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Wisniewska, J.J.

2017

document version

Publisher's PDF, also known as Version of record

[Link to publication in VU Research Portal](#)

citation for published version (APA)

Wisniewska, J. J. (2017). *Rabinowitz Floer Homology for Tentacular Hamiltonians*. [PhD-Thesis - Research and graduation internal, Vrije Universiteit Amsterdam].

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EXAMPLES OF TENTACULAR HAMILTONIANS

In this chapter we will analyze the conditions which define the set of tentacular Hamiltonians and present computable methods of verifying whether a given Hamiltonian is tentacular. Unfortunately, those tools do not give us a definite answer, but rather give us a computable set of assumptions under which a Hamiltonian is tentacular.

Definition 3.1. (*Tentacular Hamiltonians*)

For a smooth Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ we can state the following properties:

H1 *There exists a global Liouville vector field X^\dagger and constants $c_1, c_2 > 0$, $c_3 \geq 0$ such that for all $x \in \mathbb{R}^{2n}$ the following holds true*

$$\begin{aligned} \|X_x^\dagger\| &\leq c_1(\|x\| + 1), \\ dH_x(X^\dagger) &\geq c_2\|x\|^2 - c_3. \end{aligned}$$

H2 *H grows at most quadratically at infinity*

$$\sup_{x \in \mathbb{R}^{2n}} \|D^3 H_x\| \cdot \|x\| < L.$$

H3 *There exist constants $c_4, c_5, \nu > 0$ and a Liouville vector field X^\ddagger defined on $H^{-1}((-\nu, \nu))$, such that*

$$\begin{aligned} \|X^\ddagger(x)\| &\leq c_4(\|x\|^2 + 1) \quad \forall x \in \mathbb{R}^{2n}, \\ c_5 &:= \inf_{H^{-1}((-\nu, \nu))} dH(X^\ddagger) > 0. \end{aligned}$$

H4 *There exist a coercive function $\mathcal{F} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that the set*

$$\{x \in \mathbb{R}^{2n} \mid d_x(d\mathcal{F}(X^H))(X^H(x)) \leq 0\} \cap \{x \in \mathbb{R}^{2n} \mid d_x\mathcal{F}(X^H) = 0\} \cap H^{-1}(0)$$

is compact in \mathbb{R}^{2n} .

We will denote by \mathcal{H} the set of all smooth Hamiltonians satisfying Property (SG) and all the above Properties and call them **tentacular Hamiltonians**.

As already mentioned one can distinguish among the above Properties the one which depends only on the topology of the hypersurface - (SG) and the ones, which take into account also the symplectic structure of the manifold - namely (H1), (H3) and (H4). Therefore, we will analyze those properties separately in Section 4.1 and 4.2 respectively.

From Lemma 4.2 we can conclude that every quadratic Hamiltonian on \mathbb{R}^{2n} satisfies both Property (H2) and (SG). In other words for a non-degenerate quadratic form Q on \mathbb{R}^{2n} , a vector $b \in \mathbb{R}^{2n}$, a constant $c \in \mathbb{R}$ and any compact perturbation $h \in C_0^\infty(\mathbb{R}^{2n})$ the associated Hamiltonian

$$H(x) = \frac{1}{2}Q(x, x) + \langle b, x \rangle + h(x) + c$$

satisfies Property (H2) and (SG).

The case of the other properties is more complicated, since we have to take into account the symplectic structure of the manifold. We will use the Hörmander classification of quadratic forms to determine the conditions for a non-degenerate quadratic form to satisfy in order for the associated Hamiltonian to have Properties (H1) and (H3). In particular, whenever a quadratic form Q admits a coercive Liouville vector field (see Definition 4.4), then for all $c \neq 0$ the associated Hamiltonian of the form

$$H(x) = \frac{1}{2}Q(x, x) - c \tag{4.1}$$

satisfies Properties (H1) and (H3). Note that here the presented Hamiltonian we cannot add an arbitrary compact perturbation, because the contact type property may not hold true if the perturbation is too big. In particular, if after a symplectic change of coordinates we can decompose a given quadratic form Q into orthogonal subspaces each of which satisfies one of the assumptions of Proposition 4.6, then by Corollary 4.7 Q admits a coercive Liouville vector field and the associated Hamiltonian satisfies Properties (H1) and (H3).

Analogically, we use the Hörmander classification of quadratic forms to analyze the assumptions on Q which guarantee that Property (H4) holds. In particular, whenever after a symplectic change of coordinates we can decompose a given quadratic form Q into orthogonal subspaces each of which satisfies one of the assumptions of Corollary 4.11, then by Corollary 4.12 the Hamiltonian of the form (4.1) admits Property (H4). Moreover, it will also satisfy Properties (H1) and (H3), since Corollary 4.11 has stronger conditions than Proposition 4.6.

Summarizing: Let Q be a quadratic form, such that after a symplectic change of coordinates we can decompose it into orthogonal subspaces each of which satisfies one of the assumptions of Corollary 4.11. Then for $c \neq 0$ the Hamiltonian defined by

$$H(x) = \frac{1}{2}Q(x, x) - c$$

is tentacular. This result comes from the following observations:

- (i) By construction H satisfies Property (H2),
- (ii) By Lemma 4.2 H satisfies Property (SG),
- (iii) By Corollary 4.7 H satisfies Properties (H1) and (H3),
- (iv) By Corollary 4.12 H satisfies Property (H4).

The last part of this chapter is dedicated to the analysis of certain geometrical properties of the 0 level sets of tentacular Hamiltonians, which are needed in the proof of Theorem 7 in the previous chapter.

4.1 Hypersurface with cylindrical ends

In this section we analyze the geometric properties of the hypersurface given as a 0-level set of a Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$. In particular, we try to investigate what kind of analytic assumption we have to pose on H to assure that the hypersurface $H^{-1}(0)$ satisfies condition (SG). Recall, that by Proposition 2.35 the the Hamiltonian H satisfies condition (SG) if and only if there exists a coercive function on $H^{-1}(0)$, which has all its critical points in a compact set. In Remark 4.1 we state the condition which a function $\tilde{f} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ has to satisfy so that its restriction to the hypersurface $H^{-1}(0)$ has all its critical points in a compact set.

Remark 4.1. *Let $H, \tilde{f} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ and let 0 be a regular value of H . Denote the restriction of \tilde{f} to $H^{-1}(0)$ by f . Then all the critical points of f lie in a compact set $K \subseteq H^{-1}(0)$ if and only if*

$$\|\nabla H(x)\| \|\nabla \tilde{f}(x)\| > |\langle \nabla H, \nabla \tilde{f} \rangle| \quad \forall x \in H^{-1}(0) \setminus K,$$

where the gradient is taken with respect to the standard metric on \mathbb{R}^{2n} .

Proof. This follows quickly from the fact that

$$\begin{aligned} \nabla f &= \nabla \tilde{f} - \frac{\langle \nabla H, \nabla \tilde{f} \rangle}{\|\nabla H\|^2} \nabla H, \\ \|\nabla f\|^2 &= \|\nabla \tilde{f}\|^2 - \frac{|\langle \nabla H, \nabla \tilde{f} \rangle|^2}{\|\nabla H\|^2} \end{aligned}$$

so

$$\nabla f(x) \neq 0 \quad \iff \quad \|\nabla H(x)\| \|\nabla \tilde{f}(x)\| > |\langle \nabla H, \nabla \tilde{f} \rangle|.$$

□

Having proven Remark 4.1 we know the condition a function $\tilde{f} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ has to satisfy so that its restriction to the hypersurface $H^{-1}(0)$ has all its critical values in a compact set. To assure that Condition (SG) is satisfied, we have to make sure that the function is also coercive on $H^{-1}(0)$.

In the following lemma we state a condition on a Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, which assures that on $H^{-1}(0)$ there exists a coercive function, which has all its critical points in a compact set. In other words Lemma 4.2 assures that Condition (SG) is satisfied.

Lemma 4.2. *Take \mathbb{R}^{2n} with the standard metric and a function $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, such that 0 is a regular value and satisfying*

$$\begin{aligned} \|\nabla H(x)\| &\geq \gamma(\|x\| - 1) \quad \forall x \in \mathbb{R}^{2n}, \\ \text{and} \quad \left| \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) \right| &\leq \alpha(\|x\| + 1), \quad \forall x \in \mathbb{R}^{2n}, \end{aligned}$$

for some $\alpha, \gamma > 0$. Then the function

$$\begin{aligned} f : H^{-1}(0) &\rightarrow \mathbb{R}, \\ f(x) &:= \|x\|^2, \quad x \in H^{-1}(0) \end{aligned}$$

has all its critical points in a compact set and for all $a \in \mathbb{R}$ the associated set $f^{-1}((-\infty, a])$ is compact.

Proof. Extend f to \mathbb{R}^{2n} in the natural way and denote the extension \tilde{f} . Observe that for all $a \in \mathbb{R}$ the associated set $f^{-1}((-\infty, a])$ is compact since it's an intersection of $\tilde{f}^{-1}((-\infty, a])$, which is compact and $H^{-1}(0)$, which is closed.

Now we will prove that all the critical points of f lie in a compact set using result from Remark 4.1.

Denote

$$r := \frac{1}{2} + \frac{\alpha}{\gamma} + \sqrt{\left(\frac{1}{2} + \frac{\alpha}{\gamma}\right)^2 + 1},$$

then

$$f^{-1}((r^2, +\infty)) = \{x \in H^{-1}(0) \mid \|x\| > r\}$$

and by the assumptions for all $x \in f^{-1}((r^2, +\infty))$ one has

$$\begin{aligned} |\langle \nabla H(x), \nabla \tilde{f}(x) \rangle| &= |\langle \nabla H(x), 2x \rangle| \\ &= 4 \left| \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) \right| \\ &\leq 4\alpha(\|x\| + 1) \\ &< 2\gamma\|x\|(\|x\| - 1) \\ &\leq \|\nabla H(x)\| \|\nabla \tilde{f}(x)\|. \end{aligned}$$

Since

$$\|\nabla H(x)\| \|\nabla \tilde{f}(x)\| > |\langle \nabla H(x), \nabla \tilde{f}(x) \rangle| \quad \forall x \in f^{-1}((r^2, +\infty)),$$

therefore f has all its critical points in $f^{-1}((-\infty, r^2])$, which is compact. \square

Lemma 4.3. *Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, be a smooth function. Then the following are equivalent:*

1. *There exists $\alpha > 0$ such that*

$$\left| \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) \right| \leq \alpha(\|x\| + 1) \quad \forall x \in \mathbb{R}^{2n}.$$

2. *One can express the H in the form*

$$H(x) = a(x)\|x\|^2 + \langle \nabla H(0), x \rangle + H(0),$$

where the function $a : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ satisfies

$$|\langle \nabla a(x), x \rangle| \leq \beta \frac{\|x\| + 1}{\|x\|^2} \quad \forall x \in \mathbb{R}^{2n},$$

for some constant $\beta > 0$. In particular the following function is well defined

$$\begin{aligned} b : S^{2n-1} &\rightarrow \mathbb{R}, \\ b(x) &:= \lim_{t \rightarrow +\infty} \frac{H(tx)}{t^2} = \lim_{t \rightarrow +\infty} a(tx). \end{aligned}$$

Proof. One can always express H in the form

$$H(x) = a(x)\|x\|^2 + \langle \nabla H(0), x \rangle + H(0).$$

In this representation one can calculate

$$\begin{aligned} \nabla H(x) &= \|x\|^2 \nabla a(x) + 2a(x)x + \nabla H(0), \\ \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) &= \frac{1}{2} \|x\|^2 \langle \nabla a(x), x \rangle - \frac{1}{2} \langle \nabla H(0), x \rangle - H(0). \end{aligned} \quad (4.2)$$

Proof 1. \Rightarrow 2.

From equality (4.2) and the assumption 1. for all $x \in \mathbb{R}^{2n}$ we get

$$\begin{aligned} |\langle \nabla a(x), x \rangle| &\leq \frac{2}{\|x\|^2} \left(\left| \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) \right| + \left| \frac{1}{2} \langle \nabla H(0), x \rangle \right| + |H(0)| \right) \\ &\leq \frac{2}{\|x\|^2} \left(\alpha(\|x\| + 1) + \frac{1}{2} \|\nabla H(0)\| \|x\| + |H(0)| \right) \\ &\leq \beta \frac{\|x\| + 1}{\|x\|^2}, \end{aligned}$$

for some $\beta > 0$ big enough.

Proof 2. \Rightarrow 1.

From equality (4.2) and the assumption 2. for all $x \in \mathbb{R}^{2n}$ we get

$$\begin{aligned} \left| \frac{1}{2} \langle \nabla H(x), x \rangle - H(x) \right| &\leq \frac{1}{2} \|x\|^2 |\langle \nabla a(x), x \rangle| + \frac{1}{2} |\langle \nabla H(0), x \rangle| + |H(0)| \\ &\leq \frac{1}{2} \beta (\|x\| + 1) + \frac{1}{2} \|\nabla H(0)\| \|x\| + |H(0)| \\ &\leq \alpha(\|x\| + 1), \end{aligned}$$

for some $\alpha > 0$ big enough.

Since a satisfies

$$|\langle \nabla a(x), x \rangle| \leq \beta \frac{\|x\| + 1}{\|x\|^2} \quad \forall x \in \mathbb{R}^{2n}$$

therefore for every $x \in S^{2n-1}$ we have

$$\left| \frac{d}{dt} a(tx) \right| = |\langle \nabla a(tx), x \rangle| \leq \beta \frac{|t| + 1}{|t|^3}$$

Therefore

$$\lim_{t \rightarrow +\infty} \left| \frac{d}{dt} a(tx) \right| = 0$$

and

$$b(x) := \lim_{t \rightarrow +\infty} a(tx)$$

is well defined. □

4.2 Quadratic Hamiltonians

4.2.1 Symplectic classification of quadratic forms

In this section we will analyze the three first conditions defining the class of tentacular Hamiltonians. Note that those three conditions are also the ones one needs to establish bounds on moduli spaces in Theorem 7.

Suppose our Hamiltonian H is defined by a non-degenerate quadratic form Q . Then the condition (H2) is automatically satisfied. To assure that this Hamiltonian satisfies condition (H1) and (H3) we will analyze Hamiltonians defined by non-degenerate quadratic form Q and use the symplectic classification of quadratic forms presented in the paper of Lars Hörmander [24], who classifies them up to a symplectic linear change of coordinates.

Theorem 9. (Hörmander 1995)

Let S be a real symplectic vector space with symplectic form ω and let Q be a real, non-degenerate quadratic form in S . Let $Q : S \rightarrow S$ be a linear map defined by

$$\omega(\cdot, Q \cdot) = Q(\cdot, \cdot).$$

Then S is a direct sum of subspaces orthogonal with respect to Q and ω of one of the following types uniquely determined by the eigenvalues of Q :

(a) $S = T^*\mathbb{R}^m$ and with $\lambda > 0$

$$Q(q, p) = 2\lambda \sum_{j=1}^m q_j p_j + 2 \sum_{j=1}^{m-1} q_{j+1} p_j.$$

Then the Jordan decomposition of Q has one $m \times m$ box for each of the eigenvalues λ and $-\lambda$. The signature of Q is m, m .

(b) $S = T^*\mathbb{R}^{2n}$ and with $\lambda_1, \lambda_2 > 0$

$$Q(q, p) = 2 \left(\sum_{j=1}^{2m-2} q_j p_{j+2} + \lambda_1 \sum_{j=1}^{2m} q_j p_j + \lambda_2 \sum_{j=1}^m (q_{2j} p_{2j-1} - q_{2j-1} p_{2j}) \right).$$

The Jordan decomposition of Q has one $m \times m$ box for each of the eigenvalues $\pm \lambda_1 \pm i\lambda_2$. The signature of Q is $2m, 2m$.

(c) $S = T^*\mathbb{R}^m$ and with $\mu > 0, \gamma = \pm 1$,

$$Q(q, p) = \gamma \left(\mu \sum_{j=1}^m q_j q_{m+1-j} - \sum_{j=2}^m q_j q_{m+2-j} + \mu \sum_{j=1}^m p_j p_{m+1-j} - \sum_{j=1}^{m-1} p_j p_{m-j} \right).$$

The Jordan decomposition of Q has one $m \times m$ box for each of the eigenvalues $\pm i\mu$. The signature of Q is m, m if m is even and $m + \gamma, m - \gamma$ when m is odd.

This theorem assures us that if we have a quadratic Hamiltonian H then there exists a linear symplectic change of coordinates φ , such that φ^*H can be represented in the form

$$\varphi^*H(x) := \frac{1}{2}Q(x, x) - c,$$

for some non-degenerate quadratic form Q and a constant $c \in \mathbb{R}$. Moreover, we can divide \mathbb{R}^{2n} into symplectic subspaces $S_i = T^*\mathbb{R}^{m_i}$, such that

$$(\mathbb{R}^{2n}, \omega_0, J_0) = \bigoplus_i (S_i, \omega_0, J_0),$$

$$Q(x, x) = \sum_i Q^{S^i}(x_i, x_i), \quad \text{where } x = \sum_i x_i, \quad x_i \in S^i.$$

and Q^{S^i} is of one of the types (a), (b) or (c) given by the classification of Hörmander.

4.2.2 Coercive Liouville vector fields

Having the symplectic classification of quadratic forms, we will now investigate which of them define Hamiltonians satisfying properties (H1) and (H3).

Definition 4.4. Consider the symplectic manifold $(\omega_0, \mathbb{R}^{2n})$. We will say that a Hamiltonian H admits a **coercive** Liouville vector field if there exists a Liouville vector field X^\dagger such that

$$dH_x(X^\dagger) \geq c_2 \|x\|^2, \quad \& \quad \|X^\dagger\| \leq c_1 (\|x\| + 1).$$

Similarly, we will say that a non-degenerate quadratic form Q admits a coercive Liouville vector field if a Hamiltonian of the form

$$H(x) := \frac{1}{2}Q(x, x)$$

admits a coercive Liouville vector field.

Note that if a quadratic form Q admits a admits a coercive Liouville vector field, then the quadratic Hamiltonian defined by

$$H(x) := \frac{1}{2}Q(x, x) - c$$

and a constant $c \in \mathbb{R}$ satisfies condition (H1) and condition (H3) for $c \neq 0$.

Proposition 4.5. *Consider the symplectic manifold $(\omega_0, \mathbb{R}^{2n})$. Then Y is a Liouville vector field if and only if it is of the form*

$$Y(x) := \frac{1}{2}x\partial_x + X^f,$$

for some smooth function f .

Proof. Note that the radial vector field $\frac{1}{2}x\partial_x$ is a Liouville vector field. Therefore

$$d(i_{(\frac{1}{2}x\partial_x - Y)}\omega_0) = 0, \quad \Rightarrow \quad i_{(\frac{1}{2}x\partial_x - Y)}\omega_0 = df = -i_{X^f}\omega_0, \quad \Rightarrow \quad Y = \frac{1}{2}x\partial_x + X^f.$$

□

Using the classification of Hörmander we will try to classify the quadratic forms depending on whether they admit a coercive Liouville vector field.

Proposition 4.6. *Consider the symplectic space $(\omega_0, \mathbb{R}^{2n})$ together with a real, non-degenerate quadratic form Q . Let S be one of the symplectic subspaces orthogonal with respect to Q and ω_0 given by the classification of Hörmander.*

- (a) *If $S = T^*\mathbb{R}^m$ and the restriction of Q to S (denoted by Q^S) is of class (a), then Q^S admits a coercive Liouville vector field X_S^\dagger on $(\omega_0, T^*\mathbb{R}^m)$ provided $m = 1$ or $m > 1$ and $\lambda > 1$.*
- (b) *If $S = T^*\mathbb{R}^{2m}$ and Q^S is of class (b), then Q^S admits a coercive Liouville vector field X_S^\dagger on $(\omega_0, T^*\mathbb{R}^{2m})$ whenever $m = 1$ or $m > 1$ and $\lambda_1 > 1$.*
- (c) *If $S = T^*\mathbb{R}^m$ and Q^S is of class (c), then Q^S admits a coercive Liouville vector field X_S^\dagger on $(\omega_0, T^*\mathbb{R}^{2m})$ provided $m = 1$ and $\gamma = 1$.*

In other cases the existence of a coercive Liouville vector is not known.

Proof. Our proof of existence will be in fact a constructive proof. Therefore, we will first define a class of vector fields in $(T^*\mathbb{R}^m, \omega_0)$ and proof that they are Liouville.

For any $a \in \mathbb{R}$ we can define a vector field X^a as below and show that it's Liouville

$$\begin{aligned} X^a(q, p) &= \sum_{i=1}^m \left(\left(\frac{1}{2}p_i + aq_i \right) \partial_{p_i} + \left(\frac{1}{2}q_i + ap_i \right) \partial_{q_i} \right), \\ d(i_{X^a}\omega_0) &= \sum_{i=1}^m d\left(\left(\frac{1}{2}p_i + aq_i \right) dq_i - \left(\frac{1}{2}q_i + ap_i \right) dp_i \right) = \omega_0, \\ \|X^a(q, p)\| &\leq \left(\frac{1}{2} + |a| \right) \|(q, p)\|. \end{aligned}$$

Case (a): Let $m = 1$ and Q^S of class (a) with coefficient $\lambda > 0$ and the associated Hamiltonian would be

$$\begin{aligned} H(q, p) &= \frac{1}{2}Q^S((q, p), (q, p)) = \lambda qp, \\ dH(q, p) &= \lambda(pdq + qdp). \end{aligned}$$

For $a \in \mathbb{R}$ let us take a vector field X^a and calculate $dH(X^a)$

$$\begin{aligned} dH(X^a) &= \lambda p\left(\frac{1}{2}q + ap\right) + \lambda q\left(\frac{1}{2}p + aq\right) \\ &= \lambda\left(a - \frac{1}{2}\right)(q^2 + p^2) + \frac{1}{2}(q + p)^2 \\ &\geq \lambda\left(a - \frac{1}{2}\right)(q^2 + p^2). \end{aligned}$$

Therefore X^a is a coercive Liouville vector field for the quadratic form Q^S of type (a), whenever $m = 1$ and $a > \frac{1}{2}$.

Let $m > 1$ and Q^S be of class (a) with coefficient $\lambda > 0$. The associated Hamiltonian would be

$$\begin{aligned} H(q, p) &= \frac{1}{2}Q^S((q, p), (q, p)) \\ &= \lambda \sum_{j=1}^n q_j p_j + \sum_{j=1}^{n-1} q_{j+1} p_j, \\ dH(q, p) &= \lambda \sum_{j=1}^n (p_j dq_j + q_j dp_j) + \sum_{j=1}^{n-1} (p_j dq_{j+1} + q_{j+1} dp_j). \end{aligned}$$

For $a \in \mathbb{R}$ let us take a vector field X^a and calculate $dH(X^a)$

$$\begin{aligned} dH(X^a)(q, p) &= \lambda \sum_{j=1}^n \left(p_j \left(\frac{1}{2}q_j + ap_j\right) + q_j \left(\frac{1}{2}p_j + aq_j\right)\right) \\ &\quad + \sum_{j=1}^{n-1} \left(p_j \left(\frac{1}{2}q_{j+1} + ap_{j+1}\right) + q_{j+1} \left(\frac{1}{2}p_j + aq_j\right)\right) \\ &= \lambda \sum_{j=1}^n (aq_j^2 + q_j p_j + ap_j^2) + \sum_{j=1}^{n-1} (p_j q_{j+1} + ap_j p_{j+1} + aq_j q_{j+1}) \\ &\geq (a(\lambda - 1) - \frac{1}{2}(\lambda + 1)) \sum_{j=1}^n (q_j^2 + p_j^2). \end{aligned}$$

The last inequality is obtained by square completion.

Therefore X^a is a coercive Liouville vector field for the quadratic form Q^S of type (a), provided $\lambda > 1$ and $a > \frac{\lambda+1}{2(\lambda-1)}$.

Case (b): Let $m = 1$ and let Q^S be of class (b) with coefficients $\lambda_1, \lambda_2 > 0$ and the associated Hamiltonian would be

$$\begin{aligned} H(q, p) &= \frac{1}{2}Q^S((q, p), (q, p)) \\ &= \lambda_1(q_1p_1 + q_2p_2) + \lambda_2(q_2p_1 - q_1p_2), \\ dH(q, p) &= \lambda_1 \sum_{i=1}^2 (q_i dp_i + p_i dq_i) + \lambda_2(p_1 dq_2 + q_2 dp_1 - p_2 dq_1 - q_1 dp_2). \end{aligned}$$

For $a \in \mathbb{R}$ let us take a vector field X^a and calculate $dH(X^a)$

$$\begin{aligned} dH(X^a) &= \lambda_1 \sum_{i=1}^2 (q_i(\frac{1}{2}p_i + aq_i) + p_i(\frac{1}{2}q_i + ap_i)) \\ &\quad + \lambda_2(p_1(\frac{1}{2}q_2 + ap_2) + q_2(\frac{1}{2}p_1 + aq_1) - p_2(\frac{1}{2}q_1 + ap_1) - q_1(\frac{1}{2}p_2 + aq_2)) \\ &= \lambda_1 \sum_{i=1}^2 (aq_i^2 + q_i p_i + ap_i^2) + \lambda_2(p_1 q_2 - q_1 p_2) \\ &= \left(a\lambda_1 - \frac{\lambda_1 + \lambda_2}{2}\right) \sum_{i=1}^2 (q_i^2 + p_i^2) + \frac{\lambda_1}{2} \sum_{i=1}^2 (q_i + p_i)^2 \\ &\quad + \frac{\lambda_2}{2} \left((p_1 + q_2)^2 + (p_2 - q_1)^2\right) \\ &\geq \left(a\lambda_1 - \frac{\lambda_1 + \lambda_2}{2}\right) \sum_{i=1}^2 (q_i^2 + p_i^2). \end{aligned}$$

Therefore X^a is a coercive Liouville vector field for the quadratic form Q^S of type (b), whenever $m = 1$ and $a > \frac{1}{2}(1 + \frac{\lambda_2}{\lambda_1})$.

Let now $m > 1$ and Q^S be of class (b) with coefficients $\lambda_1, \lambda_2 > 0$ and the associated Hamiltonian would be

$$\begin{aligned} H(q, p) &= \frac{1}{2}Q^S((q, p), (q, p)) \\ &= \sum_{j=1}^{2m-2} q_j p_{j+2} + \lambda_1 \sum_{j=1}^{2m} q_j p_j + \lambda_2 \sum_{j=1}^m (q_{2j} p_{2j-1} - q_{2j-1} p_{2j}), \\ dH(q, p) &= \sum_{j=1}^{2n-2} (p_{j+2} dq_j + q_j dp_{j+2}) + \lambda_1 \sum_{j=1}^{2n} (q_j dp_j + p_j dq_j) \\ &\quad + \lambda_2 \sum_{j=1}^n (p_{2j-1} dq_{2j} + q_{2j} dp_{2j-1} - p_{2j} dq_{2j-1} - q_{2j-1} dp_{2j}). \end{aligned}$$

For $a \in \mathbb{R}$ let us take a vector field X^a and calculate $dH(X^a)$

$$\begin{aligned}
 dH(X^a)(q, p) &= \sum_{j=1}^{2n-2} \left(p_{j+2} \left(\frac{1}{2} q_j + a p_j \right) + q_j \left(\frac{1}{2} p_{j+2} + a q_{j+2} \right) \right) \\
 &\quad + \lambda_1 \sum_{j=1}^{2m} \left(q_j \left(\frac{1}{2} p_j + a q_j \right) + p_j \left(\frac{1}{2} q_j + a p_j \right) \right) \\
 &\quad + \lambda_2 \sum_{j=1}^m \left(p_{2j-1} \left(\frac{1}{2} q_{2j} + a p_{2j} \right) + q_{2j} \left(\frac{1}{2} p_{2j-1} + a q_{2j-1} \right) \right) \\
 &\quad - \lambda_2 \sum_{j=1}^m \left(p_{2j} \left(\frac{1}{2} q_{2j-1} + a p_{2j-1} \right) + q_{2j-1} \left(\frac{1}{2} p_{2j} + a q_{2j} \right) \right) \\
 &= \lambda_1 \sum_{j=1}^{2m} (a q_j^2 + q_j p_j + a p_j^2) + \lambda_2 \sum_{j=1}^n (p_{2j-1} q_{2j} + q_{2j-1} p_{2j}) \\
 &\quad + \sum_{j=1}^{2m-2} (p_{j+2} q_j + a(p_{j+2} p_j + q_j q_{j+2})) \\
 &\geq \left(a(\lambda_1 - 1) - \frac{1}{2}(\lambda_1 + \lambda_2 + 1) \right) \sum_{j=1}^{2m} (q_j^2 + p_j^2).
 \end{aligned}$$

The last inequality is obtained by square completion.

Therefore X^a is a coercive Liouville vector field for the quadratic form Q^S of type (b), whenever $\lambda_1 > 1$ and $a > \frac{1+\lambda_1+\lambda_2}{2(\lambda_1-1)}$.

Case (c):

Let $m = 1$ and let Q^S be of type (c). Let us first analyze the simple case $\gamma = 1$. Then for some $\mu > 0$ the associated Hamiltonian is of the form

$$\begin{aligned}
 H(q, p) &= \frac{1}{2} Q^S((q, p), (q, p)) = \frac{\mu}{2} (q_1^2 + p_1^2), \\
 dH(q, p) &= \mu (q_1 dq_1 + p_1 dp_1).
 \end{aligned}$$

Note than in this situation the vector field X^0 is in fact the coercive Liouville vector field for Q^S , provided $\gamma = 1$. □

Corollary 4.7. *Let H be a quadratic Hamiltonian on $(\omega_0, \mathbb{R}^{2n})$. The Theorem by Hörmander assures us that after a symplectic change of coordinates φ , $\varphi^* H$ can be represented in the form*

$$\varphi^* H(x) := \frac{1}{2} Q(x, x) - c,$$

for some non-degenerate quadratic form Q and a constant $c \in \mathbb{R}$. Moreover, we can

divide \mathbb{R}^{2n} into symplectic subspaces $S_i = T^*\mathbb{R}^{m_i}$, such that

$$(\mathbb{R}^{2n}, \omega_0, J_0) = \bigoplus_i (S_i, \omega_0, J_0),$$

$$Q(x, x) = \sum_i Q^{S_i}(x_i, x_i), \quad \text{where } x = \sum_i x_i, \quad x_i \in S_i.$$

Suppose for every S_i and corresponding Q^{S_i} there exists a coercive Liouville vector field $X_{S_i}^\dagger$. If we define

$$X^\dagger(x) = \sum_i X_{S_i}^\dagger(x_i), \quad \text{where } x = \sum_i x_i, \quad x_i \in S_i,$$

then $T\varphi(X^\dagger)$ is a coercive Liouville vector field for H .

Proof. Because \mathbb{R}^{2n} is a direct sum of symplectic subspaces S_i , for $x \in \mathbb{R}^{2n}$ we have

$$\sum_i \|x_i\|^2 = \sum_i \omega_0(x_i, J_0 x_i) = \omega_0\left(\sum_i x_i, \sum_i J_0 x_i\right) = \omega_0(x, J_0 x) = \|x\|^2.$$

Let us now check that such defined X^\dagger is indeed a Liouville vector field on \mathbb{R}^{2n} . Using the fact that \mathbb{R}^{2n} is a direct sum of symplectic subspaces S_i we obtain

$$d(i_{X^\dagger} \omega_0) = \sum_i d(i_{X_{S_i}^\dagger} \omega_0) = \sum_i \omega_0|_{S_i} = \omega_0.$$

Moreover, X^\dagger is coercive for Q since

$$\|X^\dagger(x)\| \leq \sum_i \|X_{S_i}^\dagger(x_i)\| \leq \sum_i c_1^i (\|x_i\| + 1) \leq c_1 (\|x\| + 1),$$

$$\text{where } \hat{c}_1 := (\max_i c_1^i) \left(\max_{\|x\|=1} \sum_i \|x_i\| \right).$$

$$\begin{aligned} Q(x, X^\dagger) &= Q\left(\sum_i x_i, \sum_i X_{S_i}^\dagger(x_i)\right) \\ &= \sum_i Q(x_i, X_{S_i}^\dagger(x_i)) \\ &= \sum_i Q^{S_i}(x_i, X_{S_i}^\dagger(x_i)) \\ &\geq \sum_i \hat{c}_2^i \|x_i\|^2 \\ &\geq \min_i c_2^i \sum_i \|x_i\|^2 \\ &= \min_i c_2^i \|x\|^2. \end{aligned}$$

Having checked that X^\dagger is coercive for Q let us show that $T\varphi(X^\dagger)$ is coercive for H .

Since φ is a symplectic change of coordinates, $T\varphi(X^\dagger)$ is also a Liouville vector field.

$$\begin{aligned}
 d_{T\varphi(X^\dagger)}\omega_0 &= d((\varphi^{-1})^*(i_{X^\dagger}\varphi^*\omega_0)) \\
 &= (\varphi^{-1})^*d(i_{X^\dagger}\varphi^*\omega_0) \\
 &= (\varphi^{-1})^*d(i_{X^\dagger}\omega_0) \\
 &= (\varphi^{-1})^*\omega_0 \\
 &= \omega_0
 \end{aligned}$$

Moreover, since X^\dagger is coercive for Q , then $T\varphi(X^\dagger)$ is coercive for H

$$\begin{aligned}
 dH_x(T\varphi(X^\dagger)) &= \varphi^*(dH)_{\varphi(x)}(X^\dagger) \\
 &= d(\varphi^*H)_{\varphi(x)}(X^\dagger) \\
 &= Q(\varphi(x), X^\dagger) \\
 &\geq c_2\|\varphi(x)\|^2 \\
 &\geq c_2\|\varphi^{-1}\|^2\|x\|^2, \\
 \|T\varphi(X^\dagger)\| &\leq \|\varphi\| \|X^\dagger(\varphi(x))\| \\
 &\leq \hat{c}_1\|\varphi\|(\|\varphi(x)\| + 1) \\
 &\leq c_1(\|x\| + 1).
 \end{aligned}$$

□

The Hörmander classification allows us to decompose a quadratic Hamiltonian into symplectic subspaces and find a coercive Liouville vector field for each subspace. Unfortunately, even with such a decomposition, the task of finding a coercive Liouville vector field may not be easy. Below we present another approach to finding coercive Liouville vector fields for quadratic Hamiltonians.

According to De la Cruz [16] an $2n \times 2n$, A is symplectically diagonalizable (i.e. conjugate by symplectic matrices to a diagonal matrix) if and only if it satisfies

$$AJ_0A^TJ_0 = J_0A^TJ_0A.$$

Corollary 4.8. *Suppose A is a symmetric matrix of rank $2n$ defining a non-degenerate quadratic form Q . If A is symplectically diagonalizable, then the Hörmander's decomposition of Q will give us n , 1-dimensional symplectic subspaces $S_i = T^*\mathbb{R}$ such that Q^{S_i} will be of type (a) or (c). In case all the Q^{S_i} of type (c) have $\gamma = 1$, then Q admits a coercive Liouville vector field.*

4.2.3 Boundedness of periodic orbits

In this subsection we will analyze Property (H4) in the definition of tentacular Hamiltonians, which, as we recall, guaranties that the periodic orbits of the Hamiltonian vector field are all contained in a compact set. In particular, in Proposition 4.9 we will find a sufficient condition for quadratic Hamiltonians to satisfy Property (H4) and then later we will use the Hörmander classification of quadratic forms to verify it for a class of quadratic Hamiltonians. This assures that the class of tentacular Hamiltonians contains a variety of interesting examples.

Proposition 4.9. *Consider the symplectic space $(\omega_0, \mathbb{R}^{2n})$ together with a real, non-degenerate matrix A . For $c \neq 0$ define an associated Hamiltonian*

$$H(x) := \frac{1}{2} \langle x, Ax \rangle - c.$$

Whenever there exist $a, b \in \mathbb{R}$, such that the matrix

$$M := A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T) + b(JA + (JA)^T) + aA$$

is positive definite, then H admits a coercive function $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, such that the set

$$\{x \in \mathbb{R}^{2n} \mid d_x(dF(X^H))(X^H(x)) \leq 0\} \cap \{x \in \mathbb{R}^{2n} \mid d_x F(X^H) = 0\} \cap H^{-1}(0)$$

is compact.

Proof. We claim the assertion holds true for

$$F(x) := \frac{1}{2} \|x\|^2.$$

Naturally, F is coercive.

Let us calculate

$$\begin{aligned} X^H(x) &= J \nabla \left(\frac{1}{2} \langle x, Ax \rangle \right) \\ &= JA x \\ dF_x(X^H) &= \langle x, JA x \rangle \\ &= \langle x, \frac{1}{2} (JA + (JA)^T) x \rangle \\ d(dF_x(X^H))(X^H) &= \langle (JA + (JA)^T) x, JA x \rangle \\ &= \langle JA + (JA)^T x, JA x \rangle \\ &= \langle x, (A^2 + (JA)^2) x \rangle \\ &= \langle x, (A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T) x \rangle \end{aligned}$$

If we now combine it together, we obtain

$$\begin{aligned}
 dF_x(dF(X^H))(X^H) + 2b dF_x(X^H) + 2aH(x) &= \langle x, (A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T))x \rangle \\
 &+ 2\beta \langle x, \frac{1}{2}(JA + (JA)^T)x \rangle + 2\alpha(\frac{1}{2}\langle x, Ax \rangle - c) \\
 &= \langle x, (A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T) + \beta(JA + (JA)^T) + \alpha A)x \rangle - 2\alpha c \\
 &= \langle x, Mx \rangle - 2\alpha c
 \end{aligned}$$

By assumption M is positive definite therefore there exists $K > 0$, such that for all $x \in \mathbb{R}^{2n}$ whenever $\|x\| > K$ then

$$dF_x(dF(X^H))(X^H) + 2b dF_x(X^H) + 2aH(x) > 0.$$

In particular, that means that the considered set

$$\begin{aligned}
 &\{x \in \mathbb{R}^{2n} \mid d_x(dF(X^H))(X^H(x)) \leq 0\} \cap \{x \in \mathbb{R}^{2n} \mid d_x F(X^H) = 0\} \cap H^{-1}(0) \\
 &\subseteq \{x \in \mathbb{R}^{2n} \mid dF_x(X^H) + 2bdF_x(X^H) + 2aH(x) \leq 0\} \\
 &\subseteq \{x \in \mathbb{R}^{2n} \mid \|x\| \leq K\}
 \end{aligned}$$

is contained in a compact set and thus, being a closed set, it is compact itself. \square

Remark 4.10. *Note that the Proposition 4.9. holds true if we add to the Hamiltonian a linear or compactly supported perturbation.*

In the poof of the Proposition below we will show how to use Hörmander classification of quadratic forms, to determine whether a quadratic Hamiltonian satisfies the assertions of Proposition 4.9. In particular the list presented in Proposition is far from complete - it is a rather rough estimate which is to argue that there exist many interesting examples which fall into our class.

Corollary 4.11. *Consider the symplectic space $(T^*\mathbb{R}^n, \omega_0)$ together with a real, non-degenerate quadratic form Q . Let*

$$H(x) := \frac{1}{2}Q(x, x) - c$$

be the Hamiltonian on $T^\mathbb{R}^n$ associated to Q and let*

$$F(x) := \frac{1}{2}\|x\|^2$$

be a coercive function on $T^\mathbb{R}^n$. Then depending on the class of Q with respect to the Hörmander classification the following holds true*

(a) *If $n = m$ and Q is of class (a), then*

$$d_x(dF(X^{H^S}))(X^{H^S}) > 0 \quad \forall x \in T^*\mathbb{R}^m, x \neq 0,$$

provided $m = 1$ or $m > 1$ and $\lambda > \frac{1+\sqrt{17}}{2}$.

(b) If $n = 2m$ and Q is of class (b), then

$$d_x(dF(X^{H^S}))(X^{H^S}) > 0 \quad \forall x \in T^*\mathbb{R}^{2m}, x \neq 0,$$

provided $m = 1$ or $m > 1$ and $\lambda_1 > 2$

(c) If $n = m$ and Q is of class (c), then

$$\frac{1}{2}Q(x, x) > 0 \quad \forall x \in T^*\mathbb{R}^m, x \neq 0 \quad \text{and} \quad d_x(dF(X^{H^S}))(X^{H^S}) = 0$$

provided $m = 1$, $\gamma = 1$.

In other cases the existence of such a function is not known.

Proof. In each case we will follow the approach presented in Proposition 4.9. Let A be the nondegenerate matrix associated to Q in the following way

$$Q(x, x) = \langle x, Ax \rangle,$$

then for all $x \in S$

$$d(dF(X^{H^S}))(X^{H^S}) = \langle x, (A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T)x \rangle.$$

Let us explore whether $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ is positive definite depending $m \in \mathbb{N}$ and on the Hörmander class.

(a) Suppose Q is of class (a), then for

$$m = 1$$

$$A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T = \begin{pmatrix} 2\lambda^2 & 0 \\ 0 & 2\lambda^2 \end{pmatrix}$$

thus $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ is positive definite for all $\lambda > 0$.

$m > 1$ verification whether $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ is positive definite becomes more difficult. One of the methods is to require it to be diagonally dominant. If we calculate $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ explicitly we will see that for $\lambda > \frac{1+\sqrt{17}}{2}$ it is diagonally dominant.

(b) Suppose Q is of class (b), then for

$$m = 1$$

$$A = \begin{pmatrix} 0 & \lambda_1 & 0 & \lambda_2 \\ \lambda_1 & 0 & -\lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_1 \\ \lambda_2 & 0 & \lambda_1 & 0 \end{pmatrix}$$

and $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ is of the form

$$\begin{pmatrix} 2\lambda_1^2 & 0 & 0 & 0 \\ 0 & 2\lambda_1^2 & 0 & 0 \\ 0 & 0 & 2\lambda_1^2 & 0 \\ 0 & 0 & 0 & 2\lambda_1^2 \end{pmatrix}$$

therefore it is positive definite for all $\lambda_1, \lambda_2 > 0$.

$m > 1$ if we calculate $A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T$ explicitly we will see that for $\lambda > 2$ it is diagonally dominant.

(c) Suppose Q is of class (c) with $\gamma = 1$, then for $m = 1$

$$A = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad A^2 + \frac{1}{2}((JA)^2 + ((JA)^2)^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

thus A itself is positive definite. □

Now we would like to be able to tell whether a quadratic Hamiltonian satisfies Property (H4) once we know it's Hörmander decomposition satisfies assertions of Corollary 4.11. This would allow us to create a variety of quadratic tentacular Hamiltonians, by combining quadratic forms from different Hörmander classes.

Corollary 4.12. *Let H be a quadratic Hamiltonian on $(\mathbb{R}^{2n}, \omega_0)$. The Theorem by Hörmander assures us that after a symplectic change of coordinates φ , φ^*H can be represented in the form*

$$\varphi^*H(x) := \frac{1}{2}Q(x, x) - c,$$

for some non-degenerate quadratic form Q and a constant $c \in \mathbb{R}$. Moreover, we can divide \mathbb{R}^{2n} into symplectic subspaces $S_i = T^*\mathbb{R}^{m_i}$, such that

$$(\mathbb{R}^{2n}, \omega_0, J_0) = \bigoplus_i (S_i, \omega_0, J_0),$$

$$Q(x, x) = \sum_i Q^{S_i}(x_i, x_i), \quad \text{where } x = \sum_i x_i, \quad x_i \in S_i.$$

Let H^i and F^i be the restriction of φ^*H and

$$F(x) := \frac{1}{2}\|x\|^2 \quad x \in \mathbb{R}^{2n}.$$

to S^i , respectively.

Suppose for every i either $d_x(dF^i(X^{H^i}))(X^{H^i})$ is positive definite or Q^{S_i} is positive definite and $d_x(dF^i(X^{H^i}))(X^{H^i})$ is zero.

Then the function φ_*F is coercive on \mathbb{R}^{2n} and the set

$$\{x \in \mathbb{R}^{2n} \mid d_x(d(\varphi_*F)(X^H))(X^H(x)) \leq 0\} \cap H^{-1}(0)$$

is compact.

Proof. By assumption the set of indexes consists of two subsets I and J such that for every $i \in I$ there exist $\alpha, \beta > 0$, such that

$$d_{x_i}(dF^i(X^{H^i}))(X^{H^i}(x_i)) \geq \alpha \|x_i\|^2 \quad \text{and} \quad |Q^{S_i}(x_i, x_i)| \leq \beta \|x_i\|^2$$

and for all $j \in J$

$$d_{x_j}(dF^j(X^{H^j}))(X^{H^j}(x_j)) = 0$$

and there exists $\gamma > 0$, such that

$$Q^{S_i}(x_i, x_i) \geq \gamma \|x_i\|^2.$$

Now if we take $a > 0$ we can estimate

$$\begin{aligned} d_x(dF(X^{\varphi^*H}))(X^{\varphi^*H}(x)) + a\varphi^*H(x) &= \\ &= \sum_{i \in I \cup J} (d_{x_i}(dF^i(X^{H^i}))(X^{H^i}(x_i)) + \frac{a}{2}Q^{S_i}(x_i, x_i)) - ac \\ &\geq (\alpha - \frac{a\beta}{2}) \sum_{i \in I} \|x_i\|^2 + \frac{a\gamma}{2} \sum_{j \in J} \|x_j\|^2 - ac \\ &\geq \min \left\{ \alpha - \frac{a\beta}{2}, \frac{a\gamma}{2} \right\} \|x\|^2 - ac. \end{aligned}$$

Therefore, whenever we choose $a \in (0, \frac{2\alpha}{\beta})$, then

$$d_x(dF(X^{\varphi^*H}))(X^{\varphi^*H}(x)) + a\varphi^*H(x) > 0$$

for all $x \in \mathbb{R}^{2n}$, such that

$$\|x\| > \frac{2c}{\min\{\frac{2\alpha}{a} - \beta, \gamma\}}.$$

Hence, the set

$$\{x \in \mathbb{R}^{2n} \mid d_x(dF(X^{\varphi^*H}))(X^{\varphi^*H}(x)) \leq 0\} \cap (\varphi^*H)^{-1}(0)$$

is compact and because φ is a linear transformation, therefore

$$\begin{aligned} \varphi^{-1} \left(\{x \in \mathbb{R}^{2n} \mid d_x(dF(X^{\varphi^*H}))(X^{\varphi^*H}(x)) \leq 0\} \cap (\varphi^*H)^{-1}(0) \right) &= \\ = \{x \in \mathbb{R}^{2n} \mid d_x(d(\varphi_*F)(X^H))(X^H(x)) \leq 0\} \cap H^{-1}(0) \end{aligned}$$

is compact and the function φ_*F is coercive as well. □

4.3 Geometry of 0-level sets of tentacular Hamiltonians

In this section we present two lemmas, which analyze the geometrical behavior of the 0-level set of a tentacular Hamiltonian, which is used in the proof of the partition of the set of infinitesimal action derivation in Proposition 3.8 and in the definition of the tubular neighborhood in subsection 3.6.1.

Lemma 4.13. *Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ satisfy (H1) and (H3). Then for all $\delta > 0$ there exists a $\mu(\delta, H) > 0$, such that*

$$H^{-1}(-\mu, \mu) \subseteq \left\{ x \in \mathbb{R}^{2n} \mid \text{dist}(x, H^{-1}(0)) < \delta \right\}.$$

Proof. In the first step we show that for $\nu > 0$ as in (H3) H has no critical points in the neighborhood $H^{-1}(-\nu, \nu)$, that is

$$\inf_{H^{-1}(-\nu, \nu)} \|\nabla H\| > 0.$$

By (H1) for every $x \in \mathbb{R}^{2n}$ we have

$$\|\nabla H(x)\| \geq \frac{c_2 \|x\|^2 - c_3}{c_1(\|x\| + 1)}.$$

On the other hand, by (H3) for all $x \in H^{-1}(-\nu, \nu)$ we have

$$\|\nabla H(x)\| \geq \frac{c_5}{c_4(\|x\|^2 + 1)}.$$

If we denote

$$\begin{aligned} f_1(r) &:= \frac{c_2 r^2 - c_3}{c_1(r + 1)}, \\ f_2(r) &:= \frac{c_5}{c_4(r^2 + 1)}, \end{aligned}$$

then we have

$$\inf_{x \in H^{-1}(-\nu, \nu)} \|\nabla H(x)\| \geq \inf_{r \geq 0} \max\{f_1(r), f_2(r)\}.$$

We want to show that the infimum is positive. When we calculate the derivatives of functions f_1 and f_2

$$\begin{aligned} f_1'(r) &= \frac{c_2 r^2 + 2c_2 r + c_3}{c_1(r + 1)^2} > 0 \quad \forall r > 0, \\ f_2'(r) &= \frac{-2c_5 r}{c_4(r^2 + 1)} < 0 \quad \forall r > 0, \end{aligned}$$

we can see that f_1 is monotonically increasing whereas f_2 is monotonically decreasing. That means that the function $\max\{f_1(r), f_2(r)\}$ obtains its minimum on $[0, +\infty)$ either at $r = 0$ or when $f_1(r) = f_2(r)$. In either case there exists $r_0 \geq 0$, such that

$$f_1(r_0) = \inf_{r \geq 0} \max\{f_1(r), f_2(r)\}$$

and this minimum is positive, since the function f_1 is everywhere positive.

4.3. GEOMETRY OF 0-LEVEL SETS OF TENTACULAR HAMILTONIANS

Denote $\Sigma = H^{-1}(0)$. On $H^{-1}(-\nu, \nu)$ we can define a flow ψ in the following way

$$\frac{d}{dt}\psi(t, x) = \frac{\nabla H(\psi(t, x))}{\|\nabla H(\psi(t, x))\|^2}.$$

Observe that

$$H(\psi(t, x)) = H(x) + t \quad \text{since} \quad \frac{d}{dt}H(\psi(t, x)) = 1.$$

That means that

$$\psi : (-\nu, \nu) \times \Sigma \rightarrow H^{-1}(-\nu, \nu)$$

is a well defined bijection. Note that for every $(t, x) \in (-\nu, \nu) \times \Sigma$, we have

$$\begin{aligned} \text{dist}(\psi(t, x), \Sigma) &\leq \|\psi(t, x) - \psi(0, x)\| \\ &\leq \int_0^t \left\| \frac{d}{d\tau}\psi(\tau, x) \right\| d\tau \\ &\leq \int_0^t \frac{1}{\|\nabla H(\psi(\tau, x))\|} d\tau \\ &\leq \frac{t}{\inf_{r \geq 0} \max\{f_1(r), f_2(r)\}} \\ &= \frac{H(\psi(t, x))}{\inf_{r \geq 0} \max\{f_1(r), f_2(r)\}} \end{aligned}$$

Therefore, for all $0 < \delta$ if denote

$$\mu(\delta, H) := \min\{\nu, \delta \inf_{r \geq 0} \max\{f_1(r), f_2(r)\}\},$$

then for all $x \in H^{-1}(-\mu, \mu)$ we have

$$\begin{aligned} \text{dist}(x, \Sigma) &\leq \frac{H(x)}{\inf_{r \geq 0} \max\{f_1(r), f_2(r)\}} \\ &\leq \frac{\mu}{\inf_{r \geq 0} \max\{f_1(r), f_2(r)\}} \\ &\leq \delta. \end{aligned}$$

□

The following lemma ensures that for every tentacular Hamiltonian there exists a tubular neighborhood with a constant radius, which is used in Section 3.6 to prove the boundedness of Floer trajectories.

Lemma 4.14. *Let $H : \mathbb{R}^m \rightarrow \mathbb{R}$ be a smooth function. Denote $\Sigma = H^{-1}(0)$ and let $N(\Sigma)$ be its normal bundle. For $\delta > 0$ denote*

$$N_\delta(\Sigma) := \{(x, v) \in N(\Sigma) \mid \|v\| < \delta\}.$$

If

$$\inf_{\Sigma} \|\nabla H\| > 0 \quad \text{and} \quad \sup_{\mathbb{R}^n} \|D^2 H\| = M < +\infty,$$

then there exists $\tilde{\delta}(H) > 0$, such that

$$\exp : N_{\tilde{\delta}(H)}(\Sigma) \rightarrow \{x \in \mathbb{R}^{2n} \mid \text{dist}(x, \Sigma) < \tilde{\delta}(H)\}$$

is a diffeomorphism.

Observe that whenever a Hamiltonian $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ satisfies Properties (H1), (H2) and (H3) then it satisfies the assumptions of the lemma above, since in Lemma 4.13 it was shown that from Properties (H1) and (H3) it follows that:

$$\inf_{\Sigma} \|\nabla H\| > 0.$$

So in view of Lemma 4.14 the 0 level set of of a tentacular Hamiltonian has a well defined tubular neighborhood.

Proof. The proof relies on Theorem 2.33 from [17], which states that $\delta(H) > 0$ exists provided Σ is a uniformly embedded manifold as defined in Definition 2.22 in [17].

A hypersurface $\Sigma = H^{-1}(0)$ is a uniformly embedded submanifold of \mathbb{R}^{2n} if there exists $\delta_0 > 0$, such that for every $x \in \Sigma$

1. the intersection of Σ with a ball of radius δ_0 , $B(x, \delta_0) \cap \Sigma$ has only one connected component.
2. there exists a function $\varphi : \text{Ker}(d_x H) \rightarrow \mathbb{R}$, such that Σ is locally a graph of φ over the tangent space $T_x \Sigma$, i.e.

$$B(x, \delta_0) \cap \Sigma = \left\{ y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x \mid y \in \text{Ker}(d_x H), \quad \|y\| < \delta_0 \right\}$$

and the functions φ corresponding to different $x \in \Sigma$ have their first and second derivatives uniformly bounded.

We will first prove 1. and then 2.

Part 1:

We will show that for every $x \in \Sigma$ the intersection of Σ with a ball of radius $\frac{1}{M} \inf_{\Sigma} \|\nabla H\|$:

$$B\left(x, \frac{1}{M} \inf_{\Sigma} \|\nabla H\|\right) \cap \Sigma$$

has only one connected component. Suppose the opposite is true i.e. there exists a $x \in \Sigma$ and connected component $\tilde{\Sigma}$ of $B\left(x, \frac{1}{M} \inf_{\Sigma} \|\nabla H\|\right) \cap \Sigma$, such that $x \notin \tilde{\Sigma}$. Then there exists $z \in \tilde{\Sigma}$, such that

$$0 < \text{dist}(\tilde{\Sigma}, x) = \|x - z\| < \frac{1}{M} \inf_{\Sigma} \|\nabla H\|. \quad (4.3)$$

4.3. GEOMETRY OF 0-LEVEL SETS OF TENTACULAR HAMILTONIANS

Since $z \in \tilde{\Sigma}$ minimizes the distance between x and $\tilde{\Sigma}$, the vector $x - z$ is perpendicular to $\tilde{\Sigma}$, i.e.

$$\frac{x - z}{\|x - z\|} = \frac{\nabla H(z)}{\|\nabla H(z)\|}.$$

Define a function

$$f : [0, 1] \rightarrow \mathbb{R}$$

$$f(t) := H\left(z + t \frac{\|x - z\|}{\|\nabla H(z)\|} \nabla H(z)\right).$$

Then its derivatives are

$$f'(t) = dH_{\left(z + t \frac{\|x - z\|}{\|\nabla H(z)\|} \nabla H(z)\right)}(\nabla H(z)) \frac{\|x - z\|}{\|\nabla H(z)\|} \quad (4.4)$$

$$f''(t) = D^2H_{\left(z + t \frac{\|x - z\|}{\|\nabla H(z)\|} \nabla H(z)\right)}(\nabla H(z), \nabla H(z)) \frac{\|x - z\|^2}{\|\nabla H(z)\|^2} \quad (4.5)$$

Observe that $f(0) = f(1) = 0$, hence there exists $t_0 \in (0, 1)$, such that $f'(t_0) = 0$. On one hand, (4.5) gives us

$$\|f'(t_0) - f'(0)\| \leq t_0 \sup_{t \in [0, t_0]} \|f''(t)\| \leq M \|x - z\|^2$$

on the other hand (4.4) gives us

$$\|f'(t_0) - f'(0)\| = \|f'(0)\| = \|\nabla H(z)\| \|x - z\|,$$

and the two inequalities above combined with (4.3) lead to a contradiction:

$$\|\nabla H(z)\| \leq M \|x - z\| < \inf_{\Sigma} \|\nabla H\|.$$

Part 2:

Fix $x \in \Sigma$. By definition Σ is a regular level set of H in \mathbb{R}^m , hence there exists a neighborhood V of 0 in $T_x \Sigma$, a neighborhood U_x of x in \mathbb{R}^m and a smooth function $\varphi : V \rightarrow \mathbb{R}$, such that

$$U_x \cap \Sigma = \left\{ y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x \mid y \in V \right\}.$$

Let us analyze φ . Since H vanishes on Σ we can find the following relations between

H and φ :

$$\begin{aligned}
 0 &= H\left(y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x\right) \\
 0 &= \frac{d}{dy_i} H\left(y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x\right) \\
 &= dH(\partial_{y_i}) + \frac{\partial \varphi}{\partial y_i}(y) dH\left(\frac{\nabla H(x)}{\|\nabla H(x)\|}\right)
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 0 &= \frac{d^2}{dy_i dy_j} H\left(y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x\right) \\
 &= \frac{d}{dy_j} \left(dH(\partial_{y_i}) + \frac{\partial \varphi}{\partial y_i}(y) dH\left(\frac{\nabla H(x)}{\|\nabla H(x)\|}\right) \right) \\
 &= D^2 H(\partial_{y_i}, \partial_{y_j}) + \frac{\partial \varphi}{\partial y_i}(y) D^2 H\left(\partial_{y_j}, \frac{\nabla H(x)}{\|\nabla H(x)\|}\right) + \frac{\partial \varphi}{\partial y_j}(y) D^2 H\left(\partial_{y_i}, \frac{\nabla H(x)}{\|\nabla H(x)\|}\right) \\
 &\quad + \frac{\partial \varphi}{\partial y_i}(y) \frac{\partial \varphi}{\partial y_j}(y) D^2 H\left(\frac{\nabla H(x)}{\|\nabla H(x)\|}, \frac{\nabla H(x)}{\|\nabla H(x)\|}\right) + \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(y) dH\left(\frac{\nabla H(x)}{\|\nabla H(x)\|}\right)
 \end{aligned} \tag{4.7}$$

To estimate $\frac{\partial \varphi}{\partial y_i}(y)$, we will show that for all $z \in B(x, \frac{1}{4M} \inf_{\Sigma})$ the following holds

$$dH_z(\nabla H(x)) > \frac{1}{2} \|\nabla H(x)\| \|\nabla H(z)\|. \tag{4.8}$$

If we calculate the first derivative of the function above, we obtain

$$\begin{aligned}
 \|\nabla(dH_z(\nabla H(x)) - \frac{1}{2} \|\nabla H(x)\| \|\nabla H(z)\|)\| &= \|D_z^2 H(\nabla H(x) - \frac{1}{2} \nabla H(z))\| \\
 &\leq M\left(\frac{3}{2} \|\nabla H(x)\| + \frac{1}{2} \|z - x\|\right).
 \end{aligned}$$

which allows us to estimate

$$dH_z(\nabla H(x)) - \frac{1}{2} \|\nabla H(x)\| \|\nabla H(z)\| \geq \frac{1}{2} \|\nabla H(x)\|^2 - \|z - x\| M\left(\frac{3}{2} \|\nabla H(x)\| + \frac{1}{2} \|z - x\|\right)$$

Therefore, (4.8) is satisfied whenever

$$z \in B\left(x, \frac{\sqrt{13} - 3}{2} M \|\nabla H(x)\|\right) \supseteq B\left(x, \frac{1}{4} M \inf_{\Sigma} \|\nabla H\|\right).$$

Now combining (4.6) and (4.8) we can conclude that for $y + \varphi(y) \frac{\nabla H(x)}{\|\nabla H(x)\|} + x \in B(x, \frac{1}{4} M \inf_{\Sigma} \|\nabla H\|)$ one has the following estimate

$$\left| \frac{\partial \varphi}{\partial y_i}(y) \right| \leq \frac{\|\nabla H(x)\| |dH(\partial_{y_i})|}{|dH(\nabla H(x))|} \leq 2$$

Using the above estimate along with equality (4.5) we obtain

$$\begin{aligned} \left| \frac{\partial^2 \varphi}{\partial y_i \partial y_j}(y) \right| &\leq M \left| dH \left(\frac{\nabla H(x)}{\|\nabla H(x)\|} \right) \right|^{-1} \left(1 + \left| \frac{\partial \varphi}{\partial y_i}(y) \right| + \left| \frac{\partial \varphi}{\partial y_j}(y) \right| + \left| \frac{\partial \varphi}{\partial y_i}(y) \right| \left| \frac{\partial \varphi}{\partial y_j}(y) \right| \right) \\ &\leq 18M (\inf_{\Sigma} \|\nabla H\|)^{-1}. \end{aligned}$$

This concludes the proof that Σ is an uniformly embedded manifold for

$$\delta_0 = \frac{1}{4M} \inf_{\Sigma} \|\nabla H\|,$$

and therefore by Theorem 2.33 from [17] there exists a uniform tubular neighborhood for Σ . \square

A Calculations on maximum principle

Lemma 4.15. *Consider F be the radial function*

$$F(x) := \frac{1}{4}\|x\|^2,$$

and H a Hamiltonian satisfying

$$\sup_{x \in \mathbb{R}^{2n}} \|D^3 H\| \|x\| \leq L.$$

Define f_1 as in (3.42)

$$f_1(x) = d(dF(X^H))(X^H) - \|\nabla H\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))\|^2 - \|\nabla(dF(X^H))\|^2.$$

Then $|f_1|$ is of order 2 in $\|x\|$ and $\|\nabla f_1(x)\|$ is linear in $\|x\|$, meaning

$$|f_1(x)| \leq (2M\|x\| + \frac{2}{3}h_1)^2, \quad (3.43)$$

$$\|\nabla f_1(x)\| \leq h_2\|x\| + h_3, \quad (3.44)$$

where $M := \|\text{Hess}_0 H\| + L$, $h_1 := \|\nabla H(0)\|$,

$$h_2 := M(8M + \frac{5}{2}L), \quad h_3 := \frac{1}{2}h_1(7M + 3L).$$

Proof. Recall the definition of f and f_1 from (3.38) and (3.42)

$$\begin{aligned} f(s, t) &= \eta^2(d(dF(X^H))(X^H) - \|\nabla H\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))\|^2 - \|\nabla(dF(X^H))\|^2) \\ &\quad + \partial_s \eta d^{\mathbb{C}}F(X^H), \\ f_1(x) &= d(dF(X^H))(X^H) - \|\nabla H\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))\|^2 - \|\nabla(dF(X^H))\|^2. \end{aligned}$$

We will find bounds on $|f_1(x)|$ and $\|\nabla f_1(x)\|$ by investigating and bounding each of the expressions separately

$$\begin{aligned}
 dF(X^H) &= \omega_0(X^H, X^F) = \langle J\nabla H, \nabla F \rangle, \\
 d^{\mathbb{C}}F(X^H) &= \omega_0(JX^H, X^F) = -\langle \nabla H, \nabla F \rangle, \\
 |d^{\mathbb{C}}F_x(X^H)| &\leq \|\nabla H\| \|\nabla F\| \leq \frac{1}{2}\|x\|(M\|x\| + h_1), \\
 d(dF(X^H))(\xi) &= \langle JHess_x H(\xi), \nabla F \rangle + \langle J\nabla H, Hess_x F(\xi) \rangle \\
 &= \langle Hess_x F(J\nabla H) - Hess_x H(J\nabla F), \xi \rangle, \\
 \|\nabla(dF(X^H))\| &\leq \|Hess_x F\| \|\nabla H\| + \|Hess_x H\| \|\nabla F\| \\
 &\leq \frac{1}{2}(M\|x\| + h_1) + \frac{1}{2}M\|x\| \\
 &= M\|x\| + \frac{1}{2}h_1,
 \end{aligned} \tag{4.9}$$

$$\begin{aligned}
 d(d^{\mathbb{C}}F(X^H))(\xi) &= -(\langle Hess_x H(\xi), \nabla F \rangle + \langle \nabla H, Hess_x F(\xi) \rangle) \\
 &= -\langle Hess_x F(\nabla H) + Hess_x H(\nabla F), \xi \rangle, \\
 \|\nabla(d^{\mathbb{C}}F(X^H))\| &\leq \|Hess_x F\| \|\nabla H\| + \|Hess_x H\| \|\nabla F\| \\
 &\leq M\|x\| + \frac{1}{2}h_1,
 \end{aligned} \tag{4.10}$$

$$\begin{aligned}
 d(\|\nabla(dF(X^H))\|^2)(\xi) &= 2\langle \nabla(dF(X^H)), Hess_x F(JHess_x H(\xi)) \rangle \\
 &\quad - 2\langle \nabla(dF(X^H)), Hess_x H(JHess_x F(\xi)) + D^3H(J\nabla F, \xi) \rangle, \\
 \|\nabla(\|\nabla(dF(X^H))\|^2)\| &\leq 2\|\nabla(dF(X^H))\|(2\|Hess_x F\| \|Hess_x H\| + \|D^3H\| \|\nabla F\|) \\
 &\leq 2(M\|x\| + \frac{1}{2}h_1)(M + \frac{1}{2}\|D^3H\| \|x\|) \\
 &\leq 2(M\|x\| + \frac{1}{2}h_1)(M + \frac{1}{2}L) \\
 d(\|\nabla(d^{\mathbb{C}}F(X^H))\|^2)(\xi) &= -2\langle \nabla(d^{\mathbb{C}}F(X^H)), Hess_x F(Hess_x H(\xi)) \rangle \\
 &\quad - 2\langle \nabla(d^{\mathbb{C}}F(X^H)), Hess_x H(Hess_x F(\xi)) + D^3H(\nabla F, \xi) \rangle, \\
 \|\nabla(\|\nabla(d^{\mathbb{C}}F(X^H))\|^2)\| &\leq 2\|\nabla(d^{\mathbb{C}}F(X^H))\|(2\|Hess_x F\| \|Hess_x H\| + \|D^3H\| \|\nabla F\|) \\
 &\leq 2(M\|x\| + \frac{1}{2}h_1)(M + \frac{1}{2}L), \\
 d(dF(X^H))(X^H) &= \langle Hess_x F(J\nabla H) - Hess_x H(J\nabla F), X^H \rangle, \\
 |d(dF(X^H))(X^H)| &\leq (M\|x\| + \frac{1}{2}h_1)(M\|x\| + h_1), \\
 d(d(dF(X^H)))(X^H)(\xi) &= \langle X^H, Hess_x F(JHess_x H(\xi)) - Hess_x H(JHess_x F(\xi)) \rangle \\
 &\quad + \langle JHess_x H(\xi), Hess_x F(J\nabla H) - Hess_x H(J\nabla F) \rangle \\
 &\quad - \langle X^H, D^3H(J\nabla F, \xi) \rangle,
 \end{aligned}$$

$$\begin{aligned}
 \|\nabla(d(dF(X^H))(X^H))\| &\leq \|\nabla H\|(2\|Hess_x F\| \|Hess_x H\| + \|D^3 H\| \|\nabla F\|) \\
 &\quad + \|Hess_x H\|(\|Hess_x F\| \|\nabla H\| + \|Hess_x H\| \|\nabla F\|) \\
 &\leq (M\|x\| + h_1)(M + \frac{1}{2}L) + M(M\|x\| + \frac{1}{2}h_1)
 \end{aligned}$$

Now taking into account all the above calculations we obtain explicit bounds on $|f_1(x)|$ and $\|\nabla f_1(x)\|$ in terms of $\|x\|$ as stated in the equations (3.43) and (3.44).

$$\begin{aligned}
 |f_1(x)| &\leq |d(dF(X^H))(X^H)| + \|\nabla H\|^2 + \|\nabla(d^{\mathbb{C}}F(X^H))\|^2 + \|\nabla(dF(X^H))\|^2 \\
 &\leq (M\|x\| + \frac{1}{2}h_1)(M\|x\| + h_1) + (M\|x\| + h_1)^2 + 2(M\|x\| + \frac{1}{2}h_1)^2 \\
 &\leq 4M^2\|x\|^2 + \frac{11}{2}Mh_1\|x\| + 2h_1^2 \\
 &\leq (2M\|x\| + \frac{2}{3}h_1)^2, \\
 \|\nabla f_1(x)\| &\leq \|\nabla(d(dF(X^H))(X^H))\| + \|\nabla(\|\nabla H\|^2)\| + \|\nabla(\|\nabla(d^{\mathbb{C}}F(X^H))\|^2)\| \\
 &\quad + \|\nabla(\|\nabla(dF(X^H))\|^2)\| \\
 &\leq (M\|x\| + h_1)(M + \frac{1}{2}L) + M(M\|x\| + \frac{1}{2}h_1) + 2M(M\|x\| + h_1) \\
 &\quad + 4(M\|x\| + \frac{1}{2}h_1)(M + \frac{1}{2}L) \\
 &\leq M(8M + \frac{5}{2}L)\|x\| + \frac{1}{2}h_1(7M + 3L).
 \end{aligned}$$

□

Lemma 4.16. *Let $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a Hamiltonian and $u : \mathbb{R} \times S^1 \rightarrow \mathbb{R}^{2n+1}$, $u(s) = (v(s), \eta(s))$ be a solution to the Floer equation corresponding to the constant almost complex structure J_0 . Then for the radial, plurisubharmonic function*

$$F(x) := \frac{1}{4}\|x\|^2$$

there exists a function $f : \mathbb{R} \times S^1 \rightarrow \mathbb{R}$, such that

$$\Delta(F \circ v) \geq f(s, t).$$

Moreover, if we assume the following

1. H satisfies (H2)
2. there exist constants $\mathbf{e}, \mathbf{v}, \boldsymbol{\eta} > 0$, such that

$$\|\nabla \mathcal{A}^H(u)\|_{L^2(\mathbb{R} \times S^1)}^2 \leq \mathbf{e} \tag{4.12}$$

$$\|v(s)\|_{L^2(S^1)} \leq \mathbf{v}, \quad |\eta(s)| \leq \boldsymbol{\eta}, \quad \forall s \in \mathbb{R} \tag{4.13}$$

3. for a constant $\varepsilon > 0$, $\Omega \subseteq \mathbb{R} \times S^1$ is a connected, open subset, such that

$$\forall s \in \Omega \quad \|\nabla \mathcal{A}^H(u(s))\| > \varepsilon.$$

If we represent $u(s) = (v(s), \eta(s)) \in C^\infty(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$, we can define f as in (3.38)

$$f(s, t) := \eta^2(s)(d_{v(s,t)}(dF(X^H))(X^H) - \|\nabla H(v(s, t))\|^2 - \|\nabla(d^{\mathbb{C}}F(X^H))(v(s, t))\|^2 - \|\nabla(dF(X^H))(v(s, t))\|^2) + \partial_s \eta(s) d_{v(s,t)}^{\mathbb{C}} F(X^H).$$

then $f \in W^{1,1}(\Omega)$ and $\|f\|_{W^{1,1}(\Omega)}$ can be bounded by constants depending only on the parameters of the Hamiltonian, the chosen value of ε and the constants $\mathbf{v}, \eta, \mathbf{e}$.

Proof. Having a quadratic bound on $|f_1(x)|$ and a linear bound on $\|\nabla f_1(x)\|$ in terms of $\|x\|$ given in Lemma 4.15. and moreover the relation between f and f_1 stated in (3.45), (3.46) and (3.47), we can now estimate the $W^{1,1}$ norm of f on Ω .

$$\begin{aligned} \|f\|_{L^1(\Omega)} &\leq \eta^2 \|f_1(v)\|_{L^1(\Omega)} + \|\partial_s \eta(s) d^{\mathbb{C}} F_{v(s,t)}(X^H)\|_{L^1(\Omega)} \\ \|\partial_t f\|_{L^1(\Omega)} &\leq \|\eta^2 \nabla f_1(v) + \partial_s \eta \nabla(d^{\mathbb{C}} F_{v(s,t)}(X^H))\|_{L^2(\Omega)} \|\partial_t v\|_{L^2(\Omega)} \\ \|\partial_s f\|_{L^1(\Omega)} &\leq \|\eta^2 \nabla f_1(v) + \partial_s \eta \nabla(d^{\mathbb{C}} F_v(X^H))\|_{L^2(\Omega)} \|\partial_s v\|_{L^2(\Omega)} \\ &\quad + 2\|\eta \partial_s \eta f_1(v)\|_{L^1(\Omega)} + \left\| d^{\mathbb{C}} F_v(X^H) \int_0^1 dH(\partial_s v) dt \right\|_{L^1(\Omega)} \end{aligned}$$

We will show the boundedness of f in $W^{1,1}(\Omega)$ by bounding each of the above expressions separately, using the relation

$$\|\nabla \mathcal{A}^H(u)\|_{L^2(\mathbb{R} \times S^1)}^2 = \left\| \int_{\mathbb{R}} \left(|\partial_s \eta(s)|^2 + \int_0^1 \|\partial_s v(s, t)\|^2 dt \right) ds \right\| \leq \mathbf{e}, \quad (4.14)$$

and Lemma 3.17 equation (3.34) which states that from the assumption

$$\forall s \in \Omega \quad \|\nabla \mathcal{A}^H(u(s))\| > \varepsilon.$$

it follows that one can bound the area of Ω by

$$area(\Omega) \leq \frac{\mathbf{e}}{\varepsilon^2} \quad (4.15)$$

Now using (3.43), (3.45), (4.9), (4.12), (4.13) and (4.15) we can estimate

$$\begin{aligned} \|f\|_{L^1(\Omega)} &\leq \eta^2 \|f_1(v(s))\|_{L^1(\Omega)} + \|\partial_s \eta d^{\mathbb{C}} F_{v(s)}(X^H)\|_{L^2(\Omega)} \\ &\leq \eta^2 area(\Omega) \max_{s \in [s_0, s_1]} (2M \|v(s)\|_{L^2(S^1)} + \frac{2}{3} h_1)^2 \\ &\quad + \sqrt{area(\Omega)} \|\partial_s \eta\|_{L^2(\Omega)} \frac{1}{2} \max_{s \in [s_0, s_1]} \left(\|v(s)\|_{L^2(S^1)} (M \|v(s)\|_{L^2(S^1)} + h_1) \right) \\ &\leq \eta^2 \frac{\mathbf{e}}{\varepsilon^2} (2M \mathbf{v} + \frac{2}{3} h_1)^2 + \frac{\sqrt{\mathbf{e}}}{\varepsilon} \sqrt{\mathbf{e}} \frac{1}{2} \mathbf{v} (M \mathbf{v} + h_1) \\ &= \frac{\mathbf{e}}{\varepsilon} \left(\frac{\eta^2}{\varepsilon} (2M \mathbf{v} + \frac{2}{3} h_1)^2 + \frac{1}{2} \mathbf{v} (M \mathbf{v} + h_1) \right) \end{aligned}$$

In view of (3.44), (4.11), (4.12), (4.13) and (4.15) we estimate

$$\begin{aligned}
 \|\eta^2 \nabla f_1(v) + \partial_s \eta \nabla(d^{\mathbb{C}} F_v(X^H))\|_{L^2(\Omega)} &\leq \eta^2 \|\nabla f_1(v)\|_{L^2(\Omega)} + \|\partial_s \eta \nabla(d^{\mathbb{C}} F_v(X^H))\|_{L^2(\Omega)} \\
 &\leq \max_{s \in [s_0, s_1]} \left(\eta^2 \sqrt{\text{area}(\Omega)} \|\nabla f_1(v(s))\|_{L^2(S^1)} + \|\partial_s \eta\|_{L^2([s_0, s_1])} \|\nabla(d^{\mathbb{C}} F_{v(s)}(X^H))\|_{L^2(S^1)} \right) \\
 &\leq \max_{s \in [s_0, s_1]} \left(\eta^2 \frac{\sqrt{\epsilon}}{\epsilon} (h_2 \|v(s)\|_{L^2(S^1)} + h_3) + \sqrt{\epsilon} \left(M \|v(s)\|_{L^2(S^1)} + \frac{1}{2} h_1 \right) \right) \\
 &\leq \sqrt{\epsilon} \left(\frac{\eta^2}{\epsilon} (h_2 \mathbf{v} + h_3) + M \mathbf{v} + \frac{1}{2} h_1 \right),
 \end{aligned}$$

Using (3.37), (3.40), (4.12), (4.13) and (4.15) we calculate

$$\begin{aligned}
 \|\partial_t v\|_{L^2(\Omega)} &\leq (\|\partial_s v\|_{L^2(\Omega)} + \eta \sqrt{\text{area}(\Omega)} \max_{s \in [s_0, s_1]} \|\nabla H(v(s))\|_{L^2(S^1)}) \\
 &\leq \sqrt{\epsilon} + \eta \frac{\sqrt{\epsilon}}{\epsilon} (M \max_{s \in [s_0, s_1]} \|v(s)\|_{L^2(S^1)} + h_1) \\
 &\leq \sqrt{\epsilon} \left(1 + \frac{\eta}{\epsilon} (M \mathbf{v} + h_1) \right),
 \end{aligned}$$

In view of (4.15), (4.12), (4.13) and (4.15) we obtain

$$\begin{aligned}
 \|\eta \partial_s \eta f_1(v)\|_{L^1(\Omega)} &\leq \eta \|\partial_s \eta\|_{L^1([s_1, s_0])} \max_{s \in [s_0, s_1]} \|f_1(v(s))\|_{L^1(S^1)} \\
 &\leq \eta \sqrt{\text{area}(\Omega)} \cdot \epsilon \left(2M \max_{s \in [s_0, s_1]} \|v(s)\|_{L^2(S^1)} + \frac{2}{3} h_1 \right)^2 \\
 &\leq \eta \frac{\epsilon}{\epsilon} \left(2M \mathbf{v} + \frac{2}{3} h_1 \right)^2,
 \end{aligned}$$

Combining (3.40), (4.9), (4.12), (4.13) and (4.15) we get

$$\begin{aligned}
 \left\| d^{\mathbb{C}} F_v(X^H) \int_0^1 dH(\partial_s v) dt \right\|_{L^1(\Omega)} &\leq \\
 &\leq \max_{s \in [s_0, s_1]} \|d^{\mathbb{C}} F_{v(s)}(X^H)\|_{L^1(S^1)} \|\nabla H(v(s))\|_{L^2(S^1)} \sqrt{\text{area}(\Omega)} \|\partial_s v\|_{L^2(\Omega)} \\
 &\leq \max_{s \in [s_0, s_1]} \left(\frac{1}{2} \|v(s)\|_{L^2(S^1)} (M \|v(s)\|_{L^2(S^1)} + h_1)^2 \right) \sqrt{\text{area}(\Omega)} \sqrt{\epsilon} \\
 &\leq \frac{\epsilon}{2\epsilon} \mathbf{v} (M \mathbf{v} + h_1)^2
 \end{aligned}$$

Now combining (3.46) and (3.47) with all the bounds above gives us that f is bounded in the $W^{1,1}(\Omega)$ by a constant, which depends only on the parameters of the Hamiltonian, the chosen value of ϵ and the constants $\mathbf{v}, \eta, \epsilon$. \square

ACKNOWLEDGMENTS

Obtaining a PhD degree is a long and challenging process and it would not be possible without the support of numerous people I had the pleasure to meet as a PhD student. They not only shared their mathematical knowledge with me, but equally importantly created good, motivating and supporting atmosphere.

First and foremost, I would like to thank my supervisor Federica Pasquotto for offering me the PhD position, trusting me with this PhD project and believing in my abilities to complete it. I would like to thank for all her patience, time, knowledge, guidance and support and that she was with me through thick and thin, whenever I have found holes and mistakes in the proofs and when I found means to fix them. Needless to say that without Federica this thesis would had never appeared.

Secondly I would like to thank my promotor, Rob van der Vorst, for suggesting me the idea of working on Rabinowitz Floer homology for the class of Hamiltonians admitting coercive Liouville vector fields, for which it is easier to obtain bounds on η for Floer trajectories along the non-compact hypersurface.

I would like to express my special thanks to Alberto Abbondandolo, whom I visited in Bochum on numerous occasions, for suggesting to me one of the most innovative ideas of this thesis namely how to bound the Floer trajectories near the non-compact hypersurface using Morse-Bott property.

I would also like to mention here other three researchers: Oliver Fabert, Will Merry and Alexander Fauck, who by sharing with me their time and knowledge contributed greatly to the project. Oliver's remark, that perturbing the almost complex structures only on an open set containing all the periodic orbits is enough to achieve transversality, moved the project forward, when we were completely stuck. Will helped me by explaining the role of the dependency of the almost complex structure on η in the transversality proof. Likewise, Alexander explained to me the role of waged spaces in the transversality proof and the relation between isolation of critical values and Morse-Bott property at the summer school in Bures-sur-Yvette and later during his visit in Amsterdam.

I am also very grateful to Kai Cieliebak and Urs Frauenfelder for hosting me in Augsburg, expressing their interest in the project (yes, it is very important for a young researcher) and turning my attention to the possibility of extending the theory of Rabinowitz Floer homology to include the restricted three body problem.

I would like to thank Berry Backer, Thomas Rot and Patrick Hafkenscheid for sharing their broad knowledge in functional analysis during all the numerous fruitful math discussions we had, which helped me out in my thesis significantly. They did not only help me with my math, but together with Eddie Nijholt, Chris Groothedde, Elena Queirolo, Gosse Overal and other PhD students and staff members created a unique, nerdy, crazy, friendly and supportive atmosphere at the VU math department. The PhD experience would not be the same without our Elfia and ball folk outings, lazy tea and cookie afternoons in our sunny office, card games and Stelling's beer meters.

One of the most amazing people I have met during the four years of my PhD at the VU was Beata Roś. We have first met when she invited me, at that time a total stranger, to live with her at Uilenstede for the first two weeks of my PhD. She soon became my close friend and later again a flatmate, who I had great fun with making our crazy ideas reality, whether it was pole dancing choreography or T-shirt painting, or spoiling each other with delicious cakes. Another amazing person I want to thank is Amber Borra, with whom I became friends much later in my PhD, but whose support, understanding and wise advice helped me survive difficult times.

Among all the marvelous friends I made during my PhD years, I want to give special thanks to the three amazing men: Gosse Overal, Frol Zapolsky and Kevin Sackel who were there for me when I was down, providing a warm hug to cheer me up or a vodka shot where the hug would not work :)

I would also like to thank to all the members of the symplectic geometry community who made me so welcome as a newbie researcher. I have met many great people there, among them Alvaro del Pino, Milena Pabiniak, Sasha Zamorzaev, Wolfgang Schmaltz, Yaniv Ganor, Doug Schultz, Ziva Myer, Catherine Cannizzo, Lucas Dahinden, Juan Ojeda, Igor Uljarevic, Roger Casals, Rob Castellano, Sara Venkatesh, Daniel Alvarez-Gavela, Kyler Siegel, Keon Choi, Sherry Gong, Agustín Moreno, Simon Schatz, Chris Gerig and Oleksandr Kravets. I would like to thank for all the beautiful moments we have shared that made my PhD experience so unique. I loved talking math with them during rainy afternoons in Toulouse, in cafes and restaurants by the canals in Amsterdam, overtwisted disks scribbled over cafe napkins, over a drink in graffiti bar in Berlin or during a math-girls-only meeting in a basement of a Parisian cafe. Somehow on these occasions they made me understand much more math than I got from most lectures. I will always remember how we have danced on the streets of Toulouse, how we have played psycho in Strasbourg, the fondue night in Zurich, our lazy talks during dinners in Münster, the summer nights with wine and guitar sitting on the grass in Bures-sur-Yvette and how we have explored Berlin's street art. We made such a big and integrated group that we had troubles fitting in any restaurant in Lisboa or in Paris. They were such interesting company that I blame them for losing my coat in Zurich, because I was so much into conversation with them, that I totally stopped paying attention to anything else going around :)

Last but not least, I would like to thank my family - my siblings, parents and grandfather for supporting me in my choice to take the PhD position and later being a great source of strength all the way through it. Even though living in different countries, I could always skype with them and share all my troubles and joys.

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