1.1 A short historic overview

1.1.1 From classical mechanics to Hamiltonian dynamics

Hamiltonian dynamics originates from physics, more exactly from classical mechanics. The main aim of classical mechanics is to describe a physical system and to investigate and predict the evolution of the system in time. Examples of such systems arise in different fields of physics like celestial mechanics or electromagnetism, describing the orbits of the planets under their gravitational force or the movement of charged particles in a magnetic field. Evolution in time of a dynamical system is governed by differential equations, which relate certain quantities to their derivatives. In case of classical mechanics a differential equation describes the relation between the acceleration of a particle and a force depending on its position and momentum.

In classical mechanics the set of all possible positions of a physical system is called the configuration space of the system. The points of the configuration space correspond to distinct positions of the system, in other words the position of the system can be uniquely determined by the coordinates in the configuration space.

The coordinates of the configuration space are so-called generalized coordinates. The coordinates are called generalized as opposed to the standard coordinates, which were Euclidean at the time of Lagrange and Hamilton, and it reflects the fact that

Figure 1.1: A numerical simulation of a trajectory in a magnetic Hamiltonian system for $H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + (p_2 - q_1^3q_2)^2 - 1)$. 
there can be given constraints on the system, such that the coordinates describing
the position of the system may not be independent, but related to each other by con-
straints. Moreover, by physical assumption the state of the physical system cannot
depend on the chosen coordinate system. This leads to the conclusion that the config-
uration space can be viewed as a smooth manifold. The dimension of the configuration
space is the number of degrees of freedom of the system, whereas the coordinates de-
scribing the manifold are precisely the generalized coordinates. We will identify the
configuration space with a smooth manifold \( Q \). However, in classical mechanics the
state of the system cannot be described precisely by the coordinates in the configura-
tion space only. The generalized coordinates carry the information about the actual
position of the system, but cannot predict how the system will evolve in short time.
On physical basis, we assume that the evolution of the system in time depends only
on the systems current position and its momentum. In the mathematical language of
classical mechanics, it means that the state of the system is fully described if all the
generalized coordinates and their generalized momenta are simultaneously specified.
In other words, the actual state of the system corresponds to a unique point in the
phase space, which is the cotangent bundle \( T^*Q \) over the configuration space \( Q \).

The evolution of the system in time can be described as a curve in the phase space,
\[ t \mapsto (q(t), p(t)) \in T^*Q. \]

The principle of Hamiltonian dynamics states that the time evolution of the system
is uniquely defined by Hamilton’s equations:
\[
\dot{q}(t) = \partial_p H(t, q(t), p(t)) \\
\dot{p}(t) = -\partial_q H(t, q(t), p(t)),
\]
where
\[ H : \mathbb{R} \times T^*Q \to \mathbb{R} \]
is called the Hamiltonian function or just the Hamiltonian and represents the energy
of the system.

Modern symplectic geometry started as a mathematical formulation of classical
mechanics introduced first by V. Arnold in 1974 in his seminal book *Mathematical
Methods in Classical Mechanics* [5]. Instead of considering the cotangent bundle \( T^*Q \),
we define the symplectic manifold \((M, \omega)\) as a manifold equipped with a closed, non-
degenerate 2-form \( \omega \) called the symplectic form. Then to every function \( H : \mathbb{R} \times M \to \mathbb{R} \) one can associate a Hamiltonian vector field, such that it satisfies
\[ dH_t = \omega(\cdot, X^H_t). \]

This assignment is unique, due to the non-degeneracy of the symplectic form. Under
this generalization, Hamilton’s equations are defined by the Hamiltonian vector field
\[
\dot{x}(t) = X^H_t(x).
\]

The cotangent bundle \( T^*Q \) with the standard symplectic form
\[ \omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i \]
CHAPTER 1. INTRODUCTION

is an example of a symplectic manifold. If we calculate the corresponding Hamiltonian vector field we will obtain the same equations as before. If a symplectic form exhibits a primitive $\omega = d\lambda$, then we call $(M, \omega = d\lambda)$ an exact symplectic manifold. For example $(T^*Q, \omega_0)$ is an exact symplectic manifold.

An important property of the Hamiltonian systems is that in the case of time independent, i.e. autonomous Hamiltonians, the flow of the Hamiltonian vector field stays on the level sets of the Hamiltonian function. Indeed, we can calculate

$$\frac{d}{dt} H(x(t)) = dH(\dot{x}) = dH(X_H) = \omega(X^H, X^H) = 0,$$

since $\omega$ is skew-symmetric. Thus for an autonomous Hamiltonian a natural setting would be to analyze the Hamiltonian flow on a given energy level set. In particular, we can try to investigate the relation between the dynamics of the Hamiltonian flow on a chosen energy level set and the geometry and topology of the energy level set in question. It is important to remark here that the Hamiltonian dynamics on a fixed hypersurface $\Sigma$ does not depend on the choice of the Hamiltonian. Indeed, if we take two Hamiltonians $H$ and $F$, such that the hypersurface $\Sigma$ is a level set of both $H$ and $F$, then the corresponding Hamiltonian vector fields $X^H$ and $X^F$ on $\Sigma$ differ only by a scalar function. This implies that the geometry of $\Sigma$ inside the symplectic manifold $(M, \omega)$ fully determines the behavior of Hamiltonian flow on $\Sigma$ up to a reparametrization. In particular, the precise choice of the Hamiltonian is not relevant to the question of existence of periodic orbits on a given hypersurface.

Natural questions to pose analyzing a Hamiltonian system would be:

- Does the Hamiltonian system have periodic orbits on a given energy level?
- Can we estimate the minimal number of the orbits?
- How does this number depend on the geometry and topology of the energy level?

An important conjecture concerning the existence of periodic orbits on a fixed hypersurface was formulated by Weinstein in 1978 in [36]. For a precise formulation we have to introduce the following notion: we say that a hypersurface $\Sigma$ in a symplectic manifold $(M, \omega)$ is of contact type if in the neighborhood of $\Sigma$ there exists a 1-form $\lambda$, called a contact form, such that $d\lambda = \omega$ and $\lambda|_\Sigma$ is nowhere vanishing on $T\Sigma$.

**Conjecture** (Weinstein, 1978)

If $\Sigma$ is a closed, contact type hypersurface in a symplectic manifold $(M, \omega)$, with vanishing first homology group $H^1(\Sigma) = 0$, then any Hamiltonian for which $\Sigma$ is a level set, admits on $\Sigma$ a periodic solution of Hamilton’s equations.

However, non-trivial Hamiltonian systems are often too complex to find explicit solutions and numerical solutions can be very complicated as depicted in Figures 1.1 and 1.2. Moreover, they may also be very sensitive to the changes of initial conditions and parameters of the system. Therefore, rather than trying to solve Hamilton’s equations directly, we will use other mathematical tools, namely variational analysis and algebraic topology, to analyze the Hamiltonian system and answer the above questions.
1.1. A SHORT HISTORIC OVERVIEW

Figure 1.2: Numerical simulations of 3 different trajectories in the restricted 3-body problem.

1.1.2 Rabinowitz action functional

The first variational tool we will introduce is the Rabinowitz action functional. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold and let $H : M \to \mathbb{R}$ be a Hamiltonian, such that $dH|_{H^{-1}(0)} \neq 0$, i.e. $\Sigma = H^{-1}(0)$ is a hypersurface in $M$. We define a functional

$$A^H : C^\infty(S^1, M) \times \mathbb{R} \to \mathbb{R},$$

$$A^H(v, \eta) = \int_{S^1} \lambda(\partial_t v) - \eta \int_{S^1} H(v),$$

where $\lambda$ is a primitive of $\omega$. Note that $A^H$ does not depend on the choice of $\lambda$. The above functional has been defined by Rabinowitz in his paper *Periodic solutions of Hamiltonian systems* [32] and thus has his name. The parameter $\eta$ in the functional can be regarded as the period of the loop. Indeed there is a correspondence between elements of $C^\infty(S^1, M) \times \mathbb{R} \setminus \{0\}$ and loops of period $\eta$ in the following way:

$$\eta \neq 0 \quad (v(t), \eta) \longleftrightarrow v\left(\frac{t}{\eta}\right) =: \tilde{v}(t) \in C^\infty(\mathbb{R}/\eta\mathbb{Z}, M),$$

7
On the other hand every point of $M$ can be viewed as a constant loop. This way we have an embedding of the manifold of constant loops $M \times \mathbb{R}$ into the domain of the action functional:

$$M \times \mathbb{R} \hookrightarrow C^\infty(S^1, M) \times \mathbb{R}.$$ 

Apart from being the period of the loop, the parameter $\eta$ has another function - it acts as a Lagrange multiplier, picking out only the periodic orbits of the Hamiltonian vector field, which lie on the 0 level set of $H$. Indeed, if we calculate the derivative of $A^H$ with respect to a variation $((\xi, \sigma) \in T_{(v, \eta)}(C^\infty(S^1, M) \times \mathbb{R}) = C^\infty(S^1, v^*TM) \times \mathbb{R}$ we obtain:

$$dA^H_{(v, \eta)}(\xi, \sigma) = \int \omega(\xi, \partial_t v - \eta X^H) - \sigma \int H(v),$$

where $X^H$ is the Hamiltonian vector field associated to $H$. We can see that the critical points of $A^H$ are precisely the periodic orbits on the energy hypersurface $H^{-1}(0)$.

Note that, in particular, all the points of $\Sigma = H^{-1}(0)$ viewed as constant loops with $\eta = 0$ are in the critical set of $A^H$. Moreover, if we define the rotation action

$$\theta^* v(t) = v(t + \theta) \quad v \in C^\infty(S^1, M), \quad t, \theta \in S^1$$

then the Rabinowitz action functional is invariant under this action, that is

$$A^H(v, \eta) = A^H(\theta^* v, \eta) \quad (v, \eta) \in C^\infty(S^1, M) \times \mathbb{R}, \quad \theta \in S^1.$$
As a result the critical set \( \text{Crit}(A^H) \) is also invariant under rotation, i.e.

\[
\forall (v, \eta) \in \text{Crit}(A^H) \quad \forall \theta \in S^1, \quad (\theta^*v, \eta) \in \text{Crit}(A^H).
\]

Another property of the critical set of \( A^H \) is its invariance under the \( \mathbb{Z} \)-action defined by

\[
k^*(v, \eta)(t) = (v(kt), k\eta) \quad (v, \eta) \in \text{Crit}(A^H), \quad k \in \mathbb{Z}.
\]

Therefore we can conclude that the critical set of \( A^H \) consists a copy of the hypersurface \( \Sigma = H^{-1}(0) \) embedded in \( C^\infty(S^1, M) \times \mathbb{R} \) as a subset \( \Sigma \times \{0\} \) of constant loops \( M \times \mathbb{R} \) and all the non-constant periodic solutions of the Hamiltonian equations, which appear in \( \text{Crit}(A^H) \) in \( S^1 \)-invariant and \( \mathbb{Z} \)-graded families. An intuitive representation of the critical set of \( A^H \) is depicted in Figure 1.3.

### 1.1.3 Morse and Morse-Bott homology

Having introduced the Rabinowitz action functional in the previous section, we will now present the tools from algebraic topology. Those ingredients we will later combine to define Rabinowitz Floer homology and show how it can be used to analyze Hamiltonian dynamics.

We will start by introducing one of the most important theories which describe the relation between a dynamical system on a manifold and its purely topological properties - Morse theory. It was Morse in 1925 [26], who first observed the relation between the number of critical points of a function on a given compact manifold and the homology of the manifold. That observation begun the development of Morse theory and Morse homology. The setting for Morse homology is as follows:

Let \( Q \) be a manifold and \( f : Q \to \mathbb{R} \) be a smooth function. We say that \( f \) is a Morse function if all its critical points are non-degenerate. Moreover, we say that \( f \) is coercive if for all \( a \in \mathbb{R} \) the corresponding sublevel sets

\[
\{ x \in Q \mid f(x) \leq a \}
\]

are compact in \( Q \). Naturally, every function on a compact manifold is coercive, so this assumption is omitted in the construction of Morse homology on compact manifolds. However, coercivity of \( f \) allows the construction to be carried out also on non-compact manifolds, which are the focus of this thesis.

If we choose a metric \( g \) on \( Q \), then to the pair \( (f, g) \) we can associate a negative gradient flow \( \phi_f^t \). Define a function \( \mu_M : \text{Crit}(f) \to \mathbb{Z} \), which to every critical point \( p \in \text{Crit}(f) \) assigns the number of negative eigenvalues of the Hessian of \( f \) at \( p \). We call \( \mu_M \) the Morse index. Now, if we assume \( f \) to be smooth, coercive and Morse, then for a generic choice of a metric \( g \), we have that the number of negative gradient flow lines between two critical points \( p, q \in \text{Crit}(f) \) of index difference 1 is finite. The \( k \)th chain group \( C_k(f) \) is generated by critical points of \( f \) with Morse index \( k \) and the boundary operator

\[
\partial_k : C_k(f) \to C_{k-1}(f)
\]
CHAPTER 1. INTRODUCTION

is defined as the linear extension of the map
\[ \partial p := \sum_{q \in \text{Crit}(f) \atop \mu_M(p) - \mu_M(q) = 1} n(p, q)q \]
where \( n(p, q) \) is the number of negative gradient flow lines from \( p \) to \( q \). One can show that \( \partial_k \) is a well defined boundary operator, that is \( \partial^2 = 0 \) and thus we can define the \( k \)th Morse homology group as follows
\[ HM_k(Q) := \text{Ker} \partial_k / \text{Im} \partial_{k+1}. \]

Morse homology is isomorphic to singular homology
\[ HM_*(Q) \cong H_*(Q). \]

In particular Morse homology does not depend on the choice of the function \( f \) or the metric \( g \). Using this relation we can extract information about the function \( f \) from the strictly topological properties of \( Q \). In particular, we are able to find lower bounds on the number of critical points of \( f \) in terms of topology of \( Q \). More precisely, the following estimate holds true
\[ \text{Crit}(f) \geq \sum_k \dim(H_k(Q)). \]

For the details on Morse theory we refer the curious reader to the articles of Bott [11] and Milnor [30] or to the book by Schwarz [35]. Also, an explicit example of Morse homology for a sphere is calculated in Figure 1.4.

Note that an important assumption we make on the function \( f \) is that it is Morse. Can we carry out a similar construction for a function, which has degenerate critical values? As it turns out the answer is - yes. However, to construct the homology we still have to make certain assumptions on the function \( f \in C^\infty(Q) \). Citing the definition introduced by Bott in [10], we say that a function \( f \) is Morse-Bott if its critical set \( \text{Crit}(f) \) is a disjoint union of connected submanifolds and the Hessian of \( f \) is non-degenerate in the direction normal to \( \text{Crit}(f) \).

For a Morse-Bott function \( f : Q \to \mathbb{R} \) and a Morse function on the critical set \( h : \text{Crit}(f) \to \mathbb{R} \) we can construct a Morse-Bott homology following the approach by Frauenfelder in Appendix C of [21]. The complex \( C_*(f, h) \) is generated by the critical points of \( h \) and the index on \( \text{Crit}(h) \) is defined as the sum of the number of negative eigenvalues of Hessians of \( f \) and \( h \), i.e. the sum of Morse indexes with respect to \( f \) and \( h \). Similar to the case of Morse homology, where we define the boundary operator by counting the negative gradient flow lines, now we count so called flow lines with cascades, which consist of sequences of flow lines of \( -\nabla f \) and \( -\nabla h \). As before, the number of flow lines with cascades between two critical points with index difference 1 is finite, therefore we can define the boundary operator
\[ \partial_k : C_k(f, h) \to C_{k-1}(f, h) \]
1.1. A SHORT HISTORIC OVERVIEW

as the linear extension of the map

\[ \partial p := \sum_{q \in \text{Crit}(f)} n(p, q)q \]

where \( n(p, q) \) is the number of flow lines with cascades from \( p \) to \( q \). One can show that \( \partial_k \) is a well defined boundary operator, that is \( \partial^2 = 0 \) and thus we can define the \( k \)th Morse-Bott homology group as follows:

\[ HMB_k(Q) := \text{Ker} \partial_k / \text{Im} \partial_{k+1}. \]

Morse-Bott homology is isomorphic to Morse homology and singular homology

\[ HMB_*(Q) \cong \text{HM}_*(Q) \cong \text{H}_*(Q), \]

Figure 1.4: Morse and Morse-Bott setting for functions \( f = z \) and \( f = z^2 \), \( h = x \) respectively.

<table>
<thead>
<tr>
<th></th>
<th>Morse</th>
<th>Morse-Bott</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_2 )</td>
<td>( \mathbb{Z}_2p )</td>
<td>( \mathbb{Z}_2p \oplus \mathbb{Z}_2q )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>0</td>
<td>( \mathbb{Z}_2s )</td>
</tr>
<tr>
<td>( C_0 )</td>
<td>( \mathbb{Z}_2q )</td>
<td>( \mathbb{Z}_2r )</td>
</tr>
</tbody>
</table>

In case of Morse homology \( \partial_2p = 0 \), whereas for Morse-Bott \( \partial_2p = \partial_2q = s \) and \( \partial_1s = 2r = 0 \). Therefore, the homologies are the same and equal to the singular homology of \( S^2 \):

<table>
<thead>
<tr>
<th></th>
<th>Morse</th>
<th>Morse-Bott</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_2(S^2) )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>( H_1(S^2) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( H_0(S^2) )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
</tbody>
</table>
in particular, it does not depend on the choices of $f, h$ or the metrics on $Q$ and $\text{Crit}(f)$. For more details on Morse-Bott homology we refer the curious reader to the survey paper by Hurtubise [23], where he discusses three different approaches to Morse-Bott homology - the presented above cascade approach from the thesis of Frauenfelder [21], the Morse perturbation approach by Banyaga and Hurtubise [7] and the multicomplex approach also by the same authors defined in [8].

1.2 Rabinowitz Floer homology

1.2.1 Construction

The main focus of this thesis is to apply algebraic topology and variational analysis to analyze Hamiltonian dynamics. As we stated before, the geometric properties of a hypersurface $\Sigma$ in a symplectic manifold $(M, \omega)$ fully determine Hamiltonian dynamics on this hypersurface up to a reparametrization. Rabinowitz Floer homology is an algebraic invariant which captures the relation between the geometry of the hypersurface and the question of existence of periodic orbits of the Hamiltonian system. In order to define it, we want to apply the ideas of Morse theory to the infinite-dimensional setting of Rabinowitz action functional.

The idea to define a homology theory using the gradient flow of the symplectic action functional was first introduced by Floer in his seminal article *Symplectic fixed points and holomorphic spheres* [20]. The Floer homology can be described as an infinite-dimensional Morse homology of the symplectic action functional. In other words, it is constructed by considering the complex generated by the critical set of the action functional and defining the boundary operator by counting the gradient flow lines of the action functional with respect to a suitable metric, which we call the *Floer trajectories*. However, the infinite dimensional setting is in many aspects very different from the finite dimensional case. For example, the gradient equation for the symplectic action functional

\[ \partial_s u = \nabla A^H \]

does not define a wellposed Cauchy problem.

One of the most important differences between the Floer and the Rabinowitz setting is the difference between the symplectic action functional and the Rabinowitz action functional. In the symplectic action functional, the Hamiltonian is time dependent and there is no Lagrange multiplier. Therefore its critical set consists of all 1-periodic solutions of Hamilton’s equations in the whole symplectic manifold. In particular, for a generic choice of Hamiltonian all the critical points are isolated and non-degenerate, i.e. the symplectic action functional is Morse and we can apply infinite-dimensional analogue of Morse theory to construct the homology. On the other hand, in the case of Rabinowitz action functional the Hamiltonian is autonomous and we have the Lagrange multiplier $\eta$, which picks out the periodic solutions to Hamilton’s equation confined to the zero level set of the Hamiltonian. As a result the critical set consists of the hypersurface $H^{-1}(0) \times \{0\}$ of constant loops and all the non-constant periodic solutions of the Hamiltonian equations, which appear
in $\text{Crit}(A^H)$ in $S^1$-invariant and $\mathbb{Z}$-graded families (see Figure 1.3). In particular, the critical points are not isolated, so there is no chance for the Rabinowitz action functional to be Morse. However, it turns out that for a generic choice of Hamiltonian the associated Rabinowitz action functional is Morse-Bott, i.e. the $\text{Crit}(A^H)$ consists of a disjoint union of manifolds and the tangent space to $\text{Crit}(A^H)$ is equal to the kernel of the Hessian of the Rabinowitz action functional. Therefore, one can say that Rabinowitz Floer homology is to Morse Bott homology, what Floer homology is to Morse homology.

Rabinowitz Floer homology was first defined by Cieliebak and Frauenfelder in [12] as an algebraic invariant of a compact, exact convex hypersurface $\Sigma$ in an exact convex symplectic manifold $(M, \omega = d\lambda)$. To construct the homology we first choose a Hamiltonian $H : M \to \mathbb{R}$, such that $\Sigma$ is its 0-level set. Next we equip the loop space with a suitable metric, coming from a 2-parameter\(^1\) family of $\omega$-compatible almost

\(^1\)Actually, in [12] Cieliebak and Frauenfelder work with only $t$-dependent families of almost complex structures and the dependence on $\eta$ was introduced later by Abbondandolo and Merry in [1] to prove transversality explicitly. However, we introduce the the $\eta$ dependence already here to be consistent throughout the entire thesis.
complex structures \( \{J_t(\cdot, \eta)\}_{(t, \eta) \in S^1 \times \mathbb{R}} \). An almost complex structure is a vector bundle isomorphism of the tangent space \( J : TM \to TM \), such that \( J^2 = -Id \). We say that an almost complex structure \( J \) is compatible with a symplectic form \( \omega \) if \( \omega(\cdot, J \cdot) \) is a Riemannian metric on \( M \). We define a metric on the loop space in the following way

\[
g_J(v, \eta)((\xi_1, \sigma_1), (\xi_2, \sigma_2)) = \int_0^1 \omega(\xi_1(t), J_t(v(t), \eta)\xi_2(t)) dt + \sigma_1 \sigma_2,
\]

for \( (v, \eta) \in C^\infty(S^1, M) \times \mathbb{R} \) and \( (\xi_1, \sigma_1), (\xi_2, \sigma_2) \in T_{(v, \eta)}C^\infty(S^1, M) \times \mathbb{R} \). Now we can calculate the gradient of the Rabinowitz action functional with respect to that metric:

\[
\nabla^J A^H(v, \eta) = \begin{pmatrix}
-J_t(v(t), \eta)(\partial_t v - \eta X_H(v)) \\
-\int H(v)
\end{pmatrix},
\]

and analyze the corresponding gradient flow lines of \( A^H \), i.e. the solutions to the Rabinowitz-Floer equations

\[
\begin{cases}
\partial_s v + J_t(v(t), \eta)(\partial_t v - \eta X_H(v)) = 0, \\
\partial_s \eta + \int H(v) = 0
\end{cases}
\]

At last, following the approach presented in the previous section for Morse-Bott homology, we choose on \( \text{Crit}(A^H) \) an auxiliary Morse function \( f : \text{Crit}(A^H) \to \mathbb{R} \). The chain complex is generated by the critical points of \( f \). We introduce grading on \( \text{Crit}(f) \) as a sum of the signature index \( \mu_\tau \) taken with respect to the gradient flow of \( f \) and the Conley-Zehnder index \( \mu_{CZ} \) introduced by Conley and Zehnder in [14], which is the equivalent of the Morse index for the gradient flow of symplectic action functional. The boundary operator is constructed by counting the gradient flow lines with cascades, i.e. sequences consisting of gradient flow lines of \( \nabla A^H \) and \( \nabla f \). The visual representation of the flow lines with cascades is depicted in Figure 1.5. One can prove that the boundary operator defined by counting the flow lines with cascades is well defined, i.e. squares to 0. The homology of this chain complex is precisely the Rabinowitz Floer homology. One can prove that it does not depend on choices of the family of almost complex structures \( \{J_t(\cdot, \eta)\}_{(t, \eta) \in S^1 \times \mathbb{R}} \), the auxiliary function \( f \) and the metric on \( \text{Crit}(A^H) \). Therefore, it is an invariant of the hypersurface \( \Sigma \) in the exact symplectic manifold \( (M, \omega) \) and is denoted by \( RFH(\Sigma, M) \).

The first important property of Rabinowitz Floer homology to observe is that if there are no periodic orbits on \( \Sigma \) then the Rabinowitz Floer homology is isomorphic to the homology of the hypersurface

\[
RFH_*(\Sigma, M) \cong H_*(\Sigma).
\]

This isomorphism shows the connection between Rabinowitz Floer homology and Hamiltonian flow on the hypersurface. More precisely, this implies existence of periodic orbits on \( \Sigma \), whenever the Rabinowitz Floer homology of \( \Sigma \) in \( (M, d\lambda) \) is different from the homology of \( \Sigma \). However, as we will show in the next subsection, this is only the first of many interesting properties of Rabinowitz Floer homology.
1.2.2 Developments in the field

After being defined by Cieliebak and Frauenfelder in [12], Rabinowitz Floer Homology has been studied by many others. Abbondandolo and Schwarz in [2] extended the definition of Rabinowitz Floer homology to Liouville domains, which are exact, compact symplectic manifolds $(M,\omega = d\lambda)$ with a smooth, contact type boundary $\Sigma$.

In [29] Merry extended the definition of Rabinowitz Floer homology to closed hypersurfaces in weakly exact, twisted cotangent bundles. A twisted cotangent bundle is the cotangent bundle $T^*Q$ with symplectic form $\omega$ defined

$$\omega := \omega_0 + \pi^*\sigma,$$

where $\omega_0$ is the standard symplectic form on $T^*Q$ and $\pi^*\sigma$ is a pull back of a 2-form $\sigma \in \Omega^2(Q)$ to the cotangent bundle $T^*Q$. We say that a 2-form is weakly exact if its pullback to its universal cover is exact. Maybe it is good to remark here that the twisted cotangent bundles are connected to the notion of magnetic Hamiltonians (mentioned in Figure 1.1). In fact, there is a one-to-one correspondence between magnetic Hamiltonians in $(T^*Q,\omega_0)$ and mechanical Hamiltonians in exact, twisted cotangent bundles.

Recently in [18] Fauck extended the definition of Rabinowitz Floer homology to closed hypersurfaces with symmetries in exact convex symplectic manifolds in order to calculate it for a class of contact type manifolds called Brieskorn manifolds.

However, in all of the above examples the Rabinowitz Floer homology was defined for compact hypersurfaces $\Sigma$ in non-compact, symplectic manifolds $(M,\omega)$.

Rabinowitz Floer homology has been found to have many connections to other parts of symplectic geometry. The first result, proven by Cieliebak and Frauenfelder already in [12] is the relation between $RFH(\Sigma,M)$ and displaceability of $\Sigma$ in $M$. We say that a compact hypersurface $\Sigma$ is displaceable in $M$ if there exists a compactly supported Hamiltonian $H \in C^\infty_0(M)$, whose Hamiltonian flow time-1-map $\phi^H : M \to M$ satisfies

$$\Sigma \cap \phi^H(\Sigma) = \emptyset.$$

The result states, that if $\Sigma$ is displaceable in $M$, then $RFH(\Sigma,M) = 0$. This means that every displaceable, compact, exact convex hypersurface $\Sigma$ in an exact convex symplectic manifold $(M,\omega = d\lambda)$ with $H(\Sigma) \neq 0$ admits a periodic orbit of the Hamiltonian flow on $\Sigma$.

Another example of an application of Rabinowitz Floer homology is its relation to the question of existence of leaf-wise intersections showed by Albers and Frauenfelder in [3]. The notion of leaf-wise intersections was first introduced by Moser in [31]. Let $\Sigma$ by a hypersurface in a symplectic manifold $(M,\omega)$. Let $U(\Sigma)$ be a neighborhood of $\Sigma$ in $M$ and let $\psi : U(\Sigma) \to M$ be a symplectomorphism. Let $\phi^\Sigma : \mathbb{R} \times \Sigma \to \Sigma$ be the Hamiltonian flow corresponding to $\Sigma$. Then a leaf-wise intersection is a point $p \in \Sigma$, such that $\psi(p) \in \phi^\Sigma(\mathbb{R},p)$. In other words, $\psi(p)$ lies on the same trajectory as $p$. The notion of leaf-wise intersections is opposite to the notion of displaceability. Indeed, in Theorem C in [3] it is proven that whenever $\Sigma$ is a compact, contact-type hypersurface in a simply connected, exact symplectic manifold $(M,\omega = d\lambda)$ and $RFH(\Sigma,M) \neq 0$, then for every compactly supported Hamiltonian diffeomorphism $\psi : M \to M$ there
always exist a leaf-wise intersection point. Moreover, Theorem B in the same paper gives the estimate for the number of leaf-wise intersection points by the sum of Betti numbers of the Rabinowitz Floer homology for small enough perturbations $\psi$.

Two important properties of Rabinowitz Floer homology have been established by Cieliebak, Fraunfelder and Oancea in [13] - one is its independence of the symplectic completion and the other is its connection to symplectic homology. Let $\Sigma$ be a compact, exact convex hypersurface in an exact convex symplectic manifold $(M, \omega = d\lambda)$, such that one of the connected components of $M \setminus \Sigma$ is compact. We denote it by $V$ and state that $\partial V = \Sigma$. By Proposition 3.1 in [13] $RFH(\Sigma, M)$ does not depend on $M \setminus V$. On the other hand, Theorem 1.2 in [13] provides a relation between Rabinowitz Floer homology and symplectic homology proving the existence of the following long exact sequence between Rabinowitz Floer homology, symplectic homology and symplectic cohomology

$$\ldots \rightarrow SH^{-*}(V) \rightarrow SH_*(V) \rightarrow RFH_*(\partial V, V) \rightarrow SH^{1-*}(V) \rightarrow \ldots$$

Another example is application of Rabinowitz Floer homology to classification of contact structures. In [18] Fauck first shows that for certain Liouville domains $(V, \Sigma)$ the associated Rabinowitz Floer homology does not depend on the filling $V$ and therefore $RFH(V, \Sigma)$ becomes itself an invariant of the contact structure $(\Sigma, \xi)$. Later he shows that the Rabinowitz Floer homology for different Brieskorn manifolds is different, thus proving that the contact structures they carry are not contactomorphic, even though topologically all the Brieskorn manifolds are spheres.

The last property of Rabinowitz Floer homology we mention here, is its isomorphism to the Floer homology on the time-energy extended phase space shown by Abbondandolo and Merry in [1]. An extended phase space of an exact symplectic manifold $(M, \omega)$ is an exact symplectic manifold of the form $(M \times T^*\mathbb{R}, \omega \times \omega_0)$, where $\omega_0$ denotes the canonical symplectic form on $T^*\mathbb{R}$. In [1] the authors show that the Rabinowitz Floer homology corresponding to a chosen, compactly supported, defining Hamiltonian $H$ for a hypersurface $\Sigma = H^{-1}(0) \subset M$ is isomorphic to the Floer homology on the extended phase space defined in terms of a Hamiltonian $\tilde{H}$ related to $H$.

As we can see there are plenty of interesting connections of Rabinowitz Floer homology to other parts of symplectic geometry. We refer a curious reader, who would like to learn more, to the survey [4] by Albers and Frauenfelder. However, in all of the mentioned examples Rabinowitz Floer homology was defined for a compact contact-type hypersurface in a non-compact exact symplectic manifold. This thesis presents the first attempt to extend the definition of Rabinowitz Floer homology to non-compact hypersurfaces.

1.3 Subject of this thesis

The result of this thesis is an extension of the definition of Rabinowitz Floer homology, which includes examples of non-compact energy hypersurfaces $\Sigma$ in exact symplectic manifolds $(M, \omega = d\lambda)$. In Chapter 2 we present a general framework
for the construction of Rabinowitz Floer homology under suitable conditions on the
hypersurface $\Sigma$ and the associated Hamiltonian system. In Chapter 3 we introduce
a class of Hamiltonians, called \textit{tentacular Hamiltonians}, which satisfy the conditions
and for which the Rabinowitz Floer homology is therefore well defined. Finally, in
Chapter 4, we show some explicit examples of tentacular Hamiltonians.

Let us first discuss the framework and the construction of Rabinowitz Floer ho-
mology presented in Chapter 2. Since we want to construct a generalization of the
existing definition, that requires the new construction to agree with the old one in the
case of compact hypersurfaces. Similarly as in the compact case we have an exact,
symplectic manifold $(M, \omega = d\lambda)$ and a hypersurface $\Sigma$ of contact type. We also
choose $\Sigma$ to be a hypersurface \textit{with cylindrical ends}. That means that there exists a
compact subset $N \subset \Sigma$ with smooth boundary, such that $\Sigma$ is diffeomorphic to

\[ \Sigma \cong N \bigcup_{\partial N} \partial N \times [0, \infty). \]

Note that every closed hypersurface satisfies the above condition with $N$ chosen triv-
ially to be the whole hypersurface. In Section 2.8 we discuss the topology of the
hypersurface $\Sigma$ in more detail. In particular, we show that requiring the hypersurface $\Sigma$
to have cylindrical ends is equivalent with the existence of coercive, Morse functions
on $\Sigma$, which have all critical points in $N$. This ensures that the constructed Rabi-
nowitz Floer homology is isomorphic to the singular homology of the hypersurface,
whenever the hypersurface admits no periodic orbits of the Hamiltonian flow.

To construct the Rabinowitz action functional we choose a Hamiltonian $H : M \to \mathbb{R}$, such that $\Sigma$ is its zero level set. The first requirement we impose on $H$ is that all its
periodic orbits of the Hamiltonian flow with bounded action, are in fact in a bounded
subset of $M \times \mathbb{R}$. This is of course trivially satisfied whenever the hypersurface $\Sigma$
is compact. The second condition we have to impose on the Hamiltonian is that
the associated Rabinowitz action functional is Morse-Bott. It is necessary in the
construction, since our aim is to construct a Morse-Bott homology of the Rabinowitz
action functional. The Morse-Bott property is satisfied generically for Hamiltonians
defining compact hypersurfaces $\Sigma$ as it was proven in [12], however the proof does
not generalize to the non-compact setting without additional requirements. Those
requirements and other implications of the Morse-Bott property are further discussed
in Section 2.7. The last condition we impose on the system, which is necessary to
define the boundary operator and to prove that it squares to 0, is the boundedness
of the corresponding Floer trajectories. More precisely, one has to assure that the
solutions to the Rabinowitz-Floer equations with action in a chosen interval, i.e. all
$u : \mathbb{R} \to C^\infty(S^1, M) \times \mathbb{R}$, such that

\[ \partial_s u(s) = \nabla A^H(u(s)) \quad \text{and} \quad A^H(u(s)) \in [a, b] \quad \forall \ s \in \mathbb{R}, \]

are uniformly bounded in the $L^\infty$ norm. This requirement is highly non-trivial, but
in Chapter 3 we present a family of Hamiltonians for which it is satisfied.

The main content of Chapter 2 is Theorem 1, which states that in the general
setting presented above the Rabinowitz Floer homology is well defined. We define the
Rabinowitz Floer homology in terms of the exact, symplectic manifold \((M,\omega)\), the Hamiltonian \(H : M \to \mathbb{R}\) and a regular almost complex structure \(J\) and denote it by

\[ RFH(H,J,M). \]

We also show that for every Hamiltonian in the general setting presented above the set of regular almost complex structures is dense. Unfortunately, to prove the independence of Rabinowitz Floer homology of the choice of almost complex structure one needs additional assumptions, which are discussed in detail in Section 2.6 along with the invariance of Rabinowitz Floer homology under small perturbations of the hypersurface \(\Sigma\). The proofs are mainly adaptations of the compact case methodology to the non-compact setting.

Similarly as in the compact setting, whenever there are no periodic orbits of the Hamiltonian flow on \(\Sigma = H^{-1}(0)\), then the corresponding Rabinowitz Floer homology is in fact equal to the singular homology of the hypersurface. More precisely, the following relation holds true

\[ RFH(H,J,M) \cong H(\Sigma). \]

This observation allows us to use the Rabinowitz Floer homology to answer the Weinstein conjecture in certain cases. Indeed, whenever the Rabinowitz Floer homology differs from the singular homology of the hypersurface, then we can infer the existence of periodic orbits on \(\Sigma\).

In Chapter 3 Definition 3.1 we introduce a class of Hamiltonians in the exact symplectic manifold \((\mathbb{R}^{2n},\omega_0)\), called tentacular Hamiltonians, which include natural examples of Hamiltonians with non-compact 0-level sets as this one:

\[ H(q_1,q_2,p_1,p_2) := \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 - q_2^2 - 1), \quad \text{on} \quad \mathbb{R}^4. \]

In Theorem 6 we prove that for tentacular Hamiltonians the Rabinowitz Floer homology is well defined by showing that the conditions from Chapter 2 are satisfied. The most challenging condition to verify is the boundedness of Floer trajectories, which is the statement of Theorem 7 and its proof occupies most of Chapter 3. Additionally, we show that the Rabinowitz Floer homology for tentacular Hamiltonians is independent of the choice of an almost complex structure and invariant under compactly supported homotopies of Hamiltonians, using the methods introduced in Section 2.6. Since the symplectic manifold for tentacular Hamiltonians is always \((\mathbb{R}^{2n},\omega_0)\) and we show that the homology is independent of the choice of an almost complex structure, for tentacular Hamiltonians we denote the Rabinowitz Floer homology by

\[ RFH(H) \]

as an invariant of the Hamiltonian alone.

Let us take a closer look at the system and discuss different techniques to prove boundedness of Floer trajectories. The trajectories \(u(s) = (v(s,t),\eta(s))\) have two components - the \(\eta\)-component, which is a function from \(\mathbb{R}\) to \(\mathbb{R}\) and the \(v\)-component
which is a function from a cylinder into \( M = \mathbb{R}^{2n} \). We have to bound the Floer trajectories in both components.

Recall, that in the paper by Cieliebak and Frauenfelder [12] the convex hypersurface \( \Sigma \) was compact so the defining Hamiltonian \( H \) satisfying \( H^{-1}(0) = \Sigma \), was chosen to be constant outside a compact set. This turned the Floer trajectories into \( J \)-holomorphic curves outside a compact set, which then could be easily bounded in \( L^\infty \) using the maximum principle. Therefore, in that case the main challenge was to bound the \( \eta \)-component of the Floer trajectory. However, in our case, since we want to include examples of non-compact hypersurfaces \( \Sigma \), we are not allowed to choose the defining Hamiltonian arbitrarily outside a compact set, without changing the hypersurface itself. This means that the boundedness of the Floer trajectories is not automatic as it was in the case of compactly supported Hamiltonians.

The inspiration for the class of Hamiltonians with non-compact level sets came from the work of van den Berg, Pasquotto and Vandervorst. In their paper [9] they prove existence of periodic orbits on non-compact hypersurfaces which arise as level sets of mechanical Hamiltonians, which admit certain limit behavior at infinity. Even though, the techniques they use in their paper are different, the conditions they impose on the Hamiltonians inspired (H1) in definition of tentacular Hamiltonians, which as a result enabled us to prove the bounds for the \( \eta \)-component of the Floer trajectories.

Another approach to the question of bounding Floer trajectories was presented by Abbondandolo and Schwarz in [2]. In their case the hypersurface \( \Sigma \) was also compact, but the defining Hamiltonian \( H^{-1}(0) = \Sigma \) did not vanish outside a compact set. Instead the Hamiltonians in question were chosen to exhibit certain radial growth on the completion of the Liouville domain, which made it possible to find the uniform \( L^\infty \) bounds on \( v \)-component of the Floer trajectories by using the Aleksandrov maximum principle. Unfortunately, this construction enables to establish bounds only in the direction transversal to the hypersurface \( \Sigma \), which in case of non-compact hypersurfaces is not enough.

The novel approach introduced here is the method to bound the Floer trajectories along the non-compact hypersurface \( \Sigma \). In Section 3.6, we analyze the behavior of Floer trajectories in the neighborhood of the non-compact component of the critical set of the Rabinowitz action functional, \( \Sigma \times \{0\} \subseteq \text{Crit}(\mathcal{A}^H) \). Due to the Morse-Bott property of the action functional the Floer trajectories near \( \Sigma \times \{0\} \) are transverse to \( \Sigma \times \{0\} \). In other words, in the neighborhood of \( \Sigma \times \{0\} \) the tangential component of the Floer trajectories can be bounded and as a result the Floer trajectories cannot escape along \( \Sigma \times \{0\} \).

The other method used in the proof is the application of Aleksandrov maximum principle in order to achieve \( L^\infty \) bounds of Floer trajectories. The Aleksandrov principle itself is not new and has been applied for example in [2], but here we use it in a non standard setting. The general idea of the method is to find a coercive plurisubharmonic function and show that its composition with the Floer trajectories satisfies a certain elliptic differential inequality. In [2] the coercive function was chosen to relate to the Hamiltonian in a specific way, namely their gradients were assumed to be parallel. Even though this condition simplifies the computations greatly, it is not necessary to establish the bounds and can be weakened as we show it in Section 3.8.
Finally, in Chapter 4 we analyze further the class of tentacular Hamiltonians, presenting some explicit examples and using Hörmander’s classification to verify which quadratic Hamiltonians are tentacular.

1.4 Conclusions and future work

This thesis is the first attempt to extend the notion of Rabinowitz Floer homology to non-compact hypersurfaces. A natural step to take from here would be to find a method to calculate such defined homology. We also strongly believe that one can extend the result by including more examples of Hamiltonians beyond the tentacular class. One could do it by trying either to generalize the setting presented in Chapter 2, or by finding more examples to fit into the existing framework.

We would like to bring your attention to the fact that Theorem 7, in which we establish the boundedness of Floer trajectories, does not require all the conditions from Definition 3.1, thus can be applied to Hamiltonians outside of the tentacular class. However, we require more properties in our definition of tentacular Hamiltonians in order to make the theory complete, in particular to establish the genericity of the Morse-Bott property, which is necessary to define the Rabinowitz Floer homology.

The Morse-Bott property is proven by adding a perturbation to the original Hamiltonian and changing it in the neighborhood of its periodic orbits, so that the Rabinowitz action functional corresponding to the perturbed Hamiltonian is Morse-Bott. In the case of compact hypersurfaces $\Sigma$ in an exact symplectic manifold $(M,\omega)$ all the periodic orbits on $\Sigma$ were obviously in a compact set. Moreover, if one would add a compact perturbation $h \in C^\infty_0(M)$ to the defining Hamiltonian $H \in C^\infty(M)$, $H^{-1}(0) = \Sigma$, then the perturbed hypersurface $(H + h)^{-1}(0)$ would be still compact and the perturbed periodic orbits would be still in a compact set. However, in the case of a non-compact hypersurface $\Sigma$ the setting is much more challenging - the periodic orbits of the Hamiltonian vector field might not be confined to a compact set and even if they are in a compact set for the original Hamiltonian then adding a compact perturbation might create a sequence of periodic orbits diverging to infinity.

There are three ways we can approach this challenge:

1. One requires that the original Hamiltonian $H$ has all periodic orbits of its Hamiltonian flow on $H^{-1}(0)$ confined to compact set and if we add a compact perturbation then the perturbed Hamiltonian $H + h$ has all periodic orbits of its Hamiltonian flow on $(H + h)^{-1}(0)$ confined to compact set. In other words, Hamiltonian $H$ satisfies (PO+) and property (PO+) persists under compact perturbations.

2. For the original Hamiltonian $H$ one requires that for every action window $[a,b] \subseteq \mathbb{R}$ all the periodic orbits of the Hamiltonian flow on $H^{-1}(0)$ with action between $a$ and $b$, are confined to a compact set. Moreover, the same property is satisfied for every compact perturbation of the original Hamiltonian. In other words, Hamiltonian $H$ satisfies (PO) and property (PO) persists under compact perturbations.
3. One proves the Morse-Bott property using non-compactly supported perturbations.

In this thesis we chose the first approach, which poses quite strong restrictions on the Hamiltonians, but has the advantage of being quite similar to the compact case. In particular, the reason we required the tentacular Hamiltonians to satisfy Property (H4) is exactly to assure that all the tentacular Hamiltonians satisfy (PO+) and that property (PO+) persists under compact perturbations, which we use to prove the genericity of the Morse-Bott property in the set of tentacular Hamiltonians. However, now we would like to discuss the idea how one could possibly extend the theory using the second approach.

Suppose we had a Hamiltonian $H$ on the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, which would satisfy Property (PO) and such that every compact perturbation of $H$ would also satisfy Property (PO). For every action window $[a, b] \subseteq \mathbb{R}$ we could compactly perturb $H$ and prove that for a generic set of compact perturbations $h^{(a,b)}$ the corresponding action functional $A^{H + h^{(a,b)}}$ is Morse-Bott in the action window $[a, b]$. Assuming additionally that $H$ satisfies assumption of Theorem 7, which would give us the boundedness of the Floer trajectories, one could then define the truncated Rabinowitz Floer homology $RFH_{s}^{(a,b)}(H + h^{(a,b)})$, which is the homology generated by the critical points with action between $a$ and $b$. If moreover this homology was invariant under compact perturbations, then we could define the truncated homology for $H$ by setting $RFH_{s}^{(a,b)}(H) = RFH_{s}^{(a,b)}(H + h^{(a,b)})$ for small enough perturbation $h^{(a,b)}$. In the end one could define the full Rabinowitz Floer homology for $H$ as the limit of truncated homologies in the following way:

$$RFH_{s}(H) = \lim_{b \to +\infty} \lim_{a \to -\infty} RFH_{s}^{(a,b)}(H).$$

This idea has been inspired by the result of Cieliebak, Frauenfelder and Oancea, who in [13] prove that the Rabinowitz Floer homology of a compact, exact convex hypersurface $\Sigma$ in an exact convex symplectic manifold $(M, \omega = d\lambda)$ is equal to the limit of truncated homologies in the way described above. In particular, that would mean that the two definitions would coincide in the compact case. However, to carry out this construction one would have first to investigate the class of Hamiltonians, which not only would satisfy Property (PO) themselves, but also every of their compact perturbation would satisfy Property (PO).

The other idea how to extend the theory in the non-compact setting would be to look for relations between Rabinowitz Floer homology and other notions such as symplectic homology, displaceability or leaf wise intersections, inspired by the way they relate in the compact setting, which was described in Subsection 1.2.2. However, as far as we know none of the above notions has been defined for non-compact hypersurfaces, yet.