|^\alpha, \tag{2.3}$$

where the c_α^δ are stochastically bounded. Then the family $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$ is asymptotically tight.

Proof. By definition of stochastic boundedness, for any $\varepsilon > 0$ there exists $M < \infty$ such that

$$\liminf_{\delta\to 0} \mathbb{P}_\delta(c_\alpha^\delta \leq M) > 1 - \varepsilon, \tag{2.4}$$

on a set $\mathcal{K}_\delta \subset \Omega_D^\delta$. For every curve configuration $F_\delta \in \mathcal{K}_\delta$, the curves $C \in F_\delta$ have simultaneous parametrizations $\gamma : [0,1] \to D$ such that for all $0 \leq t_1, t_2 \leq 1$,

$$|\gamma(t_1) - \gamma(t_2)| \leq M |t_1 - t_2|^\alpha. \tag{2.5}$$

Define $\mathcal{K}$ as the set of all curve configurations whose curves satisfy (2.5). Since the bound (2.5) is satisfied uniformly, it follows that $\mathcal{K}$ is compact by Corollary 2.5. Therefore, the family $\{\mathbb{P}_\delta\}_{0<\delta\leq\delta_0}$ is asymptotically tight. □

It remains to prove a bound of the form (2.1). The next subsection is dedicated to this.

2.4 Hölder continuity for curves in random systems

For $\delta, \varepsilon > 0$ and an integer k, let us define the following random variable.

$$r_{\varepsilon,k}^\delta := \inf\left\{0 < r \leq 1 : \begin{array}{c} \exists x \in D \text{ such that } D(x; r^{1+\varepsilon}, r) \text{ is crossed} \\ \text{by } k \text{ disjoint segments of curves in } F_\delta \end{array}\right\}. \tag{2.6}$$

If no such k crossings occur we set $r_{\varepsilon,k}^\delta = 1$. We show that, under Hypothesis 2.6, for every $\varepsilon > 0$ there exists an integer k such that the collection $\{r_{\varepsilon,k}^\delta\}_{0<\delta\leq\delta_0}$ of random variables is stochastically bounded away from zero.

Lemma 2.8. *Let $\varepsilon > 0$. Under Hypothesis 2.6, there exists an integer k so that the collection of random variables $\{r^{\delta}_{\varepsilon,k}\}_{0<\delta\leq\delta_0}$ stays stochastically bounded away from zero as $\delta \to 0$.*

Proof. Let $\varepsilon > 0$ be given. Our task is to find an integer k such that $\{r^{\delta}_{\varepsilon,k}\}_{0<\delta\leq\delta_0}$ stays stochastically bounded away from zero. That is, for $\varepsilon' > 0$ we have to choose $u = u(\varepsilon')$ such that

$$\limsup_{\delta\to 0} \mathbb{P}_{\delta}(r^{\delta}_{\varepsilon,k} \leq u) < \varepsilon'. \tag{2.7}$$

Therefore, let k be an integer and u be positive (both to be chosen later) and consider the event $\{r^{\delta}_{\varepsilon,k} \leq u\}$. By definition, if $\{r^{\delta}_{\varepsilon,k} \leq u\}$ occurs, there exist x in D and $r \leq u$ such that the annulus $D(x; r^{1+\varepsilon}, r)$ is crossed by k disjoint segments of curves in F_{δ}. We can not apply Hypothesis 2.6 directly to the former event, since there are uncountable many possibilities for x and r. Therefore, we have to find a countable set of annuli such that if $\{r^{\delta}_{\varepsilon,k} \leq u\}$ occurs at least one annulus of this set is crossed by k disjoint curves. To this end, take u so small that

$$\frac{2^{-(2+\varepsilon)}}{6} u^{-\varepsilon} \geq 2, \tag{2.8}$$

the reason for this particular choice will become clear later. Write $r_n = 2^{-n}$ and choose n so that $r_{n+1} < r \leq r_n$. Since $r \leq u$, it follows that

$$\frac{r_{n+1}}{r_n^{1+\varepsilon}} = \frac{1}{2\cdot 2^{1+\varepsilon}} \frac{r_n}{r_{n+1}^{1+\varepsilon}} > 2^{-(2+\varepsilon)} \frac{r}{r^{1+\varepsilon}} = 2^{-(2+\varepsilon)} r^{-\varepsilon} \geq 1,$$

by (2.8). Hence, the annulus $D(x; r_n^{1+\varepsilon}, r_{n+1})$ is well-defined. Observe that, if the annulus $D(x; r^{1+\varepsilon}, r)$ is crossed by k disjoint segments of curves in F_{δ}, then this is necessarily also the case for the thinner annulus $D(x; r_n^{1+\varepsilon}, r_{n+1})$.

Next, define a finite set of annuli by $A_i = D(x_i; 3r_n^{1+\varepsilon}, r_{n+1}/2)$, with $x_i \in (\frac{4r_n^{1+\varepsilon}}{\sqrt{2}})\mathbb{Z}^2$. Note that, by our choice for u in (2.8), it is the case that the ratio between the outer radius and inner radius of each A_i is at least 2. In addition, if $\{r^{\delta}_{\varepsilon,k} \leq u\}$ occurs, then at least one of the annuli A_i is crossed by k disjoint segments of curves in F_{δ}. The number of such annuli A_i in D is bounded above by $c\left(\frac{\sqrt{2}}{4r_n^{1+\varepsilon}}\right)^2$, where $c < \infty$ is some absolute constant. Now, summing over all possibilities for the annuli A_i, Hypothesis 2.6 yields

$$\begin{aligned}
&\limsup_{\delta\to 0} \mathbb{P}_\delta \begin{pmatrix} D(x; r^{1+\varepsilon}, r) \text{ is crossed by } k \text{ disjoint} \\ \text{segments of curves in } F_\delta \text{ for } r \in (r_{n+1}, r_n] \end{pmatrix} \\
&\le c \left(\frac{\sqrt{2}}{4 r_n^{1+\varepsilon}} \right)^2 c_k \left(\frac{3 r_n^{1+\varepsilon}}{r_n/4} \right)^{\lambda(k)} \\
&\le C_k r_n^{\varepsilon\lambda(k) - 2(1+\varepsilon)}.
\end{aligned}$$

Next, take k so large that the exponent on r_n is positive. We get

$$\begin{aligned}
&\limsup_{\delta\to 0} \mathbb{P}_\delta (r^\delta_{\varepsilon,k} \le u) \\
&\le \limsup_{\delta\to 0} \sum_{n:\delta\le r_n\le u} \mathbb{P}_\delta \begin{pmatrix} D(x; r^{1+\varepsilon}, r) \text{ is crossed by } k \text{ disjoint} \\ \text{segments of curves in } F_\delta \text{ for } r \in (r_{n+1}, r_n] \end{pmatrix} \\
&\le \sum_{n:0< r_n\le u} \limsup_{\delta\to 0} \mathbb{P}_\delta \begin{pmatrix} D(x; r^{1+\varepsilon}, r) \text{ is crossed by } k \text{ disjoint} \\ \text{segments of curves in } F_\delta \text{ for } r \in (r_{n+1}, r_n] \end{pmatrix} \\
&\le \sum_{n:0< r_n\le u} C_k r_n^{\varepsilon\lambda(k) - 2(1+\varepsilon)} \\
&\le u^{\varepsilon\lambda(k) - 2(1+\varepsilon)}.
\end{aligned}$$

Hence, for any $\varepsilon' > 0$ we choose u sufficiently small such that (2.7) holds. □

Let us summarize our results so far. Theorem 2.7 states that if a system of random curves, given by an assignment $\{\mathbb{P}_\delta\}_{0<\delta\le\delta_0}$, satisfies a bound of the form (2.1) the family of probability measures $\{\mathbb{P}_\delta\}_{0<\delta\le\delta_0}$ is asymptotically tight. Prohorov's theorem gives that symptotic tightness of a family of probability measures implies relative compactness of the family (i.e. existence of scaling limit(s) along subsequences). We have just shown that Hypothesis 2.6 implies that for every $\varepsilon > 0$ there exists $k < \infty$ such that the random variable $r^\delta_{\varepsilon,k}$ is stochastically bounded away from zero as $\delta \to 0$. The missing link between the former result and Theorem 2.7 is thus a result which states that if the random variable $r^\delta_{\varepsilon,k}$ is stochastically bounded away from zero as $\delta \to 0$, then the system of random curves satisfies a bound of the form (2.1). This can be formulated as the followinh theorem. We prove it in the next subsection, which correspond to Section 5.4 in [57].

Theorem 2.9. *Let F be a system of random curves such that Hypothesis 2.6 is satisfied. Then for each $\varepsilon > 0$ all the curves $C \in F^\delta(\omega)$ can be simultaneously parametrized by continuous functions $\gamma : [0,1] \to D$ such that*

$$|\gamma(t_1) - \gamma(t_2)| \le \kappa^\delta_\varepsilon(\omega) g(diam(C))^{1+\varepsilon} |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}}, \tag{2.9}$$

where the random variable k_ε^δ *stays stochastically bounded as* $\delta \to 0$, *and* $g(r) = r^{-\frac{\lambda(1)}{2-\lambda(1)}}$.

Note that this is not yet a bound of the form (2.1), since $g(r) \uparrow \infty$ as $r \downarrow 0$. However, we also have the trivial bound $|\gamma(t_1) - \gamma(t_2)| \leq \mathrm{diam}(D)$. We can interpolate between this bound and (2.9) to obtain a bound independent of diam(C):

$$\begin{aligned}|\gamma(t_1) - \gamma(t_2)| &\leq \kappa_\varepsilon^\delta(\omega) g(\mathrm{diam}(C))^{1+\varepsilon} |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}} \\ &\leq \kappa_\varepsilon^\delta(\omega) g(|\gamma(t_1) - \gamma(t_2)|)^{1+\varepsilon} |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}} \\ &= \kappa_\varepsilon^\delta(\omega) |\gamma(t_1) - \gamma(t_2)|^{\frac{-\lambda(1)(1+\varepsilon)}{2-\lambda(1)}} |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}}.\end{aligned}$$

Hence,

$$|\gamma(t_1) - \gamma(t_2)|^{1+\frac{\lambda(1)(1+\varepsilon)}{2-\lambda(1)}} \leq \kappa_\varepsilon^\delta(\omega) |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}}.$$

Thus, we finally obtain

$$\begin{aligned}|\gamma(t_1) - \gamma(t_2)| &\leq (\kappa_\varepsilon^\delta)^{\frac{2-\lambda(1)}{2+\varepsilon\lambda(1)}} |t_1 - t_2|^{\frac{2-\lambda(1)}{(2-\lambda(1)+\varepsilon)(2+\varepsilon\lambda(1))}} \\ &\leq \hat{\kappa}_\varepsilon^\delta(\omega) |t_1 - t_2|^{\frac{1}{2+\varepsilon\lambda(1)}},\end{aligned}$$

where $\hat{\kappa}_\varepsilon^\delta$ stays stochastically bounded as $\delta \to 0$. This is a bound of the form (2.1) we had in mind. The following theorem, which is our main result, follows now immediately from Theorem 2.7

Theorem 2.10. *Assume that a system of random curves in a compact set* $D \subset \mathbb{R}^2$ *satisfies Hypothesis 2.6. Then, for every sequence* $\delta \to 0$ *there exists a subsequence* $\delta_n \to 0$ *such that* $\mathbb{P}_{\delta_n}$ *converges to some probability measure* $\mathbb{P}$ *on* Ω_D.

2.5 Proof of Theorem 2.9

Define $M(C, l)$ as the minimal number of segments needed for a partition of the curve C into segments of diameter at most l. A power bound on $M(C, l)$, that is $M(C, l) \leq c l^{-s}$ as $l \to 0$ for some $c, s > 0$, will enable us to prove Hölder continuity of the curve C. This follows from the following lemma.

Lemma 2.11. *If the curve C satisfies the bound*

$$M(C, l) \leq 1/\psi(l), \tag{2.10}$$

for all $0 < l \leq 1$, *then C can be parametrized by some* $\gamma : [0, 1] \to \mathbb{R}^d$ *such that*

$$\tilde{\psi}(|\gamma(t_1) - \gamma(t_2)|) \leq |t_1 - t_2| \tag{2.11}$$

whenever $|\gamma(t_1) - \gamma(t_2)| \leq 1$, *with* $\tilde{\psi} : (0, 1] \to (0, 1]$ *strictly increasing.*

Proof. Given $M(C,l) \leq 1/\psi(l)$, we need to construct a parametrization such that (2.11) holds. Choose an auxiliary parametrization $\gamma_0(s)$ for C, and define C_s to be the curve corresponding to $\gamma_0[0,s]$. Let $l_n = 2^{-n}$, and set

$$\tilde{t}(C) = \sum_{n=0}^{\infty} \frac{\psi(l_n)M(C,l_n)}{(n+1)^2}, t(s) = \frac{\tilde{t}(C_s)}{\tilde{t}(C)}.$$

It is easy to see that t is a strictly increasing right-continuous function going from $[0,1]$ to $[0,1]$. We therefore define $s(t)$ to be the generalized inverse of $t(s)$, so that s is continuous, and reparametrize by $\gamma(t) = \gamma_0(s(t))$.

Notice that by assumption we have $\tilde{t}(C) \leq \sum_{n=0}^{\infty}(n+1)^{-2} < 2$. Now suppose $t_1 < t_2$ with $\gamma(t_1) - \gamma(t_2)| \geq l$. Write $s_i = s(t_i)$, so $\gamma(t_i) = \gamma_0(s_i)$. Then $M(C_{s_2}, l_n) - M(C_{s_1}, l_n) \geq 1$ for all $l_n < l$. Hence,

$$\begin{aligned}
t_2 - t_1 &\geq \frac{\sum_{n=0}^{\infty} \frac{\psi(l_n)M(C_{s_2},l_n)}{(n+1)^2} - \sum_{n=0}^{\infty} \frac{\psi(l_n)M(C_{s_1},l_n)}{(n+1)^2}}{\tilde{t}(C)} \\
&\geq \frac{1}{2} \sum_{n:l/2<l_n<l} \frac{\psi(l_n)(M(C_{s_2},l_n) - M(C_{s_1},l_n))}{(n+1)^2} \\
&\geq \frac{\psi(l/2)}{2} \sum_{n=\log_2(1/l)}^{\log_2(2/l)} \frac{1}{(n+1)^2} \\
&\geq \frac{\psi(l/2)(\log_2(2/l) - \log_2(1/l)}{2(\log_2(2/l)+1)^2} \\
&\geq \frac{\psi(l/2)}{2(\log_2(4/l))^2} =: \tilde{\psi}(l),
\end{aligned}$$

as desired. □

However, dealing with $M(C,l)$ directly tends to be difficult. Therefore, we relate $M(C,l)$ to another notion of roughness which we can handle. Let $N(C,l)$ be the minimal number of sets of diameter l needed to cover the trace of C. $N(C,l)$ satisfies the following nice property. For $r \leq l$ we have

$$N(c,r) \leq N(c,l)\left\lceil \frac{l}{r} \right\rceil^2. \tag{2.12}$$

Let $B_r(x)$ denote the Euclidean ball with radius $r > 0$ centered at $x \in \mathbb{R}^2$. We have the following deterministic result.

Lemma 2.12. *Consider a curve C and let $\varepsilon > 0$. If there is no shell $D(x; (l/2)^{1+\varepsilon}, l/2)$ that contains k disjoint segments of C then*

$$M(C, 2l) \leq kN(C, 2(l/2)^{1+\varepsilon}). \tag{2.13}$$

Proof. We partition the curve by the sequence $x_0, x_1, \ldots$ as follows: set $t_0 = 0$ and $x_0 = \gamma(t_0)$ (where γ is a representative for C), and recursively define $t_{i+1} = \inf\}t \geq t_i : \gamma(t) \notin B_l(x_i)\}$ and define $x_{i+1} = \gamma(t_{i+1})$. Thus x_{t+1} is the first point (if any) that leaves the ball with radius l at x_i. If γ never leaves $B_l(x_i)$, we terminate by setting $x_{i+1} = \gamma(1)$.

The segments of this partition must have diameter at most $2l$, so the number of segments is an upper bound for $M(C, 2l)$. Consider any covering of C by balls of radius $(l/2)^{1+\varepsilon}$: since C does not cross any annulus $D(x; (l/2)^{1+\varepsilon}, l/2)$ for at least k times and $|x_i - x_{i+1}| \geq l$ for each i, each ball can contain at most k of points x_i. Therefore $M(C, 2l) \leq N(C, 2(l/2)^{1+\varepsilon})$. □

Lemma 2.8 gives us a stochastic bound of $M(C, l)$ in terms of $N(C, l)$.

Lemma 2.13. *Suppose $0 < \varepsilon < 1$. Under Hypothesis 2.6, choose k as in Lemma 2.8 such that $r^{\delta}_{\varepsilon,k}$ is stochastically bounded away from zero as $\delta \to 0$. We have*

$$M(C, 2l) \leq A^{\delta}_{\varepsilon} N(C, l^{1+\varepsilon}),$$

for all $C \in F^{\delta}(\omega)$, where the random variable A^{δ}_{ε} remains stochastically bounded as $\delta \to 0$.

Proof. It follows from Lemma 2.13 that for $l < r^{\delta}_{\varepsilon,k}(\omega)$ we have $M(C, 2l) \leq 2^d kN(C, l^{1+\varepsilon})$ for all $C \in F^{\delta}(\omega)$. For $l \geq r^{\delta}_{\varepsilon,k}(\omega)$, we use the fact that $M(C, l)$ increases as l decreases,so for all $C \in F^{\delta}(\omega)$ we have, for any $r < r^{\delta}_{\varepsilon,k}(\omega)$,

$$M(C, 2l) \leq M(C, 2r) \leq 2^d kN(C, r^{1+\varepsilon}) \leq 2^d k \left[\left(\frac{l}{r} \right)^{1+\varepsilon} \right]^d N(C, l^{1+\varepsilon}).$$

By taking $r = (1 - \eta) r^{\delta}_{\varepsilon,k}(\omega)$ for small $\eta > 0$, the multiplying factor of $N(C, l^{1+\varepsilon})$ in this last expression remains stocahstically bounded as $\delta \to 0$ by Lemma 2.8. □

We proceed by finding a power bound on $N(C, l^{1+\varepsilon})$. First, we need some more definitions. Recall that we work in the domain D and for each l, let $\Pi_l(D)$ be a fixed partition (e.g. the standard grid partition) of D into sets of diameter at most l, and let $\tilde{N}^{\delta}(r, l)$ denote the number of sets $B \in \Pi_l(D)$ meeting some curve in F^{δ} of diameter at least r, where $r \geq 2l$. Clearly, $N(C, l^{1+\varepsilon}) \leq \tilde{N}(\mathrm{diam}(C), l^{1+\varepsilon})$, so it suffices to prove a bound on the latter. We have the following bound on its expectation.

Lemma 2.14. *Under Hypothesis 2.6 we have, for $r \geq 2l$,*

$$\mathbb{E}_\delta \tilde{N}^\delta(r,l) \leq c r^{-\lambda(1)} l^{-(2-\lambda(1))},$$

where c is some constant independent of r and l.

Proof. The number of sets in $\Pi_l(D)$ grows like $1/l^2$. For each $B \in \Pi_l(D)$ let $x = x_B$ be a point such that $B \subset B_{l/2}(x)$. If $B \in \Pi_l(D)$ meets a curve of diameter larger than r, we must have a crossing of the spherical shell $D(x_B; l/2, r/2)$, which by Hypothesis 2.6 occurs with probability at most $c_1(l/r)^{\lambda(1)}$. Summing over all $B \in \Pi_l(D)$ gives the result. □

We reformulate Theorem 2.9 in the following way.

Lemma 2.15. *Consider a system of random curves such that Hypothesis 2.6 is satisfied. Then for each $\varepsilon > 0$ all the curves $C \in F^\delta(\omega)$ satisfy the bound*

$$M(C,l) \leq \frac{\tilde{\kappa}^\delta_\varepsilon(\omega)}{(diam(C))^{\lambda(1)+\varepsilon}} \frac{1}{l^{2-\lambda(1)+\varepsilon}}, \tag{2.14}$$

where $\tilde{\kappa}^\delta_\varepsilon$ stays stochastically bounded as $\delta \to 0$.

The equivalence of Theorem 2.9 and (2.14) follows from Lemma 2.11. For a full proof of this equivalence we refer the reader to [57, Appendix C], since this consists of a quite messy computation

Proof of Lemma 2.15. Let $C \in F^\delta$ and $r = \text{diam}(C)$. As discussed above,

$$N(C,l) \leq \tilde{N}^\delta(r,l) = \left(\frac{\tilde{N}^\delta(r,l)}{\mathbb{E}\tilde{N}^\delta(r,l)} \right) \mathbb{E}\tilde{N}^\delta(r,l).$$

Trivially, the first factor in parentheses has expectation 1, so by the Markov inequality it is stochastically bounded for each fixed C, l and we can use Hypothesis 2.6 to bound $\mathbb{E}\tilde{N}^\delta(r,l)$ by Lemma 2.14. However, since we are interested in letting $l \to 0$ for all curves C in the random system, this does not suffice. To obtain a bound for all C, l we sum over scales $r = \text{diam}(C)$ and $l \leq r/2$ as follows: define the random variable:

$$U^\delta := \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \sum_{m: l_m < r_n/2} \frac{1}{(m+1)^2} \frac{\tilde{N}^\delta(r_n, l_m)}{\mathbb{E}\tilde{N}^\delta(r_n, l_m)}.$$

Again by the Markov inequality this is stochastically bounded as $\delta \to 0$. Writing $r_n = l_n = 2^{-n}$, we choose n, m so that $r_{n+1} < r \le r_n, l_{m+1} < l \le l_m$. Since $\tilde{N}^\delta$ decreases as r and l increase, we have

$$\begin{aligned}\tilde{N}^\delta(r,l) &\le \tilde{N}^\delta(r_{n+1}, l_{m+1}) \le (n+2)^2(m+2)^2 U^\delta \mathbb{E}\tilde{N}^\delta(r_{n+1}, l_{n+1}) \\ &\le (\log_2(4/r))^2(\log_2(4/l))^2 U^\delta \left(\frac{c}{(r/2)^{\lambda(1)}}\frac{1}{(l/2)^{2-\lambda(1)}}\right)\end{aligned} \tag{2.15}$$

Observe that $(\log(4/r))^2 \le r^{-\varepsilon'}$ and $(\log(4/l))^2 \le l^{-\varepsilon'}$ for every $\varepsilon' > 0$. Therefore, choosing k as in Lemma 2.8, for all $C \in F^\delta$ we have, by Lemma 2.13 and (2.15),

$$\begin{aligned}M(C, 2l) &\le A_\varepsilon^\delta \tilde{N}^\delta(r, l^{1+\varepsilon}) \\ &\le cA_\varepsilon^\delta U^\delta \frac{(\log_2(4/r))^2(\log_2(4/l))^2}{r^{\lambda(1)} l^{(1+\varepsilon)(2-\lambda(1))}} \\ &\le \tilde{\kappa}_\varepsilon^\delta \left(\frac{1}{r}\right)^{\lambda(1)+\varepsilon'} \left(\frac{1}{l}\right)^{(1+\varepsilon)(2-\lambda(1))+\varepsilon'},\end{aligned}$$

for any $\varepsilon' > 0$, where $\tilde{\kappa}_\varepsilon^\delta$ is a product of stochastically bounded random variables, hence stochastically bounded. The desired bound (2.14) follows by adjusting $\varepsilon, \varepsilon'$. □

3 Trivial, critical and near-critical scaling limits of two-dimensional percolation

3.1 Introduction and main results

In Bernoulli site (respectively, bond) percolation, the sites (resp., bonds) of a regular lattice with lattice spacing δ are colored black with probability p and white otherwise, independently of each other. One is then interested in the connectivity properties of the monochromatic subgraphs of the lattice, called *clusters* (see, e.g., [13, 36, 39]).

The rigorous geometric analysis of the continuum scaling limit ($\delta \to 0$) of two-dimensional critical site percolation on the triangular lattice has made tremendous progress in recent years. In particular, the work of Schramm [51] and Smirnov [53] has allowed to identify the scaling limit of critical interfaces (i.e., boundaries between black and white clusters) in terms of the Schramm-Loewner Evolution (SLE) (see also [22, 54]). Based on that, Camia and Newman have constructed [20] a process of continuum nonsimple loops in the plane, and proved [21] that it coincides with the scaling limit of the collection of all percolation interfaces (the *full* scaling limit). The use of SLE technology and computations, combined with Kesten's scaling relations [40], has also led to the derivation of important properties of percolation such as the values of some critical exponents [42, 56].

In later work [18, 19], based on heuristic arguments, Camia, Fontes and Newman have proposed an approach for obtaining a one-parameter family of *near-critical* scaling limits with density of black sites (or bonds) given by

$$p = p_c + \lambda \delta^{\alpha}, \tag{3.1}$$

where p_c is the critical density, δ is the lattice spacing, $\lambda \in (-\infty, \infty)$, and α is set equal to 3/4 to get nontrivial λ-dependence in the limit $\delta \to 0$ (see below and [1, 2, 14]). The approach proposed in [18, 19] is based on the critical full scaling limit and the "Poissonian marking" of some special ("macroscopically pivotal") points, and it leads to a conceptual framework that can in principle describe not only the scaling limit of near-critical percolation but also of related two-dimensional models such as dynamical percolation, the minimal spanning tree and invasion percolation (see [19]).

In this chapter, we consider the collection of all percolation interfaces and show how one can use known scaling relations for percolation to prove that, besides the trivial scaling limit corresponding to subcritical and supercritical percolation, there are only two other alternatives, that we call *critical* and *near-critical* scaling limits, and for which we give a geometric characterization.

We postpone precise definitions till Sections 3.2 and 3.3 (we refer to Chapter 2 for definitions concercing the space of interface curves and the topology of weak convergence), but in order to avoid delaying the statement of the main result, we present it here. We denote by $\mathbb{P}_{\delta,p}$ the probability measure corresponding to Bernoulli site percolation on the triangular lattice with lattice spacing δ and parameter p. It is well known [39] that percolation on the triangular lattice has a phase transition at $p = 1/2$. Let $H^w_\delta(n)$ denote the event that there is a black horizontal crossing in a "box" of Euclidean side length $n\delta$ on the lattice with lattice spacing δ (see Figure 3.2 and the next section for precise definitions). Due to the black/white symmetry of the model, without loss of generality, we can restrict our attention to the case $p \geq 1/2$. For $\varepsilon \in (0, 1/2)$, we define

$$p^+_\varepsilon(n) := \inf\{p : \mathbb{P}_{\delta,p}(H^w_\delta(n)) > 1/2 + \varepsilon\}.$$

(Note that $p^+_\varepsilon(n)$ is independent of δ, $p^+_\varepsilon(n) \geq 1/2 \ \forall \varepsilon \in (0, 1/2)$ and $p^+_{\varepsilon_1}(n) \leq p^+_{\varepsilon_2}(n)$ if $\varepsilon_1 \leq \varepsilon_2$.) Let $\mu_{\delta,p}$ denote the distribution of all percolation interfaces for site percolation with parameter p on the triangular lattice with lattice spacing δ.

Theorem 3.1. *Suppose that μ is the weak limit of a sequence $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$, with $\delta_j \to 0$ as $j \to \infty$ and $p_j \geq 1/2$ for all j. Then one of the following non-void scenarios holds.*

(1) Trivial scaling limit: μ-a.s. there are no loops of diameter larger than zero.

(2) Critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a countably infinite family of nested loops with radii going to zero. Moreover, every point is surrounded by a countably infinite family of nested loops with radii going to infinity.

(3) Near-critical scaling limit: μ-a.s. any deterministic point in the plane is surrounded by a largest loop and by a countably infinite family of nested loops with radii going to zero.

Moreover, the third scenario can be realized by taking $0 < \varepsilon_1 < \varepsilon_2 < 1/2$ *and (an appropriate subsequence of)* $\{p_j\}_{j\in\mathbb{N}}$ *chosen so that* $p^+_{\varepsilon_1}(1/\delta_j) \leq p_j \leq p^+_{\varepsilon_2}(1/\delta_j)$ *for every* j*.*

The above geometric characterization of near-critical scaling limits, case *(3)*, was conjectured in [18]. It shows that such limits are not scale invariant and differ qualitatively from the critical scaling limit at large scales, since in the latter case there is no largest loop around any point. At the same time, they resemble the critical scaling limit at short scales because of the presence, around any given point, of infinitely many nested loops with radii going to zero. Depending on the context, this situation is also described as *off-critical* or *massive* scaling limit (where "massive" refers to the persistence of a macroscopic correlation length, which should give rise to what is known in the physics literature as a "massive field theory").

The three regimes in Theorem 3.1 correspond to those in Proposition 4 of [48], which contains, among other things, results analogous to some of ours in the context of a single percolation interface and its scaling limit. Perhaps the most interesting results of [48] and of this chapter concern the near-critical regime (regime *(3)* of Theorem 3.1). While [48] deals with a single interface, proving that its scaling limit in the near-critical regime is singular with respect to SLE_6 (the critical scaling limit), in this chapter we consider the full scaling limit and are concerned with the geometry of the set of all interfaces, so that, in some sense, our result on the near-critical regime complements that of [48].

Our results imply that when $\alpha < 3/4$ in (3.1), the full scaling limit is trivial, when $\alpha > 3/4$ it is critical, and there is a non-empty regime where it is neither trivial nor critical. The following corollary is an immediate consequence of Theorem 3.1 and the power law (3.2) given at the end of the next section.

Corollary 3.2. *Consider site percolation on the triangular lattice with lattice spacing* δ *and parameter* $p = 1/2 + \lambda\delta^\alpha$*. Then, for every* $\lambda \in (-\infty, \infty)$*,*

- *if* $\alpha < 3/4$*, there is a unique scaling limit which is trivial in the sense of Theorem 3.1,*
- *if* $\alpha > 3/4$*, every subsequential scaling limit is critical in the sense of Theorem 3.1.*

It is natural to conjecture that the near-critical regime (case *(3)* in Theorem 3.1) corresponds to the case $\alpha = 3/4$, but at the moment this is not known. In order to prove that, one would need to show that when $\alpha = 3/4$ the correlation length $L_\varepsilon(p)$ defined in Section 3.2 below remains bounded away from zero and infinity as $\delta \to 0$ (see the proof of case *(3)* of Theorem 3.1). This is believed to be the case, and in fact the correlation length is expected to follow the power law $L_\varepsilon(p) \asymp |p - 1/2|^{-4/3}$ as

$p \to 1/2$, where $\asymp$ means that the ratio between the two quantities is bounded away from zero and infinity. However, only the weaker power law $L_\varepsilon(p) = |p-1/2|^{-4/3+o(1)}$ has been proved [56].

In a remark at the end of Section 3.4, we explain how one can combine [21] with (part of) the proof of Proposition 4 of [48] to obtain a (much) stronger version of case *(2)* of Theorem 3.1 (namely, that the scaling limit in regime *(2)* coincides with the critical full scaling limit [20, 21]). In view of this result, in the second item of Corollary 3.2, one can identify the scaling limit with $\alpha > 3/4$ with the unique critical scaling limit [20, 21].

It is our understanding that significant progress has recently been made [35] (see also [34, 33]) in proving the approach of [18, 19] to near-critical scaling limits. A consequence would be that *all* subsequential limits discussed in this chapter are in fact limits.

To conclude this section, we point out that, although our results are stated for site percolation on the triangular lattice, except for Corollary 3.2 which relies on the power law (3.2) and Remark 3.10 which relies on results from [21, 22, 48], they also apply to bond percolation and to other regular lattices like the square lattice (after replacing 1/2 with p_c when necessary). Indeed, the main tools in our proofs originated in Kesten's work [40] on the square lattice and can be used for both site and bond percolation models on a large class of lattices (see [39]). For a discussion of the range of applicability of Kesten's and related results, and consequently of the results of the present chapter, the reader is referred to Section 8.1 of [47].

3.2 Notation and some background

Consider the hexagonal lattice $\mathcal{H}_\delta$ with lattice spacing $\delta > 0$, and its dual, the triangular lattice $\mathcal{T}_\delta$, embedded in $\mathbb{R}^2$ as in Figure 3.1. A site of the triangular lattice is identified with the face of the hexagonal lattice that contains it.

Throughout this chapter, we are interested in Bernoulli site percolation on $\mathcal{T}_\delta$, defined as follows. Each site of $\mathcal{T}_\delta$ is independently declared black, and the corresponding hexagon colored black, with probability p. Sites that are not black are declared white, and the corresponding hexagons are colored white. We denote by $\mathbb{P}_{\delta,p}$ the probability measure corresponding to site percolation on $\mathcal{T}_\delta$ with parameter p. It is well known [39] that percolation on the triangular lattice has a phase transition at $p = 1/2$.

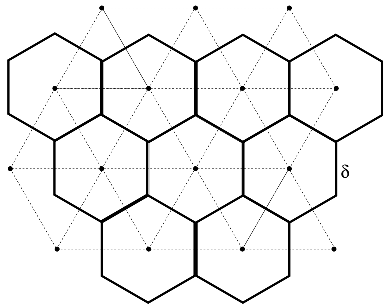


Figure 3.1: Embedding of the triangular and hexagonal lattices in $\mathbb{R}^2$.

A *path* of length n in $\mathcal{T}_\delta$ is a sequence of n distinct sites $(x_1, x_2, \dots, x_n)$ of $\mathcal{T}_\delta$ and the edges of $\mathcal{T}_\delta$ between them such that x_k and x_{k+1} are adjacent in $\mathcal{T}_\delta$ for all $k = 1, \dots, n-1$. A *circuit* of length n is a path whose first and last sites are adjacent. We define the diameter of a set $U \subset \mathbb{R}^2$ as

$$\operatorname{diam}(U) := \sup\{|x - y| : x, y \in U\},$$

where $|\cdot|$ denotes Euclidean distance. We call a path or a circuit black (resp., white) if all its sites are black (resp., white).

The edges between neighboring hexagons with different colors form *interfaces*. A concatenation of such edges will be called a *boundary path* or a *boundary loop* if it forms a closed curve. Note that boundary curves and loops are always simple (i.e., no self-touching occurs) for $\delta > 0$. However, this will not necessarily be the case in the scaling limit $\delta \to 0$.

For $n_1, n_2 > 0$, $[0, n_1] \times [0, n_2]$ will denote the closed parallelogram with Euclidean side-lengths n_1 and n_2 and sides which are parallel to two of the axes of the triangular lattice as in Figure 3.2. In particular, when $n_1 = n_2$, we call such a parallelogram a *box*. $B(x; r)$ will denote the box centered at x, obtained by translating $[0, r] \times [0, r]$ (see Figure 3.2). For $0 < r < R$, we define the *annulus* $A(x; r, R)$ as

$$A(x; r, R) := B(x; R) \setminus B^\circ(x; r),$$

where $B^\circ(x; r)$ denotes the interior of $B(x; r)$. When x is the origin, we will write $B(r)$ and $A(r, R)$, respectively. Note that boxes and annuli are defined in terms of the Euclidean metric and not relative to the lattice spacing.

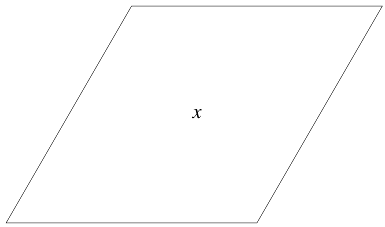


Figure 3.2: The box $B(x;r)$.

The notion of *correlation length* will be very important. Various equivalent definitions are possible; we choose the one, introduced in [24] and also used in [40], that is most suitable for our purposes. Let n be an integer and $H^w_\delta(n)$ be the event that a percolation configuration on $\mathcal{T}_\delta$ contains a black path inside $B(n\delta)$ intersecting both its "left side" and its "right side." For each $\varepsilon \in (0,1/2)$, the correlation length $L_\varepsilon(p)$ is defined as follows:

$$L_\varepsilon(p) := \min\{n : \mathbb{P}_{\delta,p}(H^w_\delta(n)) > 1/2 + \varepsilon\} \text{ when } p > 1/2,$$
$$L_\varepsilon(p) := \min\{n : \mathbb{P}_{\delta,p}(H^w_\delta(n)) < 1/2 - \varepsilon\} \text{ when } p < 1/2.$$

We also define $L_\varepsilon(1/2) = \infty$ for all $\varepsilon \in (0,1/2)$. Note that in the definition above, the correlation length is measured in lattice spacings (rather than in the Euclidean metric), and is therefore independent of δ. Below we will frequently make use of the *scaled* correlation length $\delta L_\varepsilon(p)$, which can be seen as the "macroscopic" correlation length.

An important fact about the correlation length is that the ε in the definition is unimportant, due to the following result [47] (a weaker version is proved in [40]): for any $\varepsilon, \varepsilon' \in (0,1/2)$ we have

$$L_\varepsilon(p) \asymp L_{\varepsilon'}(p),$$

where $f \asymp g$ means that the ratio between the functions f and g is bounded away from 0 and ∞ as $p \to 1/2$. In view of this, we fix some $\varepsilon \in (0,1/2)$ and work with this choice of ε throughout the rest of the chapter without loss of generality. We will also need the following five results. The first is a consequence of Theorem 26 of [47] (see also Theorem 1 of [40] for a similar result).

Lemma 3.3. *Consider percolation on $\mathcal{T}_\delta$ with parameter p and let $C^w(r,R)$ (resp., $C^b(r,R)$) be the event that the annulus $A(r,R)$ is crossed (from the inner to the outer boundary) by* *a black (resp., white) path. Then,*

$$\mathbb{P}_{\delta,p}(C^w(r,R)) \asymp \mathbb{P}_{\delta,p}(C^b(r,R)) \asymp \mathbb{P}_{\delta,1/2}(C^b(r,R)) = \mathbb{P}_{\delta,1/2}(C^w(r,R))$$

uniformly in p and $0 < r \le R \le \delta L_\varepsilon(p)$.

We interpret this result as follows: on a scale not larger than the correlation length, percolation with parameter p looks roughly like critical percolation.

The second result, stated below, is Remark 38 of [47].

Lemma 3.4. *Consider percolation on $\mathcal{T}_\delta$ with parameter $p \ge 1/2$. Let $C_H([0,n]\times[0,kn])$ denote the event that the parallelogram $[0,n]\times[0,kn]$ contains a* black *horizontal crossing. For any $k \ge 1$ there exist two constants $C_1 < \infty$ and $C_2 > 0$, both depending on k and ε, such that*

$$\mathbb{P}_{\delta,p}(C_H([0,n]\times[0,kn])) \le C_1 \exp\left(-\frac{C_2 n}{\delta L_\varepsilon(p)}\right).$$

The third result is as follows (see, e.g., [9, 47] for more explanation and references).

Lemma 3.5. *Consider percolation on $\mathcal{T}_\delta$ with parameter $p \ge 1/2$, and let*

$$D_r = \{\exists \text{ black circuit } S \text{ surrounding the origin with } \operatorname{diam}(S) \ge r\}.$$

Then, for each $\varepsilon \in (0,1/2)$ there exist two constants $C_3 = C_3(\varepsilon) < \infty$ and $C_4 = C_4(\varepsilon) > 0$ such that

$$\mathbb{P}_{\delta,p}(D_r) \le C_3 \exp\left(-\frac{C_4 r}{\delta L_\varepsilon(p)}\right).$$

The fourth result is an immediate consequence of Lemma 3.4 (see Figure 3.3 for an example of a similar argument).

Lemma 3.6. *Consider percolation on $\mathcal{T}_\delta$ with parameter $p \ge 1/2$, and let*

$$D'_r = \{\exists \text{ black path containing the origin and of diameter at least } r\}.$$

For each $\varepsilon \in (0,1/2)$ there exist two constants $C_5 = C_5(\varepsilon) < \infty$ and $C_6 = C_6(\varepsilon) > 0$ such that

$$\mathbb{P}_{\delta,p}(D'_r) \le C_5 \exp\left(-\frac{C_6 r}{\delta L_\varepsilon(p)}\right).$$

The last result is the celebrated power law for the correlation length [56] (see also [47]): as $p \to 1/2$,

$$L_\varepsilon(p) = |p - 1/2|^{-4/3+o(1)}. \quad (3.2)$$

3.3 The scaling limit

We turn our attention to the main object of study in this chapter – the scaling limit of the collection of all boundary loops. We will briefly recall the most import objects of Chapter 2.

We define a distance function $\Delta(\cdot,\cdot)$ on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$\Delta(u, v) := \inf_{\varphi} \int (1 + |\varphi(s)|^2)^{-1}\, ds,$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parametrized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere.

We regard curves as equivalence classes of continuous functions from the unit interval to $\dot{\mathbb{R}}^2$, modulo increasing, continuous reparametrizations. Below, γ will represent a particular curve and $\gamma(t)$ a parametrization of γ. Denote by $\mathcal{S}$ the complete separable metric space of curves in $\dot{\mathbb{R}}^2$ with the distance

$$\mathrm{D}(\gamma_1, \gamma_2) := \inf \sup_{t \in [0,1]} \Delta(\gamma_1(t), \gamma_2(t)), \quad (3.3)$$

where the infimum is over all choices of parametrizations of γ_1 and γ_2 from the interval $[0, 1]$. A set of curves (more precisely, a closed subset of $\mathcal{S}$) will be denoted by $\mathcal{F}$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\mathrm{Dist}(\mathcal{F}, \mathcal{F}') \leq \varepsilon \Leftrightarrow (\forall\, \gamma \in \mathcal{F},\ \exists\, \gamma' \in \mathcal{F}' \text{ with } \mathrm{D}(\gamma, \gamma') \leq \varepsilon \text{ and vice versa}). \quad (3.4)$$

The space Ω of closed sets of $\mathcal{S}$ (i.e., collections of curves in $\dot{\mathbb{R}}^2$) with the metric (3.4) is also a complete separable metric space. We denote by $\mathcal{B}$ its Borel σ-algebra.

When we talk about convergence in distribution of random curves, we always mean with respect to the uniform metric (3.3), while when we deal with closed collections of curves, we always refer to the metric (3.4). In this chapter, the space Ω of closed sets of $\mathcal{S}$ is used for collections of boundary loops and their scaling limits.

In Chapter 2 we formulated a hypothesis that implies, for every sequence $\delta_j \downarrow 0$, the existence of a scaling limit along some subsequence $\{\delta_{j_i}\}$. The hypothesis in Chapter 2 is formulated in terms of crossings of *spherical* annuli, but one can work with the annuli defined in Section 3.2 just as well. In order to state it, we need one more piece of notation. For $\delta > 0$, we denote by μ_δ any probability measure supported on collections of curves that are polygonal paths on the edges of the hexagonal lattice $\mathcal{H}_\delta$.

In our context, the hypothesis is as follows.

Hypothesis 3.7. *For all $k < \infty$ and for all annuli $A(x; r, R)$ with $\delta \leq r \leq R \leq 1$, the following bound holds uniformly in δ:*

$$\mu_\delta\left(A(x; r, R) \text{ is crossed by } k \text{ disjoint curves}\right) \leq K_k \left(\frac{r}{R}\right)^{\varphi(k)}$$

for some $K_k < \infty$ and $\varphi(k) \to \infty$ as $k \to \infty$.

The next theorem follows from Theorem 2.10.

Theorem 3.8 ([2]). *Hypothesis 3.7 implies that for any sequence $\delta_j \downarrow 0$, there exist a subsequence $\{\delta_{j_i}\}_{i \in \mathbb{N}}$ and a probability measure μ on Ω such that $\mu_{\delta_{j_i}}$ converges weakly to μ as $i \to \infty$.*

It was already remarked in the appendix of [2] that the above hypothesis can be verified for two-dimensional critical and near-critical percolation. The same conclusion follows from results in [47], and is obtained in Proposition 1 of [48]. We will need a slightly more general result, stated and proved below for completeness.

Lemma 3.9. *Let $\{\mu_{\delta_j, p_j}\}_{j \in \mathbb{N}}$ be a sequence of measures on boundary paths induced by percolation on $\mathcal{T}_{\delta_j}$ with parameters p_j. For* any *sequence $\delta_j \to 0$ and* any *choice of the collection $\{p_j\}_{j \in \mathbb{N}}$, Hypothesis 3.7 holds.*

Proof. First of all, observe that the number of boundary paths crossing an annulus is necessarily even and that, if there are k disjoint boundary paths crossing the annulus, then the annulus must also be crossed by $k/2$ disjoint white paths. For any $\delta > 0$ and $p \geq 1/2$, we have

$$\begin{aligned}&\mathbb{P}_{\delta,p}\left(A(x;r,R) \text{ is crossed by } k/2 \text{ disjoint white paths}\right)\\ &\leq \mathbb{P}_{\delta,1/2}\left(A(x;r,R) \text{ is crossed by } k/2 \text{ disjoint white paths}\right)\\ &\leq \mathbb{P}_{\delta,1/2}\left(A(x;r,R) \text{ is crossed by a white path}\right)^{k/2},\end{aligned}$$

where we have used monotonicity and the BK inequality [8]. Define l_1, l_2 as the largest, respectively smallest integer such that $1/2^{l_1} \geq r$, resp. $1/2^{l_2} \leq R$. Consider the annuli

$$A_1 = A(x;(1/2^{l_1}, 1/2^{l_1-1}), A_2 = A(x; 1/2^{l_1-1}, 1/2^{l_1-2}), \ldots, A_N = A(x; 1/2^{l_2+1}, 1/2^{l_2}),$$

where N denotes the maximal number of annuli of this type that can be placed in $A(x;r,R)$. Note that N is of order $\log(R/r)$ and hence there exists a constant $C > 0$, independent of r and R, such that $\lfloor N/2 \rfloor \geq C\log(R/r)$. Observe furthermore that if $A(x;r,R)$ is crossed by a white path then none of the annuli $A_1, \ldots, A_N$ contains a black circuit surrounding x. It follows from the RSW theorem (see, e.g., [39, 36, 47]) that the probability of the event that the annulus A_i contains a black circuit is uniformly (in i) bounded from below by some $\gamma > 0$, independent of δ. By definition of the annuli, A_{2i+1} and $A_{2i'+1}$ are disjoint for $i \neq i'$. Putting everything together we obtain

$$\begin{aligned}&\mathbb{P}_{\delta,1/2}\left(A(x;r,R) \text{ is crossed by a white path}\right)^{k/2}\\ &\leq \mathbb{P}_{\delta,1/2}\left(\bigcap_{i=0}^{\lfloor N/2\rfloor-1}\{A_{2i+1} \text{ does not contain a black circuit surrounding } x\}\right)^{k/2}\\ &= \left[\prod_{i=0}^{\lfloor N/2\rfloor-1}\mathbb{P}_{\delta,1/2}(A_{2i+1} \text{ does not contain a black circuit surrounding } x)\right]^{k/2}\\ &\leq \left[(1-\gamma)^{\lfloor N/2\rfloor}\right]^{k/2}\\ &\leq \left[(1-\gamma)^{C\log(R/r)}\right]^{k/2}\\ &= \left[(r/R)^{-C\log(1-\gamma)}\right]^{k/2}.\end{aligned}$$

Therefore, taking $K_k = 1$ and $\varphi(k) = -C\log(1-\gamma)k/2$, we obtain a bound of the desired form since $-C\log(1-\gamma) > 0$.

For any $p \leq 1/2$, the same uniform bound follows from swapping black and white in the above argument. Hence, we have the desired bound for any sequence $\delta_j \to 0$ and any $\{p_j\}_{j\in\mathbb{N}}$ and the lemma is proved.

3.4 Proof of Theorem 3.1

We first show how assuming different behaviors for the correlation length leads to the three scenarios described in the theorem. Later we will prove that those three scenarios are non-void and are the only three possibilities.

(1) Suppose that for some $\varepsilon \in (0, 1/2)$, $\delta_j L_\varepsilon(p_j) \to 0$ as $j \to \infty$. Recall that μ is the weak limit of a sequence $\{\mu_{\delta_j,p_j}\}$ with $\delta_j \to 0$ as $j \to \infty$ and $p_j \geq 1/2$. Note that it is actually the case that $p_j > 1/2$ for all but finitely many j since $\delta_j L_\varepsilon(p_j) \to 0$.

The existence of a boundary loop with positive diameter would imply that there exist $x \in \mathbb{Q}^2$ and $0 < r_1 < r_2 \in \mathbb{Q}$ such that the loop intersects both $B^o(x, r_1)$ and $\mathbb{R}^2 \setminus B(x, r_2)$, so that the annulus $A(x; r_1, r_2)$ is crossed by a boundary path. One of the four (overlapping) parallelograms with side-lengths $(r_2 - r_1)/2$ and r_2 depicted in Figure 3.3 is then necessarily crossed at least once in the "easy" direction by a boundary path (see Figure 3.3). Let E denote such a crossing event. More precisely, crossings that realize E start and end outside the parallelogram and do not intersect the short sides of the parallelogram. This makes E open in our topology. Note also that the occurrence of E implies, for $\delta_j > 0$, that the parallelogram contains a black crossing in the easy direction. Thus, the portmanteau theorem and Lemma 3.4 yield

$$\begin{aligned}
&\mu(A(x; r_1, r_2) \text{ is crossed by a boundary path}) \\
&\leq 4\mu(E) \\
&\leq 4 \liminf_{j\to\infty} \mu_{\delta_j,p_j}(E) \\
&\leq \liminf_{j\to\infty} C_1 \exp\left(-\frac{C_2(r_2 - r_1)}{2\delta_j L_\varepsilon(p_j)}\right) = 0.
\end{aligned} \tag{3.5}$$

We can then conclude that

$$\begin{aligned}
&\mu(\text{there exists a boundary loop with positive diameter}) \\
&\leq \bigcup_{x\in\mathbb{Q}^2; r_1,r_2\in\mathbb{Q}^+} \mu(A(x; r_1, r_2) \text{ is crossed by a boundary path}) = 0.
\end{aligned}$$

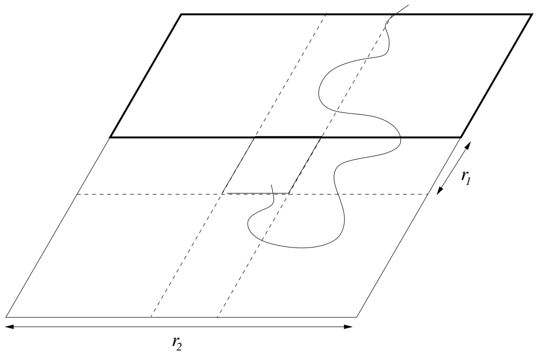


Figure 3.3: The annulus $A(x; r_1, r_2)$ contains four rectangles of side lengths $(r_2 - r_1)/2$ and r_2. If a curve γ crosses the annulus $A(x; r_1, r_2)$ then one (indicated with the heavy lines) of the four rectangles contains a crossing in the "easy" direction.

(2) Suppose that for some $\varepsilon \in (0, 1/2)$, $\delta_j L_\varepsilon(p_j) \to \infty$ as $j \to \infty$. Then $1 \leq \delta_j L_\varepsilon(p_j)$ for each j sufficiently large. To show the a.s. existence of an infinite sequence of boundary loops with radii going to zero around the origin (or any other deterministic point), we proceed as follows. Consider the sequence of annuli $A_1 = A(\frac{1}{2}, 1), A_2 = A(\frac{1}{4}, \frac{1}{2}), \ldots, A_k = A((\frac{1}{2})^k, (\frac{1}{2})^{k-1}), \ldots$, and denote by F_k the event that there is (at least) one boundary loop surrounding the origin in the annulus A_{2k+1} with $k \geq 0$. Since we defined annuli to be closed sets, the event F_k is closed in our topology. Note that in order to guarantee the presence of a boundary loop inside the annulus A_k, it suffices to have, for example, a black circuit in $A(\frac{5}{3}(\frac{1}{2})^k, (\frac{1}{2})^{k-1})$ and a white circuit in $A((\frac{1}{2})^k, \frac{1}{3}(\frac{1}{2})^{k-2})$ (note that those two annuli are disjoint). Since $\delta_j L_\varepsilon(p_j) \geq 1$ for each large j, it follows from Lemma 3.3 and the RSW theorem that the probability to find a black circuit in $A(\frac{5}{3}(\frac{1}{2})^k, (\frac{1}{2})^{k-1})$ and a white circuit in $A((\frac{1}{2})^k, \frac{1}{3}(\frac{1}{2})^{k-2})$ is bounded away from 0 as $j \to \infty$, uniformly in k. Therefore there exists $\varepsilon_0 > 0$ such that

$$\mu(F_k) \geq \limsup_{j \to \infty} \mu_{\delta_j, p_j}(F_k) \geq \varepsilon_0, \quad \text{for every } k,$$

where the first inequality follows from the portmanteau theorem. Note also that the events F_k and $F_{k'}$ are independent for $k' \neq k$. Moreover, $\sum_{k=0}^{\infty} \mu(F_k) = \infty$ and thus by the Borel-Cantelli lemma there are infinitely many boundary circuits surrounding the origin with diameter going to zero, μ-a.s.

We argue in a similar way as above to show that every point is surrounded by a countably infinite family of nested loops with radii going to infinity. Let B_k denote the annulus $A(2^k, 2^{k+1})$ and write F'_k, with $k \geq 0$, for the event that the annulus B_{2k+1} contains at least one boundary loop surrounding $B(1)$. For each k it holds that $\delta_j L_\varepsilon(p_j) \geq 2^{k+1}$ for j sufficiently large. Hence, it follows from Lemma 3.3 and the RSW theorem that the probability to find a boundary loop in B_{2k+1} is uniformly (in k) bounded away from 0 as $j \to \infty$. Again, the event F'_k is closed in our topology, thus $\mu(F'_k) \geq \limsup_{j\to\infty} \mu_{\delta_j,p_j}(F'_k) \geq \varepsilon_1$, for some $\varepsilon_1 > 0$ independent of k. Hence, the Borel-Cantelli lemma implies that there are infinitely many boundary loops surrounding $B(1)$ with diameter going to infinity μ-a.s. By translation invariance, the same is true for every $B(x;1)$ with $x \in \mathbb{Q}$ and therefore for every point of the plane.

(3) Suppose that for some $\varepsilon \in (0, 1/2)$, $\delta_j L_\varepsilon(p_j)$ stays bounded away from both 0 and ∞ as $j \to \infty$. That is, there exist $\beta > 0$ and $K < \infty$ such that $\beta \leq \delta_j L_\varepsilon(p_j) \leq K$ for each j sufficiently large. The first part of the proof in case *(2)* carries over directly to the present case, with 1 replaced by β in the lower bound for the macroscopic correlation length and the annuli $A(1/2^k, 1/2^{k-1})$ replaced by $A(\beta/2^k, \beta/2^{k-1})$. Thus, μ-a.s. there exist infinitely many boundary loops surrounding the origin, with diameter going to zero.

Our next goal is to prove the a.s. existence of a largest boundary loop surrounding the origin. Let G_L denote the event that there exists a largest boundary loop γ surrounding or containing the origin and that this loop has $\mathrm{diam}(\gamma) \leq L$. Then $G := \bigcup_{L=1}^{\infty} G_L$ is the event that there exists a largest loop surrounding or containing the origin. Note that if all black circuits around the origin have diameter smaller than L and there is no black path containing the origin of diameter larger than $L - 2\delta_j$, then G_L occurs. Therefore

$$\mu_{\delta_j,p_j}(G_L) \geq 1 - [\mathbb{P}_{\delta_j,p_j}(D_L) + \mathbb{P}_{\delta_j,p_j}(D'_{L-2\delta_j})],$$

where D_L is the event that the origin is surrounded by a black circuit of diameter at least L and $D'_{L-2\delta_j}$ is the event that there is a black path containing the origin of diameter at least $L - 2\delta_j$. Using Lemmas 3.5 and 3.6 and the fact that the event G_L

is closed in our topology, we can write

$$\begin{aligned}\mu(G_L) &\geq \limsup_{j\to\infty} \mu_{\delta_j,p_j}(G_L)\\ &\geq 1 - \liminf_{j\to\infty}[\mathbb{P}_{\delta_j,p_j}(D_L) + \mathbb{P}_{\delta_j,p_j}(D'_{L-2\delta_j})]\\ &\geq 1 - \liminf_{j\to\infty} C' \exp\left(-\frac{C''L}{\delta_j L_\varepsilon(p_j)}\right)\\ &\geq 1 - C' \exp\left(-\frac{C''L}{K}\right).\end{aligned}$$

Since the events are nested (i.e., $G_{L_1} \subset G_{L_2}$ for $L_1 < L_2$),

$$\mu(G) = \lim_{L\to\infty} \mu(G_L) \geq \lim_{L\to\infty}\left[1 - C^{'} \exp\left(-\frac{C^{''}L}{K}\right)\right] = 1.$$

Since boundary loops cannot cross each other and, by the previous part of the proof, the origin is surrounded with probability one by a sequence of infinitely many boundary loops with diameter going to zero, the largest boundary loop does not touch the origin. Hence, the event G coincides with the existence of a largest boundary circuit surrounding the origin and we are done.

To continue the proof, note that for each $\varepsilon \in (0, 1/2)$, as $j \to \infty$,

- either $\delta_j L_\varepsilon(p_j) \to 0$,
- or $\delta_j L_\varepsilon(p_j) \to \infty$,
- or $\delta_j L_\varepsilon(p_j)$ is bounded away from both 0 and ∞.

This is clearly so because we are assuming that $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$ has a limit μ, and we have proved that the three cases above give rise to three incompatible scenarios for μ. Indeed, if we are not in one of the three cases above, then there must be two different subsequences of $\{(\delta_j, p_j)\}_{j\in\mathbb{N}}$ falling in two different cases, which contradicts the existence of a limit μ. We can then conclude that there are no other possible scenarios for μ besides the three described in the theorem.

To conclude the proof, we need to show that all three scenarios are non-void. For the first two, this is obvious. To prove that the third scenario is also non-void, take $0 < \varepsilon_1 < \varepsilon_2 < 1/2$ and consider any sequence $\{(\delta_j, p_j)\}_{j\in\mathbb{N}}$ such that $\delta_j \to 0$ and $p^+_{\varepsilon_1}(1/\delta_j) \leq p_j \leq p^+_{\varepsilon_2}(1/\delta_j)$. This implies that $L_{\varepsilon_1}(p_j) \leq 1/\delta_j \leq L_{\varepsilon_2}(p_j)$ for each j. We can assume without loss of generality that the sequence $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$ has a limit

μ. (If that is not the case, by Theorem 3.8 and Lemma 3.9 we can extract a subsequence $\{\mu_{\delta_{j_k},p_{j_k}}\}_{k\in\mathbb{N}}$ that does have a limit, and rename it $\{\mu_{\delta_j,p_j}\}_{j\in\mathbb{N}}$.) Since $\delta_j L_{\varepsilon_1}(p_j)$ remains bounded as $j\to\infty$, $\delta_j L_\varepsilon(p_j)$ must remain bounded as $j\to\infty$ for every other $\varepsilon\in(0,1/2)$ because $L_\varepsilon(p)\asymp L_{\varepsilon_1}(p)$. Analogously, since $\delta_j L_{\varepsilon_2}(p_j)$ is bounded away from 0 as $j\to\infty$, $\delta_j L_\varepsilon(p_j)$ must be bounded away from 0 as $j\to\infty$ for every $\varepsilon\in(0,1/2)$. Therefore, for each $\varepsilon\in(0,1/2)$, $\delta_j L_\varepsilon(p_j)$ remains bounded away from both 0 and ∞ as $j\to\infty$, showing that μ falls in the third scenario.

Remark 3.10. *A significantly stronger version of case* (2) *can be proven, namely that μ in this case coincides with the full scaling limit of critical percolation. In order to prove this one can use the same strategy as in [21], combined with the proof of Proposition 4 of [48]. Below we briefly sketch how one can obtain the result by modifying the arguments of [21]. We stress that this is not meant to be a self-contained proof, and some familiarity with [21] is needed in order to follow the arguments outlined below.*

First of all, wherever statement (S) (concerning the convergence of the critical exploration path to SLE_6 — see p. 18 of [21]) is invoked in [21], one needs to use the proof of Proposition 4 of [48]. We remark that the statement of Proposition 4 of [48] concerns triangular domains and is therefore not sufficient for our purposes, but the proof applies in much greater generality. In particular, the reader can check that it applies to the situations that arise in [21].

Uniform bounds on "six-arm" events in the plane and "three-arm" events near a boundary (see Lemma 6.1 of [21] and its proof) are used repeatedly in [21]. Thanks to Theorem 1 of [40] and similar results described in [47] (see, e.g., Section 3.2, and Theorem 27 and the discussion following it), such uniform bounds are also available for percolation with parameter p_j on $\mathcal{T}_{\delta_j}$ inside a disc of diameter L, provided that $\delta_j L_\varepsilon(p_j)\geq L$. Due to the assumption that $\delta_j L_\varepsilon(p_j)\to\infty$ as $j\to\infty$, such a condition is satisfied for any $L<\infty$, for j sufficiently large.

Some care is also needed in the proof of the second part of Theorem 5 of [21] (see p. 19 for the statement; the proof begins on p. 27), and in particular of Lemma 6.4 (see p. 27) and Lemma 6.6 (see p. 29) used in that proof. The proof of the second part of Theorem 5 of [21] is given for critical percolation, but the only features of critical percolation that are really used are uniform bounds on certain crossing probabilities (involving crossings of a rectangle or an annulus). Due to the above considerations, similar bounds can be used in the present context (see again Section 3.2 and Theorem 27 of [47]).

4 Existence of subsequential scaling limits in random Voronoi percolation

4.1 Introduction

In random Voronoi percolation a Poisson process in the plane is generated and the cells of the associated Voronoi tiling are colored black with probability $p \in [0,1]$ or white otherwise, independently of other cells. Bollobás and Riordan established in their paper [12] a 'weak' RSW theorem for the random Voronoi percolation model and proved that the critical probability for having an infinite black cluster (precise definitions will be given in Section 4.1.1) equals 1/2. The random Voronoi percolation model allows for a gradual refinement of percolation configurations by increasing the intensity of the Poisson process, and it is therefore natural to study the scaling limit of random Voronoi percolation as the intensity of the Poisson process tends to ∞.

In 1994, Langlands, Pouliot and Saint-Aubin [41] conjectured conformal invariance of crossing probabilities for a wide range of percolation models. This form of conformal invariance was proven for Bernoulli site percolation on the triangular lattice in 2001 and for the Ising model on the square lattice in 2010, both by Smirnov [53, 55]. Benjamini and Schramm proved in 1998 a different form of conformal invariance (essentially saying that crossing probabilities are invariant under conformal transformation of the underlying metric) in [5] for the random Voronoi percolation model. Despite the results in [5, 12] and the fact that the Voronoi model exhibits more rotational invariance than the afore mentioned lattice models, a proof of conformal invariance in the sense of [41] for random Voronoi percolation is still missing. In this chapter we will derive, conditional on the assumption of a 'strong' RSW result, the existence of subsequential scaling limits for the random Voronoi percolation model.

4.1.1 Definitions

Consider $\mathbb{R}^2$ and let $\lambda > 0$. Let η be a realization of a Poisson process $\mathcal{P}$ with intensity λ. Let ν_λ denote the distribution of $\mathcal{P}$. Enumerate the points in η in a non-random way, for instance by distance to the origin: $\eta = (x_1, x_2, \ldots)$. Let $p \in [0, 1]$, and given η, flip a coin to assign a color $\sigma(x)$ to each point $x \in \eta$: $\mathbb{P}(\sigma(x) = \text{black}) = p = 1 - \mathbb{P}(\sigma(x) = \text{white})$. For each $x \in \eta$ let $V(x)$ denote its associated Voronoi cell: $V(x) = \{y \in \mathbb{R}^2 : |x - y| \leq |x' - y|, \forall x' \in \eta\}$. Each point $y \in V(x)$ is then colored according to the color of x, so points on the boundary of a Voronoi cell can be both black and white.

It is known [12] that with probability 1 each cell is a convex polygon and two cells are either disjoint or share an entire edge. The graph of the Voronoi tiling is the countably infinite graph G with η as its vertex set, in which two points are adjacent if they share an entire edge. The notions of percolation (paths, clusters, crossings, etc) on the Voronoi tiling will be the usual notions of Bernoulli site percolation on G: a *path of cells* corresponds to a path in G. Let x_0 be the a.s. unique point in η such that the origin is contained in $V(x_0)$. In the graph G, the *black vertex cluster containing* x_0 is the set C_0^G of all black vertices of G connected to v_0 by a path consisting of black vertices. Correspondingly, the *black component of the origin* C_0 is the maximal connected set of black points in $\mathbb{R}^2$ containing the origin, i.e., the union of the cells $V(x)$, with $x \in C_0^G$.

The probability measure corresponding to random Voronoi percolation with percolation parameter p and intensity parameter λ is denoted by $\mathbb{P}_{\lambda,p}$. We write $\mathbb{P}_p$ for $\mathbb{P}_{1,p}$. We define the *percolation function* $\vartheta(p, \lambda)$ by $\vartheta(p, \lambda) := \mathbb{P}_{\lambda,p}(|C_0^G| = \infty)$. The *critical probability* p_c^λ is defined as $p_c^\lambda = \inf\{p : \vartheta(p, \lambda) > 0\}$. Bollobás and Riordan [12] proved that $p_c^1 = 1/2$. Since $\lambda > 0$ is a scaling factor, $\vartheta(p, \lambda)$ is constant in λ and therefore $p_c^\lambda = 1/2$ for all $\lambda > 0$. We say that random Voronoi percolation with parameter $\lambda > 0$ is *critical* when the percolation parameter equals $1/2$. Our goal is to establish the existence of a scaling limit along a sequence of probability measures with $\lambda \to \infty$, corresponding to critical random Voronoi percolation. Therefore, for the rest of the chapter we set $p = 1/2$ and drop p from the notation. We start by defining polygonal curves and other relevant objects in the case of random Voronoi percolation.

4.1.2 Random curves

In order to establish the existence of a scaling limit along a subsequence, we apply the framework of Chapter 2 to interface curves in Voronoi percolation. First, we have to define interface curves. An *interface* is defined as the edge between two Voronoi cells of opposite color. A concatenation of interfaces is called an *interface curve*. See Figure 4.1 for an illustration of the Voronoi percolation model and its corresponding interfaces. Note that "checkerboard"-configurations are not possible since at most 3 cells meet at one point a.s., hence interface loops are well-defined and simple a.s. It is known [13] that the typical diameter of a Voronoi cell is of order $\lambda^{-1/2}$. Hence, the interface curves in random Voronoi percolation have step-sizes of order $\lambda^{-1/2}$. In analogy with Chapter 3 we consider interface curves in the plane. Therefore, we use the distance function $\Delta(\cdot,\cdot)$ on $\mathbb{R}^2 \times \mathbb{R}^2$, which is defined by

$$\Delta(u,v) := \inf_{\varphi} \int (1 + |\varphi(s)|^2)^{-1}\, ds,$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parametrized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere. We regard curves as equivalence classes of continuous functions from the unit interval to $\dot{\mathbb{R}}^2$, modulo increasing, continuous reparametrizations. Below, γ will represent a particular curve and $\gamma(t)$ a parametrization of γ. Let C^λ denote the space of interface curves in random Voronoi percolation with intensity $\lambda > 0$. The distance between two curves in C^λ is defined as follows.

$$D(\gamma_1,\gamma_2) := \inf \sup_{0\le t\le 1} \Delta(\gamma_1(t),\gamma_2(t)),$$

where the infimum is taken over all parametrizations of γ_1 and γ_2. A set of curves (more precisely, a closed subset of C^λ) will be denoted by $\mathcal{F}_\lambda$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\text{Dist}(\mathcal{F},\mathcal{F}') \le \varepsilon \Leftrightarrow (\forall\, \gamma \in \mathcal{F},\ \exists\, \gamma' \in \mathcal{F}' \text{ with } \text{D}(\gamma,\gamma') \le \varepsilon \text{ and vice versa}). \quad (4.1)$$

Let Ω^λ denote the space of closed subsets of C_λ (i.e., collections of curves in $\dot{\mathbb{R}}^2$) with the metric (4.1). The probability measure on Ω^λ that describes the polygonal paths of interfaces along the edges of the Voronoi cells with intensity of the Poisson process equal to λ is denoted by μ_λ, and is induced by $\mathbb{P}_\lambda$.

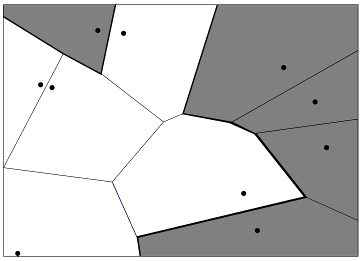


Figure 4.1: A typical Voronoi configuration (black cells are colored grey for visibility reasons). The heavy lines denote the interfaces between black and white cells.

4.1.3 RSW condition and main results

To satisfy Hypothesis 2.6 we have to impose a condition on the probability of crossing a $4n$ by n rectangle in the long direction for all n. The RSW theorem for random Voronoi percolation, proven by Bollobas and Riordan in [12] and improved by van den Berg et al. in [7], states that the limit superior of the probability of crossing a $4n$ by n rectangle in the long direction is positive. Unfortunately, this not enough for our purposes. We introduce some necessary notation to state the sufficient condition.

Let $R(m,n)$ denote a rectangle of height n and width m. Let $H(m,n)$ denote the event that the rectangle $R(m,n)$ is crossed in the horizontal direction, i.e. there exists a black connected component joining the left-hand side of $R(m,n)$ with its right-hand side. The RSW result of [7] is:

$$\text{If} \limsup_{n\to\infty} \mathbb{P}_1(H(\rho n,n)) > 0 \text{ for some } \rho > 0,$$
$$\text{then} \limsup_{n\to\infty} \mathbb{P}_1(H(\rho n,n)) > 0 \text{ for all } \rho > 0.$$

By scaling this is equivalent with

$$\text{If} \limsup_{\lambda\to\infty} \mathbb{P}_\lambda(H(\rho,1)) > 0 \text{ for some } \rho > 0,$$
$$\text{then} \limsup_{\lambda\to\infty} \mathbb{P}_\lambda(H(\rho,1)) > 0 \text{ for all } \rho > 0. \tag{4.2}$$

It is widely believed, and will also be enough for our purposes, that in (4.2) the limit superior in the conclusion can we replaced with a limit inferior. Thus, our assumption can be formulated as follows:

$$\text{If } \limsup_{\lambda\to\infty} \mathbb{P}_\lambda(H(\rho,1)) > 0 \text{ for some } \rho > 0,$$
$$\text{then } \liminf_{\lambda\to\infty} \mathbb{P}_\lambda(H(\rho,1)) > 0 \text{ for all } \rho > 0 \quad (C)$$

In order to prove our main result we need a special version of the BK inequality [8] for random Voronoi percolation, which we state here seperately since it could be of independent interest. Identifying black with 1 and white with 0, the construction of the model as carried out in Section 4.1.1 can be formalized as follows: let η be distributed according to ν_λ and enumerate the points in η according to some predetermined deterministic rule (e.g. by distance to the origin). Thus $\eta = (x_i)_{i\geq 1}$, where each x_i is a Poisson point. Consider an infinite sequence $\sigma = (\sigma_i)_{i\geq 1}$ of fair coin flips and let the outcome of the i-th coinflip denote the color of the i-th Poisson point. That is, $\sigma(x_i) = \sigma_i$, where

$$\mathbb{P}(\sigma_i = 1) = 1/2 = \mathbb{P}(\sigma_i = 0),$$

identifying black with 1 and white with 0. Note that this coloring of Poisson points is slightly different than the coloring carried out in the introduction but the colorings have the same distribution. Let $\mathcal{P}$ and $\Sigma = \{0,1\}^\infty$ be the respective outcome spaces of the Poisson process and the coloring procedure. Define $\Omega := \mathcal{P} \times \Sigma$, thus $\omega = (\eta, \sigma) \in \Omega$ is the realization of a colored Poisson process.

We call an event A increasing if A is increasing in terms of the colors of the Poisson points. In other words, A is increasing if

$$\omega \in A \text{ and } \omega' \geq \omega \Rightarrow \omega' \in A,$$

where $\omega' = (\eta', \sigma') \geq \omega = (\eta, \sigma)$ if and only if $\eta' = \eta$ and $\sigma'_i \geq \sigma_i$ for all $i \geq 1$. Let A be an event then, given a realization η of the Poisson process, $A(\eta)$ consists of those colorings which force A to occur:

$$A(\eta) := \{\sigma \in \{0,1\}^\infty : (\eta, \sigma) \in A\}.$$

We say that an event A is *defined in a bounded domain* D, when knowing the Poisson points in D and their colors allows to determine the occurence of A. Let $K(\sigma)$ denote the indices which are colored black, i.e.

$$K(\sigma) := \{i : \sigma_i = 1\}.$$

Similarly, for a set of indices I let $\sigma(\eta, I)$ denote the coloring which is black on $(x_i)_{i\in I}$ and white on the complement of I, thus

$$\sigma(\eta, I)_i := \begin{cases} 1, & i \in I, \\ 0, & i \notin I. \end{cases}$$

Given two increasing events A and B, both defined in a bounded domain, define the disjoint occurence of A and B as

$$A \circ B := \{\omega = (\eta, \sigma) : \exists I \subset K(\sigma) \text{ such that } \sigma(\eta, I) \in A(\eta), \sigma(\eta, K(\sigma) \setminus I) \in B(\eta)\}.$$

In words, $A \circ B$ describes the event that A and B occur on disjoint Voronoi cells. The BK inequality reads as follows, its proof can be found in Section 4.3.

Theorem 4.1. *Let A and B be two increasing events, both defined in a bounded domain D. Then*

$$\mathbb{P}_\lambda(A \circ B) \leq \mathbb{P}_\lambda(A)\mathbb{P}_\lambda(B).$$

We formulate Hypothesis 2.6 in the case of random Voronoi percolation. First, observe that it is sufficient to prove Hypothesis 2.6 if the spherical shells $D(x; r, R)$ are replaced by annuli of the form $A(x; r, R) := B(x; R) \setminus B(x; r)$, where $B(x; n) \subset \mathbb{R}^2$ is a square of side length n centered around $x \in \mathbb{R}^2$. Instead of the expression 'is crossed by k disjoint segments of curves in $\mathcal{F}_\lambda$' we will simply write 'is crossed by k interface curves' or 'contains k interface curves'.

Hypothesis 4.2. *Let $x \in \mathbb{R}^2$ and let $r, R > 0$ be such that $R/r \geq 2$. There exists a sequence $\{\varphi(k)\}_{k\geq 1}$, with $\varphi(k) \to \infty$ as $k \to \infty$ and each $\varphi(k)$ independent of r and R, such that*

$$\limsup_{\lambda\to\infty} \mathbb{P}_\lambda(A(x; r, R) \textit{ is crossed by } k \textit{ interface curves}) \leq c_k \left(\frac{r}{R}\right)^{\varphi(k)}, \tag{4.3}$$

where c_k is a constant depending only on k.

Our main result is then the following. We will present its proof in the next section.

Theorem 4.3. *Consider random Voronoi percolation with $p = 1/2$. Then, assuming condition (C) is true, Hypothesis 4.2 is satisfied. In particular, for every sequence $\lambda_n \to \infty$, there exists a subsequence $\lambda_{n_k} \to \infty$ such that $\mu := \lim_{k\to\infty} \mu_{\lambda_{n_k}}$ exists.*

4.2 Proof of Theorem 4.3

Proof of Theorem 4.3. Since random Voronoi percolation is translation invariant we prove, without loss of generality, the theorem when x is the origin and write $A(r,R)$ for $A(O;r,R)$. Note that when $A(r,R)$ contains $k/2$ interface crossings, the annulus $A(r,R)$ is necessarily crossed by $k/2$ disjoint white paths. These disjoint white crossings are defined in terms of disjoint Poisson points, since there are black crossings separating them from each other, hence we can apply Theorem 4.1:

$$
\begin{aligned}
&\mu_\lambda(A(r,R) \text{ contains } k \text{ disjoint interface crossings})\\
\leq\ &\mathbb{P}_\lambda(A(r,R) \text{ contains } k/2 \text{ disjoint white crossings})\\
\leq\ &\mathbb{P}_\lambda(A(r,R) \text{ contains one white crossing})^{k/2}.
\end{aligned}
$$

Let $a_i = 2^{-i}$ and write A_i for the annulus $A(a_i, 2a_i)$, where i is such that $A_i \subset A(r,R)$. There exists a constant $C_1 > 0$ such that the number of A_i that can be placed in $A(r,R)$ is at least $C_1 \log(R/r)$. We make the observation that if $A(r,R)$ is crossed by a white path, none of the A_i contain a black circuit which surrounds the origin. Since $P_\lambda(H(1,1)) = 1/2$ for all $\lambda > 0$, we can conclude from Condition (C) that there exist $\lambda_0 < \infty$ and $c > 0$ such that

$$
P_\lambda(H(4,1)) \geq c, \tag{4.4}
$$

for all $\lambda \geq \lambda_0$. It follows from the FKG inequality (see Lemma 3.3 in [12]) and (4.4) that the probability of having a black circuit in $A(2,1)$ which surrounds the origin is at least $c^4 > 0$, for λ large enough. Let $\mathcal{A}_i$ denote the event that the annulus A_i contains a black circuit which surrounds the origin. The bound in (4.3) would now follow immediately if

(1.) There exists an event B such that the collection of events $\{\mathcal{A}_i\}_{i:i \text{ even}}$ is conditionally independent on B for λ large enough and $P_\lambda(B) \to 1$ as $\lambda \to \infty$.

(2.) $P_\lambda(\mathcal{A}_i) \geq c^4 > 0$ for λ large enough and all i;

Note that one can not expect more than "asymptotic" independence for the collection of events $\{\mathcal{A}_i\}_{i:i \text{ even}}$, since for any two points $x, y \in A(r,R)$, there is a positive probability that they belong to the same Voronoi cell. Hence, crossing events defined in disjoint annuli are not independent.

We start by proving the first statement. Our proof is similar to the proof of Lemma 3.2 in [12]. Let S be a subset of $\mathbb{R}^2$ and denote by $S[s]$ the s-neighborhood of S, that is the set of all points within distance s of some point in S. Set $s_\lambda =$

$2\sqrt{\log(R\sqrt{\lambda})/\lambda}$ and cover $A(r,R)$ with squares S_i of side length s_λ, such that the S_i are disjoint and their union is in $A(r,R)[2s_\lambda]$. Let B denote the event that every S_i contains at least one Poisson point. The probability that a square S_i does not contain a Poisson point is $\exp(-\lambda s_\lambda^2) = 1/(R^4\lambda^2)$. There exists a constant C_2 such that the number of squares S_i in $A(r,R)$ can be bounded above by C_2R^2/s_λ^2. Therefore,

$$\mathbb{P}_\lambda(B) \geq (1-\exp(-\lambda s_\lambda^2))^{\frac{C_2R^2}{s_\lambda^2}} \geq \left(1-\frac{1}{R^4\lambda^2}\right)^{\frac{C_2R^2\lambda}{\log(R^4\lambda^2)}} =: p(\lambda).$$

Note that $p(\lambda) \to 1$ as $\lambda \to \infty$. Also, observe that on the event B the color of every point of A_i is determined by the restriction of the colored Poisson process $(\mathcal{P},\sigma)$ to $A_i[2s_\lambda]$, see Figure 4.2. Since $s_\lambda \to 0$ as $\lambda \to \infty$, it is the case that $A_i[2s_\lambda] \cap A_{i-2}[2s_\lambda] = \emptyset$ for all even i and λ large enough. Moreover, we have conditional independence for the collection $\{\mathcal{A}_i\}_{i:i \text{ even}}$ on the event B for λ large enough.

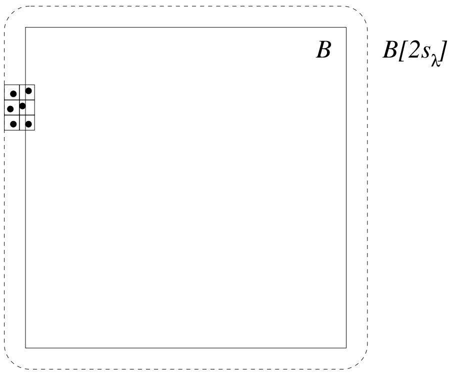


Figure 4.2: The dotted line denotes the boundary of the $2s_\lambda$-neighborhood $B[2s_\lambda]$ of a box B. The small squares denote (a portion of) the squares S_i of side length s_λ. Observe that if each S_i contains a Poisson point, no $x \in B$ can belong to a Voronoi cell corresponding to a Poisson point outside $B[2s_\lambda]$.

Next, we prove the second statement. Since $\lambda a_i^2 \geq \lambda_0$ for λ large enough, it follows from scaling and (4.4) that

$$P_\lambda(H(4a_i, a_i)) = P_{\lambda a_i^2}(H(4,1)) \geq c,$$

for λ large enough. Hence, for each i,

$$P_\lambda(\mathcal{A}_i) \geq c^4 > 0$$

for λ large enough.

It follows from statements (1.) and (2.) that

$$
\begin{aligned}
&\mathbb{P}_\lambda(A(r,R) \text{ contains one white crossing})\\
&\leq \mathbb{P}_\lambda\left(\bigcap_{i:A_i\subset A(r,R)} \mathcal{A}_i^c\right)\\
&\leq \mathbb{P}_\lambda\left(\bigcap_{i:A_i\subset A(r,R)} \mathcal{A}_i^c \middle| B\right) + \mathbb{P}_\lambda(B^c)\\
&\leq \prod_{i:i \text{ even}, A_i\subset A(r,R)} \mathbb{P}_\lambda\left(\mathcal{A}_i^c|B\right) + 1 - p(\lambda)\\
&\leq \prod_{i:i \text{ even}, A_i\subset A(r,R)} \frac{\mathbb{P}_\lambda\left(\mathcal{A}_i^c\right)}{p(\lambda)} + 1 - p(\lambda)\\
&\leq \left(\frac{1-c^4}{p(\lambda)}\right)^{C_1\log(R/r)/2-1} + 1 - p(\lambda),
\end{aligned}
$$

for $\lambda \geq \lambda_1$. Hence,

$$
\begin{aligned}
&\limsup_{\lambda\to\infty} \mu_\lambda(A(r,R) \text{ contains } k \text{ disjoint interface crossings})\\
&\leq \limsup_{\lambda\to\infty} \mathbb{P}_\lambda(A(r,R) \text{ contains one white crossing})^{k/2}\\
&\leq \left(1-c^4\right)^{(C_1\log(R/r)/2-1)k/2}\\
&= (1-c^4)^{-k/2}\left(\frac{r}{R}\right)^{-C_1\log(1-c^4)k/4}.
\end{aligned}
$$

Thus, Hypothesis 4.2 is satisfied, with $c_k = (1-c^4)^{-k/2}$ and $\varphi(k) = -C_1\log(1-c^4)k/4$. Tightness of the collection of probability measures $\{\mu_\lambda\}_{\lambda>0}$ follows from Theorem 2.10. □

4.3 Proof of the BK inequality

The proof of Theorem 4.1 was pointed out by Rob van den Berg [6].

Without loss of generality we consider the case when $\lambda = 1$ and we write $\mathbb{P}$, resp. ν, instead of $\mathbb{P}_1$, resp. ν_1. Let E be an event defined in a bounded domain (hence, the number of Voronoi cells is finite a.s. and we can therefore restrict the set of possible colorings to $\{0,1\}^n$, where n is the number of Voronoi cells) and recall the definition of $E(\eta)$:

$$E(\eta) = \{\sigma \in \{0,1\}^n : (\eta, \sigma) \in E\}.$$

For a coloring σ of a realization η, let $1-\sigma$ be the coloring obtained from σ by flipping each state: $(1-\sigma)(x) = 1-\sigma(x)$ for all $x \in \eta$. Define the "conjugate" of E and $E(\eta)$ as follows.

$$\begin{aligned} \overline{E} &:= \{\omega = (\eta,\sigma) : (\eta, 1-\sigma) \in E\}, \\ \overline{E(\eta)} &:= \{\sigma \in \{0,1\}^n : 1-\sigma \in E(\eta)\}. \end{aligned}$$

Recall that the coloring $\sigma(\eta, I)$, given any coloring σ and I a subset of $\{1,\dots,n\}$, equals σ on I and 0 otherwise. Given a Poisson realization η we define the event $A(\eta) \circ_\eta B(\eta)$ for increasing A and B as follows:

$$A(\eta) \circ_\eta B(\eta) := \left\{ \sigma \in \{0,1\}^n : \begin{array}{c} \exists I \subset \{1,\dots,n\} \text{ such that } \sigma(\eta, I) \in A(\eta) \\ \text{and } \sigma(\eta, \{1,\dots,n\} \setminus I) \in B(\eta) \end{array} \right\}$$

We start by proving an auxiliary result, which is essentially Reimers inequality. We prove this result in a more general setting, that is, let n be an integer and consider the space $\Sigma_n = \{0,1\}^n$. Write $[n]$ for $\{1,\dots,n\}$. An event $A \in \mathcal{B}(\Sigma_n)$, where $\mathcal{B}(\Sigma_n)$ is the set of all subsets of Σ_n, is increasing if $\omega \in A$ and $\omega(m)' \geq \omega(m)$, for all $m \in [n]$, implies that $\omega' \in A$. For $\omega \in \Sigma_n$ and a set $S \subset [n]$ define ω_S by $\omega_S(x) = \omega(x)$ if $x \in S$ and $\omega_S(x) = 0$ otherwise. For two increasing events E, F we define $E \square F$ as follows

$$E \square F := \{\omega \in \Sigma_n : \exists S \subset \{1,\dots,n\} \text{ such that } \omega_S \in E \text{ and } \omega_{[n]\setminus S} \in F\}.$$

Remark. Since we define disjoint occurence only for increasing events E and F in $\mathcal{B}(\Sigma_n)$, $E \circ F$ would be more appropriate to use than $E \square F$ (see [36]). However, since we already defined the operator $\circ$ for increasing events in random Voronoi percolation, we want to avoid any confusion and therefore use $E \square F$ at this place.

Let Π be the product measure with density $1/2$ on $\{1,\dots,n\}$. Write $|E|$ for the cardinality of an event E. Since 0 and 1 have equal weight, the probability $\Pi(E)$ of an event E is in this case equal to $|E|/2^n$. We formulate Reimer's result as follows.

Lemma 4.4. *Let $E, F \subset \{0,1\}^n$ be increasing events. Then*

$$|E \square F| \leq |E \cap \overline{F}|.$$

Proof. The proof goes by induction. The case $n = 1$ is easy, since $E \square F$ can only occur when E is zero (say) and F one, which equals the event $E \cap \overline{F}$.

Now, assume the claim is true for $n = m - 1$. We have to prove the claim for $n = m$. For $x \in \{0,1\}^{m-1}$, let (x, s), $s \in \{0,1\}$ denote the concatenation of the vector x with s, i.e. $(x, s) = (x_1, \ldots, x_{m-1}, s)$. Let S be an arbitrary subset of $\{0,1\}^m$, define

$$S_0 := \{x \in \{0,1\}^{m-1} : (x, 0) \in S\},$$
$$S_1 := \{x \in \{0,1\}^{m-1} : (x, 1) \in S\}.$$

Note that $|S| = |S_0| + |S_1|$ for any set S. We proceed by bounding $|(E \square F)_1|$. It is not difficult to check that

$$|(E \square F)_1| \leq |E_1 \square F_0| + |E_0 \square F_1| - |E_0 \square F_0|. \tag{4.5}$$

Next, take $x \in (E \square F)_0$. Then there exist two disjoint sets of indices $I, J \subset \{1, \ldots, m\}$ such that E holds on $(x, 0)$ restricted to I and F on $(x, 0)$ restricted to J. Since E and F are both increasing it follows that x is contained in $E_0 \square F_0$. Hence,

$$|(E \square F)_0| \leq |E_0 \square F_0|.$$

Combining this inequality with (4.5) yields

$$|(E \square F)_1| \leq |E_1 \square F_0| + |E_0 \square F_1| - |(E \square F)_0|.$$

By applying the induction hypothesis to the first two terms on the right we obtain

$$|(E \square F)_1| \leq |E_1 \cap \overline{F_0}| + |E_0 \cap \overline{F_1}| - |(E \square F)_0|. \tag{4.6}$$

It is easy to check (flip all zeros to ones and vice versa) that $(\overline{S})_0 = \overline{S_1}$ and $(\overline{S})_1 = \overline{S_0}$ for every set $S \subset \{0,1\}^{m-1}$. Starting from the original event $E \square F$ we get

$$\begin{aligned} |E \square F| &= |(E \square F)_0| + |(E \square F)_1| \\ &\leq |E_1 \cap \overline{F_0}| + |E_0 \cap \overline{F_1}| - |(E \square F)_0| + |(E \square F)_0|, \end{aligned}$$

by (4.6). Hence,

$$\begin{aligned} |E \square F| &\leq |E_1 \cap \overline{F_0}| + |E_0 \cap \overline{F_1}| \\ &= |E_1 \cap (\overline{F})_1| + |E_0 \cap (\overline{F})_0| \\ &= |(E \cap \overline{F})_1| + |(E \cap \overline{F})_0| \\ &= |E \cap \overline{F}|. \end{aligned}$$

This proves the claim. □

We can now prove the BK inequality for random Voronoi percolation.

Proof of Theorem 4.1. Let A, B be two increasing events, both defined in a finite region. Then

$$\begin{aligned}\mathbb{P}(A \circ B) &= \sum_{\eta} \nu(\eta)\mathbb{P}(A \circ B|\eta) = \sum_{\eta} \nu(\eta)\mathbb{P}(A(\eta) \circ_{\eta} B(\eta)) \\ &\leq \sum_{\eta} \nu(\eta)\mathbb{P}(A(\eta) \cap \overline{B(\eta)}),\end{aligned} \tag{4.7}$$

where the inequality follows from Lemma 4.4. Therefore,

$$\begin{aligned}\mathbb{P}(A \circ B) &\leq \sum_{\eta} \nu(\eta)\mathbb{P}(A(\eta) \cap \overline{B(\eta)}) \\ &= \sum_{\eta} \nu(\eta)\mathbb{P}(A \cap \overline{B}|\eta) \\ &= \mathbb{P}(A \cap \overline{B}) \\ &\leq \mathbb{P}(A)\mathbb{P}(\overline{B}),\end{aligned}$$

where the last inequality follows from the FKG-inequality since $\overline{B}$ is decreasing. By symmetry (recall that the coloring probability equals $1/2$), $\mathbb{P}(\overline{B}) = \mathbb{P}(B)$. Hence, we proved the theorem. □

5 Dimension (in)equalities and Hölder continuous curves in fractal percolation

5.1 Introduction and main results

In this chapter we are concerned with a percolation model, first introduced in [44], which is known as Mandelbrot's fractal percolation process and which can be informally described as follows. For any integers $d \geq 2$ and $N \geq 2$, we start by dividing the unit cube $[0,1]^d \subset \mathbb{R}^d$ into N^d closed subcubes of equal size $1/N \times 1/N \times \cdots \times 1/N$. Given $p \in [0,1]$ and a subcube, we retain the subcube with probability p and discard it with probability $1-p$. This is done independently for every subcube of the partition. Sometimes we adopt the terminology calling retained cubes *black* and deleted cubes *white*. We define the random set $C_N^1 = C_N^1(d,p) \subset [0,1]^d$ as the union of all retained subcubes. Next consider any retained (assuming that $C_N^1 \neq \emptyset$) subcube B in C_N^1. We repeat the described procedure on a smaller scale by dividing B into N^d further subcubes, retaining them with probability p independently of all other subcubes. We do this for every retained subcube of C_N^1. This yields a new random set $C_N^2 \subset C_N^1$. Iterating the procedure on every smaller scale yields an infinite sequence of random sets $[0,1]^d \supset C_N^1 \supset C_N^2 \supset \cdots$ and we define the limiting set

$$C_N := \bigcap_{n=1}^{\infty} C_N^n.$$

We will hereafter suppress the N in our notation and simply write C for C_N.

We will need a more formal definition of the model as well. Let

$$I_k^n := \left[\frac{(k-1)}{N^n}, \frac{k}{N^n}\right],$$

where $n \geq 1$ and $1 \leq k \leq N^n$. For $\mathbf{k} = (k_1, \ldots, k_d)$, consider the subcube $D^n_{\mathbf{k}}$ of $[0,1]^d$ defined by $D^n_{\mathbf{k}} := I^n_{k_1} \times I^n_{k_2} \times \ldots \times I^n_{k_d}$, and let $D := \{D^n_{\mathbf{k}} : n \geq 1, 1 \leq k_l \leq N^n\}$. A cube $D^n_{\mathbf{k}}$ will sometimes be called a *level-n* cube. We define the sample space by

$$\Omega := \{0,1\}^D,$$

and denote an element of Ω by ω. We let $\mathcal{B}$ be the Borel σ-algebra on Ω generated by the cylinders and let $\mathbb{P}_p$ denote the product measure on $\mathcal{B}$ with density $p \in [0,1]$, that is, we let $\mathbb{P}_p(\omega(D^n_{\mathbf{k}}) = 1) = p$ independently for every $D^n_{\mathbf{k}} \in D$. The limiting set is then defined to be the intersection of all $D^n_{\mathbf{k}} \in D$ such that $\omega(D^n_{\mathbf{k}}) = 1$.

Let $CR([0,1]^d)$ denote the event that C contains a connected component which intersects the left hand side $\{0\} \times [0,1]^{d-1}$ of the unit cube and also intersects the right hand side $\{1\} \times [0,1]^{d-1}$. In this case we say that a *left-right crossing* of the unit cube occurs.

We define the *percolation function* $\vartheta_{N,d}$ by

$$\vartheta_{N,d}(p) := \mathbb{P}_p(CR([0,1]^d)). \tag{5.1}$$

The *critical value* is defined as

$$\tilde{p}_c = \tilde{p}_c(N,d) := \inf\{p : \vartheta_{N,d}(p) > 0\}.$$

It has been shown in [23] that the phase transition in Mandelbrot fractal percolation is non-trivial, i.e. $0 < \tilde{p}_c(N,d) < 1$. Furthermore it was discovered in [23] that for $d = 2$, $\vartheta_{N,d}(p)$ is discontinuous at $\tilde{p}_c$ (see [29] for an easy proof). This was generalised in [16] to all $d \geq 3$ and N large enough but the result is conjectured to hold for all N in any dimension. (At this point we remark that as a corollary to the proof of Theorem 5.1 below, we obtain an explicit bound for the size of the discontinuity at the critical value p_c, defined below and conjectured to coincide with $\tilde{p}_c$, in terms of the Hausdorff dimension of the set C^c, also to be defined below.)

In $d = 2$, the set C is a.s. totally disconnected for $p < \tilde{p}_c(N,2)$. This is also known to be true in higher dimensions for the same set of N for which it is known that $\vartheta_{N,d}(p)$ is discontinuous at $\tilde{p}_c$ (see [17]) and is conjectured to be true for all d and N. It is therefore natural to work with the following critical value:

$$p_c(N,d) := \sup\{p : C \text{ is a.s. totally disconnected}\}.$$

It is known (see [17]) that for any $d \geq 2$ and $N \geq 2$,

$$\mathbb{P}_p(C \text{ is not totally disconnected}) > 0$$

if $p = p_c(N,d)$. Given this, it is an easy exercise to show that for $p \geq p_c(N,d)$, $C \neq \emptyset$ implies that the set C^c consisting of the union of all connected components larger than one point is a.s. not empty.

We now fix $N, d \geq 2$ and assume that $p_c(N,d) \leq p < 1$. For any point $x \in C$, let $C_x \subset C$ be the set of points $y \in C$ that are connected to x in C. We call C_x the *connected component* of x. It is known (see [45]) that for $p \geq p_c(N,d)$ there exist a.s. uncountably many $x \in C$ such that $C_x = \{x\}$. We partition C into two sets, $C^d := \{x \in C : C_x = \{x\}\}$ and the aforementioned $C^c := C \setminus C^d$. (To understand the notation: d is short for "dust", and c is short for "connected".)

Before we can state our results we will need some more definitions. The reader is referred to [31] for a general overview of the subject of fractal sets.

A countable collection $\{B_i\}_{i=1}^{\infty}$ of subsets of $\mathbb{R}^d$ with diameter at most ε is called an ε-cover of F if $F \subset \cup_{i=1}^{\infty} B_i$. Define the *s-dimensional Hausdorff measure* of F as follows:

$$\mathcal{H}^s(F) := \liminf_{\varepsilon \to 0} \left\{ \sum_{i=1}^{\infty} diam(B_i)^s : \{B_i\}_{i=1}^{\infty} \text{ is an } \varepsilon\text{-cover of } F \right\}.$$

The *Hausdorff dimension* $\dim_{\mathcal{H}}(F)$ of F is defined as

$$\dim_{\mathcal{H}}(F) := \inf\{s : \mathcal{H}^s(F) = 0\}, \tag{5.2}$$

which also turns out to be equal to $\sup\{s : \mathcal{H}^s(F) = \infty\}$. The Hausdorff dimension of the limiting set in fractal percolation is a.s. given by the following equation, whose proof can be found in [23] or [31], Proposition 15.4:

$$\dim_{\mathcal{H}}(C) = \begin{cases} d + \frac{\log p}{\log N} & \text{if } C \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

There are many other concepts of dimensionality and we will in particular use the following. For a bounded set $F \subset \mathbb{R}^d$ let $M_\delta(F)$ be the minimal number of closed cubes of side length δ that is needed to cover F.

The *Lower Box counting dimension* of $F \subset \mathbb{R}^d$ is given by

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log M_\delta(F)}{-\log \delta},$$

while the *Upper Box counting dimension* of $F \subset \mathbb{R}^d$ is given by

$$\overline{\dim}_B(F) := \limsup_{\delta \to 0} \frac{\log M_\delta(F)}{-\log \delta}.$$

If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$ then the common value is denoted $\dim_B(F)$ and called the *Box counting dimension* of F. It is known (see e.g. [31]) that for any bounded set $F \subset \mathbb{R}^d$

$$\dim_{\mathcal{H}}(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F). \tag{5.4}$$

The next two theorems contain our dimension results for fractal percolation.

Theorem 5.1. *For $p_c(N, d) \leq p < 1$, we a.s. have*

$$\dim_B(C^c) = \dim_B(C) = \dim_{\mathcal{H}}(C). \tag{5.5}$$

If $C \neq \emptyset$ then a.s.

$$\dim_{\mathcal{H}}(C^c) < \dim_{\mathcal{H}}(C), \tag{5.6}$$

from which it easily follows that a.s.

$$\dim_{\mathcal{H}}(C^d) = \dim_{\mathcal{H}}(C). \tag{5.7}$$

Note that if $p < p_c(N, d)$, $C^c = \emptyset$ a.s. and so equation (5.6) still holds as long as $\dim_{\mathcal{H}}(C) > 0$.

Theorem 5.2. *There exists $1 \leq \beta \leq d$, dependent on p, such that a.s. either $C = \emptyset$ or $\dim_{\mathcal{H}}(C^c) = \beta$.*

For $\varepsilon > 0$, let $C^{c,\varepsilon}$ be the union of the connected components of diameter at least ε. The following result suggests that the "small components" of C^c are the ones which actually determine its Box counting dimension.

Proposition 5.3. *We have that*

$$\mathbb{E}_p[\underline{\dim}_B(C^{c,\varepsilon})] \leq D \dim_B(C^c),$$

where $D < 1$ is independent of ε and where the right hand side refers to the a.s. value taken by $\dim_B(C^c)$ if the limiting set is not empty.

When $p \geq \tilde{p}_c$, it is natural to ask about the nature of the left-right crossings of the unit cube. For $d = 2$, it was shown in [45] that C contains at least one continuous curve crossing the square as soon as a connected component crossing the square exists. It was later established in [26] (again for $d = 2$) that any curve in C must have Hausdorff dimension strictly larger than 1.

In this chapter, focusing again on the two-dimensional version of the model, we take the issue of the existence of continuous curves in C much further, using the sophisticated machinery of Aizenman and Burchard [2]. Their paper deals with scaling limits of systems of random curves, but we will show how their results can be useful in the context of fractal percolation as well. This is perhaps somewhat surprising, since the scaling limits in [2] deal with convergence in distribution, whereas in the fractal context, the fractal limiting set is an a.s. limit. The key will be a very careful comparison between convergence in the weak sense of curves in an appropriate topology, and convergence in the a.s. sense of compact sets in another topology. From such a comparison, one can obtain information about the compact sets that make up the a.s. limit of the fractal construction.

In order to state our results, we need some definitions. First of all, we define *interface curves* in the fractal process. The complement $\mathbb{R}^2 \setminus C^n$ consist of a finite number of connected components, exactly one of which is unbounded. The boundary of any such connected component can be split into closed curves (loops). We call such loops *interface curves* and denote by $\mathcal{F}_n$ the collection of interface curves after n iterations of the fractal process. In order for our interface curves to be uniquely defined, we orient them in such a way that they have black (retained) squares on the left and white (discarded) squares on the right, and assume that they turn to the right at corners where two white and two black squares meet in a checkerboard configuration, see Figure 5.1.

A connected subset of an interface curve delimited by a starting and an ending point will be called an *interface segment*.

We continue with recalling some general definitions concerning curves, taken from Chapter 2. We regard curves in $[0,1]^2$ as equivalence classes of continuous functions from $[0,1]$ to $[0,1]^2$ modulo increasing, continuous re-parametrizations. Below, γ will represent a particular curve and $\gamma(t)$ a particular parametrization of γ. Denote by $\mathcal{S}$ the complete separable metric space of curves in $[0,1]^2$ with metric

$$\mathrm{D}(\gamma_1, \gamma_2) := \inf \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|, \tag{5.8}$$

where the infimum is over all parametrizations of γ_1 and γ_2. The distance between

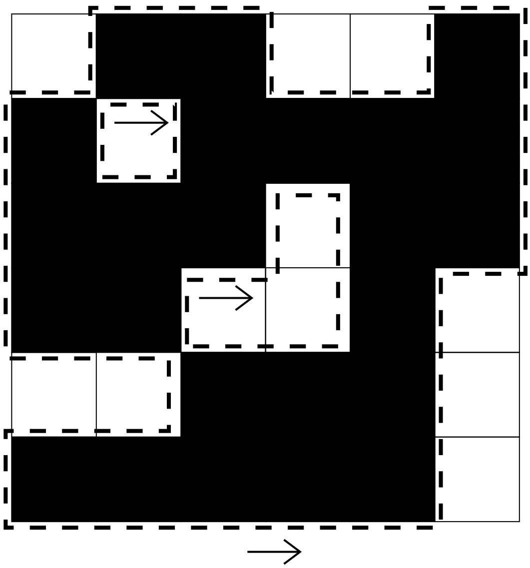


Figure 5.1: The interface curves are drawn with broken lines. Arrows indicate the orientation.

two sets $\mathcal{F}$ and $\mathcal{F}'$ of curves is defined by the Hausdorff metric induced by D, that is, $\mathrm{Dist}(\mathcal{F}, \mathcal{F}') \leq \varepsilon$ if and only if

$$\forall\, \gamma \in \mathcal{F},\ \exists\, \gamma' \in \mathcal{F}' \text{ with } \mathrm{D}(\gamma, \gamma') \leq \varepsilon \text{ and vice versa.} \tag{5.9}$$

The space Σ of closed subsets of $\mathcal{S}$ with the metric Dist is also a complete separable metric space.

The fractal process induces a probability measure μ_n on Σ, where μ_n denotes the distribution of $\mathcal{F}_n$. With this notation, we can present our main result on continuous curves in C.

Theorem 5.4. *The sequence of measures (μ_n) has subsequential weak limits. Any such weak limit μ assigns probability 1 to curve configurations in which all curves are Hölder continuous with the same exponent. The limiting set C has the same distribution as $g(\mathcal{F}) := \cup_{\gamma \in \mathcal{F}} \mathrm{Image}(\gamma)$, where $\mathcal{F}$ is a random set of curves distributed as μ. In other words, C is distributed as the union of the images of the curves in a sample from a weak limit of the (μ_n).*

We now briefly discuss this result. Since a single point is of course a Hölder continuous curve, the bare statement that a set is the union of Hölder continuous curves is in itself close to being an empty statement. However, the curves in C mentioned in Theorem 5.4 cannot be exclusively curves whose image is one point. One way to see this is to rephrase the notion of weak convergence as follows (see also [2]). A sequence of probability measures (μ_n) on Σ converges weakly to a probability μ on Σ if and only if there exists a family of probability measures ρ_n on $\Sigma \times \Sigma$ such that the first marginal of ρ_n is μ_n, the second marginal of ρ_n is μ (for all n), and

$$\int_{\Sigma\times\Sigma} \mathrm{Dist}(\mathcal{F}_n, \mathcal{F}) d\rho_n(\mathcal{F}_n, \mathcal{F}) \to 0$$

as $n \to \infty$. It is now also clear what happens to the points in the "dust set" C^d: these are accounted for as well in the theorem, since any point $x \in C^d$ can be approximated by curves in $\mathcal{F}_n$ whose diameter and distance to x converge to 0.

It is also possible to specialise to certain particular curves. As an example, we discuss the *lowest crossing* in C which we will first properly define. Condition on the existence in C^n of a left-right crossing of the unit square for all n, and consider the lowest interface segment σ_n in $\mathcal{F}_n$ connecting the left and right side of the unit square. The closure of the region in the unit square above σ_n is a compact set (in the Euclidean topology) which decreases in n and which therefore converges as $n \to \infty$. The lowest crossing in C is defined as the boundary of this limiting set.

Theorem 5.5. *If C contains a left-right crossing of the unit square, then the lowest crossing in C is a Hölder continuous curve.*

The machinery of Aizenman and Burchard also allows for a quick proof, given in Section 5.4, of the following result, first proved in [26] and extended in [49].

Theorem 5.6. *In two dimensions, there exists a constant $\kappa > 1$ such that all continuous curves in C^c have Hausdorff dimension at least κ.*

5.2 Proofs of Theorems 5.1, 5.2 and Proposition 5.3

Proof of Theorem 5.1. We start with the second equality of (5.5). Let Z_n be the number of cubes of side length N^{-n} that are retained in C^n. First observe that

$$\frac{\log Z_n}{-\log N^{-n}} = \frac{\log Z_n^{1/n}}{\log N}.$$

It is well known from the theory of branching processes (see e.g. [4, 28]) that $Z_n^{1/n} \to pN^d$ a.s. on the event $C \neq \emptyset$. Therefore, for a.e. ω such that $C(\omega) \neq \emptyset$,

$$\lim_{n\to\infty} \frac{\log Z_n}{-\log N^{-n}} = d + \frac{\log p}{\log N},$$

and hence $\overline{\dim}_B(C) \leq d + \frac{\log p}{\log N}$. Equations (5.3) and (5.4) then imply that $\dim_B(C) = \dim_{\mathcal{H}}(C)$.

For the first equality of (5.5), let $\{B_i\}_{i=1}^{M_\delta}$ be a cover of C^c using the minimal number M_δ of closed cubes of side length δ. For $A, B \subset \mathbb{R}^d$, define $d(A,B) := \inf\{|x-y| : x \in A, y \in B\}$, with $|\cdot|$ denoting Euclidean distance. Assume that there exists $x \in C$ such that $d(x, \bigcup_{i=1}^{M_\delta} B_i) > 0$. Then there must exist some $D^n_{\mathbf{k}}$ such that $d(D^n_{\mathbf{k}}, \bigcup_{i=1}^{M_\delta} B_i) > 0$ and $x \in D^n_{\mathbf{k}}$, which implies that $\omega(D^n_{\mathbf{k}}) = 1$ (i.e., $D^n_{\mathbf{k}}$ is retained). However, because of the scale invariant construction of C and the fact that, for $p \geq p_c(N,d)$, a.s. C is either empty or contains connected components larger than one point (see [17]), $C \cap D^n_{\mathbf{k}}$ must contain connected components larger than one point. This contradicts the fact that $\{B_i\}_{i=1}^{M_\delta}$ is a cover of C^c and shows that such an x cannot exist. Furthermore, since the union $\bigcup_{i=1}^{M_\delta} B_i$ is closed, it follows that, if $d(x, \bigcup_{i=1}^{M_\delta} B_i) = 0$ for $x \in C$, then $x \in \bigcup_{i=1}^{M_\delta} B_i$. Therefore, M_δ must be the minimal number of closed cubes of side length δ that covers C. This concludes the proof of (5.5).

Since $C = C^c \cup C^d$, (5.7) follows from (5.6) and the fact that (see [31])

$$\dim_{\mathcal{H}}(C) = \max(\dim_{\mathcal{H}}(C^c), \dim_{\mathcal{H}}(C^d)). \tag{5.10}$$

We proceed therefore by proving (5.6). Recall that $C^{c,\varepsilon}$ is the union of the connected components of diameter at least ε. For $p \geq p_c(N,d)$, we have that

$$\dim_{\mathcal{H}}(C^c) = \sup_{\varepsilon} \dim_{\mathcal{H}}(C^{c,\varepsilon}), \tag{5.11}$$

which is an easy consequence of the definition of Hausdorff dimension (see [31]). Therefore, it suffices to find an upper bound of $\dim_{\mathcal{H}}(C^{c,\varepsilon})$ which is uniform in ε and strictly smaller than $\dim_{\mathcal{H}}(C)$.

We will now assume that $N \geq 5$ is odd (this will avoid certain technicalities) and we leave it to the reader to adapt the proof for all cases $N \geq 2$. For $D^1_{\mathbf{k}}$ such that $(1/2, \ldots, 1/2) \in D^1_{\mathbf{k}}$, let $B(D^1_{\mathbf{k}}; 1) := [0,1]^d$ and $B(D^1_{\mathbf{k}}; 3N^{-1})$ be the two cubes concentric to $D^1_{\mathbf{k}}$ with side length 1 and $3N^{-1}$ respectively. Let $\varphi_{N,d}(p)$ be the probability that there exists a connected component of C that crosses the "shell" $B(D^1_{\mathbf{k}}; 1) \setminus B(D^1_{\mathbf{k}}; 3N^{-1})$. It follows from [17] that $\varphi_{N,d}(p) > 0$ whenever $p \geq p_c(N,d)$.

Assuming that $C^c \neq \emptyset$, fix $\varepsilon > 0$ such that C^c contains at least one component of diameter larger than ε and let l be such that $N^{-l+1} \leq \varepsilon/d$. Consider a cube $D^n_{\mathbf{k}}$ for $n \geq l$ which is intersected by a component of C^c of diameter larger than ε. Let $B(D^n_{\mathbf{k}}; 3N^{-n})$ and $B(D^n_{\mathbf{k}}; N^{-n+1})$ be two cubes which are concentric to $D^n_{\mathbf{k}}$ and have side length $3N^{-n}$ and N^{-n+1} respectively. Obviously, for $D^n_{\mathbf{k}}$ to be intersected by a connected component of diameter larger than ε, there must be a crossing of the shell $B(D^n_{\mathbf{k}}; N^{-n+1}) \setminus B(D^n_{\mathbf{k}}; 3N^{-n})$. (Note that, depending on the position of $D^n_{\mathbf{k}}$, it is possible that $B(D^n_{\mathbf{k}}; 3N^{-n})$ and/or $B(D^n_{\mathbf{k}}; N^{-n+1})$ are only partially contained in $[0,1]^d$.)

We will now construct a specific cover of $C^{c,\varepsilon}$ which we will use in our estimate for its Hausdorff dimension. Let W_n denote the set of cubes $D^n_{\mathbf{k}}$ with the following two properties:

- The intersection of all retained cubes of level n and higher contains a crossing of the shell $B(D^n_{\mathbf{k}}; N^{-n+1}) \setminus B(D^n_{\mathbf{k}}; 3N^{-n})$. In other words, if we would make all cubes black until level $n-1$ (inclusive), then there would be a connected component in C crossing the shell $B(D^n_{\mathbf{k}}; N^{-n+1}) \setminus B(D^n_{\mathbf{k}}; 3N^{-n})$,
- $D^n_{\mathbf{k}}$ is retained, that is, $\omega(D^n_{\mathbf{k}}) = 1$.

By scale invariance and independence between the two conditions, we have that $\mathbb{P}_p(D^n_{\mathbf{k}} \in W_n) \leq p\varphi_{N,d}(p)$. The inequality is due to a boundary effect since, as mentioned earlier, $B(D^n_{\mathbf{k}}; 3N^{-n})$ and/or $B(D^n_{\mathbf{k}}; N^{-n+1})$ need not be completely contained in $[0,1]^d$.

For a given cube $D^n_{\mathbf{k}}$, let B^m denote the level-m cube which contains $D^n_{\mathbf{k}}$, where $m \leq n$ (with $B^n = D^n_{\mathbf{k}}$). We make two observations:

1. If $D^n_{\mathbf{k}}$ has a non-empty intersection with $C^{c,\varepsilon}$, then we have that $B^m \in W_m$, for all $m = l, l+1, \ldots, n$.
2. The events $\{B^m \in W_m\}$ form a collection of independent events; see Figure 5.2.

This motivates us to define V_n as the collection of cubes $D^n_{\mathbf{k}}$ for which the corresponding cubes B^m are in W_m, for all $m = l, l+1, \ldots, n$. From observation 1, we have that the collection V_n forms a cover of $C^{c,\varepsilon}$. We can now write, using observation 2,

$$\begin{aligned}
\mathbb{P}_p(D^n_{\mathbf{k}} \in V_n) &= \mathbb{P}_p\left(\bigcap_{m=l}^{n} B^m \in W_m\right) \\
&= \prod_{m=l}^{n} \mathbb{P}_p(B^m \in W_m) \\
&\leq (p\varphi_{N,d}(p))^{n-l+1}.
\end{aligned}$$

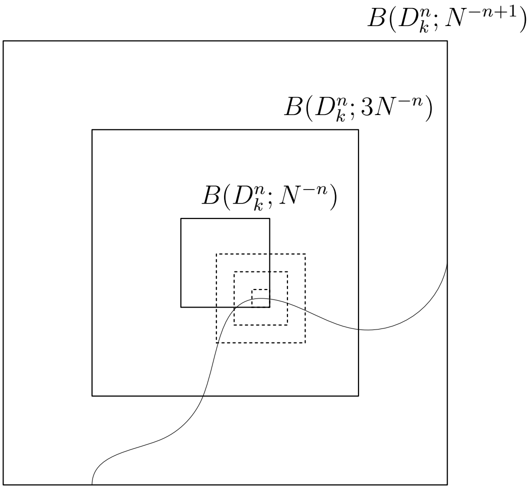


Figure 5.2: The cube D_k^n is in V_n and D_k^{n+1}, drawn with broken lines, belongs to V_{n+1}. Note that the corresponding shells are disjoint.

Using Fatou's lemma and the fact that the collection of cubes in V_n covers $C^{c,\varepsilon}$, we obtain (writing $|V_n|$ for the number of cubes in V_n)

$$\begin{aligned}\mathbb{E}_p\left(\mathcal{H}^s(C^{c,\varepsilon})\right) &\leq \liminf_{n\to\infty} \mathbb{E}_p\left(\sum_{D_{\mathbf{k}}^n \in V_n} diam(D_{\mathbf{k}}^n)^s\right) \\ &= \liminf_{n\to\infty}(\sqrt{d}N^{-n})^s \mathbb{E}_p(|V_n|) \\ &\leq \liminf_{n\to\infty} d^{s/2}N^{-sn}(p\varphi_{N,d}(p)N^d)^{n-l+1} \\ &= d^{s/2}N^{-d(l-1)}p\varphi_{N,d}(p) \lim_{n\to\infty} N^{n(d+\frac{\log(p\varphi_{N,d}(p))}{\log N}-s)}. \end{aligned} \tag{5.12}$$

The limit in (5.12) is finite if and only if

$$s \geq d + \frac{\log(p\varphi_{N,d}(p))}{\log N},$$

showing that

$$\dim_{\mathcal{H}}(C^{c,\varepsilon}) \leq d + \frac{\log(p\varphi_{N,d}(p))}{\log N} \quad \text{a.s.}$$

It follows from (5.11) that

$$\dim_{\mathcal{H}}(C^c) \le d + \frac{\log(p\varphi_{N,d}(p))}{\log N} \quad \text{a.s.}$$

and since $\varphi_{N,d}(p) < 1$ the result follows from this and (5.3). □

The proof of this theorem has an interesting corollary which links $\dim_{\mathcal{H}}(C^c)$ to the discontinuity at the critical point $p_c(N, d)$ (This is the corollary which was announced in the introduction.)

Corollary 5.7. *Let Δ denote the Hausdorff dimension of C^c when $p = p_c(N, d)$. Then,*

$$\varphi_{N,d}(p_c(N, d)) \ge \frac{1}{N^{d-\Delta}}.$$

Proof. Let $p_b(N, d) := \inf\{p \le 1 : \varphi_{N,d}(p) > 0\}$. Theorem 4.1 of [17] and the observation preceding it show that $p_b(N, d) = p_c(N, d)$. Moreover, it follows from [17] that $\varphi_{N,d}(p_c(N, d)) > 0$. Combining these observations with the last line of the proof of Theorem 5.1, we obtain that, for $p \ge p_c(N, d)$,

$$\varphi_{N,d}(p) \ge p\varphi_{N,d}(p) \ge \frac{1}{N^{d-\Delta}}.$$

□

Remark 5.8. *For Theorem 5.1 we use the result from [17] that for $p = p_c(N, d)$, $P_p(C^c \neq \emptyset) > 0$. It is possible to prove the result without this prior knowledge, as follows. We can start with the observation from [17] that $p_c = p_b$. We can then prove Theorem 5.1 in the case of $p > p_c$ and from the last line of that proof we get that $\varphi_{N,d}(p) \ge \frac{1}{N^{d-1}}$, using the fact that the Hausdorff-dimension of a connected set which consists of more than one point is always at least 1 (see e.g. [31], Proposition 4.1). Using this uniform bound and a right-continuity argument similar to the ones in [17], we conclude that in fact $\varphi_{N,d}(p_c) \ge \frac{1}{N^{d-1}}$. Hence we can* conclude *that for $p = p_c(N, d)$, $P_p(C^c \neq \emptyset) > 0$, and run through the proof once more to obtain the same result as above.*

Proof of Theorem 5.2. Let Z_n be the number of retained cubes after n steps of the fractal construction procedure and let $B_1, \ldots, B_{Z_n}$ be the retained cubes. If the event $\{\dim_{\mathcal{H}}(C^c) \ge \alpha\}$ occurs, then by (5.10), $\{\dim_{\mathcal{H}}(C^c \cap B_k) \ge \alpha\}$ for at least one $k = 1, \ldots, Z_n$. Therefore,

$$\mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \ge \alpha \Big| Z_n = l) = 1 - \prod_{k=1}^{l}(1 - \mathbb{P}_p(\dim_{\mathcal{H}}(C^c \cap B_k) \ge \alpha).$$

However, because of scale invariance, $\dim_{\mathcal{H}}(C^c \cap B_k)$, conditioned on the event that $B_k \subset C^n$, must have the same distribution as $\dim_{\mathcal{H}}(C^c)$ and so in fact

$$\mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha \Big| Z_n = l) = 1 - (1 - \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha))^l.$$

We can now write

$$\begin{aligned}
\mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha) &= \sum_{l=1}^{N^{dn}} \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha \Big| Z_n = l)\mathbb{P}_p(Z_n = l) \\
&= \sum_{l=1}^{N^{dn}} \left(1 - (1 - \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha))^l\right) \mathbb{P}_p(Z_n = l) \\
&\geq [1 - (1 - \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha))^n]\mathbb{P}_p(Z_n \geq n).
\end{aligned}$$

This last quantity is bounded below by

$$[1 - (1 - \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha))^n]\mathbb{P}_p(Z_n \geq n \Big| C \neq \emptyset)\mathbb{P}_p(C \neq \emptyset). \tag{5.13}$$

As mentioned above, a.s. $Z_n^{1/n} \to pN^d > 1$ as $n \to \infty$ if $C \neq \emptyset$. Therefore, if $\mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha) > 0$ for some α, then by taking the limit in inequality (5.13) we conclude that $\mathbb{P}_p(\dim_{\mathcal{H}}(C^c \geq \alpha)) \geq \mathbb{P}_p(C \neq \emptyset)$. If $\alpha > 0$, it easily follows that in fact $\mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha) = \mathbb{P}_p(C \neq \emptyset)$. Now note that, if $C^c \neq \emptyset$, then $\dim_{\mathcal{H}}(C^c) \geq 1$ (see, e.g., Proposition 4.1 of [31]), and define $\beta := \sup\{\alpha : \mathbb{P}_p(\dim_{\mathcal{H}}(C^c) \geq \alpha) > 0\}$. The statement holds with this choice of β. □

Proof of Proposition 5.3. Recall the definition of M_δ. We again use the fact that $C^{c,\varepsilon}$ is contained in the union of the cubes in V_n, defined in the proof of Theorem 5.1, and therefore, $M_{N^{-n}} = M_{N^{-n}}(C^{c,\varepsilon}) \leq |V_n|$. First observe that

$$\liminf_{\delta \to 0} \frac{\log M_\delta}{-\log \delta} = \liminf_{n \to \infty} \frac{\log M_{N^{-n}}}{-\log N^{-n}}.$$

By Fatou's lemma and Jensen's inequality we get along the same lines as in the last part of the proof of Theorem 5.1 (using the same l as in that proof) that

$$\begin{aligned}
&\mathbb{E}_p\left(\liminf_{n \to \infty} \frac{\log M_{N^{-n}}}{-\log N^{-n}}\right) \\
&\leq \liminf_{n \to \infty} \mathbb{E}_p\left(\frac{\log M_{N^{-n}}}{-\log N^{-n}}\right) \leq \liminf_{n \to \infty} \frac{\log \mathbb{E}_p(M_{N^{-n}})}{-\log N^{-n}} \\
&\leq \liminf_{n \to \infty} \frac{\log \mathbb{E}_p\left(|V_n|\right)}{n \log N} \leq \liminf_{n \to \infty} \frac{\log(p\varphi_{N,d}(p)N^d)^{n-l+1}}{n \log N} \\
&= \frac{\log(p\varphi_{N,d}(p)N^d)}{\log N} = d + \frac{\log(p\varphi_N(p))}{\log N}.
\end{aligned}$$

Since $\varphi_{N,d}(p) < 1$ it follows that $d + \log(p\varphi_{N,d}(p))/\log N < d + \log p/\log N$. □

5.3 Proof of Theorems 5.4 and 5.5

In this section and the next one we work in two dimensions, that is, $d = 2$. We will be using the machinery in [2] concerning scaling limits of systems of random curves.

One part of the argument is to apply some of the machinery developed in [2] in order to show that the sequence of measures μ_n defined in the introduction has subsequential weak limits, and we first deal with this issue. After that, we combine the existence of weak limits with the a.s. limit behaviour of the fractal process in order to draw the final conclusions.

Let $B(x, r)$ denote a closed square centered at x with side length r and $B^\circ(x; r)$ its interior. For $R > r$, let $A(x; r, R) := B(x, R) \setminus B^\circ(x, r)$ be an annulus. The basic estimate is the following, from which everything else will follow.

Lemma 5.9. *Let $p \geq p_c(N, 2)$. There exists a sequence $\lambda(1), \lambda(2), \ldots$ with $\lim_{k\to\infty} \lambda(k) = \infty$ and finite constants K_k such that, for u small enough, the following bound holds uniformly for all $r \leq R \leq u$ with r small enough, and all x:*

$$\limsup_{n\to\infty} \mathbb{P}_p(\mathcal{F}_n \textit{ contains } k \textit{ disjoint crossings of } A(x; r, R)) \leq K_k \left(\frac{r}{R}\right)^{\lambda(k)}. \tag{5.14}$$

Proof. We are looking for a collection of mutually disjoint annuli (all contained in $B(x, R)$ and "surrounding" $B(x, r)$) of the form $A(y; 2N^{-n}, 4N^{-n})$, where y is a corner point of some square $D^n_{\mathbf{k}}$ of the fractal construction. It is not hard to see that for any $x \in [0, 1]^2$, we can find such a collection with at least $M := c \log(R/r)$ elements, for a suitable uniform positive constant c. For small enough r, M is at least 1. For a given x, we denote these annuli by $A_{n_1}, A_{n_2}, \ldots, A_{n_M}$, where the indices $n_1 < n_2 < \cdots < n_M$ refer to the n associated with the annuli. The idea is that if there are k disjoint crossings of $A(x; r, R)$, then each of the annuli A_{n_i} must also be crossed by k disjoint crossings in $\mathcal{F}_n$. This is exponentially unlikely in the number of annuli, as we will show now.

We first consider the annulus A_{n_1}. Let $n > n_1$, and perform the fractal process until level n_1 (inclusive). The annulus A_{n_1} consists of 12 level-n_1 squares, some of which might be in C^{n_1} and some of which might not. Now observe that the collection of black (retained) squares of level n_1 in A_{n_1} is partitioned into at most 6 "level-n_1 components", where two neighbouring retained level-n_1 squares are in the same component if they share an edge.

Now let k be large enough (exactly how large will become clear soon). If A_{n_1} is, after n iterations, crossed by k disjoint interface segments, then at least one of the level-n_1 components must be crossed by at least $k/6$ such segments (we should write $\lfloor k/6 \rfloor$ but we ignore these details for the sake of notational convenience). This implies that this component is, after n iterations, also crossed by at least $k/12-1$ white crossings, namely the white crossings between the extremal interface crossings of the component.

The point of considering the level-n_1 components introduced above is that they are disjoint and separated by vacant squares, so that the fractal constructions inside them from level n_1 on are independent of each other, and none of the interface segments crossing A_{n_1} can intersect more than one of them. We can therefore bound the probability of having at least $k/6$ interface crossings in one such component after n iterations by the probability of having at least $k/12-1$ white crossings after n iterations in A_{n_1}, conditioned on having full retention up to level n_1. This means that also the probability of having at least k interface crossings in A_{n_1} after n iterations is bounded above by the probability of having at least $k/12-1$ white crossings after n iterations, conditioned on having full retention in A_{n_1} up to level n_1. However, scaling tells us that, if we condition on retention until level n_1, the probability in question is the same as the probability of having at least $k/12-1$ disjoint white components crossing $\bar{A} := A((0,0);2,4)$ when we perform $n - n_1$ iterations of the fractal process in $[-2,2]^2$ rather than in $[0,1]^2$, seen as the union of 16 independent fractal processes on the 16 unit squares making up $[-2,2]^2$. For these white crossing components we can use the BK inequality (see [45] - a similar BK inequality for black crossing is not available) and deduce that the probability of having $k/12-1$ of such white crossing components is bounded above by the probability of having at least one, raised to the power $k/12-1$. This then finally leads to the estimate that

$$\mathbb{P}_p(\mathcal{F}_n \text{ contains } k \text{ disjoint crossings of } A(x;r,R))$$

is bounded above by

$$\mathbb{P}_p(\bar{A} \text{ is crossed by a white component after } n-n_1 \text{ steps})^{k/12-1}. \tag{5.15}$$

If there is a white component crossing $\bar{A}$, then there is no black circuit surrounding the origin in $\bar{A}$. The probability of having such a black circuit after $n-n_1$ iterations is at least as large as the probability to have such a circuit in the limit, and by the weak RSW theorem for fractal percolation in [29] and the FKG inequality, we have that for $p \geq p_c(N, 2)$ this probability is strictly positive. It follows that there exists $\alpha < 1$ such that (5.15) is bounded above by $\alpha^{k/12-1}$, uniformly in $n > n_1$.

Next we consider A_{n_1} and A_{n_2} simultaneously. Take $n > n_2$. The probability to have k interface crossings in A_{n_1} and also in A_{n_2} is the probability that this happens in A_{n_1} multiplied by the probability that this happens in A_{n_2} conditioned on the fact that it happens in A_{n_1}. We can treat this conditional probability exactly as above: we can change the conditioning into one involving complete retention inside A_{n_2} until level n_2, to get an upper bound. It follows that the probability that both A_{n_1} and A_{n_2} have k interface crossings is bounded above by the square of the individual bounds, that is, by $(\alpha^{k/12-1})^2$.

We continue in the obvious way now, leading to the conclusion that for $n > n_M$, the probability that *all* annuli $A_{n_1}, \ldots, A_{n_M}$ are crossed by k interface crossings, is bounded above by

$$\left(\alpha^{k/12-1}\right)^{c \log(R/r)}.$$

A little algebra shows that this is equal to

$$\left(\frac{r}{R}\right)^{c \log(\alpha^{1-k/12})},$$

and this is a bound of the required form, with $\lambda(k) = c \log(\alpha^{1-k/12})$. □

We now describe how the existence of subsequential weak limits of the sequence μ_n follows from this lemma. (This is well known but perhaps not immediately obvious from the literature, hence our summary for the convenience of the reader.) Let

$$r^n_{\varepsilon,k} := \inf\left(\left\{0 < r \leq 1 : \begin{array}{c}\text{some annulus } A(x; r^{1+\varepsilon}, r), x \in [0,1]^2, \text{ is} \\ \text{crossed by } k \text{ disjoint crossings in } \mathcal{F}_n\end{array}\right\}, 1\right).$$

It follows exactly as in Chapter 2 that, as a consequence of Lemma 5.9, the random variables $r^n_{\varepsilon,k}$ are stochastically bounded away from zero as $n \to \infty$, that is, for any $\varepsilon > 0$,

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathbb{P}_p(r^n_{\varepsilon,k} \leq u) = 0. \tag{5.16}$$

(Note that in Lemma 5.9 we have the result only for r small enough, while the corresponding result in Chapter 2 is stated without that restriction. Our Lemma 5.9 however is sufficient to prove (5.16), as the reader can verify by checking the proof of Lemma 3.1 in [2].)

As shown in Chapter 2 equation (5.16) implies the following result.

Theorem 5.10 ([2]). *For any $\varepsilon > 0$, all curves $\Gamma \in \mathcal{F}_n$ can be parametrized by continuous functions $\gamma : [0,1] \to [0,1]^2$ such that for each curve, for all $0 \leq t_1 \leq t_2 \leq 1$*

$$|\gamma(t_1) - \gamma(t_2)| \leq k_\varepsilon^n |t_1 - t_2|^{\frac{1}{2+\varepsilon}},$$

where the random variables k_ε^n are stochastically bounded as $n \to \infty$, that is,

$$\lim_{u\to\infty} \limsup_{n\to\infty} \mathbb{P}_p(k_\varepsilon^n \geq u) = 0.$$

Once we have this result, we use once more the corresponding results from Chapter 2 to conclude that the sequence of measures $\{\mu_n\}_{n\geq 1}$ is tight. Since Σ is separable, it then follows from Prohorov's theorem that for every sequence $n_k \to \infty$ there exists a subsequence $n_{k_l} \to \infty$ such that $\mu_{n_{k_l}}$ converges weakly to a probability measure on Σ.

Finally, from the fact that the k_ε^n are stochastically bounded and the fact that the collection of curves with a given Hölder exponent is compact, we have that if we sample from any such weak limit, all curves γ in the sample can be parametrized in such a way that

$$|\gamma(t_1) - \gamma(t_2)| \leq M|t_1 - t_2|^\alpha, \tag{5.17}$$

where M is a random number common to all curves in the same sample, and α is a (non-random) constant.

Next we combine the above weak convergence with the a.s. convergence of the retained squares (as compact sets) in the fractal process. Let (S, H) denote the metric space of compact subsets of $[0,1]^d$ with the Hausdorff distance H and let (Σ, Dist) be as defined after equation (5.9). Furthermore let the function $g : \Sigma \mapsto S$ be defined by $g(\mathcal{F}) = \cup_{\gamma\in\mathcal{F}} \text{Image}(\gamma)$ and define $F_n = g(\mathcal{F}_n)$.

Our next result concerns weak convergence of $(\mathcal{F}_n, F_n)$ where we use the product topology on $\Sigma \times S$.

Lemma 5.11. *The distribution of $(\mathcal{F}_n, F_n)$ converges weakly along a subsequence. Furthermore, any pair $(\mathcal{F}, F)$ of random variables sampled from any such weak limit a.s. satisfies $F = g(\mathcal{F})$.*

Proof. We already know that the distribution μ_n of $\mathcal{F}_n$ converges weakly along a subsequence. Using (5.8) and (5.9), if $\mathrm{Dist}(\mathcal{F}, \mathcal{F}') \leq \delta$, this immediately implies that $H(F, F') \leq \delta$ since the images of any two curves γ_1, γ_2 such that $D(\gamma_1, \gamma_2) \leq \delta$ are within Hausdorff distance δ. This proves that g is continuous.

The convergence in distribution of $\mathcal{F}_n$ along some subsequence n_k to a limit $\mathcal{F}$ implies the existence of coupled versions X_k and X of $\mathcal{F}_{n_k}$ and $\mathcal{F}$ respectively, such that X_k converges to X in probability as $k \to \infty$ (see, e.g. Corollary 1 in [10]). Moreover, since g is continuous, $g(X_k) \overset{dist.}{=} F_{n_k}$ converges in probability to $g(X) \overset{dist.}{=} g(\mathcal{F})$ as $k \to \infty$. This implies convergence in probability of the vector $(X_k, g(X_k))$ to $(X, g(X))$, which yields the joint convergence in distribution of $(\mathcal{F}_{n_k}, F_{n_k})$ to some limit $(\mathcal{F}, F)$ with $F = g(\mathcal{F})$ a.s. □

Proof of Theorem 5.4. According to Lemma 5.11 there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $(\mathcal{F}_{n_k}, F_{n_k})$ converges weakly to some limit $(\mathcal{F}, F)$ where F is a.s. the union of the images of $\mathcal{F}$. Furthermore, we claim that a.s.

$$H(F_n, C) \to 0 \text{ as } n \to \infty, \tag{5.18}$$

where $F_n = g(\mathcal{F}_n)$. To see this, let F_n^ε denote the ε-neighbourhood of F_n and note that $C \not\subset F_n^\varepsilon$ implies that there exists an $x \in C$ such that $B(x, \varepsilon) \cap F_n = \emptyset$ and so $B(x, \varepsilon) \subset C_n$, otherwise an interface curve would be closer to x than ε. It is however easy to prove that the probability that there exists a ball of radius ε in C_n goes to 0. For the other direction, let $D = D(C, \varepsilon)$ be the complement of the open ε-neighbourhood of C. Obviously $D \cap C_n$ is a compact set (if nonempty) and furthermore $D \cap C_{n+1} \subset D \cap C_n$ for every n. Therefore, by compactness, if $D \cap C_n \neq \emptyset$ for every n then $\cap_{n=1}^{\infty} D \cap C_n \neq \emptyset$ and so there are points in C that are also in D which is a contradiction. Therefore, the open ε-neighbourhood of C will eventually contain C_n and hence also F_n.

Since F_{n_k} converges weakly to F and a.s. to C (because of (5.18)), we conclude that F and C have the same distribution, and hence C has the same distribution as $g(\mathcal{F})$. The fact that (as noted above) a.s., a realisation from $\mathcal{F}$ only contains Hölder continuous curves satisfying (5.17) finishes the proof. □

Proof of Theorem 5.5. Let ν_n be the distribution of σ_n. We repeat the proof of Theorems 5.4, with the distance (5.9) replaced by (5.8). This shows that conditioned on the existence of a left-right crossing for all n, ν_n has subsequential weak limits as $n \to \infty$ and in addition that any such limit assigns probability 1 to Hölder continuous curves. □

5.4 Proof of Theorem 5.6

We will start by showing that the fractal percolation process satisfies Hypothesis H2 in [2], from which Theorem 5.6 follows. The hypothesis concerns probabilistic bounds on crossings in the long direction of certain rectangles.

A collection of sets $\{A_i\}$ is called *well-separated* if for all i the distance of each set A_i to the other sets $\{A_j\}_{j \neq i}$ is at least as large as the diameter of A_i. Hypothesis H2 in [2] reads as follows.

Hypothesis 5.12. *There exist $\sigma > 0$ and some $\rho < 1$ such that for every collection of k well-separated rectangles $A_1, \dots, A_k$ of width $l_1, \dots, l_k$ and length $\sigma l_1, \dots, \sigma l_k$, the following inequality holds:*

$$\limsup_{n \to \infty} \mathbb{P}_p\Big(\mathcal{F}_n \text{ contains a long crossing in each of } A_1, \dots, A_k \Big) \leq \rho^k.$$

Lemma 5.13. *Hypothesis 5.12 holds for interface curves in the fractal process.*

Proof. We assume without loss of generality that $l_1 \geq l_2 \geq \cdots \geq l_k$. Let n_i be the smallest integer n for which *all* rectangles of dimensions l_i and σl_i contain an n-level square of the fractal construction, and define $\mathcal{A}_i$ as the event of complete retention until iteration step n_i. Let $CR_n^I(A_i)$ denote the event that the closed rectangle A_i is crossed in the long direction by an interface segment after n iterations of the fractal process, and define $CR_n^B(A_i)$ similarly for a black crossing.

Note that by the fact that $A_1, \dots, A_k$ are well-separated and by the choice of n_1, given $\mathcal{A}_1$, the event $CR_n^B(A_1)$ is, when $n > n_1$, conditionally independent of the events $CR_n^B(A_2), \dots, CR_n^B(A_k)$, since no level-$n$ square intersects more than one rectangle. Note also that, if a rectangle is crossed in the long direction by an interface segment, then it also contains a black crossing in the long direction. Using these two facts and the FKG inequality gives, for $n > n_1$,

$$\begin{aligned}
\mathbb{P}_p(\cap_{i=1}^k CR_n^I(A_i)) &\leq \mathbb{P}_p(\cap_{i=1}^k CR_n^B(A_i)) \\
&\leq \mathbb{P}_p(\cap_{i=1}^k CR_n^B(A_i) | \mathcal{A}_1) \\
&= \mathbb{P}_p(CR_n^B(A_1) | \mathcal{A}_1) \mathbb{P}_p(\cap_{i=2}^k CR_n^B(A_i) | \mathcal{A}_1).
\end{aligned}$$

Since $l_1 \geq l_2 \geq \cdots \geq l_k$, we have $n_1 \leq n_2 \leq \cdots \leq n_k$ and hence $\mathcal{A}_1 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k$. It follows as before that for $n > n_2$ we have

$$\begin{aligned}\mathbb{P}_p(\cap_{i=2}^k CR_n^B(A_i)|\mathcal{A}_1) &\leq \mathbb{P}_p(\cap_{i=2}^k CR_n^B(A_i)|\mathcal{A}_2) \\ &= \mathbb{P}_p(CR_n^B(A_2)|\mathcal{A}_2)\mathbb{P}_p(\cap_{i=3}^k CR_n^B(A_i)|\mathcal{A}_2).\end{aligned}$$

We repeat this procedure $k-2$ more times, finally obtaining, for $n > n_k$, that

$$\mathbb{P}_p(\cap_{i=1}^k CR_n^I(A_i)) \leq \prod_{i=1}^k \mathbb{P}_p(CR_n^B(A_i)|\mathcal{A}_i).$$

It remains to show that $\mathbb{P}_p(CR_n^B(A_i)|\mathcal{A}_i)$ is uniformly bounded above by some $\rho < 1$. This can be seen as follows. Let, for each i, W_i be a smallest collection of level-$(n_i + 1)$ squares in the fractal process with the property that if each of the squares in W_i is white, then A_i cannot be crossed by a black path in the long direction. It is easy to see from the choice of n_i that the cardinality of W_i is uniformly bounded above and hence that the probability that all squares in W_i are white is uniformly bounded below by some positive number. It follows that the probability of a long black crossing in A_i is uniformly bounded above by some number strictly smaller than 1. □

Proof of Theorem 5.6. The result follows from Lemma 5.13 and Theorem 1.3 in [2]. □

6 Fat fractal percolation and k-fractal percolation

6.1 Introduction

In [44] Mandelbrot introduced the following fractal percolation model. Let $N \geq 2, d \geq 2$ be integers and consider the unit cube $[0,1]^d$. Divide the unit cube into N^d subcubes of side length $1/N$. Each subcube is retained with probability p and discarded with probability $1-p$, independently of other subcubes. The closure of the union of the retained subcubes forms a random subset D_p^1 of $[0,1]^d$. Next, each retained subcube in D_p^1 is divided into N^d cubes of side length $1/N^2$. Again, each smaller subcube is retained with probability p and discarded with probability $1-p$, independently of other cubes. We obtain a new random set $D_p^2 \subset D_p^1$. Iterating this procedure in every retained cube at every smaller scale yields an infinite decreasing sequence of random subsets $D_p^1 \supset D_p^2 \supset D_p^3 \supset \cdots$ of $[0,1]^d$. We define the limit set $D_p := \bigcap_{n=1}^{\infty} D_p^n$. We will refer to this model as the Mandelbrot fractal percolation (MFP) model with parameter p. In this chapter we are concerned with two related percolation models which have the same fractal spirit, but in which subcubes are retained according to a different probabilistic mechanism. The models considered here are natural extensions of the classical Mandelbrot model, and are in fact mentioned in the literature (see below). We will next introduce the models and state our main results.

6.1.1 k-fractal percolation

Let $N \geq 2$ be an integer and divide the unit cube $[0,1]^d$, $d \geq 2$, into N^d subcubes of side length $1/N$. Fix an integer $0 < k \leq N^d$ and retain k subcubes in a uniform way, that is, all configurations where k cubes are retained have equal probability, other configurations have probability 0. Let D_k^1 denote the random set which is obtained by taking the closure of the union of all retained cubes. Iterating the described procedure in retained cubes and on all smaller scales yields a decreasing sequence of random sets $D_k^1 \supset D_k^2 \supset D_k^3 \supset \cdots$. We are mainly interested in the connectivity properties of the limiting set $D_k := \bigcap_{n=1}^{\infty} D_k^n$. This model was called the *micro-canonical fractal percolation process* by Lincoln Chayes in [25] and both *correlated fractal percolation* and *k out of N^d fractal percolation* by Dekking and Don [30]. We will adopt the terms *k-fractal percolation* and *k-model*.

For $F \subset [0,1]^d$, we say that the unit cube is *crossed by F* if there exists a connected component of F which intersects both $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. Define $\vartheta(k,N,d)$ as the probability that $[0,1]^d$ is crossed by D_k. Similarly, $\sigma(p,N,d)$ denotes the probability that $[0,1]^d$ is crossed by D_p. Let us define the critical probability $p_c(N,d)$ for ordinary Mandelbrot fractal percolation and the critical threshold value $k_c(N,d)$ for the k-model by

$$p_c(N,d) := \inf\{p : \sigma(p,N,d) > 0\}, \quad k_c(N,d) := \min\{k : \vartheta(k,N,d) > 0\}.$$

Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $(x_1,\ldots,x_d) = x \sim y = (y_1,\ldots,y_d)$ if and only if $x \neq y$, $|x_i - y_i| \leq 1$ for all i and $x_i = y_i$ for at least one value of i. Let $p_c(d)$ denote the critical probability for site percolation on $\mathbb{L}^d$. It is known (see [32]) that $p_c(N,d) \to p_c(d)$ as $N \to \infty$. We have the following analogous result for the k-model.

Theorem 6.1. *For all $d \geq 2$, we have that*

$$\lim_{N\to\infty} \frac{k_c(N,d)}{N^d} = p_c(d).$$

Remark 6.2. *Note that the choice for the unit cube in the definitions of $\vartheta(k,N,d)$ and $\sigma(p,N,d)$ (and thus implicitly also in the definitions of $k_c(N,d)$ and $p_c(N,d)$) is rather arbitrary: We could define them in terms of crossings of other shapes such as annuli, for example, and obtain the same conclusion, i.e. $k_c(N,d)/N^d \to p_c(d)$ as $N \to \infty$, where $\vartheta(k,N,d)$ and $k_c(N,d)$ are defined using the probability that D_k crosses an annulus. One advantage of using annuli is that the percolation function $\sigma(p,N,d)$ is known to have a discontinuity at $p_c(N,d)$ for all N,d and any choice of annulus [17, Corollary 2.6]. (This is known to be the case also when $p_c(N,d)$ is defined using the unit cube if $d = 2$ [23, 29], but for $d \geq 3$ it is proven only for N sufficiently large [16].) We stick to the "traditional" choice of the unit cube.*

For the MFP model it is the case that, for $p > p_c(d)$,

$$\sigma(p, N, d) \to 1, \tag{6.1}$$

as $N \to \infty$. This is part (b) of Theorem 2 in [32]. During the course of the proof of Theorem 6.1 we will prove a similar result for the k-model, see Theorem 6.10.

Next, consider the following generalization of both the k-model and the MFP model. Let $d \geq 2, N \geq 2$ be integers and let $Y = Y(N, d)$ be a random variable taking values in $\{0, \ldots, N^d\}$. Divide the unit cube into N^d smaller cubes of side length $1/N$. Draw a realization y according to Y and retain y cubes uniformly. Let D_Y^1 denote the closure of the union of the retained cubes. Next, every retained cube is divided into N^d smaller subcubes of side length $1/N^2$. Then, for every subcube C in D_Y^1 (where we slightly abuse notation by viewing D_Y^1 as the set of retained cubes in the first iteration step) draw a new (independent) realization $y(C)$ of Y and retain $y(C)$ subcubes in C uniformly, independently of all other subcubes. Denote the closure of the union of retained subcubes by D_Y^2. Repeat this procedure in every retained subcube at every smaller scale and define the limit set $D_Y := \bigcap_{n=1}^{\infty} D_Y^n$. We will call this model the *generalized fractal percolation model* (GFP model) *with generator* Y. Define $\varphi(Y, N, d)$ as the probability of the event that $[0, 1]^d$ is crossed by D_Y.

By taking Y equal to an integer k, resp. to a binomially distributed random variable with parameters N^d and p, we obtain the k-model, resp. the MFP model with parameter p. If Y is stochastically dominated by a binomial random variable with parameters N^d and p, where $p < p_c(d)$, then by standard coupling techniques it follows that $\varphi(Y, N, d) = 0$. Likewise, if $Y(N, d)$ dominates a binomial random variable with parameters N^d and p, where $p > p_c(d)$, then $\varphi(Y(N, d), N, d) \to 1$ as $N \to \infty$. The following theorem, which generalizes (6.1), shows that the latter conclusion still holds if for some $p > p_c(d)$, $\mathbb{P}(Y(N, d) \geq pN^d) \to 1$ as $N \to \infty$.

Theorem 6.3. *Consider the GFP model with generator $Y(N, d)$. Let $p > p_c(d)$. Suppose that $\mathbb{P}(Y(N, d) \geq pN^d) \to 1$ as $N \to \infty$. Then*

$$\lim_{N \to \infty} \varphi(Y(N, d), N, d) = 1.$$

Remark 6.4. *Observe that by Chebychev's inequality the condition of Theorem 6.3 is satisfied if, for some $p > p_c(d)$, $\mathbb{E}Y(N, d) \geq pN^d$, for all $N \geq 2$ and $\mathrm{Var}(Y(N, d))/N^{2d} \to 0$ as $N \to \infty$.*

Remark 6.5. *It is a natural question to ask whether a "symmetric version" of Theorem 6.3 is true. That is, if e.g. $\mathbb{P}(Y(N, d) \leq pN^d) \to 1$ as $N \to \infty$, for some $p < p_c(d)$, implies $\varphi(Y(N, d), N, d) \to 0$ as $N \to \infty$. The proof of Theorem 6.3 can not be adapted to this situation, but we could not find a counterexample.*

6.1.2 Fat fractal percolation

Let $(p_n)_{n\geq 1}$ be a non-decreasing sequence in $(0,1]$ such that $\prod_{n=1}^{\infty} p_n > 0$. We call *fat fractal percolation* a model analogous to the MFP model, but where at every iteration step n a subcube is retained with probability p_n and discarded with probability $1-p_n$, independently of other subcubes. Iterating this procedure yields a decreasing sequence of random subsets $D^1_{\text{fat}} \supset D^2_{\text{fat}} \supset D^3_{\text{fat}} \supset \cdots$ and we will mainly study connectivity properties of the limit set $D_{\text{fat}} := \bigcap_{n=1}^{\infty} D^n_{\text{fat}}$. In [27] it is shown that if $p_n \to 1$ and $\prod_{n=1}^{\infty} p_n = 0$, then the limit set does not contain a directed crossing from left to right.

For a point $x \in D_{\text{fat}}$, let C^x_{fat} denote its *connected component*:

$C^x_{\text{fat}} := \{y \in D_{\text{fat}} : y \text{ connected to } x \text{ in } D_{\text{fat}}\}.$

We define the set of "dust" points by $D^d_{\text{fat}} := \{x \in D_{\text{fat}} : C^x_{\text{fat}} = \{x\}\}$. Define $D^c_{\text{fat}} := D_{\text{fat}} \setminus D^d_{\text{fat}}$, which is the union of connected components larger than one point. Let λ denote the d-dimensional Lebesgue measure. It is easy to prove that $\lambda(D_{\text{fat}}) > 0$ with positive probability, see Proposition 6.16. Moreover, we can show that the Lebesgue measure of the limit set is positive a.s. given non-extinction, i.e. $D_{\text{fat}} \neq \emptyset$.

Theorem 6.6. *We have that $\lambda(D_{fat}) > 0$ a.s. given non-extinction.*

It is a natural question to ask whether both D^c_{fat} and D^d_{fat} have positive Lebesgue measure. The following theorem shows that they cannot simultaneously have positive Lebesgue measure.

Theorem 6.7. *Given non-extinction of the fat fractal process, it is the case that either*

$$\lambda(D^d_{fat}) = 0 \text{ and } \lambda(D^c_{fat}) > 0 \text{ a.s.} \tag{6.2}$$

or

$$\lambda(D^d_{fat}) > 0 \text{ and } \lambda(D^c_{fat}) = 0 \text{ a.s.} \tag{6.3}$$

Remark 6.8. *Part (ii) of Theorem 6.17 shows that if $\prod_{n=1}^{\infty} p_n^{N^n} > 0$, then the Lebesgue measure of the dust set is zero, and so we conclude that in this case (6.2) holds. However, we do not have an example for which (6.3) holds, and we do not know whether (6.3) is possible at all.*

Let us now outline the rest of the chapter. The next section will be devoted to a formal introduction of the fractal percolation processes in the unit cube. We also define an ordering on the subcubes which will facilitate the proof of Theorem 6.1 in Section 6.3. In Section 6.4 we prove Theorem 6.3, using results and techniques from Section 6.3. We then turn our attention to the fat fractal model. Section 6.5 contains the proofs of Theorem 6.6 and Theorem 6.7. In the last section we prove some additional results for the fat fractal model.

6.2 Preliminaries

In this section we set up an ordering for the subcubes of the fractal processes in the unit cube which will turn out to be very useful during the course of the proofs. We also give a formal probabilistic definition of the different fractal percolation models. We follow [32] almost verbatim in this section; a simple reference to [32] would however not be very useful for the reader, so we repeat some definitions here.

Order $J^d := \{0, 1, \dots, N-1\}^d$ in some way, say lexicographically by coordinates. For a positive integer n, write $J^{d,n} := \{(\mathbf{i}_1, \dots, \mathbf{i}_n) : \mathbf{i}_j \in J^d, 1 \le j \le n\}$ for the set of n vectors with entries in J^d. Set $J^{d,0} := \{\emptyset\}$. With $\mathbf{I} = (\mathbf{i}_1, \dots, \mathbf{i}_n) = ((i_{1,1}, \dots, i_{1,d}), \dots, (i_{n,1}, \dots, i_{n,d}))$ we associate the subcube of $[0,1]^d$ given by

$$C(\mathbf{I}) = c(\mathbf{I}) + [0, N^{-n}]^d,$$

where

$$c(\mathbf{I}) = \left(\sum_{j=1}^{n} N^{-j} i_{j,1}, \dots, \sum_{j=1}^{n} N^{-j} i_{j,d} \right)$$

and $c(\emptyset)$ is defined to be the origin. Such a cube $C(\mathbf{I})$ is called a *level-n cube*. A concatenation of $\mathbf{I} \in J^{d,n}$ and $\mathbf{j} \in J^d$ is denoted by $(\mathbf{I}, \mathbf{j})$, which is in $J^{d,n+1}$. We define the set of indices for all cubes until (inclusive) level-n as $\mathcal{J}^{(n)} := J^{d,0} \cup J^{d,1} \cup \dots \cup J^{d,n}$ and we order them in the following way. We declare $\mathbf{I} = (\mathbf{i}_1, \dots, \mathbf{i}_a) < \mathbf{I}' = (\mathbf{i}'_1, \dots, \mathbf{i}'_b)$ if and only if

- *either* $\mathbf{i}_r < \mathbf{i}'_r$ (according to the order on J^d) where $r \le \min\{a, b\}$ is the smallest index so that $\mathbf{i}_r \ne \mathbf{i}'_r$ holds;
- *or* $a > b$ and $\mathbf{i}_r = \mathbf{i}'_r$ for $r = 1, \dots, b$.

To clarify this ordering we give a short example, see Figure 6.1. Suppose $N = 2$, $d = 2$ and J^2 is ordered by $(1,1) > (1,0) > (0,1) > (0,0)$, then the ordering of $\mathcal{J}^{(2)}$

starts with

$$\emptyset > ((1,1)) > ((1,1),(1,1)) > ((1,1),(1,0))$$
$$> ((1,1),(0,1)) > ((1,1),(0,0)) > ((1,0)) > \dots$$

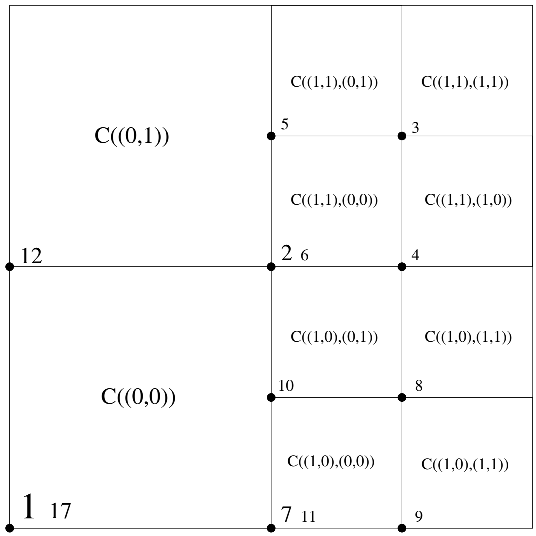


Figure 6.1: Illustration of the ordering of subcubes in $\mathcal{J}^{(2)}$, for $N = 2$ and $d = 2$. A black dot denotes the corner point $c(\mathbf{I})$ of a subcube $C(\mathbf{I})$. The number in the lower left corner of a subcube indicates the rank of the subcube in the ordering: e.g. the unit cube, i.e. $C(\emptyset)$, has rank 1 and $C((0,0))$ has rank 17.

For $\mathbf{I} = (\mathbf{i}_1, \dots, \mathbf{i}_n) = ((i_{1,1}, \dots, i_{1,d}), \dots, (i_{n,1}, \dots, i_{n,d}))$ we write $|\mathbf{I}| = n$. We introduce the following formal probabilistic definition of the fractal percolation models. As noted before, the k-model and MFP model can be obtained from the GFP model with generator Y by setting $Y \equiv k$, resp. Y binomially distributed with parameters N^d and $p \in [0,1]$. Therefore, we only provide a formal probabilistic definition of the GFP model and the fat fractal percolation model. Define the index set $\mathcal{J} := \bigcup_{n=0}^{\infty} J^{d,n}$. We define a family of random variables $\{Z_{\text{model}}(\mathbf{I})\}$, where $\mathbf{I} \in \mathcal{J}$ and – here as well as in the rest of the section – "model" stands for either p, fat, k or Y.

1. GFP model with generator Y: For every $\mathbf{I} \in \mathcal{J}$, let $y(\mathbf{I})$ denote a realization of Y, independently of other $\mathbf{I}'$. We define $J(\mathbf{I})$ as a uniform choice of $y(\mathbf{I})$ different

indices of J^d, independently of other $J(\mathbf{I}')$. For $\mathbf{j} \in J^d$ define

$$Z_Y(\mathbf{I}, \mathbf{j}) = \begin{cases} 1, & \mathbf{j} \in J(\mathbf{I}), \\ 0, & \text{otherwise.} \end{cases}$$

2. Fat fractal percolation with parameters $(p_n)_{n \geq 1}$: For every $\mathbf{I} \in \mathcal{J}$ and $\mathbf{j} \in J^d$, let $n = |\mathbf{I}|$ and define

$$Z_{\text{fat}}(\mathbf{I}, \mathbf{j}) = \begin{cases} 1, & \text{with probability } p_{n+1}, \\ 0, & \text{with probability } 1 - p_{n+1}, \end{cases}$$

independently of all other $Z_{\text{fat}}(\mathbf{I}')$.

For each $\mathbf{I} \in \mathcal{J}$ we define the indicator function $1(\mathbf{I})$ by

$$1(\emptyset) = 1, 1(\mathbf{I}) = Z_{\text{model}}(\mathbf{i}_1) \cdot Z_{\text{model}}(\mathbf{i}_1, \mathbf{i}_2) \cdots Z_{\text{model}}(\mathbf{I}),$$

where $\mathbf{I} = (\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_n) \in J^{d,n}$. We retain the subcube $C(\mathbf{I})$ if $1(\mathbf{I}) = 1$ and we write D^n_{model} for the set of retained level-n cubes. Note that $D^1_{\text{model}}, D^2_{\text{model}}, \ldots$ correspond to the sets informally constructed in the introduction. We denote by $\mathbb{P}_{\text{model}}$ the distribution of the corresponding model on $\Omega = \{0,1\}^C$, where $C := \{C(\mathbf{I}) : \mathbf{I} \in \mathcal{J}\}$ denotes the collection of all subcubes, endowed with the usual sigma algebra generated by the cylinder events. To simplify the notation, we will drop the subscripts fat, k, p, Y when there is no danger of confusion.

6.3 Proof of Theorem 6.1

In this section we prove Theorem 6.1. The proof is divided in two parts. First we treat the subcritical case and show that $\liminf_{N \to \infty} k_c(N,d)/N^d \geq p_c(d)$.

Theorem 6.9. *Consider the k-model. We have*

$$\liminf_{N \to \infty} k_c(N,d)/N^d \geq p_c(d).$$

In the supercritical case, we prove that $\vartheta(k, N, d) \to 1$ as $N \to \infty$. Again, for future reference we state this as a theorem.

Theorem 6.10. *Let $p > p_c(d)$ and let $(k(N))_{N \geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$, for all $N \geq 2$. We have*

$$\lim_{N \to \infty} \vartheta(k(N), N, d) = 1.$$

Theorem 6.1 follows immediately from these two theorems.

6.3.1 Proof of Theorem 6.9

Let $p < p_c(d)$ and consider a sequence $(k(N))_{N\geq 2}$ such that $k(N)/N^d \leq p$, for all $N \geq 2$, and $k(N)/N^d \to p$ as $N \to \infty$. Our goal is to show that the probability that the unit cube is crossed by $D_{k(N)}$, is equal to zero for all N large enough. Let $N \geq 2$ and let D_{p_0} be the limit set of an MFP process with parameters p_0 and N, where $p < p_0 < p_c(d)$. First, part (a) of Theorem 2 in [32] states that

$$p_c(d) \leq p_c(N, d), \tag{6.4}$$

for all N. Hence, the MFP process with parameter $p_0 < p_c(d)$ is subcritical. Therefore, a natural approach to prove that the probability that $D_{k(N)}$ crosses the unit cube equals zero for N large enough would be to couple the limit set $D_{k(N)}$ to the limit set D_{p_0} in such a way that $D_{k(N)} \subset D_{p_0}$. However, a "direct" coupling between the limit sets $D_{k(N)}$ and D_{p_0} is not possible, since with fixed positive probability at each iteration of the MFP process the number of retained subcubes is less than $k(N)$. We therefore need to find a more refined coupling.

The following is an informal strategy of the proof. We will define an event E on which the MFP process contains an infinite tree of retained subcubes, such that each subcube in this tree contains at least $k(N)$ retained subcubes in the tree. Next, we perform a construction of two auxiliary random subsets of the unit cube, from which it will follow that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E. In particular, the probability that $D_{k(N)}$ crosses $[0,1]^d$ is less than or equal to the conditional probability that D_{p_0} crosses the unit cube, given E. The latter probability is zero for N large enough, since the event E has positive probability for N large enough and the MFP process is subcritical.

Let us start by defining the event E. Consider an MFP process with parameters p_0 and N. For notational convenience we call the unit cube the level-0 cube. A level-n cube, $n \geq 0$, is declared *0-good* if it is retained and contains at least $k(N)$ retained level-$(n+1)$ subcubes. (We adopt the convention that $[0,1]^d$ is automatically retained.) Recursively, we define the notion *m-good*, for $m \geq 0$. A level-n cube, for $n \geq 0$, is $(m+1)$-good if it is retained and contains at least $k(N)$ m-good subcubes. We say that the unit cube is *∞-good* if it is m-good for every $m \geq 0$. Define the following events

$$\begin{aligned} E_m &:= \{[0,1]^d \text{ is } m\text{-good}\}, \\ E &:= \{[0,1]^d \text{ is } \infty\text{-good}\}. \end{aligned} \tag{6.5}$$

The following lemma states that we can make the probability of E arbitrary close to 1, for N large enough. In particular, E has positive probability for large enough N, which will be sufficient for the proof of Theorem 6.9.

Lemma 6.11. *Let $p_0 < p_c(d)$. Let $(k(N))_{N\geq 2}$ satisfy $\limsup_{N\to\infty} k(N)/N^d < p_0$. Consider an MFP model with parameters p_0 and N. For all $\varepsilon > 0$ there exists N_0 such that $\mathbb{P}_{p_0}(E) > 1-\varepsilon$ for all $N \geq N_0$.*

Proof. Let $\delta > 0$ and N_0 be such that $k(N)/N^d \leq p_0 - 2\delta =: p$ for all $N \geq N_0$. Choose $N_1 \geq N_0$ so large that $p_0/(4\delta^2 N^d) < \delta$ for $N \geq N_1$. We will show that

$$\mathbb{P}_{p_0}(E_m) \geq 1 - \frac{1}{4\delta^2 N^d}, \tag{6.6}$$

for all $m \geq 0$ and $N \geq N_1$. Since E_m decreases to E as $m \to \infty$, it follows that

$$\mathbb{P}_{p_0}(E) = \lim_{m\to\infty} \mathbb{P}_{p_0}(E_m) \geq 1 - \frac{1}{4\delta^2 N^d},$$

for $N \geq N_1$. Now take $N_2 \geq N_1$ so large that $1 - \frac{1}{4\delta^2 N^d} > 1-\varepsilon$ for all $N \geq N_2$. It remains to show (6.6).

We prove (6.6) by induction on m. Consider the event E_0, i.e. the event that the unit cube contains at least $k(N)$ retained level-1 subcubes. Let $X(n,p)$ denote a binomially distributed random variable with parameters $n \in \mathbb{N}$ and $p \in [0,1]$. Since the number of retained level-1 cubes has a binomial distribution with parameters N^d and p_0, it follows from Chebychev's inequality that, for every $N \geq N_1$, we have (writing $\mathbb{P}$ for the probability measure governing the binomially distributed random variables)

$$\begin{aligned}
\mathbb{P}_{p_0}(E_0) &= \mathbb{P}(X(N^d, p_0) \geq k(N)) \\
&\geq \mathbb{P}(X(N^d, p_0) \geq pN^d) \\
&\geq 1 - \frac{\operatorname{Var} X(N^d, p_0)}{4\delta^2 N^{2d}} \\
&= 1 - \frac{p_0(1-p_0)N^d}{4\delta^2 N^{2d}} \\
&\geq 1 - \frac{1}{4\delta^2 N^d}.
\end{aligned}$$

Next, let $m \geq 0$ and $N \geq N_1$ and suppose that (6.6) holds for this m and N. Recall that E_{m+1} is the event that the unit cube contains at least $k(N)$ m-good level-1 cubes. The probability that a level-1 cube is m-good, given that it is retained, is equal to $\mathbb{P}_{p_0}(E_m)$. Using the induction hypothesis, we get

$$\begin{aligned}\mathbb{P}_{p_0}(E_{m+1}) &= \mathbb{P}(X(N^d, p_0\mathbb{P}_{p_0}(E_m)) \geq k(N)) \\ &\geq \mathbb{P}(X(N^d, p_0(1 - \tfrac{1}{4\delta^2 N^d})) \geq k(N)).\end{aligned}$$

By our choices for δ and N it follows that $p_0(1 - \frac{1}{4\delta^2 N^d}) > p + \delta$. Hence, using Chebychev's inequality, we get

$$\begin{aligned}\mathbb{P}(X(N^d, p_0(1 - \tfrac{1}{4\delta^2 N^d})) \geq k(N)) &\geq \mathbb{P}(X(N^d, p+\delta) \geq k(N)) \\ &\geq \mathbb{P}(X(N^d, p+\delta) \geq pN^d) \\ &\geq 1 - \frac{\operatorname{Var} X(N^d, p+\delta)}{\delta^2 N^{2d}} \\ &\geq 1 - \frac{1}{4\delta^2 N^d}.\end{aligned}$$

Therefore, the induction step is valid and we have proved (6.6). □

Proof of Theorem 6.9. Let p, p_0 be such that $p < p_0 < p_c(d)$. Let $(k(N))_{N\geq 2}$ be a sequence such that $k(N)/N^d \leq p$, for all $N \geq 2$, and $k(N)/N^d \to p$ as $N \to \infty$. Consider an MFP model with parameters p_0 and N and define the event E as in (6.5). Henceforth, we assume that N is so large that $\mathbb{P}_{p_0}(E) > 0$, which is possible by Lemma 6.11. In order to prove Theorem 6.9 we will use E to construct two random subsets, $\tilde{D}_{p_0}$ and $\tilde{D}_{k(N)}$, of the unit cube, on a common probability space and with the following properties:

(i) $\tilde{D}_{k(N)} \subset \tilde{D}_{p_0}$;
(ii) the law of $\tilde{D}_{p_0}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E;
(iii) the law of $\tilde{D}_{k(N)}$ is the same as the law of $D_{k(N)}$.

It follows that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_{p_0}, conditioned on the event E. Hence, the probability that the unit cube is crossed by $D_{k(N)}$ is at most the conditional probability that D_{p_0} crosses the unit cube, conditioned on the event E. By (6.4) the MFP process with parameter p_0 is subcritical, thus the latter probability equals zero. Using the fact that $k(N)/N^d \to p$ as $N \to \infty$, we conclude that

$$\liminf_{N\to\infty} \frac{k_c(N)}{N^d} \geq p.$$

Since $p < p_c(d)$ was arbitrary, we get

$$\liminf_{N\to\infty} \frac{k_c(N)}{N^d} \geq p_c(d).$$

It remains to construct random sets $\tilde{D}_{p_0}, \tilde{D}_{k(N)}$ with the properties (i)-(iii). First we construct two sequences $(\tilde{D}^n_{p_0})_{n\geq1}, (\tilde{D}^n_{k(N)})_{n\geq1}$ of decreasing random subsets. Let $\mathcal{L}$ be the conditional law of the number of ∞-good level-1 cubes of the MFP process, conditioned on the event E. Note that the support of $\mathcal{L}$ is $\{k(N), k(N)+1, \ldots, N^d\}$. Furthermore, for a fixed level-n cube $C(\mathbf{I})$, $\mathcal{L}$ is also equal to the conditional law of the number of ∞-good level-$(n+1)$ subcubes in $C(\mathbf{I})$, conditioned on $C(\mathbf{I})$ being ∞-good.

Choose an integer l according to $\mathcal{L}$ and choose l level-1 cubes uniformly. Define $\tilde{D}^1_{p_0}$ as the closure of the union of these l level-1 cubes. Choose $k(N)$ out of these l cubes in a uniform way and define $\tilde{D}^1_{k(N)}$ as the closure of the union of these $k(N)$ cubes. For each level-1 cube $C(\mathbf{I}) \subset \tilde{D}^1_{p_0}$, pick an integer $l(\mathbf{I})$ according to $\mathcal{L}$, independently of other cubes, and choose $l(\mathbf{I})$ level-2 subcubes of $C(\mathbf{I})$ in a uniform way. Define $\tilde{D}^2_{p_0}$ as the closure of the union of all selected level-2 cubes. For each level-1 cube $C(\mathbf{I}) \subset \tilde{D}^1_{k(N)}$, uniformly choose $k(N)$ out of the $l(\mathbf{I})$ selected level-2 subcubes. Define $\tilde{D}^2_{k(N)}$ as the closure of the union of the $k(N)^2$ selected level-2 cubes of $C(\mathbf{I})$. Iterating this procedure yields two infinite decreasing sequences of random subsets $(\tilde{D}^n_{p_0})_{n\geq1}, (\tilde{D}^n_{k(N)})_{n\geq1}$.

Now define

$$\tilde{D}_{p_0} := \bigcap_{n\geq1} \tilde{D}^n_{p_0}, \quad \tilde{D}_{k(N)} := \bigcap_{n\geq1} \tilde{D}^n_{k(N)}.$$

By construction, for each $n \geq 1$, we have that (1) $\tilde{D}^n_{k(N)} \subset \tilde{D}^n_{p_0}$, (2) the law of $\tilde{D}^n_{p_0}$ is stochastically dominated by the conditional law of $D^n_{p_0}$ given E and (3) the law of $\tilde{D}^n_{k(N)}$ is equal to the law of $D^n_{k(N)}$. It follows that the limit sets $\tilde{D}_{p_0}, \tilde{D}_{k(N)}$ satisfy properties (i)-(iii). □

6.3.2 Proof of Theorem 6.10

Let us start by outlining the proof. The first part consists mainly of setting up the framework, where we use the notation of Falconer and Grimmett [32], which will enable us in the second part to prove that the subcubes of the k-fractal process satisfy certain "good" properties with probability arbitrarily close to 1 as $N \to \infty$. Informally, a subcube is good when there exist many connections inside the cube between its faces and when it is also connected to other good subcubes. Therefore, the probability of crossing the unit cube converges to 1 as N tends to ∞.

Although we will partly follow [32], it does not seem possible to use Theorem 2.2 of [32] directly. First, we state (a slightly adapted version of) Lemma 2 of [32], which concerns site percolation with parameter π on $\mathbb{L}^d$. We let every vertex of $\mathbb{L}^d$ be colored *black* with probability π and *white* otherwise, independently of other vertices. We write P_π for the ensuing product measure with density $\pi \in [0, 1]$. We call a subset C of $\mathbb{L}^d$ a *black cluster* if it is a maximal connected subset (with respect to the adjacency relation on $\mathbb{L}^d$) of black vertices.

Denote the cube with vertex set $\{1, 2, \ldots, N\}^d$ by B_N. Let $\mathcal{L}$ be the set of edges of the unit cube $[0, 1]^d$, that is $\mathcal{L}$ contains all sets of the form

$$L_r(\mathbf{a}) = \{a_1\} \times \{a_2\} \times \cdots \times \{a_{r-1}\} \times [0, 1] \times \{a_{r+1}\} \times \cdots \times \{a_d\}$$

as r ranges over $\{1, \ldots, d\}$ and $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ ranges over $\{0, 1\}^d$. For each $L = L_r(\mathbf{a}) \in \mathcal{L}$ we write

$$L_N = \{\mathbf{x} \in B_N : x_i = \max\{1, a_i N\} \text{ for } 1 \le i \le d, i \neq r\}$$

for the corresponding edge of B_N. The result is the following.

Lemma 6.12. *Suppose $\pi > p_c(d)$, $\varepsilon > 0$ and let q be a positive integer. There exist positive integers u and N_1 such that the following holds for all $N \ge N_1$. Let $U(1), \ldots, U(q)$ be subsets of vertices of B_N such that for each $r \in \{1, \ldots, q\}$, (i) $|U(r)| \ge u$ and (ii) there exists $L \in \mathcal{L}$ such that $U(r) \subset L_N$. Then,*

$$P_\pi\begin{pmatrix} \text{there exists a black cluster } C_N \text{ such that } |C_N \cap L_N| \ge u \\ \text{for all } L \in \mathcal{L}, \text{ and } |C_N \cap U(r)| \ge 1, \text{ for all } r \in \{1, \ldots, q\} \end{pmatrix} \ge 1 - \frac{\varepsilon}{2}. \tag{6.7}$$

Our goal is to show that the following holds uniformly in n: With probability arbitrarily close to 1 as $N \to \infty$, there is a sequence of cubes in D_k^n, each with at least one edge in common with the next, which crosses the unit cube. In order to prove this we examine the cubes $C(\mathbf{I})$ in turn according to the ordering on $\mathcal{J}^{(n)}$, and declare some of them to be good according to the rule given below. Since the probabilistic bounds on the goodness of cubes will hold uniformly in n, the desired conclusion follows.

Fix integers $n, u, k \geq 1$ until Lemma 6.15. Identify a level-n cube with a vertex in $B_{N^n} \subset \mathbb{L}^d$ in the canonical way. A set $\{C(\mathbf{I}_1), \ldots, C(\mathbf{I}_l)\}$ of level-n cubes is called *edge-connected* if they form a connected set with respect to the adjacency relation of $\mathbb{L}^d$. Whether a cube $C(\mathbf{I})$ is (n, u)-*good* or not is determined by the following inductive procedure. Let $\mathbf{I} \in \mathcal{J}^{(n)}$, and assume that the goodness of $C(\mathbf{I}')$ has been decided for all $\mathbf{I}' < \mathbf{I}$. We have the following possibilities:

(a) $|\mathbf{I}| = n$. Then $C(\mathbf{I})$ is always declared (n, u)-good.
(b) $0 \leq |\mathbf{I}| = m < n$.

In the latter case we act as follows. Note that the subcubes $C(\mathbf{I}, \mathbf{j})$ with $\mathbf{j} \in J^d$ have already been examined, since $(\mathbf{I}, \mathbf{j}) < \mathbf{I}$. Define the following set of level-$(m + 1)$ subcubes of $C(\mathbf{I})$,

$$\mathcal{D}(\mathbf{I}) := \{C(\mathbf{I}, \mathbf{j}) : \mathbf{j} \in J^d \text{ with } C(\mathbf{I}, \mathbf{j})\, (n, u)\text{-good and } Z_k(\mathbf{I}, \mathbf{j}) = 1\}. \tag{6.8}$$

$\mathcal{D}(\mathbf{I})$ is the set of retained (n, u)-good subcubes of $C(\mathbf{I})$. We declare $C(\mathbf{I})$ to be (n, u)-good if there exists an edge-connected set $\mathcal{H}(\mathbf{I}) \subset \mathcal{D}(\mathbf{I})$ such that

(i) Each edge of $C(\mathbf{I})$ intersects at least u cubes of $\mathcal{H}(\mathbf{I})$;
(ii) For every (n, u)-good level-m cube $C(\mathbf{I}')$ with $\mathbf{I}' < \mathbf{I}$ that has (at least) one edge in common with $C(\mathbf{I})$, there are a cube of $\mathcal{H}(\mathbf{I}')$ and a cube of $\mathcal{H}(\mathbf{I})$ with a common edge.

(If there is more than one candidate for $\mathcal{H}(\mathbf{I})$ we use some deterministic rule to choose one of them.) This procedure determines whether $C(\mathbf{I})$ is (n, u)-good for each $\mathbf{I}$ in turn. Note that it is easier for higher level cubes to be (n, u)-good than for lower level cubes. In particular, for the unit cube, i.e. $C(\emptyset)$, it is the hardest to be (n, u)-good.

The next lemma shows that if the unit cube is (n, u)-good then there is a sequence of cubes in D_k^n, each with at least one edge in common with the next, which connects the "left-hand side" of $[0, 1]^d$ with its "right-hand side". If such a sequence of cubes exists in D_k^n we say that *percolation occurs in* D_k^n.

Lemma 6.13. *Suppose* $[0, 1]^d$ *is* (n, u)*-good, then percolation occurs in* D_k^n.

Proof. Assume that the unit cube, i.e. $C(\emptyset)$, is (n, u)-good. We will show, with a recursive argument, that for $1 \leq m \leq n$ there exists an edge-adjacent chain of retained (n, u)-good level-m cubes which joins $\{0\} \times [0, 1]^{d-1}$ and $\{1\} \times [0, 1]^{d-1}$. In particular, this holds for $m = n$ and hence percolation occurs in D_k^n.

Since the unit cube is assumed to be (n, u)-good, $\mathcal{D}(\emptyset)$ contains by definition an edge-connected subset $\mathcal{H}(\emptyset)$ of retained (n, u)-good level-1 subcubes, such that each edge of $C(\emptyset)$ intersects at least u cubes of $\mathcal{H}(\emptyset)$. In particular, there is a sequence of retained (n,u)-good edge-adjacent level-1 cubes that connects the left-hand side of $[0,1]^d$ with its right-hand side.

Let $1 \leq m < n$ and assume that there exists a chain $C(\mathbf{I}_1), \dots, C(\mathbf{I}_l)$ of retained (n,u)-good level-m cubes which connects the left-hand side of $[0,1]^d$ with its right-hand side. For each i, $1 \leq i \leq l$, either $\mathbf{I}_i < \mathbf{I}_{i+1}$ or $\mathbf{I}_{i+1} < \mathbf{I}_i$. By condition (ii), there exist level-$(m+1)$ cubes of $\mathcal{H}(\mathbf{I}_{i+1})$ which are edge-adjacent to level-$(m+1)$ cubes of $\mathcal{H}(\mathbf{I}_i)$. These level-$(m+1)$ cubes $C(\mathbf{J})$ are all (n,u)-good and have $Z_k(\mathbf{J}) = 1$, by (6.8) and the definition of the $\mathcal{H}(\mathbf{I})$. It follows that there is an edge-adjacent chain of retained (n,u)-good level-$(m+1)$ cubes $C(\mathbf{J})$ which joins $\{0\} \times [0,1]^{d-1}$ and $\{1\} \times [0,1]^{d-1}$. □

Next, we consider the probabilities of various events. For $\mathbf{I} \in \mathcal{J}^{(n)}$, define the index $\mathbf{I}^- \in \mathcal{J}^{(n)}$ by

$$\mathbf{I}^- = \max\{\mathbf{I}' : \mathbf{I}' < \mathbf{I} \text{ and } |\mathbf{I}'| \leq |\mathbf{I}|\}.$$

If there is no such index, $\mathbf{I}^-$ is left undefined. For each $\mathbf{I} \in \mathcal{J}^{(n)}$ we let $\mathcal{F}(\mathbf{I})$ denote the σ-field

$$\mathcal{F}(\mathbf{I}) = \sigma(Z(\mathbf{I}', \mathbf{j}) : |\mathbf{I}'| \leq n-1, \mathbf{I}' \leq \mathbf{I}, \mathbf{j} \in J^d).$$

If $\mathbf{I}^-$ is undefined, we take $\mathcal{F}(\mathbf{I}^-)$ to be the trivial σ-field. Note that $\mathcal{F}(\mathbf{I})$ is generated by those Z that have been examined prior to deciding whether $C(\mathbf{I})$ is (n,u)-good. In particular, by virtue of the ordering on the cubes as introduced in Section 6.2, $\mathcal{F}(\mathbf{I}^-)$ does *not* contain any information about subcubes of $\mathbf{I}$.

For $p > p_c(d)$, let $(k(N))_{N \geq 2}$ be a sequence such that $k(N)/N^d \geq p$, for all $N \geq 2$. We want to prove that, for every $\varepsilon > 0$, the probability that $[0,1]^d$ is (n,u)-good in the $k(N)$-model is at least $1-\varepsilon$, for $N \geq N_0$, where N_0 is an integer which has to be taken sufficiently large to satisfy certain probabilistic bounds but is independent of n. We prove this using a recursive argument. Let us first give a sketch of the proof. The smallest level-n cube according to the ordering is by definition (n,u)-good. Then, for fixed $N \geq N_0$, assuming that $\mathbb{P}_{k(N)}(C(\mathbf{I}') \text{ is } (n,u)\text{-good} | \mathcal{F}(\mathbf{I}'^-)) > 1-\varepsilon$ for all $\mathbf{I}' < \mathbf{I}$, we have to prove that, given $\mathcal{F}(\mathbf{I}^-)$, $C(\mathbf{I})$ is (n,u)-good with probability at least $1-\varepsilon$. The proof of this consists of a coupling between a product measure with density $\pi \in (p_c(d), (1-\varepsilon)p)$ in the box B_N and the distribution of the set of good and retained subcubes. Applying Lemma 6.12 to the product measure combined with the coupling yields that the subcubes satisfy properties (i) and (ii) with probability at least $1-\varepsilon$ for $N \geq N_0$. Therefore, the cube $C(\mathbf{I})$ is then (n,u)-good with probability at least $1 - \varepsilon$ for $N \geq N_0$. Iterating this argument then yields that the unit cube is (n, u)-good with probability at least $1 - \varepsilon$, for $N \geq N_0$.

The proof in [32] of the analogous result that $\sigma(p, N, d) \to 1$ as $N \to \infty$ for $p > p_c(d)$ is considerably less involved. In the context of [32], subcubes are retained with probability p independently of other cubes, which is not the case in k-fractal percolation. Therefore, they can directly show that there exists $\pi > p_c(d)$ such that the distribution of the set of (n, u)-good and retained subcubes in $C(\mathbf{I})$ dominates an i.i.d. process on the box B_N with density π.

We need the following result for binomially distributed random variables, which we state as a lemma for future reference. Since the result follows easily from Chebychev's inequality, we omit the proof.

Lemma 6.14. *Let $p > p_c(d)$ and let $(k(N))_{N\geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$ for all $N \geq 2$. Let $\varepsilon > 0$ be such that $(1-\varepsilon)p > p_c(d)$, let $\pi \in (p_c(d), (1-\varepsilon)p)$ and define $M := ((1-\varepsilon)p + \pi)N^d/2$. There exists N_2 such that*

$$\mathbb{P}(\{X(k(N), 1-\varepsilon) \geq M\} \cap \{X'(N^d, \pi) \leq M\}) \geq 1 - \varepsilon/2,$$

for $N \geq N_2$, where X and X' are independent, binomially distributed random variables with the indicated parameters.

We now prove that, for any $\varepsilon > 0$, given $\mathcal{F}(\mathbf{I}^-)$ the unit cube is (n, u)-good with probability at least $1 - \varepsilon$, for N large enough but independent of n.

Lemma 6.15. *Let $p > p_c(d)$ and let $(k(N))_{N\geq 2}$ be a sequence of integers such that $k(N)/N^d \geq p$, for all $N \geq 2$. Let $\varepsilon > 0$ be such that $(1-\varepsilon)p > p_c(d)$. Take $\pi \in (p_c(d), (1-\varepsilon)p)$ and set $q = 3^d$. Let u and N_1 be given by Lemma 6.12. Let N_2 be given by Lemma 6.14. Set $N_0 = \max\{N_1, N_2\}$. Then, for all $n \geq 1$,*

$$\mathbb{P}_{k(N)}([0,1]^d \textit{ is } (n, u)\textit{-good }) \geq 1 - \varepsilon, \tag{6.9}$$

for all $N \geq N_0$.

Proof. Let $\varepsilon > 0$ be given. Our aim is to show that

$$\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n, u)\text{-good}|\mathcal{F}(\mathbf{I}^-)) \geq 1 - \varepsilon \tag{6.10}$$

holds for all $\mathbf{I} \in \mathcal{J}^{(n)}$. Taking $\mathbf{I} = \emptyset$ then yields (6.9). We prove this with a recursive argument. Let $\mathbf{I}_0$ be the smallest index in $\mathcal{J}^{(n)}$, according to the ordering on $\mathcal{J}^{(n)}$. By virtue of the ordering, we have $|\mathbf{I}_0| = n$. Hence, by definition, $C(\mathbf{I}_0)$ is (n,u)-good. In particular, (6.10) holds for $\mathbf{I}_0$.

Next, fix $N \geq N_0$ and consider the $k(N)$-fractal model. The recursive step is as follows. Take an index $\mathbf{I} \in \mathcal{J}^{(n)}$ and assume that

$$\mathbb{P}_{k(N)}(C(\mathbf{I}') \text{ is } (n,u)\text{-good}|\mathcal{F}(\mathbf{I}'^{-})) \geq 1-\varepsilon, \tag{6.11}$$

has been established for all indices $\mathbf{I}'$ in $\mathcal{J}^{(n)}$ less than $\mathbf{I}$. We have to show that (6.10) holds for $\mathbf{I}$ given this assumption.

For $\mathbf{I} \in \mathcal{J}^{(n)}$ we have two cases

(a) $|\mathbf{I}| = n$; then $\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n,u)\text{-good}) = 1$ and (6.10) is true.
(b) $0 \leq |\mathbf{I}| = m < n$.

For case (b), given $\mathcal{F}(\mathbf{I}^{-})$, the goodness of $C(\mathbf{I}')$ is determined (in particular) for all $\mathbf{I}' < \mathbf{I}$ with $|\mathbf{I}| = m$. Let

$$Q = \left\{ \mathbf{I}' : \begin{array}{c} \mathbf{I}' < \mathbf{I} \text{ and } C(\mathbf{I}') \text{ is a } (n,u)\text{-good level-}m \\ \text{cube with an edge in common with } C(\mathbf{I}) \end{array} \right\}.$$

For each $\mathbf{I}' \in Q$, let $E(\mathbf{I}')$ be some common edge of $C(\mathbf{I})$ and $C(\mathbf{I}')$. Since $C(\mathbf{I}')$ is (n,u)-good, there are at least u level-$(m+1)$ subcubes in $\mathcal{H}(\mathbf{I}')$ which intersect $E(\mathbf{I}')$; call this set of subcubes $\mathcal{U}(\mathbf{I}')$. To see whether $C(\mathbf{I})$ is (n,u)-good, we look at $C(\mathbf{I},\mathbf{j}(l))$ where $\mathbf{j}(l), 1 \leq l \leq N^d$, are the vectors of J^d arranged in order. We have $(\mathbf{I},\mathbf{j}(l)) < \mathbf{I}$, so by the induction hypothesis (6.11) we have

$$\mathbb{P}_{k(N)}(C(\mathbf{I},\mathbf{j}(l)) \text{ is } (n,u)\text{-good } |\mathcal{F}((\mathbf{I},\mathbf{j}(l))^{-})) \geq 1-\varepsilon, \tag{6.12}$$

for all l. We identify each subcube of $C(\mathbf{I})$ in the canonical way with a vertex in B_N. Choose $\pi \in (p_c(d),(1-\varepsilon)p)$. We will construct three random subsets G_1, G_2, G_3 of B_N on a common probability space with the following properties:

(I) the law of G_1 equals the law of the good and retained subcubes in $C(\mathbf{I})$;
(II) G_2 is obtained by first selecting $k(N)$ vertices of B_N uniformly and then retaining each selected vertex with probability $1-\varepsilon$, independently of other vertices;
(III) the law of G_3 is the Bernoulli product measure with density π on B_N;
(IV) $G_1 \supset G_2$;
(V) $\mathbb{P}(G_2 \supset G_3) \geq 1-\varepsilon/2$.

From (6.12) and a standard coupling technique, sometimes referred to as sequential coupling, the construction of G_1 and G_2 with properties (I), (II) and (IV) is straightforward. The construction of G_3 such that properties (III) and (V) hold is given below. Let $|G_2|$ denote the cardinality of the set G_2. Define $M = ((1-\varepsilon)p + \pi)N^d/2$ and let R be a number drawn from a binomial distribution with parameters N^d and π, independently of G_1 and G_2. If $|G_2| \geq M$ and $M \geq R$ we select R vertices uniformly out of the $|G_2|$ retained vertices of G_2 and call this set G_3. Otherwise, we select, independently of G_1 and G_2, R vertices of B_N in a uniform way and call this set G_3. From the construction (note that also G_2 was obtained in a uniform way) it is clear that G_3 satisfies property (III). Observe that $|G_2|$ has a binomial distribution with parameters $k(N)$ and $1-\varepsilon$. From Lemma 6.14 it follows that

$$\mathbb{P}(\{|G_2| \geq M\} \cap \{R \leq M\}) \geq 1 - \varepsilon/2.$$

Hence, property (V) also holds.

Let us now return to the goodness of $C(\mathbf{I})$. As before, we identify the random subsets G_1, G_2, G_3 of B_N with the corresponding sets of subcubes of $C(\mathbf{I})$ in the canonical way. It then follows from property (III) and Lemma 6.12 (note that Q has cardinality at most $3^d = q$) that G_3 has an edge-connected subset which satisfies the following properties with probability at least $1 - \varepsilon/2$:

(i) intersects every edge of $C(\mathbf{I})$ with at least u cubes;
(ii) contains a cube that is edge-adjacent to a cube of $\mathcal{U}(\mathbf{I}')$, for all $\mathbf{I}' \in Q$.

Combining properties (IV), (V) and the previous paragraph we obtain

$$\begin{aligned} &\mathbb{P}_{k(N)}(C(\mathbf{I}) \text{ is } (n,u)\text{-good}|\mathcal{F}(\mathbf{I}^-)) \\ &\geq \mathbb{P}(\{G_1 \supset G_3\} \cap \{G_3 \text{ satisfies properties (i) and (ii)}\}) \\ &\geq 1 - \varepsilon. \end{aligned}$$

Therefore, (6.10) holds for the index $\mathbf{I}$ given that (6.11) holds for all indices $\mathbf{I}' < \mathbf{I}$. A recursive use of this argument – recall that (6.11) is valid for $\mathbf{I}_0$ (the smallest index according to the ordering) – yields that (6.10) holds for all $\mathbf{I}$. Taking $\mathbf{I} = \emptyset$ in (6.10) proves the lemma. □

We are now able to conclude the proof of Theorem 6.10.

Proof of Theorem 6.10. Let $p > p_c(d)$ and consider a sequence $(k(N))_{N\geq 2}$ such that $k(N)/N^d \geq p$, for all $N \geq 2$. We get, using both Lemma 6.15 and Lemma 6.13, that for any $\varepsilon > 0$ such that $(1-\varepsilon)p > p_c(d)$, there exists N_0, depending on ε, such that

$$\mathbb{P}_{k(N)}(\text{percolation in } D^n_{k(N)}) \geq \mathbb{P}_{k(N)}([0,1]^d \text{ is } (n,u)\text{-good}) \geq 1 - \varepsilon, \tag{6.13}$$

for $N \geq N_0$. It is well known (see e.g. [32]) that

$$\{[0,1]^d \text{ is crossed by } D_{k(N)}\} = \bigcap_{n=1}^{\infty} \{\text{percolation in } D^n_{k(N)}\}.$$

Hence, taking the limit $n \to \infty$ in (6.13) yields that for $\varepsilon > 0$ small enough

$$\mathbb{P}_{k(N)}([0,1]^d \text{ is crossed by } D_{k(N)}) \geq 1 - \varepsilon, \tag{6.14}$$

for $N \geq N_0$. Therefore,

$$\vartheta(k(N), N, d) \to 1,$$

as $N \to \infty$. □

6.4 Proof of Theorem 6.3

Proof of Theorem 6.3. We use the idea of the proof of Theorem 6.9 and the result of Theorem 6.10. Fix some p_0 such that $p_c(d) < p_0 < p$ and set $k(N) := \lfloor p_0 N^d \rfloor$. Consider the event F that in the GFP model with generator Y there exists an infinite tree of retained subcubes such that each subcube in the tree contains at least $k(N)$ retained subcubes in the tree. Similar to the proof of Lemma 6.11, we prove that $\mathbb{P}(F) \to 1$ as $N \to \infty$. We then show that the law of $D_{k(N)}$ is stochastically dominated by the conditional law of D_Y, conditioned on the event F. By Theorem 6.10 we can then conclude that $\varphi(Y(N,d), N, d) \to 1$ as $N \to \infty$.

Consider the construction of D_Y. We will use the same definition of m-good as in Section 6.3.1, that is, if a level-n cube contains at least $k(N)$ retained subcubes, we call this level-n cube 0-good. Recursively, we say that a level-n cube is m-good if it contains at least $k(N)$ retained $(m-1)$-good level-$(n+1)$ cubes. We call the unit cube ∞-good if it is m-good for every $m \geq 1$. Define the following events

$$F_m := \{[0,1]^d \text{ is } m\text{-good}\},$$
$$F := \{[0,1]^d \text{ is } \infty\text{-good}\}.$$

We will show that for every $\varepsilon > 0$ such that $(1-\varepsilon)p > p_0$ there exists $N_0 = N_0(\varepsilon)$ such that, for all $m \geq 0$,

$$\mathbb{P}(F_m) > 1 - \varepsilon, \quad \text{for all } N \geq N_0. \tag{6.15}$$

The proof of (6.15) is similar to the proof of Lemma 6.11. Take $\delta > 0$ such that $(1-\varepsilon)p > p_0 + \delta$. Then, take N_0 so large that

$$1 - \frac{1}{4\delta^2 N} > 1 - \varepsilon/2 \quad \text{and} \tag{6.16}$$

$$\mathbb{P}(Y \geq pN^d) > 1 - \varepsilon/2, \tag{6.17}$$

for all $N \geq N_0$. We prove (6.15) by induction. Since $k(N) = \lfloor p_0 N^d \rfloor \leq pN^d$ it follows from (6.17) that $\mathbb{P}(F_0) > 1 - \varepsilon$.

Next, assume that (6.15) holds for some $m \geq 1$. The probability that a level-1 cube is m-good, given that it is retained, is equal to $\mathbb{P}(F_m)$. It follows that, given that the number of retained level-1 cubes equals y, the number of m-good level-1 cubes has a binomial distribution with parameters y and $\mathbb{P}(F_m)$. By our choices for N_0 and δ we get

$$\begin{aligned}
\mathbb{P}(F_{m+1}) &= \sum_{y \geq k(N)} \mathbb{P}(X(y, \mathbb{P}(F_m)) \geq k(N)) \cdot \mathbb{P}(Y = y) \\
&\geq \mathbb{P}(X(\lfloor pN^d \rfloor, \mathbb{P}(F_m)) \geq p_0 N^d) \cdot \mathbb{P}(Y \geq \lfloor pN^d \rfloor) \\
&\geq \mathbb{P}(X(\lfloor pN^d \rfloor, 1-\varepsilon) \geq p_0 N^d)(1 - \varepsilon/2) \\
&\geq \left(1 - \frac{\operatorname{Var} X(\lfloor pN^d \rfloor, 1-\varepsilon)}{(p_0 - (1-\varepsilon)p)^2 N^{2d}}\right)(1 - \varepsilon/2) \\
&\geq \left(1 - \frac{(1-\varepsilon)\varepsilon p N^d}{\delta^2 N^{2d}}\right)(1 - \varepsilon/2) \\
&\geq \left(1 - \frac{1}{4\delta^2 N^d}\right)(1 - \varepsilon/2) \\
&\geq (1 - \varepsilon/2)(1 - \varepsilon/2) > 1 - \varepsilon,
\end{aligned}$$

for all $N \geq N_0$. Hence, the induction step is valid.

Analogously to the proof of Theorem 6.9 we use the event $F = \bigcap_{m=1}^{\infty} F_m$ to construct two random subsets $\tilde{D}_{k(N)}$ and $\tilde{D}_Y$ on a common probability space, with the following properties:

(i) $\tilde{D}_{k(N)} \subset \tilde{D}_Y$;
(ii) the law of $\tilde{D}_Y$ is stochastically dominated by the conditional law of D_Y, conditioned on the event F;
(iii) the law of $\tilde{D}_{k(N)}$ is equal to the law of $D_{k(N)}$.

This construction is the same (modulo replacing the binomial distribution with Y) as in the proof of Theorem 6.9 and is therefore omitted. From properties (i)-(iii) and Theorem 6.10 we get

$$\begin{aligned}
&\mathbb{P}([0,1]^d \text{ is crossed by } D_{Y(N,d)}|F)\\
&\geq \mathbb{P}([0,1]^d \text{ is crossed by } \tilde{D}_{Y(N,d)})\\
&\geq \mathbb{P}([0,1]^d \text{ is crossed by } \tilde{D}_{k(N)})\\
&= \mathbb{P}([0,1]^d \text{ is crossed by } D_{k(N)}) \to 1,
\end{aligned}$$

as $N \to \infty$. Since (6.15) implies that $\mathbb{P}(F) \to 1$ as $N \to \infty$, we obtain

$$\mathbb{P}([0,1]^d \text{ is crossed by } D_{Y(N,d)}) \to 1,$$

as $N \to \infty$. □

6.5 Proofs of the main fat fractal results

In this section we prove our main results concerning fat fractal percolation. First, we state an elementary property of the fat fractal percolation model; it follows immediately from Fubini's theorem and we omit the proof.

Proposition 6.16. *The expected Lebesgue measure of the limit set of fat fractal percolation is given by*

$$\mathbb{E}\lambda(D_{fat}) = \prod_{n=1}^{\infty} p_n.$$

6.5.1 Proof of Theorem 6.6

Since $\prod_{n=1}^{\infty} p_n > 0$ it follows from Proposition 6.16 that with positive probability the limit set has positive Lebesgue measure given $D_{\text{fat}} \neq \emptyset$. Theorem 6.6 states that the latter holds with probability 1.

Proof of Theorem 6.6. Let Z_n denote the number of retained level-n cubes after iteration step n and set $Z_0 := 1$. Since the retention probabilities p_n vary with n, the process $(Z_n)_{n\geq 1}$ is a so-called branching process in a time-varying environment. Following the notation of Lyons in [43] let L_n be a random variable, having the distribution of Z_n given that $Z_{n-1} = 1$. Note that L_n has a binomial distribution with parameters N^d and p_n.

Define the process $(W_n)_{n\geq 1}$ by

$$W_n := \frac{Z_n}{\prod_{i=1}^n p_i N^d}.$$

It is straightforward to show that $(W_n)_{n\geq 1}$ is a martingale:

$$\begin{aligned}\mathbb{E}[W_n|W_{n-1}] &= \frac{\mathbb{E}[Z_n|Z_{n-1}]}{\prod_{i=1}^n p_i N^d} = \frac{Z_{n-1}}{\prod_{i=1}^n p_i N^d}\mathbb{E}[Z_n|Z_{n-1}=1] \\ &= \frac{Z_{n-1}p_n N^d}{\prod_{i=1}^n p_i N^d} = W_{n-1}.\end{aligned}$$

The Martingale Convergence Theorem tells us that W_n converges almost surely to a random variable W. Theorem 4.14 of [43] states that if

$$A := \sup_n \|L_n\|_\infty < \infty,$$

then $W > 0$ a.s. given non-extinction. It is clearly the case that $A < \infty$, because L_n can take at most the value N^d. Therefore, W_n converges to a random variable W which is stricly positive a.s. given non-extinction.

The Lebesgue measure of the retained cubes at each iteration step n is equal to Z_n/N^{dn}. We have

$$\lambda(D^n_{\text{fat}}) = \frac{Z_n}{N^{dn}} = \frac{\left(\prod_{i=1}^n p_i N^d\right) W_n}{N^{dn}} = \left(\prod_{i=1}^n p_i\right) W_n. \tag{6.18}$$

Letting $n \to \infty$ in (6.18) yields $\lambda(D_{\text{fat}}) = (\prod_{i=1}^\infty p_i)W$. Since $\prod_{i=1}^\infty p_i > 0$ and $W > 0$ a.s. given non-extinction, we get the desired result. □

6.5.2 Proof of Theorem 6.7

We start with a heuristic strategy for the proof. For a fixed configuration $\omega \in \Omega$, let us call a point x in the unit cube *conditionally connected* if the following property holds: If we change ω by retaining all cubes that contain x, then x is contained in a connected component larger than one point. We show that for almost all points x it is the case that x is conditionally connected with probability 0 or 1. We define an ergodic transformation T on the unit cube. The transformation T enables us to prove that the probability for a point x to be conditionally connected has the same value for λ-almost all x. From this we can then conclude that either the set of dust points or the set of connected components contains all Lebesgue measure.

Proof of Theorem 6.7. First, we have to introduce some notation. Let U be the collection of points in $[0,1]^d$ not on the boundary of a subcube. For each $x \in U$ there exists a unique sequence $(C(\mathbf{x}_1,\ldots,\mathbf{x}_n))_{n\geq 1}$ of cubes of the fractal process, where $\mathbf{x}_j \in J^d$ for all j, such that $\bigcap_{n\geq 1} C(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \{x\}$. Therefore, we can define an invertible transformation $\varphi : U \to (J^d)^{\mathbb{N}}$ by $\varphi(x) = (\mathbf{x}_1,\mathbf{x}_2,\ldots)$. For each $n \in \mathbb{N}$ let μ_n be the uniform measure on $(X_n,\mathcal{F}_n)$, where $X_n = J^d$ and $\mathcal{F}_n$ is the power set of X_n. Let $(X,\mathcal{F},\mu) = \bigotimes_{n=1}^{\infty}(X_n,\mathcal{F}_n,\mu_n)$ be the product space. Since $\varphi : (U,\mathcal{B}(U),\lambda) \to (X,\mathcal{F},\mu)$ is an invertible measure-preserving transformation, we have that $(X,\mathcal{F},\mu)$ is by definition isomorphic to $(U,\mathcal{B}(U),\lambda)$. Here $\mathcal{B}(U)$ denotes the Borel σ-algebra.

Next, we define the transformation $T : U \to U$, which will play a crucial role in the rest of the proof. Define the auxiliary shift transformation $T^* : X \to X$ by $T^*((\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\ldots)) = (\mathbf{x}_2,\mathbf{x}_3,\ldots)$, for $(\mathbf{x}_1,\mathbf{x}_2,\ldots) \in X$. The transformation T^* is measure preserving with respect to the measure μ and also ergodic, see for instance [59]. Let $T := \varphi^{-1} \circ T^* \circ \varphi$ be the induced transformation on U and note that T is isomorphic to T^* and hence also ergodic. Informally, T sends a point $x \in U$ to the point Tx, in such a way that the relative position of Tx in the unit cube is the same as the relative position of x in its level-1 cube $C(\mathbf{x}_1)$; see Figure 6.2.

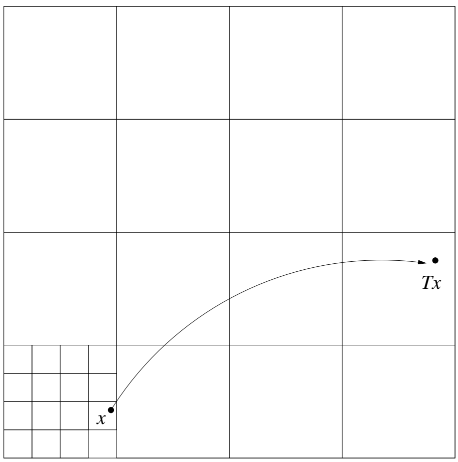


Figure 6.2: Illustration of the transformation T. Note that the relative position of x in the level-1 cube is the same as the relative position of Tx in the unit cube.

Recall that $\omega \in \Omega$ denotes a particular realization of the fat fractal percolation process. For $x \in U$, we define the following event.

$$A^x := \left\{\omega : \text{if we set } \omega(C(\mathbf{x}_1, \dots, \mathbf{x}_n)) = 1 \text{ for all } n \geq 1, \text{ then } C^x_{\text{fat}} \neq \{x\}\right\}.$$

In other words, A^x consists of those configurations ω such that when we change the configuration by retaining all $C(\mathbf{x}_1, \dots, \mathbf{x}_n)$, then in this new configuration, x is in the same connected component as some $y \neq x$. Observe that

$$A^x \cap \{x \in D_{\text{fat}}\} = \{x \in D^c_{\text{fat}}\}. \tag{6.19}$$

It is easy to see that A^x is a tail event. Hence, by Kolmogorov's 0-1 law we get $\mathbb{P}(A^x) \in \{0, 1\}$ for all $x \in U$.

However, a priori it is not clear that for almost all x in the unit cube $\mathbb{P}(A^x)$ has the same value. To this end, define the set $V := \{x \in U : \mathbb{P}(A^x) = 0\}$. We will show that $\lambda(V) \in \{0, 1\}$. Recall that the relative position of Tx in the unit cube is the same as the relative position of x in the level-1 cube $C(\mathbf{x}_1)$. It is possible to construct a coupling between the fractal process in the unit cube and the fractal process in $C(\mathbf{x}_1)$, given that $C(\mathbf{x}_1)$ is retained, with the following property: For every cube $C(\mathbf{I})$ in $C(\mathbf{x}_1)$, it is the case that if $TC(\mathbf{I})$ is retained in the fractal process in the unit cube, then $C(\mathbf{I})$ is also retained in the fractal process in $C(\mathbf{x}_1)$, given that $C(\mathbf{x}_1)$ is retained. It is straightforward that such a coupling exists since the retention probabilities p_n are non-decreasing in n. Hence,

$$\mathbb{P}(A^{Tx}) \leq \mathbb{P}(A^x | C(\mathbf{x}_1) \text{ is retained}). \tag{6.20}$$

Furthermore, since A^x is a tail event, we have

$$\mathbb{P}(A^x) = \mathbb{P}(A^x | C(\mathbf{x}_1) \text{ is retained}). \tag{6.21}$$

It follows from (6.20) and (6.21) that $\mathbb{P}(A^{Tx}) \leq \mathbb{P}(A^x)$ for all x. This implies that $V \subset T^{-1}V$. Because T is measure preserving it follows that

$$\lambda(V \Delta T^{-1}V) = \lambda(V \setminus T^{-1}V) + \lambda(T^{-1}V \setminus V) = 0 + \lambda(T^{-1}V) - \lambda(V) = 0.$$

Ergodicity of T now yields that $\lambda(V) \in \{0, 1\}$.

Suppose $\lambda(V) = 0$. Then $\mathbb{P}(x \in D^d_{\text{fat}}) = \mathbb{P}(\{x \in D_{\text{fat}}\} \setminus A^x) = 0$ for almost all $x \in [0, 1]^d$, by (6.19). Applying Fubini's theorem gives

$$\begin{aligned}
\mathbb{E}\lambda(D^d_{\text{fat}}) &= \int_\Omega \int_{[0,1]^d} 1_{D^d_{\text{fat}}}(x, \omega) d\lambda d\mathbb{P} \\
&= \int_{[0,1]^d} \int_\Omega 1_{D^d_{\text{fat}}}(x, \omega) d\mathbb{P} d\lambda \\
&= \int_{[0,1]^d} \mathbb{P}(x \in D^d_{\text{fat}}) d\lambda = 0.
\end{aligned}$$

Therefore $\lambda(D^d_{\text{fat}}) = 0$ a.s. By Theorem 6.6 we have $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction.

Next suppose that $\lambda(V) = 1$. Then with a similar argument we can show that $\lambda(D^c_{\text{fat}}) = 0$ and $\lambda(D^d_{\text{fat}}) > 0$ a.s. given non-extinction. □

6.6 More results on fat fractal percolation

The following theorem shows that the limit set either has an empty interior or can be written as the union of finitely many cubes a.s. Furthermore, it gives a sufficient condition under which the Lebesgue measure of the dust set is 0 a.s.

Theorem 6.17. *We have that*

(*i*) *If* $\prod_{n=1}^{\infty} p_n^{N^{dn}} = 0$, *then* D_{fat} *has an empty interior a.s.;*

(*ii*) *If* $\prod_{n=1}^{\infty} p_n^{N^n} > 0$, *then the Lebesgue measure of the dust set is 0 a.s.;*

(*iii*) *If* $\prod_{n=1}^{\infty} p_n^{N^{dn}} > 0$, *then* D_{fat} *can be written as the union of finitely many cubes a.s.*

Proof. (i) Suppose that D_{fat} has a non-empty interior with positive probability. Then we have

$$\begin{aligned} 0 &< \mathbb{P}(D_{\text{fat}} \text{ has non-empty interior}) \\ &= \mathbb{P}(\exists n, \exists \mathbf{i}_1, \ldots, \mathbf{i}_n : C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}) \\ &\leq \sum_{n, \mathbf{i}_1, \ldots, \mathbf{i}_n} \mathbb{P}(C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}). \end{aligned}$$

Since we sum over countably many cubes, there must exist n and $\mathbf{i}_1, \ldots, \mathbf{i}_n$ such that $\mathbb{P}(C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}) > 0$. Hence, by translation invariance, $\mathbb{P}(C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}) > 0$ for this specific n and all $\mathbf{i}_1, \ldots, \mathbf{i}_n$. We can apply the FKG inequality to obtain $\mathbb{P}(D_{\text{fat}} = [0,1]^d) = \mathbb{P}(C(\mathbf{i}_1, \ldots, \mathbf{i}_n) \subset D_{\text{fat}}\ \forall \mathbf{i}_1, \ldots, \mathbf{i}_n) > 0$. Since $\mathbb{P}(D_{\text{fat}} = [0,1]^d) = \prod_{n=1}^{\infty} p_n^{N^{dn}}$, this proves the first part of the theorem.

(ii) Suppose $\prod_{n=1}^{\infty} p_n^{N^n} > 0$. Then for each $x \in [0,1]^{d-1}$ we have $\mathbb{P}(\{x\} \times [0,1] \subset D_{\text{fat}}) \geq \prod_{n=1}^{\infty} p_n^{N^n} > 0$. Let λ_{d-1} denote $(d-1)$-dimensional Lebesgue measure.

Applying Fubini's theorem gives

$$\begin{aligned}
&\mathbb{E}\lambda_{d-1}(\{x \in [0,1]^{d-1} : \{x\} \times [0,1] \subset D_{\text{fat}}\}) \\
&= \int_\Omega \int_{[0,1]^{d-1}} 1_{\{x\}\times[0,1]\subset D_{\text{fat}}} \, d\lambda_{d-1} d\mathbb{P} \\
&= \int_{[0,1]^{d-1}} \int_\Omega 1_{\{x\}\times[0,1]\subset D_{\text{fat}}} \, d\mathbb{P} d\lambda_{d-1} \\
&= \int_{[0,1]^{d-1}} \mathbb{P}(\{x\} \times [0,1] \subset D_{\text{fat}}) d\lambda_{d-1} > 0.
\end{aligned}$$

Hence,

$$\lambda_{d-1}(\{x \in [0,1]^{d-1} : \{x\} \times [0,1] \subset D_{\text{fat}}\}) > 0 \tag{6.22}$$

with positive probability. Observe that

$$D_{\text{fat}}^c \supset \bigcup_{x\in[0,1]^{d-1}:\{x\}\times[0,1]\subset D_{\text{fat}}} \{x\} \times [0,1].$$

In particular,

$$\lambda(D_{\text{fat}}^c) \geq \lambda_{d-1}(\{x \in [0,1]^{d-1} : \{x\} \times [0,1] \subset D_{\text{fat}}\}).$$

From (6.22) we conclude that $\lambda(D_{\text{fat}}^c) > 0$ with positive probability. It now follows from Theorem 6.7 that the Lebesgue measure of the dust set is 0 a.s.

(iii) Next assume that $\prod_{n=1}^\infty p_n^{N^{dn}} > 0$. For each level n, we have $\mathbb{P}(D_{\text{fat}}^n = D_{\text{fat}}^{n-1}) \geq p_n^{N^{dn}}$. Since $\prod_{n=1}^\infty p_n^{N^{dn}} > 0$ is equivalent to $\sum_{n=1}^\infty (1 - p_n^{N^{dn}}) < \infty$, we have

$$\sum_{n=1}^\infty \mathbb{P}(D_{\text{fat}}^n \neq D_{\text{fat}}^{n-1}) \leq \sum_{n=1}^\infty (1 - p_n^{N^{dn}}) < \infty.$$

Applying the Borel-Cantelli lemma gives that, with probability 1, $\{D_{\text{fat}}^n \neq D_{\text{fat}}^{n-1}\}$ occurs for only finitely many n. Hence, with probability 1 there exists an n such that D_{fat} can be written as the union of level-n cubes. □

In two dimensions, we have the following characterizations of $\lambda(D_{\text{fat}}^c)$ being positive a.s. given non-extinction of the fat fractal process.

Theorem 6.18. *Let $d = 2$. The following statements are equivalent.*

(i) $\lambda(D_{fat}^c) > 0$ *a.s., given non-extinction of the fat fractal process;*

(*ii*) *There exists a set* $U \subset [0,1]^2$ *with* $\lambda(U) > 0$ *such that for all* $x, y \in U$ *it is the case that* $\mathbb{P}(x$ *is in the same connected component as* $y) > 0$*;*
(*iii*) *There exists a set* $U \subset [0,1]^2$ *with* $\lambda(U) = 1$ *such that for all* $x, y \in U$ *it is the case that* $\mathbb{P}(x$ *is in the same connected component as* $y) > 0$*;*

Proof. (*iii*) $\Rightarrow$ (*ii*). Trivial.

(*ii*) $\Rightarrow$ (*i*). Suppose $\mathbb{P}(x \text{ connected to } y) > 0$ for all $x, y \in U$, for some set $U \subset [0,1]^2$ with $\lambda(U) > 0$. Fix $y \in U$. By Fubini's theorem

$$\begin{aligned}
\mathbb{E}\lambda(D^c_{\text{fat}}) &= \int_\Omega \int_{[0,1]^2} 1_{D^c_{\text{fat}}}(x,\omega) d\lambda(x) d\mathbb{P}(\omega) \\
&= \int_{[0,1]^2} \int_\Omega 1_{D^c_{\text{fat}}}(x,\omega) d\mathbb{P}(\omega) d\lambda(x) \\
&= \int_{[0,1]^2} \mathbb{P}(x \in D^c_{\text{fat}}) d\lambda(x) \\
&\geq \int_{U \setminus \{y\}} \mathbb{P}(x \text{ connected to } y) d\lambda(x) > 0.
\end{aligned}$$

Hence $\lambda(D^c_{\text{fat}}) > 0$ with positive probability. By Theorem 6.7 it follows that $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction of the fat fractal process.

(*i*) $\Rightarrow$ (*iii*). Next suppose that $\lambda(D^c_{\text{fat}}) > 0$ a.s. given non-extinction of the fat fractal process. For points $x \in [0,1]^2$ not on the boundary of a subcube, define the event A^x as in the proof of Theorem 6.7. It follows from the proof of Theorem 6.7 that $\mathbb{P}(A^x) = 1$ for all $x \in V$, for some set $V \subset [0,1]^2$ with $\lambda(V) = 1$. By (6.19) we have for all $x \in V$

$$\mathbb{P}(x \in D^c_{\text{fat}}) = \mathbb{P}(x \in D_{\text{fat}}) > 0.$$

Let $x \in V$. Then

$$0 < \mathbb{P}(x \in D^c_{\text{fat}}) \leq \sum_{n=1}^{\infty} \mathbb{P}(\text{diam}(C^x_{\text{fat}}) > \tfrac{1}{n}),$$

where $\text{diam}(C^x_{\text{fat}})$ denotes the diameter of the set C^x_{fat}. So there exists a natural number n_x such that $\mathbb{P}(\text{diam}(C^x_{\text{fat}}) > \frac{1}{n_x}) > 0$. Hence

$$\mathbb{P}(x \text{ connected to } S(x, \tfrac{1}{2n_x})) > 0,$$

where $S(x, \frac{1}{2n_x})$ is a circle centered at x with radius $\frac{1}{2n_x}$. Write $x = (x_1, x_2)$ and define the following subsets of $\mathbb{R}^2$

$$
\begin{aligned}
H_1 &= [0,1] \times [x_2 - \tfrac{1}{4n_x}, x_2],\\
H_2 &= [0,1] \times [x_2, x_2 + \tfrac{1}{4n_x}],\\
V_1 &= [x_1 - \tfrac{1}{4n_x}, x_1] \times [0,1],\\
V_2 &= [x_1, x_1 + \tfrac{1}{4n_x}] \times [0,1].
\end{aligned}
$$

Note that for every $x \in [0,1]^2$ it is the case that at least one horizontal strip H_i and at least one vertical strip V_j is entirely contained in $[0,1]^2$. Define the event Γ_x by

$$
\begin{aligned}
\Gamma_x = &\bigcap_{i\in\{1,2\}:H_i\subset[0,1]^2} \{\text{horizontal crossing in } H_i\}\\
&\cap \bigcap_{j\in\{1,2\}:V_j\subset[0,1]^2} \{\text{vertical crossing in } V_j\}.
\end{aligned}
$$

See Figure 6.3 for an illustration of the event Γ_x. From Theorem 2 in [23] it follows that in the MFP model with parameter $p \geq p_c(N,2)$, the limit set D_p connects the left-hand side of $[0,1]^2$ with its right-hand side with positive probability. It then follows from the RSW lemma (e.g. Lemma 5.1 in [29]) and the FKG inequality that $\mathbb{P}_p(\Gamma_x) > 0$. Let A_n denote the event of complete retention until level n, i.e. $\omega(C(\mathbf{I})) = 1$ for all $\mathbf{I} \in \mathcal{J}^{(n-1)}$. Since $\prod_{n=1}^{\infty} p_n > 0$ there exists an integer n_0 such that $p_n \geq p_c(N,2)$ for all $n \geq n_0$. Hence, the probability measure $\mathbb{P}_{\text{fat}}(\cdot|A_{n_0})$ dominates $\mathbb{P}_{p_c(N,2)}(\cdot)$. Since $\mathbb{P}_{\text{fat}}(A_{n_0}) > 0$ it follows that $\mathbb{P}_{\text{fat}}(\Gamma_x) > 0$.

Observe that for $x, y \in V$

$$
\begin{aligned}
&\{x \text{ connected to } y\}\\
&\supset \{x \text{ connected to } S(x, \tfrac{1}{2n_x})\} \cap \Gamma_x \cap \{y \text{ connected to } S(y, \tfrac{1}{2n_y})\} \cap \Gamma_y.
\end{aligned}
$$

Since all four events on the right-hand side are increasing and have positive probability, we can apply the FKG inequality to conclude that for all $x, y \in V$ we have $\mathbb{P}(x \text{ connected to } y) > 0$. □

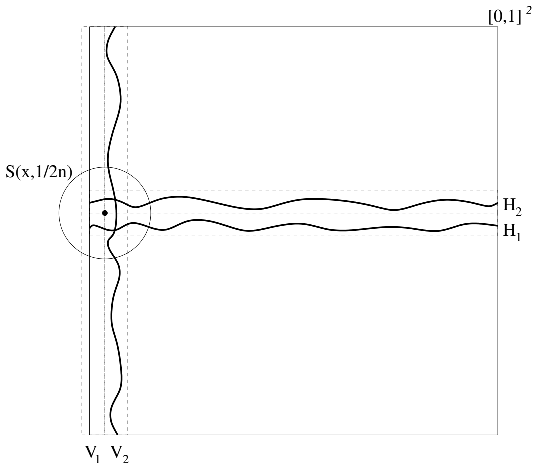


Figure 6.3: Realization of the event Γ_x.

Samenvatting

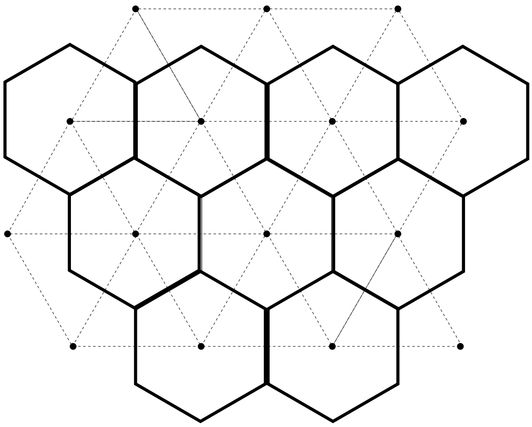
Toevalsfractals en schalingslimieten in percolatie

De titel van dit proefschrift is *Random fractals and scaling limits in percolation*. Vermoedelijk werpt deze titel geen (of anders een heel zwak) licht op de wiskundige inhoud voor de meeste Nederlandstalige lezers. Daarom zal ik hieronder in zo begrijpelijk mogelijk Nederlands uitleggen waar mijn proefschrift over gaat.

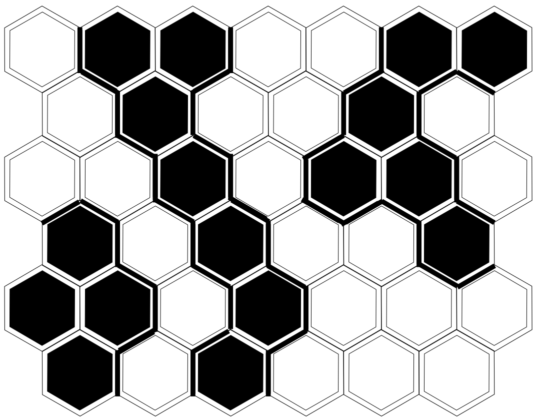
Laten we beginnen met het introduceren van de hoofdrolspeler van dit proefschrift: percolatie. Beschouw het driehoeksrooster in het platte vlak, zie Figuur 6.4 voor een illustratie van dit rooster. Merk op dat ieder punt in het driehoeksrooster correspondeert met een zeshoekig vlak uit het "honingraatrooster". Om de visualisatie van het model te vergemakkelijken werken we in deze samenvatting alleen met hexagons uit het honingraatrooster.

Voor iedere hexagon gooien we een munt (waarbij de kans p is dat kop bovenkomt): bij kop kleuren we de hexagon zwart, bij munt wit, onafhankelijk van de kleur van andere hexagons. Als we nu alle buurhexagons met dezelfde kleur verbinden krijgen we zwarte en witte clusters van hexagons. In Hoofdstuk 3 bekijken we het gedrag van de grenzen tussen zwarte en witte clusters. Deze grenzen vormen een curve, zie Figuur 6.5 voor enkele voorbeelden van zulke curves. In het Engels worden deze curves interface curves genoemd en bij gebrek aan een goede Nederlandse vertaling zal ik deze term hier ook gebruiken.

Vanuit een wis- en natuurkundig standpunt is de schalingslimiet van percolatie heel interessant. Onder schalingslimiet verstaan we het gedrag van een



Figuur 6.4: Het driehoeksrooster. De vetgedrukte punten vorm de punten van het driehoeksrooster en zijn met elkaar verbonden via de gestippelde lijnen. Ieder punt van het driehoeksrooster is het middelpunt van een hexagon. Deze hexagons vormen het honingraatrooster.



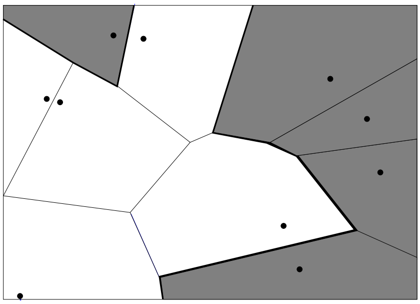
Figuur 6.5: Voorbeeld van percolatie met parameter $p = 1/2$. Hexagons worden zwart gekleurd met kans 1/2 en wit met kans 1/2, onafhankelijk van andere hexagons. De dikke lijnen tussen zwarte en witte clusters zijn de interface curves.

(nog nader te bepalen) object uit het percolatiemodel wanneer we de grootte van de hexagons naar nul sturen. Voor dit object zijn vele keuzes mogelijk en iedere keuze heeft zo zijn voor- en nadelen: we willen zo veel mogelijk informatie hebben over het percolatiemodel, maar het object moet ook niet te veel informatie bevatten, want dan wordt het wiskundig onbeheersbaar. Wij kiezen ervoor om de kansverdeling van interface curves als uitgangspunt te nemen. In Hoofdstuk 2 zetten we het wiskundig geraamte van kansverdelingen van interface curves op. Voor dit hoofdstuk geldt helaas een negatief leesadvies voor niet-wiskundigen: hen wordt aangeraden dit hoofdstuk alleen onder begeleiding van een professionele wiskundige te lezen.

Voor een schalingslimiet sturen we de grootte van de hexagons naar nul, laat daarom δ de grootte van een hexagon aangeven (bijv. de lengte van een zijde van een hexagon). Verder laten we de parameter p, die de kans op een zwarte hexagon weergeeft, afhangen van de parameter δ. Dus bij een rijtje $\{\delta_n\}$ zodanig dat $\delta_n \to 0$ hoort een rijtje $\{p_n\}$, waarbij p_n van δ_n afhangt. We geven de kansverdeling van interface curves op het honingraatrooster met schaalgrootte δ en percolatieparameter p aan met $\mu_{\delta,p}$. Voor iedere keuze van $\{\delta_n\}$ en $\{p_n\}$ convergeert de rij μ_{δ_n,p_n} naar een kansverdeling μ op een ruimte van lussen. (Dat wil zeggen, de ruimte van lussen bevat mogelijkheden voor het "uiterlijk" van de lussen en de kansverdeling μ kent een kans aan deze mogelijkheden toe.) Dit is een toepassing van een resultaat van Aizenman en Burchard [2]. In Hoofdstuk 3 bewijzen we dat er slechts drie mogelijkheden zijn voor deze kansverdeling μ en we geven ook een kwalitatieve beschrijving van deze mogelijkheden.

We laten zien dat de schalingslimiet triviaal, kritisch of bijna-kritisch is. Met een triviale schalingslimiet wordt bedoeld dat alle lussen een diameter van 0 hebben met kans 1. In de kritische schalingslimiet is ieder punt in het platte vlak met kans 1 omringd door oneindig veel lussen, waarvan de diameters willekeurig klein en groot worden. In het bijna-kritische geval is ieder punt in het platte vlak met kans 1 omringd door een grootste lus en oneindig veel lussen met willekeurig kleine diameter.

In Hoofdstuk 4 tonen we het bestaan aan van een schalingslimiet voor Voronoipercolatie, onder een (heel natuurlijke) aanname. Het Voronoipercolatie-model wordt via een andere procedure verkregen dan percolatie op het honingraatrooster. In plaats van punten op een rooster werken we met punten verkregen via een stochastisch proces, het zogenaamde Poisson proces. Dit is een verzameling punten in het platte vlak met o.a. de eigenschap dat het verwachte aantal punten in een gebied proportonieel is aan de oppervlakte van het gebied en dat deze punten uniform verdeeld zijn binnen dit gebied. Laat $\mathcal{P}$ zo'n Poisson proces zijn. Ieder punt x uit $\mathcal{P}$ krijgt via de inmiddels bekende percolatiemethode een kleur: met kans p wordt x zwart, met kans $1-p$ wit, onafhankelijk van andere punten. Voor ieder Poisson punt x bepalen we welke punten y uit het platte vlak dichter bij x dan bij andere Poisson punten liggen. Deze punten vormen de Voronoicel van x en we geven ieder punt in de Voronoicel van x dezelfde kleur als x, zie Figuur 6.6 voor een illustratie.



Figuur 6.6: Voorbeeld van Voronoipercolatie. De punten stellen de Poisson punten voor en de lijnen zijn de grenzen van de Voronoicellen.

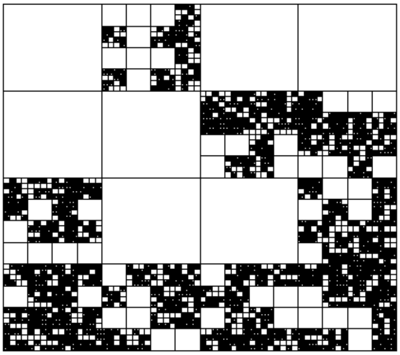
Als we wederom Voronoicellen van dezelfde kleur met elkaar verbinden verkrijgen we zwarte en witte clusters. Ook interface curves zijn in Voronoipercolatie de curves tussen witte en zwarte clusters in. Een Poisson proces wordt geparametriseerd door de intensiteitsparameter λ: hoe hoger λ hoe hoger het verwachte aantal punten in een gebied. Het is duidelijk dat als λ groeit de Voronoicellen kleiner worden. We geven de kansverdeling van interface curves in Voronoipercolatie met parameter λ dan ook aan met μ_λ. In Hoofdstuk 4 bewijzen we een eigenschap voor μ_λ, onder een zekere aanname voor Voronoipercolatie, en uit deze eigenschap volgt het bestaan van een kansverdeling μ als limiet van μ_λ.

De Franse wiskundige Benoît Mandelbrot introduceerde het begrip fractal in zijn monumentale boek *The fractal geometry of nature*. Fractal is afgeleid van het Latijnse woord fractus, dat gebroken betekent. Mandelbrot vond dat wiskundigen eeuwenlang hun aandacht hadden besteed aan mooie, reguliere objecten terwijl deze objecten niet in het wild voorkomen: bergen zijn geen kegels, wolken geen bollen en kustlijnen geen rechte lijnen, om maar een paar voorbeelden te noemen. Het is daarom veel ”natuurlijker“ om fractale meetkunde te bestuderen. Dat is een meetkunde waarin “gebroken” objecten met een zekere mate van zelfvormigheid als uitganspunt worden genomen, dit zijn de zogenaamde fractals. Simpel gezegd is een object zelfvormig wanneer inzoomen op het object weer (ongeveer) hetzelfde object oplevert. Voorbeelden van fractals in het dagelijks leven zijn broccoli en bloemkool, de Britse kustlijn, riviernetwerken, kristallen en beurskoersen. De wiskundige beschrijving van fractals kent twee hoofdvarianten: deterministische fractals of toevalsfractals. In Hoofdstukken 5 en 6 bestuderen we enkele toevalsfractals.

Het model dat we in Hoofdstuk 5 bestuderen is bedacht door Mandelbrot en wordt daarom wel Mandelbrot fractalpercolatie genoemd. Laat d en N twee gehele getallen groter dan of gelijk aan 2 zijn. Bekijk de eenheidskubus $[0,1]^d$ (ook in dimensies groter dan drie spreken we over kubus) en deel dit op in N^d gelijke kubusssen (met zijdelengte $1/N$). Voor iedere kubus passen we de percolatieprocedure toe: met kans p kleuren we het zwart, met kans $1-p$ wit, onafhankelijk van andere kubussen. Dit was de eerste stap van de constructie van de fractal. In stap twee herhalen we dezelfde procedure alleen in de zwarte kubussen: iedere zwarte kubus wordt opgedeeld in N^d gelijke subkubussen (met zijdelengte $1/N^2$) waarna iedere subkubus met kans p zwart blijft, of met kans $1-p$ wit wordt. We blijven dit in iedere zwarte kubus op iedere kleinere schaal herhalen, zie Figuur 6.7 voor een voorbeeld van Mandelbrot fractalpercolatie in dimensie $d = 2$ tot en met de derde stap. In iedere stap van de fractalconstructie hebben we een verzameling van zwarte kubussen, waarbij de verzameling in stap $n+1$ een deelverzameling is van de verzameling in stap n. We zijn geïnteresseerd in de limietverzameling van de zwarte kubussen. Dat wil zeggen, de verzameling die overblijft wanneer n willekeurig groot wordt. We noteren deze limietverzameling met D_p.

Het begrip dimensie van een object zal voor veel mensen een duidelijke (intuïtieve) betekenis hebben: het minimum aantal coördinaten waarmee we een punt van dit object kunnen beschrijven. Een plat vlak heeft twee dimensies en de ruimte om ons heen drie. In de wiskunde zijn er echter talrijke objecten, waaronder fractals, waarvoor dit dimensiebegrip ontoereikend is. We hanteren daarom (vooral) het begrip Hausdorff-dimensie wanneer we over fractals praten. Een precieze definitie kan de enthousiaste lezer in Hoofdstuk 5 vinden. De Hausdorff-dimensie komt in "gewone" ruimtes overeen met het intuïtieve idee van dimensie: de Hausdorff-dimensie van een rechte lijn is 1, van een vierkant 2, etc. Verder geldt dat de Hausdorff-dimensie een mate van fractaliteit is: hoe groter de Hausdorff-dimensie hoe fractaler het object.

De limietverzameling D_p kan in twee disjuncte verzamelingen worden opge-



Figuur 6.7: Voorbeeld van Mandelbrot fractalpercolatie in twee dimensies tot en met de derde stap van de constructie.

splitst: een totaal onsamenhangende verzameling van louter punten, die ook wel de "stof"-verzameling wordt genoemd, en een verzameling van samenhangende componenten. Simpel gezegd is een component samenhangend als we de component niet kunnen scheiden in twee aparte stukken. En een object is onsamenhangend als dit wel kan. Voorbeelden van samenhangende componenten zijn curves en intervallen. We laten in Hoofdstuk 5 zien dat de Hausdorff dimensie van de samenhangende verzameling kleiner is dan de Hausdorff dimensie van D_p. In het bijzonder impliceert dit dat de Hausdorff dimensie van de stof-verzameling groter is dan de Hausdorff dimensie van de samenhangende verzameling. Verder bewijzen we dat de verzameling van samenhange componenten een vereniging is van mooie, gladde curves (Hölder continue curves voor de liefhebbers).

In Hoofdstuk 6 bekijken we twee variaties op het fractal-thema. Laat N en d weer twee gehele getallen strict groter dan 1 zijn. Neem een geheel k tussen 0 en N^d. De eerste stap van ons nieuwe model, dat k-fractalpercolatie heet, is als volgt. Deel de eenheidskubus $[0,1]^d$ weer op in N^d kubussen met zijdelengte $1/N$. We kiezen nu k kubussen op een uniforme manier, dat wil zeggen dat alle verzamelingen van k kubussen gelijke kans hebben om gekozen te worden. De gekozen kubussen kleuren we zwart. Vervolgens herhalen we deze procedure weer in alle zwarte kubussen en op iedere kleinere schaal. Wederom zijn we geïnteresseerd in de limietverzameling, die we ditmaal met D_k aanduiden. We definiëren de drempelwaarde $k_c(N,d)$ als de kleinste waarde k waarvoor geldt dat de limietverzameling D_k met positieve kans de linkerkant van de eenheidskubus met de rechterkant verbindt. We definiëren de analoge drempelwaarde $p_c(N,d)$ voor Mandelbrot fractalpercolatie (deze is uiteraard gedefinieerd in termen van de kansparameter p). We bewijzen in Hoofdstuk 6 dat $k_c(N,d)/N^d$ naar dezelfde (bekende) waarde convergeert als $p_c(N,d)$. Het voert te ver om deze waarde hier te introduceren en geïnteresseerde lezer wordt hiervoor naar het desbetreffende hoofdstuk verwezen.

In de andere variant verandert de kansparameter p per stap in de fractalconstructie. Laat p_n de kans zijn waarmee we een kubus zwart kleuren in stap n en neem aan dat $\prod_{n=1}^{\infty} p_n > 0$. We laten zien dat de Lebesgue-maat van de limietverzameling van de fractal die we op deze manier verkrijgen met kans 1 positief is. De Lebesgue-maat is de standaard manier om volume (of oppervlak dan wel lengte) toe te kennen aan verzamelingen. We kunnen de limietverzameling wederom op twee manieren opsplitsen: de stof-verzameling en de verzameling met samenhangende componenten. We bewijzen dat óf de stof-verzameling positieve Lebesgue-maat heeft óf de verzameling met samenhangende componenten, maar niet allebei tegelijk.

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