Chapter 1

Introduction

Monotone variational recurrence relations arise as mathematical formalizations of seemingly unrelated physical systems, via the application of a variational principle. This powerful mathematical tool is based on the observation that many natural systems have a conserved quantity and that the state or the evolution of such systems can be described as a critical point of this quantity. This general principle appears in many forms in mathematical modeling and throughout this thesis we will see some of its manifestations.

An important case arises from the observation that many natural systems seem to evolve in some sense optimally in time. In the analysis of such systems this optimal behavior then corresponds to a (local) minimizer of a particular functional called the free energy, or the action. The solutions of the mathematical model that attain minimal values of the free energy are called minimizers. For example, physical particles or light often follow geodesics, which are (locally) the shortest paths in the appropriate geometry. The free energy in this case is the length of the path.

In section 1.1, we roughly present the formal setting of this thesis, by introducing monotone variational recurrence relations and explaining the variational principle that they obey. A more precise setting is presented in each of the following chapters separately. The wide range of applications of monotone variational recurrence relations will become apparent via an example in section 1.2 and because of their tight relation to Hamiltonian twist maps, explained in section 1.3.

The general variational principle that we consider in this text is slightly awkward, since the free energy is undefined. More precisely, the free energy in our setting is given only formally by a divergent sum. Nevertheless, it is still possible to introduce a concept of a minimizing solution, the so-called global minimizer. The mathematical theory dealing with the qualitative and quantitative investigation of such solutions is called Aubry-Mather Theory. We give a brief overview of Aubry-Mather theory in section 1.4, while an overview of other relevant results for this thesis can be found in section 1.5. Finally, we give a short description of the results in the other chapters in section 1.6.
1.1 Formal setting

1.1.1 Recurrence relations for configurations

Throughout this thesis, we are interested in recurrence relations for bi-infinite real-valued sequences, \( x : \mathbb{Z} \to \mathbb{R} \), and more generally, for real-valued functions on \( d \)-dimensional lattices \( x : \mathbb{Z}^d \to \mathbb{R} \), where \( d \in \mathbb{N} \) is a positive natural number. We call such a generalized sequence \( x \in \mathbb{R}^{\mathbb{Z}^d} \) a configuration.

A recurrence relation for a configuration is a rule that relates coordinates with nearby indices. We first recall the simplest form of a recurrence relation for real-valued sequences \( x \in \mathbb{R}^\mathbb{N} \). Let \( f : \mathbb{R}^r \to \mathbb{R} \) be a function and let \( r \in \mathbb{N} \) be a positive natural number. Then we may define a recurrence relation of order \( r \), by requiring that for all \( n \geq r \),

\[
x_n = f(x_{n-1}, \ldots, x_{n-r})
\]

holds. A sequence \( x \), which satisfies equation (1.1.1) for all \( n \geq r \), is called a solution of the recurrence relation defined by \( f \).

Assume now that \( x \in \mathbb{R}^\mathbb{Z} \) is a configuration over \( \mathbb{Z} \). In this case it makes more sense to represent a recurrence relation in the following way. Let \( R : \mathbb{R}^{2r+1} \to \mathbb{R} \) be a function and let \( r \in \mathbb{N} \) be a non-negative natural number. Then \( R \) defines a recurrence relation by requiring that for all \( i \in \mathbb{Z} \),

\[
R_i(x) := R(x_{i-r}, \ldots, x_{i+r}) = 0
\]

holds. Again, a (bi-infinite) sequence \( x \), which satisfies equation (1.1.2) for all \( i \in \mathbb{Z} \), is called a solution of the recurrence relation defined by \( R \). Obviously, (1.1.2) gives us a more general representation of a recurrence relation than (1.1.1), since it allows for recurrence relations where \( x_i \) is not given as a function of \( \{x_{i-1}, \ldots, x_{i-r}\} \).

Finally, it is not difficult to generalize (1.1.2) to configurations over finite dimensional lattices \( \mathbb{Z}^d \), where \( d > 1 \). Roughly speaking, let \( d > 1 \), and let

\[
B_i(r) := \{ j \in \mathbb{Z}^d \mid \|i - j\| \leq r \},
\]

where \( \| \cdot \| \) is some norm on the lattice \( \mathbb{Z}^d \). Moreover, let us introduce for every \( i \in \mathbb{Z}^d \) the shift \( \sigma_i : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d} \) by \( \sigma_i(x)_j := x_{j-i} \). Let now \( R : \mathbb{R}^{B_0(r)} \to \mathbb{R} \) be a function, which defines, via the shifts, for every \( i \in \mathbb{Z}^d \) the function \( R_i : \mathbb{R}^{B_i(r)} \to \mathbb{R} \) by \( R_i(x) := R(\sigma_i(x)) \). Then \( R \) defines a recurrence relation, if we require that for all \( i \in \mathbb{Z}^d \), \( R_i(x) = 0 \). A more thorough presentation of such recurrence relations can be found in chapter 2.

1.1.2 The variational principle, monotonicity, and global minimizers

In this section, we briefly explain the properties of the recurrence relations that we study in this thesis. In view of clarity, we limit ourselves to recurrence relations for configurations over \( \mathbb{Z} \). A more rigorous presentation of the variational recurrence relations
relations, together with the generalization to configurations for lattices where \( d > 1 \), can be found in chapter 2.

Let us first explain what we mean by a variational recurrence relation.

**Definition 1.1.1.** Let \( x \in \mathbb{R}^\mathbb{Z} \) be a configuration, let \( r \in \mathbb{Z} \) be a non-negative integer and let \( S : \mathbb{R}^{r+1} \to \mathbb{R} \) be a differentiable function. We denote for all \( i \in \mathbb{Z} \),

\[
S_i(x) := S(x_i, \ldots, x_{i+r}).
\]

Moreover, we denote for all \( j \in \{i, \ldots, i + r\} \), the derivatives \( \partial_j S_i(x) := \frac{\partial}{\partial x_j} S_i(x) \), whereas for all \( j \notin \{i, \ldots, i + r\} \), we set \( \partial_j S_i(x) := 0 \).

We then say that \( S \) defines a variational recurrence relation \( R \) of range \( r \) with a solution \( x \), if it holds for all \( i \in \mathbb{Z} \) that

\[
R_i(x) := \sum_{j=i-r}^{i} \partial_i S_j(x) = 0. \tag{1.1.3}
\]

The variational principle from definition 1.1.1 does not allow us to define the usual concept of the free energy. In fact, let us formally write the possibly divergent sum

\[
W(x) := \sum_{i \in \mathbb{Z}} S_i(x). \tag{1.1.4}
\]

It then holds that formally

\[
\partial_i W(x) = \sum_{j=i-r}^{i} \partial_i S_j(x),
\]

which is well defined, and that (1.1.3) holds for all \( i \in \mathbb{Z} \), if and only if \( \nabla W(x) = 0 \). This shows that the formal sum \( W \) can be seen as the formal action of the variational principle.

It is thus clear that the concept of a minimizing configuration does not make sense. The standard way to circumvent this problem, is to find a configuration \( x \) that is minimal with respect to compactly supported variations. More precisely, let \( v \in \mathbb{R}^\mathbb{Z} \) be a configuration with compact support. It then holds for any configuration \( x \) that the expression \( W(x + v) - W(x) \) is well defined as a finite sum (see (1.1.4)). This allows us to introduce the following definition.

**Definition 1.1.2.** We call a configuration \( x \in \mathbb{R}^\mathbb{Z} \) a global minimizer, if it holds for every compactly supported configuration \( v \in \mathbb{R}^\mathbb{Z} \) that

\[
W(x + v) - W(x) \geq 0. \tag{1.1.5}
\]

Let us mention here that originally global minimizers were introduced as “class A minimizers” by Morse in [60]. It is easy to see that any global minimizer \( x \in \mathbb{R}^\mathbb{Z} \) solves the recurrence relation (1.1.3).

Next, we explain what kind of variational recurrence relations we consider.
**Definition 1.1.3.** Let \( x \in \mathbb{R}^\mathbb{Z} \) be a configuration, let \( r \in \mathbb{N} \) be a non-negative integer and let \( S \) be a twice continuously differentiable function. Assume that \( S \) defines a recurrence relation \( R \) on \( \mathbb{R}^\mathbb{Z} \) as in definition 1.1.1, via the functions \( S_j \). We call \( S_j \) local potentials, if the following conditions are satisfied for all \( j \in \mathbb{Z} \).

**A.** Periodicity:
\[
S_j(x + 1) = S(x_j + 1, \ldots, x_{j+r} + 1) = S_j(x).
\]

**B.** Monotonicity:
\[
\partial_{i,k} S_j \leq 0 \text{ for all } i \neq k, \text{ while } \partial_{j,j+1} S_j < 0.
\]
Condition B is also called a twist condition or a ferromagnetic condition and it implies monotonicity of the recurrence relation \( R_i \) with respect to variables \( x_j \), where \( j \in \{i+1, \ldots, i+r\} \).

**C.** Coercivity: it holds for all \( k \in \{j+1, \ldots, j+r\} \) that
\[
\lim_{|x_j - x_k| \to \infty} S_j(x) = \infty.
\]

As in definition 1.1.1, the local potentials \( S_j \) define by (1.1.3) a variational recurrence relation. It is called a monotone variational recurrence relation.

The existence of global minimizers of monotone variational recurrence relations satisfying conditions A-C from definition 1.1.3, will become apparent in chapter 2.

### 1.2 An example: the Frenkel-Kontorova crystal model

The Frenkel-Kontorova model is the pivotal example of a monotone variational recurrence relation. It models a one-dimensional infinite crystal in a periodic potential field and was first introduced in [34], as a model of dislocations in metals and is by now used to describe a wide range of physical systems. For example, the Frenkel-Kontorova model and slight adaptations of it, such as the Frenkel-Kontorova-Tomilson model, are used for describing ferro- and antiferro-magnetics, some conductivity problems, the DNA-structure, and dry friction in tribology (see for example [20], [77], [32]).

Let the configuration \( x \in \mathbb{R}^\mathbb{Z} \) represent an infinite sequence of coupled crystal particles lying in a periodic potential field and let \( c \in \mathbb{R} \) be a real number (see figure 1.1). The defining Newton’s equation of motion for \( i \)th particle is then given by
\[
\frac{d^2 x_i(t)}{dt^2} = x_{i+1} - 2x_i + x_{i-1} - c \sin(2\pi x_i).
\]
(1.2.6)

We shall be interested in steady states of the crystal. They are solutions of the recurrence relation
\[
x_{i+1} - 2x_i + x_{i-1} - c \sin(2\pi x_i) = 0 \text{ for all } i \in \mathbb{Z}.
\]
(1.2.7)
Figure 1.1: The Frenkel-Kontorova model: neighboring particles of a crystal attract each other via linear elastic forces and the crystal lies in a periodic background potential.

It is not difficult to see that (1.2.7) is a monotone variational recurrence relation satisfying definition 1.1.3, with the local potentials defined by

\[ S_i(x) := \frac{1}{2}(x_{i+1} - x_i)^2 + \frac{c}{2\pi} \cos(2\pi x_i). \]

Despite its simple structure, (1.2.7) exhibits surprisingly rich behavior. Some kind of chaotic behavior of (1.2.7) can be anticipated from its natural correspondence to the so-called Chirikov standard map, well known for its chaotic properties (see section 1.3.2). This correspondence has been first mentioned in [34] and is described more precisely in section 1.3.1. On the other hand, the continuum counterpart of (1.2.7) is a completely integrable nonlinear hyperbolic partial differential equation called the sine-Gordon equation, the dynamics of which is determined by solitary waves (see [73]).

Another interesting phenomenon that crystals modeled by the Frenkel-Kontorova model often exhibit are so-called incommensurations (see [8]). An incommensurate crystal corresponds to a solution of (1.2.7), where the spacing of the particles \( x_i \) does not coincide with the periodicity of the background potential. The first rigorous proof of existence of such solutions for the Frenkel-Kontorova model was constructed by Aubry and Le Daeron in [8]. They show that there exist global minimizers of any average spacing between the particles. This was one of the two launching points of Aubry-Mather theory, which we will explain more precisely in section 1.4. The other one came from the realm of twist maps, which we briefly introduce in the following section.

The generalizations of the Frenkel-Kontorova model considered in this thesis and introduced in section 1.1, correspond to modeling further-range and non-linear interactions between particles of a more-dimensional crystal lying in a periodic background potential.

1.3 Hamiltonian twist maps

In this section we briefly introduce Hamiltonian twist maps of the cylinder and explain their relation to monotone variational recurrence relations. We furthermore give some classical examples of Hamiltonian twist maps.
1.3.1 Definition and relation to monotone variational recurrence relations

A more precise, but still rather brief overview of the content below can be found in the appendix of [68]. For detailed proofs, we refer to [37] or [57].

Let us denote the one-dimensional cylinder by $A := \mathbb{R}/\mathbb{Z} \times \mathbb{R}$, with coordinates $(x \mod 1, y)$ and let $T : A \to A$ be a smooth map. Roughly speaking, $T$ is a Hamiltonian twist map, if it is area preserving, has zero flux, and “twists” the cylinder.

To be more precise, let us state the following definition.

**Definition 1.3.1.** Let $T$ be a smooth map of the cylinder and denote the lift of $T$ by $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2$. Moreover, write $\tilde{T}(x, y) = (X(x, y), Y(x, y))$. Then we call the map $T$ a Hamiltonian positive twist map if $T$ and $\tilde{T}$ satisfy the following conditions.

1. **Degree one:** $\tilde{T}(x + 1, y) = \tilde{T}(x, y) + (1, 0)$.

2. **Exact symplectic:** The one-form $Y dx - y dX$ is exact on $A$, i.e. $T$ is an area preserving map with zero flux.

   (Note that conditions 1 and 2 hold if and only if $T$ is a so-called Hamiltonian map, i.e. $T$ is the time-1 flow of a time-1-periodic Hamiltonian vector field $X_{H(t)}$ on $A$ (see [37]).)

3. **Positive Twist:** The map $(x, y) \mapsto (x, X(x, y)) : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. This implies that $\partial_y X \neq 0$. We require that $\partial_y X > 0$.

Condition 3 says that $T$ twists each fiber $\{x \mod 1\} \times \mathbb{R} \subset A$ around the cylinder $A$ “in the positive direction”. In other words, $\tilde{T}$ maps the fiber $\{x\} \times \mathbb{R} \subset \mathbb{R}^2$ to a graph over the $y$-axis (see figure 1.2).

![Figure 1.2: A twist map twist each fiber around the cylinder and its lift maps a fiber into a graph.](image)

Let us now explain the correspondence between Hamiltonian twist maps and monotone variational recurrence relations. First of all, recall that a sequence $i \mapsto (x_i, y_i) \in \mathbb{R}^2$ is called an orbit of $T$, if $\tilde{T}(x_i, y_i) = (x_{i+1}, y_{i+1})$ for all $i \in \mathbb{Z}$. It follows from property 2 in definition 1.3.1, that corresponding to every twist map $T$, there exists a so-called generating function $S : \mathbb{R}^2 \to \mathbb{R}$, satisfying $dS := Y dX - y dX$.  

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Such a generating function $S$ then satisfies conditions A-C from definition 1.1.3, thus generating a variational monotone recurrence relation (see [37]). The following well-known theorem is crucial in the theory of twist maps.

**Theorem 1.3.2.** The sequence $i \mapsto (x_i, y_i) \in \mathbb{R}^2$ is an orbit of $\tilde{T}$ if and only if the following two conditions are satisfied:

- $\{x_i\}_{i \in \mathbb{Z}}$ solves the variational monotone recurrence relation generated by $S$, i.e.
  \[
  \partial_i S(x_{i-1}, x_i) + \partial_i S(x_i, x_{i+1}) = 0 \quad \text{for all } i \in \mathbb{Z},
  \]

- it holds for all $i \in \mathbb{Z}$ that $y_i = -\partial_i S(x_i, x_{i+1})$.

**Proof.** For a proof of this theorem, see the appendix to [68].

This theorem proves that finding an orbit of a Hamiltonian twist map is equivalent to solving a second order monotone recurrence relation.

Next, we give some examples of Hamiltonian twist maps of the cylinder.

### 1.3.2 Examples of Hamiltonian twist maps of the cylinder

#### The restricted three body problem

The study of Hamiltonian twist maps of the cylinder goes back to the work of Poincaré on the restricted circular planar three body problem in [72]. This is a model representing three bodies moving in their gravitational field in a plane, where one has negligible mass compared to the other two. A Hamiltonian twist map in this model arises as a Poincaré return map in the vicinity of an elliptic equilibrium point.

In fact, under generic conditions, the Poincaré return map of any 2 degree of freedom Hamiltonian system near an elliptic equilibrium point is a Hamiltonian twist map (see [57]). Poincaré postulated that every Hamiltonian twist map has at least two fixed points. This result is now known as Poincaré’s geometric theorem, and was proved by Birkhoff in [15].

#### Convex billiards

A toy example of twist maps arises in the context of convex billiards (see figure 1.3). This was first noted by Birkhoff in [17].

The configuration space of such a billiard consists of the arclength parameter $x \in \mathbb{R}/\mathbb{Z}$ that describes the position of the billiard ball along the boundary of the billiard at the moment of a reflection and the angle $y \in (0, \pi)$ measuring the direction of the outgoing billiard trajectory with respect to the line tangent to the billiard at $x$ (see figure 1.3). Then the motion of a billiard ball is described by a Hamiltonian positive twist map $T : (x_i, y_i) \mapsto (x_{i+1}, y_{i+1})$. The twist condition $\frac{\partial x_{i+1}}{\partial y_i} > 0$ simply means that increasing the angle at a point of reflection increases the position of the next reflection point. Birkhoff proved in [17] by a simple variational argument that periodic billiard trajectories of all periods exist. For more details on billiards see [76].
Figure 1.3: The billiard ball bounces by the law of reflection: the angle of reflection is the same as the angle of incidence.

Chirikov standard map

Perhaps the most famous concrete example of a Hamiltonian twist map is the Chirikov standard map, $T_c : \mathbb{A} \rightarrow \mathbb{A}$, defined by

$$T_c(x, y) = \left( x + y - c \sin(2\pi x) \mod 1, y - c \sin(2\pi x) \right).$$

It turns out that the generating function of $T_c$ is $S(x, X) := \frac{1}{2} (x - X)^2 + \frac{c}{2\pi} \cos(2\pi x)$. In other words, the variational monotone recurrence relation corresponding to $T_c$ is exactly the Frenkel-Kontorova recurrence relation (1.2.7). This was observed already in [34]. However, an independent study of the Chirikov standard map was first attempted in [24, 25] followed by extensive numerical studies in the seventies by Chirikov, Greene, Percival and others, because of its chaotic properties. For an overview of these results see [44]). Observe that for $c = 0$, the orbits of the map are contained in invariant circles of constant coordinate $y$. With increasing constant $c$ it seems that more and more of these circles break and chaotic regions form. This transition to chaos can be seen in figure 1.4, where the coordinate $y$ is also taken periodic.

Figure 1.4: Chirikov standard map $T_c$ with values $c = -0.5$, $c = -0.971635$ and $c = -5$. 

Monotone Variational Recurrence Relations
1.4 Aubry-Mather Theory

In sections 1.2 and 1.3, we tried to illuminate the intricate behavior of solutions of monotone variational recurrence relations. Not surprisingly, the rigorous mathematical analysis of such recurrence relations is involved and complicated. In this section we focus on an interesting set of results, called Aubry-Mather theory, which is at the basis of this thesis.

Aubry-Mather theory grew on one hand out of the work of Aubry et al. in [6-8] on the generalized Frenkel-Kontorova model, and on the other hand out of a series of papers by Mather on Hamiltonian twist maps, starting with [50]. The two approaches were independent and quite different, but the conclusions are similar. We first explain Aubry’s approach and conclusions.

1.4.1 Ground states of the Frenkel-Kontorova model

In this section we focus on Aubry’s approach to the problem of finding solutions to the Frenkel-Kontorova model. We roughly explain the physical motivation and the main ideas behind the results.

As mentioned in section 1.2, physical systems modeled by the Frenkel-Kontorova recurrence relation often display modulations of the period, which are incommensurate with respect to the lattice spacing [8]. In the Frenkel-Kontorova chain (1.2.7), this corresponds to a non-periodic solution \( x \). An example of a non-periodic solution \( x \) of the Frenkel-Kontorova recurrence relation (1.2.7) is a sequence \( x \) for which the rotation number

\[
\rho(x) := \lim_{i \to \pm \infty} \frac{x_i}{i} \in \mathbb{R}
\]

exists and is irrational.

![Figure 1.5: The single crossing principle: Aubry graphs of two global minimizers can cross at most once.](image)

The groundbreaking results on incommensurations of slightly generalized Frenkel-Kontorova models were published by Aubry and Le Daeron in [8]. Most of their proofs are heavily based on the so-called fundamental lemma of Aubry, or the single crossing...
principle. This principle states that the Aubry graphs of two global minimizers can cross in at most one point, where for a configuration \( x \in \mathbb{R}^2 \), its Aubry graph is the piecewise linear graph connecting the points \((i, x_i) \in \mathbb{R}^2\) by line segments (see figure 1.5).

The following line of reasoning, leading to the first big result in [8], is worked out quite precisely in sections 2.3 and 2.4, so we keep it brief here. First of all, it can be shown via a minimization argument that periodic minimizers of any period exist. In particular, for any rational number \( \frac{p}{q} \in \mathbb{Q} \), we can find a solution \( x \) of (1.1.3) for which every \( q \) subsequent coordinates lie in \( p \) periods of the background potential.

Because of Aubry’s lemma, periodic minimizers are totally ordered, which means that if \( x \neq y \) are minimizers of the same periodicity, then it either holds that \( x_i > y_i \) for all \( i \in \mathbb{Z} \), or that \( y_i > x_i \) for all \( i \in \mathbb{Z} \). In particular, this implies that any periodic minimizer \( x \) is ordered with respect to its translates \( \tau_{k,l} x \), which are defined by \( (\tau_{k,l} x)_i := x_{i-k} + l \). In other words, every periodic minimizer is a so-called Birkhoff configuration. The important property of Birkhoff configurations is that, roughly speaking, they form a compact set when viewed in the proper topology. The idea now is to take for any irrational number \( \omega \in \mathbb{R} \setminus \mathbb{Q} \), a sequence of rational approximations of \( \omega \), i.e., \( \frac{q_n}{p_n} \to \omega \), for \( n \to \infty \), and to show that the corresponding periodic minimizers converge along a subsequence to a global minimizer with rotation number \( \omega \). This is the first main result in [8].

The second big result in [8], proved by involved use of the single crossing principle, is that all global minimizers have a rotation number and are in fact Birkhoff. This is a surprising result, since it means for example that no orbits of the Chirikov standard map corresponding to global minimizers behave chaotically, even if they are in chaotic regions of the map.

Next, Aubry and Le Daeron show that all global minimizers of the same irrational rotation number are totally ordered with respect to each other and that slightly weaker but similar ordering properties hold for global minimizers of the same rational rotation number. An important concept in the proof of this result is the concept of a ground state, which is a recurrent global minimizer, i.e., a global minimizer that can be approximated by its own translates. It turns out that the ground states of a specific irrational rotation number form a unique minimal set called the Aubry-Mather set, and it follows from a Poincaré-classification theorem that these sets are either connected, or Cantor sets (this is discussed more precisely in section 2.4). Connected Aubry-Mather sets are called minimal foliations and are related to the effect of “sliding” in the crystal model. Cantor Aubry-Mather sets, on the other hand, are called minimal laminations and are connected to the effect of “pinning” in a crystal. This terminology is due to Moser [61, 63, 64].

Finally, Aubry and Le Daeron show in [8] also the existence of incommensurate global minimizers of rational rotation number, which emerge as heteroclinic connections between subsequent periodic minimizers.

All these results also have important implications for Hamiltonian twist maps, which will be discussed in the subsequent section.
1.4.2 Action-minimizing sets for Hamiltonian twist maps of the cylinder

In this section we explain an alternative approach to Aubry-Mather theory from the one introduced by Aubry and described in the previous section. A thorough presentation of the following results can be found in [57].

Ground states, or recurrent global minimizers of monotone variational recurrence relations, correspond to a specific class of orbits of Hamiltonian twist maps. These orbits were discovered by Mather in [50], via a variational approach proposed by Percival in [70, 71]. A rigorous connection between the approach of Aubry and Le Daeron in [8], and Mather in [50], was obtained by Bangert in [13], where he weakened the conditions from definition 1.1.3, so that they apply also to problems with a rather wide class of local potentials. A connection between Birkhoff’s proof of Poincaré’s geometric theorem and Mather’s existence result is described in [41].

In [50], Mather constructs for every Hamiltonian twist map \( T \) and rotation number \( \omega \in \mathbb{R} \), a so-called “action minimizing set” \( M_\omega \subset A \). Roughly speaking, \( M_\omega \) can be characterized as the support of an invariant measure that minimizes the average action (i.e. the Percival Lagrangian) within the class of invariant measures of rotation number \( \omega \). In other words, the set \( M_\omega \subset \mathbb{S}^1 \times \mathbb{R} \) is invariant under the map \( T \) and it corresponds to the Aubry-Mather set of rotation number \( \omega \) of the related monotone variational recurrence relation.

A generalization of the famous invariant curve theorem by Birkhoff from [16], shows that every \( M_\omega \) lies on a Lipschitz graph over \( \mathbb{S}^1 \times \{0\} \). Moreover, the map \( T \) restricted to \( M_\omega \) is semi-conjugate to a circle rotation of angle \( \omega \). If the corresponding Aubry-Mather set is connected, i.e., a minimal foliation, then \( M_\omega \) defines an invariant topological circle. If, on the other hand, the Aubry-Mather set is a minimal lamination, then \( M_\omega \) defines a so-called “remnant circle” or a cantorus, which is a Cantor set embedded in the Lipschitz graph. This latter terminology is due to Percival [71].

Invariant circles are interesting for the dynamics of a Hamiltonian twist map, since they split the cylinder into invariant regions, thus forming “energy-transport barriers” and implying some stability of the map. On the other hand, an open subset of the cylinder bounded by two invariant circles, in which no invariant circles of the Hamiltonian twist map exist, is called a stochastic region or a region of Birkhoff instability. It has been shown by Mather in [56] that such regions contain so-called “wandering” orbits, which approach the bounding circles arbitrarily well (see [56]). Nevertheless, numerical experiments suggest that the cantori in the Birkhoff regions of instability form partial barriers, through which the orbits pass slowly (see [47]).

It follows from an argument by Moser that any invariant circle in fact corresponds to a foliation by minimizers. Moreover, it is easy to see that an Aubry-Mather set that is a Cantor set, is generically not interpolated by minimizers. It is thus an interesting question in view of energy transport, when an Aubry-Mather set is a Cantor set, and when it is connected. We will give a short overview of these results in section 1.5.1.

To conclude the current section, let us mention that Aubry-Mather theory in its full generality cannot be extended to twist maps on more-dimensional annuli. In fact, in...
more dimensions several new phenomena occur, such as the famous Arnol’d diffusion (see [3]), which would show that invariant tori no longer form energy-transfer barriers. Nevertheless, the existence of action-minimizing sets from Mather’s approach sketched above, can be generalized to positive definite Lagrangian systems of more degrees of freedom, also called Tonelli Lagrangians, as was shown in [55]. This generalization is based on the work of Moser [62] and it opened the lively field of weak-KAM theory (see [31]).

1.4.3 Moser-Bangert theory

Even though the base space of the problems discussed in this thesis is discrete, many ideas and results come from a continuum counterpart of Aubry-Mather theory, sometimes referred to as Moser-Bangert theory. Here we give a brief overview of the main results of this very interesting and nontrivial extension. For details and more references we refer to [74].

As described in section 1.4.2, by weakening Aubry’s conditions on local potentials, Bangert showed in [13] the compatibility of Aubry’s and Mather’s approach. But more importantly, this formal approach allowed him to apply Aubry-Mather theory to the question of existence of geodesics on a two-dimensional torus with an arbitrary Riemannian metric. This result connects Aubry-Mather theory to Hedlund’s work on geodesics in [39], which in turn stems from the famous results of Morse in [60]. More precisely, in [13] Bangert shows that for every irrational rotation number the “globally minimal” geodesics, or Bangert-Hedlund geodesics, either form a unique foliation of the torus, or define a minimal lamination on the torus.

Similarly as in the case of Aubry-Mather theory for twist maps, also these results on geodesics cannot be extended to more-dimensional tori. Nevertheless, Moser managed to generalize this theory to variational nonlinear elliptic PDEs on more-dimensional tori, by studying co-dimension one hypersurfaces instead of trajectories. He showed in [61, 64] that there exist minimal hypersurfaces of arbitrary rotation vectors, which either form minimal foliations, or minimal laminations. In this case it is not difficult to see that there exist non-Birkhoff global minimizers. However, it follows from a result by Bangert in [12] that all Birkhoff global minimizers of a fixed irrational rotation vector are ordered, if the rotation vector is rationally independent. Moser’s results were further extended by de la Llave and Valdinoci in [28] and others, to variational non-linear and possibly degenerate elliptic partial differential equations and pseudo-differential equations, also on other manifolds than $T^n$.

1.4.4 Generalized crystal models

Finally, let us mention the extensions of Aubry-Mather theory to more general crystal models, which also appear in this thesis.

The extensions of Aubry-Mather Theory to finite range monotone variational recurrence relations on $\mathbb{Z}^d$, was investigated first by Blank in [18, 19]. He proved the existence of Birkhoff global minimizers of arbitrary rotation vectors. Moreover, in this
manuscripts Blank discusses the existence of non-Birkhoff global minimizers. These results were extended to monotone variational recurrence relations with infinite interactions and over more complicated lattices by de la Llave et al. in [23, 29].

1.5 Other relevant results

Aubry-Mather theory comprises the qualitative and quantitative study of global minimizers, as described in section 1.4. Of course, this is not the end of the story. For example, a number of related results on existence and stability of minimal foliations and laminations exist. Moreover, other tools for solving monotone variational recurrence relations have been developed, some building on the results from Aubry-Mather theory and some quite independent. In this section we briefly overview these results.

1.5.1 More on minimal foliations and laminations

We have already discussed in section 1.4.1 how minimal foliations and minimal laminations correspond to different physical effects in crystal models. Also, we discussed in section 1.4.2 the implications of invariant circles to the stability of a Hamiltonian twist map. Here we give a brief account of some major results on the existence of minimal foliations and laminations, their genericity, and their stability properties.

Let us first consider the existence of minimal laminations. A rather crude way of constructing cantori in the Frenkel-Kontorova model is from the so-called anti-integrable limit, which describes the transition from no coupling to weak coupling of particles in the recurrence relation (see [46]). Also, it is not difficult to see that minimal foliations in quite general crystal models can be destroyed into minimal laminations, by adding a large enough oscillation to the local potential (see theorem 2.4.19). A similar statement holds also for Moser’s PDE setting (see [11]). In the special case of the Chirikov standard map $T_c$ from section 1.3.2, it holds that when the constant $c$ is large enough, then there are no invariant circles. In fact, much research has been devoted to the question of finding the lowest value of $c$ for which $T_c$ has no invariant circles (see [49] for the sharpest current estimate).

On the other hand, there are no minimal laminations in integrable systems. For example, the standard map with constant $c = 0$ (and hence also the corresponding Frenkel-Kontorova recurrence relation) has invariant circles of all rotation numbers.

The question is, what kind of transition happens for $T_c$, where the constant $c$ runs from 0 to $l$. This is a very delicate question, which can be approached from two angles.

The first approach is based on a sophisticated version of the implicit function theorem, which implies stability of some minimal foliations. This is the famous KAM theorem, developed by Kolmogorov, Arnol’d and Moser (see [26] for a nice overview). In its most general form ([75]) it states that invariant tori of a Hamiltonian twist map, which have a “very irrational”, for example a Diophantine rotation vector, persist under small perturbations of the twist map. The “rationality degree” here is measured...
by how well an irrational number can be approximated by rational numbers. KAM theory for example shows the stability of the elliptic equilibria of the restricted three body problem (see section 1.3.2 above), which can be viewed as a close-to integrable Hamiltonian twist map. A KAM theorem also exists for Moser’s PDE setting (see [63, 64]), but for the most general monotone variational recurrence relations has yet to be proved.

The second approach goes under the name *converse KAM theory* and works for minimal foliations of not-too irrational rotation numbers, for example *Liouville numbers*. These are irrational numbers that can be approximated by rationals relatively well. This approach was developed for Hamiltonian twist maps of the cylinder by Mather in [54] and is based on explicitly constructing a $C^k$ small perturbation of the local potentials which destroys a minimal foliation of a specific Liouville rotation number. This result was generalized to the analytic setting by Forni in [33]. The generalization of this destruction result to finite range monotone variational recurrence relations is one of the subjects of this thesis.

### 1.5.2 Non-minimizing solutions

Quite some research has been done on the question of finding non-minimizing solutions of monotone recurrence relations. A first obvious question in this direction is whether there is an Aubry-Mather-type theory for monotone recurrence relations without a variational structure. A partial answer for maps on $S^1 \times \mathbb{R}$ is given by a topological argument by Hall in [38]. An extension to maps on $S^1 \times \mathbb{R}^n$ is due to Angenent (see [2]).

Even in the variational setting, however, it is important to keep in mind that for twist maps the concept of global minimizers, useful as it is in Aubry-Mather theory, does not correspond to stable orbits. In fact, it has been shown by MacKay and Meiss in [45] that periodic orbits that correspond to nondegenerate local minima of the variational principle are dynamically hyperbolic and hence hard to observe. Moreover, also in crystal modeling the property of being a global minimizer is not always of large physical importance. This is quite evident from the work of Baesens and MacKay on the aforementioned anti-integrable limit (see [10]). Namely, the authors show by an implicit function theorem that all recurrent Birkhoff minimal configurations continue to Birkhoff “local” minimizers, which form cantori. The class of such cantori can be parameterized by a disc in the “vague topology”. This shows that the specific cantorus corresponding to global minimizers is somewhat indistinguishable from other solutions of the crystal model. Mather proved a sharper version of these results in [51], showing that whenever the Aubry-Mather set is a Cantor set, then there is a more-dimensional disc in the vague topology that consists of other Cantor sets of Birkhoff solutions.

To get a full picture of the dynamics of Hamiltonian twist maps and of the kind of properties a typical crystal (found in nature) carries, it is thus important also to study non-minimizing solutions. It turns out that a very fruitful approach to this problem uses the concept of a gradient flow.

Gradient flow techniques were first applied to the study of periodic orbits of mono-
1.6 Outline of the thesis

Chapter 2, 3, 4 and 5 of this thesis present the results from the papers [68, 67, 69, 66], respectively. The exposition of these results is largely unchanged, which makes all the chapters self-contained. Nevertheless, some changes to the papers have been made in view of a better presentation of the material in this thesis, as a whole. In particular, the appendix of chapter 2 contains a result on the ordering of Birkhoff global minimizers, whereas the content of the appendix of [68] is now included in section 1.3 of this introduction.

In chapter 2, with the title *Ghost circles in lattice Aubry-Mather theory*, we present the paper [68]. It deals with finite range recurrence relations over more-dimensional lattices, generalizing Golé’s construction of ghost circles (see [36]). In the setting of finite-range recurrence relations the original proof of Golé fails, since the single crossing principle does not apply. Our proof is based on a Harnack inequality for the gradient flow instead. Furthermore, with the use of ghost circles, we show the existence of non-minimizing solutions lying in the gaps of any minimal lamination, giving an alternative proof to the result in [27]. The only changes made to [68] are the exclusion of the appendix 2.A, the results of which are presented already in section 1.3, and the inclusion of a new appendix, where we provide a proof of the statement that Aubry-Mather sets of rationally independent rotation vectors are unique. This is an adaptation of Bangert’s proof in [12] to monotone variational recurrence relations over \( \mathbb{Z}^d \), and it implies that the collection of Birkhoff global minimizers of any fixed rationally independent rotation vector is ordered.

Chapter 3 contains the paper [67] with the title *A dichotomy theorem for minimizers of monotone recurrence relations* and provides a classification of global minimizers for finite range monotone variational recurrence relations over \( \mathbb{Z} \). When the range of interaction of the recurrence relation is larger than one, non-Birkhoff global

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minimizers may exist. Under a slightly sharper twist condition than in definition 1.1.3, we prove the following dichotomy for global minimizers: either they are Birkhoff, and thus very regular, or they oscillate and grow exponentially. In this case they are very irregular and quite non-physical. In the appendix 3.A, we investigate the ordering properties of Birkhoff global minimizers of finite-range recurrence relations over Z, which have the same rational rotation number. We show that the same ordering properties hold, as in the case of range-one interactions. Furthermore, we construct heteroclinic connections between successive periodic minimizers. All proofs in appendix 3.A are adjusted versions of proofs in [57].

Chapter 4, titled *On the destruction of invariant foliations*, follows the paper [69]. Here we prove a converse KAM theorem for finite range variational monotone recurrence relations over Z. More precisely, we show that minimal foliations with Liouville rotation numbers can be destroyed into minimal laminations, by an arbitrarily small $C^k$ perturbation of the local action. That is, for any Liouville rotation number, the set of local actions that do not admit a minimal foliation is dense in any $C^k$ topology. This generalizes a result of Mather from [54] for second order recurrence relations. His proofs rely heavily on the single crossing property of global minimizers, which does not hold for higher order relations, while our proofs are based on a new strategy for comparing minimizers of close-by rotation numbers.

The main result of chapter 4 is sharpened in chapter 5, which corresponds to the paper [66]. Here we show that for any Liouville rotation number the set of local actions that do not admit a minimal foliation is not only dense, but also open, and thus generic, in any $C^k$ topology for $k \geq 2$. More precisely, chapter 5 with title *Continuity of the Peierls Barrier and Robustness of Minimal Laminations* contains a new continuity theorem for the Peierls barrier function, which is a major ingredient for the following robustness result: for any irrational rotation number, the set of local actions that do not admit a minimal foliation is open in the $C^2$ topology.